Journal of Inequalities and Special Functions
ISSN: 2217-4303, URL: www.ilirias.com/jiasf
Volume 10 Issue 3(2019), Pages 26-32.

CHEBYSHEV-DUBINER NORMING WEBS ON STARLIKE POLYGONS

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Abstract. We construct web-shaped polynomial norming meshes on starlike polygons by radial and boundary Chebyshev points, via the approximation theoretic notion of Dubiner distance. As an application, we get a $(1-\varepsilon)$-approximation to the minimum of an arbitrary polynomial of degree n by $\mathcal{O}(n^2/\varepsilon)$ sampling points.

1. Introduction

In some recent papers [10, 11, 13, 14, 15] we applied, in the framework of polynomial optimization, *discrete polynomial norming inequalities* on multidimensional compact sets with different geometries. The basic notions were that of *polynomial norming mesh* and of *Dubiner distance* of a compact set.

Let $K \subset \mathbb{R}^d$ be a compact set. For convenience, we restrict here to polynomial determining compact sets, i.e., a real d-variate polynomial vanishing there vanishes everywhere in $\mathbb{R}^d$ (the definition below can be easily extended to nondetermining compact sets, for example subsets of an algebraic variety).

A *polynomial norming mesh* of K is a sequence of finite norming sets $X_n \subset K$, such that

$$\|p\|_K \leq C\,\|p\|_{X_n}\ ,\ \ \forall p \in \mathbb{P}_n^d\ ,\ \ card(X_n) = \mathcal{O}(n^\beta)\ , \tag{1.1}$$

for some constant $C \geq 1$, where $\mathbb{P}_n^d$ denotes the subspace of polynomials of total-degree not exceeding n with dimension $N = dim(\mathbb{P}_n^d) = \binom{n+d}{d}$, and $\|p\|_Y$ the uniform norm on a continuous or discrete compact set Y). Observe that $\beta \geq d$, since X_n is $\mathbb{P}_n^d$-determining and thus $card(X_n) \geq N \sim n^d/d!$.

When $\beta = d$ the polynomial mesh is termed *optimal* in the literature, since it has the lowest possible order of growth with respect to n; cf., e.g., [8, 9].

Among their properties, we recall that polynomial norming meshes are invariant under affine transformations, can be extended from known instances by algebraic transformation, finite union and finite product, are near-optimal for least-square

2010 *Mathematics Subject Classification.* 41A17, 65K05, 90C26.
Key words and phrases. discrete polynomial norming inequalities; starlike polygon; Chebyshev points; Dubiner distance; polynomial optimization.

Submitted November 13, 2018. Published June 3, 2019.
Work partially supported by the DOR funds and the biennial project BIRD181249 of the University of Padova, and by the GNCS-INdAM. This research has been accomplished within the RITA "Research ITalian network on Approximation".
Communicated by Valmir Krasniqi.

approximation and contain good unisolvent sets for polynomial interpolation. On the theory and applications of polynomial norming meshes, mainly developed in the last decade, we may refer the reader to [2, 4, 8, 9, 12] with the references therein.

Concerning polynomial optimization, the general result in [13] implies that if K admits an optimal norming mesh, for any given $\varepsilon \in (0,1)$ we can construct a (optimal) norming mesh $X_n(\varepsilon)$ with $card(X_n(\varepsilon)) = \mathcal{O}((n/\varepsilon)^d)$, such that

$$\|p\|_K \leq (1+\varepsilon)\,\|p\|_{X_n(\varepsilon)}\ ,\ \ \forall p \in \mathbb{P}_n^d\ . \tag{1.2}$$

From (1.2) it is easy to prove that

$$\min_{X_n(\varepsilon)} p - \min_K p \leq \varepsilon\,\left(\max_K p - \min_K p\right)\ ,\ \ \forall p \in \mathbb{P}_n^d\ , \tag{1.3}$$

i.e., discrete minimization on $X_n(\varepsilon)$ provides a $(1-\varepsilon)$-approximation to the global minimum (the notion is relative to the range of p, as usual in the optimization context, cf. e.g. [5]).

The cardinality can be reduced for special classes of compact sets, by resorting to the geometric structure of the domain. In [10], it is shown that (1.2)-(1.3) hold for multidimensional boxes on suitable Chebyshev grids, with

$$card(X_n(\varepsilon)) = \mathcal{O}((n/\sqrt{\varepsilon})^d) \tag{1.4}$$

(a similar result in the tensor-product setting was already proved in [16], whereas uniform rational grids are considered, e.g., in [5, 6]). Similar results have been recently obtained also for sections of sphere, ball and torus [14, 15], and on convex (or even starlike) bodies with smooth boundary [11].

