

THE AXIOM OF CHOICE FOR COLLECTIONS
OF FINITE SETS

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Title of Thesis: The Axiom of Choice for Collections
of Finite Sets

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ABSTRACT

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Some implications among finite versions of the Axiom of Choice are considered. In the first of two chapters some theorems are proven concerning the dependence or independence of these implications on the theory ZFU, the modification of ZF which permits the existence of atoms. The second chapter outlines proofs of corresponding theorems with "ZFU" replaced by "ZF". The independence proofs involve Mostowski type permutation models in the first chapter and Cohen forcing in the second chapter.

The finite axioms considered are C^n , "Every collection of n-element sets has a choice function"; W^n , "Every well-orderable collection of n-element sets has a choice function"; D^n , "Every denumerable collection of n-element sets has a choice function"; and $A^n(X)$, "Every collection Y of n-element sets, with $Y \approx X$, has a choice function". The conjunction $C^{n_1} \& \dots \& C^{n_k}$ is denoted by C_Z where $Z = \{n_1, \dots, n_k\}$. Corresponding

conjunctions of other finite axioms are denoted similarly by W_Z , D_Z and $A_Z(X)$.

Theorem: The following are provable in ZFU:

$$W^{k_1 n_1 + \dots + k_r n_r} \rightarrow W^{n_1} \vee \dots \vee W^{n_r},$$

$$D^{k_1 n_1 + \dots + k_r n_r} \rightarrow D^{n_1} \vee \dots \vee D^{n_r}, \text{ and}$$

$$C^{k_1 n_1 + \dots + k_r n_r} \rightarrow C^{n_1} \vee W^{n_2} \vee \dots \vee W^{n_r}.$$

The principal result involves the condition

$T_{n,Z}$: For every subgroup G of S_n without fixed points, there is a finite sequence $\langle H_1, \dots, H_m \rangle$ of proper subgroups of G such that $\frac{|G|}{|H_1|} + \dots + \frac{|G|}{|H_m|} \in Z$.

Theorem T: If ZF is consistent, then $T_{n,Z}$ is necessary and sufficient for

- (I) $\overline{\text{ZFU}} (D_Z \rightarrow D^n)$ and sufficient for
- (II) $\overline{\text{ZFU}} (C_Z \rightarrow C^n)$. Furthermore (I) is equivalent to each of the following:
 - (Ia) $\overline{\text{ZFU}} (W_Z \rightarrow D^n)$,
 - (Ib) $\overline{\text{ZFU}} (W_Z \rightarrow W^n)$,
 - (Ic) $\overline{\text{ZFU}} (A_Z(X) \rightarrow A^n(X))$.

It can be shown that $T_{\{2\},4}$ fails. Tarski has shown that $C_{\{2\}} \rightarrow C^4$ is provable in ZFU. Hence it follows from the above theorem that (I) is not always necessary for (II).

Another main result involves Mostowski's condition

$M_{n,Z}$: For every decomposition of n into a sum of (not

necessarily distinct) positive primes, $n = p_1 + \dots + p_r$,
 there exist non-negative integers k_1, \dots, k_r such that
 $k_1 p_1 + \dots + k_r p_r \in \mathbb{Z}$.

Theorem: $M_{n, \mathbb{Z}}$ is sufficient for (III) $\overline{\text{ZFU}} (C_{\mathbb{Z}} \rightarrow W^n)$.

Mostowski has proven that if ZF is consistent, $M_{n, \mathbb{Z}}$
 is necessary for (IIIa) $\overline{\text{ZFU}} (C_{\mathbb{Z}} \rightarrow D^n)$. Hence, the following
 result:

Theorem M: If ZF is consistent, $M_{n, \mathbb{Z}}$ is necessary and
 sufficient for (III) and also for (IIIa).

It follows from Theorems T and M that there is an
 effective procedure for determining whether (I) holds
 and whether (III) holds.

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TABLE OF CONTENTS

Chapter	Page
ACKNOWLEDGMENTS	ii
INTRODUCTION	1
PRELIMINARIES	2
I. FINITE AXIOMS OF CHOICE AND THE THEORY ZFU	4
1. Finite Axioms of Choice	4
2. The Role of Well-Orderings	6
3. The Main Theorems	9
4. Proofs of Sufficiency	12
5. Models of ZFU	21
6. Proofs of Necessity	26
II. CORRESPONDING RESULTS IN THE THEORY ZF	33
BIBLIOGRAPHY	46

INTRODUCTION

For a set theory without the Axiom of Choice it is natural to ask what sort of implications involving various restricted versions of the Axiom of Choice are provable. Mostowski [2] considers implications among axioms of the form C^n : "Every collection of n -element sets has a choice function" where n is a natural number. Mostowski proved the necessity of a certain number theoretic condition M for a proof of $C^{n_1} \&\dots\& C^{n_r} \rightarrow C^n$ and raised the question of its sufficiency. Mostowski actually shows that M is necessary for a proof of $C^{n_1} \&\dots\& C^{n_r} \rightarrow D^n$ where D^n means "Every denumerable collection of n -element sets has a choice function". One of the main results of this paper is that M is also sufficient for a proof of $C^{n_1} \&\dots\& C^{n_r} \rightarrow D^n$. The principal result of this paper is that a certain group theoretic condition T is necessary and sufficient for a proof of $D^{n_1} \&\dots\& D^{n_r} \rightarrow D^n$.

PRELIMINARIES

Let ZFU be the modification of ZF (Zermelo-Fraenkel set theory) which permits the existence of atoms (also called urelemente or individuals). The axioms of ZFU are given by Suppes [4]. Statements and proofs will be given informally, and the symbol " \vdash_{ZFU} " will frequently be omitted where a proof involves only conventional mathematical methods. However, except where another theory is specified, all statements and proofs may be formalized in ZFU.

We let \emptyset denote the empty set. We say x is an atom if $x \neq \emptyset$, but x has no elements. x is a set if x is not an atom. A pure set is a set having no atoms in its transitive closure. We let ω be the set of finite ordinals. A natural number n is an element of ω , so we have $n = \{0, \dots, n-1\}$. We write $x \approx y$ if there is a 1-1 mapping from x onto y . For finite x , we write $|x|$ for the cardinal of x . We write UX for the set of all elements of elements of X . A function F is called a choice function if for each x in the domain of F we have $F(x) \in x$. F is called a subset function if for each x in the domain of F , $F(x)$

is a non-empty proper subset of x . A collection Y is said to have a choice function (subset function) if Y is the domain of some choice function (subset function).

CHAPTER I

FINITE AXIOMS OF CHOICE

AND THE THEORY ZFU

1. Finite Axioms of Choice

For each natural number n we let $A^n(X)$ denote the sentence "For every collection Y of n -element sets with $Y \approx X$, Y has a choice function" (more precisely $A^n(X)$ is some formula of ZFU, with one free variable X , having the same meaning as the given sentence). Let C^n be the statement "Every collection of n -element sets has a choice function", let W^n denote "Every well-orderable collection of n -element sets has a choice function" and let D^n denote "Every denumerable collection of n -element sets has a choice function". Letting α vary over ordinals, we then have $C^n \leftrightarrow (\forall X)A^n(X)$, $W^n \leftrightarrow (\forall \alpha)A^n(\alpha)$ and $D^n \leftrightarrow A^n(\omega)$.

For each finite set $Z = \{n_1, \dots, n_k\} \subseteq \omega$ define $A_Z(X)$ as the conjunction $A^{n_1}(X) \& \dots \& A^{n_k}(X)$. Similarly we define conjunctions C_Z , W_Z and D_Z so we have $C_Z \leftrightarrow (\forall X)A_Z(X)$, $W_Z \leftrightarrow (\forall \alpha)A_Z(\alpha)$ and $D_Z \leftrightarrow A_Z(\omega)$. Throughout the paper n will vary over natural numbers, Z will vary over finite sets of positive natural numbers and α will vary over ordinals.

Lemma 1 (Tarski): $C^2 \rightarrow C^4$.

Proof: Suppose C^2 holds and Y is a collection of 4-element sets. We wish to show that Y has a choice function. We have from C^2 that there is a choice function f on the set of 2-element subsets of UY . For each $y \in Y$, y has exactly 4 elements so the set y^* of 2-element subsets of y has exactly $\binom{4}{2}=6$ elements. Each $x \in y^*$ is in the domain of f so $f(x) \in y$ and hence y^* determines exactly 6 choices from y (i.e., $|\{\langle x, v \rangle \in f: x \in y^*, v \in y\}| = 6$). Let $g(y)$ be the set of elements of y chosen most often (i.e., $g(y) = \{v \in y: |\{\langle x, v \rangle \in f: x \in y^*\}| \text{ is maximum}\}$). Since there are exactly 4 elements in y , they cannot be chosen equally often so $g(y) \neq y$. Hence $|g(y)| = 1, 2$ or 3 . We now define a choice function F on Y as follows: for each $y \in Y$, if $g(y)$ has one element, let $F(y)$ be that element, if $g(y)$ has 2 elements, let $F(y) = f(g(y))$ and if $g(y)$ has 3 elements, let $F(y)$ be the element of $y \sim (g(y))$.

Lemma 2 (Tarski): For natural numbers k and n we have $C^{kn} \rightarrow C^n$.

Proof: Assume C^{kn} and suppose Y is a collection of n -element sets. The number $k = \{0, 1, \dots, k-1\}$ is a set with k elements and for each $y \in Y$, y has n elements so the Cartesian product $k \times y$ has kn elements. It follows from C^{kn} that there is a choice function f on $\{k \times y: y \in Y\}$.

Thus $f(kXy)$ is an ordered pair $\langle i, v \rangle$ for some $i \in k$ and $v \in y$. We define a choice function F on Y as follows: for each $y \in Y$, $F(y)$ is the second element of the ordered pair $f(kXy)$.

As a special case of lemma 2 we have $C^4 \rightarrow C^2$ and hence by lemma 1 we have the following

Corollary: $C^2 \leftrightarrow C^4$.

The proof of lemma 2 can easily be generalized to obtain

Lemma 3: $A^{kn}(X) \rightarrow A^n(X)$, $W^{kn} \rightarrow W^n$ and $D^{kn} \rightarrow D^n$.

2. The Role of Well-Orderings

The existence of a well-ordering on a collection Y is no assurance of the existence of a choice function on Y . However a well-ordering can be relevant as the proof of the following lemma illustrates.

Lemma 4: Suppose X has a well-ordering and $\{x \cup y : x \in X, y \in Y\}$ has a choice function. Then either X has a choice function or Y has a choice function.

