On $\mathfrak{m}$-adic Higher Differentials in Commutative Rings

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Abstract. In [1], it was shown that quotient ring R of affine k-algebra with respect to a separable prime ideal is regular if and only if $D_N^e(R,k)$ is free R-algebra for $N \neq \mathbf{N}$. In this paper, we shall define the algebra $\widehat{D}_N(R,P)$ of $\mathfrak{m}$-adic P-differential of rank N in R. When R is a local ring of equal characteristic, we have the following result under some assumptions: $\widehat{D}_N(R,k)$ is $\mathfrak{m}$-adic free algebra if and only if R is regular.

In this paper, all $\mathfrak{m}$-adic ring R are assumed to satisfy the conditions $\bigcap_{r \geqq 1} \mathfrak{m}^r = 0$ unless otherwise stated.

§ 1. Generalities.

In the present paper, all rings will be assumed to be commutative and have identities. Let R be a ring and let $\mathfrak{m}$ be an ideal of R. A ring R will be called an $\mathfrak{m}$-adic ring if R is topologized by taking $\mathfrak{m}^r (r=1,2,\cdots)$ as a fundamental system of neighborhoods of zero. Let R be an $\mathfrak{m}$-adic ring.

An R module E will be called an $\mathfrak{m}$-adic R-module if E is endowed with the topology in which $\mathfrak{m}^r E (r=1,2,\cdots)$ form a fundametal system of neighborhoods. An $\mathfrak{m}$-adic R-module is not necessarily a Hausdorff space. An $\mathfrak{m}$-adic R-module E is a Hausdorff if and only if $\bigcap_{r \geqq 0} \mathfrak{m}^r E = 0$.

The following lemma are well known.

LEMMA 1.1. *Let* $(R,\mathfrak{m})$ *be a Zariski ring and let E be a finite R-module. Then E is a Hausdorff* $\mathfrak{m}$*-adic R-module and any submodule F of E is a closed set. Moreover the* $\mathfrak{m}$*-adic topology of F coincides with the induced* $\mathfrak{m}$*-adic topology of E.* (Cartan, H. and chevally, C. : Géométrie algébrique, Seminaire de E.N.S., 8^e année, 1955/1956. Th. 1 of Exposé 18)

Let N be a set $\{1,2,\cdots,n\}$ or the set $\mathbf{N}$ of natural numbers and let N_0 be $N \cup \{0\}$.

Let P be a ring and let R be a P-algebra.

DEFINITION 1.2. A family $\mathbf{d} = \{d^i\}_{i \in N_0}$ of P-linear mapping from R into an R-algebra A is called a *P-derivation of rank* N from R into A if the following conditions are satisfied:

(i) $d^0(x) = x \cdot 1_A$ for every $x \in R$, where 1_A is the identity element of A.

(ii) $d^i(xy) = \sum_{0 \leqq s \leqq i} d^s x d^{i-s} y$ for every $x, y \in R$ and all $i \in N$.

DEFINITION 1.3. A P-derivation $\{d^i\}_{i \in N_0}$ of rank N from R into A is called *universal* if the following universal mapping property is satisfied: For any R-algebra E and any P-derivation $\{\delta^i\}_{i\in N_0}$ of rank N from R into E, there exists a unique R-algebra homomorhism $\varphi: A \longrightarrow E$ such that for all $i \in N_0$, $\varphi \circ d^i = \delta^i$.

It is known that for any ring P and any P-algebra R, there exists a universal P-derivation of rank N from R into A and it is unique up to an R-algebra isomorphism.

Henceforce, we shall denote by $D_N(R,P)$ the R-algebra A for such an universal P-derivation of rank N from R into A. The associated universal P-derivation of rank N from R into $D_N(R,P)$ will be denoted by $\mathbf{d}_N = \{d^i_{R,P}\}_{i\in N_0}$.

Let $A = \bigoplus_{i\geqq 0} A_i$ be a graded ring.

DEFINITION 1.4. A P-derivation $\mathbf{d}_N = \{d^i_{R,P}\}_{i\in N_0}$ of rank N from R into A is called *universal-finite* if the following conditions are satisfied:

(1) $d^i_{R,P}(R) \subset A_i$ for all $i \in N_0$

(2) For all $i \in N_0$, A_i is a finitely generated R-module by virture of $d^0_{R,P}$.

