A discrete mutualism model: analysis and exploration of a financial application

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Abstract

We perform a stability analysis on a discrete analogue of a known, continuous model of mutualism. We illustrate how the introduction of delays affects the asymptotic stability of the system's positive nontrivial equilibrium point. In the second part of the paper we explore the insights that the model can provide when it is used in relation to interacting financial markets. We also note the limitations of such an approach.

Keywords: discrete system, stability, delay, finance, stock markets, modelling, ecology, mutualism.

1. Introduction

In ecology, mutualism is an interaction between two or more species in which both or all species involved derive benefit. An example of this is the classic flower/bumblebee pollinator mutualism. Mathematical modelling of mutualistic relationships has received comparatively less historical attention than the interactions of predator-prey or competition. We present the following discrete model (with delays in the realisation of mutualistic benefits) representing facultative (beneficial but not necessary for species survival)

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mutualism.

$$N_1(k+1) = N_1(k)\exp\left(r_1 - \frac{b_1 N_1(k)}{1+\alpha_{12}N_2(k-\tau_2)} - d_1 N_1(k)\right), \tag{1a}$$

$$N_2(k+1) = N_2(k)\exp\left(r_2 - \frac{b_2 N_2(k)}{1+\alpha_{21}N_1(k-\tau_1)} - d_2 N_2(k)\right), \tag{1b}$$

where $N_1(k), N_2(k)$ represent abundancies of species 1 and 2 respectively at discrete time point k, for $k = 0, 1, 2, \dots$, delay terms τ_1, τ_2 are nonnegative integers and associated initial functions are

$$N_1(-\theta+1) = \phi_1(N_1(-\theta), N_2(-\theta)), \tag{1c}$$

$$N_2(-\theta+1) = \phi_2(N_1(-\theta), N_2(-\theta)), \tag{1d}$$

for $\theta = 1, 2, \dots, \tau_{\text{max}}$, where $\tau_{\text{max}} = \max\{\tau_1, \tau_2\}$, $N_1(-\tau_{\text{max}}) > 0$ and $N_2(-\tau_{\text{max}}) > 0$ are known. Constant parameters $r_1, r_2, b_1, b_2, d_1, d_2 > 0$ represent species 1 and 2's intrinsic growth, birth and death rates respectively, $\alpha_{12} \geq 1$ represents the strength of the mutualistic effect species 2 has upon species 1 and vice-versa for $\alpha_{21} \geq 1$. This model is a discrete analogue of a continuous model which is well understood and is described without delay below:

$$\frac{dN_1(t)}{dt} = \left(r_1 - \frac{b_1 N_1(t)}{1+\alpha_{12}N_2(t)} - d_1 N_1(t)\right) N_1(t), \tag{2a}$$

$$\frac{dN_2(t)}{dt} = \left(r_2 - \frac{b_2 N_2(t)}{1+\alpha_{21}N_1(t)} - d_2 N_2(t)\right) N_2(t). \tag{2b}$$

Model (2) represents an indirect mutualistic interaction in which the presence of a species decreases the density dependence in the other's per capita birth rate; it is discussed in works by Wolin and Lawlor [18] and Kot [8]. The model differs to other mutualism models in that it eliminates the so-called "orgy of mutual benefaction" as coined by May [12] whereby the populations of both species exhibit unbounded growth. In [15], Roberts and Joharjee introduced nonnegative delays in the interaction terms of (2) to represent a time-lag in the realisation of beneficial effects felt by the both species. It was proved analytically in [15] that (2) possesses an asymptotically stable positive nontrivial equilibrium (i.e. the point in the positive quadrant of $N_1 \times N_2$ phase space at which neither species is extinct and coexistence occurs) for $r_1, r_2, b_1, b_2, d_1, d_2 \in (0, \infty)$, $\alpha_{12}, \alpha_{21} \in [1, \infty)$ and that this result is independent to the size of delays. In the following sections, we analyse whether this is the case for a discrete analogue of (2), namely (1).

2. Construction of discrete model

Nonlinear, continuous equations such as those in (2) may often be difficult, if not impossible to solve analytically. Thus, in order to obtain solutions to such systems it is not uncommon to apply a numerical method to the component differential equations; thereby creating discrete approximations of solutions to the original continuous system. Schemes which are sometimes used are those based on a simple Euler's method approach or, if one wants to take advantage of ready-made routines in packages such as ODE 45 in MATLAB, a Runge-Kutta type method may be employed [15]. There is, however, another approach; it has been argued, in Li [10] for example that discrete time models governed by difference equations are more appropriate than continuous ones when the populations have nonoverlapping generations. As such, we choose a discretisation approach which is not based around simply applying a numerical method. In [5], piecewise constant arguments are introduced by Fan and Wang to discretise a predator-prey system. In this section we follow a parallel approach to create such a discrete analogue of the continuous mutualistic system (2). We outline the approach next.

2.1. Piecewise constant arguments approach

The piecewise constant arguments approach is a method by which differential equations are discretised into nonlinear difference equations. Akmet [1] describes a system consisting of these types of equation as "roughly" a hybrid of discrete and continuous time systems, they have also been said to occupy a position midway between differential equations and difference equations [5]. Let us assume that the average growth rates in (2) change at regular intervals of time. We can incorporate this aspect in (2a) and obtain the following differential equation with piecewise constant arguments;

$$\frac{1}{N_1(t)}\frac{dN_1(t)}{dt} = r_1 - \frac{b_1 N_1([t])}{1+\alpha_{12}N_2([t])} - d_1 N_1([t]), \tag{3}$$

where $[t]$ denotes the integer part of $t,\ t \in (0,\infty)$. To be a solution of (3), N_1 must possess the following properties:

1. N_1 is continuous on $[0,\infty)$.
2. The derivative $\frac{dN_1(t)}{dt}$ exists at each point $t \in [0,+\infty)$ with the possible exception of the points $t \in \{0,1,2,\dots\}$, where left-sided derivatives exist.
3. (3) is satisfied on each interval $[k,k+1)$ with $k = 0,1,2,\dots$

[5]. A solution N_2 must possess symmetrical properties. On any closed interval $[k,k+1), k = 0,1,2,\dots$, we may integrate (3) between $k+1$ and k, i.e.

