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Надійшла (received) 26.03.2019

Відомості про авторів / Сведения об авторах / Information about authors

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UDC 519:537.81

P. P. VOROBIYENKO, I. YU. DMITRIEVA

ANALYTIC SOLUTION OF THE DIFFERENTIAL MAXWELL SYSTEM AND ITS NUMERICAL IMPLEMENTATION

The differential Maxwell equations are solved constructively under the specific requirements in the spatial Cartesian coordinate system. The expressions of the unknown electromagnetic field vector intensities are found explicitly as the solutions of the general wave equation regarding all scalar components of the initially unknown vector field functions. The aforesaid equation is equivalent to the original Maxwell system. The present results are obtained using two new efficient operator analytical methods which application is shown also for the heterogeneous media. The numerical implementation for the particular case of the considered electrodynamic mathematical model is proposed here as well.

Key words: general wave equation, analytic operator methods, constructive solution, mathematical model.

П. П. ВОРОБІЄНКО, І. Ю. ДМИТРІЄВА

АНАЛІТИЧНЕ РОЗВ'ЯЗАННЯ ДИФЕРЕНЦІАЛЬНОЇ СИСТЕМИ МАКСВЕЛЛА ТА ЙОГО ЧИСЛОВА РЕАЛІЗАЦІЯ

Запропоновано конструктивне розв'язання диференціальних рівнянь Максвелла за певних умов у просторовій декартовій системі координат. Явні вирази шуканих вектор-функцій напруженості електромагнітного поля знайдено як розв'язки загального хвильового рівняння, еквівалентного вихідній максвелловській системі. Дані результати отримано завдяки двох нових ефективних операторних аналітичних методів, застосовування яких продемонстровано також для неоднорідних середовищ. Частковий випадок розглянутої математичної моделі електродинаміки реалізовано чисельно.

Ключові слова: загальне хвильове рівняння, аналітичні операторні методи, конструктивне розв'язання, математична модель.

П. П. ВОРОБИЕНКО, И. Ю. ДМИТРИЕВА

АНАЛИТИЧЕСКОЕ РЕШЕНИЕ ДИФФЕРЕНЦИАЛЬНОЙ СИСТЕМЫ МАКСВЕЛЛА И ЕГО ЧИСЛЕННАЯ РЕАЛИЗАЦИЯ

Предложено конструктивное решение дифференциальных уравнений Максвелла при определенных условиях в пространственной декартовой системе координат. Явные выражения искомых вектор-функций напряженностей электромагнитного поля найдены как решения общего волнового уравнения, эквивалентного исходной максвелловской системе. Данные результаты получены с помощью двух новых эффективных операторных аналитических методов, применимость которых продемонстрирована также для неоднородных сред. Частный случай рассмотренной математической модели электродинамики реализован численно.

Ключевые слова: общее волновое уравнение, аналитические операторные методы, конструктивное решение, математическая модель.

Introduction. The majority of current problems connected with the mathematical simulation of the electromagnetic phenomena in the guided structures and space are described by *the systems of partial differential equations* (PDEs). It is natural, because the original fundamental mathematical model in the electromagnetic field theory is represented by *the differential Maxwell system* [1].

The resent years of the computer expansion directed the scientific tendencies to the numerical and approximate study. However, the challenge of search of the new analytic solving techniques for the applied and engineering problems remains urgent till now. Really, the constructive methods enhance the development of the relevant theory; reveal the principal patterns, rules and laws which are formulated as the general statements and theorems.

Moreover, the engineering can not manage without appropriate computations which become essentially simpler dealing with the explicit solutions and exact formulae. The latter often promote faster getting of the required numerical result as well as determination of the calculated functional dependencies.

Therefore, even nowadays the creation and the improvement of the existing analytic procedures for the research of the respective applied problems are prior.

Analysis of recent studies. It is known that each vector function is determined uniquely by its scalar components which usually are *hidden* in some matrix problem.

Pursuing the twofold goal of analytical as well as effective numerical study, it is better to diagonalize at first the original matrix statement. This operation means the reduction of the initial system to the equivalent union of the equations or problems each of which depends on one scalar component of the unknown vector field function only.

Actually, investigation of the aforesaid scalar statements is either well-known, or the development of their solving procedures is incomparably simpler with respect to the aforesaid vector-matrix problem. Hence, the first step of the present challenge implies the creation of the analytic diagonalization technique which will allow getting the explicit results in the electromagnetic field theory and in the research of the corresponding engineering phenomena as well.