(3) As an R-algebra, A is generated by $\{d^i_{R,P}x \mid i \in N_0.\ x \in R\}$

(4) Let $B = \bigoplus_{i\geqq 0} B_i$ be a graded ring and let $\delta = \{\delta^i_{R,P}\}_{i\in N_0}$ be a P-derivation of rank N from R into B such that $\delta^i_{R,P}(R) \subset B_i$. Let B_i be finitely generated R-module for all $i \in N_0$ by virture of $\delta^0_{R,P}$. Then there exists a ring homomorphism φ from A into B which satisfies $\varphi \circ d^i_{R,P} = \delta^i_{R,P}$ for all $i \in N_0$.

By (3), the ring homomorphism in (4) is uniquely determined. When there exists a universal-finite P-derivation of rank N from R into A, A is uniquely determined up to an R-isomorphism by (3) and (4). Then we shall call A an *algebra of finite P-differential of rank N* in R and we shall denote it by $D^e_N(R,P)$. The associated universal-finite P-derivation of rank N from R into $D^e_N(R,P)$ will be denoted by $\mathbf{d}_N = \{d^i_{R,P}\}_{i\in N_0}$.

Since $d^i_{R,P}(\mathfrak{m}^k) \subset \mathfrak{m}^{k-i} D_N(R,P)$ for all $k \geqq i$, all $d^i_{R,P}$ are continuous in $\mathfrak{m}$-adic topology. When in Definition 1.2 R is an $\mathfrak{m}$-adic ring and A is an $\mathfrak{m}$-adic R-algebra, a P-derivation of rank N from R into A will be denoted by $\{\widehat{d}^i_{R,P}\}_{i\in N_0}$.

DEFINITION 1.5. Let R be an $\mathfrak{m}$-adic ring. An $\mathfrak{m}$-adic R-algebra A is called an *algebra of* $\mathfrak{m}$-*adic P-differenlials of rank N* in R if the following conditions are satisfied:

(1) $\widehat{d}^i_{R,P}(R) \subset A_i$ for all $i \in N_0$.

(2) As an R-algebra, A is generated by $\{\widehat{d}^i_{R,P}x \mid i \in N,\ x \in R\}$

(3) A is Hausdorff $\mathfrak{m}$-adic R-algebra.

(4) Let $B = \bigoplus_{i\geqq 0} B_i$ be an any Hausdorff $\mathfrak{m}$-adic R-algebra and let $\widehat{\delta} = \{\widehat{\delta}^i_{R,P}\}_{i\in N_0}$ be a P-derivation of rank N from R into B such that $\widehat{\delta}^i_{R,P}(R) \subset B_i$. Then there exists an R-algebra homomorphism $\widehat{\varphi}$ from A into B which satisfies $\widehat{\varphi} \circ \widehat{d}^i_{R,P} = \widehat{\delta}^i_{R,P}$ for all $i \in N_0$

Clearly the R-algebra homomorphism in (4) is uniquely determined and A is uniquely determined up to an R-isomorphism. Then we shall denote A by $\widehat{D}_N(R,P)$ and associated universal P-derivation of rank N from R into $\widehat{D}_N(R,P)$ will be denoted by $\widehat{\mathbf{d}}_N = \{\widehat{d}^i_{R,P}\}_{i\in N_0}$.

PROPOSITION 1.6. *Let R be a P algebra and let* $\mathfrak{m}$ *be an ideal of R. Assume that R is an* $\mathfrak{m}$*-adic ring. Then $\hat{D}_N(R, P)$ is given by*

$$\hat{D}_N(R,P) = D_N(R,P) / \bigcap_{r\geq 0} \mathfrak{m}^r D_N(R,P)$$

PROOF. We shall show that $D_N(R,P)/\bigcap_{r\geq 0} \mathfrak{m}^r D_N(R,P)$ satisfies the four properties in Definition 1.5. Let ρ be a natural homomorphism