$$\int_k^{k+1}\frac{dN_1(t)}{N_1(t)} = \int_k^{k+1}\left(r_1 - \frac{b_1 N_1([t])}{1+\alpha_{12}N_2([t])} - d_1 N_1([t])\right)dt,$$

and obtain

$$N_1(t) = N_1(k) \exp\left(\left[r_1 - \frac{b_1 N_1(k)}{1 + \alpha_{12} N_2(k)} - d_1 N_1(k)\right](t - k)\right),$$

for $k \leq t < k+1$. Letting $t \to k+1$, performing the same process with (2b) we obtain a discrete analogue of system (2) and introducing integer delays in the interaction terms $\alpha_{12}N_2(k)$ and $\alpha_{21}N_1(k)$ we arrive at the system of difference equations (1). We can also consider the nondelay system (i.e. when $\tau_1 = \tau_2 = 0$) as being a special case of (1). For computations in the next section, we set the initial data (1c), (1d) to be the solutions of the nondelay system with identical parameter values (i.e. we consider the nondelay system until enough time has passed for the delay to kick in) with initial values $N_1(1) = 1, N_2(1) = 1$. We note that the introduction of delays increases the order of the system (1) such that we are now considering a system of difference equations of order $\tau_1 + \tau_2 + 2$. Next, we shall state the equilibrium points of the system (1).

2.2. Equilibria of the discrete model

The discrete system (1) possesses identical equilibria to that of the continuous system (2) from which it is derived. These four equilibria are:

- $N^*_{(1)} = (N^*_1, N^*_2) = (0, 0)$
- $N^*_{(2)} = (N^*_1, N^*_2) = \left(\frac{r_1}{b_1+d_1}, 0\right)$
- $N^*_{(3)} = (N^*_1, N^*_2) = \left(0, \frac{r_2}{b_2+d_2}\right)$
- $N^*_{(4)} = (N^*_1, N^*_2) = \left(\frac{-B_1 + \sqrt{B_1^2 - 4A_1C_1}}{2A_1}, \frac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}\right)$ where,

$$\begin{aligned}
&A_1 = b_1d_2a_{21} + d_1d_2a_{21} + d_1r_2a_{12}a_{21}, \;\; A_2 = b_2d_1a_{12} + d_1d_2a_{12} + d_2r_1a_{12}a_{21},\\
&B_1 = b_1b_2 + b_1d_2 + d_1b_2 + d_1d_2 + d_1a_{12}r_2 - r_1d_2a_{21} - r_1r_2a_{12}a_{21},\\
&B_2 = b_1b_2 + b_1d_2 + d_1b_2 + d_1d_2 + d_2a_{21}r_1 - r_2d_1a_{12} - r_1r_2a_{12}a_{21},\\
&C_1 = -r_1b_2 - r_1d_2 - r_1r_2a_{12}, \;\; C_2 = -r_2b_1 - r_2d_1 - r_1r_2a_{21}.
\end{aligned}$$

Focusing upon the equilibrium within the positive quadrant, we shall determine circumstances under which $N^*_{(4)}$ is asymptotically stable. Since the system (1) is nonlinear, we must perform a stability analysis by linear approximation and thus we may only check for local rather than global asymptotic stability. Local asymptotic stability means that both solutions $N_1(k)$ and $N_2(k)$ converge to the point $N^*_{(4)}$ as $k \to \infty$ given that they begin sufficiently close to it.

3. Stability analysis of discrete model

It is known that for a system of nonlinear difference equations to have a local asymptotically stable equilibrium point, then $|\lambda| < 1$ must be true for all characteristic roots (λ) of the linearised system when determined around that equilibrium.

3.1. Finding (1)*'s characteristic equation around* $N^*_{(4)}$

Consider (1) in the following vector form,

$$\mathbf{N}(k+1) = \begin{pmatrix} N_1(k+1) \\ N_2(k+1) \end{pmatrix} = \begin{pmatrix} f_1(N_1, N_2) \\ f_2(N_1, N_2) \end{pmatrix}, \tag{4}$$

where $f_i(N_1, N_2) = N_i(k) \exp\left(r_i - \frac{b_i N_i(k)}{1+\alpha_{ij} N_j(k-\tau_j)} - d_j N_i(k)\right),\ r_i, b_i, d_i > 0, \alpha_{ij} \geq 1,\ i = 1, 2,\ j = 1, 2,\ i \neq j$. A linear approximation of the nonlinear system (1) around its equilibrium $N^*_{(4)}$ may be written in the form

$$\mathbf{N}(k+1) = \mathbf{J}_{N^*_{(4)}} \mathbf{N}(k) + \tilde{R}, \tag{5}$$

where,

$$\mathbf{J}_{N^*_{(4)}} = \begin{pmatrix} \frac{\partial f_1\left(N^*_{(4)}\right)}{\partial N_1} & \frac{\partial f_1\left(N^*_{(4)}\right)}{\partial N_2} \\ \frac{\partial f_2\left(N^*_{(4)}\right)}{\partial N_1} & \frac{\partial f_2\left(N^*_{(4)}\right)}{\partial N_2} \end{pmatrix} = \begin{pmatrix} 1 - r_1 & \frac{\alpha_{12} b_1 \left(N_1^*\right)^2 \lambda^{-\tau_2}}{\left(1+\alpha_{12} N_2^*\right)^2} \\ \frac{\alpha_{21} b_2 \left(N_2^*\right)^2 \lambda^{-\tau_1}}{\left(1+\alpha_{21} N_1^*\right)^2} & 1 - r_2 \end{pmatrix} \tag{6}$$

is the Jacobian matrix of (1) around the system's equilibrium $N^*_{(4)}$ and $\tilde{R}$ is a column vector of residual terms. (6) is important in assessing the qualitative local behaviour of solutions to the system (1) around $N^*_{(4)}$. Steps in determining the partial derivatives which compose the elements of (6) can be found in detail in [16]. The eigenvalues (λ) of (6) are the roots of the following characteristic equation of the system (1) around $N^*_{(4)}$;

$$\det\left(\mathbf{J}_{N^*_{(4)}} - \lambda \mathbf{I}\right) = 0, \tag{7}$$

where $\mathbf{I}$ is the 2×2 identity matrix. We may now expand (7), collect like powers of λ and multiply both sides by $\lambda^{\tau_1+\tau_2}$ to obtain the following characteristic polynomial of degree $\tau_1 + \tau_2 + 2$;

$$\lambda^{\tau+2} - Q\lambda^{\tau+1} + R\lambda^{\tau} - S = 0, \tag{8}$$

where

$$Q = 2 - (r_1 + r_2), \quad R = (1 - r_1)(1 - r_2), \quad S = \frac{\alpha_{12}\alpha_{21}b_1b_2\left(N_1^*N_2^*\right)^2}{\left(1 + \alpha_{12}N_2^*\right)^2\left(1 + \alpha_{21}N_1^*\right)^2}$$

and $\tau = \tau_1 + \tau_2$. Since $\tau + 2$ is the highest power of the polynomial (8), then it is clear to see that as the delays within system (1) grow large then it will become more and more difficult to determine the roots λ analytically. However, it is possible to determine when $|\lambda| < 1$ is true without solving (8). Jury conditions (the discrete time version of Routh-Hurwitz conditions) are a set of necessary and sufficient conditions (often in the form of inequalities involving coefficients within the polynomial) which, if held true, guarantee that all roots of an n^{th} order polynomial are less than one in absolute value. The conditions were developed by Jury [7], yet we shall take advantage of a formulation of the conditions found in Murray [14].

