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Circumferences of 3-connected claw-free graphs, II

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Abstract

For a graph H, the circumference of H, denoted by $c(H)$, is the length of a longest cycle in H. It is proved in [4] that if H is a 3-connected claw-free garph of order n with $\delta \geq 8$, then $c(H) \geq \min\{9\delta - 3, n\}$. In [11], Li conjectured that every 3-connected k-regular claw-free graph H of order n has $c(H) \geq \min\{10k - 4, n\}$. Later, Li posed an open problem in [12]: how long is the best possible circumference for a 3-connected regular claw-free graph? In this paper, we study the circumference of 3-connected claw-free graphs without the restriction on regularity and provide a solution to the conjecture and the open problem above. We determine five families $\mathcal{F}_i$ ($1 \leq i \leq 5$) of 3-connected claw-free graphs which are characterized by graphs contractible to the Petersen graph and show that if H is a 3-connected claw-free graph of order n with $\delta \geq 16$, then one of the following holds:

(a) either $c(H) \geq \min\{10\delta - 3, n\}$ or $H \in \mathcal{F}_1$.
(b) either $c(H) \geq \min\{11\delta - 7, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2$.
(c) either $c(H) \geq \min\{11\delta - 3, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.
(d) either $c(H) \geq \min\{12\delta - 10, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$.
(e) if $\delta \geq 23$ then either $c(H) \geq \min\{12\delta - 7, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$.

This is also an improvement of the prior results in [4, 10, 13, 14].

Keywords: Claw-free graph; Circumference; Minimum degree; Petersen graph

1 Introduction

Graphs considered in this paper are finite and loopless. A graph is called a *multigraph* if it contains multiple edges. A graph without multiple edges is called a *simple graph* or simply a graph. As in [1], $\kappa'(G)$ and $d_G(v)$ denote the edge-connectivity of G and the degree of a vertex v in G, respectively. The minimum degree of a graph G is denoted by $\delta(G)$ or δ. For a vertex $v \in V(G)$, let $E_G(v)$ be the set of edges in G incident with v. Thus, when G is a simple graph, $|E_G(v)| = d_G(v)$. An edge cut X of a graph G is *essential* if each of the components of $G - X$ contains an edge. A graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut of size less than k. A vertex set $U \subseteq V(G)$ is called a *covering* of G if every edge of G is incident with a vertex in U. The minimum number of vertices in a covering of G is called the *covering number* of G and denoted by $\beta(G)$. An edge $e = uv$ is called a *pendant edge* if $\min\{d_G(u), d_G(v)\} = 1$.

A *trail* T is a finite sequence $T = u_0e_1u_1e_2u_2 \cdots e_ru_r$, whose terms are alternately vertices and edges, with $e_i = u_{i-1}u_i$ ($1 \leq i \leq r$), where the edges are distinct. A trail T is a *closed trail* if $u_0 = u_r$

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and is called a (u, v)-trail if $u = u_0$ and $v = u_r$. A trail or closed trail T in a graph G is called a *spanning trail* (ST) or a *spanning closed trail* (SCT) of G if $V(G) = V(T)$ and is called a *dominating trail* (DT) or a *dominating closed trail* (DCT) if $E(G - V(T)) = \emptyset$. The family of graphs with SCTs is denoted by $\mathcal{SL}$. A graph G is called a *DCT graph* if G has a DCT.

The circumference of a graph H, denoted by $c(H)$, is the length of a longest cycle in H. A graph H is *claw-free* if H does not contain an induced subgraph isomorphic to $K_{1,3}$. In this paper, we will be concerned with the circumference of 3-connected claw-free graphs.

In [14], Matthews and Sumner proved that every 2-connected claw-free graph H of order n has $c(H) \geq \min\{n, 2\delta + 4\}$. Li, et al. [13] proved that every 3-connected claw-free graph H of order n has $c(H) \geq \min\{n, 6\delta - 15\}$. Solving a conjecture posed in [13], we proved the following.

Theorem 1.1 ([4])**.** *If H is a 3-connected claw-free graph of order n and $\delta \geq 8$, $c(H) \geq \min\{n, 9\delta - 3\}$.*

Theorem 1.1 is best possible in the sense that if $H_r = L(G_r)$ where G_r is obtained from the Petersen graph P by adding $r > 0$ pendant edges at each vertex of P, then $c(H_r) = 9\delta(H_r) - 3$.

For regular claw-free graphs, Li posed the following conjecture in [11].

Conjecture 1.2 (Li, Conjecture 6 [11])**.** *Every 3-connected k-regular claw-free graph H on n vertices has $c(H) \geq \min\{10k - 4, n\}$.*

In [12], Li restated the conjecture with a different lower bound on $c(H)$.

Conjecture 1.3 (Li, Conjecture 5.17 [12])**.** *Every 3-connected k-regular claw-free graph H on n vertices has $c(H) \geq \min\{12k - 7, n\}$.*

It was stated in [12] that Conjecture 1.3 was from [11]. However, Conjecture 1.2 is the only conjecture in [11]. We don't know why "$10k - 4$" is changed to "$12k - 7$" in Conjecture 1.3. Maybe it is more proper to treat them as open problems. In fact, Li posed an open problem in [12].

Problem 1.4 (Li, Problem 5.18 [12])**.** *How long is the best possible circumference for a 3-connected regular claw-free graph?*

Note that H_r mentioned above is a non-regular claw-free graph. These conjectures and the open problem suggest a more general problem: how long is the best possible circumference for a 3-connected claw-free graph H if $H \neq H_r$?

In this paper, using much improved techniques employed in [4], we provide solutions to these open problems and conjectures. Our results are given in next section.

2 Main results and Ryjáček's closure concept

For a graph G, the line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. As we know that all line graphs are claw-free and a connected line graph $H \neq K_3$ has a unique graph G with $H = L(G)$. We call G the preimage graph of H. Ryjáček [16] defined the closure $cl(H)$ of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H as long as this is possible, and H is said to be *closed* if $H = cl(H)$.

Theorem 2.1. *(Ryjáček [16]). Let H be a claw-free graph and $cl(H)$ its closure. Then*

(a) $cl(H)$ is well defined, and $\kappa(cl(H)) \geq \kappa(H)$;

(b) there is a K_3-free simple graph G such that $cl(H) = L(G)$;

(c) for every cycle C_0 in $L(G)$, there exists a cycle C in H with $V(C_0) \subseteq V(C)$.

Let P be the Petersen graph. Let Φ_a and Φ_b be two connected K_3-free simple graphs. Let $P(\Phi_a, \Phi_b)$ be an essentially 3-edge-connected K_3-free simple graph obtained from P by replacing a vertex v_a in P by Φ_a and replacing a vertex v_b in P by Φ_b, and by adding at least $r > 0$ pendant edges at each vertex of $V(P) - \{v_a, v_b\}$ and subdividing m edges of P for $m = 0, 1, \cdots, 15$.

Let Π_a and Π_b be two families of K_3-free graphs. Define $\mathcal{P}(\Pi_a, \Pi_b)$ be the family of graphs below: $\mathcal{P}(\Pi_a, \Pi_b) = \{G \mid G = P(\Phi_a, \Phi_b)$ where $\Phi_a \in \Pi_a$ and $\Phi_b \in \Pi_b \}$ (see Fig. 2.1. for examples).

Here is a list of families of K_3-free graphs that will be used for Π_a or Π_b.

- Let $\mathcal{K}_{1,r}$ be the family of stars $K_{1,r}$ with $r \geq 1$ edges.
- Let $\mathcal{K}_{2,r}$ be the family of spanning connected subgraphs of $K_{2,r}$ for some $r \geq 2$.
- Let $\mathcal{Q}_t$ be the family of K_3-free connected simple graphs G with $\alpha'(G) = t$.

Note that $K_{t,s} \in \mathcal{Q}_t$ for $t \leq s$ and $\mathcal{K}_{t,s} = \mathcal{Q}_t$ for $t \in \{1, 2\}$ and $s \geq t$ (see Proposition 3.3).

For essentially 3-edge-connected K_3-free simple graphs, we define the following families:

- $\mathcal{P}_1 = \mathcal{P}(\mathcal{K}_{1,r}, \mathcal{K}_{1,r})$.
- $\mathcal{P}_2 = \mathcal{P}(\mathcal{K}_{2,r}, \mathcal{K}_{1,r})$.
- $\mathcal{P}_3 = \mathcal{P}(\mathcal{Q}_3, \mathcal{K}_{1,r})$.
- $\mathcal{P}_4 = \mathcal{P}(\mathcal{K}_{2,r}, \mathcal{K}_{2,r})$.
- $\mathcal{P}_5 = \mathcal{P}(\mathcal{Q}_4, \mathcal{K}_{1,r})$.
- $\mathcal{P}_6 = \mathcal{P}(\mathcal{Q}_3, \mathcal{K}_{2,r})$.

For each i ($1 \leq i \leq 6$), we define a family $\mathcal{F}_i$ of 3-connected claw-free graphs according to $\mathcal{P}_i$:

$\mathcal{F}_i = \{H$: H is a 3-connected claw-free graph with $cl(H) = L(G)$ and $G \in \mathcal{P}_i\}$.

Here is our main result.

Theorem 2.2. *Let H be a 3-connected claw-free simple graph of order n with $\delta(H) \geq 16$.*

(a) Either $c(H) \geq \min\{10\delta(H) - 3, n\}$ or $H \in \mathcal{F}_1$.

(b) Either $c(H) \geq \min\{11\delta(H) - 7, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2$.

(c) Either $c(H) \geq \min\{11\delta(H) - 3, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

(d) Either $c(H) \geq \min\{12\delta(H) - 10, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$.

(e) If $\delta(H) \geq 23$, then either $c(H) \geq \min\{12\delta(H) - 7, n\}$ or $H \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$.

The theorem below shows a relationship between DCTs and Hamiltonian cycles.

Theorem 2.3. *(Harary and Nash-Williams [9]). The line graph $H = L(G)$ of a graph G with at least three edges is Hamiltonian if and only if G has a DCT.*

For a graph G, define

$$\overline{\sigma}_2(G) = \min\{d_G(u) + d_G(v) \mid \text{for every edge } uv \in E(G)\}. \tag{1}$$

If $cl(H) = L(G)$ is k-connected and $L(G)$ is not complete, then G is essentially k-edge-connected and $\delta(cl(H)) = \min\{d_G(x) + d_G(y) - 2 \mid xy \in E(G)\}$. Thus, $\overline{\sigma}_2(G) = \delta(cl(H)) + 2 \geq \delta(H) + 2$.

By Theorems 2.1 and 2.3, to prove Theorem 2.2, it suffices to show the following.

Theorem 2.4. *Let G be an essentially 3-edge-connected K_3-free simple graph with $|E(G)| = n$ and $\overline{\sigma}_2(G) \geq 18$.*

(a) Either G has a DCT subgraph Θ with $|E(\Theta)| \geq \min\{10\overline{\sigma}_2(G) - 23, n\}$ or $G \in \mathcal{P}_1$.

(b) Either G has a DCT subgraph Θ with $|E(\Theta)| \geq \min\{11\overline{\sigma}_2(G) - 29, n\}$ or $G \in \mathcal{P}_1 \cup \mathcal{P}_2$.

(c) Either G has a DCT subgraph Θ with $|E(\Theta)| \geq \min\{11\overline{\sigma}_2(G) - 25, n\}$ or $G \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$.

(d) Either G has a DCT subgraph Θ with $|E(\Theta)| \geq \min\{12\overline{\sigma}_2(G) - 34, n\}$ or $G \in \bigcup_{i=1}^{4} \mathcal{P}_i$.

(e) If $\overline{\sigma}_2(G) \geq 25$, then either G has a DCT subgraph Θ with $|E(\Theta)| \geq \min\{12\overline{\sigma}_2(G) - 31, n\}$ or $G \in \bigcup_{i=1}^{5} \mathcal{P}_i$.

With Theorem 2.4 we can prove Theorem 2.2.

Proof of Theorem 2.2. We prove the case (a) only. The other cases can be proved in the same way.

Let H be a 3-connected claw-free simple graph of order n with $\delta(H) \geq 16$ and $cl(H)$ its closure. By Theorem 2.1, $cl(H)$ is 3-connected and there is a K_3-free simple graph G such that $cl(H) = L(G)$. Then G is essentially 3-edge-connected and has size $|E(G)| = n$ and $\overline{\sigma}_2(G) = \delta(cl(H)) + 2 \geq \delta(H) + 2 \geq 18$. By Theorem 2.4, one of the following holds.

Case 1. G has a DCT subgraph Θ with $|E(\Theta)| \geq \min\{10\overline{\sigma}_2(G) - 23, n\}$.

Let $H_1 = L(\Theta)$, the line graph of Θ. Then H_1 is a subgraph of $L(G) = cl(H)$ and $V(H_1) \subseteq V(cl(H)) = V(H)$ and $|V(H_1)| = |E(\Theta)|$. Since Θ has a DCT, by Theorem 2.3, H_1 has a Hamiltonian cycle C_0, which is a cycle with length $|E(\Theta)|$ in $L(G)$. By Theorem 2.1, there is a cycle C in H such that $V(C_0) \subseteq V(C)$. Therefore, since $\overline{\sigma}_2(G) \geq \delta(H) + 2$, $c(H) \geq |V(C)| \geq |V(C_0)| = |E(\Theta)| \geq \min\{10\overline{\sigma}_2(G) - 23, n\} \geq \min\{10\delta(H) - 3, n\}$.

Case 2. $G \in \mathcal{P}_1$. Then $H \in \mathcal{F}_1$. This proves Theorem 2.2(a). □

Remark 2.5. *For a claw-free graph H, no matter whether H is regular or not, its closure $cl(H)$ can be obtained in polynomial time [16] and the preimage graph G of a line graph $L(G)$ can be obtained in linear time [15]. Thus, we can compute G efficiently for $cl(H) = L(G)$. Theorems 2.2 and 2.4 show that the lower bound of $c(H)$ of a 3-connected claw-free graph H with $cl(H) = L(G)$ can be obtained by checking if the graph G is in $\mathcal{P}_i$ for some i . Since the size of a maximum matching of a graph can be determined in polynomial time, one can find the expected lower bound of $c(H)$ by checking if the graph G is in $\mathcal{P}_i$ in polynomial time.*

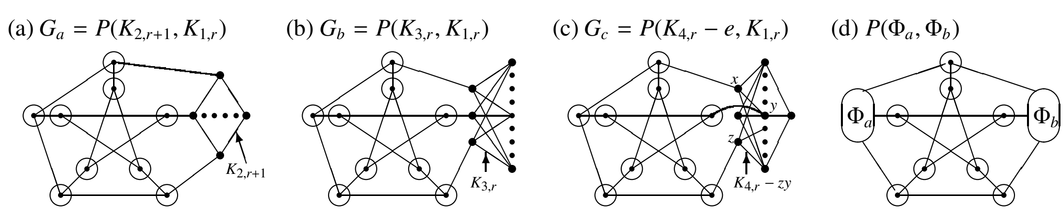


Fig. 2.1: Graphs in $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_5$ and $\mathcal{P}(\Pi_a, \Pi_b)$, respectively.