All these constructions, which typically produce nonuniform meshes clustering at the boundary, make use of another relevant notion of multivariate polynomial approximation theory, the *Dubiner distance* on a compact set K (introduced in the seminal paper [7])

$$dub_K(x,y) = \sup_{deg(p)\geq 1,\,\|p\|_K\leq 1} \left\{\frac{1}{deg(p)}\ |\arccos(p(x)) - \arccos(p(y))|\right\}\ . \tag{1.5}$$

Among its basic properties, we recall that it is invariant under invertible affine transformations, i.e., if $\sigma(x) = Ax + b$, $det(A) \neq 0$, then

$$dub_K(x,y) = dub_{\sigma(K)}(\sigma(x),\sigma(y))\ . \tag{1.6}$$

Moreover, it is monotone nonincreasing with respect to set inclusion, namely, if $x, y \in K \subseteq H$ then $dub_H(x,y) \leq dub_K(x,y)$.

The notion of Dubiner distance plays a deep role in multivariate polynomial approximation, cf. e.g. [3, 7]. Unfortunately, such a distance is explicitly known only in the univariate case on intervals (where it is the *arccos* distance by the Van der Corput-Schaake inequality), and on cube, simplex, sphere and ball (in any dimension), cf. [3, 7]. On the other hand, it can be often estimated, for example on smooth convex bodies via a tangential Markov inequality on the boundary, cf. [11]. Its connection with the theory of polynomial norming meshes is given by the following elementary but powerful lemma (proved essentially in [1], see also [10]); for the reader's convenience, we recall also the simple proof.

Lemma 1.1. *Let X be a compact subset of a compact set $K \subset \mathbb{R}^d$ whose covering radius $r(X)$ with respect to the Dubiner distance does not exceed θ/n, where $\theta \in$*

$(0, \pi/2)$ *and* $n \geq 1$*, i.e.*

$$r(X) = \max_{x \in K} dub_K(x, X) = \max_{x \in K} \min_{y \in X} dub_K(x, y) \leq \frac{\theta}{n} . \qquad (1.7)$$

Then, the following inequality holds

$$\|p\|_K \leq \frac{1}{\cos\theta} \|p\|_X , \quad \forall p \in \mathbb{P}_n^d . \qquad (1.8)$$

Proof. First, possibly normalizing and/or multiplying p by -1, we can assume that $\|p\|_K = p(\hat{x}) = 1$ for a suitable $\hat{x} \in K$. Since (1.7) holds for X, there exists $\hat{y} \in X$ such that

$$|\arccos(p(\hat{x})) - \arccos(p(\hat{y}))| = |\arccos(p(\hat{y}))| \leq \frac{\theta\, deg(p)}{n} \leq \theta < \frac{\pi}{2} .$$

Since the arccos function is monotonically decreasing and nonnegative, we have that $p(\hat{y}) \geq \cos(\theta) > 0$, and thus

$$\|p\|_K = 1 \leq \frac{p(\hat{y})}{\cos\theta} \leq \frac{1}{\cos\theta} \|p\|_X . \quad \square$$

Notice that X is not necessarily discrete, for example in [1] the notion is applied to suitable Lissajous curves of the cube. Now, fix $\varepsilon \in (0, 1)$. In view of the inequality

$$\frac{1}{\cos(\theta)} - 1 = \frac{1 - \cos(\theta)}{\cos(\theta)} \leq \frac{\theta^2}{2} \frac{1}{1 - \theta^2/2} = \frac{\theta^2}{2 - \theta^2} , \qquad (1.9)$$

valid for $\theta < \sqrt{2} < \pi/2$, let $\theta(\varepsilon)$ be the angle such that $\theta^2/(2 - \theta^2) = \varepsilon$. If we are able to construct a mesh $X_n(\varepsilon)$ with covering radius

$$r(X_n(\varepsilon)) \leq \frac{\theta(\varepsilon)}{n} , \quad \theta(\varepsilon) = \sqrt{\frac{2\varepsilon}{1 + \varepsilon}} \sim \sqrt{2\varepsilon} , \quad \varepsilon \to 0^+ , \qquad (1.10)$$

we get a $(1 - \varepsilon)$-approximation to the global minimum of any polynomial in $\mathbb{P}_n^d$, in view of (1.2)-(1.3).

