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Limits of Julia Sets for Sums of Power Maps and Polynomials

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LIMITS OF JULIA SETS FOR SUMS OF POWER MAPS AND POLYNOMIALS

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1. Abstract

Suppose $f_{n,c}$ is a complex-valued mapping of one complex variable given by

$$f_{n,c}(z) = z^n + p(z) + c,$$

where p is a polynomial such that $p(0) = 0$ and c is a complex parameter such that $|c| < 1$. We provide necessary and sufficient conditions that the geometric limit, as $n \to \infty$, of the set of points that remain bounded under iteration by $f_{n,c}$ is the disk of radius 1 centered at the origin.

2. Background

Complex dynamics is a relatively new field, considering much of what is known in dynamics has only been discovered since the 1980s. The roots of dynamics, however, lie in functional analysis in the early 19th century, where the groundwork was laid by the English mathematician Charles Babbage in his 1815 paper *An Essay Towards the Calculus of Functions* [2]. Continuing through the rest of the 19th and early 20th century, contributions from mathematicians such as Ernst Schröder, Lucjan Böttcher, Pierre Fatou, and Gaston Julia continued to bolster the foundation of dynamics. According to an overview of the history of dynamics, (see [1]), Ernst Schröder was the first mathematician to explore iteration, which is the backbone of modern dynamics. In 1871, he published a paper that discussed the concept of conjugating mappings to much simpler maps to allow for easier study of the dynamics (see [9]). Classifying when these conjugations could occur was worked on in part by

I would like to graciously thank Scott Kaschner for his work on this project and his mentoring throughout this wonderful introduction to mathematical research.

Böttcher in the early 1900s (see [3]). Most of these conjugations only occurred locally, which limited analysis to small subsets of the mapping's domain. As research in the field of dynamics progressed, however, it became increasingly important to understand how iterations affect points in a global context. Fatou and Julia's contributions included new definitions for useful sets of tools to examine iterations globally. Two of the main objects of their studies are now aptly named the Fatou and Julia Sets of a mapping, the latter of which is an integral component of this paper's research.

After the 1940s, research on dynamics tapered off, mainly because the field is deeply entwined with a geometric intuition that quickly became intractable. It was not until the 1980s that the field resurfaced as computers became powerful enough to allow for visualization and experimentation. Throughout the 1980s and 1990s, the field of dynamics truly blossomed into what it is today due to the work of mathematicians such as Douady, Hubbard, Milnor, and McMullen (see [4], [6], [7], and [8] for examples).

2.1. **Notation and Terminology.** In general, the study of complex dynamics explores iterative mappings in the complex plane. Before a closer examination of dynamics and our research, it will be helpful to explain certain notation and terminology used throughout the paper.

Complex numbers are of the form $z = x + iy$, where the real numbers x and y are respectively the real and imaginary parts of z, with $i = \sqrt{-1}$. Because complex numbers contain a real and an imaginary part, visualizing complex numbers is rather simple: associate the real part, x, of a complex number to the x-axis of a Cartesian plane and the imaginary part, y, to the y-axis. By doing this for the entire set of complex numbers, which is denoted by $\mathbb{C}$, the **complex plane** is generated.

The complex numbers are a two-real-dimensional set; consequently, we must use two-dimensional notions of size for the numbers and the distance between them. When looking at only real numbers (i.e. numbers without any imaginary part), the notion of size and distance is simple because it is a one-dimensional ordered field. The size of a real number is just the absolute value of the number, and the distance between two numbers is the absolute value of the difference of the two numbers. However, the complex numbers are not an ordered field, and therefore absolute value cannot be used. Instead, we define the **modulus** of a complex point $z = x + iy$ to be $|z| = \sqrt{x^2 + y^2}$. We use the same notation as real absolute value because the modulus of a real number is equal to the absolute value. Similar to absolute value, the modulus of a complex number is the distance from its representative point to the origin at $z = 0$ in the complex plane. Furthermore, the distance between two non-zero complex points is similar: take $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$; the distance between the two points is $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Studying how these points in the complex plane behave under repeated compositions of a mapping is a main focus of complex dynamics. In our research, we studied complex mappings, which take complex numbers and map them to other complex numbers. Let $f : \mathbb{C} \to \mathbb{C}$ denote such a mapping. Repeatedly composing a mapping with itself is called **iteration**. For example, the k^{th} iterate of a mapping is the mapping composed with itself k times. We will use the following notation to describe this:

$$f^k = f \circ \cdots \circ f.$$

The sequence of iterates of a point z_0 is as follows:

$$\{z_0, z_1 = f(z_0), z_2 = f^2(z_0), ...\},$$

where z_k denotes the k^{th} iterate of z_0 by f. For example, $z_2 = f(f(z_0)) = f(z_1)$. These sequences of iterates, called **orbits**, define the mapping's dynamics.