Remark 2.6. *For the graphs in Fig. 2.1, each vertex marked by ⊙ is incident with $r > 0$ pendant edges. Each of them has a DCT subgraph Θ that contains all the edges except r pendant edges incident with a ⊙ vertex. Thus, Theorem 2.2 and Theorem 2.4 are the best possible in some sense.*

(a) Graph G_a is a graph of order $n = 11r + 17$ in $\mathcal{P}_2$ that has a DCT subgraph Θ_a with $|E(\Theta_a)| = 10r + 17 = 10\overline{\sigma}_2(G_a) - 23$ where $\overline{\sigma}_2(G_a) = r + 4$. Then $H_a = L(G_a)$ has $c(H_a) = 10\delta(H_a) - 3$.

(b) Graph G_b is a graph in $\mathcal{P}_3$ with $\overline{\sigma}_2(G_b) = r + 4$ and has a DCT subgraph Θ_b with $|E(\Theta_b)| = 11r + 15 = 11\overline{\sigma}_2(G_b) - 29$. Then $H_b = L(G_b)$ has $c(H_b) = 11\delta(H_b) - 7$.

(c) For graph G_c in Fig. 2.1(c), edge yz is deleted from $K_{4,r}$ ($r \geq 4$), and y and z are incident with two of the three edges connecting $K_{4,r} - yz$ and $G_c - V(K_{4,r} - yz)$. Then G_c is in $\mathcal{P}_5$ with $\overline{\sigma}_2(G_c) = 4 + r$ and has a DCT subgraph Θ_c with $|E(\Theta_c)| = 12\overline{\sigma}_2(G_c) - 34$. Then $H_c = L(G_c)$ has $c(H_c) = 12\delta(H_c) - 10$.

(d) Let $G_d = P(K_{2,r+1}K_{2,r+1})$ (Fig. 2.1(d) with $\Phi_a = \Phi_b = K_{2,r+1}$). Then $G_d \in \mathcal{P}_4$ with $\overline{\sigma}_2(G_d) = r + 4$ and has a DCT subgraph Θ_d with $|E(\Theta_d)| = 11\overline{\sigma}_2(G_d) - 25$. Then $H_d = L(G_d)$ has $c(H_d) = 11\delta(H_d) - 3$.

(e) Let $G_e = P(K_{3,r}, K_{2,r+1})$ (Fig 2.1(d) with $\Phi_a = K_{3,r}$ and $\Phi_b = K_{2,r+1}$). Then $G_e \in \mathcal{P}_6$ and has a DCT subgraph Θ_e with $|E(\Theta_e)| = 12\overline{\sigma}_2(G_e) - 31$. Then $H_e = L(G_e)$ has $c(H_e) = 12\delta(H_e) - 7$.

The following corollary of Theorem 2.2 is an improvement of a main result in [10].

Corollary 2.7. *If H is a 3-connected claw-free simple graph of order $n \geq 148$ and if $\delta(H) \geq \frac{n+3}{10}$, then either H is Hamiltonian, or $H \in \mathcal{F}_1$.*

Proof. Since $n \geq 148$ and $\delta(H) \geq \frac{n+3}{10} > 15$, $\delta(H) \geq 16$ and $10\delta(H) - 3 \geq n$. By Theorem 2.2, either H has $c(H) \geq n$ and so H is Hamiltonian, or $H \in \mathcal{F}_1$. □

Remark 2.8. *Lai, et al., in [10] prove Corollary 2.7 for $n \geq 196$ and $\delta(H) \geq \frac{n+5}{10}$. More results on conditions involved δ for the Hamiltonicity of 3-connected claw-free graphs can be found in [8, 12].*

3 Graph contraction and Catlin's reduction method

Let G be a connected multigraph. For $X \subseteq E(G)$, the *contraction* G/X is the multigraph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. Note that multiple edges may arise by the identification even G is a simple graph. If Γ is a connected subgraph of G, we write G/Γ for $G/E(\Gamma)$ and say that G/Γ is obtained from G by contracting Γ.

Let G and G_T be two connected graphs. We say that G is *contractible* to G_T if G_T is a graph obtained from G by successively contracting a collection of pairwise vertex disjoint connected subgraphs, and call G_T the *contraction graph* of G. For a vertex $v \in V(G_T)$, there is a connected subgraph $G(v)$ in G such that v is obtained by contracting $G(v)$. We call $G(v)$ the preimage of v in G and call v the contraction image of $G(v)$ in G_T.

Let $O(G)$ be the set of vertices of odd degree in G. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph Γ_R of G with $O(\Gamma_R) = R$. Note that if $R = \{x, y\}$ then Γ_R is a spanning (x, y)-trail; and if $R = \emptyset$ then Γ_R is an SCT in G.

Catlin [2] showed that every multigraph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_c$ such that $V(G) = \cup_{i=1}^{c} V(\Gamma_i)$. The *reduction* of G is a graph obtained from G by contracting each Γ_i into a vertex v_i ($1 \leq i \leq c$) and is denoted by G'. Thus, the reduction G' of G is a special type of contraction graph of G. Although multiple edges may arise by contracting an edge, contracting a maximal collapsible graph will not generate multiple edges.

We regard the edges in $E(G')$ as the edges in $E(G)$. Thus, $E(G) = E(G') \cup_{i=1}^{c} E(\Gamma_i)$. For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph $\Gamma_0(v)$ in G such that v is the contraction image of $\Gamma_0(v)$ and $\Gamma_0(v)$ is the *preimage* of v. A vertex $v \in V(G')$ is a *contracted vertex* if $\Gamma_0(v) \neq K_1$. A graph is *reduced* if $G = G'$. We regard K_1 as a closed trail with $\kappa'(K_1) = \infty$.

Let G be a connected simple graph. Define

$$\begin{aligned} D_i(G) &= \{v \in V(G) \mid d_G(v) = i\}; \\ D_i^*(G) &= \{v \in V(G) \mid d_G(v) \geq i\}. \end{aligned}$$

Some results on Catlin's reduction method that will be needed are summarized below:

Theorem 3.1. *Let G be a connected multigraph and let G' be the reduction of G. Let Γ be a collapsible subgraph in G. Then each of the following holds:*

(a) ([2]). $G \in \mathcal{SL}$ if and only if $G/\Gamma \in \mathcal{SL}$. In particular, $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.

(b) ([2]). G has a DCT (or DT) if and only if G' has a DCT (or a DT) containing all the contracted vertices of G'.

(c) ([2, 3]). G' is simple and K_3-free with $\delta(G') \leq 3$, and any subgraph of G' is reduced. Furthermore, if $G' \notin \{K_1, K_2, K_{2,s}\}$ $(s \geq 2)$, then $|E(G')| \leq 2|V(G')| - 5$.

(d) ([6]). If $G \neq K_1$ is reduced with $|V(G)| \leq 7$ and $\kappa'(G) \geq 2$, then $|D_2(G)| \geq 3$. Furthermore, if $|D_2(G)| = 3$, then $G \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}$ (see Fig 3.1).

(e) ([5]). Let G be a connected reduced graph of order n with $\delta(G) \geq 2$ and $G \neq K_{2,b}$ $(b \geq 2)$. Let M be a maximum matching in G and $|D_2(G)| = l$. Then $|M| \geq \min\{\frac{n-1}{2}, \frac{n+5-l}{3}\}$.

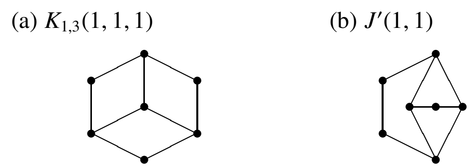


Fig. 3.1: Two reduced graphs G of order 7 with $|D_2(G)| = 3$.

Let G be an essentially 3-edge-connected simple graph. Then $D_1(G) \cup D_2(G)$ is an independent set. Let E_1 be the set of pendant edges in G. For each $x \in D_2(G)$, there are two edges e_x^1 and e_x^2 incident with x. Let $X_2(G) = \{e_x^1 \mid x \in D_2(G)\}$. Thus $|X_2(G)| = |D_2(G)|$. Define

$$G_1 = G/E_1 \text{ and } G_0 = G_1/X_2(G).$$

Since G is essentially 3-edge-connected, G_1 is essentially 3-edge-connected and 2-edge-connected, and G_0 is 3-edge-connected.

In [17], Shao defined G_0 for essentially 3-edge-connected graphs G and called G_0 the **core** of G. Although G is simple, G_0 may not be simple. But by Theorem 3.1, G_0' is simple and K_3-free.

For a vertex $v \in V(G_0')$, let $\Gamma_0(v)$ be the collapsible preimage of v in G_0, let $\Gamma_1(v)$ be the preimage of v in G_1 and let $\Gamma(v)$ be the preimage of v in G. Then $\Gamma(v)$ is a subgraph induced by $E(\Gamma_0(v))$ and some edges in $E_1 \cup X_2(G)$. By the definitions, we have the following:
(a) $\Gamma_1(v) = \Gamma(v)/(E_1 \cap E(\Gamma(v)))$ (it is still K_3-free);
(b) $\Gamma_0(v) = \Gamma_1(v)/(X_2(G) \cap E(\Gamma_1(v)))$ (it may not be K_3-free).

A vertex $v \in V(G_0')$ (or $V(G_0)$) is a *contracted* vertex if $|V(\Gamma(v))| > 1$. A vertex $v \in V(G_0')$ (or $V(G_0)$) is *nontrivial* in G_0' (or in G_0) if $|V(\Gamma(v))| > 1$ or $|V(\Gamma(v))| = 1$ and v is adjacent to a vertex in $D_2(G)$. A vertex v in G_0' is *trivial* if $d_{G_0'}(v) = d_G(v)$ and v is not adjacent to a vertex in $D_2(G)$. For instance, if $x \in D_2(G)$ with $N_G(x) = \{u, v\}$, and if u_x is a vertex in G_0 obtained by contracting ux, then both u_x and v are nontrivial in G_0 but u_x is a contracted vertex and v is not a contracted vertex in G_0.

Using Theorem 3.1(b), Shao [17] proved the following:

Theorem 3.2. *([17]). Let G be an essentially 3-edge-connected graph and $L(G)$ is not complete. Let G_0 be the core of graph G, and let G'_0 be the reduction of G_0, then the following holds:*

(a) G_0 is well defined, nontrivial and $\delta(G_0) = \kappa'(G_0) \geq 3$ and so $\kappa'(G'_0) \geq \kappa'(G_0) \geq 3$;

(b) G has a DCT if and only if G'_0 has a DCT containing all the nontrivial vertices in G'_0.

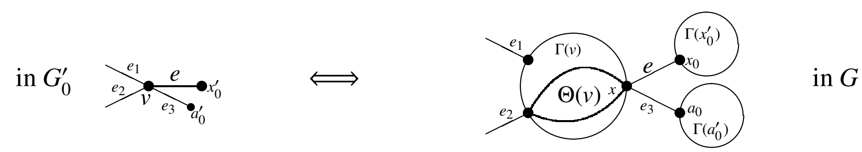
Let G_T be a contraction graph of G. Let v be a vertex in G_T and let $G(v)$ be the preimage of v in G. Let $\Theta(v)$ be a connected subgraph of $G(v)$. Define

$$\mathcal{E}_{G(v)}(\Theta(v)) = \{e \in E(G(v)) \mid e \text{ is incident with some vertices in } \Theta(v)\}. \tag{2}$$

For a vertex $x \in V(G(v))$, let $i(x)$ be the number of edges in $E(G_T)$ incident with x in G (see Fig. 3.2). For a vertex subset $S \subseteq V(G(v))$, let $i(S) = \sum_{x \in S} i(x)$, which is the number of edges in $E(G_T)$ that are incident with some vertices in S. When $\Theta(v)$ is a subgraph of $G(v)$, we use $i(\Theta(v))$ for $i(V(\Theta(v)))$. Then for any $x \in V(\Theta(v)) \subseteq V(G(v))$,

$$d_G(x) = i(x) + |N_{G(v)}(x)| = i(x) + d_{G(v)}(x) \quad \text{and} \quad i(x) \leq \sum_{w \in V(\Theta(v))} i(w) = i(\Theta(v)) \leq d_{G_T}(v). \tag{3}$$

When $G_T = G'_0$ and $G(v) = \Gamma(v)$ with a subgraph $\Theta(v)$, $i(x) \leq i(\Theta(v)) \leq d_{G'_0}(v)$ (See Fig. 3.2).



$i(x) = 2 < i(\Theta(v)) = 3 < d_{G'_0}(v) = 4$

Fig. 3.2. Description of edges and vertices in G'_0 and G that are related to $i(x)$ and $i(\Theta(v))$

Proposition 3.3. *Let G be an essentially 3-edge-connected K_3-free simple graph with $\overline{\sigma}_2(G) \geq 7$. Let G_T be the contraction graph of G. For a vertex $v \in V(G_T)$ with $d_{G_T}(v) = 3$, let $E_{G_T}(v)$ be the set of the three edges incident with v in G_T and let $G(v)$ be the preimage of v in G. If $\alpha'(G(v)) \in \{1, 2\}$, then for any two edges in $E_{G_T}(v)$, $G(v)$ has a dominating (x, y)-trail T_v where x, y are incident with the two edges and that each of the following holds:*

(a) if $\alpha'(G(v)) = 1$, then $G(v) \in \mathcal{K}_{1,r}$ and $|E(G(v))| = |\mathcal{E}_{G(v)}(T_v)| \geq \overline{\sigma}_2(G) - 4$;

(b) if $\alpha'(G(v)) = 2$, then $G(v) \in \mathcal{K}_{2,r}$ and $|E(G(v))| = |\mathcal{E}_{G(v)}(T_v)| \geq 2\overline{\sigma}_2(G) - 3 - i(T_v)$.

Proof. If $\alpha'(G(v)) = 1$, then since G is essentially 3-edge-connected, K_3-free and simple, $G(v) = K_{1,r}$. Then $T_v = K_1$. Let $V(T_v) = \{x\}$. Then $|E(G(v))| = |\mathcal{E}_{G(v)}(T_v)| = |N_{G(v)}(x)| = r$ and $i(x) = d_{G_T}(v) = 3$. Let xy be an edge in $E(G(v))$ with $d_{G(v)}(x) = r$ and $d_G(y) = 1$. Since $d_G(x) = d_{G(v)}(x) + i(x) = r + 3$ and $d_G(x) + d_G(y) \geq \overline{\sigma}_2(G)$, $r \geq \overline{\sigma}_2(G) - 4$ and (a) is proved.

Next, we assume that $\alpha'(G(v)) = 2$. Then $G(v)$ has a cycle. Let $C_s = u_1u_2 \cdots u_su_1$ be a cycle in $G(v)$. Since $G(v)$ is simple and K_3-free and $\alpha'(G(v)) = 2$, $4 \leq s \leq 5$.

