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Profinite groups with restricted centralizers of commutators

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A group G has restricted centralizers if for each g in G the centralizer $C_G(g)$ either is finite or has finite index in G. A theorem of Shalev states that a profinite group with restricted centralizers is abelian-by-finite. In the present article we handle profinite groups with restricted centralizers of word-values. We show that if w is a multilinear commutator word and G a profinite group with restricted centralizers of w-values, then the verbal subgroup $w(G)$ is abelian-by-finite.

Keywords: group words, profinite groups, centralizers, FC-groups

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Dedicated to Aner Shalev on the occasion of his 60*th birthday.*

1. Introduction

A group G is said to have restricted centralizers if for each g in G the centralizer $C_G(g)$ either is finite or has finite index in G. This notion was introduced by Shalev in [25] where he showed that a profinite group with restricted centralizers is finite-by-abelian-by-finite. Note that a finite-by-abelian profinite group is necessarily abelian-by-finite so Shalev's theorem essentially states that a profinite group with restricted centralizers is abelian-by-finite.

In the present article we handle profinite groups with restricted centralizers of word-values. Given a word w and a group G, we denote by G_w the set of all values of w in G and by $w(G)$ the subgroup generated by G_w. In the case where G is a profinite group $w(G)$ denotes the subgroup topologically generated by G_w.

Recall that multilinear commutator words are words which are obtained by nesting commutators, but using always different variables. Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials.

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let w be a multilinear commutator word and G a profinite group in which all centralizers of w-values are either finite or open. Then $w(G)$ is abelian-by-finite.*

From the above theorem we can deduce the following results.

Corollary 1.2. *Under the hypothesis of Theorem 1.1, the group G has an open subgroup T such that $w(T)$ is abelian. In particular G is soluble-by-finite.*

Corollary 1.3. *Let w be a multilinear commutator word and G a profinite group in which every nontrivial w-value has finite centralizer. Then either $w(G) = 1$ or G is finite.*

The proof of Theorem 1.1 is fairly complicated. We will now briefly describe some of the tools employed in the proof.

Recall that a group G is an FC-group if the centralizer $C_G(g)$ has finite index in G for each $g \in G$. Equivalently, G is an FC-group if each conjugacy class g^G is finite. A group G is a BFC-group if all conjugacy classes in G are finite and have bounded size. A famous theorem of B. H. Neumann says that the commutator subgroup of a BFC-group is finite [21]. Shalev used this to show that a profinite FC-group has finite commutator subgroup [25].

In Section 4 we generalize Shalev's result by showing that if w is a multilinear commutator word and G is a profinite group in which all w-values are FC-elements, then $w(G)$ has finite commutator subgroup. In fact, we establish a much stronger result involving the marginal subgroup introduced by P. Hall (see Section 4 for details). The results of Section 4 enable us to reduce Theorem 1.1 to the case where all w-values have finite order.

A famous result by Zelmanov says that periodic profinite groups are locally finite [34]. Recall that a group is said to locally have some property if all its finitely generated subgroups have that property. There is a conjecture stating that for any word w and any profinite group G in which all w-values have finite order, the verbal subgroup $w(G)$ is locally finite. The conjecture is known to be correct in a number of particular cases (see [28, 17, 3]). In Section 5 we obtain another result in this direction. Namely, let p be a prime, w a multilinear commutator word and G a profinite group in which all w-values have finite p-power order. We prove that the abstract subgroup generated by all w-values is locally finite.

The proof of the above result relies on the techniques created by Zelmanov in his solution of the Restricted Burnside Problem [35]. While the result falls short of proving that $w(G)$ is locally finite, it will be shown to be sufficient for the purposes of the present paper. Indeed, in Section 6 we prove that if a profinite group G satisfies the hypotheses of Theorem 1.1 and has all w-values of finite order, then $w(G)$ is locally finite. This is achieved by combining results of previous sections with the ones obtained in [17] and [3].

In Section 7 we finalize the proof of Theorem 1.1. At this stage without loss of generality we can assume that $w(G)$ is locally finite and at least one w-value has finite centralizer. With these assumptions, a whole range of tools (in particular, those using the classification of finite simple groups) become available. We appeal to Wilson's theorem on the structure of compact torsion groups which implies that in our situation $w(G)$ has a finite series of closed characteristic subgroups in which each factor either is a pro-p group for some prime p or is isomorphic (as a topological group) to a Cartesian product of finite simple groups.

Recall that the famous Ore's conjecture, stating that every element of a nonabelian finite simple group is a commutator, was proved in [20]. It follows that for each multilinear commutator word w every element of a nonabelian finite simple group is a w-value. If a group K is isomorphic to a Cartesian product of nonabelian finite simple groups and has restricted centralizers of w-values, then actually all centralizers of elements in K are either finite or of finite index and so, by Shalev's theorem [25], K is finite. We use this observation to conclude that under our assumptions the verbal subgroup is (locally soluble)-by-finite. Finally, an application of the results on FC-groups obtained in Section 4 completes the proof of Theorem 1.1.

The next section contains a collection of mostly well-known auxiliary lemmas which are used throughout the paper. In Section 3 we describe combinatorial techniques developed in [8, 3, 4] for handling multilinear commutator words. We also prove some new lemmas which are necessary for the purposes of the present article. Throughout the paper, unless explicitly stated otherwise, subgroups of profinite groups are assumed closed.

2. Auxiliary lemmas

Multilinear commutator words are words which are obtained by nesting commutators, but using always different variables. More formally, the word $w(x) = x$ in one variable is a multilinear commutator; if u and v are multilinear commutators involving different variables then the word $w = [u, v]$ is a multilinear commutator, and all multilinear commutators are obtained in this way.

An important family of multilinear commutator words is formed by so-called derived words δ_k, on 2^k variables, defined recursively by

$$\delta_0 = x_1, \qquad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$ is the k-th term of the derived series of G.

We recall the following well-known result (see for example [27, Lemma 4.1]).

Lemma 2.1. *Let G be a group and let w be a multilinear commutator word on n variables. Then each δ_n-value is a w-value.*

The following is Lemma 4.2 in [27]

Lemma 2.2. *Let w be a multilinear commutator word and G a soluble group in which all w-values have finite order. Then the verbal subgroup $w(G)$ is locally finite.*

If x is an element of a group G, we write x^G for the conjugacy class of x in G. More generally, if S is a subset of G, we write S^G for the set of conjugates of elements of S. On the other hand, if K is a subgroup of G, then K^G denotes the normal closure of K in G, that is, the subgroup generated by all conjugates of K in G, with the usual convention that if G is a topological group then K^G is a closed subgroup.

Recall that if G is a group, $a \in G$ and H is a subgroup of G, then $[H, a]$ denotes the subgroup of G generated by all commutators of the form $[h, a]$, where $h \in H$. It is well-known that $[H, a]$ is normalized by a and H.

We will denote by $\Delta(G)$ the set of FC-elements of G, i.e.

$$\Delta(G) = \{x \in G \mid |x^G| < \infty\}.$$

Obviously $\Delta(G)$ is a normal subgroup of G. Note that if G is a profinite group, $\Delta(G)$ needs not be closed.