3.2. Local asymptotic stability of the positive nontrivial equilibrium

For the nondelay case of $\tau_1 = \tau_2 = 0$, (8) becomes the following quadratic characteristic equation;

$$\lambda^2 - Q\lambda + R - S = 0. \tag{9}$$

We require both roots of (9) to be less than one in absolute value for local asymptotic stability of the equilibrium $N^*_{(4)}$ to be confirmed. Since we have a quadratic, we know from Brauer and Chaves [3] that the Jury conditions required for local asymptotic stability of $N^*_{(4)}$ in this case can be written as

$$|-Q| < R - S + 1 < 2, \tag{10}$$

which is

$$|r_1 + r_2 - 2| < (1 - r_1)(1 - r_2) - \frac{\alpha_{12}\alpha_{21}b_1b_2\left(N_1^*N_2^*\right)^2}{\left(1 + \alpha_{12}N_2^*\right)^2\left(1 + \alpha_{21}N_1^*\right)^2} + 1 < 2. \tag{11}$$

Next, we may consider the $\tau = 1$ case. This is, of course when either $\tau_1 = 1, \tau_2 = 0$ or $\tau_1 = 0, \tau_2 = 1$. The characteristic equation of the now, third order system (1) around $N^*_{(4)}$ in this case is the following cubic polynomial,

$$\lambda^3 - Q\lambda^2 + R\lambda - S = 0. \tag{12}$$

Here, of course, we require all three of the roots of equation (12) to have magnitude less than 1 for local asymptotic stability of $N^*_{(4)}$ to be confirmed. Again from [2], the Jury conditions required for this to be true are

$$
\begin{gathered}
1-Q+R-S>0,\ \ 1+Q+R+S>0, \\
3-Q-R+3S>0 \ \text{ and } \ 1+QS-R-S^2>0.
\end{gathered} \tag{13}
$$

These conditions immediately seem to be different to the conditions required in the nondelay case. If this is true, then it means that changing the value of the delays within (1) can affect the stability of its nontrivial coexistence equilibrium; a property which is not present in the analogous continuous system of DDEs. Next, we consider the case of when $\tau = 2$, this yields the following characteristic polynomial of degree 4,

$$
P(\lambda) = \lambda^4 - Q\lambda^3 + R\lambda^2 - S = 0. \tag{14}
$$

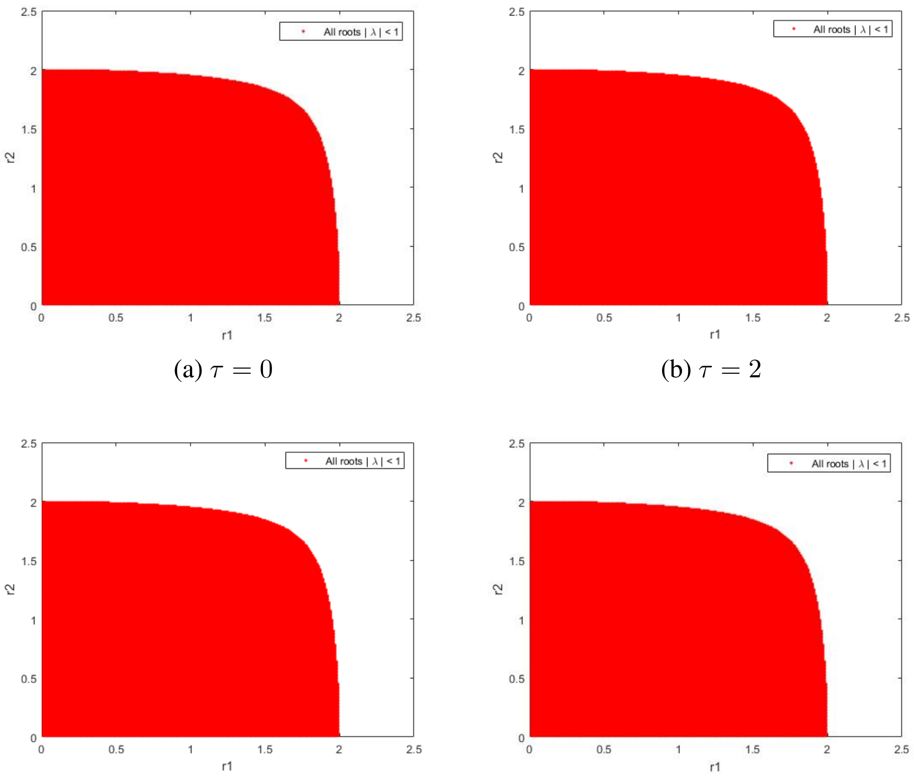
This case has five Jury conditions required for stability; they are

$$
\begin{gathered}
1-Q+R-S>0,\ 1+Q+R-S>0, \\
\left|(-S)^2\right| < 1, \quad \left|1-S^2\right| > |-QS|\,, \\
\left|(1-S^2)^2 - Q^2S^2\right| > \left|(1-S^2)(R+RS) - Q^2S\right|.
\end{gathered} \tag{15}
$$

Even though the Jury conditions are expressed differently in each case; (11), (13) and (15), we may not yet conclude that they are not equivalent. We are interested in determining how these conditions manifest themselves on the stability of the equilibrium, or more specifically, how changing the value of the delay within (1) affects this. Analytically, this proves difficult since it becomes a longer and more arduous task to develop such conditions as the degree of the characteristic polynomial (8) increases. However, by using computer programs, we can gather insight into what is happening to the asymptotic stability of $N^*_{(4)}$ when τ increases. We do this by considering the system's parameter space next.