The classical diagonalization methods exist in the general algebra [2] and in *the ordinary differential equations* (ODEs) [3] theory. Still, until the recent years the systems of PDEs mathematically describing the electrodynamic processes were diagonalized mostly in the particular cases [4]. In general, the majority of the electrodynamic problems are solved using the approximate numerical methods of the previous century [5].

Nowadays, the practice confirms that the successful study of the electromagnetic field behavior basing only on the approximate numerical algorithms rather often becomes insufficient [6, 7]. Thus, the general necessity in the development of the effective analytic apparatus for this mentioned scientific branch appears again as extremely claimed.

Problem statement. The general detailed operator diagonalization procedure for a system of PDEs was suggested in [8], its preceding algorithmic scheme as the relevant flow-chart was described in [9], and the particular application to a specific engineering version of the differential Maxwell equations was considered in [10].

The aim of the present article concerns the analytic study of the differential Maxwell statement in the Cartesian coordinate system regardless the initial and boundary conditions in terms of a unified wave equation. The latter is derived by the above mentioned diagonalization technique, contains all scalar components of the unknown electromagnetic field vector intensities and is equivalent to the original system of PDEs. The given wave PDE is solved explicitly using the improved version of the integral transform method [11].

The research proposed in the paper generalizes essentially the results from [12] uniting the improved integral transform apparatus [13] and the aforesaid operator diagonalization procedure

Mathematical model. Let the classical differential Maxwell equations as the base mathematical model in the Cartesian coordinate system be given

$$\begin{cases} \mathbf{rot}\,\vec{H} = \partial_0 \vec{D} + \vec{i}; \\ \mathbf{rot}\,\vec{E} = -\partial_0 \vec{B}; \\ \operatorname{div}\vec{D} = \rho; \;\; \vec{D} = \varepsilon\vec{E}; \\ \operatorname{div}\vec{B} = 0; \;\; \vec{B} = \mu\vec{H}; \\ \vec{i} = \sigma\vec{E}. \end{cases} \tag{1}$$

In (1): $\vec{E},\vec{H} = \vec{E},\vec{H}\left(x,y,z,t\right)$ are the unknown electromagnetic field vector intensities with the scalar components $E_i, H_i = E_i, H_i(x,y,z,t),\ (i=\overline{1,3})$; $\vec{D},\vec{B} = \vec{D},\vec{B}\left(x,y,z,t\right)$ describe the electric and magnetic field induction respectively; $\vec{i} = \vec{i}\left(x,y,z,t\right)$ and $\rho = \rho\left(x,y,z,t\right)$ determine the current (charges) and charge density; σ, $\mu = \mu_0 > 0$, $\varepsilon = \varepsilon_0 > 0$ denote the specific conductivity, relative magnetic and electric permeability of the medium; $\partial_0 = \dfrac{\partial}{\partial t}$;

$\mathbf{rot}\ F_k = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_{k1} & F_{k2} & F_{k3} \end{bmatrix}$, $\operatorname{div}\vec{F}_k = \sum_{i=1}^{3}\partial_i F_{ki}$ represent the fundamental field operations, where

$\partial_1 = \dfrac{\partial}{\partial x},\ \partial_2 = \dfrac{\partial}{\partial y},\ \partial_3 = \dfrac{\partial}{\partial z},$ and $F_{ki} = F_{ki}(x, y, z, t)$ $(k = 1, 2;\ i = \overline{1, 3})$ are the corresponding scalar components of the electromagnetic field vector intensities $\vec{F}_1, \vec{F}_2 = \vec{F}_1, \vec{F}_2(x, y, z, t)$, $\vec{F}_1, \vec{F}_2 = \vec{E}, \vec{H}$.

Writing (1) in the equivalent form

$$\begin{cases} \mathbf{rot}\,\vec{H} = \varepsilon\partial_0\vec{E} + \sigma\vec{E}; \\ \mathbf{rot}\,\vec{E} = -\mu\partial_0\vec{H}; \\ \varepsilon\,\mathrm{div}\,\vec{E} = \rho; \\ \mu\,\mathrm{div}\,\vec{H} = 0. \end{cases} \Leftrightarrow \begin{cases} (\varepsilon\partial_0 + \sigma)\vec{E} - \mathbf{rot}\,\vec{H} = 0; \\ \mathbf{rot}\,\vec{E} + \mu\partial_0\vec{H} = 0; \\ ---------- \\ \mathrm{div}\,\vec{E} = \rho/\varepsilon; \\ \mathrm{div}\,\vec{H} = 0 \end{cases} \tag{2}$$

and using the diagonalization technique from [12] we reduce the first part of (2) to the following system

$$\begin{cases} \left(\mathbf{rot^2} + \hat{\partial}_0^2\right)\vec{E} = 0; \\ \mu\partial_0\left(\mathbf{rot^2} + \hat{\partial}_0^2\right)\vec{H} = 0 \end{cases} \tag{3}$$

with the partial differential operator

$$\hat{\partial}_0^2 = \mu\partial_0\left(\varepsilon\partial_0 + \sigma\right). \tag{4}$$