Lemma 2.3. *Let G be a group. For every $x \in \Delta(G)$ the subgroup $[\Delta(G), x]^G$ is finite.*

Proof. Let $\Delta = \Delta(G)$. Note that Δ' is locally finite (see [24, Section 14.5]). The subgroup $[\Delta, x]$ is generated by finitely many commutators $[y, x]$ where $y \in \Delta$. Hence $[\Delta, x]$ is finite. Further, each commutator $[y, x]$ is an FC-element and so $C_G([\Delta, x])$ has finite index in G. Consequently, $[\Delta, x]^G$ is a product of finitely many conjugates of $[\Delta, x]$. The conjugates of $[\Delta, x]$ normalize each other so $[\Delta, x]^G$ is finite. □

Lemma 2.4. *Let G be a locally nilpotent group containing an element with finite centralizer. Suppose that G is residually finite. Then G is finite.*

Proof. Choose $x \in G$ such that $C_G(x)$ is finite. Let N be a normal subgroup of finite index such that $N \cap C_G(x) = 1$. Assume that $N \neq 1$ and let $1 \neq y \in N$. The subgroup $\langle x, y\rangle$ is nilpotent and so the center of $\langle x, y\rangle$ has nontrivial intersection with N. This is a contradiction since $N \cap C_G(x) = 1$. □

Lemma 1.6.1 in [16] states that if G is a finite group, N is a normal subgroup of G and x an element of G, then $|C_{G/N}(xN)| \leq |C_G(x)|$. We will need a version of this lemma for locally finite groups.

Lemma 2.5. *Let G be a locally finite group and x an element of G such that $C_G(x)$ is finite of order m. If N is a normal subgroup of G, then $|C_{G/N}(xN)| \leq m$.*

Proof. Arguing by contradiction, assume that $C_{G/N}(xN)$ contains $m+1$ pairwise distinct elements $b_1N, \ldots, b_{m+1}N$. Let $K = \langle x, b_1, \ldots, b_{m+1}\rangle$ and $N_0 = N \cap K$. Note that K is a finite group and $C_{K/N_0}(xN_0)$ contains the $m+1$ distinct elements $b_1N_0, \ldots, b_{m+1}N_0$. This contradicts Lemma 1.6.1 in [16]. □

Lemma 2.6. *Let d, r, s be positive integers. Let G be a soluble group of derived length d generated by a set X such that every element in X has finite order dividing r and has at most s conjugates in G. Then G has finite exponent bounded by a function of d, r, s.*

Proof. The proof is by induction on the derived length of G. If G is abelian then G has exponent dividing r. Note that G' is generated by all conjugates of the set $\{[y, z] | y, z \in X\}$. As $y, z \in X$ have at most s conjugates in G it follows that $[y, z]$ has at most s^2 conjugates. Note that the center of $\langle y, z\rangle$ coincides with $C_{\langle y,z\rangle}(y) \cap C_{\langle y,z\rangle}(z)$ so it has index at most s^2, thus the order of the derived subgroup of $\langle y, z\rangle$ is bounded by a function of s by Schur's theorem [24, 10.1.4]. By induction, the exponent of G' is finite and bounded by a function of d, r, s. As G/G' has exponent at most r, the result follows. □

Throughout the paper, we will use without explicit references the following result.

Lemma 2.7. *Let G be a finite-by-abelian profinite group. Then G is central-by-finite.*

Proof. Let T be a finite normal subgroup of G such that G/T is abelian and let N be an open normal subgroup of G such that $N \cap T = 1$. Then $N \cap G' = 1$ and so N is central in G. □

3. Combinatorics of commutators

We will need some machinery concerning combinatorics of commutators, so we now recall some notation from the paper [4].

Throughout this section, $w = w(x_1, \ldots, x_n)$ will be a fixed multilinear commutator word. If $A_1, \ldots, A_n$ are subsets of a group G, we write

$$\mathcal{X}_w(A_1, \ldots, A_n)$$

to denote the set of all w-values $w(a_1, \ldots, a_n)$ with $a_i \in A_i$. Moreover, we write $w(A_1, \ldots, A_n)$ for the subgroup $\langle \mathcal{X}_w(A_1, \ldots, A_n) \rangle$. Note that if every A_i is a normal subgroup of G, then $w(A_1, \ldots, A_n)$ is normal in G.

Let I be a subset of $\{1, \ldots, n\}$. Suppose that we have a family $A_{i_1}, \ldots, A_{i_s}$ of subsets of G with indices running over I and another family $B_{l_1}, \ldots, B_{l_t}$ of subsets with indices running over $\{1, \ldots, n\} \setminus I$. We write

$$w_I(A_i; B_l)$$

for $w(X_1, \ldots, X_n)$, where $X_k = A_k$ if $k \in I$, and $X_k = B_k$ otherwise. On the other hand, whenever $a_i \in A_i$ for $i \in I$ and $b_l \in B_l$ for $l \in \{1, \ldots, n\} \setminus I$, the symbol $w_I(a_i; b_l)$ stands for the element $w(x_1, \ldots, x_n)$, where $x_k = a_k$ if $k \in I$, and $x_k = b_k$ otherwise.

The following lemmas are Lemma 2.4, Lemma 2.5 and Lemma 4.1 in [4].

Lemma 3.1. *Let $w = w(x_1, \ldots, x_n)$ be a multilinear commutator word. Assume that H is a normal subgroup of a group G. Let $g_1, \ldots, g_n \in G$, $h \in H$ and fix $s \in \{1, \ldots, n\}$. Then there exist $y_j \in g_j^H$, for $j = 1, \ldots, n$, such that*

$$w_{\{s\}}(g_s h; g_l) = w(y_1, \ldots, y_n) w_{\{s\}}(h; g_l).$$

Lemma 3.2. *Let G be a group and let $w = w(x_1, \ldots, x_n)$ be a multilinear commutator word. Assume that $M, A_1, \ldots, A_n$ are normal subgroups of G such that for some elements $a_i \in A_i$, the equality*

$$w(a_1(A_1 \cap M), \ldots, a_n(A_n \cap M)) = 1$$

holds. Then for any subset I of $\{1, \ldots, n\}$ we have

$$w_I(A_i \cap M; a_l(A_l \cap M)) = 1.$$

Lemma 3.3. *Let G be a group and let $w = w(x_1, \dots, x_n)$ be a multilinear commutator word. Let $A_1, \dots, A_n$ and M be normal subgroups of G. Let I be a subset of $\{1, \dots, n\}$. Assume that*

$$w_J(A_i; A_l \cap M) = 1$$

for every proper subset J of I. Suppose we are given elements $g_i \in A_i$ for $i \in I$ and elements $h_k \in A_k \cap M$ for $k \in \{1, \dots, n\}$. Then we have

$$w_I(g_ih_i; h_l) = w_I(g_i; h_l).$$

Lemma 3.4. *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator word. Assume that T is a normal subgroup of a group G and $a_1, \dots, a_n$ are elements of G such that every element in $\mathcal{X}_w(a_1T, \dots, a_nT)$ has at most m conjugates in G. Then every element in T_w has at most m^{2^n} conjugates in G.*

Proof. We will first prove the following statement:

$(*)$ Assume that for some $g_1, \dots, g_n \in G$ every element in the set $\mathcal{X}_w(g_1T, \dots, g_nT)$ has at most t conjugates in G, and let $s \in \{1, \dots, n\}$. Then every element of the form $w_{\{s\}}(h_s; g_lh_l)$, where $h_1, \dots, h_n \in T$, has at most t^2 conjugates.

Choose an element $z = w_{\{s\}}(h_s; g_lh_l)$ as above. By Lemma 3.1

$$w_{\{s\}}(g_sh_s; g_lh_l) = w(y_1, \dots, y_n)w_{\{s\}}(h_s; g_lh_l),$$

where $y_j \in (g_jh_j)^T \subseteq g_jT$, for $j = 1, \dots, n$.

As both $w_{\{s\}}(g_sh_s; g_lh_l)$ and $w(y_1, \dots, y_n)$ belong to $\mathcal{X}_w(g_1T, \dots, g_nT)$, they have at most t conjugates in G. Thus

$$z = w(y_1, \dots, y_n)^{-1}w_{\{s\}}(g_sh_s; g_lh_l)$$

has at most t^2 conjugates in G. This proves $(*)$.

We will now prove that every element in

$$\mathcal{X}_w(T, \dots, T, a_iT, \dots, a_nT)$$

has at most m^{2^i} conjugates, by induction on i. The lemma will follow by taking $i = n$.