Note that $E_{G_T}(v)$ is the set of edges outside of $G(v)$ incident with some vertices in $G(v)$. Since $|E_{G_T}(v)| = 3$ and $|V(C_s)| = s \geq 4$, a vertex (say u_1) in $V(C_s)$ is not incident with any edge in $E_{G_T}(v)$. Then $d_G(u_1) = d_{G(v)}(u_1)$. Since G is an essentially 3-edge-connected K_3-free simple graph and $\alpha'(G(v)) = 2$, $N_G(u_1) = N_{G(v)}(u_1) = \{u_2, u_s\}$ and $d_G(u_1) = 2$.

Since $i(u_2)+i(u_s) \leq d_{G_T}(v) = 3$, we may assume that $i(u_2) \leq 1$. Since $\overline{\sigma}_2(G) \geq 7$, $d_G(u_2) \geq \overline{\sigma}_2(G) - d_G(u_1) \geq \overline{\sigma}_2(G) - 2 \geq 5$. Then $|N_{G(v)}(u_2)| = d_{G(v)}(u_2) = d_G(u_2) - i(u_2) \geq 4$. Let $z \in N_{G(v)}(u_2) - V(C_s)$. If $s = 5$, then $\{u_1u_5, u_2z, u_3u_4\}$ is a matching in $G(v)$, a contradiction. Thus $s = 5$ is impossible.

Hence, $s = 4$. If there is a vertex z_1 in $N_{G(v)}(u_3) - \{u_2, u_4\}$, then $\{u_1u_4, zu_2, z_1u_3\}$ is a matching in $G(v)$, a contradiction. Thus, $N_{G(v)}(u_3) = \{u_2, u_4\}$.

Let $X = \{u_2, u_4\}$ and $Y = N_{G(v)}(u_2) \cup N_{G(v)}(u_4)$. Then since G is K_3-free, $G(v) \in \mathcal{K}_{2,r}$ with $V(G(v)) = X \cup Y$. Since $d_G(u_1) = 2$, only u_2, u_3 and u_4 are the possible nontrivial vertices in $G(v)$ and may be incident with the edges in $E_{G_T}(v)$. By inspection, for any given two edges in $E_{G_T}(v)$, $G(v)$ has a dominating (x, y)-trail T_v containing all the nontrivial vertices of $G(v)$ where x and y are incident with the two given edges. Thus, $i(T_v) \geq 2$ and $\{u_2, u_4\} \subseteq V(T_v)$. Next, we shall prove that $|\mathcal{E}_{G(v)}(T_v)| \geq 2\overline{\sigma}_2(G) - 3 - i(T_v)$.

Since $\{u_2, u_4\} \subseteq V(T_v)$ and $E_{G(v)}(u_2) \cap E_{G(v)}(u_4) = \emptyset$, $E_{G(v)}(u_2) \cup E_{G(v)}(u_4) \subseteq \mathcal{E}_{G(v)}(T_v)$ and

$$|\mathcal{E}_{G(v)}(T_v)| \quad \geq \quad |E_{G(v)}(u_2)| + |E_{G(v)}(u_4)| = d_{G(v)}(u_2) + d_{G(v)}(u_4). \tag{4}$$

For $u \in \{u_2, u_4\}$ and a vertex $w \in N_{G(v)}(u)$, since $d_G(w) + d_G(u) \geq \overline{\sigma}_2(G)$, by (3),

$$d_{G(v)}(u) = d_G(u) - i(u) \geq \overline{\sigma}_2(G) - d_G(w) - i(u). \tag{5}$$

For each $z \in N_{G(v)}(u)$ where $u \in \{u_2, u_4\}$, since G is K_3-free and $\alpha'(G(v)) = 2$, $N_{G(v)}(z) \subseteq \{u_2, u_4\}$, and either $d_{G(v)}(z) = 1$ or $d_{G(v)}(z) = 2$.

Case 1. There is a vertex z in $N_{G(v)}(u)$ where $u \in \{u_2, u_4\}$ (say $u = u_2$) such that $d_{G(v)}(z) = 1$.

We have the following two sub cases:

Subcase 1.1. $d_G(z) = d_{G(v)}(z) = 1$. Then zu_2 is a pendant edge. By (5) with $u = u_2$ and $w = z$, $d_{G(v)}(u_2) \geq \overline{\sigma}_2(G) - 1 - i(u_2)$. Since $u_1 \in N_{G(v)}(u_4)$ and $d_G(u_1) = 2$, by (5) with $u = u_4$ and $w = u_1$, $d_{G(v)}(u_4) \geq \overline{\sigma}_2(G) - 2 - i(u_4)$. By (4) and $i(u_2) + i(u_4) \leq i(T_v)$,

$$|\mathcal{E}_{G(v)}(T_v)| \geq d_{G(v)}(u_2) + d_{G(v)}(u_4) \geq (\overline{\sigma}_2(G) - 1 - i(u_2)) + (\overline{\sigma}_2(G) - 2 - i(u_4)) \geq 2\overline{\sigma}_2(G) - 3 - i(T_v).$$

In the following, we assume that no vertices in $V(C_4)$ are incident with a pendant edge in G.

Subcase 1.2. $d_{G(v)}(z) = 1$ and $d_G(z) \neq 1$.

Since G is essentially 3-edge-connected and $\alpha'(G(v)) = 2$, z must be incident with an edge in $E_{G_T}(v) \cap X_2(G)$ and $d_G(z) = 2$ and $i(z) = 1$.

Let Z_i be the set of vertices in $N_{G(v)}(u_i)$ that are incident with an edge in $E_{G_T}(v) \cap X_2(G)$ $(i = 2, 4)$. Then $|Z_2| + |Z_4| \geq i(z) = 1$ and $|Z_2| + i(u_2) + |Z_4| + i(u_4) \leq d_{G_T}(v) = 3$. Without loss of generality, we assume that $|Z_2| + i(u_2) \leq 1$.

Let $W = N_{G(v)}(u_2) \cap N_{G(v)}(u_4)$. Then $|N_{G(v)}(u_i)| = |W| + |Z_i|$ and $|N_G(u_i)| = |N_{G(v)}(u_i)| + i(u_i)$ for $i \in \{2, 4\}$. By (5) with $u = u_2$ and $w = u_1$, $d_{G(v)}(u_2) \geq \overline{\sigma}_2(G) - 2 - i(u_2)$. Hence, $|W| \geq \overline{\sigma}_2(G) - 2 - |Z_2| - i(u_2)$. Then by (4) and $2(|Z_2| + i(u_2)) \leq 2 \leq i(T_v)$,

$$\begin{aligned} |\mathcal{E}_{G(v)}(T_v)| \quad &\geq \quad |N_{G(v)}(u_2)| + |N_{G(v)}(u_4)| = 2|W| + |Z_2| + |Z_4| \geq 2|W| + 1 \\ &\geq \quad 2(\overline{\sigma}_2(G) - 2 - |Z_2| - i(u_2)) + 1 = 2\overline{\sigma}_2(G) - 3 - 2(|Z_2| + i(u_2)) \geq 2\overline{\sigma}_2(G) - 3 - i(T_v). \end{aligned}$$

We are done for this case.

Case 2. For any z in $N_{G(v)}(u_2) \cup N_{G(v)}(u_4)$, $d_{G(v)}(z) = 2$.

Then $N_{G(v)}(z) = \{u_2, u_4\}$ and $N_{G(v)}(u_2) = N_{G(v)}(u_4)$. We have a T_v trail containing the vertices that are incident with the three edges in $E_{G_T}(v)$. Thus, $i(u_2) + i(u_4) \leq i(T_v) = 3$. We assume $i(u_2) \leq 1$.

By (5) with $u \in \{u_2, u_4\}$ and $w = u_1$, and by $i(u_2) \leq 1$ and $d_G(u_1) = 2$, $d_{G(v)}(u_2) = d_G(u_2) - i(u_2) \geq \overline{\sigma}_2(G) - d_G(u_1) - 1 = \overline{\sigma}_2(G) - 2 - i(u_2)$. Therefore, by (4) and $2i(u_2) + 1 \leq 3 = i(T_v)$,

$$\begin{aligned} |\mathcal{E}_{G(v)}(T_v)| \quad &\geq \quad |E_{G(v)}(u_2)| + |E_{G(v)}(u_4)| = |N_{G(v)}(u_2)| + |N_{G(v)}(u_4)| = 2|N_{G(v)}(u_2)| \\ &\geq \quad 2(\overline{\sigma}_2(G) - 2 - i(u_2)) = 2\overline{\sigma}_2(G) - 3 - (1 + 2i(u_2)) \geq 2\overline{\sigma}_2(G) - 3 - i(T_v). \end{aligned}$$

The proof is complete. □

4 Associated Theorems and the proof of Theorem 2.4

The following theorem plays an important role in our approach to prove Theorem 2.4.

Theorem 4.1. *([7]). Let G be a 3-edge-connected simple graph. Let $S \subseteq V(G)$ be a vertex subset with $|S| \le 12$. Then either G has a closed trail C such that $S \subseteq V(C)$, or G can be contracted to P in such a way that the preimage of each vertex of P contains at least one vertex in S.*

We shall choose a subset S of $V(G_0')$ that allow us to find a DCT subgraph in G with large size according to whether G_0' is contractible to the Petersen graph or G_0' has a closed trail containing S.

Let G be an essentially 3-edge-connected K_3-free graph. We will use the following notation:

- $S_0 = \{v \in V(G_0') \mid v$ is a contracted vertex in G_0', i.e., $\Gamma(v) \neq K_1\}$;
- $S_1 = V(G_0') - S_0$, (then $d_G(v) = d_{G_0}(v) = d_{G_0'}(v)$ if $v \in S_1$);
- $S_1^* = \{v \in S_1 - D_3(G_0') \mid d_{G_0'}(v) \ge \overline{\sigma}_2(G) - 3\}$;
- $S_2 = V(G_0') - (S_0 \cup S_1^*)$;
- $\Phi = G_0'[S_2]$, the subgraph induced by S_2 in G_0';
- let M_Φ be a maximum matching in Φ and let S_M be the set of end vertices of the edges in M_Φ;
- let $S_3 = V(G_0') - (S_0 \cup S_1^* \cup S_M)$, and so $S_3 = V(\Phi) - S_M = S_2 - S_M$;
- let $V_a = S_0 \cup S_1^* \cup S_M$.

Theorem 2.4 can be proved by establishing the following two associated theorems.

Theorem 4.2. *Let G be an essentially 3-edge-connected K_3-free simple graph with $|E(G)| = n$. Let G_0' be the reduction of G_0. Suppose that $G_0' \notin \mathcal{SL} \cup \{P\}$ and G_0' can not be contracted to P in such a way that the preimage of each vertex in P contains at least one vertex in V_a. Then each of the following holds:*

(a) if $\overline{\sigma}_2(G) \ge 18$, then G has a DCT subgraph Θ with $|E(\Theta)| \ge \min\{12\overline{\sigma}_2(G) - 34, n\}$;
(b) if $\overline{\sigma}_2(G) \ge 25$, then G has a DCT subgraph Θ with $|E(\Theta)| \ge \min\{12\overline{\sigma}_2(G) - 31, n\}$.

Theorem 4.3. *Let G be an essentially 3-edge-connected K_3-free simple graph with $|E(G)| = n$ and $\overline{\sigma}_2(G) \ge 8$. Let G_0' be the reduction of G_0. Let V_a be the set defined above. If $G_0' = P$ or G_0' can be contracted to P in such a way that the preimage of each vertex in P contains at least one vertex in V_a, then each of the following holds:*

(a) either G has a DCT subgraph Θ with $|E(\Theta)| \ge \min\{10\overline{\sigma}_2(G) - 23, n\}$ or $G \in \mathcal{P}_1$;
(b) either G has a DCT subgraph Θ with $|E(\Theta)| \ge \min\{11\overline{\sigma}_2(G) - 29, n\}$ or $G \in \mathcal{P}_1 \cup \mathcal{P}_2$;
(c) if $\overline{\sigma}_2(G) \ge 9$, then either G has a DCT subgraph Θ with $|E(\Theta)| \ge \min\{11\overline{\sigma}_2(G) - 25, n\}$ or $G \in \bigcup_{i=1}^{3} \mathcal{P}_i$;
(d) either G has a DCT subgraph Θ with $|E(\Theta)| \ge \min\{12\overline{\sigma}_2(G) - 34, n\}$ or $G \in \bigcup_{i=1}^{4} \mathcal{P}_i$;
(e) if $\overline{\sigma}_2(G) \ge 12$, then either G has a DCT subgraph Θ with $|E(\Theta)| \ge \min\{12\overline{\sigma}_2(G) - 31, n\}$ or $G \in \bigcup_{i=1}^{5} \mathcal{P}_i$.

With Theorems 4.3 and 4.2 we can prove Theorem 2.4.

Proof of Theorem 2.4. By Theorem 3.2, G_0 and G_0' are 3-edge-connected. If $G_0' \in \mathcal{SL}$, then by Theorem 3.1, $G_0 \in \mathcal{SL}$. By Theorem 3.2, G has a DCT. Theorem 2.4 is proved for this case.

Next, we assume that $G_0' \notin \mathcal{SL}$. Let $V_a = S_0 \cup S_1^* \cup S_M$ be the subset of $V(G_0')$ defined above.

If $G_0' = P$ or G_0' can be contracted to P in the way stated in Theorem 4.3, then Theorem 2.4 follows from Theorem 4.3. Otherwise, Theorem 2.4 follows from Theorem 4.2. □

5 Technical lemmas

The following lemma will be needed which can be proved easily and a proof can be found in [4].

Lemma 5.1 ([4])**.** *Let G be a 2-edge-connected graph. Let $\{x, y, z\}$ be a set of vertices in G (possibly $x = y$ or $x = z$). Then for any two vertices (say x and y) in $\{x, y, z\}$, G has a (x, y)-trail containing z.*

Lemma 5.2. *Let G be a connected K_3-free simple graph. Let G_T be a contraction graph of G. For a vertex $v \in V(G_T)$, let $G(v)$ be the preimage of v in G and let M be a matching of size t in $G(v)$. Let $\Theta(v)$ be a connected subgraph of $G(v)$ and $\mathcal{E}_{G(v)}(\Theta(v))$ contains all the edges of M. Then*

$$|\mathcal{E}_{G(v)}(\Theta(v))| \geq t\overline{\sigma}_2(G) - t^2 - i(\Theta(v)) \geq t\overline{\sigma}_2(G) - t^2 - d_{G_T}(v). \tag{6}$$

Furthermore, each of the following holds:

(a) if $G_T = G'_0$ and M is a matching of size $t \geq 3$ in $G(v)$ and all the edges in M are in G'_0, then $|\mathcal{E}_{G(v)}(\Theta(v))| \geq t\overline{\sigma}_2(G) - 4t + 5 - i(\Theta(v))$;

(b) if $\Theta(v)$ is a connected dominating subgraph of $G(v)$ with $i(\Theta(v)) \geq 2$, $d_{G_T}(v) \geq 3$ and $t = \alpha'(G(v)) \geq 4$, then $|\mathcal{E}_{G(v)}(\Theta(v))| \geq t\overline{\sigma}_2(G) - t^2 - i(\Theta(v)) + 2$.