Further, the identity of the classical field theory $\mathbf{rot^2} = \mathbf{grad}\,\mathrm{div} - \Delta$, $\mathbf{grad}\,F\left(x, y, z, t\right) = \partial_1 F \cdot \vec{i} + \partial_2 F \cdot \vec{j} + \partial_3 F \cdot \vec{k}$, $\Delta = \sum_{i=1}^{3} \partial_i^2$ and the second part from (2) applied to (3) give

$$\begin{cases} \left(\hat{\partial}_0^2 - \Delta\right)\vec{E} = -\dfrac{1}{\varepsilon}\mathbf{grad}\rho; \\ \mu\partial_0\left(\hat{\partial}_0^2 - \Delta\right)\vec{H} = 0. \end{cases} \tag{5}$$

Then, applying the inverse operator $\partial_0^{-1} = \int \mathrm{d}t$ to the second equation from (5), which right-hand part is the zero vector, we arrive at the following system:

$$\begin{cases} \left(\hat{\partial}_0^2 - \Delta\right)\vec{E} = -\dfrac{1}{\varepsilon}\mathbf{grad}\rho; \\ \left(\hat{\partial}_0^2 - \Delta\right)\vec{H} = \dfrac{1}{\mu}\vec{g}\left(x, y, z\right). \end{cases} \tag{6}$$

The vector function $\vec{g}\left(x, y, z\right)$ from (6) is the integration result mentioned above determined by the physical character of the specific engineering problem statement.

It is clear that (6) represents diagonalization of (1) at the vector field level and can be written as the general vector wave PDE regarding both electromagnetic field intensities $\vec{E}$ and $\vec{H}$:

$$\left(\hat{\partial}_0^2 - \Delta\right)\vec{F}_k = \vec{f}_k,\ (k = 1, 2);\quad \vec{F}_1 = \vec{E},\ \vec{F}_2 = \vec{H};\quad \vec{f}_1 = -\frac{1}{\varepsilon}\mathbf{grad}\rho,\quad \vec{f}_2 = \frac{1}{\mu}\vec{g}\left(x, y, z\right). \tag{7}$$

Simultaneously, it is easy to notice that (7) can be rewritten in its scalar form including all components of $\vec{E}$ and $\vec{H}$:

$$\left(\hat{\partial}_0^2 - \Delta\right)F_{ki} = f_{ki},\ \left(k = 1, 2;\ i = \overline{1, 3}\right), \tag{8}$$

where

$$F_{1i} = \left\{E_i\right\}_{i=1}^{3},\ F_{2i} = \left\{H_i\right\}_{i=1}^{3};\quad f_{1i} = -\frac{1}{\varepsilon}\partial_i\rho,\quad f_{2i} = \frac{1}{\mu}g_i\left(x, y, z\right),\ \left(i = \overline{1, 3}\right);\ \mathbf{grad}\,\rho = \begin{bmatrix} \partial_1\rho\left(x, y, z, t\right) \\ \partial_2\rho\left(x, y, z, t\right) \\ \partial_3\rho\left(x, y, z, t\right) \end{bmatrix},$$

$$\vec{g} = \begin{bmatrix} g_1\left(x, y, z\right) \\ g_2\left(x, y, z\right) \\ g_3\left(x, y, z\right) \end{bmatrix}. \tag{9}$$

The explicit solution of (8), (9) and the formulations of the corresponding boundary problems describing investigated electrodynamic phenomena are simpler even in comparison with (7), not mentioning (1) where both the statements and the constructive results are almost unobservable.

Summarizing the ideas of the present paragraph we conclude that the original mathematical model (1) of the electromagnetic field behaviour is reduced to equivalent form (8), (9) which is easier to deal with. Hence, the exact study of (8), (9) presented below solves (1) explicitly as well.