If $i = 1$ the statement is true by the hypotheses. So assume that $i \geq 2$ and every element in $\mathcal{X}_w(T, \dots, T, a_{i-1}T, \dots, a_nT)$ has at most $m^{2^{i-1}}$ conjugates. By applying $(*)$ with $g_1 = \cdots = g_{i-1} = 1$, $t = 2^{i-1}$ and $s = i$ we get the result. □

Lemma 3.5. *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator word. Assume that H is a normal subgroup of a group G. Then there exixst a positive integer t_n depending only on n such that for every $g_1, \dots, g_n \in G$, $h_1, \dots, h_n \in G$ the w-value $w(g_1h_1, \dots, g_nh_n)$ can be written in the form: $w(g_1h_1, \dots, g_nh_n) = ah$, where a is a product of at most t_n conjugates of elements in $\{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ and $h \in H_w$.*

Proof. The proof is by induction on the number n of variables appearing in w. If $n = 1$ then $w = x$ and the result is true.

If $n > 1$, then w is of the form $w = [u, v]$, where $u = u(x_1, \dots, x_r)$, $v = v(x_{r+1}, \dots, x_n)$ are multilinear commutator words. By induction, $u(g_1h_1, \dots, g_rh_r) =$

a_1h_1, $v(g_{r+1}h_{r+1,}\dots,g_nh_n) = a_2h_2$, where a_1 (resp. a_2) is a product of at most t_r (resp. t_{n-r}) conjugates of elements in $S = \{g_1^{\pm1},\dots,g_n^{\pm1}\}$, $h_1 \in H_u$ and $h_2 \in H_v$. By the standard commutator formulas we have that:

$$w(g_1h_1,\dots,g_nh_n) = [a_1h_1, a_2h_2] = [a_1, a_2h_2]^{h_1}[h_1, a_2h_2] =$$

$$[([a_1,h_2][a_1,a_2]^{h_2})^{h_1}[h_1,h_2][h_1,a_2]^{h_2} =$$

$$[a_1,h_2]^{h_1}[a_1,a_2]^{h_2h_1}[h_1,a_2]^{h_2}[h_1,h_2]^{[h_1,a_2]^{h_2}},$$

where $[a_1,h_2] = a_1^{-1}a_1^{h_2}$, $[a_1,a_2] = a_1^{-1}a_1^{a_2}$ are products of at most $2t_r$ conjugates of elements in S, $[h_1,a_2] = (a_2^{-1})^{h_1}a_2$ is a product of at most $2t_{n-r}$ conjugates of elements in S and $[h_1,h_2]^{[h_1,a_2]^{h_2}} \in H_w$. So the result follows taking t_n to be the maximum of the set $\{4t_r + 2t_{n-r} | r = 1,\dots,n-1\}$. ☐

4. Profinite groups in which w-values are FC-elements.

The famous theorem of B. H. Neumann says that the commutator subgroup of a BFC-group is finite [21]. This was recently extended in [5] as follows. Let w be a multilinear commutator word and G a group in which $|x^G| \le m$ for every w-value x. Then the derived subgroup of $w(G)$ is finite of order bounded by a function of m and w. The case where $w = [x,y]$ was handled in [6].

In the present article we require a profinite (non-quantitative) version of the above result. We show that if G is a profinite group in which all w-values are FC-elements, then the derived subgroup of $w(G)$ is finite. In fact we establish a stronger result, which uses the concept of marginal subgroup.

Let G be a group and $w = w(x_1,\dots,x_n)$ a word. The marginal subgroup $w^*(G)$ of G corresponding to the word w is defined as the set of all $x \in G$ such that

$$w(g_1,\dots,xg_i,\dots,g_n) = w(g_1,\dots,g_ix,\dots,g_n) = w(g_1,\dots,g_i,\dots,g_n)$$

for all $g_1,\dots,g_n \in G$ and $1 \le i \le n$. It is well known that $w^*(G)$ is a characteristic subgroup of G and that $[w^*(G), w(G)] = 1$.

Note that marginal subgroups in profinite groups are closed.

Let S be a subset of a group G. Define the w^*-residual of S of G to be the intersection of all normal subgroups N such that SN/N is contained in the marginal subgroup $w^*(G/N)$.

For multilinear commutator words the w^*-residual of a normal subgroup has the following characterization.

Lemma 4.1. *Let w be a multilinear commutator word, G a group and N a normal subgroup of G. Then the w^*-residual of N in G is the subgroup generated by the elements $w(g_1,\dots,g_n)$ where at least one of $g_1,\dots,g_n$ belongs to N.*

This follows from [29, Theorem 2.3]. For the reader's convenience, we will give here a proof in the spirit of Section 3.

Proof. Let $N_i = \langle w(g_1,\dots,g_n) | g_1,\dots,g_n \in G \text{ and } g_i \in N\rangle$ and let $R = N_1N_2\dots N_n$. Clearly, if M is a normal subgroup of G such that N/M is contained in $w^*(G/M)$

then $N_i \leq M$ for every $i = 1, \ldots, n$. Therefore R is contained in the w^*-residual of N

On the other hand, it follows from Lemma 3.3 that if $N_i = 1$, then

$$w(g_1, \ldots, g_i h, \ldots, g_n) = w(g_1, \ldots, g_i, \ldots, g_n)$$

for every $g_1, \ldots, g_n \in G$ and every h in N. Thus we have

$$w(g_1, \ldots, g_i h, \ldots, g_n)R = w(g_1, \ldots, g_i, \ldots, g_n)R$$

for every $i = 1, \ldots, n$, for every $g_1, \ldots, g_n \in G$ and every $h \in N$. So N/R is contained in $w^*(G/R)$. This implies the result. □

It follows form Lemma 4.1 that if w is a multilinear commutator word and N is a normal subgroup of a group G which does not contain nontrivial w-values, then N is contained $w^*(G)$ and, in particular, it centralizes $w(G)$. Indeed in this case, by Lemma 4.1, the w^*-residual of N in G is trivial.

A word w is concise if whenever G is a group such that the set G_w is finite, it follows that also $w(G)$ is finite. Conciseness of multilinear commutators was proved by J. C. R. Wilson in [31] (see also [8]).

Lemma 4.2. *Let w be multilinear commutator word, G a profinite group and N an open normal subgroup of G. Then the w^*-residual of N is open in $w(G)$.*

Proof. Let K be the w^*-residual of N. As N/K is contained in $w^*(G/K)$ and it has finite index in G/K, we deduce that the set of w-values of G/K is finite. It follows from the above result of Wilson that $w(G/K)$ is finite, as desired. □

As above, $\Delta(G)$ denotes the set of FC-elements of G. In what follows we will denote by H the topological closure of $\Delta(G)$ in a profinite group G.

The goal of this section is to prove the following theorem.

Theorem 4.3. *Let w be a multilinear commutator word, G a profinite group and T a normal subgroup of G such that every w-value of G contained in T is an FC-element. Then the w^*-residual of T has finite commutator subgroup.*

It is straightforward that the w^*-residual of G is precisely $w(G)$. Thus Theorem 4.3 has the following consequence.

Corollary 4.4. *Let w be a multilinear commutator word and G a profinite group in which every w-value is an FC-element. Then $w(G)$ has finite commutator subgroup.*

The key result of the remaining part of this section is the next proposition, from which Theorem 4.3 will be deduced.

Proposition 4.5. *Let $w = w(x_1, \ldots, x_n)$ be a multilinear commutator word, G a profinite group and H the topological closure of $\Delta(G)$ in G. Assume that $A_1, \ldots, A_n$ are normal subgroups of G with the property that*

$$\mathcal{X}_w(A_1, \ldots, A_n) \subseteq \Delta(G).$$

Then $[H, w(A_1, \ldots, A_n)]$ is finite.

The following lemma can be seen as a development related to Lemma 2.4 in [6] and Lemma 4.5 in [30].