Proof: Suppose we have (*) for each $y \in Y$, there is $x \in X$ such that $f(x \cup y) \in y$. Then for each $y \in Y$ let x_y be the first element of X such that $f(x_y \cup y) \in y$. The function g defined by $g(y) = f(x_y \cup y)$ is a choice function on Y . Now suppose (*) fails. Then there is

$y_0 \in Y$ such that for all $x \in X$, $f(x \cup y_0) \in x$. Hence the function F defined by $F(x) = f(x \cup y_0)$ is a choice function on X .

We wish to use lemma 4 to prove that for natural numbers n and m we have $W^{n+m} \rightarrow W^n \vee W^m$, $C^{n+m} \rightarrow C^n \vee W^m$ and $D^{n+m} \rightarrow D^n \vee D^m$. One difficulty involved is that $|x| = n$ and $|y| = m$ do not imply $|x \cup y| = n + m$ unless x and y are disjoint. We can overcome this difficulty by replacing each $v \in x$ by $\langle v, j \rangle$ and each $v \in y$ by $\langle v, k \rangle$ where $j \neq k$. The set of all $\langle v, k \rangle$ with $v \in y$ is the Cartesian product $y \times \{k\}$ so we introduce the following definition.

Definition: For each collection Y and each natural number k we write $\mathcal{D}_k(Y) = \{y \times \{k\} : y \in Y\}$ and we call $\mathcal{D}_k(Y)$ a copy of Y .

The next lemma summarizes some trivial consequences of the above definition.

Lemma 5: For any collection Y and any k there is a one to one mapping f from Y onto $\mathcal{D}_k(Y)$ such that for each $y \in Y$, $|f(y)| = |y|$. For any collection X and any $j \neq k$, each element of $\mathcal{D}_k(Y)$ is disjoint from each element of $\mathcal{D}_j(Y)$.

Remark 1: The usual proof, that the Cartesian product of finitely many denumerable sets is denumerable, does not involve the axiom of choice and may be formalized

in ZFU. Similarly we can prove in ZFU that the Cartesian product of finitely many well-orderable sets has a well-ordering. We also have in ZFU that if a set X can be indexed by a well-orderable set Y (i.e., if X is the range of a function on Y), then X has a well-ordering.

We are now ready to prove

Lemma 6: For natural numbers n and m we have

$$W^{n+m} \rightarrow W^n \vee W^m \text{ and } D^{n+m} \rightarrow D^n \vee D^m.$$

Proof: Suppose W^{n+m} holds and W^n fails so there is a well-orderable collection X of n -element sets such that X has no choice function. To show W^m holds, suppose Y is a well-orderable collection of m -element sets. It follows from Lemma 5 that $\mathcal{D}_1(X)$ and $\mathcal{D}_2(Y)$ have well-orderings.

Then by Remark 1, the collection $W = \{x \cup y : \langle x, y \rangle \in \mathcal{D}_1(X) \times \mathcal{D}_2(Y)\}$

$$W = \{x \cup y : \langle x, y \rangle \in \mathcal{D}_1(X) \times \mathcal{D}_2(Y)\}$$

has a well-ordering. For each $x \in \mathcal{D}_1(X)$ and $y \in \mathcal{D}_2(Y)$ we have, from Lemma 5, $|x| = n$, $|y| = m$ and $x \cap y = \emptyset$.

Thus each member of W has $n+m$ elements. Hence by W^{n+m} ,

W has a choice function so by Lemma 4 either $\mathcal{D}_1(X)$ has a choice function or $\mathcal{D}_2(Y)$ has a choice function. Then by Lemma 5 either X or Y has a choice function. Since X has none, Y must have a choice function. Thus we have

established $W^{n+m} \rightarrow W^n \vee W^m$. Similarly we can show

$$D^{n+m} \rightarrow D^n \vee D^m.$$

We can also establish by a similar proof

$$\text{Lemma 7: } C^{n+m} \rightarrow C^n \vee W^m.$$

An easy consequence of Lemma 7 is the following

$$\text{Corollary: } C^{n+m} \rightarrow C^n \vee C^m \vee (W^n \& W^m).$$

By Lemmas 2, 3, 6 and 7 and induction we have the following result.

Theorem 1: For any natural numbers k_1, \dots, k_r and n_1, \dots, n_r we have

$$\begin{aligned} W^{k_1 n_1 + \dots + k_r n_r} &\rightarrow W^{n_1} \vee \dots \vee W^{n_r}, \\ D^{k_1 n_1 + \dots + k_r n_r} &\rightarrow D^{n_1} \vee \dots \vee D^{n_r}, \text{ and} \\ C^{k_1 n_1 + \dots + k_r n_r} &\rightarrow C^{n_1} \vee W^{n_2} \vee \dots \vee W^{n_r}. \end{aligned}$$

3. The Main Theorems

Mostowski [2] considered the problem of deciding for given n and Z whether the implication $C_Z \rightarrow C^n$ is provable, without arriving at a complete solution. We shall not give a solution here to this problem, but we shall give complete solutions of the corresponding problems for the implications $D_Z \rightarrow D^n$, $W_Z \rightarrow W^n$, $W_Z \rightarrow D^n$, $C_Z \rightarrow W^n$ and $C_Z \rightarrow D^n$.

Let S_n be the symmetric group on $n = \{0, \dots, n-1\}$. If G is a subgroup of S_n and $k \in n$, we say that k is a fixed point of G if for every permutation $\pi \in G$, we have $\pi(k) = k$.

Definition: $T_{n,Z}$ means that for every subgroup G of S_n without fixed points, there is a finite sequence $\langle H_1, \dots, H_m \rangle$ of proper subgroups of G such that $\frac{|G|}{|H_1|} + \dots + \frac{|G|}{|H_m|} \in Z$.

We now state the principal result of this paper.

Theorem T: If ZF is consistent, then $T_{n,Z}$ is necessary and sufficient for

- (I) $\vdash_{ZFU} (D_Z \rightarrow D^n)$ and sufficient for
 (II) $\vdash_{ZFU} (C_Z \rightarrow C^n)$. Furthermore (I) is equivalent to each of the following:
- (Ia) $\vdash_{ZFU} (W_Z \rightarrow D^n)$,
 (Ib) $\vdash_{ZFU} (W_Z \rightarrow W^n)$,
 (Ic) $\vdash_{ZFU} (A_Z(X) \rightarrow A^n(X))$.

The proof of Theorem T has two main parts, the proofs of $T_{n,Z} \Rightarrow$ (Ic) and (Ia) $\Rightarrow T_{n,Z}$ (Theorems 2 and 8). After these are established we have $T_{n,Z} \Rightarrow$ (Ic) $\Rightarrow \vdash_{ZFU} (A_Z(w) \rightarrow A^n(w)) \Rightarrow$ (I) \Rightarrow (Ia) $\Rightarrow T_{n,Z} \Rightarrow$ (Ic) $\Rightarrow \vdash_{ZFU} (\forall \alpha) (A_Z(\alpha) \rightarrow A^n(\alpha)) \Rightarrow \vdash_{ZFU} ((\forall \alpha) A_Z(\alpha) \rightarrow (\forall \alpha) A^n(\alpha)) \Rightarrow$ (Ib) \Rightarrow (Ia). We also have (Ic) $\Rightarrow \vdash_{ZFU} (\forall X) (A_Z(X) \rightarrow A^n(X)) \Rightarrow \vdash_{ZFU} ((\forall X) A_Z(X) \rightarrow (\forall X) A^n(X)) \Rightarrow$ (II) so the proof of Theorem T will be complete.

In order to test $T_{n,Z}$ it is sufficient to check the finitely many subgroups G of S_n and the finitely many sequences $\langle H_1, \dots, H_m \rangle$ of subgroups of G with m not greater

than the largest number in Z . Hence by Theorem T we can effectively determine for given n and Z whether (I) holds.

The alternating subgroup G of S_4 has no fixed points and has order 12. Each proper subgroup H of G has order at most 4 so $\frac{|G|}{|H|} > 2$. It follows that $T_{4,\{2\}}$ fails so we have from Theorem T the following

Corollary: If ZF is consistent, we do not have $\vdash_{ZFU} (D_{\{2\}} \rightarrow D^4)$.

We have from Lemma 1, $\vdash_{ZFU} (C_{\{2\}} \rightarrow C^4)$ so it follows from the above Corollary that if ZF is consistent, (I) is not always necessary for (II).

Definition: $M_{n,Z}$ means for every decomposition of n into a sum of (not necessarily distinct) positive primes, $n = p_1 + \dots + p_r$, there exist non-negative integers k_1, \dots, k_r such that $k_1 p_1 + \dots + k_r p_r \in Z$.

One of the main results of this paper will be Theorem 3: $M_{n,Z}$ is sufficient for (III) $\vdash_{ZFU} (C_Z \rightarrow W^n)$. Mostowski has proven that if ZF is consistent, $M_{n,Z}$ is necessary for (IIIa) $\vdash_{ZFU} (C_Z \rightarrow D^n)$. Since (III) implies (IIIa) we shall have the following result.

Theorem M: If ZF is consistent, $M_{n,Z}$ is necessary and sufficient for $\vdash_{ZFU} (C_Z \rightarrow D^n)$ and also for $\vdash_{ZFU} (C_Z \rightarrow W^n)$.

It follows from Theorem M that for given n and Z , we can effectively determine whether (III) holds.

4. Proofs of Sufficiency

In order to show that $T_{n,Z}$ implies $\overline{ZFU} (A_Z(X) \rightarrow A^n(X))$, we must first prove two lemmas.

Lemma 8: Suppose we have $A^n(X)$ and $Y \approx X' \subseteq X$ where each member of Y is an n -element set. Then Y has a choice function.

Proof: Let $W = \{\{x\} X^n : x \in X \sim X'\}$. Then $W \approx X \sim X'$ and each member of W has n elements. It follows from Lemma 5 that $\mathcal{D}_1(Y) \approx X'$, $\mathcal{D}_2(W) \approx X \sim X'$, each element of $\mathcal{D}_1(Y) \cup \mathcal{D}_2(W)$ has n elements and $\mathcal{D}_1(Y)$ is disjoint from $\mathcal{D}_2(W)$. Thus $\mathcal{D}_1(Y) \cup \mathcal{D}_2(W) \approx X' \cup (X \sim X') = X$ and hence by $A^n(X)$, $\mathcal{D}_1(Y) \cup \mathcal{D}_2(W)$ has a choice function. It follows that $\mathcal{D}_1(Y)$ has a choice function and thus by Lemma 5, Y has a choice function.

Lemma 9: Suppose $A_Z(X)$ and $Y \approx X' \subseteq X$ where each element of Y is a set with cardinal in Z . Then Y has a choice function.