Main analytic results. The explicit solution of (8), (9) is obtained using the improved method of integral transforms [11] on all spatial variables (x, y, z) but the time argument t, which is accepted as the main one. To simplify the further computation, the following table of symbols is proposed here: the new notations for the spatial variables from $\mathbf{R}_3$ are $x = x_1$, $y = x_2$, $z = x_3$; $K_i = K_i(x_i, p_i)$ stands for the kernel of the i-th integral transform on the argument x_i with the parameter p_i; and the direct integral operator transform is determined by the formula

$$S_i = \int_{a_i}^{b_i} K_i(x_i, p_i)\,dx_i \,,$$

where a_i, b_i are the initial and the endpoint respectively of the open integration contour L_i, $\left(i = \overline{1,3}\right)$. The values of a_i, b_i can be either finite or infinite real, or complex [11]. To analyze the influence of the integral transform on $\Delta F = \sum_{i=1}^{3} \partial_i^2 F$ the spatial coordinates are rewritten in the form $(x, y, z) = \left\{x_i, \left(i = \overline{1,3}\right)\right\}$, which makes the function look as follows: $F = F(x, y, z, t) = F\left(x_i, \left(i = \overline{1,3}\right); t\right)$.

The application of the i-th integral transform to ΔF followed by double integration by parts gives

$$\int_{a_i}^{b_i} \left(\partial_i^2 F\right) K_i(x_i, p_i)\,dx_i = \left(K_i(\partial_i F) - (\partial_i K_i) F\right)\Big|_{x_i = a_i}^{b_i} + \int_{a_i}^{b_i} \left(\partial_i^2 K_i\right) F dx_i = s_i\left(p_i, x_\nu \left(\nu \neq i;\ \nu = \overline{1,3}\right); t\right) +$$

$$+ \eta_i(p_i)\,{}_iF_{tr}\,, \quad \left(i = \overline{1,3}\right), \tag{10}$$

where

$$s_i = s_i\left(p_i, x_\nu \left(\nu \neq i;\ \nu = \overline{1,3}\right); t\right) = \left(K_i(\partial_i F) - (\partial_i K_i) F\right)\Big|_{x_i = a_i}^{b_i} = s_i(p_i, x_\nu, x_l, t),\ (\nu, l \neq i;\ l > \nu), \tag{11}$$

and the right or the left inferior index «tr» everywhere in this paper stands for conversion of the respective transform. Though in general $i, \nu, l = \overline{1,3}$, two last inequalities in (11) imply that ν, l take on only two values from the possible three. Moreover, the second item $\eta_i(p_i)\,{}_iF_{tr}$ in the right-hand part of (10) has the factor $\eta_i(p_i)$ dependent only on the parameter p_i of the i-th integral transform. This item results from the operation $\partial_i^2 K_i$, $\left(i = \overline{1,3}\right)$.

Further, the *incomplete* i-th transform of F on the argument x_i – ${}_iF_{tr}$ and the corresponding *complete* one on all spatial variables $(x, y, z) = \left(x_i,\ i = \overline{1,3}\right)$ – F_{tr} are determined by the following formulae:

$${}_iF_{tr} = {}_iF_{tr}\left(p_i, x_\nu \left(\nu \neq i;\ \nu = \overline{1,3}\right); t\right) = \int_{a_i}^{b_i} F\left(x_i \ \left(i = \overline{1,3}\right); t\right) K_i(x_i, p_i)\,\mathrm{d}\,x_i = \int_{a_i}^{b_i} F K_i\,\mathrm{d}\,x_i,\ \left(i = \overline{1,3}\right); \tag{12}$$

$$F_{tr} = F_{tr}\left(p_i \ \left(i = \overline{1,3}\right); t\right) = F_{tr}(p, t) = \left(\prod_{\substack{\nu=1 \\ \nu \neq i}}^{3} \int_{a_\nu}^{b_\nu} K_\nu\,\mathrm{d}\,x_\nu\right) {}_iF_{tr} =$$

$$= \int_{a_l}^{b_l} \int_{a_\nu}^{b_\nu} K_\nu(x_\nu, p_\nu) K_l(x_l, p_l)\,{}_iF_{tr}(p_i, x_\nu, x_l, t)\,\mathrm{d}\,x_\nu\,\mathrm{d}\,x_l\ ; \quad p = \bigcup_{i=1}^{3} p_i = (p_1, p_2, p_3),\ \left(i = \overline{1,3}\right), \tag{13}$$

where the conditions for ν, l are the same as in (11).

Problem (8), (9) is solved using the technique from [12, 13] developed in [11], but regardless of the specific boundary and initial conditions. Application of the required integral transforms (12), (13) to the spatial variables $(x, y, z) = \left(x_i,\ i = \overline{1,3}\right)$ reduces the original general wave PDE (8), (9) to the ODE of the second order regarding transforms dependent on the time argument t

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\sigma}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\Delta_{tr}}{\mu\varepsilon}\right) {}_{tr}F_{ki} = {}_{tr}f_{ki}^{*},\ \left(k = 1, 2;\ i = \overline{1,3}\right). \tag{14}$$