Lemma 4.6. *Assume the hypotheses of Proposition 4.5, with $A_1, \dots, A_n$ being normal subgroups of G with the property that $\mathcal{X}_w(A_1, \dots, A_n) \subseteq \Delta(G)$. Let M be an open normal subgroup of G and $a_i \in A_i$ for $i = 1, \dots, n$. Then there exist elements $\tilde{a}_i \in a_i(A_i \cap M)$ and an open normal subgroup $\tilde{M}$ of M, such that the order of*

$$[H, w(\tilde{a}_1(A_1 \cap \tilde{M}), \dots, \tilde{a}_n(A_n \cap \tilde{M}))]^G$$

is finite.

Proof. Throughout the proof, whenever K is a subgroup of G we write K_i for $A_i \cap K$.

For each natural number j consider the set Δ_j of elements $g \in G$ such that $|G : C_G(g)| \leq j$. Note that the sets Δ_j are closed (see for instance [19, Lemma 5]). Consider the sets

$$C_j = \{(y_1, \dots, y_n) \mid y_i \in a_i M_i \text{ and } w(y_1, \dots, y_n) \in \Delta_j\}.$$

Each set C_j is closed, being the inverse image in $a_1M_1 \times \cdots \times a_nM_n$ of the closed set Δ_j under the continuous map $(g_1, \dots, g_n) \mapsto w(g_1, \dots, g_n)$. Moreover the union of the sets C_j is the whole $a_1M_1 \times \cdots \times a_nM_n$. By the Baire category theorem (cf. [15, p. 200]) at least one of the sets C_j has nonempty interior. Hence, there exist a natural number m, some elements $z_i \in a_iM_i$ and a normal open subgroup Z of G such that

$$w(z_1Z_1, \dots, z_nZ_n) \subseteq \Delta_m.$$

By replacing Z with $Z \cap M$, if necessary, we can assume that $Z \leq M$.

Choose in $\mathcal{X}_w(z_1Z_1, \dots, z_nZ_n)$ an element $a = w(\tilde{a}_1, \dots, \tilde{a}_n)$ such that the number of conjugates of a in H is maximal among the elements of $\mathcal{X}_w(z_1Z_1, \dots, z_nZ_n)$, that is, $|a^H| \geq |g^H|$ for any $g \in \mathcal{X}_w(z_1Z_1, \dots, z_nZ_n)$.

Since $\Delta(G)$ is dense in H, we can choose a right transversal $b_1, \dots, b_r$ of $C_H(a)$ in H consisting of FC-elements. Thus $a^H = \{a^{b_i} | i = 1, \dots, r\}$, where $a^{b_i} \neq a^{b_j}$ if $i \neq j$. Let $\tilde{M}$ be the intersection of Z and all G-conjugates of $C_G(b_1, \dots, b_r)$:

$$\tilde{M} = \left(\bigcap_{g \in G} C_G(b_1, \dots, b_r)^g \right) \cap Z$$

and note that $\tilde{M}$ is open in G.

Consider the element $w(\tilde{a}_1v_1, \dots, \tilde{a}_nv_n)$ where $v_i \in \tilde{M}_i$ for $i = 1, \dots, n$. As $w(\tilde{a}_1v_1, \dots, \tilde{a}_nv_n)\tilde{M}_i = a\tilde{M}_i$ in the quotient group $G/\tilde{M}_i$, we have

$$w(\tilde{a}_1v_1, \dots, \tilde{a}_nv_n) = va,$$

for some $v \in \tilde{M} \leq C_G(b_1, \dots, b_r)$. It follows that $(va)^{b_i} = va^{b_i}$ for each $i = 1, \dots, r$. Therefore the elements va^{b_i} form the conjugacy class $(va)^H$ because they are all different and their number is the allowed maximum. So, for an arbitrary element $h \in H$ there exists $b \in \{b_1, \dots, b_r\}$ such that $(va)^h = va^b$ and hence $v^ha^h = va^b$.

Therefore $[h,v]=v^{-h}v=a^ha^{-b}$ and so $[h,v]^a=a^{-1}a^ha^{-b}a=[a,h][b,a]\in[H,a]$. Thus $[H,v]^a\le[H,a]$ and

$$[H,va]=[H,a][H,v]^a\le[H,a].$$

Therefore $[H,w(\tilde a_1\tilde M,\dots,\tilde a_n\tilde M)]\le[H,a]$. Lemma 2.3 states that the abstract group $[\Delta(G),a]^G$ has finite order and thus the same holds for $[H,a]^G$. The result follows. □

For the reader's convenience, the most technical part of the proof of Proposition 4.5 is isolated in the following proposition.

Proposition 4.7. *Assume the hypotheses of Proposition 4.5, with $A_1,\dots,A_n$ being normal subgroups of G with the property that $\mathcal{X}_w(A_1,\dots,A_n)\subseteq\Delta(G)$. Let I be a nonempty subset of $\{1,\dots,n\}$ and assume that there exist a normal subgroup U of G of finite order and an open normal subgroup M of G such that*

$$[H,w_J(A_i;A_l\cap M)]\le U\quad\textit{for every } J\subsetneq I.$$

Then there exist a finite normal subgroup U_I of G containing U and an open normal subgroup M_I of G contained in M such that

$$[H,w_I(A_i;A_l\cap M_I)]\le U_I.$$

Proof. For each $i=1,\dots,n$ consider a right transversal C_i of $A_i\cap M$ in A_i, and let Ω be the set of n-tuples $\underline{c}=(c_1,\dots,c_n)$ where $c_r\in C_r$ if $r\in I$ and $c_r=1$ otherwise. Note that the set Ω is finite, since C_r is finite for every r. For any n-tuple $\underline{c}=(c_1,\dots,c_n)\in\Omega$, by Lemma 4.6, there exist elements $d_i\in c_i(A_i\cap M)$ and an open normal subgroup $M_{\underline{c}}$ of G such that the order of

$$[H,w(d_1(A_1\cap M_{\underline{c}}),\dots,d_n(A_n\cap M_{\underline{c}}))]^G$$

is finite. Let

$$\begin{aligned} M_I &= M\cap\Big(\bigcap_{\underline{c}\in\Omega}M_{\underline{c}}\Big),\\ U_I &= U\prod_{\underline{c}\in\Omega}[H,w(d_1(A_1\cap M_{\underline{c}}),\dots,d_n(A_n\cap M_{\underline{c}}))]^G. \end{aligned}$$

As Ω is finite, it follows that M_I is open in G and U_I has finite order.

Let Z/U_I be the center of HU_I/U_I in the quotient group G/U_I and let $\bar G=G/Z$. We will use the bar notation to denote images of elements or subgroups in the quotient group $\bar G$.