3.3. Stability regions in the $r_1 \times r_2$ plane

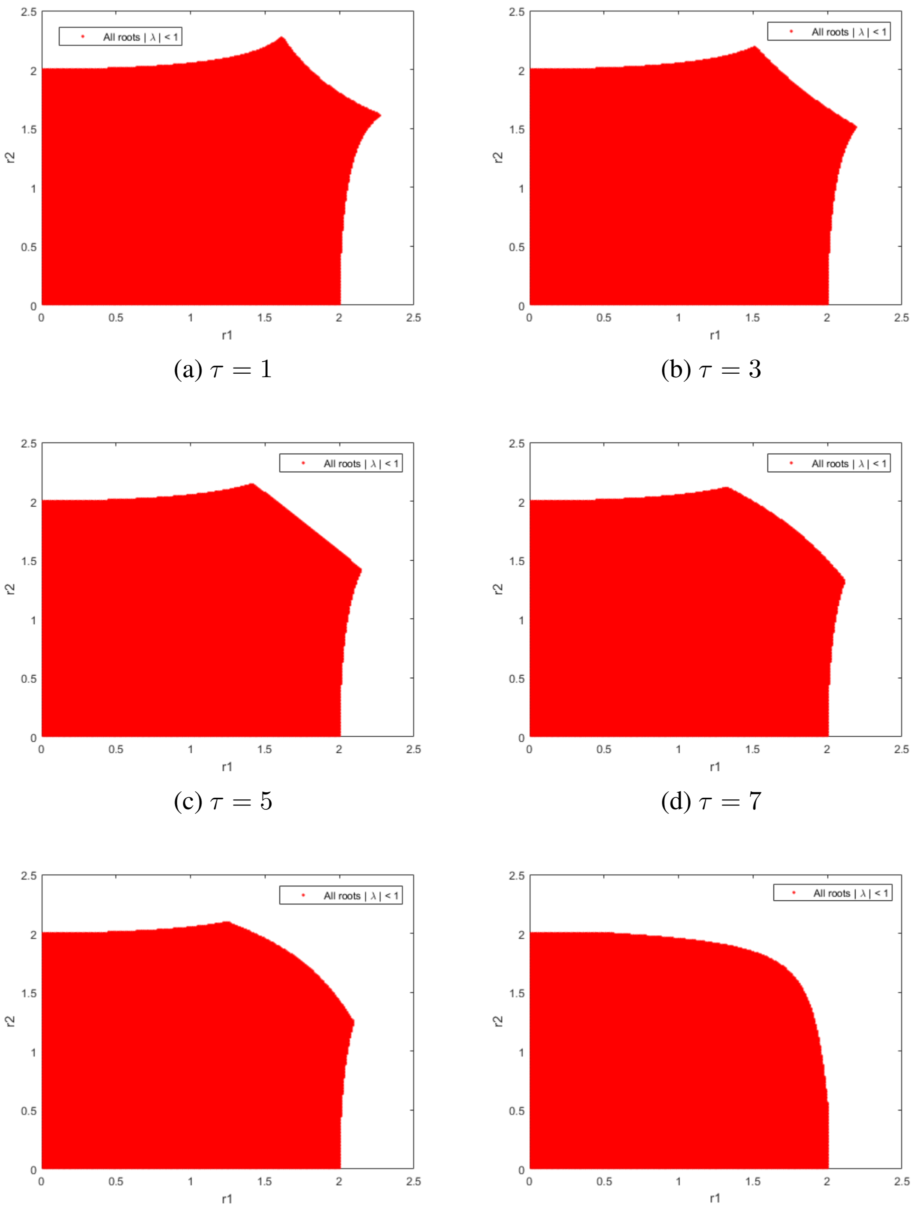
By increasing and decreasing each parameter by different percentages from a baseline set of values, then measuring the percentage change in the resultant solution vector, we determined that the parameters r_1 and r_2 are the most influential upon $N^*_{(4)}$'s local asymptotic stability property. This is known as performing a one-at-a-time sensitivity analysis. With this knowledge, we fixed the system's other six parameter constants to $b_1, b_2, d_1, d_2, \alpha_{12}, \alpha_{21} = 1$ and used MATLAB to develop images of the $r_1 \times r_2$ plane at different τ values. Points (r_1, r_2) within this plane which satisfy the local asymptotic stability conditions of $N^*_{(4)}$ make up a 2-dimensional, asymptotic stability region. Images are presented below, we separate even and odd images for visual purposes.



(a) $\tau = 0$ (b) $\tau = 2$

(c) $\tau = 4$ (d) $\tau = 10$

Figure 1: Images of $r_1 \times r_2$ planes of the system (1) with different even τ values and all other parameter values fixed. Area is red when all roots of the corresponding system's characteristic polynomial determined around $N^*_{(4)}$ have magnitude less than 1. (NB. images of intermediate values of $\tau \in (4, 10)$ are identical and thus have been omitted).



(a) $\tau = 1$ (b) $\tau = 3$

(c) $\tau = 5$ (d) $\tau = 7$

(e) $\tau = 9$ (f) $\tau = 101$

Figure 2: Images showing $r_1 \times r_2$ planes of the system (1) with different odd τ values and all other parameter values fixed. Area is red within an image when all roots of the corresponding system's characteristic polynomial determined around $N^*_{(4)}$ have magnitude less than 1.

In the images in the odd τ case (Figure 2), we seem to see a reduction in the asymptotic stability region as the τ value gets larger but in the even τ case (Figure 1) these regions do not change. With a suitably chosen mesh size, we present a line graph plotting the number of r_1, r_2 pairs which satisfy our stability criteria against an increasing τ value within the system (1). This allows us to visualise how the regions are changing in Figure 2 up to a large τ value without producing a large number of images. The resultant image is shown below. Again, trends in odd and even τ cases have been separated.