In order to gain deeper intuition of the dynamics of a mapping, being able to understand iterations of many points at once is helpful. One of the mathematical objects that we used heavily in our research does that by separating all the points whose orbits stay bounded from the points whose orbits stay unbounded.

Definition 1. The **filled Julia set**, $K(f)$, for some mapping f, is the set of all points whose orbits stay bounded under iteration by f.

Understanding how the Julia set is built will help motivate our result later, so let us explore a simple example. In this example, the following standard notation will be used. Let $a \in A$ denote that a is an element of the set A. Let $A \subset B$ denote that a set A is a subset of a set B, meaning that all the elements of A are contained within B. Let $\mathbb{D}$ denote the unit disk, which is the set of all complex numbers with modulus less than one, that is, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Subsequently, $\overline{\mathbb{D}}$ will denote the closed unit disk, which is the set of all complex numbers with modulus less than or equal to one. In our proofs, we subscript the disk notation to denote a disk of radius r as follows,

$$\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\} \quad \text{and} \quad \overline{\mathbb{D}}_r = \{z \in \mathbb{C} : |z| \leq r\}.$$

For simplicity, we omit the subscript when $r = 1$ to align with the standard notation of the unit disk.

Example 1. Let $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$. If we take $z_0 = 1$, iterating it will leave $z_0 = 1$ with its orbit as follows:

$$\{z_0 = 1, z_1 = 1 = f(z_0), z_2 = 1 = f^2(z_0), ...\}.$$

Because the iterates remain at $z_0 = 1$, the point $z_0 = 1$ is called fixed. Moreover, any $z \in \mathbb{C}$ such that $|z| = 1$ will have its orbit remain on the unit circle, meaning

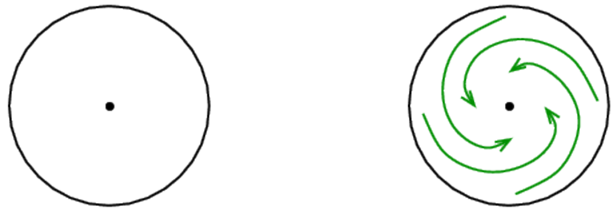

FIGURE 1. Left: Points of modulus 1 remain modulus 1 after iteration. Right: Points less than 1 in modulus iterate down to zero by f.

that each iterate's modulus will be one (see Figure 1). This happens because $|z| = 1$ implies $|f(z)| = |z^2| = |z|^2 = 1^2 = 1$. Now, if we take $z_0 = 0.5$, its orbit converges to 0:

$$\{z_0 = 0.5, z_1 = 0.25 = f(z_0), z_2 = 0.125 = f^2(z_0), ...\}.$$

Not surprisingly, any $z \in \mathbb{C}$ such that $|z| < 1$ will have an orbit that converges to 0 (see Figure 1). Finally, any $z \in \mathbb{C}$ with $|z| > 1$ will have an unbounded orbit. Take $z_0 = 2$ for example. Then the orbit is as follows:

$$\{z_0 = 2, z_1 = 4 = f(z_0), z_2 = 16 = f^2(z_0), ...\}.$$

Compiling all of this, we can describe the filled Julia set for f. Any point that is less than or equal to 1 in modulus has a bounded orbit while any point that is greater than 1 in modulus has an unbounded orbit. Therefore, $K(f)$ is the closed unit disk, that is,

$$K(f) = \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

This example is rather trivial in comparison to the majority of filled Julia sets. By simply adding a complex parameter c, the filled Julia sets of the family of mappings $f(z) = z^2 + c$ become much more intricate. In Figure 2, notice the complicated nature

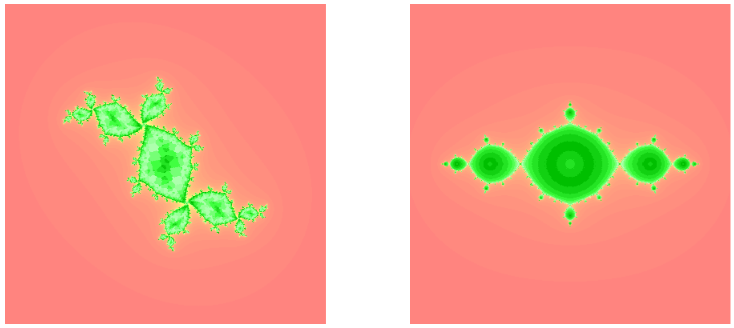

FIGURE 2. Left: Douady Rabbit with $c = -0.1 + 0.75i$. Right: Basilica with $c = -1$.

of two famous filled Julia sets generated from this family of mappings. These complicated structures are one of the main reasons why research in the field of dynamics experienced such a long hiatus in the mid-20$^{\text{th}}$ century — computers are needed to generate their images, and this technology did not become widely available until the 1980s.