Let us consider an arbitrary generator $w_I(k_i,h_l)$ of $w_I(A_i;A_l\cap M_I)$, where $k_i\in A_i$ and $h_l\in A_l\cap M_I$. Let $\underline{c}=(c_1,\dots,c_n)\in\Omega$ be the n-tuple such that

$$k_i\in c_i(A_i\cap M)$$

if $i\in I$ and $c_i=1$ otherwise. Let $d_1,\dots,d_n$ be the elements as above, corresponding to the n-tuple $\underline{c}$. Then, by definition of U_I,

$$[H,w(d_1(A_1\cap M_I),\dots,d_n(A_n\cap M_I))]\le U_I,$$

that is

$$\overline{w(d_1(A_1 \cap M_I), \dots, d_n(A_n \cap M_I))} = 1,$$

in the quotient group $\bar{G} = G/Z$. We deduce from Lemma 3.2 that

$$\overline{w_I(d_i(A_i \cap M_I); (A_l \cap M_I))} = 1. \tag{4.1}$$

Moreover, as $c_i(A_i \cap M) = d_i(A_i \cap M)$, we have that $k_i = d_i v_i$ for some $v_i \in A_i \cap M$. It also follows from our assumptions that

$$\overline{w_J(A_i; A_l \cap M)} = 1$$

for every proper subset J of I. Thus we can apply Lemma 3.3 and obtain that

$$w_I(\overline{k}_i; \overline{h}_l) = w_I(\overline{d}_i \overline{v}_i; \overline{h}_l) = w_I(\overline{d}_i; \overline{h}_l) = 1,$$

where in the last equality we have used (4.1). Since $w_I(k_i, h_l)$ was an arbitrary generator of $w_I(A_i; A_l \cap M_I)$, it follows that

$$\overline{w_I(A_i; A_l \cap M_I)} = 1,$$

that is

$$[H, w_I(A_i; A_l \cap M_I)] \leq U_I,$$

as desired. □

Proof of Proposition 4.5. Recall that $w = w(x_1, \dots, x_n)$ is a multilinear commutator word, G is a profinite group, H is the closure of $\Delta(G)$ and $A_1, \dots, A_n$ are normal subgroup of G with the property that

$$\mathcal{X}_w(A_1, \dots, A_n) \subseteq \Delta(G).$$

We want to prove that $[H, w(A_1, \dots, A_n)]$ is finite.

We will prove that for every $s = 0, \dots, n$ there exist a finite normal subgroup U_s of G and an open normal subgroup M_s of G such that whenever I is a subset of $\{1, \dots, n\}$ of size at most s we have

$$[H, w_I(A_i; A_l \cap M_s)] \leq U_s.$$

Once this is done, the proposition will follow taking $s = n$.

Assume that $s = 0$. We apply Lemma 4.6 with $M = G$ and $a_i = 1$ for every $i = 1, \dots, n$. Thus there exist $\tilde{a}_1, \dots, \tilde{a}_n \in G$ and an open normal subgroup M_0 of G, such that the order of

$$U_0 = [H, w(\tilde{a}_1(A_1 \cap M_0), \dots, \tilde{a}_n(A_n \cap M_0)]^G$$

is finite.

Let Z/U_0 be the center of HU_0/U_0 in the quotient group G/U_0 and let $\bar{G} = G/Z$. We have that

$$\overline{w(\tilde{a}_1(A_1 \cap M_0), \dots, \tilde{a}_n(A_n \cap M_0))} = 1,$$

so it follows from Lemma 3.2 that

$$\overline{w(A_1 \cap M_0, \ldots, A_n \cap M_0)} = 1,$$

that is, $[H, w(A_1 \cap M_0, \ldots, A_n \cap M_0)] \leq U_0$. This proves the proposition in the case where $s = 0$.

Now assume that $s \geq 1$. Choose $I \subseteq \{1, \ldots, n\}$ with $|I| = s$. By induction, the hypotheses of Proposition 4.7 are satisfied with $U = U_{s-1}$ and $M = M_{s-1}$, so there exist a finite normal subgroup U_I of G containing U_{s-1} and an open normal subgroup M_I of G contained in M_{s-1} such that

$$[H, w_I(A_i; A_l \cap M_I)] \leq U_I.$$

Let

$$M_s = \bigcap_{|I|=s} M_I, \quad U_s = \prod_{|I|=s} U_I,$$

where the intersection (resp. the product) ranges over all subsets I of $\{1, \ldots, n\}$ of size s.

As there is a finite number of choices for I, it follows that U_s (resp. M_s) has finite order (resp. finite index in G). Note that $M_s \leq M_{s-1}$ and $U_{s-1} \leq U_s$. Therefore

$$[H, w_I(A_i; A_l \cap M_s)] \leq U_s$$

for every $I \subseteq \{1, \ldots, n\}$ with $|I| \leq s$. This completes the induction and the proof of the proposition. ☐

Proof of Theorem 4.3. Let $w = w(x_1, \ldots, x_n)$ be a multilinear commutator word, G a profinite group and T a normal subgroup of G. For $i = 1, \ldots n$, let X_i be the set of w-values $w(g_1, ..., g_n)$ such that g_i belongs to T. Obviously $X_i \subseteq T$ and therefore $X_i \subseteq \Delta(G)$ for every i. It follows from Proposition 4.5 that $[H, \langle X_i \rangle]$ is finite for every i. By Lemma 4.1, the w^*-residual of T is the subgroup N generated by the set $X = X_1 \cup \cdots \cup X_n$. Thus $[H, N] = \prod_{i=1}^{n} [H, \langle X_i \rangle]$ is finite. Finally, note that $N \leq H$ and so $N' \leq [H, N]$ is also finite. ☐

Corollary 4.8. *Let w be a multilinear commutator word and let G be a profinite group with restricted centralizers of w-values. If G has a w-value of infinite order, then $w(G)$ is abelian-by-finite.*

Proof. Let x be a w-value of G of infinite order. As $C_G(x)$ is open, it contains an open normal subgroup C of G. Let K be the w^*-residual of C in G. Since all w-values contained in C have infinite centralizers, we apply Theorem 4.3 and conclude that K' is finite. Being finite-by-abelian, K is also abelian-by-finite. It follows from Lemma 4.2 that K has finite index in $w(G)$ and so $w(G)$ is abelian-by-finite. ☐

5. Pronilpotent groups with restricted centralizers of w-values

In the present section we use the techniques created by Zelmanov to deduce a theorem about pronilpotent groups with restricted centralizers of w-values (see

Theorem 5.7). A combination of this result with Corollary 4.8 yields a proof of Theorem 1.1 for pronilpotent groups.

For the reader's convenience we collect some definitions and facts on Lie algebras associated with groups (see [26] or [35] for further information). Let L be a Lie algebra over a field. We use the left normed notation; thus if $l_1, \dots, l_n$ are elements of L then

$$[l_1, \dots, l_n] = [\dots [[l_1, l_2], l_3], \dots, l_n].$$

An element $y \in L$ is called ad-nilpotent if $ad\, y$ is nilpotent, i.e. there exists a positive integer n such that $[x, {}_n y] = 0$ for all $x \in L$. If n is the least integer with the above property then we say that y is ad-nilpotent of index n. Let X be any subset of L. By a commutator in elements of X we mean any element of L that could be obtained from elements of X by repeated operation of commutation with an arbitrary system of brackets, including the elements of X. Here the elements of X are viewed as commutators of weight 1. Denote by F the free Lie algebra over the same field as L on countably many free generators $x_1, x_2, \dots$. Let $f = f(x_1, \dots, x_n)$ be a nonzero element of F. The algebra L is said to satisfy the identity $f \equiv 0$ if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in L$. In this case we say that L is PI. We are now in position to quote a theorem of Zelmanov [35, 36] which has numerous important applications to group theory. A detailed proof of this result recently appeared in [37].

Theorem 5.1. *Let L be a Lie algebra generated by finitely many elements $a_1, \dots, a_m$ such that all commutators in $a_1, \dots, a_m$ are ad-nilpotent. If L is PI, then it is nilpotent.*

Let G be a group. Recall that the lower central word $[x_1, \dots, x_k]$ is usually denoted by γ_k. The corresponding verbal subgroup $\gamma_k(G)$ is the familiar kth term of the lower central series of the group G. Given a prime p, a Lie algebra can be associated with the group G as follows. We denote by

$$D_i = D_i(G) = \prod_{jp^k \geq i} (\gamma_j(G))^{p^k}$$

the ith dimension subgroup of G in characteristic p (see for example [12, Chap. 8]). These subgroups form a central series of G known as the Zassenhaus-Jennings-Lazard series. Set $L(G) = \bigoplus D_i/D_{i+1}$. Then $L(G)$ can naturally be viewed as a Lie algebra over the field $\mathbb{F}_p$ with p elements. For an element $x \in D_i \setminus D_{i+1}$ we denote by $\tilde{x}$ the element $xD_{i+1} \in L(G)$.