Proof. Let $M = \{y_1z_1, y_2z_2, \cdots, y_tz_t\}$ be a matching in $G(v)$ such that $\mathcal{E}_{G(v)}(\Theta(v))$ contains all the edges in M. Let $Y = \{y_1, \cdots, y_t\}$ and $Z = \{z_1, \cdots, z_t\}$ and let $G_M = G[Y \cup Z]$. Note that each edge in G_M occurs in exactly two of the edge sets of $\{E_G(y_i), E_G(z_i) \mid 1 \leq i \leq t\}$. Thus,

$$\sum_{i=1}^{t}(|E_G(y_i)| + |E_G(z_i)|) - |E(G_M)| \leq |\bigcup_{i=1}^{t}(E_G(y_i) \cup E_G(z_i))|. \tag{7}$$

Let $E_{G_T}(v)$ be the set of edges in $E(G) - E(G(v))$ incident with some vertices in $\Theta(v)$. Let $A(v) = E_{G_T}(v) \cap \left(\bigcup_{i=1}^{t}(E_G(y_i) \cup E_G(z_i))\right)$. Then $\bigcup_{i=1}^{t}(E_G(y_i) \cup E_G(z_i)) \subseteq A(v) \cup \mathcal{E}_{G(v)}(\Theta(v))$ and $|A(v)| = i(\Theta(v))$. Since $d_G(y_i) + d_G(z_i) \geq \overline{\sigma}_2(G)$, by (7),

$$t\overline{\sigma}_2(G) - |E(G_M)| \leq \sum_{i=1}^{t}(|E_G(y_i)| + |E_G(z_i)|) - |E(G_M)| \leq i(\Theta(v)) + |\mathcal{E}_{G(v)}(\Theta(v))|. \tag{8}$$

Now, we need to find $|E(G_M)|$ in terms of t, which is depended on how the edges in M are selected.

Since G is K_3-free simple graph, G_M is K_3-free and simple. By Turán's Theorem, G_M has at most t^2 edges. Since $|A(v)| = i(\Theta(v)) \leq d_{G_T}(v)$ and $|E(G_M)| \leq t^2$, (6) follows from (8).

If all the edges in M are the edges in G'_0, we have a better estimate on $|E(G_M)|$ for $t \geq 3$.

Note that we regard $E(G'_0) \subseteq E(G)$. Let $M' = \{y'_1z'_1, y'_2z'_2, \cdots, y'_tz'_t\}$ be a matching in G'_0, which are the edges in $G(v)$. Let $\Gamma(y'_i)$ and $\Gamma(z'_i)$ be the preimages of y'_i and z'_i ($1 \leq i \leq t$) in G, respectively. Then for each $y'_iz'_i$ in M', there are y_i in $\Gamma(y'_i)$ and z_i in $\Gamma(z'_i)$ such that y_iz_i is the edge in G corresponding to $y'_iz'_i$ in G'_0. Thus, $M = \{y_1z_1, y_2z_2, \cdots, y_tz_t\}$ is a matching in $G(v)$. Let $Y' = \{y'_1, \cdots, y'_t\}$ and $Z' = \{z'_1, \cdots, z'_t\}$. Let $G'_{M'} = G'_0[Y' \cup Z']$. Since $y_i \in V(\Gamma(y'_i))$, the number of edges in $E_G(y_i)$ (or $E_G(z_i)$) incident with vertices in $Y \cup Z$ is no more than the number of edges in $E_{G'_0}(y'_i)$ incident with vertices in $Y' \cup Z'$. Thus, $|E(G_M)| \leq |E(G'_{M'})|$. Since $G'_{M'}$ is a subgraph of G'_0, $G'_{M'}$ is reduced. Since $t \geq 3$, $G'_{M'} \notin \{K_1, K_2, K_{2,s}\}$. By Theorem 3.1, $|E(G'_{M'})| \leq 2|V(G'_{M'})| - 5 = 4t - 5$. By (8) and $|A(v)| = i(\Theta(v))$, $|\mathcal{E}_{G(v)}(\Theta(v))| \geq t\overline{\sigma}_2(G) - 4t + 5 - i(\Theta(v))$. Case (a) is proved.

For (b), $\Theta(v)$ is a dominating subgraph of $G(v)$ with $i(\theta(v)) \geq 2$, $d_{G_T}(v) \geq 3$ and $t = \alpha'(G(v)) \geq 4$.

To the contrary, suppose that (b) is false, i.e.,

$$|\mathcal{E}_{G(v)}(\Theta(v))| \le t\overline{\sigma}_2(G) - t^2 - i(\Theta(v)) + 1. \tag{9}$$

By (8) and (9), $|E(G_M)| \ge t^2 - 1$. We further assume that M is a maximum matching in $G(v)$ with $|E(G_M)|$ as small as possible.

Since $|E(G_M)| \ge t^2 - 1$ and $t \ge 4$, the total number of edge incidents in G_M is $\sum_{i=1}^{t}(d_{G_M}(y_i) + d_{G_M}(z_i)) = 2|E(G_M)| \ge 2t^2 - 2$. At least one vertex in Y is adjacent to all the vertices in Z (otherwise, we relabel them). Since G is K_3-free, Z is an independent set in G. Similarly, at least one vertex in Z is adjacent to all the vertices in Y and so Y is an independent set in G.

Let $U = V(\Theta(v)) - (Y \cup Z)$. Then we have the following facts:

Claim 1. (a) U is an independent set and so $E_{G(v)}(u_1) \cap E_{G(v)}(u_2) = \emptyset$ for any $u_1 \neq u_2$ in U;

(b) each vertex v in $Y \cup Z$ is adjacent to at most one end of each edge in M and so $d_{G_M}(v) \le t$.

(c) each $u \in U$ is adjacent to one end of each edge in M and so $d_G(u) = t$.

Proof of Claim 1. Since $\Theta(v)$ is a dominating subgraph of $G(v)$, G is K_3-free and M is a maximum matching in $\Theta(v)$, (a) and (b) are trivially true. Thus, we only need to prove case (c).

To the contrary, suppose that u is not adjacent to either ends of an edge e, say $e = y_1z_1$.

Since U is independent, each $u \in U$ is only adjacent to vertices in $Y \cup Z$. Furthermore, $u \in U$ is adjacent to at least $t - 1$ vertices in $Y \cup Z$. Otherwise, if u is only adjacent to at most $t - 2$ vertices in $Y \cup Z$ (say u is adjacent to y_3), then $M_1 = (M - \{y_3z_3\}) \cup \{uy_3\}$ is a maximum matching. Since at least two edge-incidents at u are missing, $|E(G_{M_1})| \le t^2 - 2 < |E(G_M)|$, a contradiction.

Thus, u is adjacent to one end of each of the edges in $\{y_2z_2, \cdots, y_tz_t\}$. We may assume that $uy_2 \in E(G)$ and y_2 is adjacent to all the vertices in Z. Since G is K_3-free, u cannot adjacent to any vertex in Z. Thus, u is adjacent to all the vertices in $Y - \{y_1\}$.

If $|E(G_M)| = t^2$, then $M_1 = (M - \{y_2z_2\}) \cup \{uy_2\}$ is a maximum matching. Since u is not adjacent to y_1 and z_1, $|E(G_{M_1})| < |E(G_M)|$, a contradiction.

If $|E(G_M)| = t^2 - 1$, then a vertex $y \in Y$ is not adjacent to a vertex $z \in Z$. If $y \neq y_1$ (say $y = y_3$), then $M_b = (M - \{y_3z_3\}) \cup \{uy_3\}$ is a maximum matching. Since one edge-incident is missing at u and one edge-incident is missing at y_3 and $uy_3 \in M_b$, $|E(G_{M_b})| < |E(G_M)|$, a contradiction.

If $y = y_1$, the $M_b = (M - \{y_4z_4\}) \cup \{uy_4\}$ is a maximum matching. Again, since one edge-incident is missing at u and one edge-incident is missing at y_4 and $uy_4 \in M_b$, $|E(G_{M_b})| < |E(G_M)|$, a contradiction.

We reach contradiction for all the possible cases. Claim 1 is proved.

Let $W = V(G(v)) - V(\Theta(v))$. Since $\Theta(v)$ is a dominating subgraph of $G(v)$, an edge in $G(v)$ incident with a vertex in W must be incident with a vertex in $\Theta(v)$ and W is an independent set. Thus, $E_{G(v)}(w_1) \cap E_{G(v)}(w_2) = \emptyset$ for any $w_1 \neq w_2$ in W and $\cup_{w \in W} E_{G(v)}(w) \subseteq \mathcal{E}_{G(v)}(\Theta(v))$. If $i(\Theta(v)) < d_{G_T}(v)$, then $W \neq \emptyset$. Since $d_{G_T}(v) \ge 3$ and $i(\Theta(v)) \ge 2$,

$$i(\Theta(v)) + |W| \ge 3. \tag{10}$$

By Claim 1 and W is an independent set with $\cup_{w \in W} E_{G(v)}(w) \subseteq \mathcal{E}_{G(v)}(\Theta(v))$, we have

$$|\mathcal{E}_{G(v)}(\Theta(v))| = |E(G_M)| + \sum_{u \in U} d_{G(v)}(u) + \sum_{w \in W} d_{G(v)}(w) \ge |E(G_M)| + t|U| + |W|. \tag{11}$$

For each y_iz_i in M, by Claim 1(c),

$$d_G(y_j) + d_G(z_j) = d_{G_M}(y_j) + d_{G_M}(z_j) + i(y_j) + i(z_j) + |U|. \tag{12}$$

Since $t \geq 4$, at least one edge (say y_4z_4) in M is not adjacent to any edges in $A(v)$. Thus $i(y_4) = i(z_4) = 0$. Since $\max\{d_{G_M}(y_4), d_{G_M}(z_4)\} \leq t$, by (12)

$$\overline{\sigma}_2(G) \leq d_G(y_4) + d_G(z_4) = d_{G_M}(y_4) + d_{G_M}(z_4) + i(y_4) + i(z_4) + |U| \leq 2t + |U|. \tag{13}$$

Since $|E(G_M)| \geq t^2 - 1$, by (9), (11) and (13),

$$\begin{aligned} t^2 - 1 + t|U| + |W| &\leq |\mathcal{E}_{G(v)}(\Theta(v))| \leq t\overline{\sigma}_2(G) - t^2 - i(\Theta(v)) + 1 \\ &\leq t(2t + |U|) - t^2 - i(\Theta(v)) + 1 = t^2 + t|U| - i(\Theta(v)) + 1, \end{aligned}$$

which yields $|W| + i(\Theta(v)) \leq 2$, contrary to (10). The proof is completed. □

Lemma 5.3. *Let G be an essentially 3-edge-connected K_3-free graph with $\overline{\sigma}_2(G) \geq 7$. Let G_0' be the reduction of G_0. For each $v \in V(G_0')$, let $\Gamma(v)$ be the preimage of v in G. Let S_0, S_1, S_1^*, S_2 and S_3 be the sets defined in Section 4. Then each of the following holds:*

(a) For each $v \in S_0$ and $1 \leq t \leq \alpha'(\Gamma(v))$, $|E(\Gamma(v))| \geq t\overline{\sigma}_2(G) - t^2 - d_{G_0'}(v)$.

(b) For each $v \in D_3(G_0') \cap S_1$, $N_{G_0'}(v) \subseteq S_0 \cup S_1^*$.

(c) S_3 is an independent set.

(d) All the vertices in S_2 are trivial vertices in G_0' and so all the nontrivial vertices are in $S_0 \cup S_1^*$.

Proof. (a) For each $v \in S_0$, since v is a contracted vertex in G_0', $\alpha'(\Gamma(v)) \geq 1$. This is the special case of Lemma 5.2 with $G_T = G_0'$ and $\Theta(v) = G(v) = \Gamma(v)$.

(b) If $v \in D_3(G_0') \cap S_1$, then $d_{G_0'}(v) = d_G(v) = 3$. If $u \in N_{G_0'}(v)$ and $u \notin S_0$, then $d_{G_0'}(u) = d_G(u)$ and $d_{G_0'}(v) + d_{G_0'}(u) = d_G(v) + d_G(u) \geq \overline{\sigma}_2(G)$. Thus, $d_{G_0'}(u) \geq \overline{\sigma}_2(G) - 3 \geq 4$ and so $u \in S_1^*$. (b) is proved.

(c) Since $S_3 = S_2 - S_M$ and M_Φ is a maximum matching in $G_0'[S_2]$, no edge has two ends in S_3.

(d) If $v \in S_2 = V(G_0') - (S_0 \cup S_1^*)$, then v is not a contracted vertex and so $d_G(v) = d_{G_0'}(v)$. To the contrary, suppose that v is nontrivial. Then v is adjacent to a vertex u in $D_2(G)$. Then $d_{G_0'}(v) + 2 = d_G(v) + d_G(u) \geq \overline{\sigma}_2(G) \geq 7$ and $d_{G_0'}(v) \geq \overline{\sigma}_2(G) - 2 > \overline{\sigma}_2(G) - 3$. Hence, $v \in S_1^*$, a contradiction. □

6 Proof of Theorem 4.2

We prove the following lemma first.

Lemma 6.1. *Let G_0' be the reduction of the core G_0 of an essentially 3-edge-connected graph G. Let Φ be the subgraph of G_0' defined in section 4, and let M_Φ be a maximum matching in Φ. Then*

$$|D_3(G_0')| \geq 10 + |M_\Phi|(\overline{\sigma}_2(G) - 8). \tag{14}$$

Proof. Since $\delta(G_0') \geq 3$, $G_0' \notin \{K_1, K_2, K_{2,s} (s \geq 2)\}$. By Theorem 3.1, $|E(G_0')| \leq 2|V(G_0')| - 5$. Since $2|E(G_0')| = \sum_{v \in V(G_0')} d_{G_0'}(v) = \sum_{i=3} i|D_i(G_0')|$ and $|V(G_0')| = \sum_{i=3} |D_i(G_0')|$, we have

$$\begin{aligned} 2|E(G_0')| &\leq 4|V(G_0')| - 10; \\ 3|D_3(G_0')| + 4|D_4(G_0')| \cdots + i|D_i(G_0')| + \cdots &\leq 4(|D_3(G_0')| + |D_4(G_0')| \cdots + |D_i(G_0)| \cdots) - 10; \\ |D_5(G_0')| + 2|D_6(G_0')| \cdots + (i-4)|D_i(G_0')| \cdots &\leq |D_3(G_0')| - 10. \end{aligned} \tag{15}$$

Recall that S_M is the set of the vertices in M_Φ. Let $D_i^M = D_i(G_0') \cap S_M$. By the definition of M_Φ, for each $uv \in M_\Phi$, $d_G(u) = d_{G_0'}(u) \geq 4$, $d_G(v) = d_{G_0'}(v) \geq 4$, and so $d_G(u) + d_G(v) \geq \overline{\sigma}_2(G)$. By (15),

$$\begin{aligned} |M_\Phi|(\overline{\sigma}_2(G) - 8) &\leq \sum_{uv \in M_\Phi} (d_G(u) - 4 + d_G(v) - 4) \\ &= \sum_{x \in S_M} (d_G(x) - 4) = |D_5^M| + 2|D_6^M| + \cdots + (i-4)|D_i^M| + \cdots \\ &\leq |D_5(G_0')| + 2|D_6(G_0')| \cdots + (i-4)|D_i(G_0')| \cdots \leq |D_3(G_0')| - 10. \end{aligned}$$

This proves Lemma 6.1. □

Proof of Theorem 4.2. Let $V_a = S_0 \cup S_1^* \cup S_M$ which are defined in Section 4. By Lemma 5.3, $S_3 = V(G_0') - V_a$ is an independent set and all the nontrivial vertices are in $S_0 \cup S_1^*$. Then V_a is a vertex covering of G_0' containing all the nontrivial vertices of G_0'.