$$\rho : D_N(R,P) \longrightarrow D_N(R,P) / \bigcap_{r\geq 0} \mathfrak{m}^r D_N(R,P)$$

and let us put $\hat{\delta}^i_{R,P}\, x = \rho\,(d^i_{R,P}\, x)$ for all $i \in N_o$. Then $\{\hat{\delta}^i_{R,P}\}_{i\in N_o}$ is a P-derivation of rank N from R into R-algebra $D_N(R,P)/\bigcap_{r\geq 0} \mathfrak{m}^r D_N(R,P)$. Thus properties (1) (2) and (3) are easily satisfied. Since $D_N(R,P)/\bigcap_{r\geq 0} \mathfrak{m}^r D_N(R,P)$ is a Hausdorff $\mathfrak{m}$-adic R-algebra, there exists an R-algebra homomorphism

$$\hat{g} : D_N(R,P) / \bigcap_{r\geq 0} \mathfrak{m}^r D_N(R,P) \longrightarrow \hat{D}_N(R,P)$$

such that $\hat{d}^i_{R,P} = \hat{g}\,\hat{\delta}^i_{R,P}$ for all $i \in N_o$. Hence by the universal mapping property, $\hat{g}$ is an R-algebra isomorphism and property is satisfied.

COROLLARY 1. *If $D_N(R,P)$ is a Hausdorff* $\mathfrak{m}$*-adic R-algebra, we have*

$$\hat{D}_N(R,P) = D_N(R,P)$$

COROLLARY 2. *If R is a field, we have*

$$\hat{D}_N(R,P) = D_N(R,P)$$

CROPOSITION 1.7. *$\hat{D}_N(R,P)$ is a direct sum of $\{\hat{D}_N(R,P)_i\}_{i\geq o}$;*

$$\hat{D}_N(R,P) = \bigoplus_{i\geq o} \hat{D}_N(R,P)_i$$

where $\hat{D}_N(R,P)_i$ is the $\mathfrak{m}$*-adic R-submodule of $\hat{D}_N(R,P)$ generated by the elements $\hat{d}^{k_1}_{R,P} x_1 \cdots \hat{d}^{k_r}_{R,P}\, x_r$ such that $x_1, \cdots, x_r \in R$ and $k_1 + \cdots + k_r = i$ for $i \geq o$.*

PROOF. Let $\hat{D} = \bigoplus_{i\geq o} \hat{D}_N(R,P)_i$. Then $\hat{D}$ is easily seen to have a structure of graded R-algebra. Further, the P-derivation $\{\hat{d}^i_{R,P}\}_{i\in N_o}$ of rank N defines a P-derivation $\{\hat{\delta}^i\}_{i\in N_o}$ of rank N from R into $\hat{D}$ by the rule

$$\hat{\delta}^i x = \hat{d}^i_{R,P}\, x \text{ for any } x \in R \text{ and all } i \in N_o.$$

Since $\hat{D}_N(R,P)$ is generated by $\hat{d}^i_{R,P} x$'s $(x \in R, i \in N_o)$ over R, $\hat{D}$ is generated by $\hat{\delta}^i x$'s $(x \in R, i \in N_o)$ over R. Since $\hat{D}_N(R,P)_i$ is a Hausdorff $\mathfrak{m}$-adic R-submodule of $\hat{D}_N(R,P)$ we have

$$\bigcap_{r\geq o} \mathfrak{m}^r \Big(\bigoplus_{i\geq o} \hat{D}_N(R,P)_i \Big) = \bigoplus_{i\geq o} \Big(\bigcap_{r\geq o} \mathfrak{m}^r \hat{D}_N(R,P)_i \Big) = o$$

Therefore $\hat{D}$ is a Hausdorff $\mathfrak{m}$-adic R-algebra. Clearly there exists an R-algebra homomorphism $\hat{f} : \hat{D} \longrightarrow \hat{D}_N(R,P)$ such that $\hat{f}\,\hat{\delta}^i = \hat{d}^i_{R,P}$ for all $i \in N_o$. Thus by the universal mapping property of $\hat{D}_N(R,P)$ and $\{\hat{d}^i_{R,P}\}_{i\in N}$, $\hat{f}$ must be an R-algebra isomorphism.