Lemma 5.2 (Lazard, [18])**.** *For any $x \in G$ we have $(ad\, \tilde{x})^p = ad\, (\tilde{x^p})$.*

The next proposition follows from the proof of the main theorem in the paper of Wilson and Zelmanov [33].

Proposition 5.3. *Let G be a group satisfying a coset identity. Then $L(G)$ is PI.*

Let $L_p(G)$ be the subalgebra of $L(G)$ generated by D_1/D_2. Often, important information about the group G can be deduced from nilpotency of the Lie algebra $L_p(G)$.

Proposition 5.4. *[26, Corollary 2.14] Let G be a group generated by elements $a_1, a_2, \ldots, a_m$ such that every γ_k-value in $a_1, a_2, \ldots, a_m$ has finite order, for every k. Assume that $L_p(G)$ is nilpotent. Then the series $\{D_i\}$ becomes stationary after finitely many steps.*

Let P be a Sylow subgroup of a finite group G. An immediate corollary of the Focal Subgroup Theorem [9, Theorem 7.3.4] is that $G' \cap P$ is generated by commutators. A weaker version of this fact for multilinear commutator words was proved in [1, Theorem A].

Proposition 5.5. *Let G be a finite group and P a Sylow subgroup of G. If w is a multilinear commutator word, then $w(G) \cap P$ is generated by powers of w-values.*

Proposition 5.6. *Let p be a prime, w a multilinear commutator word and G a profinite group in which all w-values have finite p-power order. Let K be the abstract subgroup of G generated by all w-values. Then K is a locally finite p-group.*

Proof. It follows from Proposition 5.5 that $w(G)$ is a pro-p group. Indeed if Q is a Sylow q-subgroup of $w(G)$, then the image of Q in any finite continuous image of G is generated by powers of w-values, which are p-elements, hence $Q = 1$ unless $q = p$.

By Lemma 2.1 there exists an integer k such that each δ_k-value is a w-value. It is sufficient to prove that the abstract subgroup R generated by all δ_k-values is locally finite. Indeed, the abstract group G/R is a soluble group such that all w-values have finite order. Hence $w(G/R)$ is locally finite by Lemma 2.2.

Let X be the set of δ_k-values of G. Every finitely generated subgroup of R is contained in a subgroup generated by a finite subset of X. So we choose finitely many elements $a_1, \ldots, a_s$ in X and consider the subgroup H topologically generated by $a_1, \ldots, a_s$. It is sufficient to prove that H is finite.

Note that H is a pro-p group, since it is a subgroup of $w(G)$. For every positive integer t, consider the set

$$S_t = \{(h_1, \ldots, h_{2^k}) \mid h_i \in H \text{ and } \delta_k(h_1, \ldots, h_{2^k})^{p^t} = 1\}.$$

These sets are closed and their union is the whole Cartesian product of 2^k copies of H. By the Baire category theorem at least one of the sets S_t has nonempty interior. Hence, there exist a natural number m, some elements $y_i \in H$ and a normal open subgroup Z of H such that

$$\delta_k(y_1 Z, \ldots, y_{2^k} Z)^{p^m} = 1.$$

In particular H satisfies a coset identity.

Let $L = L_p(H)$ be the Lie algebra associated with the Zassenhaus-Jennings-Lazard series $\{D_i\}$. Then L is generated by $\tilde{a}_i = a_i D_2$ for $i = 1, \ldots, s$. Let b any Lie-commutator in $\tilde{a}_1, \ldots, \tilde{a}_s$ and let c be the group-commutator in $a_1, \ldots, a_s$ having the same system of brackets as b. Since X is commutator closed, c is a δ_k-value and so it has finite order. By Lemma 5.2 this implies that b is ad-nilpotent. As H satisfies a coset identity, it follows from Proposition 5.3 that L satisfies some nontrivial polynomial identity. By Theorem 5.1 we conclude that L is nilpotent. As every γ_k-value in $a_1, \ldots, a_s$ has finite order, Proposition 5.4 shows that the series $\{D_i\}$ has only finitely many nontrivial terms. Since H is a pro-p group, it follows that the intersection of all D_i's is trivial. Taking into account that each D_i has finite index in H, we deduce that H is finite. This proves that R is locally finite and the proposition follows. □

Theorem 5.7. *Let w be a multilinear commutator word and let G be a pronilpotent group with restricted centralizers of w-values in which every w-value has finite order. Then the derived subgroup of $w(G)$ is finite.*

Proof. First assume that G is a pro-p group. Let K be the abstract subgroup of G generated by all w-values. By Proposition 5.6 K is a locally finite p-group. If a w-value of G has finite centralizer, then K is finite by Lemma 2.4. Since K is dense in $w(G)$, we conclude that $w(G)$ is finite. Therefore we can assume that every w-value in G is an FC-element and so the result follows from Corollary 4.4.

When G is pronilpotent, it is the Cartesian product of its Sylow subgroups. Let $\mathcal{P}$ be the set of primes p such that $w(P) \neq 1$ where P is the Sylow p-subgroup of G. If $\mathcal{P}$ is infinite, then G has a w-value of infinite order, against our assumption. Thus $\mathcal{P}$ is finite. If P is a Sylow p-subgroup of G, then the derived subgroup of $w(P)$ is finite by what we proved above. Therefore the derived subgroup of $w(G) = \prod_{p\in\mathcal{P}} w(P)$ is finite, as desired. □

6. Local finiteness of $w(G)$

The goal of the present section is to show that if the hypotheses of Theorem 1.1 hold and all w-values have finite order, then $w(G)$ is locally finite. There is a long-standing conjecture stating that each torsion profinite group has finite exponent (cf. Hewitt and Ross [11]). The conjecture can be easily proved for soluble groups (cf. [23, Lemma 4.3.7]). In [3] this was extended as follows.

Proposition 6.1. *[3, Theorem 3] Let w be a multilinear commutator word and G a soluble-by-finite profinite group in which all w-values have finite order. Then $w(G)$ is locally finite and has finite exponent.*

We remark that the above result does not follow from Lemma 2.2 and its proof is significantly more complicated.

Given a word w and a subgroup P of a profinite group G, we denote by $W(P)$ the closed subgroup generated by all elements of P that are conjugate in G to elements of P_w:

$$W(P) = \langle {P_w}^G \cap P \rangle.$$

Let $\mathcal{Y}_w$ be the class of all profinite groups G in which all w-values have finite order and the subgroup $W(P)$ is periodic for any Sylow subgroup P of G.

The following theorem was implicitly established in [17]. We will now reproduce the proof.

Theorem 6.2. *Let w be a multilinear commutator word and let G be a profinite group in the class $\mathcal{Y}_w$. Then $w(G)$ is locally finite.*

Proof. Recall that finite groups of odd order are soluble by the Feit-Thompson theorem [7]. Combining this with [17, Theorem 1.5] (applied with $p = 2$), we deduce that G has a finite series of closed characteristic subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_s = 1 \tag{6.1}$$

in which each factor either is prosoluble or is isomorphic to a Cartesian product of nonabelian finite simple groups. There cannot be infinitely many nonisomorphic nonabelian finite simple groups in a factor of the second kind, since this would give a w-value of infinite order. Indeed, by a result of Jones [13], any infinite family of finite simple groups generates the variety of all groups; therefore, the orders of w-values cannot be bounded on such an infinite family. Thus, we can assume in addition that each nonprosoluble factor in (6.1) is isomorphic to a Cartesian product of isomorphic nonabelian finite simple groups. We use induction on s. If $s = 0$, then $G = 1$ and the result follows. Let $s \geq 1$. By induction, $w(G_1)$ is locally finite. Passing to the quotient $G/w(G_1)$, we can assume that G_1 is soluble. If G/G_1 is isomorphic to a Cartesian product of isomorphic nonabelian finite simple groups, then G/G_1 is locally finite and the result follows from [17, Lemma 5.6]. If G/G_1 is prosoluble, then so is G, and then by [17, Proposition 5.12] G has a series of finite length with pronilpotent quotients. In this case, $w(G)$ is locally finite by [17, Lemma 5.7], as required. □

Proposition 6.3. *Let w be a multilinear commutator word and let G be a profinite group with restricted centralizers of w-values. Assume that every w-value has finite order. Then $w(G)$ is locally finite.*

Proof. By Lemma 2.1 there exists an integer k such that each δ_k-value is a w-value. Set $u = \delta_{2k}$. Let us show that $G \in \mathcal{Y}_u$, that is,

$$U(P) = \langle P_u^G \cap P \rangle$$

is periodic for every Sylow subgroup of P of G.