Proof: For each $k \in Z$ we have $A^k(X)$. Let $Y_k = \{y \in Y : |y| = k\}$. Then $Y_k \subseteq Y \approx X' \subseteq X$ so $Y_k \approx X'' \subseteq X$ for some X'' . Since each element of Y_k has k elements, it follows from Lemma 8 that Y_k has a choice function f_k . Since Z is finite we can define $f = \cup \{f_k : k \in Z\}$. It is easy to verify that f is a choice function on Y .

We shall now show that $T_{n,Z}$ implies $\overline{\text{ZFU}} (A_Z(X) \rightarrow A^n(X))$. Let X be fixed and assume $T_{n,Z}$ and $A_Z(X)$. We wish to show $A^n(X)$: "For every collection Y of n -element sets with $Y \approx X$, Y has a choice function". Let Y be fixed, let x vary over X and let y vary over Y . For each x let y_x be the image of x under a 1-1 mapping from X onto Y . For each y , let \hat{y} be the set of 1-1 mappings from $n = \{0, 1, \dots, n-1\}$ onto y .

For any subgroup G of S_n and any mappings f, g belonging to some \hat{y} , f will be called G -equivalent to g if there is $p \in G$ such that $f = g \circ p$. It is easy to verify that G -equivalence is an equivalence relation. We call v an equivalence class if for some subgroup G of S_n and some y , $v \subseteq \hat{y}$ and v is a G -equivalence class. Let V be the set of equivalence classes and let v, w vary over V .

Lemma 10: For each v there is exactly one subgroup G of S_n such that v is a G -equivalence class. Furthermore $|G| = |v|$.

Proof: There is at least one such G by definition of V . Suppose v is a G' equivalence class and choose $g \in v$. Then $p \in G \rightarrow g \circ p \in v \rightarrow g \circ p' = g \circ p$ for some $p' \in G' \rightarrow p' = p \rightarrow p \in G'$. Hence $G \subseteq G'$. Similarly $G' \subseteq G$ so that $G' = G$. The mapping which sends each $p \in G$ to $g \circ p$ is 1-1 from G onto v . Hence $|G| = |v|$.

Let G_v be the unique subgroup of S_n such that v is a G_v -equivalence class. We have $|G_v| = |v|$.

Lemma 11: For each v and each subgroup H of G_v , there are exactly $\frac{|G_v|}{|H|}$ H -equivalence classes included

in v , i.e., letting $k = |\{w \in V : w \subseteq v \text{ and } G_w = H\}|$ we

$$\text{have } k = \frac{|G_v|}{|H|}.$$

Proof: v is partitioned by H -equivalence classes,

each of which has cardinal $|H|$ by the previous lemma. Thus

$$|v| = k |H| \text{ and hence } k = \frac{|v|}{|H|} = \frac{|G_v|}{|H|}.$$

Let H^* be a function such that for each subgroup G of S_n without fixed points, $H^*(G)$ is a finite sequence $\langle H_1, \dots, H_m \rangle$ of proper subgroups of G satisfying $\sum_{i=1}^m \frac{|G|}{|H_i|} \in \mathbb{Z}$. The

existence of H^* follows from $T_{n, \mathbb{Z}}$. Let V' be the set of all v such that G_v has no fixed points.

Lemma 12: Y has a choice function.

Proof: It is sufficient to show (1) there is a mapping F which sends each x to an element of y_x . Each \hat{y}_x is an equivalence class so we have (2) there is a mapping E which sends each x to an equivalence class $v_x \subseteq \hat{y}_x$. We show that (2) \rightarrow (1) by induction on $K_E = \max \{|v_x| : v_x \in V'\}$. For $K_E = 1$ we have for each x either (i) $|v_x| = 1$ or (ii) $v_x \notin V'$.

In case (i) v_x has just one element $f \in \hat{Y}_x$. f is a mapping from n onto y_x so we let $F(x) = f(0)$. In case (ii) G_{v_x} has a smallest fixed point $r \in n$. Let $F(x) = f(r)$ where $f \in v_x$. $f(r)$ is independent of the choice of f since for any $f' \in v_x$ there is $p \in G_{v_x}$ such that $f' = f \circ p$ so that $f'(r) = (f \circ p)(r) = f(p(r)) = f(r)$. It follows that (2) \rightarrow (1) holds for $K_E = 1$. Now suppose (2) \rightarrow (1) holds

for all E' with $K_{E'} < K_E$. For each x with $v_x \in V'$, let $H^*(G_{v_x}) = \langle H_1, \dots, H_m \rangle$ so that $\sum_{i=1}^m \frac{|G_{v_x}|}{|H_i|} \in \mathbb{Z}$. Let

$$U_x = \{ \langle w, i \rangle : w \subseteq v_x \text{ and } G_w = H_i \text{ for some } i = 1, \dots, m \}$$

when $v_x \in V'$. We have $|U_x| = \sum_{i=1}^m |\{w : w \subseteq v_x \text{ and } G_w = H_i\}|$

$$= \sum_{i=1}^m \frac{|G_{v_x}|}{|H_i|} \text{ by the previous lemma. Hence } |U_x| \in \mathbb{Z}.$$

It follows from $A_{\mathbb{Z}}(X)$ and Lemma 9 that there is a choice

function Q on the set of all U_x with $v_x \in V'$. Q sends

U_x to an ordered pair $\langle w_x, i \rangle$ such that $w_x \subseteq v_x$ and

$$|w_x| = |G_{w_x}| = |H_i| < |G_{v_x}| = |v_x|. \text{ Thus when } v_x \in V',$$

$$|w_x| < |v_x| \leq K_E \text{ and } w_x \subseteq v_x \subseteq \hat{Y}_x. \text{ When } v_x \notin V' \text{ let } w_x = v_x.$$

The mapping E' which sends each x to w_x satisfies (2) and

$$K_{E'} = \max \{ |w_x| : w_x \in V' \} < K_E \text{ so (1) follows from the}$$

induction hypothesis.

We have proven that $T_{n,Z}$ and $A_Z(X)$ imply $A^n(X)$. The proof can be carried out in ZFU. Furthermore, when $T_{n,Z}$ holds, it can be proven in ZFU so we have

Theorem 2: $T_{n,Z}$ is sufficient for $\overline{\text{ZFU}}(A_Z(X) \rightarrow A^n(X))$.

Our next objective is to show that $M_{n,Z}$ implies $\overline{\text{ZFU}} (C_Z \rightarrow W^n)$.

Lemma 13: Let A be a collection of m -element sets and let p be a prime factor of m . Suppose the collection B of all p -element subsets of elements of A has a choice function. Then A has a subset function.

Proof: Let f be a choice function on B . For each $X \in A$, let \hat{X} be the set of p -element subsets of X . Thus f maps \hat{X} into X so $\{\hat{X} \cap f^{-1}(\{x\}) : x \in X\}$ is a collection of pairwise disjoint sets whose union is \hat{X} . We have

$$\sum_{x \in X} |\hat{X} \cap f^{-1}(\{x\})| = |\hat{X}| = \binom{m}{p}. \text{ For each } X \in A, \text{ let}$$

$g(X) = \{x \in X : |\hat{X} \cap f^{-1}(\{x\})| \text{ is maximum}\}$. We will show

that g is a subset function. Clearly $g(X) \subseteq X$ and

$g(X) \neq \emptyset$. Suppose $g(X) = X$. Then $|\hat{X} \cap f^{-1}(\{x\})|$ has a

constant value k as x varies over X . Since $|X| = m$, we

have $\binom{m}{p} = \sum_{x \in X} |\hat{X} \cap f^{-1}(\{x\})| = mk$. Hence

$$k = \binom{m}{p} / m = \frac{(m-1) \dots (m-[p-1])}{p!} \text{ so } p \text{ divides one of the}$$

numbers $m-1, \dots, m-[p-1]$. Since p divides m it follows

that p divides one of the numbers $1, \dots, p-1$ which is

impossible. Hence $g(X) \neq X$ for any $X \in A$ so g is a

subset function.

Definition: A collection A is called seperable if for some $n \in \omega$ we have (*) there is a well orderable

collection B such that each member of B has not more than n elements and $A = \cup B$. The smallest n such that (*) holds is called the index of A .

Lemma 14: Suppose B is a collection of well-orderable sets such that B has no choice function. Suppose A is separable and $\{a \cup b : \langle a, b \rangle \in A \times B\}$ has a choice function f . Then A has a choice function.

Proof: Otherwise there is a smallest n such that the lemma fails for some A of index n . Then for some ordinal α , A is expressible as $A = \cup \{E_\gamma : \gamma \in \alpha\}$ where $|E_\gamma| \leq n$ for all $\gamma \in \alpha$. There must exist $b \in B$ such that for all $\gamma \in \alpha$, $\{f(a \cup b) : a \in E_\gamma\}$ is not a one element subset of b (otherwise we can construct a choice function g on B by selecting for each $b \in B$, the first $\gamma \in \alpha$ such that $\{f(a \cup b) : a \in E_\gamma\} = \{c\}$ for some $c \in b$ and defining $g(b) = c$). We can express $b = \{c_\delta : \delta \in \beta\}$ where β is some ordinal. For each $\gamma \in \alpha$, $\delta \in \beta$, let $C_{\gamma\delta} = \{a \in E_\gamma : f(a \cup b) = c_\delta\}$. $C_{\gamma\delta} \neq E_\gamma$ for otherwise $\{f(a \cup b) : a \in E_\gamma\} = \{c_\delta\}$ is a one element subset of b . Hence $|C_{\gamma\delta}| < |E_\gamma| \leq n$. The collection $C = \{C_{\gamma\delta} : \gamma \in \alpha, \delta \in \beta\}$ has a well ordering since it is indexed by $\alpha \times \beta$. Therefore $\cup C$ is separable with index $< n$. We have $\cup C \subseteq A$ so f is a choice function on $\{a \cup b : a \in \cup C, b \in B\}$. It follows that $\cup C$ has a choice function g . We now define a choice function h on A .

For each $a \in A$, if $f(a \cup b) \in a$, let $h(a) = f(a \cup b)$.

Otherwise $f(a \cup b) \in b$ and hence $f(a \cup b) = c_\delta$ for some

$\delta \in \beta$. Since $a \in E_\gamma$ for some $\gamma \in \alpha$ it follows that

$a \in C_{\gamma\delta}$ and hence $a \in UC$. We then let $h(a) = g(a)$.

Lemma 15: Suppose B_1, \dots, B_n are separable collections of well-orderable sets. Suppose

$\{b_1 \cup \dots \cup b_n : \langle b_1, \dots, b_n \rangle \in B_1 \times \dots \times B_n\}$ has a choice function. Then one of the B_i has a choice function.