In the next Section, we apply such a construction to planar starlike polygons.

2. Chebyshev-Dubiner norming webs

We focus now on 2-dimensional instances, namely on *planar starlike polygons* with respect to an internal center. A polygon K is starlike with respect to a point $c \in int(K)$, that we may term a star center of the polygon, if for ever $x \in K$ the closed segment $[c, x]$ connecting c and x is in contained in K (a convex polygon as any convex body is obviously starlike with respect to any of its points).

Proposition 2.1. *Let* $K \subset \mathbb{R}^2$ *be a simple polygon, starlike with respect to a center point* $c \in int(K)$*, with (clock- or counterclockwise) ordered vertices* $\{v_i\}$*,* $1 \leq i \leq \ell$*. Let* $\{t_j\}$ *be the nm Chebyshev points of* $(0, 1)$*, namely* $t_j = \frac{1}{2}\tau_j + \frac{1}{2}$*, where* $\tau_j = \cos((2j - 1)\pi/(2nm))$*,* $1 \leq j \leq nm$*, and*

$$\xi_{is} = \frac{v_{i+1} - v_i}{2} \tau_s + \frac{v_{i+1} + v_i}{2} , \quad 1 \leq s \leq nm , \qquad (2.1)$$

the nm Chebyshev points of the side (v_i, v_{i+1})*,* $1 \leq i \leq \ell$ *(where we have put* $v_{\ell+1} = v_1$*).*

Then for every $m > 2$ the point set (Chebyshev-Dubiner web)

$$Z_{nm} = \bigcup_{j=1}^{nm} (c + t_j(W_{nm} - c)) \ , \ \ W_{nm} = \bigcup_{i=1}^{\ell} \bigcup_{s=1}^{nm} \xi_{is} \subset \partial K \ , \tag{2.2}$$

satisfies the inequality

$$\|p\|_K \leq \frac{1}{\cos(\pi/m)} \|p\|_{Z_{nm}} \ , \ \ \forall p \in \mathbb{P}_n^d \ . \tag{2.3}$$

Proof. Denote by $K_c = K - c$ the translation of K by the center point $c \in int(K)$, by

$$\phi_{K_c}(u) = \inf\{\lambda > 0 : \ u \in \lambda K_c\} \ , u \in K_c$$

its Minkowski functional, $\phi_{K_c}(u) \in [0, 1]$, and by

$$x_{\partial K} = c + (x - c)/\phi_{K_c}(x - c)$$

the intersection point of the ray exiting from c and containing x with the boundary ∂K.

First, we prove the following estimate of the Dubiner distance (that is valid on any starlike body)

$$dub_K(x, y) \leq |\arccos(2\phi_{K_c}(x - c) - 1) - \arccos(2\phi_{K_c}(y - c) - 1)| + dub_{\partial K}(x_{\partial K}, y_{\partial K}) \ , \ \ \forall x, y \in K \ . \tag{2.4}$$

Consider the intersection point $y(x)$ of the ray exiting from c and contaning x with the Minkowski level set of y, say $S_y = \{w \in K : \ \phi_{K_c}(w - c) = \phi_{K_c}(y - c)\} = c + \phi_{K_c}(y - c)(\partial K - c)$. Then by the metric triangle inequality we can write

$$dub_K(x, y) \leq dub_K(x, y(x)) + dub_K(y(x), y) \ . \tag{2.5}$$

Concerning the first summand on the right-hand side, since $y(x) \in [c, x_{\partial K}]$

$$dub_K(x, y(x)) \leq dub_{[c, x_{\partial K}]}(x, y(x)) = dub_{[0,1]}(\phi_{K_c}(x - c), \phi_{K_c}(y(x) - c)) = dub_{[0,1]}(\phi_{K_c}(x - c), \phi_{K_c}(y - c)) \ , \tag{2.6}$$

where the inequality comes from the nonincreasing monotonicity of the Dubiner distance with respect to set inclusion and the first equality from its affine invariance (cf. (1.6)). Notice that $dub_{[0,1]}(s, t) = |\arccos(2s - 1) - \arccos(2t - 1)|$ since the Dubiner distance on $[-1, 1]$ is known to be the arccos distance in view of the Van der Corput-Schaake inequality, cf. e.g. [3].