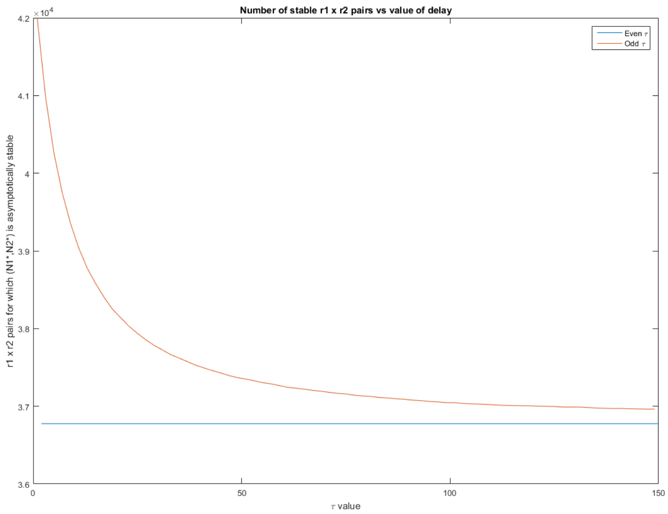


Figure 3: Line graph of the number of coupled r_1, r_2 values for which the equilibrium $N^*_{(4)}$ of the system (1) is asymptotically stable when all other parameters are fixed at 1 vs the value of the sum of integer delays $\tau \in [1, 150]$.

Figure 3 indicates that when even, the sum of the delay values ($\tau = \tau_1 + \tau_2$) within system (1) has no effect upon the subset of parameter space for which the equilibrium $N^*_{(4)}$ is asympotically stable. It also indicates that when odd, this region changes with the value of τ. Moreover, small odd τ values produce larger stability regions whereas large odd τ values produce regions more similar to those produced when τ is even, this is also seen in Figure 2f. We discuss this finding further at the end of the paper.

4. Fitting ecological models to financial data

In the paper by Lee, Lee and Oh [9], a model designed for the purposes of population dynamics is used in an attempt to better understand stock market fluctuations in Korea. The authors attempt to analyse the dynamic relationship between two stock markets; Korean Securities Dealers Automated Quotations (KOSDAQ) and Korean Stock Exchange (KSE, presently known as KOSPI), via a comparison with a discrete version of a Lotka-Volterra model. With financial data from after the latter market's emergence in 1997 following a Korean economic crisis, it is claimed that the two stock markets could be understood well by viewing them as species competing for the same investor's resources during this time period and by analysing through the use of ecological population models. The software EViews was used with data of the closing values of KSE and KOSDAQ through the period 1997-2001, together with a least squares method to estimate coefficients within their chosen ecological model. Once the coefficient's sign was determined, the type of the model which best fitted with their data was identified (different signs of the interaction coefficient within the Lotka-Voltera model has been considered to represent different ecological interactions). The paper introduces a novel approach to understanding financial markets and we have built upon this approach here. In doing so, we note some limitations in such an approach which we shall describe later.

4.1. Our approach

4.1.1. Models under consideration

We have chosen two more existing discrete time models to compare with our mutualism model (1) when fitting them to data. Firstly, the following model of species competition which is derived from the Lotka-Volterra formulation,

$$N_1(k+1) = N_1(k)\exp\Big(r_1 - a_{11}N_1(k) - a_{12}N_2(k-\tau_2)\Big), \tag{16a}$$

$$N_2(k+1) = N_2(k)\exp\Big(r_2 - a_{21}N_1(k-\tau_1) - a_{22}N_2(k)\Big), \tag{16b}$$

where constant parameters r_1, r_2 are the intrinsic growth rates of species 1 and 2 respectively. Here, $a_{11} = \frac{r_1}{K_1}, a_{12} = \frac{\alpha_{12}r_1}{K_1}, a_{21} = \frac{\alpha_{21}r_2}{K_2}$ and $a_{22} = \frac{r_2}{K_2}$, where α_{12}, α_{21} represent the strength of the competitive effect of species 2 upon species 1 and species 1 upon species 2 respectively. K_1, K_2 represent species 1 and 2's carrying capacities and τ_1, τ_2 are nonegative integers which represent a delay in competitive effect felt by each species respectively. Papers [4] by Chen and [11] by Li et al. study nondelay and delay versions of this form of competition model respectively. Secondly, we consider the following predator-prey model with Holling type-II functional response which is studied, in its nondelay form, by Fan and Wang [5]:

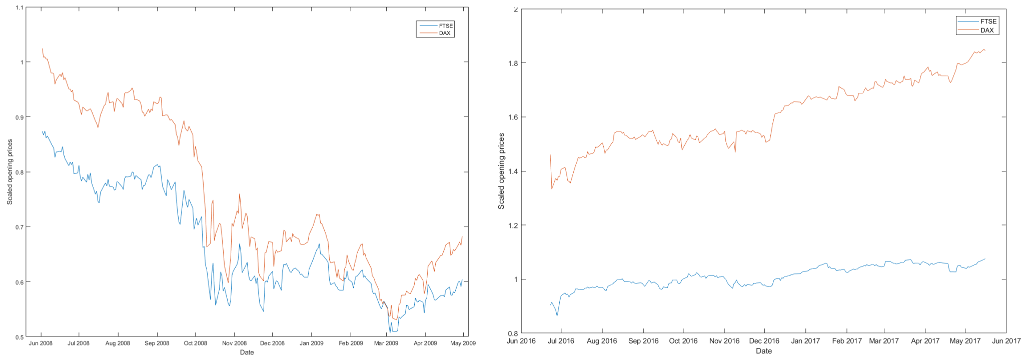
$$N_1(k+1) = N_1(k)\exp\left(a - bN_1(k) - \frac{cN_2(k-\tau_2)}{mN_2(k)+N_1(k)}\right) \text{ (Prey)}, \tag{17a}$$

$$N_2(k+1) = N_2(k)\exp\left(-d + \frac{fN_1(k-\tau_1)}{mN_2(k)+N_1(k)}\right) \text{ (Predator)}. \tag{17b}$$

In (17), parameter a represents the intrinsic growth rate for the prey, c represents the capture rate, d represents the predator's mortality rate and f represents the conversion rate of consumed prey into new predators. The parameter $b = \frac{a}{K}$ where K is the prey's carrying capacity and m is half the saturation constant (saturation is the idea that predators can only ever consume a certain amount of highly abundant prey meaning that the prey will not go extinct). We choose to introduce integer delay values (τ_1, τ_2) to the model at the main interaction terms; namely, in the capture rate term (cN_2) in (17a) and the conversion rate term (fN_1) in (17b). Initial functions for systems (16) and (17) are similar to those we impose for system (1) as stated in Section 2.