Proof: For each $k = 1, \dots, n$, let

$B^{(k)} = \{b_1 \cup \dots \cup b_k : \langle b_1, \dots, b_k \rangle \in B_1 \times \dots \times B_k\}$. Clearly each $B^{(k)}$ is a collection of well-orderable sets. Suppose

B_1 has no choice function. Then, since $B^{(1)} = B_1$, $B^{(1)}$

has no choice function so there is a largest $r < n$ such

that $B^{(r)}$ has no choice function. Then $B^{(r+1)} =$

$\{b \cup b_{r+1} : \langle b, b_{r+1} \rangle \in B^{(r)} \times B_{r+1}\}$ has a choice function.

It follows from Lemma 14 that B_{r+1} has a choice function.

Lemma 16: Suppose we have C_Z and $M_{n,Z}$ and suppose $m_1 + \dots + m_r = n$ where $m_i \geq 2$ for each $i = 1, \dots, r$. Suppose for each $i = 1, \dots, r$, A_i is a well-orderable collection and each element of A_i has cardinal m_i . Then one of the A_i has a subset function.

Proof: For each i , we can write $m_i = c_i p_i$ where p_i is a positive prime. Then $\underbrace{c_1}_{p_1 + \dots + p_1} + \dots + \underbrace{c_r}_{p_r + \dots + p_r} = n$ is a decomposition of n into a sum of positive primes.

It follows from $M_{n,Z}$ that there exist non-negative integers k_1, \dots, k_r such that $t = k_1 p_1 + \dots + k_r p_r \in Z$. By C_Z we have C^t . For each i the set B_i of p_i -element subsets of elements of A_i is a separable collection of well-orderable sets.

We replace each B_i by the copy $B_i' = \delta_i(B_i)$. Then it follows from Lemma 5 that each B_i' is a separable collection of p_i -element sets and for $j \neq i$ each element b_j of B_j' is disjoint from each element b_i of B_i' . Thus $b_i \times k_i$ has $k_i p_i$ elements and is disjoint from $b_j \times k_j$. It follows that each member of the set

$$D = \{(b_1 \times k_1) \cup \dots \cup (b_r \times k_r) : b_1 \in B_1', \dots, b_r \in B_r'\}$$

has t elements. Since we have C^t , D has a choice function

f . Define $g(b_1 \cup \dots \cup b_r)$ as the first element of the ordered pair $f((b_1 \times k_1) \cup \dots \cup (b_r \times k_r))$. It is clear

that g is a choice function on

$\{b_1 \cup \dots \cup b_r : b_1 \in B_1', \dots, b_r \in B_r'\}$. Hence, by Lemma 15,

one of the B_i' has a choice function. It follows from

Lemma 5 that B_i has a choice function and hence by Lemma 13,

A_i has a subset function.

Lemma 17: Suppose C_Z and $M_{n,Z}$. Given r collections

A_1, \dots, A_r each having a well ordering, suppose for each

$i = 1, \dots, r$ each member of A_i has m_i elements where $m_i \geq 1$

and $m_1 + \dots + m_r = n$. Then one of the A_i has a choice

function.

Proof: Clearly $1 \leq r \leq n$. If $r = n$, then $m_1 = 1$ and hence A_1 has a choice function. Thus the lemma holds for $r = n$. Assuming the lemma holds for $r = t + 1$ we shall show that it must hold for $r = t$. Suppose for $i = 1, \dots, t$, A_i has a well ordering, each member of A_i has m_i elements and $m_1 + \dots + m_t = n$. Suppose none of the A_i has a choice function. Then $m_i \geq 2$ for all i . By Lemma 16, one of the A_i , say A_t , has a subset function f . Since each member x of A_t has m_t elements, we have $1 \leq |f(x)| < m_t$. It follows that A_t is a union of finitely many sets of the form $A_t^k = \{x \in A_t : |f(x)| = k\}$ where $1 \leq k < m_t$. In order to show that A_t has a choice function, it is sufficient to show that each A_t^k has a choice function. For each $x \in A_t^k$, $|f(x)| = k \geq 1$ and since $|x| = m_t$ and f is a subset function we have $|x \sim f(x)| = m_t - k \geq 1$. Let $B_t^k = \{f(x) : x \in A_t^k\}$ and let $B_{t+1}^k = \{x \sim f(x) : x \in A_t^k\}$. Both B_t^k and B_{t+1}^k have well orderings since each is the range of a function on a subset of A_t . Each member of B_t^k has cardinal $k \geq 1$ and each member of B_{t+1}^k has cardinal $m_t - k \geq 1$. Furthermore $m_1 + \dots + m_{t-1} + k + (m_t - k) = m_1 + \dots + m_t = n$. Thus the collections $A_1, \dots, A_{t-1}, B_t^k, B_{t+1}^k$ satisfy the hypothesis of the lemma with $r = t + 1$ so it follows that one of them has a choice function. Since we have assumed that none of the A_i has a choice function, it follows that there is a choice function g with domain either B_t^k or B_{t+1}^k . For

each $x \in A_t^k$ define $h(x)$ as follows: if the domain of g is B_t^k , let $h(x) = g(f(x)) \in f(x) \subseteq x$; if the domain of g is B_{t+1}^k , let $h(x) = g(x \sim f(x)) \in x \sim f(x) \subseteq x$. Thus h is a choice function on A_t^k as required. \vee

Letting $r = 1$ in Lemma 17 we obtain

Lemma 18: If C_Z and $M_{n,Z}$ hold, then we have W^n .

Lemmas 13-18 can be formalized and proven in ZFU.

Also, when $M_{n,Z}$ holds, it can easily be proven in ZFU.

Thus from Lemma 18 we have

Theorem 3: $M_{n,Z}$ is sufficient for $\vdash_{ZFU} (C_Z \rightarrow W^n)$.

5. Models of ZFU

The theory ZFU does not guarantee the existence of atoms. In fact every model of ZF is a model of ZFU which excludes atoms. However, if we let ZFU^+ be the theory ZFU plus AC (the Axiom of Choice) plus "There is a denumerable set of atoms" we have the following theorem.

Theorem 4: If ZF is consistent, then so is ZFU^+ .

We sketch the proof. Assuming ZF is consistent, it follows that ZF plus AC is also consistent by Gödel's well known proof. Using ZF plus AC we construct a model of ZFU^+ by fixing the universe and altering the membership relation. Let F be a 1-1 ZF definable function on the entire universe whose range excludes the elements

of some denumerable set, e.g., $F(x) = \{x\}$. Define $x \hat{e} y \leftrightarrow (\exists z)(x \in z \ \& \ y = F(z))$ so that we have $x \hat{e} F(z) \leftrightarrow x \in z$. For each axiom A of ZFU^+ let \hat{A} be the formula obtained from A by replacing each occurrence of "e" by " \hat{e} ". It is then possible to show $\frac{}{ZF+AC} \hat{A}$.

The following construction may be carried out in ZFU^+ .

Let V be a denumerable set of atoms and let Π be a set of permutations on V such that for each $\pi_1, \pi_2 \in \Pi$ we have $\pi_1^{-1} \in \Pi$ and $\pi_2^{-1} \circ \pi_1 \in \Pi$. For each $\pi \in \Pi$ we define $\pi(x)$ for all x by set induction as follows: for $x \in V$, $\pi(x)$ is already defined, for an atom $x \notin V$ let $\pi(x) = x$ and for a set x let $\pi(x) = \{\pi(y) : y \in x\}$. Then we always have $y \in x \leftrightarrow \pi(y) \in \pi(x)$.

Definition: For any x and any finite set $E \subseteq V$, E is called a support of x if $\pi(x) = x$ for all $\pi \in \Pi$ satisfying $\pi(e) = e$ for all $e \in E$.

We now define a model \mathcal{M} inductively as follows.

Definition: $\mathcal{M}(x)$ ($x \in \mathcal{M}$, x is hereditarily symmetric) if x has a support and for all $y \in x$, $\mathcal{M}(y)$.

The following result is well known.

Theorem 5: \mathcal{M} is a transitive model of ZFU.

We shall prove that the power set and replacement axioms hold in \mathcal{M} . It is easy to show by set induction

that for any x and any $\pi_1, \pi_2 \in \Pi$, we have $\pi_1 \circ \pi_2(x) = \pi_1(\pi_2(x))$.

We use this fact to prove the following lemma.

Lemma 19: Suppose E is a support of x and $\pi_1 \in \Pi$.

Then $\pi_1(E)$ is a support of $\pi_1(x)$.

Proof: E is a finite subset of V and π_1 is a permutation on V so $\pi_1(E)$ is a finite subset of V . Suppose $\pi \in \Pi$ and $\pi(e) = e$ for all $e \in \pi_1(E)$. Then we have $e' \in E \rightarrow \pi_1(e') \in \pi_1(E) \rightarrow \pi(\pi_1(e')) = \pi_1(e') \rightarrow \pi_1^{-1} \circ \pi \circ \pi_1(e') = e'$. Thus for all $e' \in E$, $\pi_1^{-1} \circ \pi \circ \pi_1(e') = e'$. Then, since E is a support of x and $\pi_1^{-1} \circ \pi \circ \pi_1 \in \Pi$, we have $\pi_1^{-1} \circ \pi \circ \pi_1(x) = x$. Hence $\pi(\pi_1(x)) = \pi_1(x)$. It follows that $\pi_1(E)$ is a support of $\pi_1(x)$.

By the above lemma and set induction we have

Lemma 20: If $\pi \in \Pi$, then $\mathcal{M}(x) \leftrightarrow \mathcal{M}(\pi(x))$.

We can now prove

Lemma 21: The power set axiom holds in \mathcal{M} .

Proof: Suppose $X \in \mathcal{M}$ and let E be a support of X .

Let $Y = \{W \subseteq X : W \in \mathcal{M}\}$. To show that E is a support of

Y suppose $\pi \in \Pi$ and $\pi(e) = e$ for all $e \in E$. Then $\pi(X) = X$

and it is easy to verify that for all W we have

$W \subseteq X \leftrightarrow \pi(W) \subseteq X$. Then by the previous lemma we have

$W \in Y \leftrightarrow W \subseteq X \ \& \ W \in \mathcal{M} \leftrightarrow \pi(W) \subseteq X \ \& \ \pi(W) \in \mathcal{M} \leftrightarrow \pi(W) \in Y$

$\leftrightarrow W \in \pi^{-1}(Y)$. Thus $Y = \pi^{-1}(Y)$ so $\pi(Y) = Y$. It follows that

E is a support of Y and since for all $W \in Y$, $W \in \mathcal{M}$ we have

$Y \in \mathcal{M}$. For each $W \in \mathcal{M}$ we have $(\forall v \in W)(v \in \mathcal{M})$ and hence

$$W \in Y \leftrightarrow W \subseteq X \leftrightarrow (\forall v)(v \in W \rightarrow v \in X) \leftrightarrow$$

$(\forall v \in \mathcal{M})(v \in W \rightarrow v \in X)$. Thus we have

$(\forall X \in \mathcal{M})(\exists Y \in \mathcal{M})(\forall W \in \mathcal{M})(W \in Y \leftrightarrow (\forall v \in \mathcal{M})(v \in W \rightarrow v \in X))$

which is the relativization of the power set axiom to \mathcal{M} .