Claim 1. If G_0' has a vertex covering V_c with $|V_c| \leq 12$ and V_c contains all the nontrivial vertices of G_0', then G has a DCT.

By Theorem 3.2, $\kappa'(G_0') \geq 3$. Since G_0' can not be contracted to the Petersen graph in the way stated in Theorem 4.1 with $S = V_c$, G_0' has a closed trail Θ_c such that $V_c \subseteq V(\Theta_c)$. Since V_c is a vertex covering of G_0', Θ_c is a DCT of G_0'. Since V_c contains all the nontrivial vertices of G_0', Θ_c contains all the nontrivial vertices of G_0'. By Theorem 3.2, G has a DCT. Claim 1 is proved.

If $|V_a| \leq 12$, then by Claim 1, G has a DCT. We are done for this case.

In the following, we assume that $|S_0| + |S_1^*| + |S_M| = |V_a| \geq 13$.

Case 1. $|S_0| + |S_1^*| \leq 11$.

Since $|S_0| + |S_1^*| + |S_M| = |V_a| \geq 13$, $|S_M| \geq 2$. Thus, $|M_\Phi| \geq 1$. By Lemma 6.1 and $\overline{\sigma}_2(G) \geq 18$, $|D_3(G_0')| \geq 10 + |M_\Phi|(\overline{\sigma}_2(G) - 8) \geq 20$.

Let $S_0^3 = D_3(G_0') \cap S_0$, let $S_0^* = S_0 - S_0^3$ and let $S_1^3 = D_3(G_0') - S_0^3$. Then $|S_0| = |S_0^3| + |S_0^*|$ and

$$|S_1^3| = |D_3(G_0')| - |S_0^3|. \tag{16}$$

Note that $S_1^3 = D_3(G_0') \cap S_1$. Since $\overline{\sigma}_2(G) \geq 18$, by Lemma 5.3(b), for each $v \in S_1^3$, $N_{G_0'}(v) \subseteq S_0 \cup S_1^*$. Thus S_1^3 is an independent set in G_0'. Let $Y = \cup_{v \in S_1^3} N_{G_0'}(v)$. Then $Y \subseteq S_0 \cup S_1^*$ and so

$$|Y| \leq |S_0| + |S_1^*|. \tag{17}$$

Let Θ_b be the subgraph in G_0' induced by the edges between S_1^3 and Y. Then $|V(\Theta_b)| = |S_1^3| + |Y|$. Since $d_{G_0'}(v) = 3$ for each $v \in S_1^3$ and S_1^3 is an independent set, $|E(\Theta_b)| = 3|S_1^3|$. Since $|S_0^3| \leq |S_0| \leq 11$ and $|D_3(G_0')| \geq 20$, $|S_1^3| = |D_3(G_0')| - |S_0^3| \geq 9$ and so $\Theta_b \notin \{K_1, K_2, K_{2,s}\}$. By Theorem 3.1, $|E(\Theta_b)| \leq 2|V(\Theta_b)| - 5$. By (16), (17) and $|S_0| = |S_0^3| + |S_0^*|$,

$$\begin{aligned} 3|S_1^3| &= |E(\Theta_b)| \leq 2|V(\Theta_b)| - 5 = 2|S_1^3| + 2|Y| - 5; \\ 5 + |S_1^3| &\leq 2|Y| \leq 2|S_0| + 2|S_1^*|; \\ 5 + |D_3(G_0')| - |S_0^3| &\leq 2|S_0^3| + 2|S_0^*| + 2|S_1^*|; \\ 5 + |D_3(G_0')| &\leq 3|S_0^3| + 2|S_0^*| + 2|S_1^*| \leq 3|S_0| + 2|S_1^*|. \end{aligned} \tag{18}$$

By Lemma 6.1, $|D_3(G_0')| \geq 10 + |M_\Phi|(\overline{\sigma}_2(G) - 8)$. By (18), $\overline{\sigma}_2(G) \geq 18$ and $|S_0| + |S_1^*| \leq 11$,

$$\begin{aligned} 5 + (10 + |M_\Phi|(\overline{\sigma}_2(G) - 8)) &\leq 5 + |D_3(G_0')| \leq 3|S_0| + 2|S_1^*| \leq 3(|S_0| + |S_1^*|); \\ 15 + 10|M_\Phi| &\leq 33. \end{aligned}$$

Since $|M_\Phi| > 0$ is an integer, $|M_\Phi| = 1$.

Let $e = ab$ be the edge in M_Φ. Since M_Φ is a maximum matching in $\Phi = G'_0[S_2]$, at most one (say b) of the vertices of $\{a, b\}$ may be adjacent to some vertices in $S_2 - \{a, b\}$ and the other one (say a) is not adjacent to vertices in $S_2 - \{a, b\}$. Thus, $S_2 - \{b\}$ is an independent set.

Let $V_b = S_0 \cup S_1^* \cup \{b\}$. Then V_b is a vertex covering of G'_0 and contains all the nontrivial vertices in G'_0. Since $|S_0| + |S_1^*| \le 11$, $|V_b| \le 12$. By Claim 1, G has a DCT. We are done for this case.

Case 2. $|S_0| + |S_1^*| \ge 12$.

We prove the following claim first.

Claim 2. $|S_0| \ge 11$. Furthermore if $\overline{\sigma}_2(G) \ge 25$, $|S_0| \ge 12$.

If $|S_1^*| = 0$, then $|S_0| \ge 12$. Claim 2 is true trivially. In the following, we assume that $S_1^* \ne \emptyset$.

Combining (15) and (18), and by the definitions of $D_i(G'_0)$ and $D_i^*(G'_0)$, for $i \ge 5$, we have

$$15 + (i - 4)|D_i^*(G'_0)| \le 3|S_0| + 2|S_1^*|. \tag{19}$$

Since $\overline{\sigma}_2(G) \ge 18$, for each $v \in S_1^*$, $d_{G'_0}(v) = d_G(v) \ge \overline{\sigma}_2(G) - 3 \ge 15$ and so $v \in D_{15}^*(G'_0)$. Thus, $|S_1^*| \le |D_{15}^*(G'_0)|$. By (19) with $i = 15$ and $|S_1^*| \ge 12 - |S_0|$,

$$\begin{aligned} 15 + 9|S_1^*| \le 15 + 11|D_{15}^*(G'_0)| - 2|S_1^*| &\le 3|S_0|; \\ 15 + 9(12 - |S_0|) &\le 3|S_0|. \end{aligned}$$

Thus, $123 \le 12|S_0|$ and so $|S_0| \ge 11$.

Similarly, if $\overline{\sigma}_2(G) \ge 25$, then $i = 25$ and so $243 \le 22|S_0|$. Thus, $|S_0| \ge 12$. The claim is proved.

Let V_{12} be a subset of V_a with $|V_{12}| = 12$ in which the vertices are chosen in the following way: first pick vertices from S_0, then if $|S_0| = 11$ pick a vertex from S_1^*.
By Claim 2, V_{12} contains at most one vertex in S_1^*.

By Theorem 4.1, G'_0 has a closed trail T_b such that $V_{12} \subseteq V(T_b)$. We assume that

$$T_b \text{ is a closed trail with } V_{12} \subseteq V(T_b) \text{ and with as many vertices of } V(G'_0) \text{ as possible.} \tag{20}$$

Let $Z_0 = V_{12} \cap S_0$, and let $Z_1 = V_{12} \cap S_1^*$. Then $V_{12} = Z_0 \cup Z_1$ and $|Z_1| \le 1$. Let $V_T = V(T_b) - V_{12}$. Then $V(T_b) = V_{12} \cup V_T$, $V_T \subseteq S_1$ and

$$|V(T_b)| = |V_{12}| + |V_T| = 12 + |V_T|, \quad |Z_0| + |Z_1| = |V_{12}| = 12 \text{ and } |Z_0| \ge 11. \tag{21}$$

Let $\Phi_0 = G'_0[V(T_b)]$, the graph induced by the vertex set $V(T_b)$. Then $V(\Phi_0) = V(T_b)$, $E(T_b) \subseteq E(\Phi_0)$, and T_b is a spanning closed trail of Φ_0. Thus, $\Phi_0 \in \mathcal{SL}$.

For $v \in Z_0$, let $\Gamma_0(v)$ be the collapsible preimage of v in G_0. Let $\Phi_1 = G[E(\Phi_0) \cup_{v \in Z_0} E(\Gamma_0(v))]$. Then the reduction of $\Phi_1 = \Phi_1/(\cup_{v\in Z_0} E(\Gamma_0(v))) = \Phi_0 \in \mathcal{SL}$. By Theorem 3.1, $\Phi_1 \in \mathcal{SL}$ with $(\cup_{v\in Z_0} V(\Gamma_0(v))) \cup Z_1 \cup V_T \subseteq V(\Phi_1)$.

For $v \in V(T_b) \subseteq V(G'_0)$, let $E_0(v)$ be the set of edges incident with v in Φ_0. Then $|E_0(v)| = d_{\Phi_0}(v)$. Let $\Gamma_+(v)$ be the subgraph induced by the edges of $E(\Gamma(v))$ and all the edges incident with v in G'_0. Then $|E(\Gamma_+(v))| = |E(\Gamma(v))| + d_{G'_0}(v)$. For any $u, v \in Z_0$ and $u \ne v$,

$$(E(\Gamma_+(u)) - E_0(u)) \cap (E(\Gamma_+(v)) - E_0(v)) = \emptyset. \tag{22}$$

For $v \in Z_0$, by Lemma 5.3(a), $|E(\Gamma(v))| \ge \overline{\sigma}_2(G) - d_{G'_0}(v) - 1$. Then

$$|E(\Gamma_+(v)) - E_0(v)| \ge (|E(\Gamma(v))| + d_{G'_0}(v)) - d_{\Phi_0}(v) \ge \overline{\sigma}_2(G) - 1 - d_{\Phi_0}(v).$$

Hence,

$$\sum_{v\in Z_0}|E(\Gamma_+(v))-E_0(v)| \quad\geq\quad |Z_0|(\overline{\sigma}_2(G)-1)-\sum_{v\in Z_0}d_{\Phi_0}(v). \tag{23}$$

For $v\in Z_1\cup V_T$, $d_{G_0'}(v)=d_G(v)$. For any $u,v\in Z_1\cup V_T$ and $u\neq v$,

$$(E_G(u)-E_0(u))\cap(E_G(v)-E_0(v))=\emptyset. \tag{24}$$

For $v\in Z_1$, $d_G(v)\geq\overline{\sigma}_2(G)-3$ and $|E_G(v)-E_0(v)|=d_G(v)-d_{\Phi_0}(v)\geq(\overline{\sigma}_2(G)-3)-d_{\Phi_0}(v)$. Then

$$\sum_{v\in Z_1}|E_G(v)-E_0(v)|=\sum_{v\in Z_1}(d_G(v)-d_{\Phi_0}(v))\geq|Z_1|(\overline{\sigma}_2(G)-3)-\sum_{v\in Z_1}d_{\Phi_0}(v). \tag{25}$$

For $v\in V_T\subseteq S_1$, since $d_G(v)=d_{G_0}(v)\geq 3$,

$$\sum_{v\in V_T}d_G(v)\geq 3|V_T|. \tag{26}$$

Let $\Phi_2=G[E(\Phi_0)\cup_{v\in Z_0}E(\Gamma_+(v))\cup_{v\in Z_1\cup V_T}E_G(v)]$. Then Φ_1 is a dominating subgraph in Φ_2. Since Φ_1 has a SCT, Φ_2 has a DCT and

$$E(\Phi_2)\supseteq E(\Phi_0)\cup_{v\in Z_0}(E(\Gamma_+(v))-E_0(v))\cup_{v\in Z_1\cup V_T}(E_G(v)-E_0(v)). \tag{27}$$

By (27), (22) and (24), and by (23) and (25),

$$\begin{aligned}|E(\Phi_2)| &\geq |E(\Phi_0)|+\sum_{v\in Z_0}|E(\Gamma_+(v))-E_0(v)|+\sum_{v\in Z_1\cup V_T}|E_G(v)-E_0(v)|\\ &\geq |E(\Phi_0)|+|Z_0|(\overline{\sigma}_2(G)-1)-\sum_{v\in Z_0}d_{\Phi_0}(v)\\ &\qquad +|Z_1|(\overline{\sigma}_2(G)-3)-\sum_{v\in Z_1}d_{\Phi_0}(v)+\sum_{v\in V_T}(d_G(v)-d_{\Phi_0}(v)).\end{aligned} \tag{28}$$

Therefore, by (28), $\sum_{v\in V(\Phi_0)}d_{\Phi_0}(v)=2|E(\Phi_0)|$ and $V(\Phi_0)=Z_0\cup Z_1\cup V_T$,

$$\begin{aligned}|E(\Phi_2)| &\geq (|Z_0|+|Z_1|)\overline{\sigma}_2(G)-|Z_0|-3|Z_1|+|E(\Phi_0)|-\sum_{v\in V(\Phi_0)}d_{\Phi_0}(v)+\sum_{v\in V_T}d_G(v);\\ |E(\Phi_2)| &\geq (|Z_0|+|Z_1|)\overline{\sigma}_2(G)-|Z_0|-3|Z_1|-|E(\Phi_0)|+\sum_{v\in V_T}d_G(v).\end{aligned} \tag{29}$$

Since $|V(\Phi_0)|\geq|V_{12}|=12$, $\Phi_0\notin\{K_1,K_2\}$. As a subgraph of G_0', Φ_0 is a reduced graph. By Theorem 3.1(c), $|E(\Phi_0)|\leq 2|V(\Phi_0)|-4$. Since $|V(\Phi_0)|=|V(T_b)|=|V_{12}|+|V_T|$,

$$|E(\Phi_0)| \quad\leq\quad 2|V(\Phi_0)|-4=2|V(T_b)|-4=2|V_{12}|+2|V_T|-4=20+2|V_T|. \tag{30}$$

By (29), (30), (26), (21) $|Z_0|+|Z_1|=12$ and $|Z_1|\leq 1$,

$$\begin{aligned}|E(\Phi_2)| &\geq (|Z_0|+|Z_1|)\overline{\sigma}_2(G)-|Z_0|-3|Z_1|-|E(\Phi_0)|+\sum_{v\in V_T}d_G(v)\\ &\geq 12\overline{\sigma}_2(G)-12-2|Z_1|-(20+2|V_T|)+3|V_T|\\ &\geq 12\overline{\sigma}_2(G)-32-2|Z_1|+|V_T|\geq 12\overline{\sigma}_2(G)-34.\end{aligned}$$

Thus, Φ_2 is a DCT subgraph Θ of G with $|E(\Theta)| \geq 12\overline{\sigma}_2(G) - 34$. Theorem 4.2(a) is proved.