Let P be a Sylow subgroup of G. It follows from Theorem 5.7 that $w(P)'$ is a finite p-group, so $w(P)$ is soluble. In view of Lemma 2.1 we have $P^{(k)} \leq w(P)$ and so P is soluble. By Proposition 6.1, $P^{(k)}$ is locally finite and has finite exponent. In particular $P^{(k)}$ is locally nilpotent.

If P is finite then also $U(P)$ is finite so we can assume that P is infinite.

If some element $x \in P_{\delta_k}$ has finite centralizer we get a contradiction, because on the one hand x^P is infinite, on the other hand x^P is contained in $P^{(k)}$, which is finite by Lemma 2.4. Thus we can assume that the centralizer of each element in P_{δ_k} is infinite. As G has restricted centralizers of w-values and every δ_k-value is also a w-value, it follows that each element in P_{δ_k} has centralizer of finite index in G. Consider the sets

$$C_j = \{(y_1, \dots, y_{2^k}) \mid y_i \in P \text{ and } |\delta_k(y_1, \dots, y_{2^k})^G| \leq j\}.$$

Note that each set C_j is closed. Moreover their union is the whole Cartesian product of 2^k copies of P. By the Baire category theorem at least one of the sets C_j has nonempty interior. Hence, there exist a natural number m, some elements $a_i \in P$ and an open normal subgroup T of P such that

$$\mathcal{X}_{\delta_k}(a_1T, \dots, a_{2^k}T) \subseteq C_m.$$

We deduce from Lemma 3.4 that there exists a positive integer m_1 such that each element in T_{δ_k} has at most m_1 conjugates. Let $T_0 = T \cap P^{(k)}$. As $P^{(k)}$ is topologically generated by P_{δ_k}, we can choose a right transversal $b_1, \dots, b_r$ of T_0 in $P^{(k)}$ consisting of finite products of elements in P_{δ_k}. Of course $b_1, \dots, b_r$ are FC-elements and thus there exists a positive integer m_2 such that each b_i has at most m_2 conjugates. Let $x \in P_u$. We have

$$x = \delta_k(c_1, \dots, c_{2^k}),$$

where $c_i \in P_{\delta_k}$ for $i = 1, \dots, 2^k$. Now each c_i is of the form $c_i = g_ih_i$ where $g_i \in \{b_1, \dots, b_r\}$ and $h_i \in T_0$.

It follows from Lemma 3.5 that $x = ah$ where a is the product of at most t_{2^k} conjugates of elements in $\{b_1^{\pm 1}, \dots, b_r^{\pm 1}\}$ and $h \in T_{\delta_k}$.

As each b_i has at most m_2 conjugates and h has at most m_1 conjugates it follows that x has at most m_3 conjugates for some positive integer m_3 which does not depend on x. So each $x \in P_u$ has order dividing e, where e is the exponent of $P^{(k)}$, and has at most m_3 conjugates.

Recall that $U(P) = \langle P_u^G \cap P \rangle$. It follows from Lemma 2.6 that $U(P)$ has finite exponent. This proves that $G \in \mathcal{Y}_u$.

We deduce from Theorem 6.2 that $G^{(2k)}$ is locally finite. Thus we can pass to the quotient group $G/G^{(2k)}$ and assume that $G^{(2k)} = 1$. Now the result follows from Proposition 6.1. □

7. Proof of Theorem 1.1

We recall that the Hirsch-Plotkin radical of an (abstract) group is defined as the maximal normal locally nilpotent subgroup. In a profinite group the Hirsch-Plotkin radical need not be closed. However, in the particular case where the profinite group is locally finite, the Hirsch-Plotkin radical is closed. Indeed the closure of an abstract locally nilpotent subgroup is pronilpotent in any profinite group, and so it is locally nilpotent if the group is locally finite.

An important result about profinite torsion groups is the following theorem due to J. S. Wilson.

Theorem 7.1. *[32, Theorem 1] Let G be a compact Hausdorff torsion group. Then G has a finite series*

$$1 = G_0 \leq G_1 \leq \cdots \leq G_s \leq G_{s+1} = G$$

of closed characteristic subgroups, in which each factor G_{i+1}/G_i *either is a pro-p group for some prime p or is isomorphic (as a topological group) to a Cartesian product of finite simple groups.*

In particular, a profinite locally soluble torsion group has a finite series of characteristic subgroups in which each factor is a pro-p group for some prime p.

Proof of Theorem 1.1. Recall that w is a multilinear commutator word and G a profinite group with restricted centralizers of w-values. We want to prove that $w(G)$ is abelian-by-finite.

If G has a w-value of infinite order, then by Corollary 4.8 the subgroup $w(G)$ is abelian-by-finite. So we can assume that every w-value has finite order. It follows from Proposition 6.3 that $w(G)$ is locally finite.

By Theorem 7.1, $w(G)$ has a finite series of characteristic subgroups

$$1 = A_0 \leq A_1 \leq \cdots \leq A_s \leq A_{s+1} = w(G)$$

in which each factor either is a pro-p group for some prime p or is isomorphic to a Cartesian product of finite simple groups. Let A/B be a factor in the series which is isomorphic to a Cartesian product of finite simple groups. Recall that the famous Ore's conjecture, stating that every element of a nonabelian finite simple group is a commutator, was proved in [20]. It follows that every element of a nonabelian finite simple group is a w-value, therefore every element in A/B is a w-value. We deduce from Lemma 2.5 that A/B is a profinite group with restricted centralizers. By Shalev's result [25], A/B is abelian-by-finite and therefore finite.

Since all non-pronilpotent factors in the above series are finite, we derive that $w(G)$ is prosoluble-by-finite. Moreover $w(G)$ has an open characteristic subgroup K, which in turn has a finite characteristic series

$$1 = F_0 \leq F_1 \leq F_2 \leq \cdots \leq F_r \leq F_{r+1} = K$$

where F_{i+1}/F_i is the Hirsch-Plotkin radical of K/F_i, for every i.

Alternatively, the existence of such a subgroup K could be shown using theorems of Hartley [10] and Dade [2].

Let j be the maximal index such that all w-values contained in F_j are FC-elements. If $j = r + 1$, then by Corollary 4.4 we conclude that $w(G)$ is finite-by-abelian, hence abelian-by-finite.

So assume now that $j \leq r$. Then there exists a w-value whose centralizer in G is finite. As $w(G)$ is locally finite, Lemma 2.5 guarantees that F_{j+1}/F_j has an element with finite centralizer. Thus F_{j+1}/F_j satisfies the hypothesis of Lemma 2.4, hence it is finite. Since F_{j+1}/F_j is the Hirsch-Plotkin radical of K/F_j, it contains its centralizer in K/F_j. Taking into account that F_{j+1}/F_j is finite, we conclude that its centralizer in K has finite index. Therefore F_{j+1} has finite index in K. We deduce that F_j has finite index in $w(G)$.