4.1.2. Data and time periods under consideration

In, [13] Mikdashi uses the words “mutualistic" and “competition" in the same sense as in ecological interactions. Of course, competition for resources in the form of trading is clear to see between stock markets, but the formation of alliances (i.e. mutualism) between them may seem a slightly less obvious feature. A key quote from “The Transformation of Stock Exchanges" by Tirez [17] (within [13]) which inspired our choice of stock markets is; “in the immediate future, Europe is expected to see a consolidation of exchanges around three major poles – Frankfurt, Paris Euronext and London - and further merger and cooperation agreements that will eliminate the smaller exchanges". So, the idea of mutualistic relationships being formed and predatory behaviour manifesting as a consequence seems to be prominent in stock exchange dynamics around Europe. As a result of this, we choose to use data from arguably the two major European stock exchanges which are relatively close geographically; London’s Financial Times Stock Exchange (FTSE) and Frankfurt’s Deutscher Aktienindex (DAX). Our investigation is concerned with whether any of the models we consider; (1), (16) or (17) may best suit such financial data during times of economic stress and thus be useful in describing a stock market interaction as either mutualistic, competitive or predatory. To this end, we consider two periods of recent history during which such stress has occurred; the global financial crisis circa 2008 and the UK’s EU membership referendum circa 2016. Obtained from [6], we consider the daily opening prices of FTSE and DAX stock exchanges on the dates 2$^{\text{nd}}$ June 2008 to the 30$^{\text{th}}$ April 2009 and 23$^{\text{rd}}$ June 2016 to the 16$^{\text{th}}$ May 2017 to encompass these respective events. We omit any dates which are not trading days for both indices, this gives us 234 trading days (data points in our time series) for the period during the financial crisis and 226 trading days for the period immediately after the EU referendum. Below, we present line graphs of our obtained data to a suitable scaling, dividing each value by the first data point in the FTSE time series in the year 2000 provides this scaling.



(a) 2^{nd} June 2008 to 30^{th} April 2009 (b) 23^{rd} June 2016 to 16^{th} May 2017.

Figure 4: Scaled opening values of the FTSE and DAX stock markets during two time periods of interest.

4.1.3. Parameter fitting

In search of a best fit between a set of data and a nonlinear model with variable parameters, [9] used EViews software to compute estimations of coefficients within their model. We take a different approach and use the optimisation solver add-in from Microsoft Excel; it uses a generalised reduced gradient method to perform its optimisation tasks. With N_1 and N_2 denoting the solution vectors for our choice of model and $\tilde{N}_1$ and $\tilde{N}_2$ denoting scaled data from FTSE and DAX respectively. We set the initial values to each of the models as the first two values of our collected data, i.e. $N_1(1) = \tilde{N}_1(1), N_2(1) = \tilde{N}_2(1)$ and residuals (R) are calculated as the sum of squares of the difference between data and model solutions at each discrete time point k for $k = 1, \dots, n$, i.e.

$$R = \sum_{k=1}^{n} \left[\left(\tilde{N}_1(k) - N_1(k) \right)^2 + \left(\tilde{N}_2(k) - N_2(k) \right)^2 \right], \tag{18}$$

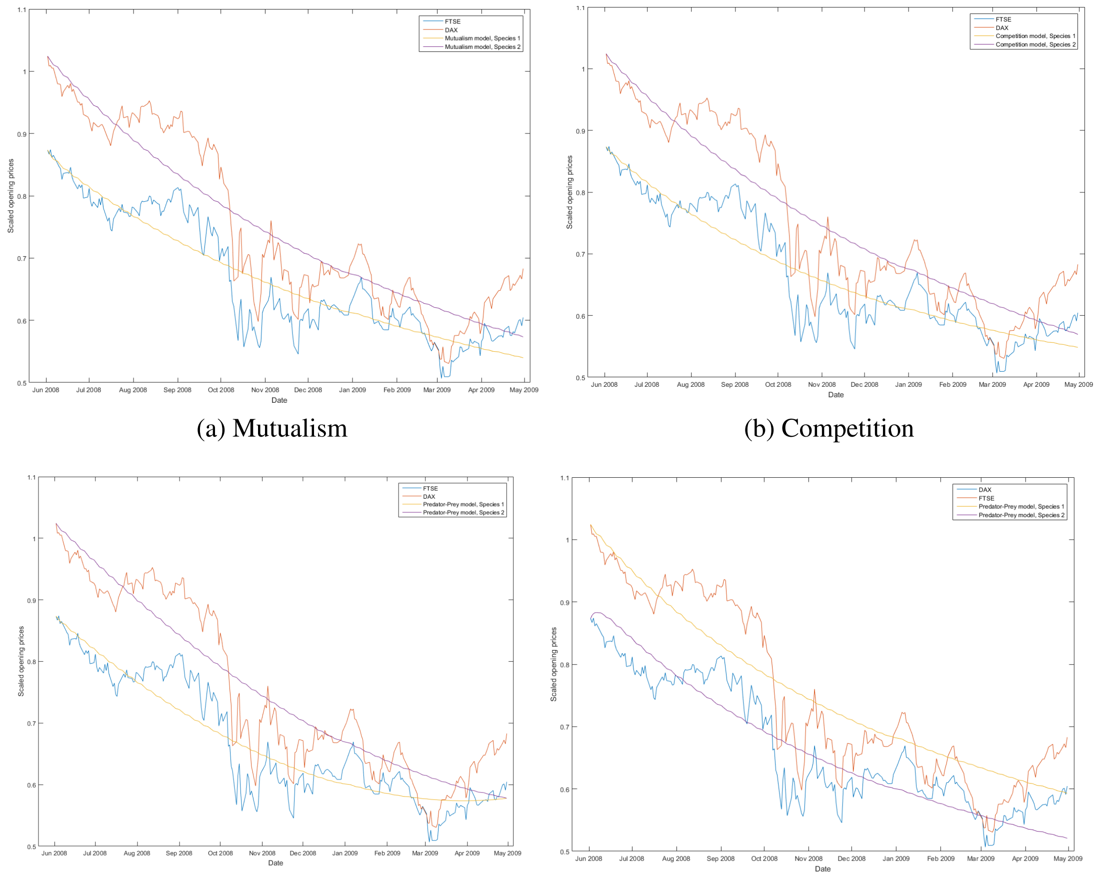
where n is the size of the particular time series we consider. We optimise parameter values (from a starting point of 1) for each model at both time periods and obtain minimised residual values R for each test. The smallest R value after each model's optimisation may be seen as the best fit for that time period. We note at this point that we have 10 parameters (including delays) in the system (1) meaning approximately 23 data points per parameter when fitting this model. Similarly, (16) and (17) contain 8 parameters each: this means that we have approximately 28 and 29 data points per parameter for 2008-09 and 2016-17 time periods respectively.