For each formula φ of ZFU we write $\varphi^{\mathcal{M}}$ for the relativization of φ to \mathcal{M} .

Lemma 22: If $\pi \in \Pi$ and $\varphi(x_1, \dots, x_n)$ is a formula of ZFU with free variables x_1, \dots, x_n then

$$\varphi^{\mathcal{M}}(x_1, \dots, x_n) \leftrightarrow \varphi^{\mathcal{M}}(\pi(x_1), \dots, \pi(x_n)).$$

Proof (by induction on the length of φ): Suppose the lemma holds for all formulas shorter than φ . We may assume φ is atomic, a conjunction, a negation or of the form $(\exists y)\psi(x_1, \dots, x_n, y)$. The conclusion follows easily in the first three cases. In the last case we have by the induction hypothesis

$$\psi^{\mathcal{M}}(x_1, \dots, x_n, y) \leftrightarrow \psi^{\mathcal{M}}(\pi(x_1), \dots, \pi(x_n), \pi(y)).$$

Every y' is of the form $\pi(y)$, where $y = \pi^{-1}(y')$, so using

Lemma 20 we have $\varphi^{\mathcal{M}}(x_1, \dots, x_n) \leftrightarrow$

$$(\exists Y)(\mathcal{M}(Y) \ \& \ \psi^{\mathcal{M}}(x_1, \dots, x_n, Y)) \leftrightarrow$$

$$(\exists Y)(\mathcal{M}(\pi(Y)) \ \& \ \psi^{\mathcal{M}}(\pi(x_1), \dots, \pi(x_n), \pi(Y))) \leftrightarrow$$

$$(\exists Y')(\mathcal{M}(Y') \ \& \ \psi^{\mathcal{M}}(\pi(x_1), \dots, \pi(x_n), Y')) \leftrightarrow$$

$$\varphi^{\mathcal{M}}(\pi(x_1), \dots, \pi(x_n)).$$

Lemma 23: The replacement axioms hold in \mathcal{M} .

Proof: Suppose X, v_1, \dots, v_k are in \mathcal{M} and we have

$$(1) (\forall x \in X) (\mathcal{M}(x) \rightarrow (\exists! y) (\mathcal{M}(y) \& \varphi^{\mathcal{M}}(x, y, v_1, \dots, v_k))).$$

Then X has a support E_X and each v_i has a support E_i for $i = 1, \dots, k$. Then $E = E_X \cup E_1 \cup \dots \cup E_k$ is a support of v_1, \dots, v_k and X . Since we have $\mathcal{M}(X)$ we also have

$$(\forall x \in X) \mathcal{M}(x) \text{ so (1) is equivalent to}$$

$$(2) (\forall x \in X) (\exists! y) (\mathcal{M}(y) \& \varphi^{\mathcal{M}}(x, y, v_1, \dots, v_k)). \text{ Hence by}$$

an axiom of replacement we also have

$$(3) (\exists Y) (\forall y) (y \in Y \leftrightarrow (\exists x \in X) (\mathcal{M}(y) \& \varphi^{\mathcal{M}}(x, y, v_1, \dots, v_k))).$$

To show that E is a support of Y , suppose $\pi \in \Pi$ and

$\pi(e) = e$ for all $e \in E$. Then we have $\pi(X) = X$ and

$\pi(v_i) = v_i$ for $i = 1, \dots, k$. Using Lemmas 20 and 22 we

obtain from (3), $y \in Y \leftrightarrow$

$$(\exists x \in \pi(X)) (\mathcal{M}(\pi(y)) \& \varphi^{\mathcal{M}}(x, \pi(y), \pi(v_1), \dots, \pi(v_k))) \leftrightarrow$$

$$(\exists x \in X) (\mathcal{M}(\pi(y)) \& \varphi^{\mathcal{M}}(x, \pi(y), v_1, \dots, v_k)) \leftrightarrow \pi(y) \in Y$$

$$\leftrightarrow y \in \pi^{-1}(Y). \text{ Hence } Y = \pi^{-1}(Y) \text{ so } \pi(Y) = Y. \text{ Thus } E$$

is a support of Y and from (3) we have $(\forall y \in Y) \mathcal{M}(y)$ so

$\mathcal{M}(Y)$ follows. Since $X \in \mathcal{M}$, then $(\forall x \in X) \mathcal{M}(x)$ so from

(3) we obtain

$$(4) (\exists Y \in \mathcal{M}) (\forall y \in \mathcal{M}) (y \in Y \leftrightarrow$$

$$(\exists x \in X) (\mathcal{M}(x) \& \varphi^{\mathcal{M}}(x, y, v_1, \dots, v_k))).$$

Thus we have proven $(\forall X, v_1, \dots, v_k \in \mathcal{M}) ((1) \rightarrow (4))$ which

is the general form of a replacement axiom relativized to \mathcal{M} .

6. Proofs of Necessity

We now show that for a suitable choice of V and Π , D^n fails in \mathcal{M} .

Let G be a subgroup of S_n without fixed points.

Let $B = \{\langle i, A_i \rangle : i \in \omega\}$ be a sequence of pairwise disjoint n -element sets of atoms. For each A_i let G_i be a group of permutations on A_i with G_i isomorphic to G and having no fixed points. Let $V = \cup\{A_i : i \in \omega\}$. Let $\Pi = \cup\{G_i : i \in \omega\}$ (more precisely Π is the set of all permutations π on V such that for some $i \in \omega$, the restriction of π to A_i is a member of G_i and $\pi(x) = x$ for each $x \in V \sim A_i$). Let \mathcal{M} be the corresponding model. We say that π is of type i if $\pi \in G_i$. The following lemma summarizes some easily verifiable facts.

Lemma 24: Given $i, j \in \omega$, $\pi \in G_i$, $\pi' \in G_j$ and any x and y we have (i) $\pi(\langle x, y \rangle) = \langle \pi(x), \pi(y) \rangle$, (ii) if x is a pure set $\pi(x) = x$, (iii) if π fixes every element of the domain of a function F then $\pi(F) = F$ if and only if π fixes every element of the range of F , (iv) $(\pi \circ \pi')(x) = \pi(\pi'(x))$, (v) if $i \neq j$ then $\pi \circ \pi' = \pi' \circ \pi$, i.e., permutations of distinct types commute.

For $\pi_1, \pi_2 \in \Pi$ we have $\pi_1^{-1} \in \Pi$ and by (v) of the above lemma we have $\pi_2^{-1} \circ \pi_1 \circ \pi_2 \in \Pi$. Hence by Theorem 5, \mathcal{M} is a model of ZFU.

Lemma 25: x has a support if and only if there is $j \in \omega$ such that for all π of type greater than j , $\pi(x) = x$.

Proof: If E is a support of x , E is finite so for some $j \in \omega$, $E \subseteq U\{A_i : i \leq j\}$. Then for all π of type greater than j , $\pi(e) = e$ for all $e \in E$ so $\pi(x) = x$. Conversely if $\pi(x) = x$ for all π of type greater than j then $U\{A_i : i \leq j\}$ is a support of x .

Theorem 6 (Mostowski): D^n fails in \mathcal{M} .

Proof: It follows from (i), (ii) and (iii) of Lemma 24 that $B \in \mathcal{M}$ and hence that the set $A^* = \{A_i : i \in \omega\}$ is denumerable in \mathcal{M} . We wish to show that A^* has no choice function in \mathcal{M} . Suppose $F = \{\langle A_i, a_i \rangle : i \in \omega\}$ is a choice function on A^* so we have $a_i \in A_i$ for all $i \in \omega$. If $F \in \mathcal{M}$ there is, by the above lemma, $j \in \omega$ such that for all $\pi \in G_{j+1}$, $\pi(F) = F$ and hence by (iii) of Lemma 24, $\pi(a_{j+1}) = a_{j+1}$ so a_{j+1} is a fixed point of G_{j+1} which is impossible. It follows that A^* has no choice function in \mathcal{M} . Since A^* is denumerable in \mathcal{M} and contains only n -element sets, it follows that D^n fails in \mathcal{M} .

Mostowski [2] has proven that when $M_{n,Z}$ does not hold, G may be chosen such that C_Z holds in \mathcal{M} . Hence we have

Theorem 7 (Mostowski): If ZF is consistent, then $M_{n,Z}$ is necessary for $\vdash_{ZF} (C_Z \rightarrow D^n)$.

We now show that when $T_{n,Z}$ does not hold, G may be chosen so that W_Z holds in \mathcal{M} . Assuming $T_{n,Z}$ fails, we

may construct \mathcal{M} from a subgroup G of S_n without fixed points such that (1) for every finite sequence $\langle H_1, \dots, H_m \rangle$ of proper subgroups of G we have $\frac{|G|}{|H_1|} + \dots + \frac{|G|}{|H_m|} \notin Z$.

To show W_Z holds in \mathcal{M} , suppose $k \in Z$, α is an ordinal,

$|B_\gamma| = k$ for all $\gamma \in \alpha$ and suppose the set

$Q = \{\langle \gamma, B_\gamma \rangle : \gamma \in \alpha\}$ is a function in \mathcal{M} . We wish to show that the set $B^* = \{B_\gamma : \gamma \in \alpha\}$ has a choice function in \mathcal{M} .

Since $Q \in \mathcal{M}$ there is, by Lemma 25, $j \in \omega$ such that for all π of type greater than j ; $\pi(Q) = Q$ and hence by (ii) and (iii) of Lemma 24 we have (2) $\pi(B) = B$ for all $B \in B^*$.

Lemma 26: For each $B \in B^*$, there is $b \in B$ such that $\pi(b) = b$ for all π of type greater than j .