In (14),

$$\Delta_{tr} = \Delta_{tr}(p) = \sum_{i=1}^{3} \eta_i(p_i)\,; \quad {}_{tr}f_{ki}^{*} = {}_{tr\,t}\tilde{f}_{ki} + \sum_{i=1}^{3}\left(\prod_{\substack{\nu=1 \\ \nu \neq i}}^{3} \int_{a_\nu}^{b_\nu} K_\nu\,\mathrm{d}\,x_\nu\right) s_i\,; \quad {}_{tr}\tilde{f}_{ki} = \frac{{}_{tr}f_{ki}}{\mu\varepsilon};\ \left(i = \overline{1,3};\ k = 1,2\right), \tag{15}$$

the symbols in (15) are either introduced in (10) – (13) or described at the beginning of the present paragraph, and

$$\left(\prod_{\substack{\nu=1\\ \nu\neq i}}^{3}\int_{a_\nu}^{b_\nu} K_\nu \,\mathrm{d}\,x_\nu\right) s_i = \int_{a_l}^{b_l}\int_{a_\nu}^{b_\nu} K_\nu\left(x_\nu, p_\nu\right) K_l\left(x_l, p_l\right) s_i\left(p_i, x_\nu, x_l, t\right)\mathrm{d}\,x_\nu\,\mathrm{d}\,x_l\ ,\quad \left(\nu, l\neq i;\ l>\nu;\ i,\nu,l=\overline{1,3}\right).$$

The general solution for (14), (15)

$$_{tr}F_{ki} = {}_{tr}F_{ki}\left(t, p\right) = C_1\left(t, p\right)\exp\left(\omega_1 t\right) + C_2\left(t, p\right)\exp\left(\omega_2 t\right) \tag{16}$$

is sought by the method of variation of constants [3], where the unknown functions $C_j\left(t\right) = C_j\left(t, p\right)$, $\left(j=1, 2\right)$ represent the solution of the system

$$\begin{cases} C_1^{'}\exp\left(\omega_1 t\right) + C_2^{'}\exp\left(\omega_2 t\right) = 0; \\ C_1^{'}\omega_1\exp\left(\omega_1 t\right) + C_2^{'}\omega_2\exp\left(\omega_2 t\right) = {}_{tr}f_{ki}^{*}; \end{cases}\qquad C_j^{'} = \frac{\partial C_j}{\partial t},\quad \left(j=1, 2\right),$$

and looks like

$$C_{1,2}\left(t, p\right) = \pm\frac{1}{\omega_1-\omega_2}\int\exp\left(-\omega_{1,2}t\right){}_{tr}f_{ki}^{*}\left(t, p\right)\mathrm{d}\,t + C_{1,2}^{*}\left(p\right)\ . \tag{17}$$

In (17), the unknown functions $C_{1,2}^{*}\left(p\right)$ are found from the corresponding transformed initial conditions of the specific boundary value problem, which is the mathematical simulation of the studied physical or engineering phenomenon.

It should be noted, that in (16), (17)

$$\omega_{1,2} = \frac{1}{2}\left(-\frac{\sigma}{\varepsilon}\pm\sqrt{D}\right) \tag{18}$$

are the roots of the performance (characteristic) equation $\omega^2 + \frac{\sigma}{\varepsilon}\omega - \frac{\Delta_{tr}}{\varepsilon\mu} = 0$ with the discriminant

$$D = \left(\frac{\sigma}{\varepsilon}\right)^2 + \frac{4\Delta_{tr}}{\varepsilon\mu}\ . \tag{19}$$

The substitution of (17) for (16) gives the required general explicit solution of (14)

$$_{tr}F_{ki} = {}_{tr}F_{ki}\left(t, p\right) = \sum_{j=1}^{2} C_j\left(t, p\right)\exp\left(\omega_j t\right) = \sum_{j=1}^{2}\exp\left(\omega_j t\right)\left(\frac{\left(-1\right)^{j+1}}{\sqrt{D}}\int\exp\left(-\omega_j t\right){}_{tr}f_{ki}\left(t, p\right)\mathrm{d}\,t + C_j^{*}\left(p\right)\right), \tag{20}$$

where ω_j, $\left(j=1, 2\right)$ and D are determined in (18), (19). The direct check confirms that (20) undoubtedly represents the general solution of (14).

The further inversion of the original inverse transform regarding (20) gives the function

$$F_{ki} = F_{ki}\left(x, y, z, t\right) = \prod_{l=1}^{3} S_l^{-1}\,{}_{tr}F_{ki}\ ,\quad \left(k=1, 2;\ i=\overline{1,3}\right), \tag{21}$$

where S_l^{-1}, $\left(l=\overline{1,3}\right)$ are the integral transforms inverse to the initially used one, and $_{tr}F_{ki}$ is from (20).