For Theorem 4.2(b), we have $\overline{\sigma}_2(G) \geq 25$. By Claim 2 above, $|Z_0| = |V_{12}| = 12$ and $|Z_1| = 0$. Note that by Theorem 3.1(c) either $\Phi_0 = K_{2,r}$ or $|E(\Phi_0)| \leq 2|V(\Phi_0)| - 5$.

If $|E(\Phi_0)| \leq 2|V(\Phi_0)| - 5$, then by (29) with $|Z_1| = 0$, $|Z_0| = 12$ and $|V(\Phi_0)| = |V(T_b)| = 12 + |V_T|$,

$$\begin{aligned} |E(\Phi_2)| &\geq |Z_0|\overline{\sigma}_2(G) - |Z_0| - |E(\Phi_0)| + \sum_{v \in V_T} d_G(v) \\ &\geq 12\overline{\sigma}_2(G) - 12 - (24 + 2|V_T| - 5) + 3|V_T| = 12\overline{\sigma}_2(G) - 31 + |V_T| \geq 12\overline{\sigma}_2(G) - 31. \end{aligned}$$

Theorem 4.2(b) is proved for this case.

Next, we assume that $\Phi_0 = K_{2,r}$ where $r = |V(\Phi_0)| - 2$.

Claim 3. $|V_T| > 0$.

To the contrary, suppose that $|V_T| = 0$. Then $|V(\Phi_0)| = |V(T_b)| = |V_{12}| = 12$ and so $\Phi_0 = K_{2,10}$.

Let $V(\Phi_0) = \{x_1, x_2, \cdots, x_{10}, y_1, y_2\}$ where $d_{\Phi_0}(x_i) = 2$ $(1 \leq i \leq 10)$ and $d_{\Phi_0}(y_j) = 10$ $(j = 1, 2)$. Since G'_0 is simple and K_3-free with $\kappa'(G'_0) \geq 3$, x_1 is adjacent to a vertex $z \notin \{x_1, \cdots, x_{10}, y_1, y_2\}$. Furthermore, $G'_0 - zx_1$ is 2-edge-connected. Therefore, there is a path P_z in $G'_0 - zx_1$ joining z to a vertex in $V(\Phi_0)$. We assume that P_z is a shortest path joining z to a vertex in $V(\Phi_0)$.

If P_z is a path from z to x_1 in $G'_0 - zx_1$, then $T_z = G'_0[E(T_b) \cup E(P_z) \cup \{zx_1\}]$ is a closed trail with $V(T_z) \supseteq V(T_b) \cup \{z\} \supset V_{12}$, contrary to (20).

If P_z is a path from z to y_i $(i = 1, 2)$ (say y_1) in $G'_0 - zx_1$, then $T_z = G'_0[(E(T_b) - \{x_1y_1\}) \cup E(P_z) \cup \{zx_1\}]$ is a closed trail with $V(T_z) \supseteq V(T_b) \cup \{z\} \supset V_{12}$. contrary to (20).

If P_z is a path from z to x_j $(2 \leq i \leq 10)$ (say x_2) in $G'_0 - zx_1$, then $T_z = G'_0[(E(T_b) - \{x_1y_1, x_2y_1\}) \cup E(P_z) \cup \{zx_1\}]$ is a closed trail with $V(T_z) \supseteq V(T_b) \cup \{z\} \supset V_{12}$, contrary to (20).

We reach contradictions for all the cases. Claim 3 is proved.

Since $\Phi_0 = K_{2,r}$ and $|V(\Phi_0)| = 12 + |V_T|$, $|E(\Phi_0)| = 2|V(\Phi_0)| - 4 = 20 + 2|V_T|$. By (29), $|Z_0| = 12$, $|Z_1| = 0$ and by Claim 3 $|V_T| \geq 1$,

$$\begin{aligned} |E(\Phi_2)| &\geq (|Z_0| + |Z_1|)\overline{\sigma}_2(G) - |Z_0| - 3|Z_1| - |E(\Phi_0)| + \sum_{v \in V_T} d_G(v) \\ &\geq 12\overline{\sigma}_2(G) - 12 - (20 + 2|V_T|) + 3|V_T| \geq 12\overline{\sigma}_2(G) - 12 - 20 + |V_T| \geq 12\overline{\sigma}_2(G) - 31. \end{aligned}$$

Thus, Φ_2 is a DCT subgraph of G for Theorem 4.2(b). The proof is complete. □

7 Graphs that are contractible to the Petersen graph

In the following, we assume that G is an essentially 3-edge-connected K_3-free simple graph with $\overline{\sigma}_2(G) \geq 7$. Let P_0 be the Petersen graph with $V(P_0) = \{v_1, \cdots, v_{10}\}$. When we say P_0 is a contraction graph of a graph G, it means that P_0 is obtained from G by the following sequence of contractions:

1) $G_1 = G/E_1$;
2) $G_0 = G_1/X_2(G)$;
3) $G'_0 = G_0/(E(\Gamma_1^0) \cup \cdots \cup E(\Gamma_c^0))$ where Γ_i^0 $(1 \leq i \leq c)$ is a maximum collapsible subgraph of G_0;
4) $P_0 = G'_0/(E(\Gamma_0^1(v_1)) \cup \cdots \cup E(\Gamma_0^1(v_{10})))$ where $\Gamma_0^1(v_i)$ is connected reduced subgraph of G'_0.

For each $v \in V(P_0)$, we define the following:

- $\Gamma_0^1(v)$ is the preimage of v in G'_0 (a reduced subgraph of G'_0).
- For each $u \in V(\Gamma_0^1(v))$, let $\Gamma_0(u)$ be the collapsible preimage of u in G_0.
- $\Gamma_0^2(v) = G_0[\cup_{u \in V(\Gamma_0^1(v))} V(\Gamma_0(u))]$ and so $\Gamma_0^1(v)$ is the reduction of $\Gamma_0^2(v)$.

- $\Gamma_1^2(v)$ is the preimage of v in G_1. Thus, $\Gamma_0^2(v) = \Gamma_1^2(v)/(X_2(G) \cap E(\Gamma_1^2(v))$.
- $\Gamma^*(v)$ is the preimage of v in G, which is the subgraph in G induced by the edges in $E(\Gamma_1^2(v))$ and the edges in E_1 that are incident with some vertices in $\Gamma_1^2(v)$.
- $\partial(\Gamma^*(v)) = \{u \in V(\Gamma^*(v)) \mid u$ is incident with an edge of P_0 $\}$, the set of vertices in $V(\Gamma^*(v))$ that are incident with some edges in $E_P(v)$. Then $|\partial(\Gamma^*(v))| \leq 3$.

If $\Gamma_0^1(v) = K_1$, then $\Gamma_0^2(v) = \Gamma_0(v)$, $\Gamma_1^2(v) = \Gamma_1(v)$ and $\Gamma^*(v) = \Gamma(v)$. Fig. 7.1 shows the contraction process from G to P_0.

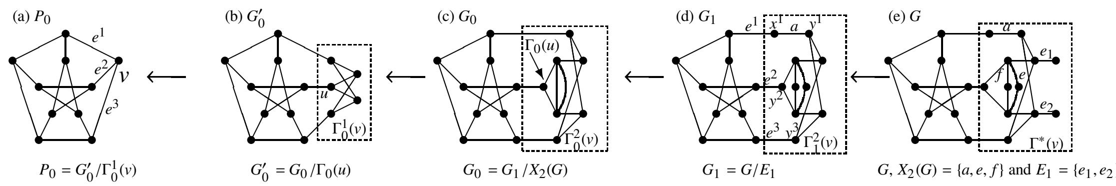


Fig. 7.1: A contraction process from G to a vertex v in P_0.

Fact 7.1. *For a vertex $v \in V(P_0)$, if $\Gamma_0^j(v) \neq K_1$ ($j = 1, 2$), each of the following holds:*

(i) $\Gamma_0^j(v)$ is 2-edge-connected and so $d_{\Gamma_0^j(v)}(x) \geq 2$ for any $x \in V(\Gamma_0^j(v))$.

(ii) $D_2(\Gamma_0^j(v)) \subseteq \partial(\Gamma^*(v))$ and so $|D_2(\Gamma_0^j(v))| \leq 3$.

Proof. Since G is essentially 3-edge-connected, by Theorem 3.2 $\kappa'(G_0') \geq \kappa'(G_0) \geq 3$. Since $\Gamma_0^1(v)$ is the reduction of $\Gamma_0^2(v)$, $\kappa'(\Gamma_0^1(v)) \geq \kappa'(\Gamma_0^2(v))$. We only need to prove (i) for the case $\Gamma_0^2(v)$.

To the contrary, suppose that $\kappa'(\Gamma_0^2(v)) = 1$. Let Φ_1 and Φ_2 be the two components of $\Gamma_0^2(v) - e$ where e is an edge-cut. Since $d_{P_0}(v) = 3$, only three edges of G_0 outside of $\Gamma_0^2(v)$ incident with some vertices in $\Gamma_0^2(v)$. Of these three edges, at most one of them is incident with one of Φ_i ($i = 1, 2$). Thus, G_0 is at most 2-edge-connected, contrary to that $\kappa'(G_0) \geq 3$. Case (i) is proved.

Case (ii) follows from the definition and the fact that $\kappa'(G_0) \geq 3$ and $|\partial(\Gamma^*(v))| \leq d_{P_0}(v) = 3$. □

With P_0 as a contraction graph of G, to find a DCT subgraph of G with large size, it is a reverse process of the contraction sequence above. The following lemma will be needed when $\Gamma_0^1(v) \neq K_1$.

Lemma 7.2. *For a vertex $v \in V(P_0)$, let $\Gamma_0^1(v)$ be the preimage of v in G_0' and $\Gamma_0^1(v) \neq K_1$. Then $D_2(\Gamma_0^1(v)) \subseteq \partial(\Gamma^*(v))$ and $|\partial(\Gamma^*(v))| \leq 3$. Furthermore, for any $x, y, z \in \partial(\Gamma^*(v))$ (x, y and z may not be distinct) there is a (x, y)-trail T_v containing z such that $\alpha'(T_v) \geq 2$ and one of the following holds:*

(a) $\Gamma_0^1(v) \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}$ and

$$|\mathcal{E}_{\Gamma^*(v)}(T_v)| \geq \begin{cases} 2\overline{\sigma}_2(G) - 2 & \text{if } \Gamma_0^1(v) = K_{2,3} \text{ and } \alpha(\Gamma^*(v)) = 2; \\ 3\overline{\sigma}_2(G) - 6 & \text{if } \Gamma_0^1(v) \in \{K_{2,3}, K_{1,3}(1,1,1)\} \text{ and } \alpha(\Gamma^*(v)) = 3; \\ 4\overline{\sigma}_2(G) - 10 & \text{if } \Gamma_0^1(v) \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\} \text{ and } \alpha(\Gamma^*(v)) \geq 4. \end{cases}$$

(b) $|V(\Gamma_0^1(v))| \geq 8$. Then $\alpha'(\Gamma^*(v)) \geq \alpha'(\Gamma_0^1(v)) \geq 4$ and $\Gamma_0^1(v)$ has an (x, y)-trail T_v^0 where $x, y \in V(\Gamma_0^1(v))$ that are incident with two of the edges in $\{e_v^1, e_v^2, e_v^3\}$ and $|\mathcal{E}_{\Gamma^*(v)}(T_v^0)| \geq 4\overline{\sigma}_2(G) - 14$.

Proof. Since G_0' is 3-edge-connected and K_3-free and $\Gamma_0^1(v)$ is a subgraph of G_0', $\Gamma_0^1(v)$ is reduced and K_3-free. By Fact 7.1, $\Gamma_0^1(v)$ is 2-edge-connected, $D_2(\Gamma_0^1(v)) \subseteq \partial(\Gamma^*(v))$ and $|\partial(\Gamma^*(v))| \leq 3$. Let $\partial(\Gamma^*(v)) = \{x, y, z\}$ (x, y and z may not be distinct). By Lemma 5.1, $\Gamma_0^1(v)$ has a (x, y)-trail T_v containing z. We assume that T_v is a longest one.

We prove $\alpha'(T_v) \geq 2$ first.

To the contrary, suppose that $\alpha'(T_v) = 1$. Then one of the following holds.

(1) $T_v = xy$ (and so $z \in \{x, y\}$, say $z = y$); (2) $T_v = xzy$.

(1) $T_v = xy$ with $z = y$. Since $\Gamma_0^1(v)$ is 2-edge-connected and K_3-free, there is a longer path in $\Gamma_0^1(v)$ joining x and y in $\Gamma_0^1(v) - \{xy\}$, contrary to that T_v is a longest one.

(2) $T_v = xzy$.

Since $\Gamma_0^1(v)$ is 2-edge-connected, x is adjacent to a vertex (say w) in $N_{\Gamma_0^1(v)}(x) - \{z\}$. Since G_0' is 3-edge-connected, by Menger's Theorem, there are at least three edge-disjoint paths joining w and a vertex (say u) in $G_0' - V(\Gamma_0^1(v))$. Since $\{x, y, z\}$ is a vertex cut of G_0' that separates w and u, there are at least two edge-disjoint paths (say P_w^1 and P_w^2) joining w to vertices in $\{x, y, z\}$ in $\Gamma_0^1(v) - \{xw\}$. We assume that P_w^i ($i = 1, 2$) is a shortest path joining w to a vertex in $\{x, y, z\}$.

If P_w^1 (or P_w^2) is a (w, x)-path in $\Gamma_0^1(v) - \{xw\}$, then xwP_w^1 is a cycle and so $T_x = G_0'[\{xw\} \cup E(P_w^1) \cup \{zy\}]$ is a (x, y)-trail containing z in $\Gamma_0^1(v)$, contrary to that T_v is a longest one.

If P_w^1 (or P_w^2) is a (w, z)-path in $\Gamma_0^1(v) - \{xw\}$, then $T_x = G_0'[\{xw\} \cup E(P_w^1) \cup \{zy\}]$ is a (x, y)-trail containing z in $\Gamma_0^1(v)$, contrary to that T_v is a longest one.

If none of the P_w^1 and P_w^2 is a (w, x)- or (w, z)-path, then P_w^1 and P_w^2 are edge-disjoint (w, y)-paths and so $G_0'[E_{G_0'}(P_w^1) \cup E_{G_0'}(P_w^2)]$ is a closed trail containing y. Then $T_y = G_0'[\{xz, zy\} \cup E(P_w^1) \cup E(P_w^2)]$ is a (x, y)-trail containing z in $\Gamma_0^1(v)$ which has more vertices than T_v has, a contradiction.

Thus, $\alpha'(T_v) \geq 2$.

Next, we find the size of $\mathcal{E}_{\Gamma^*(v)}(T_v)$ which is defined by (2) with $\Gamma^*(v) = G_T(v)$ and $T_v = \Theta(v)$.

Case (a) $|V(\Gamma_0^1(v))| \leq 7$.