Proof: Suppose there is $B \in B^*$ such that each $b \in B$ is moved by some π of type greater than j . Call two elements b, b' of B $*$ -equivalent if for some product $\pi_1 \circ \dots \circ \pi_t$ of permutations of distinct types greater than j , we have $b' = (\pi_1 \circ \dots \circ \pi_t)(b)$. For each $i > j$, call b and b' i -equivalent if for some π of type i , $b' = \pi(b)$. It is easy to verify that $*$ -equivalence and i -equivalence are equivalence relations. Let C be a $*$ -equivalence class so C contains an element $b_0 \in B$. By our supposition there is $i > j$ such that b_0 is moved by some π_0 of type i . No $b \in C$ is fixed by all π of type i for otherwise, using Lemma 24, (iv) and (v), we may write $b_0 = (\pi_1 \dots \pi_t)(b)$

where only π_t is of type i and obtain

$$\pi_0(b_0) = (\pi_0 \circ \pi_1 \dots \circ \pi_t)(b) = (\pi_0 \circ \pi_1 \dots \circ \pi_{t-1})(b)$$

$$= (\pi_1 \circ \dots \circ \pi_{t-1} \circ \pi_0)(b) = (\pi_1 \circ \dots \circ \pi_{t-1})(b)$$

$$= (\pi_1 \circ \dots \circ \pi_t)(b) = b_0 \text{ which is impossible since } b_0 \text{ is}$$

moved by π_0 . Hence each $b \in C$ is moved by some π of type

i . It follows that C is partitioned by i -equivalence

classes each having more than one element. For each

i -equivalence class $D \subseteq C$, choose $d \in D$ and let

$H = \{\pi \in G_i : \pi(d) = d\}$. It is easy to verify that H is

a subgroup of G_i and since D has more than one element,

H is a proper subgroup of G_i . Let f be the mapping which

sends each left coset πH (with $\pi \in G_i$) to $\pi(d)$. f is

well-defined and 1-1 since for any $\pi_1, \pi_2 \in G_i$ we have

$$\pi_1 H = \pi_2 H \iff \pi_1^{-1} \circ \pi_2 \in H \iff \pi_1^{-1} \circ \pi_2(d) = d \iff \pi_2(d) = \pi_1(d).$$

f is onto D by definition of an i -equivalence class. It

follows that $|D|$ is the number of left cosets of H included

in G_i . Thus $|D| = \frac{|G_i|}{|H|}$. Since G_i is isomorphic to G

we can choose a proper subgroup H_D of G such that

$$|D| = \frac{|G|}{|H_D|}. \text{ Hence } B \text{ has a partitioning } \mathcal{D} \text{ such that for each}$$

$D \in \mathcal{D}$ we have $|D| = \frac{|G|}{|H_D|}$ for some proper subgroup

H_D of G . Then $\sum_{D \in \mathcal{D}} \frac{|G|}{|H_D|} = \sum_{D \in \mathcal{D}} |D| = |B| = k \in \mathbb{Z}$

contradicting (1).

It follows from the above lemma, the Axiom of Choice and (2) that there is a choice function F on B^* such that whenever $\langle B, b \rangle \in F$ we have $\pi(B) = B$ and $\pi(b) = b$ for all π of type greater than j . It follows from Lemma 24, (i), (ii), and (iii) that $\pi(F) = F$ for all π of type greater than j . Then, by Lemma 25, F has a support and using the fact that $B^* \in \mathcal{M}$ it is easy to verify that $F \in \mathcal{M}$. Hence W_Z holds in \mathcal{M} so we have proven

Theorem 8: If ZF is consistent, then $T_{n,Z}$ is necessary for $\frac{\vdash}{ZFU} (W_Z \rightarrow D^n)$.

The proof of Theorem T is now complete.

It is already known that $W_Z \rightarrow C^n$ is not provable in ZFU unless $n = 1$. In fact the following much stronger result is known.

Theorem 9: If ZF is consistent, there is a model of ZFU in which W^n holds and C^n fails for all $n > 1$.

Proof: We use the fact that the group of all permutations on a denumerable set has no proper subgroups of finite index. Using the theory ZFU^+ we let V be a denumerable set of atoms and let Π be the set of all permutations on V . Let \mathcal{M} be the corresponding model. To show C^n fails in \mathcal{M} for $n > 1$, let X be the set of all n -element subsets of V . $X \in \mathcal{M}$ since 0 is a support of X and each member of X is a support of itself. Now suppose

f is a choice function on X and let E be any finite subset of V . We choose an n -element set $B \subseteq V$ disjoint from E . $B \in X$ so $\langle B, b \rangle \in f$ for some $b \in B$. Since $n > 1$, there is $b' \in B$ such that $b' \neq b$ so $\langle B, b' \rangle \notin f$. Let π be the permutation on V which interchanges b and b' and fixes all other atoms so $\pi(B) = B$. We have $\pi(e) = e$ for all $e \in E$ and $\langle B, b' \rangle = \pi(\langle B, b \rangle) \in \pi(f)$ so $\pi(f) \neq f$. It follows that no E is a support for f so $f \notin \mathcal{M}$. Hence C^n fails in \mathcal{M} . To show W^n holds in \mathcal{M} suppose α is an ordinal and $Y = \{ \langle \beta, y_\beta \rangle : \beta \in \alpha \}$ is a function in \mathcal{M} where $|y_\beta| = n$ for all $\beta \in \alpha$. Then Y has a support E . By the Axiom of Choice (of ZFU^+) there is a choice function f on $\{y_\beta : \beta \in \alpha\}$. We wish to show that $f \in \mathcal{M}$. Since E is finite, $V \sim E$ is denumerable so the group Π^* of all $\pi \in \Pi$ such that $\pi(e) = e$ for all $e \in E$ has no proper subgroups of finite index. For each $\pi \in \Pi^*$ we have $\pi(e) = e$ for all $e \in E$ so $\pi(Y) = Y$. Hence, $\langle \beta, y_\beta \rangle \in Y \rightarrow \pi(\langle \beta, y_\beta \rangle) \in Y \rightarrow \langle \beta, \pi(y_\beta) \rangle \in Y \rightarrow \pi(y_\beta) = y_\beta$ since Y is a function. Thus $\{ \pi \in \Pi^* : \pi(y_\beta) = y_\beta \} = \Pi^*$. Then for $x \in y_\beta$, $\Pi_x = \{ \pi \in \Pi^* : \pi(x) = x \}$ is a subgroup of Π^* having finite index (the cosets being of the form $\{ \pi \in \Pi^* : \pi(x) = x' \}$ for $x' \in y_\beta$). Since Π_x cannot be a proper subgroup of Π^* , we must have $\Pi_x = \Pi^*$. Now suppose $\pi(e) = e$ for all $e \in E$ and $\langle y_\beta, x \rangle \in f$ so $x \in y_\beta$. Then $\pi \in \Pi^* = \Pi_x$ so we have

$\pi(y_\beta) = y_\beta$ and $\pi(x) = x$. Hence every element of f is fixed by π so $\pi(f) = f$. Thus E is a support for f and it follows that $f \in \mathcal{W}$. Hence W^n holds in \mathcal{W} .

CHAPTER II

CORRESPONDING RESULTS

IN THE THEORY ZF

The theorems of Chapter I (except Theorems 4, 5, and 6) remain valid when "ZFU" is replaced by "ZF". The analogues of Theorems 1, 2, and 3 are immediate since every formula provable in ZFU is also provable in ZF.

We sketch a proof of the analogue of Theorem 8:

If ZF is consistent, then $T_{n,Z}$ is necessary for $\frac{\vdash}{ZF} (W_Z \rightarrow D^n)$. Suppose $T_{n,Z}$ fails so there is a group $G \subseteq S_n$ without fixed points such that for any proper subgroups H_1, \dots, H_m of G , we have $\frac{|G|}{|H_1|} + \dots + \frac{|G|}{|H_m|} \notin Z$.

We shall use the techniques and terminology of

P. Cohen [1].

Let \mathcal{M} be the minimal model of ZF. For each ordinal $\alpha \in \mathcal{M}$ we construct a label space S_α . Let $S_0 = \omega$ and $S_1 = \{a_{ijk} : i, k \in \omega, j \in \{1, \dots, n\}\}$ where the a 's are distinct infinite sets which will be labels for generic subsets of ω . Let $S_2 = \{d_{ij} : i \in \omega, j \in \{1, \dots, n\}\}$ where

$d_{ij} = \{a_{ijk} : k \in \omega\}$. The d 's will play the role of atoms. We define S_3 inductively as follows: $x \in S_3$ if x is finite and for all $y \in x$, $y \in S_2 \cup S_3$. $S_4 = \{B, G^*, E^*\}$ where $B = \{\langle i, A_i \rangle : i \in \omega, A_i = \{d_{i1}, \dots, d_{in}\}\}$, $G^* = \{\langle i, G_i \rangle : i \in \omega\}$ where G_i is a group of permutations on A_i isomorphic to G and having no fixed points, and $E^* = \{\langle i, E_i \rangle : i \in \omega\}$ is an enumeration of all $E_i \in S_3$ such that $\pi(E_i) = E_i$ for all $\pi \in U\{G_i : i \in \omega\}$ (for $\pi \in G_i$, $d \in S_2 \sim A_i$, $c \in S_3$ we define $\pi(d) = d$ and $\pi(c) = \{\pi(c') : c' \in c\}$). For $\alpha > 4$, S_α is a set of labels $c_{\alpha\varphi}$ corresponding to formulas φ with one free variable and relativized to $X_\alpha = U\{S_\beta : \beta \in \alpha\}$ possibly involving constants from X_α . Let $S = U\{S_\alpha : \alpha \in \mathcal{M}\}$ be the set of all labels.

A forcing condition P is a finite consistent set of statements of the form $m \in a_{ijk}$ or $m \notin a_{ijk}$. We use Cohen's definition of forcing with the following modification: for $c_1 \in S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$ and $c_2 \in S_0 \cup S_2 \cup S_3 \cup S_4$, P forces $c_1 \in c_2$ if $c_1 \in c_2$. We say P weakly forces φ if P forces $\sim \sim \varphi$. It is easy to verify that P weakly forces φ if and only if every extension of P has an extension which forces φ .

We choose a complete sequence of forcing conditions and define a correct condition to be a forcing condition having an extension in the complete sequence. We then define \bar{c} for each label c . For $m \in S_0$ let $\bar{m} = m$. For $a_{ijk} \in S_1$

let $\bar{a}_{ijk} = \{m \in \omega : \{m \in a_{ijk}\} \text{ is a correct condition}\}$.

For $c \in S_2 \cup S_3 \cup S_4$ let $\bar{c} = \{\bar{c}' : c' \in c\}$. For

$c_{\alpha\varphi} \in S_\alpha$ where $\alpha > 4$, let c_1, \dots, c_k be the constants

occurring in φ and let $\bar{X}_\alpha = \{\bar{c} : c \in X_\alpha\}$. Let

$\bar{c}_{\alpha\varphi} = \{x \in \bar{X}_\alpha : \varphi(x, \bar{c}_1, \dots, \bar{c}_k, X_\alpha)\}$. We also let

$\bar{S}_\alpha = \{\bar{c} : c \in S_\alpha\}$.