Hence, the required explicit solution (21) of the general scalar wave PDE (8), (9) is obtained and describes all scalar components of the electromagnetic field vector intensities. Formulae (20), (21) can be effectively used when the mathematical modeling for the engineering / physical problem statement is specified and the appropriate exact analytic result is sought for.

Heterogeneous media. It is known that when studying the real electrodynamic engineering problems in heterogeneous media it is often acceptable to consider the coefficients σ, ε, μ in the relevant systems of PDEs as piece-wise constant functions, where ε, μ can be also negative [14, 15]. Such structures are very important in the metamaterials which are used for the construction of the absolutely new types of antennae, filters, etc. However, the necessary condition of the electrodynamic equations states that the electromagnetic wave propagation is possible only in those media where $\operatorname{sgn}\varepsilon = \operatorname{sgn}\mu$. In such cases, the reverse waves appear and the parameters of the medium become controlled owing to the magnetic field modification [14, 15].

Therefore, turning back to (1), (2), the electromagnetic field parameters σ, ε, μ represent now the piece-wise constant functions regarding the spatial coordinates $\left(x, y, z\right)$

$$\begin{bmatrix}\sigma\\ \varepsilon\\ \mu\end{bmatrix} = \begin{bmatrix}\sigma\\ \varepsilon\\ \mu\end{bmatrix}\left(x, y, z\right) = \sum_{l=1}^{m}\begin{bmatrix}{}_l\sigma\\ {}_l\varepsilon\\ {}_l\mu\end{bmatrix}\delta\left(x, y, z;\ {}_lV\right), \tag{22}$$

where: $\delta(x,y,z;\ {}_lV)=\begin{cases}1, & (x,y,z)\in {}_lV;\\ 0, & (x,y,z)\in V\setminus {}_lV,\end{cases}$ $\left(l=\overline{1,l}\right)$ is *the Cronecker symbol*; $V=\bigcup\limits_{l=1}^{m} {}_lV$ is the finite union of various media ${}_lV={}_lV(x,y,z)$; ${}_lV\bigcap {}_{\nu}V=\varnothing,\ l\neq\nu,\ \left(l,\nu=\overline{1,m}\right)$, and each of them has its own special field characters ${}_l\sigma,\ {}_l\varepsilon,\ {}_l\mu=\mathrm{const}$. In terms of (22), the scalar action of the partial differential operators from (1), (2) looks like

$$\partial_j\begin{bmatrix}\sigma\\ \varepsilon\\ \mu\end{bmatrix}F_{ki}(x,y,z,t)=\partial_j\left(\sum_{l=1}^{m}\begin{bmatrix}{}_l\sigma\\ {}_l\varepsilon\\ {}_l\mu\end{bmatrix}\delta(x,y,z;\ {}_lV)\right)F_{ki}(x,y,z,t)=$$

$$=\left(\sum_{l=1}^{m}\begin{bmatrix}{}_l\sigma\\ {}_l\varepsilon\\ {}_l\mu\end{bmatrix}\delta(x,y,z;\ {}_lV)\right)\partial_jF_{ki}(x,y,z,t),\ \ \left(k=1,2;\ i=\overline{1,3};\ j=\overline{0,3}\right). \tag{23}$$

Dealing with the vector field function which is formed by its scalar components, (23) is applied sequentially to each scalar summand in the expression

$$\vec{F}_k(x,y,z,t)=\sum_{i=1}^{3}F_{ki}(x,y,z,t)\vec{e}_i\ ;\ \ \vec{e}_1=\vec{i},\ \vec{e}_2=\vec{j},\ \vec{e}_3=\vec{k}\ ;\ \ (k=1,2).$$

Namely,

$$\partial_j\begin{bmatrix}\sigma\\ \varepsilon\\ \mu\end{bmatrix}\vec{F}_k(x,y,z,t)=\partial_j\left(\sum_{l=1}^{m}\begin{bmatrix}{}_l\sigma\\ {}_l\varepsilon\\ {}_l\mu\end{bmatrix}\delta(x,y,z;\ {}_lV)\right)\vec{F}_k(x,y,z,t)=\left(\sum_{l=1}^{m}\begin{bmatrix}{}_l\sigma\\ {}_l\varepsilon\\ {}_l\mu\end{bmatrix}\delta(x,y,z;\ {}_lV)\right)\partial_j\vec{F}_k(x,y,z,t)=$$

$$=\left(\sum_{l=1}^{m}\begin{bmatrix}{}_l\sigma\\ {}_l\varepsilon\\ {}_l\mu\end{bmatrix}\delta(x,y,z;\ {}_lV)\right)\partial_j\left(\sum_{i=1}^{3}F_{ki}(x,y,z,t)\vec{e}_i\right)=\left(\sum_{l=1}^{m}\begin{bmatrix}{}_l\sigma\\ {}_l\varepsilon\\ {}_l\mu\end{bmatrix}\delta(x,y,z;\ {}_lV)\right)\left(\sum_{i=1}^{3}\partial_jF_{ki}(x,y,z,t)\vec{e}_i\right). \tag{24}$$