The final concept plays a minor role in this paper's research; however, it plays a major role in defining the dynamics of mappings, so it is worth discussing. A **fixed point** of a mapping f is a point that gets mapped to itself by f; in other words, z_* is a fixed point if and only if $f(z_*) = z_*$. This will be true for any number of iterations by f on the fixed point. There are multiple types of fixed points: attracting, repelling, and indifferent. First, let us examine attracting fixed points. If the modulus of the derivative[1] of the mapping at the fixed point is less than one, then the fixed point is called attracting; in other words,

$$|f'(z_*)| < 1,$$

where z_* is a fixed point of f. The attracting fixed point theorem brings this behavior into a more geometric sense. If the fixed point is attracting, then orbits nearby will converge to it (after a number of iterations). In Example 1, we saw that $f(z) = z^2$

[1]The derivative of a complex mapping f at a point z_0 is defined to be the limit $f'(z_0) = \lim_{z \to z_0}(f(z) - f(z_0))/(z - z_0)$.

had an attracting fixed point at $z_* = 0$. First, we confirm that z_* is a fixed point of f by testing $f(0) = 0^2 = 0$. Second, we confirm that z_* is attracting by finding the derivative of f at z_* and testing whether it is less than one in modulus:

$$|f'(z_*)| = |2z_*| = 0 < 1.$$

Now, notice how any point of modulus less one will eventually iterate to $z_* = 0$. The set of all points of modulus less than one are therefore in the so-called *basin of attraction* for z_*. The **basin of attraction** for an attracting fixed point is the set of all points whose orbits converge to that fixed point.

A fixed point is called repelling when the modulus of the derivative at the fixed point is greater than one. The repelling fixed point theorem characterizes the resulting behavior more geometrically: after some iterations, points near the repelling fixed point will be pushed away from the fixed point. However, this behavior is more complicated than that of attracting fixed points. The theorem only guarantees that points near the fixed point will pushed away up to a certain number of iterations (specific to each point). After that, the points can stay away from or return to the fixed point, depending on the global dynamics of the mapping itself. In Example 1, it is simple to show that $z_* = 1$ is a repelling fixed point for $f(z) = z^2$. Most points close to z_* iterate away from z_*; however, there are certain points near z_* that exhibit the more complicated behavior previously mentioned. Take $z_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ for example. It is close to z_* and has a modulus of one. After two iterations, $f^2(z_0) = z_2 = -1$, which is far from z_*. Nevertheless, the next iteration of z_0 returns back to z_* as $f(-1) = 1 = z_*$. After two iterations, the repelling nature of the fixed point has lost its power and the global properties of the mapping brought z_0 back near to the repelling fixed point.

Finally, a fixed point is defined to be indifferent when the modulus of the derivative at the fixed point is equal to one. Geometrically, the indifferent fixed point could either be what is called weakly repelling or weakly attracting, or neither of the two. If it is weakly repelling or attracting, then points close to it either diverge from or converge to the fixed point very slowly, meaning that it takes many iterations to diverge or converge.

3. Motivation

Let us explore the result from which this paper's research sprang. In 2012, S. Hruska Boyd and M. Schulz proved the following theorem (see [5]):

Theorem 3.1. *Let $c \in \mathbb{C}$ be a complex parameter. For the family of complex mappings, $P_{n,c}(z) = z^n + c$, where n is a positive integer,*

(1) *If $|c| > 1$, then*
$$\lim_{n\to\infty} K(P_{n,c}) = S^1,$$
where S^1 is the unit circle.

(2) *If $|c| < 1$, then*
$$\lim_{n\to\infty} K(P_{n,c}) = \mathbb{D}.$$

Before discussing the result in further detail, we will again need another notion of distance since we are dealing with limits of sets. In this case, it will be Hausdorff distance.