PROPOSITION 1.8. *Let R be a noetherian ring and let* $\mathfrak{m}$ *be an ideal of R. If there exists an algebra $D^e_N(R,P)$ of finite P-differential of rank N in R, we assume that one of the following conditions is satisfied:*

(1) *$(R, \mathfrak{m})$ is a Zariski ring.*

(2) *Any element a such that* $a-1 \in \mathfrak{m}$ *is not zero divisor of* $D_N^e(R,P)_i$ *for all* $i \in N$.

Then we have

$$\widehat{D}_N^e(R,P) = D_N^e(R,P)$$

where $\widehat{D}_N^e(R,P)$ *is algebra of finite* $\mathfrak{m}$*-adic P-differential of rank N in R.*

PROOF. (1) follows from Lemma 1.1 and the assumption.

(2) is a consequence of the Theorem of Artin-Rees.

Now for an ideal $\mathfrak{m}$ of R, we shall define an ideal $\widehat{I}_N(\mathfrak{m})$ of $\widehat{D}_N(R,P)$ with the following homogeneous components:

$\widehat{I}_N(\mathfrak{m})_0 = \mathfrak{m}$

$\widehat{I}_N(\mathfrak{m})_i =$ the submodule of $\widehat{D}_N(R,P)_i$ generated by all elements of the form

$w_k \widehat{d}^{i-k}\, m$ such that $m \in \mathfrak{m}, w_k \in \widehat{D}_N(R,P)_k$ and $k=1,\cdots,i$ for $i>0$.

Then $\widehat{I}_N(\mathfrak{m}) = \bigoplus_{i \geqq 0} \widehat{I}_N(\mathfrak{m})_i$ is a ideal of $\widehat{D}_N(R,P)$ which contained in $\widehat{\mathfrak{d}}_N(\mathfrak{m})$, and $\widehat{I}_N(\mathfrak{m})_i = \widehat{\mathfrak{d}}_N(\mathfrak{m})_i$ for $i \notin N$.

Hence we can obtain an exact sequence for each $i \in N$ in the same way as 1.5 in [1]

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \widehat{D}_N(R,P)_i \,/\, \widehat{I}_N(\mathfrak{m})_i \longrightarrow \widehat{D}_N(R/\mathfrak{m}, P)_i \longrightarrow 0$$

If there exists an algebra $\widehat{D}_N^e(R,P)$ of finite P-differential of rank N in R, we obtain the correspondent sequence with $\widehat{D}_N^e$ and $\widehat{I}_N^e$.

Suppose now that R is a local ring and $\mathfrak{m}$ is a maximal ideal which is finitely generated. Let $(u_1,\cdots,u_t)$ be a minimal set of generators of $\mathfrak{m}$. Let $\widehat{D}_N(R/\mathfrak{m},P)$ be generated by $\{\widehat{d}^i_{R/\mathfrak{m},P}\bar{z}_1,\cdots,\widehat{d}^i_{R/\mathfrak{m},P}\bar{z}_s \mid i \in N\}$ $(\bar{z}_j = z_j + \mathfrak{m})$. Then $\widehat{D}_N(R,P)$ is generated by $\{\widehat{d}^i_{R,P}u_1,\cdots,\widehat{d}^i_{R,P}u_t, \widehat{d}^i_{R,P}z_1,\cdots,\widehat{d}^i_{R,P}z_s \mid i \in N\}$.

Let R be an $\mathfrak{m}$-adic P-algebra with a ring homomorphism $f: P \longrightarrow R$ and let S be an $\mathfrak{n}$-adic R-algebra with a ring homomorphism $g: R \longrightarrow S$, Then S is naturally a P-algebra with the ring homomorphism $h = g \circ f : P \longrightarrow S$. Moreover we assume that