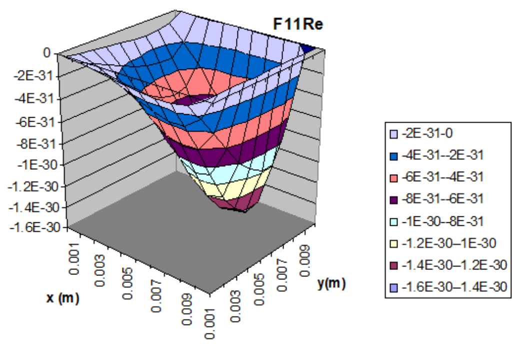

Fig. 1 – The real part of the first scalar component of the electric vector intensity.

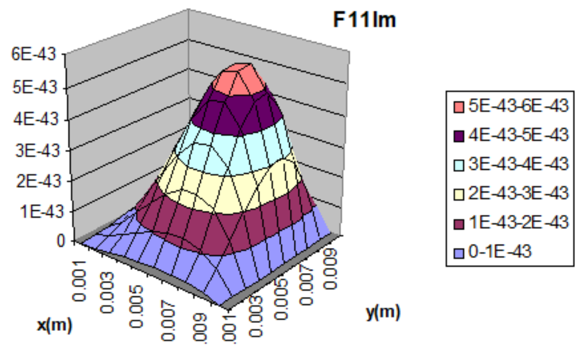

Fig. 2 – The imaginary part of the first scalar component of the electric vector intensity.

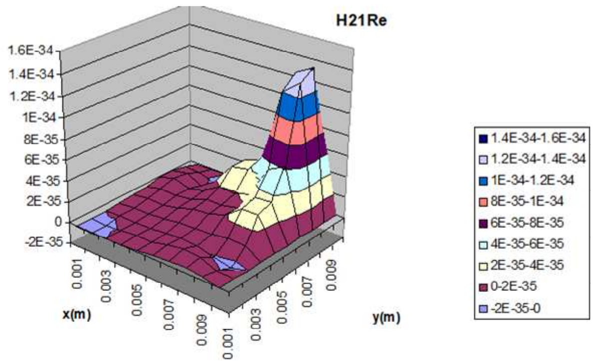

Fig. 3 – The real part of the first scalar component of the magnetic vector intensity.

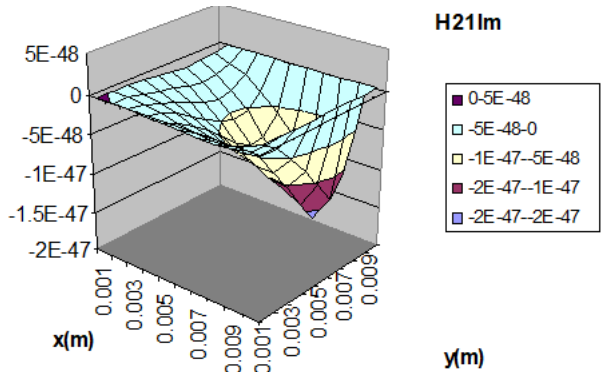

Fig. 4 – The imaginary part of the first scalar component of the magnetic vector intensity.

It is obvious that the final action of the partial differential operators (24) remain here the same, as for the homogeneous medium, accurate within the complete operation result regarding all considered media's bodies, where each of them has its own features ${}_l\sigma,\ {}_l\varepsilon,\ {}_l\mu=\mathrm{const}$. Mathematically it means the correctness of the diagonalization operator procedure [8, 9] for the PDEs systems with the piece-wise constant coefficients. The last fact gives possibility using the method [8, 9], [12] in the heterogeneous electromagnetic field media as well.

Partial numerical implementation. The particular case of the constructive results from this article was realized partially for (1), (2) when the electromagnetic field vector intensities were assumed to be harmonic in the time argument t, i.e. $\vec{E} = \vec{E}(x, y, z)\exp(i\omega t)$, $\vec{H} = \vec{H}(x, y, z)\exp(i\omega t)$, where $i = \sqrt{-1}$, and ω was the vibration frequency.