4.2. Findings

Below, we present images of the resultant solution trajectories of each model using their optimised parameter values alongside data from Figures 4a and 4b for reference. We also present summary tables containing the normalised residual values.

4.2.1. *Period of June 2008 - April 2009*

The following four images are the fitted models along with the data from Figure 4a.



(a) Mutualism

(b) Competition

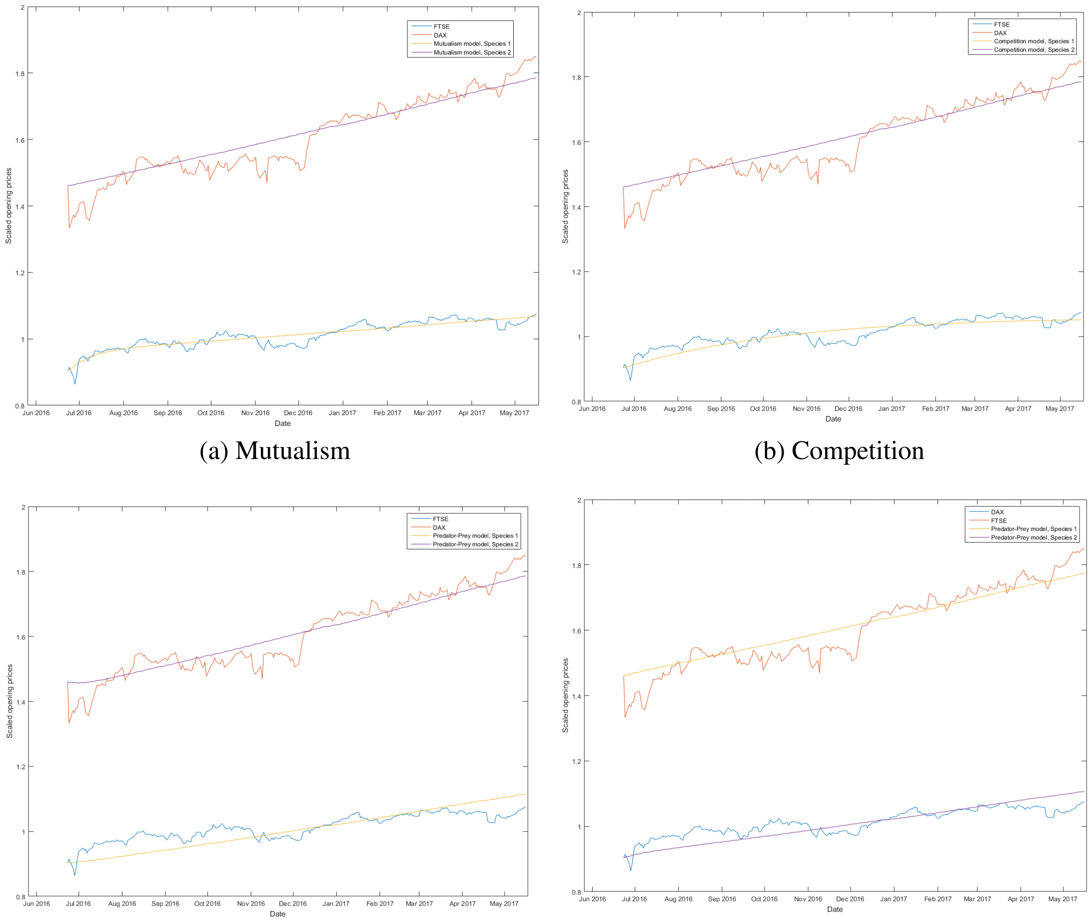
(c) Predator (FTSE) - Prey (DAX)

(d) Predator (DAX) - Prey (FTSE)

Figure 5: Images of FTSE and DAX daily opening prices from June 2008 to April 2009 and the fitted solutions to the mutualistic system (1), competition system (16) and predator-prey system (17) with optimised parameter values.

4.2.2. *Period of June 2016 - May 2017*

The four images of the fitted models along with the data from Figure 4b are presented below.



(a) Mutualism

(b) Competition

(c) Predator (FTSE) - Prey (DAX)

(d) Predator (DAX) - Prey (FTSE)

Figure 6: Images of FTSE and DAX daily opening prices from June 2016 to May 2017 and the fitted solutions to the mutualistic system (1), competition system (16) and predator-prey system (17) with optimised parameter values.

4.2.3. Summary of residual values

Below we present a table of normalised residual values obtained from the above optimal fits of each model. We require normalisation for comparative purposes between time series since the time periods contain a different number of data points. These values are normalised by dividing the minimised value R, defined by (18), by the number of data points in that time series.

	Normalised residual value	
Model	08-09 period	16-17 period
Mutualism	0.004410054226	0.002081775314
Competition	0.004417731453	0.002248030956
Predator-Prey (FTSE-DAX)	0.003944355827	0.002631759597
Predator-Prey (DAX-FTSE)	0.004772988598	0.00275144836

Table 1: Table of normalised minimised residual values for each model when fitted to both times series.

4.2.4. Can we forecast a trend?

We preface the following by saying that we are not setting out here to use our models as any kind of forecasting aid to be used for predictions, e.g. for trading. We see that all residual values in Table 1 are very small, yet the clear best fitting model to the 2008-2009 time period is (17) where FTSE is considered the predator and DAX the prey. In the following image, we extend time in Figure 5c for a further 6 months.

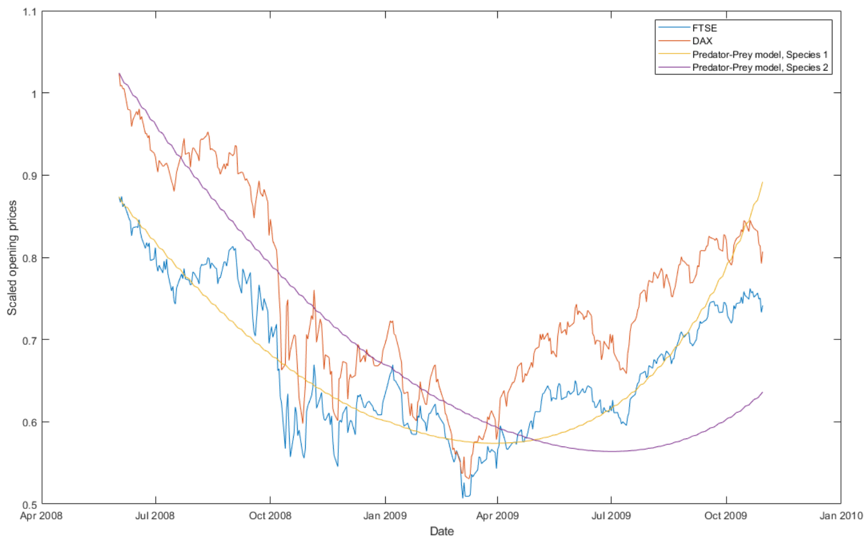


Figure 7: Image of FTSE and DAX daily opening prices from June 2008 to April 2010 and the solutions to the predator-prey system (17) with parameter values used for solution in Figure 5c.