Definition 2. For two sets A, B in a metric space (X, d), the **Hausdorff distance** $d_{\mathcal{H}}(A, B)$ between the sets is defined as

$$\begin{aligned} d_{\mathcal{H}}(A,B) &= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \\ &= \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}. \end{aligned}$$

At first, this definition may seem daunting; however, the idea behind it is simple. Take two sets A, B in a metric space (X, d). A metric space is just some set X that has a well-defined notion of distance d on the set, such as the complex numbers whose defined distance is in terms of modulus. Begin by walking around in set A: at each point a in A, find the shortest distance from a to set B and add it to a list. Then find the supremum, or the least upper bound, of the list. Intuitively, this will be the largest of the minimum distances from set A to set B, though the maximum may technically not exist[2]. Do the same process while walking around in B to find the largest minimum distance from set B to set A. While it may seem counterintuitive, more often than not, the largest minimum distance from B to A is not equal to the largest minimum distance from A to B. Refer to Figure 3. The

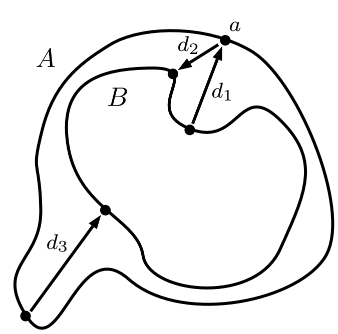

FIGURE 3. Hausdorff Distance

largest minimum distance from B to A is d_1, terminating at a; however, note that

[2]If a maximum of a set is not included in the set, then the least upper bound of the set, called the supremum, will suffice.

the shortest distance from a back to set B, denoted by d_2, is much smaller than the true largest minimum distance from A to B, denoted by d_3, due to the shape of B. Finally, to finish the process, compare the largest minimum distance from A to B with the largest minimum distance from B to A. Whichever distance is larger is the Hausdorff distance between the sets.

Now, let us discuss Theorem 3.1. We focus on the second part of the theorem, which concerns the family of mappings $P_{n,c}(z) = z^n + c$ where $|c| < 1$. Notice as the degree n of $P_{n,c}$ is increased, $|z|^n$ gets either very large for $|z| > 1$ or very small for $|z| < 1$. As long as z is chosen to be inside the unit disk, for sufficiently large n, the orbit of z will remain bounded, similar to Example 1. If z is chosen to be outside unit disk, then its orbit will be unbounded. Putting these two facts together, the filled Julia set, $K(P_{n,c})$, approaches $\overline{\mathbb{D}}$ for sufficiently large n (see Figure 4).

It is not surprising that this phenomena can be easily disrupted. Just by incorporating fixed nonconstant terms to the map, the dynamics at the limit become much more complicated. Consider the following example in which we simply add a quadratic term to the formula for $P_{n,c}$.

Example 2. Let $f : \mathbb{C} \to \mathbb{C}$ by $f_{n,c}(z) = z^n + z^2 + c$. Pick a parameter c_0 in the unit disk but not in the Mandelbrot set. The Mandelbrot set, denoted by $\mathcal{M}$, is the set of all complex parameters c whose associated quadratic polynomial $(z^2 + c)$ have a connected filled Julia set, meaning that it is all in one piece instead of being

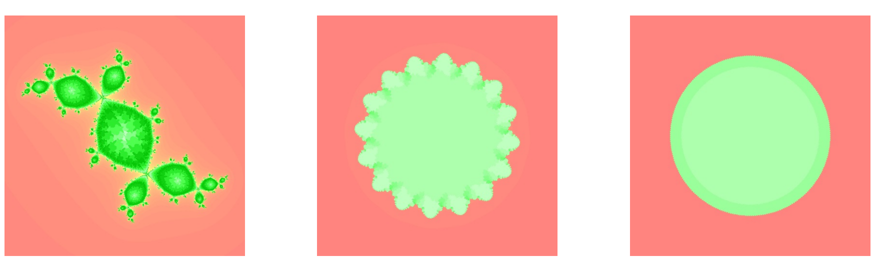

FIGURE 4. From left to right, $K(P_{n,c})$ where $n = 2$, $n = 16$, and $n = 256$

made up of multiple smaller pieces that do not intersect. By picking a parameter that is not in the Mandelbrot set, the filled Julia set of $g(z) = z^2 + c_0$ is disconnected, thereby giving a strong chance of finding points of small modulus whose orbits remain unbounded. Consequently, with the $z^2 + c_0$ terms in the formula for $f_{n,c_0}(z)$, many points z will leave the disk after one iteration; then, as before, their orbits will remain unbounded as z^n expedites their increasing moduli. See Figure 5, where only a slight adjustment of the parameter c to be outside of the Mandelbrot set causes almost every single point's orbit to remain unbounded. For reference, the different shades of blue represent the number of iterates a point requires for its modulus to exceed a given threshold (chosen large enough to guarantee an unbounded orbit). This number of iterates is often interpreted as the "speed" with which an orbit "escapes." Darker shades correspond to a larger number of iterations required for an orbit escape from the filled Julia set, and black corresponds to the filled Julia set.