The set $N = \{\bar{c} : c \in S_\alpha \text{ for some } \alpha \in \omega\}$ is a model of ZF. It is easy to verify that $\bar{X}_\alpha = \bar{c}_{\alpha, (x=x)}$ and $\bar{S}_\alpha = \bar{X}_{\alpha+1} \sim \bar{X}_\alpha$ so \bar{X}_α and \bar{S}_α are in N . A statement is true in N if and only if it is forced by some correct condition.

In order to show that W_Z holds in N and D^n fails in N we introduce permutations on S . A label is said to be of type i if it is of the form a_{ijk} or d_{ij} .

A permutation π on S is said to be of type i if all of the following hold: $\pi(a) = a$ for all $a \in S_1$ of type different from i , the restriction of π to A_i is a member of the group G_i , $\pi(c) = \{\pi(c') : c' \in c\}$ for all $c \in S_0 \cup S_2 \cup S_3 \cup S_4$, and $\pi(c_{\alpha\varphi}) = c_{\alpha\pi(\varphi)}$ for $\alpha > 4$ where $\pi(\varphi) = \varphi(x, \pi(c_1), \dots, \pi(c_r))$, c_1, \dots, c_r being the constants occurring in φ . We also define $\pi(S_\alpha) = S_\alpha$ and $\pi(X_\alpha) = X_\alpha$.

Remark: For each $k \in \omega$, $\alpha \in \omega$ and π of any type, we have $\pi(k) = k$, $\pi(G_k) = G_k$, $\pi(A_k) = A_k$, $\pi(E_k) = E_k$ and hence $\pi(B) = B$, $\pi(G^*) = G^*$ and $\pi(E^*) = E^*$.

If P is a forcing condition and $\varphi = \varphi(x_1, \dots, x_k, c_1, \dots, c_r)$ is a formula and π is a permutation of any type we define

$$\begin{aligned} \pi(P) &= \{ "m \in \pi(a_{ijk})" : "m \in a_{ijk}" \in P \} \cup \\ &\{ "m \notin \pi(a_{ijk})" : "m \notin a_{ijk}" \in P \} \text{ and} \\ \pi(\varphi) &= \varphi(x_1, \dots, x_k, \pi(c_1), \dots, \pi(c_r)). \end{aligned}$$

We state without proof two lemmas corresponding to lemmas proven by Cohen ([1] pages 137-138).

Lemma 1: For each forcing condition P , each statement φ and π of any type we have P (weakly) forces φ if and only if $\pi(P)$ (weakly) forces $\pi(\varphi)$.

Lemma 2: For each label c , forcing condition P and formula φ there is $s \in \omega$ such that for all π of type greater than s , $\pi(c) = c$, $\pi(P) = P$ and $\pi(\varphi) = \varphi$.

Lemma 3: D^n fails in N .

Proof: Otherwise for some label f there is a forcing condition P which forces " f is a choice function on $\{A_i : i \in \omega\}$ ". There is $s \in \omega$ such that $\pi(P) = P$ and $\pi(f) = f$ for all π of type greater than s . Let $i = s + 1$. There is an extension Q of P which forces $\langle A_i, d_{ij} \rangle \in f$ for some j . Since G_i has no fixed points there is π of type i such that $\pi(d_{ij}) \neq d_{ij}$ and $\pi(a)$ does not occur in Q for all a of type i occurring in Q . Then $\pi(Q)$ is compatible with Q and $\pi(Q)$ forces $\langle A_i, \pi(d_{ij}) \rangle \in \pi(f) = f$. Thus $Q \cup \pi(Q)$ forces $\langle A_i, \pi(d_{ij}) \rangle \in f$, $\langle A_i, d_{ij} \rangle \in f$ and " f is a function" which is impossible.

Each $\bar{c} \in N$ has a definition of the form

$$\bar{c} = \{x : x \in \bar{X}_\alpha \ \& \ \varphi(x)\}$$

where φ may involve constants from \bar{X}_α and has quantifiers restricted to \bar{X}_α . If $\alpha > \beta > 4$, and $\bar{c}_{\beta\theta} \in \bar{S}_\beta$ occurs in φ , we can use θ to replace φ by a formula φ' involving the same constants occurring in φ except that φ' does not involve $\bar{c}_{\beta\theta}$ but may involve constants \bar{c}_i for c_i occurring in θ . Each of the c_i is a member of some S_γ with $\gamma < \beta$. Thus φ may eventually be replaced by a formula, with quantifiers restricted to \bar{X}_α , but otherwise involving only constants of the form \bar{X}_β with $\beta \in \alpha$ or of the form \bar{c}_i with $c_i \in S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$. Furthermore each label in $S_0 \cup S_3$ may be defined in terms of finitely many labels of S_2 and B may be defined in terms of G^* . It follows that each $\bar{c} \in N$ has a definition of the form $\bar{c} = \{x : x \in \bar{X}_\alpha \ \& \ \varphi(x)\}$ where φ has all quantifiers restricted to \bar{X}_α but otherwise involves only constants of the form \bar{X}_β , with $\beta \in \alpha$, \bar{a}_{ijk} , \bar{d}_{ij} , G^* or E^* .

A formula φ is called reduced if it involves only constants of the form \bar{X}_α , \bar{a}_{ijk} , \bar{d}_{ij} , G^* or E^* . For each reduced formula φ we define $\bar{\varphi}$ as the formula obtained from φ by replacing each constant c occurring in φ by \bar{c} . It follows from the last statement in the above paragraph that each $\bar{c} \in N$ has a definition of the form $\bar{c} = \{x : \bar{\varphi}\}$ where φ is reduced. We also call such a definition reduced.

There is no mapping in N which takes each $c \in S$ to \bar{c} so we need the following definitions. A \bar{d} of type i is an element of \bar{A}_i , the i^{th} term of \bar{B} . An \bar{a} of type i is an element of a \bar{d} of type i .

Lemma 4: If a reduced statement φ is true in N , then there is a correct condition P involving no a 's except those occurring in φ such that P weakly forces φ .

Proof: If φ is true in N , there is a correct condition Q which forces φ . Let P be the part of Q involving only a 's occurring in φ . We claim that P weakly forces φ . Otherwise, there is an extension R of P which forces $\sim \varphi$. Let s be the maximum type of the a 's occurring in R but not in φ . For each $i = 1, \dots, s$ we can choose a permutation π_i of type i which fixes all d 's and moves the a 's of type i occurring in R , but not in φ , to a 's not occurring in Q . Then $\pi_1 \dots \pi_s (R)$ is compatible with Q and $\pi_1 \dots \pi_s (\sim \varphi) = \sim \varphi$. Hence, $\pi_1 \dots \pi_s (R) \cup Q$ forces $\sim \varphi$ & φ which is impossible.

The next lemma shows that for each $\bar{c} \in N$ there is a minimal finite set of \bar{a} 's such that \bar{c} has a reduced definition involving those \bar{a} 's and no others.

Lemma 5: If $\bar{c} \in N$, φ is reduced and $\bar{c} = \{x : \bar{\varphi}(x)\}$ is a reduced definition of \bar{c} involving as few \bar{a} 's as possible, then for all reduced ψ such that $\bar{c} = \{x : \bar{\psi}(x)\}$, every \bar{a} occurring in $\bar{\varphi}$ also occurs in $\bar{\psi}$.

Proof: Suppose a_{ijk_0} occurs in φ but not in ψ .
 Let $\theta = (\forall x)(\varphi \leftrightarrow \psi)$ so θ is reduced and is true in N .
 Hence there is a correct P involving only a 's occurring
 in θ such that P weakly forces θ . The part of P involving
 only a_{ijk_0} may be expressed as $Q(a_{ijk_0})$ where $Q(x)$ is of
 the form $Q(x) = \{n_1 \in x, \dots, n_s \in x, m_1 \notin x, \dots, m_r \notin x\}$.
 Suppose a_{ijk_1} does not occur in φ and suppose $Q(a_{ijk_1})$
 is correct. We wish to show that $\bar{c} = \bar{c}'$ where

$\bar{c}' = \{x : \bar{\varphi} \left(\begin{array}{c} \bar{a}_{ijk_0} \\ \bar{a}_{ijk_1} \end{array} \right)\}$. Suppose $\bar{c} \neq \bar{c}'$. Then there is a
 correct extension R of P such that R forces

$\sim (\forall x)(\varphi \leftrightarrow \varphi \left(\begin{array}{c} a_{ijk_0} \\ a_{ijk_1} \end{array} \right))$. Choose a_{ijk_2} not occurring in
 R or θ . Let $\pi_2 = (a_{ijk_1} \ a_{ijk_2})$, i.e., the permutation of
 type i which interchanges a_{ijk_1} and a_{ijk_2} . Then $\pi_2(P)$ is
 compatible with R . We can write $P = Q(a_{ijk_0}) \cup P'$ where

P' does not involve a_{ijk_0} so $\pi_2(P) = Q(a_{ijk_0}) \cup \pi_2(P')$.

Thus $\pi_2(P')$ is compatible with R and does not involve

a_{ijk_0} or a_{ijk_1} . Let $\pi_1 = (a_{ijk_0} \ a_{ijk_1})$ so

$\pi_1 \pi_2(P) = Q(a_{ijk_1}) \cup \pi_2(P')$. Since $\pi_2(P')$ is compatible

with R and since $R, Q(a_{ijk_0})$ and $Q(a_{ijk_1})$ are correct, it

follows that $R, \pi_2(P)$ and $\pi_1 \pi_2(P)$ are compatible. Since

a_{ijk_1} does not occur in φ and a_{ijk_2} does not occur in θ

and since P forces θ , we have $\pi_2(P)$ forces $\pi_2(\theta) = \theta \left(\begin{smallmatrix} a_{ijk_1} \\ a_{ijk_2} \end{smallmatrix} \right) =$

(i) $(\forall x) (\varphi \leftrightarrow \psi \left(\begin{smallmatrix} a_{ijk_1} \\ a_{ijk_2} \end{smallmatrix} \right))$. Thus a_{ijk_1} does not occur in

(i) and since a_{ijk_0} does not occur in ψ , a_{ijk_0} does not occur in $\psi \left(\begin{smallmatrix} a_{ijk_1} \\ a_{ijk_2} \end{smallmatrix} \right)$. Since $\pi_2(P)$ forces (i), it follows

that $\pi_1\pi_2(P)$ forces (ii) $(\forall x) (\varphi \left(\begin{smallmatrix} a_{ijk_0} \\ a_{ijk_1} \end{smallmatrix} \right) \leftrightarrow \psi \left(\begin{smallmatrix} a_{ijk_1} \\ a_{ijk_2} \end{smallmatrix} \right))$.