We do the same for images in Figures 6a and 6b since the mutualism model (1) and competition model (16) were the two lower residuals for the 2016-2017 time period.

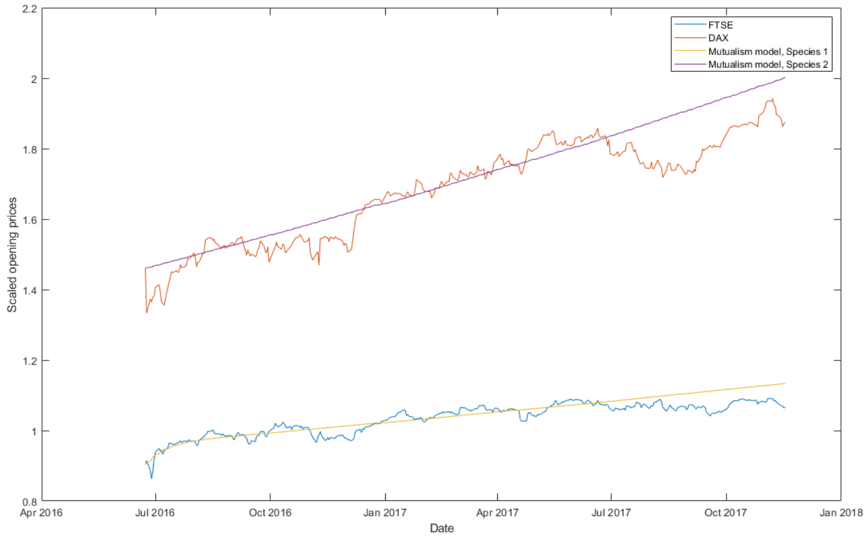


Figure 8: Image of FTSE and DAX daily opening prices from June 2016 to May 2018 and the solutions to the mutualistic system (1) with parameter values used for solution in Figure 6a.

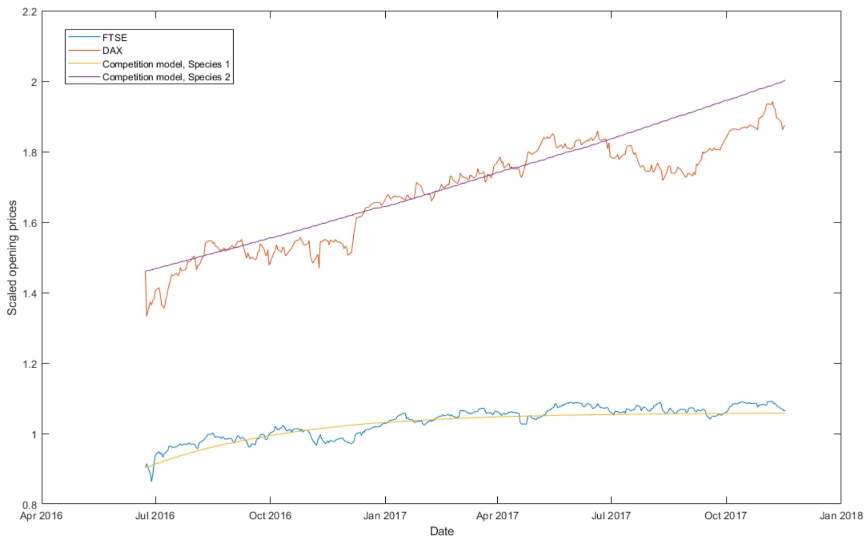


Figure 9: Image of FTSE and DAX daily opening prices from June 2016 to May 2018 and the solutions to the competition system (16) with parameter values used for solution in Figure 6b.

We can see, in Figure 7, that the predator-prey model does a reasonably good job of predicting the upward trend which occurs in FTSE and DAX throughout mid-2009. Also, the mutualism model and the competition model display some agreement with the trends in the stock market data after May 2018 as shown in Figures 8 and 9.

5. Conclusions

It can be seen in Figure 3 that the number of asymptotically stable r_1, r_2 pairs in the odd delay case appears to be converging to that of the even delay case which remains constant. This is an interesting finding with regards to the study of difference equations: it offers the proposition that when considering high order systems of such equations, certain qualitative behaviours can be produced which are very similar if not the same to those produced with the corresponding lower order systems. For this reason, it may simply not be necessary to perform difficult stability analyses on higher order systems since the lower order cases work just as well in this regard and are easier to work with; this knowledge has the potential to save time and effort for those working with such systems. We can see this exemplified in Figure 2f which is remarkably similar to that of Figure 1a. We see in Section 4 that overall, all three of our models give a good fit as trend lines to both time series with very low residual values. Further, we have shown that the models which produced the smallest of these residual values exhibit similar trends to that of the actual time series data when projected 6 months into the future. However, it is clear that dramatic conclusions, although enticing, may be naive to make here. One reason for this could be that our number of data points per parameter may be inadequate for the purposes of fitting. Another may be that there is a great deal of uncertainty from a projection point of view: we acknowledge that the differences between R values in Table 1 are far too marginal to distinguish between types of interaction i.e. there is no true “better" fit. This is also an issue we discovered in the paper [9]. So despite some positive findings in our attempts at projecting forwards, we don’t claim to have any tool for identifying in advance which model gives the best prediction. The work we have produced in this paper can provide useful assistance in the complex world of financial modelling, yet it is important that future researchers use this work to understand the following. It may be unwise to directly fit ecologically designed population models to financial data as they are not constructed for this purpose. However, the principles of ecological study, e.g. the concept of incorporating species (i.e. stock market) interactions, etc. can be extremely useful in helping modellers develop purpose-built, sophisticated financial models from the ground up.

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