We have then to estimate the second summand in the right-hand side of (2.5). Let $K(y) = c + \phi_{K_c}(y - c)(K - c)$ be the convex subset whose boundary is the level set S_y, which is clearly an affine transformation of K (being the composition of a translation with an homothetic transformation of $K - c$). Again by set monotonicity and affine invariance of the Dubiner distance

$$dub_K(y(x), y) \leq dub_{K(y)}(y(x), y) = dub_K(x_{\partial K}, y_{\partial K}) \leq dub_{\partial K}(x_{\partial K}, y_{\partial K}) \ , \tag{2.7}$$

which together with (2.5)-(2.6) gives (2.4). Focusing on starlike polygons, notice that if $x_{\partial K}, y_{\partial K}$ are in the same polygon side we have the further estimate

$$dub_{\partial K}(x_{\partial K}, y_{\partial K}) \leq dub_{[v_i, v_{i+1}]}(x_{\partial K}, y_{\partial K}) \ , \ \ x_{\partial K}, y_{\partial K} \in [v_i, v_{i+1}] \ .$$

Now, fix $x \in K$. Observing that the boundary point $x_{\partial K}$ belongs to a side $[v_i, v_{i+1}]$ for a certain i (at least one and at most two if it is a vertex), let ξ_{is} be the closest point to $x_{\partial K}$ in the Dubiner distance on $[v_i, v_{i+1}]$. Moreover, let t_j be the closest Chebyshev level to $\phi_{K_c}(x-c)$ in the Dubiner distance on $[0,1]$. Since nm (with $m>1$) Chebyshev points of any segment have covering radius $\pi/(2m)$ in the Dubiner distance of such a segment (by affine invariance, such a property being valid on $[-1,1]$), we finally get

$$dub_K(x, c+t_j(\xi_{is}-c)) \leq |\arccos(2\phi_{K_c}(x-c)-1) - \arccos(2t_j-1)|$$
$$+dub_{[v_i,v_{i+1}]}(x_{\partial K}, \xi_{is}) \leq \frac{\pi}{2m} + \frac{\pi}{2m} = \frac{\pi}{m}\,,$$

from which (2.3) follows for $m>2$ by Lemma 1. □

Remark. Observe that $Z_{nm} \subset int(K)$ (with $c \notin Z_{nm}$) and $card(Z_{nm}) = \ell n^2 m^2$. A similar construction can be made also with $nm+1$ Chebyshev-Lobatto points instead of nm Chebyshev points on sides and rays. In such a case there are mesh points on ∂K, the origin is a mesh point and the cardinality is slightly bigger, namely $\ell n^2 m^2 + \mathcal{O}(nm)$.

It is also worth observing that in the case of *centrally symmetric* starlike polygons, with an *even* number of sides, all the construction can be repeated exploiting the symmetry, by working with "diameters" instead of rays. In such a way the mesh cardinality is essentially halved and the points do not cluster at the center (see Figure 1 for an example with a regular octagon).

From Proposition 1, we obtain the following corollary on polynomial optimization.

Corollary 2.2. *Let $K \subset \mathbb{R}^2$ be a simple polygon, starlike with respect to a center point $c \in int(K)$. For any fixed $\varepsilon \in (0,1)$, K possesses a Chebyshev-Dubiner web $\{X_n(\varepsilon)\}$ such that (1.2) and (1.3) hold, with $card(X_n(\varepsilon)) = \mathcal{O}(n^2/\varepsilon)$.*

Proof. In view of (1.9) and (1.10), take $m = m(\varepsilon)$ in (2.3) such that

$$\frac{\pi}{m(\varepsilon)} \leq \sqrt{\frac{2\varepsilon}{1+\varepsilon}}\,,$$

i.e. $m(\varepsilon) = \lceil \frac{\sqrt{1+\varepsilon}}{\pi\sqrt{2\varepsilon}} \rceil = \mathcal{O}(1/\sqrt{\varepsilon})$. Then, $X_n(\varepsilon) = Z_{nm(\varepsilon)}$ fulfills (1.2) and (1.3), and has cardinality $\ell n^2 m^2(\varepsilon) = \mathcal{O}(n^2/\varepsilon)$, where ℓ is the number of polygon vertices. □

Observe that our approach is tailored to the geometry of polygons. On general bidimensional convex bodies, we can construct norming grids for polynomial optimization having $\mathcal{O}(n^4/\varepsilon^2)$ cardinality, with a constant of the $\mathcal{O}$-symbol independent of the body aspect ratio (diameter/width), by resorting to Markov polynomial inequality together with two cornerstones of convex geometry, Bieberbach and Leichtweiss inequalities. On the other hand, a sampling cardinality $\mathcal{O}(n^2/\varepsilon)$ (like that in Corollary 1) can be obtained on smooth bidimensional convex bodies, by resorting again to the notion of Dubiner distance and to another cornerstone of convex geometry, the Rolling Ball Theorem; cf. [11] for the detailed constructions.