We can further generalize this family of mappings with the addition of a general degree $d \geq 1$ polynomial p such that $p(0) = 0$; define $f_{n,c} : \mathbb{C} \to \mathbb{C}$ as

$$f_{n,c}(z) = z^n + p(z) + c.$$

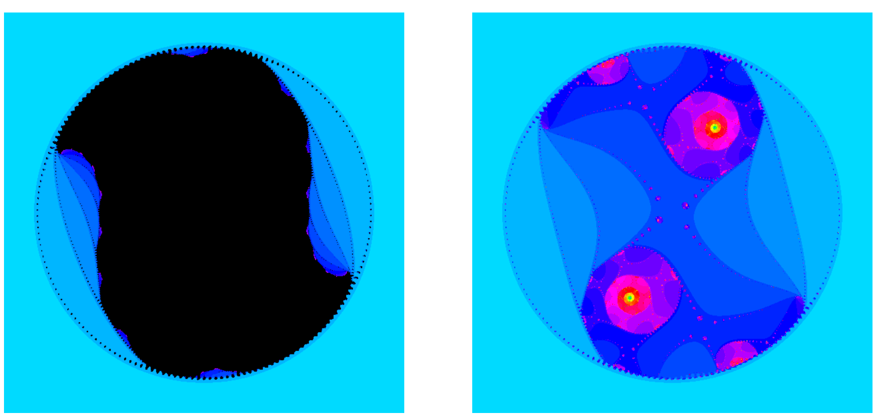

FIGURE 5. $K(f_{200,c_i})$ with $p(z) = z^2$, $c_1 = 0.25 + 0.25i \in \mathcal{M}$ (left), and $c_2 = 0.45 + 0.25i \notin \mathcal{M}$ (right)

Characterizing when this family of maps follows Boyd and Schulz's result is the main focus of this paper.

4. Main Research

First, notice that $f_{n,c} = z^n + p(z) + c$ is the sum of a power map (whose power we increase in the limit) and a fixed degree d polynomial $q(z) = p(z) + c$. Second, if we pick z_0 such that $|z_0| > 1$, we can still expect the image (the first iterate) of z_0 by $f_{n,c}$ to have large modulus for large enough n. Guided by this intuition, we find a generalization of Lemma 3.1 from [5], which provides an upper bound for the size of the filled Julia sets of $f_{n,c}$.

Lemma 4.1. *For any $c \in \mathbb{C}$ and any $\epsilon > 0$, there is an $N \geq 2$ such that for all $n \geq N$,*

$$K(f_{n,c}) \subset \mathbb{D}_{1+\epsilon}$$

We will provide the proof of this and all further lemas and theorems in the next section. When $c = 0$, the situation is fairly simple. Without the addition of a constant, we need only concern ourselves with the possibility that the image of the unit disk under the polynomial p is large. If the image of the disk is large compared to the disk, then there are points within the disk that will iterate outside of the disk by $f_{n,0}$. In those cases, the filled Julia set of $f_{n,0}$ cannot be the whole unit disk, thus diverging from the result from Boyd-Schulz. With the addition of a parameter c, the dynamics become more complicated due to a relationship between p and c. It turns out that we can find a lower bound on c in relation to p that will guarantee the existence of at least one point in the disk that will iterate outside of the disk by $f_{n,c}$. This idea is formalized in the following lemma.

Lemma 4.2. *Let* $m := \inf_{|z|=1} |p(z)|$*, and let* $r = \min\{m, 1\}$*. Then for any* $c \in \mathbb{C}$ *such that* $|c| > 1 - r$*, there exist* $z_0 \in \mathbb{D}$ *and* $N \geq 0$ *such that for all* $n \geq N$*,*

$$z_0 \notin K(f_{n,c}).$$

Our aim is to use c to translate the image of the unit circle by p outside the unit disk. Since the unit circle is the boundary of the disk, the image of the unit circle is the boundary of the image of the disk, so even if just part of the image of the unit circle lies outside the disk, we have points inside the disk whose images by p lie outside the disk. Once this happens, we can make n large enough that these points will have unbounded orbits by $f_{n,c}$. To achieve this, we first find the smallest point in modulus on the image of the circle by p. Translating this point outside the disk will guarantee that at least one point in the image of the disk lies outside of the disk. If we take $|c|$ to be at least 1 minus the modulus of that smallest point, then we will have the desired translation, and the result of the lemma follows.