By Theorem 3.1(d), $\Gamma_0^1(v) \in \{K_{2,3}, K_{1,3}(1, 1, 1), J'(1, 1)\}$. For each edge $zw \in E(\Gamma_0^1(v))$, since $d_{G_0'}(z) + d_{G_0'}(w) \leq 6$ and $\overline{\sigma}_2(G) \geq 7$, either z or w is a contracted vertex of G_0'. Let W be the set of the contracted vertices in $\Gamma_0^1(v)$. Let $\beta = \beta(\Gamma_0^1(v))$ be the covering number of $\Gamma_0^1(v)$. Then $|W| \geq \beta$.

For a vertex $w \in W$, either $\Gamma_0(w)$ is a nontrivial collapsible preimage of w in G_0 or $\Gamma(w) = K_{1,s}$. Then $E(\Gamma(w)) \subseteq \mathcal{E}_{\Gamma^*(v)}(T_v)$. Since $W \subseteq S_0$, by Lemma 5.3(a) with $t = 1$, $|E(\Gamma(w))| \geq \overline{\sigma}_2(G) - 4$.

For $\Gamma_0^1(v) \in \{K_{2,3}, K_{1,3}(1, 1, 1), J'(1, 1)\}$, since $E(T_v^0) \bigcup_{w\in W} E(\Gamma(w)) \subseteq \mathcal{E}_{\Gamma^*(v)}(T_v)$

$$|\mathcal{E}_{\Gamma^*(v)}(T_v)| \geq |E(\Gamma_0^1(v))| + \sum_{w\in W} |E(\Gamma(w))| \geq |E(\Gamma_0^1(v))| + |W|\overline{\sigma}_2(G) - 4|W|. \tag{31}$$

Picking an edge from $\Gamma(w)$ for each $w \in W$, we have a matching of $\Gamma^*(v)$. Thus, $\alpha'(\Gamma^*(v)) \geq |W|$. Since $\beta(K_{2,3}) = 2$, $\beta(K_{1,3}(1, 1, 1)) = 3$ and $\beta(J'(1, 1)) = 4$, by (31), $\alpha'(\Gamma^*(v)) \geq |W| \geq \beta$ and $6 = |E(K_{2,3})| < 9 = |E(K_{1,3}(1, 1, 1))| = |E(J'(1, 1))|$,

$$\begin{aligned} |\mathcal{E}_{\Gamma^*(v)}(T_v)| &\geq |W|\overline{\sigma}_2(G) + |E(\Gamma_0^1(v))| - 4|W| \geq |W|\overline{\sigma}_2(G) + 6 - 4|W| \\ &\geq \begin{cases} 2\overline{\sigma}_2(G) - 2 & \text{if } \Gamma_0^1(v) = K_{2,3} \text{ and } \alpha'(\Gamma^*(v)) = |W| = 2; \\ 3\overline{\sigma}_2(G) - 6 & \text{if } \Gamma_0^1(v) \in \{K_{2,3}, K_{1,3}(1,1,1)\} \text{ and } \alpha'(\Gamma^*(v)) = |W| = 3; \\ 4\overline{\sigma}_2(G) - 10 & \text{if } \Gamma_0^1(v) \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\} \text{ and } \alpha'(\Gamma^*(v)) \geq 4. \end{cases} \end{aligned}$$

For $|W| = 4 + j$ ($j \geq 0$), $|W|\overline{\sigma}_2(G) + 6 - 4|W| = 4\overline{\sigma}_2(G) - 10 + j(\overline{\sigma}_2(G) - 4) \geq 4\overline{\sigma}_2(G) - 10$. Thus, $|\mathcal{E}_{\Gamma^*(v)}(T_v)| \geq 4\overline{\sigma}_2(G) - 10$ if $\alpha'(\Gamma^*(v)) \geq |W| \geq 4$. Lemma 7.2(a) is proved.

Case (b) $n_1 = |V(\Gamma_0^1(v))| \geq 8$.

Since $l = |D_2(\Gamma_0^1(v))| \leq 3$, $\Gamma_0^1(v) \neq K_{2,s}$. By Theorem 3.1(e),

$$\alpha'(\Gamma_0^1(v)) \geq \min\left\{\frac{n_1 - 1}{2}, \frac{n_1 + 5 - l}{3}\right\} \geq \min\left\{\frac{8 - 1}{2}, \frac{8 + 5 - 3}{3}\right\} > 3.$$

Thus, $\alpha'(\Gamma_0^1(v)) \geq 4$.

Let M be a matching of size 4 in $\Gamma_0^1(v)$ and let V_8 be the set of the 8 vertices in M. Let v_p be a vertex in $V(P_0) - V(\Gamma_0^1(v))$ and let $S = V_8 \cup \{v_p\}$. Then $|S| \leq 9$. By Theorem 4.1, G_0 has a closed trail T_1 containing all the vertices in S. Then T_1 contain exactly two of the edges in $\{e_v^1, e_v^2, e_v^3\}$. We may assume $e_i^1 = v_2x_i, e_i^2 = v_3y_i \in E(T_1)$ where x_i and y_i are in $\Gamma_0^1(v)$. Thus, edges in $E(T_1) \cap E(\Gamma_0^1(v))$ induced a (x_i, y_i)-trail T_v^0 containing all the vertices in V_8. Since T_v^0 contains the vertices of a matching of size 4, by Lemma 5.2(a) with $t = 4$ and $i(T_v^0) \leq d_{G_T}(v) = d_{G_0'}(v) = 3$, $|\mathcal{E}_{\Gamma^*(v)}(T_v^0)| \geq 4\overline{\sigma}_2(G) - 14$. □

Remark 7.3. *When we say we have a T_v trail with the estimated size $|\mathcal{E}_{\Gamma^*(v)}(T_v)|$ in Lemma 5.2 or Lemma 7.2, it means that such trail T_v exists for any preselected two edges in $E_P(v)$. For the T_v^0 trail in case (b) above, we only know that T_v^0 is incident with two edges in $E_P(v)$, not for any preselected two edges. When $|V(\Gamma_0^1(v))| \geq 8$, for any two preselected edges in $E_P(v)$, we can have a trail T_v for the two selected edges but we only know that $\alpha'(T_v) \geq 2$ and by Lemma 5.2 $|\mathcal{E}_{\Gamma^*(v)}(T_v)| \geq 2\overline{\sigma}_2(G) - 7$, that is smaller than $|\mathcal{E}_{\Gamma^*(v)}(T_v^0)| \geq 4\overline{\sigma}_2(G) - 14$. In the proof of Theorem 4.3, if a vertex $v \in V(P_0)$ has its preimage $\Gamma_0^1(v)$ with $|V(\Gamma_0^1(v))| \geq 8$, we use a (x, y)-trail T_v^0 with $|\mathcal{E}_{\Gamma^*(v)}(T_v^0)| \geq 4\overline{\sigma}_2(G) - 14$ given in Lemma 7.2(b) and so we know the two edges in $E_P(v)$ incident with x and y in T_v^0, respectively. Thus, to select a dominating cycle Θ_0 in P_0, we pick the vertex v and the two edges first, and then pick the rest of the vertices and edges to form the dominating cycle Θ_0 in P_0.*

For each $v \in V(P_0)$, let $E_P(v) = \{e_v^1, e_v^2, e_v^3\}$ be the set of three edges in P_0 incident with v, which is considered as a subset of $E(G)$. We assume that e_v^i is incident with x_v^i in $\Gamma_1^2(v)$ $(i = 1, 2, 3)$ (note that x_v^1, x_v^2 and x_v^3 may not be distinct). If $x_v^i \in D_2(G) \cap V(\Gamma_1^2(v))$, then let $y_v^ix_v^i$ be an edge in $X_2(G)$ with $y_v^i \in V(\Gamma_1^2(v))$. Then y_v^i is a nontrivial vertex and $d_{\Gamma_1^2(v)}(x_v^i) = 1$. Since G is essentially 3-edge-connected, $d_{\Gamma_1^2(v)}(y_v^i) \geq 3$. If $x_v^i \notin D_2(G)$, we use $y_v^i = x_v^i$ in $V(\Gamma_1^2(v))$ (see Fig. 7.1 (a) and (d)).

The following is the procedures to construct a DCT in G from P_0:

(a) Pick a 9 vertex cycle Θ_0.
We assume that $V(\Theta_0) = \{v_1, v_2, \cdots, v_9\}$ and $E(\Theta_0) = \{e_{v_j}^1, e_{v_j}^2 \mid j = 1, \cdots 9\}$. By Lemma 7.2 and Remark 7.3, we assume that v_1 is the vertex with largest $\alpha'(\Gamma_0^1(v))$.

(b) For each $v \in V(\Theta_0)$ with $\Gamma_0^2(v) \neq K_1$ and with y_v^1 and y_v^2 in $\Gamma_0^2(v)$ that are incident with the two edges e_v^1 and e_v^2 in Θ_0, we construct a (y_v^1, y_v^2)-trail T_v according to $\Gamma_0^1(v) = K_1$ or not:
(b1) If $\Gamma_0^1(v) = K_1$ then $\Gamma_0^2(v) = \Gamma_0(v)$, a collapsible graph. Let $R = \{y_v^1, y_v^2\}$ if $y_v^1 \neq y_v^2$; and let $R = \emptyset$ if $y_v^1 = y_v^2$. Since $\Gamma_0^2(v)$ is collapsible, $\Gamma_0^2(v)$ has a spanning connected subgraph Ψ_v such that $O(\Psi_v) = R$. Then $T_v = \Psi_v$ is a spanning (y_v^1, y_v^2)-trail in $\Gamma_0^2(v)$. Thus, T_v is a dominating (y_v^1, y_v^2)-trail in $\Gamma^*(v)$ and $E(\Gamma^*(v)) = \mathcal{E}_{\Gamma^*(v)}(T_v)$.
(b2) If $\Gamma_0^1(v) \neq K_1$, then we construct a (y_v^1, y_v^2)-trail T_v as discussed in Lemma 7.2.
Since $\Gamma_0^2(v) = \Gamma_1^2(v)/(X_2(G) \cap E(\Gamma_1^2(v))$, T_v can be extended as (x_v^1, x_v^2)-trail containing y_v^3 in $\Gamma_1^2(v)$ (and in $\Gamma^*(v)$). If $y_v^3 \neq x_v^3$, then y_v^3 is a nontrivial vertex and $i(T_v) = 2$ since the edge incident with x_v^3 in $E_P(v)$ is not incident with a vertex in T_v, but $y_v^3x_v^3 \in \mathcal{E}_{\Gamma^*(v)}(T_v)$. In the following, for each $v_j = v \in V(P_0)$, we use T_j for T_v as the (x_j^1, x_j^2)-trail containing y_j^3 in $\Gamma^*(v_j)$ with $|\mathcal{E}_{\Gamma^*(v_j)}(T_j)|$ as large as possible. (See Fig. 7.2 for an example).

(c) Let $\Theta_1 = G[E(\Theta_0) \cup_{j=1}^9 E(T_j)]$ where $T_j = T_{v_j}$ found in step (b). Then Θ_1 is a closed trail.

(d) Let Θ be the graph induced by all the edges in $E(\Theta_1)$ and all the edges incident with vertices in $V(\Theta_1)$. Then Θ is a DCT subgraph of G since Θ_1 is a DCT in Θ.

In Lemmas 5.2, $i(T_j)$, the number of the edges outside of $\Gamma^*(v_j)$ incident with some vertices in $V(T_j)$, is not counted for $|\mathcal{E}_{\Gamma^*(v_j)}(T_j)|$. If $i(T_j) = 3$, then $E_P(v_j) \subseteq E(\Theta)$. If $i(T_j) = 2$, then maybe only two of the edges in $E_P(v_j)$ are in $E(\Theta)$ but by Lemmas 5.2 the lower bound on $|\mathcal{E}_{\Gamma(v_j)}(T_j)|$ is one more than the case of $i(T_j) = 3$. Thus, for counting the number of edges in $E(\Theta)$, we can assume that $i(T_j) = 3$ and so we assume $E(P_0) \subseteq E(\Theta)$. Therefore,

$$E(\Theta) \supseteq E(P_0) \cup_{j=1}^{9} \mathcal{E}_{\Gamma^*(v_j)}(T_j). \tag{32}$$

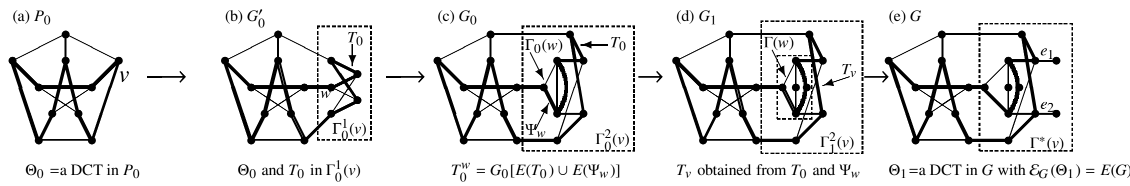


Fig. 7.2: A process to obtain a DCT (marked by thick-lines) from P_0 to G.

8 Proof of Theorem 4.3

Proof of Theorem 4.3. Without loss of generality, we assume that G does not have a DCT.

Let P_0 be the Petersen graph with $V(P_0) = \{v_1, \cdots, v_{10}\}$ as the contraction graph of G_0' as stated.

Without loss of generality, we assume that $|V(\Gamma^*(v_{10}))| = \min\{|V(\Gamma^*(v_i))| \; v_i \in V(P_0)\}$.

Claim 1. $|V(\Gamma^*(v_{10}))| > 1$.

To the contrary, suppose that $|V(\Gamma^*(v_{10}))| = 1$. Then $d_{G_0'}(v_{10}) = d_G(v_{10}) = d_{P_0}(v_{10}) = 3$.

Case 1. $G_0' = P_0$, i.e., $|V(\Gamma_0^1(v_i))| = 1$ for all $v_i \in V(P_0)$ $(1 \le i \le 10)$.

Since $|V(\Gamma^*(v_{10}))| = 1$, $d_G(v_{10}) = d_{G_0'}(v_{10}) = 3$. Since $\overline{\sigma}_2(G) \ge 7$, v_{10} is not adjacent to any vertex in $D_2(G)$ and so v_{10} is a trivial vertex in G_0'. By inspection, G_0' has a cycle Θ_0 containing v_i $(1 \le i \le 9)$. Then Θ_0 contains all the nontrivial vertices of G_0'. By Theorem 3.2(b), G has a DCT, a contradiction.

Case 2. G_0' is contracted to P_0 such that $V(\Gamma_0^1(v_i)) \cap V_a \ne \emptyset$ for all $v_i \in V(P_0)$ $(1 \le i \le 10)$.

Since $|V(\Gamma^*(v_{10}))| = 1$, $V(\Gamma_0^1(v_{10})) = \{v_{10}\}$. Thus, $\{v_{10}\} = V(\Gamma_0^1(v_{10})) \cap V_a \subseteq V_a = S_0 \cup S_1^* \cup S_M$.

If $v_{10} \in S_0$, then v_{10} is a contracted vertex and so $|V(\Gamma^*(v_{10}))| > 1$, a contradiction.