Let T be the w^*-residual of F_j. Since every w-value in F_j is an FC-element, we can apply Theorem 4.3 and we obtain that T' is finite. Hence, T is abelian-by-finite. Note that F_j/T is contained in $w^*(G/T)$, hence it centralizes $w(G/T)$. By Lemma 2.5 the verbal subgroup $w(G/T)$ has an element with finite centralizer, so we deduce that F_j/T is finite. Thus T is open in $w(G)$ and we conclude that $w(G)$ is abelian-by-finite, as desired. □

In the sequel, we will use the fact that an abelian-by-finite group contains a characteristic abelian subgroup of finite index (see [22, Ch. 12, Lemma 1.2] or [14, Lemma 21.1.4]).

Proof of Corollary 1.2. Recall that w is a multilinear commutator word and G a profinite group in which centralizers of w-values are either finite or open. If follows from Theorem 1.1 that $w(G)$ is abelian-by-finite. In particular $w(G)$ has an open characteristic abelian subgroup N. As $w(G)/N$ is finite, there exists an open normal subgroup T of G containing N, such that T/N intersects $w(G)/N$ trivially. Since $w(T) \leq T \cap w(G) \leq N$, we conclude that $w(T)$ is abelian, as desired. The solubility of T is immediate from Lemma 2.1. $\square$

Proof of Corollary 1.3. Recall that w is a multilinear commutator word and G a profinite group in which every w-value has finite centralizer. Assume that $w(G) \neq 1$. If follows from Theorem 1.1 that $w(G)$ is abelian-by-finite. In particular, $w(G)$ has an open characteristic abelian subgroup N. If N contains a nontrivial w-value, then N is finite, by assumption. Therefore we can assume that $N \cap G_w = 1$. It follows from the remark following Lemma 4.1 that N is contained in $w^*(G)$. Since the marginal subgroup centralizes $w(G)$, we deduce that N is finite. This proves that $w(G)$ is finite. Hence, $C_G(w(G))$ has finite index in G. We see that $C_G(w(G))$ is both finite and of finite index, which proves that G is finite. $\square$

As a final remark, we point out that in [25] Shalev actually proved that if G is a profinite group with restricted centralizers then $\Delta(G)$ has finite index in G and finite commutator subgroup. Our proof of Theorem 1.1 implies that if w is a multilinear commutator word and G a profinite group with restricted centralizers of w-values, then the closed subgroup generated by $G_w \cap \Delta(G)$ has finite index in $w(G)$ and finite commutator subgroup.

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References

1 C. Acciarri, G. A. Fernández-Alcober, P. Shumyatsky. A focal theorem for outer commutator words. *J. Group Theory* **15** (2012), 397–405

2 E. C. Dade. Carter subgroups and Fitting heights of finite solvable groups. *Illinois J. Math.* **13** (1969), 449–514.

3 E. Detomi, M. Morigi, P. Shumyatsky. On countable coverings of word values in profinite groups. *J. Pure Appl. Algebra* **219** (2015), 1020–1030.

4 E. Detomi, M. Morigi, P. Shumyatsky. On profinite groups with word values covered by nilpotent subgroups. *Israel J. Math.* **226** (2018), 993–1008.

5 E. Detomi, M. Morigi, P. Shumyatsky. BFC-theorems for higher commutator subgroups. Q. J. Math., to appear, arXiv:1803.04202.

6 G. Dierings, P. Shumyatsky. Groups with boundedly finite conjugacy classes of commutators, Q. J. Math. 69 (2018), no.3, 1047–1051.

7 W. Feit and J. G. Thompson. Solvability of groups of odd order. *Pacific J. Math.* **13** (1963), 773-1029.

8 G. A. Fernández-Alcober, M. Morigi. Outer commutator words are uniformly concise. *J. Lond. Math. Soc. (2)* **82** (2010), 581–595.

9 D. Gorenstein. *Finite Groups*, Chelsea Publishing Company, New York, 1980.

10 B. Hartley. A general Brauer-Fowler theorem and centralizers in locally finite groups. *Pacific J. Math.* **152** (1992), 101–117

11 E. Hewitt, K. A. Ross. *Abstract harmonic analysis.* Vol. II ,152 Springer-Verlag, New York-Berlin 1970.

12 B. Huppert, N. Blackburn. *Finite Groups II.* Springer Verlag, Berlin, 1982.

13 G. A. Jones. Varieties and simple groups. *J. Aust. Math. Soc.* **17** (1974), 163–173.

14 M. I. Kargapolov, Yu. I. Merzlyakov. *Fundamentals of the theory of groups.* Nauka, Moscow (1982); Engl. transl. of 2nd ed. Springer, New York (1979).

15 J. L. Kelley. *General Topology.* Van Nostrand, Toronto - New York - London, 1955.

16 E. I. Khukhro. *Nilpotent groups and their automorphisms.* De Gruyter Expositions in Mathematics, 8. Walter de Gruyter & Co., Berlin, 1993

17 E. I. Khukhro, P. Shumyatsky. Words and pronilpotent subgroups in profinite groups. *J. Aust. Math. Soc.* **97** (2014), 343–364.

18 M. Lazard. Sur les groupes nilpotents et les anneaux de Lie, *Ann. Sci. École Norm. Supr.* **71** (1954), 101–190.

19 L. Lévai, L. Pyber. Profinite groups with many commuting pairs or involutions. *Arch. Math. (Basel)* **75** (2000), 1–7.

20 M. W. Liebeck, E. A. O'Brien, A. Shalev, P. H. Tiep. The Ore conjecture. *J. Eur. Math. Soc.* (4) **12** (2010), 939 –1008.

21 B. H. Neumann. Groups covered by permutable subsets. *J. London Math. Soc.* **29** (1954), 236–248.

22 D. S. Passman. *The Algebraic Theory of Group Rings.* Wiley-Interscience, New York (1977).

23 L. Ribes, P. Zalesskii. *Profinite groups.* A Series of Modern Surveys in Mathematics, 40. Springer-Verlag, Berlin (2000).

24 D. J. S. Robinson. *A course in the theory of groups.* Second edition. Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1996.

25 A. Shalev. Profinite groups with restricted centralizers. *Proc. Amer. Math. Soc.* **122** (1994), 1279–1284.

26 P. Shumyatsky. *Applications of Lie ring methods to group theory*, in " Nonassociative Algebra and Its Applications", (Eds R. Costa et al.), Marcel Dekker, New York, (2000), 373–395.

27 P. Shumyatsky. Verbal subgroups in residually finite groups. *Q. J. Math.* **51** (2000), 523–528.

28 P. Shumyatsky. On profinite groups in which commutators are Engel. *J. Aust. Math. Soc.* **70** (2001), 1–9.

29 R. F. Turner-Smith. Marginal subgroup properties for outer commutator words. *Proc. London Math. Soc.* (3) **14** (1964), 321–341.

30 J. Wiegold. Groups with boundedly finite classes of conjugate elements. *Proc. Roy. Soc. London Ser. A* **238** (1957), 389–401.

31 J. C. R. Wilson. On outer-commutator words. *Can. J. Math.* **26** (1974), 608–620.

32 J. S. Wilson. On the structure of compact torsion groups. *Monatsh. Math.* **96** (1983), 57–66.

33 J. S. Wilson, E. Zelmanov. Identities for Lie Algebras of Pro-p Groups. *J. Pure Appl. Algebra* **81** (1992), 103–109.

34 E. I. Zelmanov. On periodic compact groups. *Israel J. Math.* **77**, 83–95 (1992).

35 E. Zelmanov. *Nil Rings and Periodic Groups*, The Korean Math. Soc. Lecture Notes in Math., Seoul, 1992.

36 E. I. Zelmanov. *Lie ring methods in the theory of nilpotent groups*, in "Proc. Groups'93/St. Andrews, vol. 2", London Math. Soc. Lecture Note Ser. **212**, Cambridge Univ. Press, 1995, 567–585.

37 E. I. Zelmanov. Lie algebras and torsion groups with identity. *Journal of Combinatorial algebra* **1** (2017), 289–340.