The interplay between p and c demonstrated by Lemma 4.2 makes it clear that any hope of understanding the dynamics of $f_{n,c}(z) = z^n + p(z) + c$ in terms of only c is lost. Consequently, the limiting behavior of $K(f_{n,c})$ depends more sensitively on the dynamics of q. The details of this dependence and the preceding lemmas lead to our main result, which is the following theorem.

Theorem 4.1. *Let* $q\colon \mathbb{C} \to \mathbb{C}$ *be the map given by* $q(z) = p(z) + c$*, and let* $f_{n,c}(z) = z^n + q(z)$*. Under the Hausdorff metric,*

$$\lim_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}}$$

if and only $q(\mathbb{D}) \subset \mathbb{D}$*.*

The theorem provides a biconditional statement that supplies necessary and sufficient conditions for the limit of the filled Julia set of $f_{n,c}$ to converge to the unit disk. Consider the following example.

Example 3. Suppose $f_{n,c}(z) = z^n + az^d + c$, where $a \in \mathbb{C}$. To analyze $f_{n,c}$ as previously mentioned, we look to the dynamics of $p(z) = az^d$. The mapping p is dynamically simple in the sense that its filled Julia set is a scaled disk whose radius is related to the modulus of a and the degree d:

$$K(p) = \overline{\mathbb{D}}_{|a|^{(-1/d)}}.$$

Additionally, the image of the unit circle by $p(z) = az^d$ is simply a circle whose radius is scaled by $|a|$. For example, if $p(z) = 4z^2$, then the image of the unit circle, $p(S^1)$, would be a circle of radius 4. Because the image of the unit circle is a circle, we can use Lemma 4.2 to help find exactly which c will translate the image outside of the disk. These c will determine when $q(\mathbb{D}) \subset \mathbb{D}$ and, consequently, whether the filled Julia set of $f_{n,c}$ will converge to the disk or be strictly contained within it, (i.e. there is at least one point in $\mathbb{D}$ that is not in $K(f_{n,c})$).

There are three cases that arise: when $|a| > 1$, $|a| = 1$, or $|a| < 1$. The first two are simple and do not require the lemma for support. If $|a| > 1$, then $\mathbb{D} \subset p(\mathbb{D}) = \mathbb{D}_{|a|}$. Consequently, the choice of c does not matter because even with $c = 0$,

$$q(\mathbb{D}) = p(\mathbb{D}) = \mathbb{D}_{|a|} \not\subseteq \mathbb{D}.$$

Applying Theorem 4.1 with Lemma 4.1, we conclude for all $c \in \mathbb{C}$ that if $|a| > 1$, then

$$\lim_{n\to\infty} K(f_{n,c}) \subsetneq \overline{\mathbb{D}}.$$

The case where $|a| = 1$ is identical except when $c = 0$. Note that when $c = 0$,

$$q(\mathbb{D}) = p(\mathbb{D}) = \mathbb{D} \subset \mathbb{D},$$

so Theorem 4.1 applies and we get $\lim_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}}$.

The final case in which $|a| < 1$ requires support from Lemma 4.2.

The lemma constructs a minimum radius r based on the minimum modulus of the image of the unit circle $p(S^1)$. When choosing parameters c such that $|c| > 1 - r$, the image of the unit circle is translated enough so that part of it lies outside the unit disk. Generally speaking, this r is an overestimate for certain c due to the geometry of the image of the unit circle; however, in this case, because the image of the unit circle is a circle of radius $|a|$ and the modulus of a circle map is constant and equal to its radius, $r = |a|$ will be an exact minimum radius such that for all c, where $|c| > 1 - |a|$, the image of the unit circle will be translated enough so that it is not contained in the disk. Subsequently, $q(\mathbb{D}) \not\subseteq D$. Again applying Theorem 4.1 with help from Lemma 4.1, we conclude that for all $|c| > 1 - |a|$ where $|a| < 1$,

$$\lim_{n\to\infty} K(f_{n,c}) \subsetneq \overline{\mathbb{D}}.$$

Alternatively, if we take $|c| \leq 1 - |a|$, then the image of the disk is not translated outside of the disk. In other words, $q(\mathbb{D}) \subset \mathbb{D}$ and Theorem 4.1 applies directly:

$$\lim_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}}.$$

The results of this example can be compiled neatly into a corollary of Theorem 4.1. The proof of the corollary is very similar to that of Theorem 4.1 and is therefore left to the reader.