If $v_{10} \in S_1^*$, then since $\overline{\sigma}_2(G) \ge 7$, $d_{G_0'}(v_{10}) \ge \overline{\sigma}_2(G) - 3 \ge 4$, contrary to that $d_{G_0'}(v_{10}) = 3$.

If $v_{10} \in S_M$, then there is a non-contracted vertex z in S_M such that $zv_{10} \in E(G_0') \subseteq E(G)$ with $d_G(v_{10}) = d_{G_0'}(v_{10})$ and $d_G(z) = d_{G_0'}(z)$. Therefore, $d_{G_0'}(v_{10}) + d_{G_0'}(z) = d_G(v_{10}) + d_G(z) \ge \overline{\sigma}_2(G)$. Since $d_G(v_{10}) = 3$, $d_G(z) \ge \overline{\sigma}_2(G) - 3$. Thus, $z \in S_1^*$, contrary to that $z \in S_M \subseteq V(G_0') - (S_0 \cup S_1^*)$.

We reach a contradiction for each of the cases above. Claim 1 is proved.

If $\Gamma^*(v_i) = K_{1,r}$ for all $v_i \in V(P_0)$, then $G \in \mathcal{P}_1$. Theorem 4.3(a) is proved.

In the following we assume that $\Gamma^*(v_1) \ne K_{1,r}$. Then $\Gamma^*(v_1)$ is not a tree and $\Gamma_1^2(v_1)$ is a nontrivial connected subgraph in G_1 and so $\alpha'(T_1) \ge 1$. But $\Gamma_0^1(v_1)$ may be either trivial or nontrivial.

Let Θ_0, Θ_1 and Θ be the subgraphs defined in Section 7. If there is a vertex $v \in V(P_0)$ with $|V(\Gamma_0^1(v))| \ge 8$, Θ_0 is the one defined in Remark 7.3 after Lemma 7.2.

For each $v_i \in V(P_0)$, if $\Gamma^*(v_i) = K_{1,r}$. then $T_i = K_1$. By Proposition 3.3, $|\mathcal{E}_{\Gamma^*(v_i)}(T_i)| \ge \overline{\sigma}_2(G) - 4$. If $\Gamma^*(v_i) \ne K_{1,r}$, then by Lemmas 5.2 and 7.2, $\Gamma^*(v)$ has a trail T_i as a part of the subgraph Θ_1 with

$|\mathcal{E}_{\Gamma^*(v_i)}(T_i)| \geq 2\overline{\sigma}_2(G) - 7 \geq \overline{\sigma}_2(G)$. Thus, for each $v_i \in V(P_0)$, in the worst case,

$$|\mathcal{E}_{\Gamma^*(v_i)}(T_i)| \geq \overline{\sigma}_2(G) - 4. \tag{33}$$

If $|V(\Gamma_0^1(v_1))| \geq 8$, then by Lemma 7.2(b), $\Gamma_0^1(v_1)$ has a trail $T_{v_1}^0$ with $|\mathcal{E}_{\Gamma^*(v_1)}(T_{v_1}^0)| \geq 4\overline{\sigma}_2(G) - 14$. Hence, by (32), (33) and $|\mathcal{E}_{\Gamma^*(v_1)}(T_{v_1}^0)| \geq 4\overline{\sigma}_2(G) - 14$,

$$\begin{aligned} |E(\Theta)| &\geq |E(P_0)| + |\mathcal{E}_{\Gamma^*(v_1)}(T_{v_1}^0)| + \sum_{i=2} |\mathcal{E}_{\Gamma^*(v_i)}(T_i)| \\ &\geq 15 + (4\overline{\sigma}_2(G) - 14) + 8(\overline{\sigma}_2(G) - 4) = 12\overline{\sigma}_2(G) - 31. \end{aligned}$$

Thus, Theorem 4.3 holds.

In the following, we assume that $|V(\Gamma_0^1(v_i))| \leq 7$ for all $v_i \in V(P_0)$ and

$$\alpha'(\Gamma^*(v_1)) \geq \alpha'(\Gamma^*(v_2)) \cdots \geq \alpha'(\Gamma^*(v_{10})). \tag{34}$$

Claim 2. For $v \in V(\Theta_0)$, if $\Gamma^*(v) \neq K_{1,r}$, then $\alpha'(\Gamma^*(v)) \geq 2$ and Θ contains a subgraph $\Psi(v)$ of $\Gamma^*(v)$ such that $\mathcal{E}_{\Gamma^*(v)}(\Psi(v)) \subseteq \mathcal{E}_{\Gamma^*(v)}(T_v) \subseteq E(\Theta)$ where T_v is the trail as a part of Θ defined above and

$$|\mathcal{E}_{\Gamma^*(v)}(\Psi(v))| \geq \begin{cases} 2\overline{\sigma}_2(G) - 6 & \text{if } \alpha'(\Gamma^*(v)) = 2; \\ 3\overline{\sigma}_2(G) - 12 & \text{if } \alpha'(\Gamma^*(v)) = 3; \\ 4\overline{\sigma}_2(G) - 17 & \text{if } \alpha'(\Gamma^*(v)) \geq 4; \\ 4\overline{\sigma}_2(G) - 14 & \text{if } \alpha'(\Gamma^*(v)) \geq 5 \text{ and } \overline{\sigma}_2(G) \geq 12. \end{cases} \tag{35}$$

Case A. $\Gamma_0^1(v) = K_1$. Then $\Gamma^*(v) = \Gamma(v)$ and $\Gamma_1^2(v) = \Gamma_1(v)$.

Since $\Gamma^*(v) \notin \{K_1, K_{1,r}\}$, $\Gamma_0^2(v) = \Gamma_0(v)$ is a nontrivial collapsible subgraph of G_0. Let $\Psi(v) = \Gamma_0(v)$. By the definition of Θ, $\mathcal{E}_{\Gamma^*(v)}(\Psi(v)) = E(\Gamma^*(v)) = \mathcal{E}_{\Gamma^*(v)}(T_v) \subseteq E(\Theta)$. Since G is K_3-free and simple, $\Gamma(v)$ and $\Gamma_1(v)$ are K_3-free and simple. Hence, $\alpha'(\Gamma^*(v)) \geq \alpha'(\Gamma_1^2(v)) \geq 2$. By Proposition 3.3 when $\alpha'(\Gamma^*(v)) = 2$, by (6) of Lemma 5.2 when $\alpha'(\Gamma^*(v)) = 3$ and by Lemma 5.2(b) when $\alpha'(\Gamma^*(v)) \geq 4$,

$$|\mathcal{E}_{\Gamma^*(v)}(\Psi(v))| = |E(\Gamma^*(v))| \geq \begin{cases} 2\overline{\sigma}_2(G) - 6 & \text{if } \alpha'(\Gamma^*(v)) = 2; \\ 3\overline{\sigma}_2(G) - 12 & \text{if } \alpha'(\Gamma^*(v)) = 3; \\ t\overline{\sigma}_2(G) - t^2 - 1 & \text{if } t = \alpha'(\Gamma^*(v)) \geq 4. \end{cases} \tag{36}$$

Case B. $\Gamma_0^1(v) \neq K_1$ and $|V(\Gamma_0^1(v))| \leq 7$. Then $\Gamma_0^1(v)$ is a nontrivial reduced subgraph.

Since $|V(\Gamma_0^1(v))| \leq 7$, by Lemma 7.2, $\Gamma^*(v)$ has a trail T_v as a part of Θ with

$$|\mathcal{E}_{\Gamma^*(v)}(T_v)| \geq \begin{cases} 2\overline{\sigma}_2(G) - 2 & \text{if } \alpha'(\Gamma^*(v)) = 2; \\ 3\overline{\sigma}_2(G) - 6 & \text{if } \alpha'(\Gamma^*(v)) = 3; \\ 4\overline{\sigma}_2(G) - 10 & \text{if } \alpha'(\Gamma^*(v)) \geq 4. \end{cases} \tag{37}$$

Thus, for $2 \leq \alpha'(\Gamma^*(v)) \leq 4$, (37) implies (36). If $\alpha'(\Gamma^*(v)) \geq 5$ and $\overline{\sigma}_2(G) \geq 12$, then by (36) with $t = 5$, $5\overline{\sigma}_2(G) - 26 \geq 4\overline{\sigma}_2(G) - 14$. Then T_v is the subgraph $\Psi(v)$. Claim 2 is proved.

Let n_0 be the number of $\Gamma^*(v_i) \neq K_{1,r}$. By Claim 2 and (34), for $1 \leq i \leq n_0$, Θ contains a subgraph $\Psi(v_i)$ in $\Gamma^*(v_i)$ with $|\mathcal{E}_{\Gamma^*(v_i)}(T_i)| = |\mathcal{E}_{\Gamma^*(v_i)}(\Psi(v_i))| \geq 2\overline{\sigma}_2(G) - 6$. By (32) and by (33) (for $i > n_0$),

$$|E(\Theta)| \geq |E(P_0)| + \sum_{i=1}^{n_0} |\mathcal{E}_{\Gamma^*(v_i)}(T_i)| + \sum_{i=n_0+1}^{9} |\mathcal{E}_{\Gamma^*(v_i)}(T_i)| \tag{38}$$

$$\geq 15 + n_0(2\overline{\sigma}_2(G) - 6) + (9 - n_0)(\overline{\sigma}_2(G) - 4) = (n_0 + 9)\overline{\sigma}_2(G) - 21 - 2n_0. \tag{39}$$

If $n_0 \geq 3$, then $G \notin \bigcup_{i=1}^{5} \mathcal{P}_i$. Let $n_0 = 3 + j$ and $j \geq 0$. Then by (39) and $\overline{\sigma}_2(G) \geq 7$,

$$|E(\Theta)| \geq (n_0 + 9)\overline{\sigma}_2(G) - 21 - 2n_0 = 12\overline{\sigma}_2(G) - 27 + j(\overline{\sigma}_2(G) - 2) > 12\overline{\sigma}_2(G) - 31.$$

Thus, Theorem 4.3(d) is proved for this case.

If $n_0 = 2$ and $\max\{\alpha'(\Gamma^*(v_1)), \alpha'(\Gamma^*(v_2))\} \geq 3$, then $G \notin \bigcup_{i=1}^{5} \mathcal{P}_i$. By (38), (33) and Claim 2,

$$\begin{aligned} |E(\Theta)| &\geq |E(P_0)| + |\mathcal{E}_{\Gamma^*(v_1)}(T_1)| + |\mathcal{E}_{\Gamma^*(v_2)}(T_2)| + \sum_{i=3}^{9} |\mathcal{E}_{\Gamma^*(v_i)}(T_i)| \\ &\geq 15 + (3\overline{\sigma}_2(G) - 12) + (2\overline{\sigma}_2(G) - 6) + 7(\overline{\sigma}_2(G) - 4) = 12\overline{\sigma}_2(G) - 31. \end{aligned}$$

Again, Theorem 4.3(d) holds.

Thus, we only need to consider the cases $n_0 = 1$ and $n_0 = 2$ with $\alpha'(\Gamma^*(v_1)) = \alpha'(\Gamma^*(v_2)) = 2$.

Now, we can complete our proof by checking on each of the cases of Theorem 4.3.

(a) $G \notin \mathcal{P}_1$. By (39) with $n_0 \in \{1, 2\}$, in the worst case, $|E(\Theta)| \geq 10\overline{\sigma}_2(G) - 23$. Theorem 4.3(a) holds.

(b) $G \notin \mathcal{P}_1 \cup \mathcal{P}_2$. Then either $n_0 = 2$ or $n_0 = 1$ and $\alpha'(\Gamma^*(v_1)) \geq 3$.

If $n_0 = 2$, then by (39) with $n_0 = 2$, $|E(\Theta)| \geq 11\overline{\sigma}_2(G) - 25 > 11\overline{\sigma}_2(G) - 29$. Case (b) is proved.

If $n_0 = 1$ and $\alpha'(\Gamma^*(v_1)) \geq 3$, then by Claim 2, $|\mathcal{E}_{\Gamma^*(v_1)}(T_1)| \geq 3\overline{\sigma}_2(G) - 12$. By (38) and (33), $|E(\Theta)| \geq 15 + (3\overline{\sigma}_2(G) - 12) + 8(\overline{\sigma}_2(G) - 4) = 11\overline{\sigma}_2(G) - 29$. Theorem 4.3(b) is proved.

(c) $G \notin \bigcup_{i=1}^{3} \mathcal{P}_i$ and $\overline{\sigma}_2(G) \geq 9$. Then either $n_0 = 2$ or $n_0 = 1$ and $\alpha'(\Gamma^*(v_1)) \geq 4$.

If $n_0 = 2$, then by (39) with $n_0 = 2$, $|E(\Theta)| \geq 11\overline{\sigma}_2(G) - 25$. Case (c) is proved.

If $n_0 = 1$ and $\alpha'(\Gamma^*(v_1)) \geq 4$, then by Claim 2, $|\mathcal{E}_{\Gamma^*(v_1)}(T_1)| \geq 4\overline{\sigma}_2(G) - 17$. By (38), (33) and $\overline{\sigma}_2(G) \geq 9$, $|E(\Theta)| \geq 15 + (4\overline{\sigma}_2(G) - 17) + 8(\overline{\sigma}_2(G) - 4) = 12\overline{\sigma}_2(G) - 34 \geq 11\overline{\sigma}_2(G) - 25$. (c) is proved.

(d) $G \notin \bigcup_{i=1}^{4} \mathcal{P}_i$. If $n_0 = 2$ with $\alpha'(\Gamma^*(v_i)) = 2$ $(i = 1, 2)$, then $G \in \mathcal{P}_4$, a contradiction. Thus, $n_0 = 1$.

Since $G \notin \bigcup_{i=1}^{4} \mathcal{P}_i$, $\alpha'(\Gamma^*(v_1)) \geq 4$. By Claim 2, $|\mathcal{E}_{\Gamma^*(v_1)}(T_1)| \geq 4\overline{\sigma}_2(G) - 17$. By (38) and (33), $|E(\Theta)| \geq 15 + (4\overline{\sigma}_2(G) - 17) + 8(\overline{\sigma}_2(G) - 4) = 12\overline{\sigma}_2(G) - 34$. Theorem 4.3(d) is proved.

(e) $G \notin \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \mathcal{P}_5$ and $\overline{\sigma}_2(G) \geq 12$.

Using the same argument for case (d) above, we have $n_0 = 1$. Since $G \notin \bigcup_{i=1}^{5} \mathcal{P}_i$, $\alpha'(\Gamma^*(v_1)) \geq 5$. By Claim 2 for $\alpha'(\Gamma^*(v_1)) \geq 5$ and $\overline{\sigma}_2(G) \geq 12$, $|\mathcal{E}_{\Gamma^*(v_1)}(T_1)| \geq 4\overline{\sigma}_2(G) - 12$. By (38) and (33), $|E(\Theta)| \geq 15 + (4\overline{\sigma}_2(G) - 12) + 8(\overline{\sigma}_2(G) - 4) = 12\overline{\sigma}_2(G) - 31$. Theorem 4.3(e) is proved. □

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