Several frames of the computer simulation for the specific engineering problem regarding study of the spatial rectangular resonator are shown in Figs. 1 – 4 below.

Conclusions. The generalized operator diagonalization procedure evoked by the Gauss algebraic method was applied here to the analytic investigation of the finite dimensional systems of PDEs representing the original mathematical models in the electromagnetic field theory. The proposed apparatus allowed reducing the relevant differential Maxwell equations in the Cartesian coordinate system to the general wave PDEs regarding all scalar components of the electromagnetic field vector intensities.

The aforesaid technique gives opportunity to obtain the required unified scalar equations regardless the specific boundary and initial conditions. The last fact essentially simplifies the mathematical simulation in terms of the corresponding boundary value problem whose explicit solution is got here by the generalization of the integral transform method.

Future research concerns the complete numerical implementation and computer modeling for the various types of the engineering problem mentioned in the preceding paragraph. The heterogeneous media will be considered as well.

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Received (поступила) 16.03.2019

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УДК 532.526;542

Г. А. ВОРОПАЕВ, А. А. БАСКОВА

МОДЕЛИРОВАНИЕ ПРОЦЕССА ПЕРЕХОДА В ТРУБАХ СО СПИРАЛЬНЫМ ГОФРИРОВАНИЕМ

Проведено прямое численное моделирование неизотермического течения на начальных участках гладкой трубы и трубы с гофрированными вставками различной геометрии при переходных числах Рейнольдса. Проанализировано возникновение и развитие колебательных процессов в гладкой трубе и трубах с гофрированными вставками. Исследованы особенности структуры вихревого движения и изменения гидродинамических параметров в гофрированных вставках разной геометрии. Определена степень влияния угла наклона гофрирования к оси трубы на гидродинамические процессы в потоке в следе после гофрированной вставки.

Ключевые слова: течение в трубе, переходные числа Рейнольдса, вихревые возмущения, неизотермический поток, прямое и витое гофрирование, частичное гофрирование.

Г. О. ВОРОПАЄВ, О. О. БАСКОВА

МОДЕЛЮВАННЯ ПРОЦЕСУ ПЕРЕХОДУ В ТРУБАХ ІЗ СПІРАЛЬНИМ ГОФРУВАННЯМ

Проведено пряме чисельне моделювання неізотермічної течії на початкових ділянках гладкої труби і труби з гофрованими вставками різної геометрії при перехідних числах Рейнольдса. Проаналізовано виникнення та розвиток коливальних процесів в гладкій трубі і трубах з гофрованими вставками. Досліджені особливості структури вихрового руху та зміни гідродинамічних параметрів в гофрованих вставках різної геометрії. Визначено ступінь впливу кута нахилу гофрування до осі труби на гідродинамічні процеси в сліді після гофрованої вставки.

Ключові слова: течія в трубі, перехідні числа Рейнольдса, вихрові збурення, неізотермічний потік, пряме та вите гофрування, часткове гофрування.

G. A. VOROPAIEV, A. A. BASKOVA

MODELING OF TRANSITION PROCESS IN TUBES WITH SPIRAL CORRUGATION

A direct numerical simulation of non-isothermal flow in the initial sections of a smooth pipe and a pipe with corrugated inserts of various geometry at transitional Reynolds numbers was carried out. The formation and development of a three-dimensional unsteady flow structure in a smooth tube and tubes with corrugated inserts were analyzed. The structure of the flow and the nature of changes in the hydrodynamic parameters inside the corrugated insert were investigated. The influence of corrugation geometry on the flow mixing and the nature of hydrodynamic parameters distribution in the flow after the corrugated insert were analyzed. An increase in hydraulic losses of up to 9% in tubes with corrugated inserts compared to a smooth tube was observed and its dependence on the geometry of corrugation was studied.

Key words: flow in a tube, transitional Reynolds numbers, vortex disturbances, non-isothermal flow, straight and spiral corrugation, partial corrugation.

Введение. Общеизвестно, что термогидравлические характеристики внутренних течений при *переходных числах Рейнольдса* определяются *режимом течения (ламинарный* или *турбулентный*), формирующимся в устройствах в зависимости от условий на входе, качества и вида геометрии поверхности, а также изменяемости физических параметров текущей среды. В протяженных устройствах при изменении проходных сечений режим течения может непредсказуемо меняться, что приводит к резкому изменению термо-гидравлических характеристик теплообменных устройств.

Вместе с тем, теоретический классический анализ течения в трубах постоянного диаметра не дает пороговых чисел Рейнольдса потери устойчивости течения, в отличие от устойчивости пограничных слоев, где теоретический линейный анализ потери устойчивости качественно и количественно подтверждается экспериментом.