Corollary 4.1.1. *Let $f_{n,c}(z) = z^n + az^d + c$ and $r = \min\{|a|, 1\}$. Under the Hausdorff metric,*

(1) *For $c \in \overline{\mathbb{D}}_{1-r}$,*
$$\lim_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}}$$

(2) *For $c \in \mathbb{D} \setminus \overline{\mathbb{D}}_{1-r}$,*
$$\lim_{n\to\infty} K(f_{n,c}) \subsetneq \overline{\mathbb{D}},$$
if the limit exists.

5. Proof of Main Results

Proof of Lemma 4.1. The goal of this proof is to show that any point outside of the unit disk, that is, $z \in \mathbb{C}\setminus\overline{\mathbb{D}}_{1+\epsilon}$, will produce an orbit that is unbounded under iteration by $f_{n,c}(z) = z^n + p(z) + c$ for large enough n. To achieve this, we will use induction. Before we can begin induction, however, it will be helpful to have a way to control the size of $p(z)$, so, let $z \in \mathbb{C} \setminus \overline{\mathbb{D}}_{1+\epsilon}$. One can find a degree $d-1$ polynomial $\hat{p}$ such that
$$p(z) = z^d\,\hat{p}(1/z).$$
Note that $1/z \in \mathbb{D}_{1-\epsilon}$ since we chose $|z| > 1+\epsilon$, so we can apply the Maximum Modulus Principle on $\hat{p}$. The Maximum Modulus Principle guarantees the existence of some $M \geq 0$ such that outputs of a holomorphic mapping (a mapping that is complex and differentiable) will always be less than M in modulus; so for $\hat{p}$, there is an $M \geq 0$ such that $|\hat{p}(z)| \leq M$ for all $z \in \mathbb{C}$. Subsequently, we can find a maximum modulus for p as such: $|p(z)| \leq M|z|^d$. Now we choose $B > \max\{1, |c|, M\}$ to be a lower boundary such that $|f_{n,c}(z)| > B$. Then choose $N > d+2$ large enough that
$$|z|^N > \max\{4B, 2M|z|^d + |c|\}$$

and $B^{N-d-1} > 3$. Now start the induction. Let $n \geq N$. We claim $|f_{n,c}^m(z)| \geq B^m$ for all $m \geq 1$. Observe that

$$\begin{aligned} |f_{n,c}(z)| &= |z^n + p(z) + c| \geq |z|^n - |p(z) + c| \\ &\geq |z|^n - M|z|^d - |c| \\ &\geq |z|^n - \frac{1}{2}|z|^n - |c| \geq 4B - 2B - B = B. \end{aligned}$$

Now suppose for some $m \geq 1$, we know $|f_{n,c}^m(z)| \geq B^m$. Let $z_m = f_{n,c}^m(z)$, and note that $|p(z_m) + c| \leq B|z_m|^d + B \leq |z_m|^{d+2} < |z_m|^N$. Then for any $n \geq N$,

$$\begin{aligned} \left|f_{n,c}^{m+1}(z)\right| &= |z_m^n + p(z_m) + c| \geq |z_m^n| - |p(z_m) + c| \\ &\geq |z_m|^n - M\,|z_m|^d - |c| \\ &\geq B^{mn} - B^{md}B - B \\ &\geq B^{m+1}\left(B^{mn-m-1} - B^{md-m} - 1\right) \geq B^{m+1}, \end{aligned}$$

where the last inequality follows (eventually) from the fact that $B^n > 3B^{d+1}$. By induction, we have that $|f_{n,c}^m(z)| \geq B^m$ for all $m \geq 1$. Since $B > 1$, the orbit of z under $f_{n,c}$ is not bounded. Thus, $z \notin K(f_{n,c})$. □

Proof of Lemma 4.2. The goal of this proof is to show that as long as $c \in \mathbb{D} \backslash \mathbb{D}_{1-r}$, we can always find one point in the disk with an unbounded orbit for $f_{n,c}(z) = z^n + p(z) + c$. To do this, it will be sufficient to find a z_0 that leaves the disk after one iterate. To begin, since $c \in \mathbb{D} \backslash \mathbb{D}_{1-r}$, we have that $\frac{1}{2}(|c| + r - 1) > 0$. From this it follows that $\frac{1}{2}(3r - |c| + 1) < r$. Using the open mapping theorem[3], we know $\mathbb{D}_r \subset p(\mathbb{D})$. Therefore, there exists $z_0 \in \mathbb{D}$ such that $|p(z_0)| = \frac{1}{2}(3r - |c| + 1)$. Choose N large enough that $|z_0|^N < \frac{1}{4}(3r - |c| + 1)$. Now we have everything we need to

[3]The open mapping theorem states that if f is a holomorphic mapping, then it maps open subsets of its domain in $\mathbb{C}$ to open subsets of $\mathbb{C}$.