Therefore, $\pi_2(P) \cup \pi_1\pi_2(P) \cup R$ forces (i), (ii) and

$\sim (\forall x) (\varphi \leftrightarrow \varphi \left(\begin{smallmatrix} a_{ijk_0} \\ a_{ijk_1} \end{smallmatrix} \right))$ which is impossible. Thus

$\bar{c} = \bar{c}' = \{x : \bar{\varphi} \left(\begin{smallmatrix} \bar{a}_{ijk_0} \\ \bar{a}_{ijk_1} \end{smallmatrix} \right)\}$ whenever a_{ijk_1} does not occur in

φ and $Q(a_{ijk_1})$ is correct. Thus \bar{a}_{ijk_0} is not essential to

the definition of \bar{c} since \bar{a}_{ijk_0} may be replaced by any

$y \in \bar{d}_{ij}$ not occurring in $\bar{\varphi}$ and satisfying

$Q^*(y) : n_1 \in y \ \&\dots\ \& \ n_s \in y \ \& \ m_1 \notin y \ \&\dots\ \& \ m_t \notin y$. Hence,

letting $\bar{c}_1, \dots, \bar{c}_k$ be the constants in $\bar{\varphi}$ other than \bar{a}_{ijk_0} we have

$\bar{c} = \{x : (\forall y \in \bar{d}_{ij}) (y \neq \bar{c}_1 \ \&\dots\ \& \ y \neq \bar{c}_k \ \& \ Q^*(y) \rightarrow \bar{\varphi} \left(\begin{smallmatrix} \bar{a}_{ijk_0} \\ y \end{smallmatrix} \right))\}$,

which is a reduced definition of \bar{c} involving fewer \bar{a} 's than

$\{x : \bar{\varphi}(x)\}$ contrary to the hypothesis.

Hence each $\bar{c} \in N$ determines a finite set of \bar{a} 's essential to the definition of \bar{c} . There is a natural ordering on this finite set (since the \bar{a} 's are subsets of ω) which we denote by $\bar{a}_{\bar{c}}$.

Lemma 6: Suppose P forces " F is a function on α and $\langle \beta, c \rangle \in F$ " where β is an ordinal, c corresponds to a reduced formula involving only d 's of type greater than s and F corresponds to a reduced formula involving no a 's or d 's of type greater than s . Then P weakly forces $\pi(c) = c$ for π of any type.

Proof: If π is of type $\leq s$ then π fixes the formula corresponding to c so $\pi(c) = c$. If π is of type i greater than s , π fixes the formula corresponding to F so $\pi(F) = F$. Suppose P does not weakly force $\pi(c) = c$. Then P has an extension Q which forces $\pi(c) \neq c$. We can choose π' of type i whose action on the d 's is the same as that of π but which moves a 's of type i occurring in Q to a 's not occurring in Q . Then $\pi'(Q)$ is compatible with Q and since c corresponds to a formula involving only d 's, $\pi'(c) = \pi(c)$. π' also fixes the formula corresponding to F so $\pi'(F) = F$. Q is an extension of P so Q forces $\langle \beta, c \rangle \in F$. Hence $\pi'(Q)$ forces $\langle \beta, \pi'(c) \rangle \in \pi'(F)$. Since $\pi'(c) = \pi(c)$ and $\pi'(F) = F$, it follows that $Q \cup \pi'(Q)$ forces " F is a function, $\pi(c) \neq c$, $\langle \beta, c \rangle \in F$ and $\langle \beta, \pi(c) \rangle \in F$ " which is not possible.

In order to show that W_Z holds in N we first show that for $k \in Z$, every well-orderable collection of k -element sets of a special kind has a choice function.

Lemma 7: Suppose $k \in \mathbb{Z}$, α and γ are ordinals in \mathbb{N} , $\bar{F} \in \mathbb{N}$ and $\bar{F} = \{\gamma : \bar{\varphi}(\gamma)\}$ where φ is reduced. Suppose s is the maximum type of the a 's and d 's occurring in φ . Suppose it is true in \mathbb{N} that " F is a function on α such that for each $\beta \in \alpha$, $|F(\beta)| = k$ and each member of $F(\beta)$ is of the form $\langle \delta, b \rangle$ for some $\delta \in \gamma$ and $b \in S_3$ where b is a finite set of finite sequences of d 's of type greater than s (i.e., of d 's belonging to $U\{A_i : i > s, i \in \omega\}$ ". Then there is a function $f \in \mathbb{N}$ such that $f(\beta) \in \bar{F}(\beta)$ for each $\beta \in \alpha$.

We sketch the proof. For each $\beta \in \alpha$, $F(\beta)$ corresponds to a formula involving only d 's of type greater than s . There is a correct P which forces " F is a choice function and $\langle \beta, F(\beta) \rangle \in F$ ". It follows from the previous lemma that for π of any type P weakly forces $\pi(F(\beta)) = F(\beta)$ so $\pi(F(\beta)) = F(\beta)$ is true in \mathbb{N} . For each i , each $\pi' \in \bar{G}_i$ is a permutation on \bar{d} 's and hence induces a 1-1 mapping on the elements of $\bar{F}(\beta)$. The induced mapping corresponds to the restriction of some π of type i to labels corresponding to the elements of $F(\beta)$. Since $\pi(F(\beta)) = F(\beta)$ is true in \mathbb{N} , we can verify that $\pi'(\bar{F}(\beta)) = \bar{F}(\beta)$. Using the failure of $T_{n, \mathbb{Z}}$ we can show, as in the proof of Lemma 26 of Chapter I, that there is $\bar{c} \in \bar{F}(\beta)$ such that $\pi'(\bar{c}) = \bar{c}$. Then we must have $\bar{c} = \langle \delta, \bar{E}_i \rangle$ for some $\delta \in \gamma$ and \bar{E}_i in the

range of \bar{E}^* . There is a well-ordering on $\gamma \times \text{Range}(\bar{E}^*)$ in \mathbb{N} so we can define $f(\beta)$ to be the first $\langle \delta, \bar{E}_1 \rangle \in \gamma \times \text{Range}(\bar{E}^*)$ belonging to $\bar{F}(\beta)$.

We now show that W_Z holds in \mathbb{N} by showing that for $k \in \mathbb{Z}$, each well-orderable collection of k -element sets may be replaced by a collection of the kind described in the previous lemma.

Lemma 8: Suppose $k \in \mathbb{Z}$, $F \in \mathbb{N}$ and $F = \{y : \varphi(y)\}$ is a reduced definition of F . Suppose F is a function on α such that $|F(\beta)| = k$ for all $\beta \in \alpha$. Then there is a function $f \in \mathbb{N}$ such that $f(\beta) \in F(\beta)$ for all $\beta \in \alpha$.

Proof: Let s be the maximum type of the sets occurring in φ . There are only finitely many \bar{d} 's of type $\leq s$ so there is a well-ordering on them which induces a well-ordering on all formulas involving no \bar{d} 's of type $> s$ and no \bar{a} 's. For each $x \in Y = U\{F(\beta) : \beta \in \alpha\}$ we choose the first such formula ψ_x for which x has a definition $x = \{y : \psi_x(y, \vec{a}_x, \vec{d})\}$ where \vec{d} is a finite sequence of \bar{d} 's of type $> s$. Let b_x be the set of all \vec{d} such that $x = \{y : \psi_x(y, \vec{a}_x, \vec{d})\}$ and such that the maximum type of the \bar{d} 's occurring in \vec{d} is minimum. Thus b_x is finite so $b_x \in \bar{S}_3$ and involves only \bar{d} 's of type $> s$. We now replace each $x \in Y$ by the ordered pair $\langle \langle \psi_x, \vec{a}_x \rangle, b_x \rangle$. There is a linear ordering on the set $\Omega = \{\langle \psi_x, \vec{a}_x \rangle : x \in Y\}$ induced by the

well-ordering of the ψ_x 's and the linear ordering of the \bar{a} 's. Ω is a union of a well-orderable collection of finite sets (since Y is such a union) so it follows that Ω has a well ordering. Thus each $x \in Y$ may be replaced by an ordered pair $\langle \delta_x, b_x \rangle$ where δ_x is an ordinal. Since $b_x \in \bar{S}_3$ and involves only \bar{d} 's of type greater than s , the existence of the required function f follows from the previous lemma.

Hence W_Z holds in N . Thus if $T_{n,Z}$ fails, there is a model N of ZF in which D^n fails and W_Z holds so we have the analogue of Theorem 8. It follows from the analogue of Theorem 2 that we also have that of Theorem T.

The analogue of Theorem 7 may be established by constructing the model N from the subgroup G of S_n chosen by Mostowski for a proof of Theorem 7. The analogue of Theorem M then follows from that of Theorem 3.

The analogue of Theorem 9 may be proven by constructing a model N' from label spaces defined as follows. Let $S_0 = \omega$. Let S_1 be a set of labels a_{ij} , for $i, j \in \omega$, corresponding to generic subsets of ω . Let $S_2 = \{d_i : i \in \omega\}$ where $d_i = \{a_{ij} : j \in \omega\}$. For $\alpha > 2$ let S_α be a set of labels $c_{\alpha\varphi}$ corresponding to formulas φ relativized to X_α possibly involving constants from X_α . Permutations on the labels are of just one type, those which map each d_i 1-1 onto some d_j .

It is possible to prove that for $n > 1$, C^n fails in N' (the set of n -element subsets of \overline{S}_2 has no choice function in N') and W^n holds in N' . The proofs may be carried out by the methods of this chapter without using the fact, used in the proof of Theorem 9, that the group of all permutations on a denumerable set has no proper subgroups of finite index.

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INDEX

- \vec{a} , 40
- $A^n(X)$, 4
- Atom, 2
- $A_Z(X)$, 4
- Choice function, 2-3
- C^n , 4
- C_Z , 4
- $\beta_1(X)$, 7
- D^n , 4
- D_Z , 4
- Fixed point, 9
- Index of a seperable collection, 17
- $M_{n,Z}$, 11
- Pure set, 2
- Reduced definition, 37
- Reduced formula, 37
- Seperable, 16
- Set, 2
- Subset function, 2-3
- Support, 22
- $T_{n,Z}$, 10
- Type of an \bar{a} , 38
- Type of a \bar{d} , 38
- Type of a label, 35
- Type of a permutation, 26, 35
- W^n , 4
- W_Z , 4