In order to illustrate the present construction, in Figure 1 we show two examples, a 14-side nonconvex starlike polygon and a regular octagon, with the corresponding Chebyshev-Dubiner webs for degree $n=4$ and $\varepsilon = 0.05$. As discussed in Remark 1, on the octagon we have exploited the symmetry, working by diameters instead of rays.

In Table 1 we display the average range-relative errors (100 trials) of approximate minimization by Chebyshev-Dubiner webs of a random combination of the Chebyshev bivariate basis for degree $n = 4$ (15 basis polynomials, with variables scaled to the minimal rectangle containing the polygon), on the starlike polygons of Figure 1 for some values of the tolerance ε in the range $[10^{-3}, 10^{-1}]$. The reference values of the minimum and maximum have been computed on the intersection with a uniform grid of 10^8 points in the quoted minimal rectangle.

We see that the error behavior is consistent with Corollary 1 and quite satisfactory. In particular, the tolerance ε turns out to be an overestimate of the actual error by at least two orders of magnitude (a phenomenon that has been already observed in other numerical examples on polynomial optimization by polynomial meshes, cf. e.g. [10, 15]).

TABLE 1. Average range-relative errors (100 trials) for Chebyshev-Dubiner web minimization of a random combination of the Chebyshev bivariate basis for degree $n = 4$ on the starlike polygons of Fig. 1.

	ε	1.0e-1	5.0e-2	1.0e-2	5.0e-3	1.0e-3
14-sides	avg err	2.4e-4	1.5e-4	2.0e-5	3.9e-5	3.8e-6
octagon	avg err	1.0e-3	3.3e-4	7.9e-5	5.3e-5	9.7e-6

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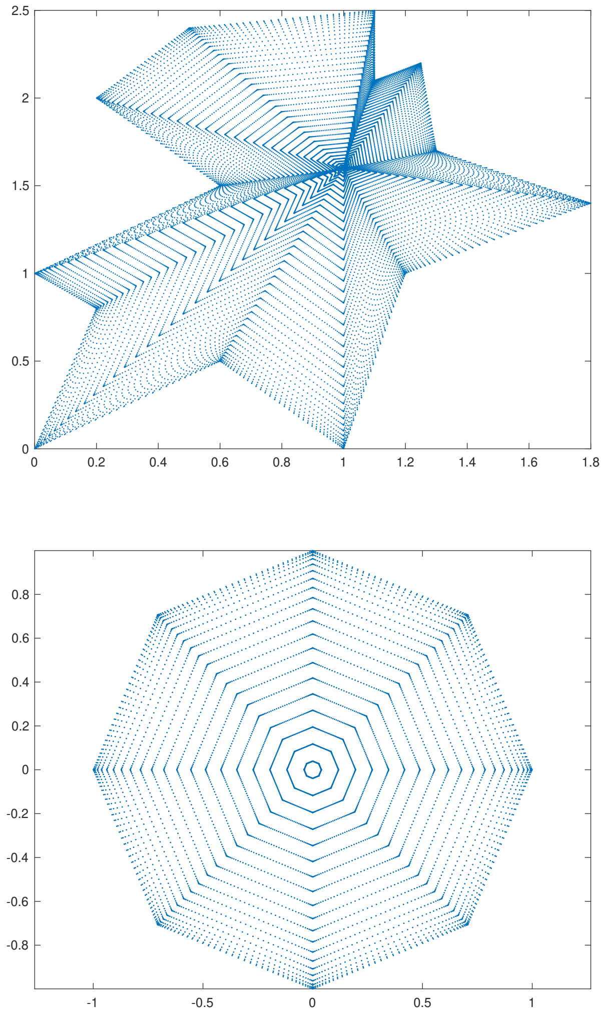

FIGURE 1. Chebyshev-Dubiner webs for polynomial minimization of degree $n = 4$ with a range-relative error less than 5%, on a 14-side starlike polygon (around 21000 points) and on a regular octagon (around 7700 points).

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