show that the first iterate of z_0, denoted by z_1, lies outside the disk. For $n \geq N$,

$$\begin{aligned}
|f_{n,c}(z_0)| &= |z_0^n + p(z_0) + c| \\
&\geq |c| + |p(z_0)| - |z_0|^n \\
&> |c| + \frac{1}{2}(3r - |c| + 1) - \frac{1}{4}(3r - |c| + 1) \\
&> \frac{3}{4}|c| + \frac{1}{4}(3r + 1) \\
&> \frac{3}{4}(1 - r) + \frac{1}{4}(3r + 1) \ > \ 1.
\end{aligned}$$

Thus, $|f_{n,c}(z_0)| > 1$. Then by Lemma 4.1, we may choose N large enough so that

$$K(f_{n,c}) \subset \mathbb{D}_{1+\frac{1}{2}(|z_1|-1)}.$$

It follows that the orbit of z_0 does not remain bounded. □

Proof of Theorem 4.1. To prove a biconditional statement, we must prove both $q(\mathbb{D}) \subset \mathbb{D}$ implies $\lim_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}}$ and $\lim_{n\to\infty} K(f_{n,c}) = \overline{\mathbb{D}}$ implies $q(\mathbb{D}) \subset \mathbb{D}$. Suppose first that the image of $\mathbb{D}$ under q is contained in $\mathbb{D}$. Let

$$s = \max_{z\in\overline{\mathbb{D}}}\{|q(z)|\},$$

so $0 < s < 1$.

To see $\lim_{n\to\infty} K(f_{n,c}) = \mathbb{D}$, let $0 < \epsilon < 1 - s$ and $\mathcal{K}$ be a compact set such that $\mathbb{D}_{1-\epsilon} \subset \mathcal{K} \subset \mathbb{D}$. We choose $\mathcal{K}$ to be compact to guarantee it is closed and bounded, meaning we include the boundary of the set and all the points inside it are close together. Since $\mathcal{K}$ is compact, we may choose this N so that for any $z \in D_{1-\epsilon}$, we have $|z|^n < 1 - \epsilon - s$. Then for any $z \in \mathbb{D}_{1-\epsilon}$, we also have

$$\begin{aligned} |f_{n,c}(z)| &\leq |z|^n + |q(z)| \\ &< (1 - \epsilon - s) + s \; < \; 1 - \epsilon. \end{aligned}$$

It follows that the orbit of any point in $\mathbb{D}_{1-\epsilon}$ never leaves the disk $\mathbb{D}_{1-\epsilon}$. Thus, we have $\mathbb{D}_{1-\epsilon} \subset K(f_{n,c})$. Combining this with Lemma 4.1, for any $\epsilon > 0$, we may choose N large enough such that

$$\mathbb{D}_{1-\epsilon} \subset K(f_{n,c}) \subset \mathbb{D}_{1+\epsilon}.$$

We now prove the converse, using proof by way of contradiction. Suppose the image of $\mathbb{D}$ under q is not contained in $\mathbb{D}$, so $q(\mathbb{D}) \setminus \mathbb{D}$ is nonempty. By the open mapping theorem, $q(\mathbb{D})$ is an open set, so $q(\mathbb{D}) \setminus \mathbb{D}$ is also open. Thus, there is some $z_0 \in \mathbb{D}$ such that $|q(z_0)| > 1$. In this case one can pick N large enough that for any $n \geq N$, we have $|f_{n,c}(z_0)| > 1$, so for all $n \geq N$, $z_0 \notin K(f_n)$ The result follows. $\square$

6. Conclusion

Because complex dynamics is such a new field relative to most fields in mathematics, there is still much to be discovered. This paper's research is significant in that the results are brand new and therefore expand the horizons of the field, particularly in the realm of polynomial dynamics. Providing conditions for when a large family of mappings follows a certain characteristic is one of the best ways to expand the field. This paper provides conditions for geometric limits of the filled Julia sets to equal the unit disk for the family of mappings, which consist of the sum of power maps and fixed-degree polynomials. These conditions are proved to be necessary and sufficient, and therefore provide a complete summary of the characteristic.

For further study, one question concerns the limits of the filled Julia sets for this family of mappings when they do not converge to a unit disk. The conjecture is that if the limits exist, then the filled Julia sets will converge to some complicated, difficult-to-define structure. While difficult, this problem seems tractable. It is expected, based on extensive experimental evidence, that such limits do exist, in general. Beyond families of polynomial mappings, one could also extend this study to rational mappings. Another avenue for study would be to explore the situation with mappings of more than one complex variable.

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