(1) $$g(\mathfrak{m}) \subset \mathfrak{n}.$$

By definition, $\widehat{D}_N(S,P)$ is a Hausdorff $\mathfrak{n}$-adic S-algebra. At the same time $\widehat{D}_N(S,P)$ is an R-algebra and we can introduce in $\widehat{D}_N(S,P)$ an $\mathfrak{m}$-adic topology. Since $\mathfrak{m}^r\widehat{D}_N(S,P) = g(\mathfrak{m})^r\widehat{D}_N(S,P)$, $\widehat{D}_N(S,P)$ is also a Hausdorff $\mathfrak{m}$-adic R-algebra under the condition (1). Let us now define a family $\{\widehat{\delta}^i\}_{i \in N_0}$ of mappings from R into $\widehat{D}_N(S,P)$ by $\widehat{\delta}^i x = \widehat{d}^i_{S,P}x$ for all $i \in N_0$ and $x \in R$. This is clearly a P-derivation of rank N from R into $\widehat{D}_N(S,P)$, hence there exists an R-homomorphism $\alpha: \widehat{D}_N(R,P) \longrightarrow \widehat{D}_N(S,P)$ such that $\widehat{\delta}^i x = \widehat{d}^i_{S,P}x = \alpha\, \widehat{d}^i_{R,P}x$ for $x \in R$. From this we can define an S-homomorphism

$$\varphi'_{P;R,S} : S \otimes_R \widehat{D}_N(R,P)_i \longrightarrow \widehat{D}_N(S,P)_i$$

by the rule $\varphi'_{P;R,S}(\sum_j s_j \otimes \widehat{d}^i_{R,P}x_j) = \sum_j s_j\, \widehat{d}^i_{S,P}x_j$ for $s_j \in S$, $x_j \in R$ and all $i \in N_0$.

Moreover we can define an S-homomophism

(2) $\varphi_{P;R,S} : S \otimes_R \widehat{D}_N(R,P) \longrightarrow \widehat{D}_N(S,P)$.

Since $\widehat{D}_N(S,P)$ is a Hausdorff $\mathfrak{n}$-adic S-algebra, we can define the homomorphism

$$\hat{\varphi}_{P;R,S} : S\otimes_R \hat{D}_N(R,P)/ \bigcap_{r\geqq o} \mathfrak{n}^r(S\otimes_R\hat{D}_N(R,P))\longrightarrow\hat{D}_N(S,P).$$

Denoting by $\hat{N}_{P;R,S}$ and $\hat{D}_{R,S}$ the kernel and cokernel of $\hat{\varphi}_{P;R,S}$, we have the following exact sequence

$$(3)\qquad o\longrightarrow \hat{N}_{P;R,S}\longrightarrow S\otimes_R\hat{D}_N(R,P)/\bigcap_{r\geqq o}\mathfrak{n}^r(S\otimes_R\hat{D}_N(R,P))\longrightarrow\hat{D}_N(S,P)\longrightarrow\hat{D}_{R,S}\longrightarrow o$$

Let us denote by $S\hat{D}_N(R)_i$ the submodule of $\hat{D}_N(R,P)_i$ generated over S by the elements $\hat{d}^{\,i}_{S,P}\, x$ for $x\in R$ and let us set $S\hat{D}_N(R)=\bigoplus_{i\geqq o} S\hat{D}_N(R)_i$.
Then we have the following proposition.

PROPOSITION 1.9. *Notations being above, assume that $g(\mathfrak{m})\subset\mathfrak{n}$. Then we have*

$$\hat{D}_N(S,R)\cong\hat{D}_{R,S}/\bigcap\mathfrak{n}^r\hat{D}_{R,S}$$

$$\hat{D}_N(S,R)\cong\hat{D}_N(S,P)/\bigcap_{r\geqq o}(\mathfrak{n}^r\hat{D}_N(S,P)+S\hat{D}_N(R)).$$

PROOF. The assertion can be proved in a similar way as in the proof of Prop. 5 in [2].

COROLLARY 1. *Let $(S,\mathfrak{n})$ is a Zariski ring. If there exists an algebra of finite P-differential of rank N in R then we have*

$$\hat{D}^e_N(S,R) = \hat{D}^e_N(S,P)/S\hat{D}^e_N(R).$$

PROOF. As is easily seen, we have

$$\hat{D}^e_N(S,P)/\bigcap_{r\geqq o}(\mathfrak{n}^r\hat{D}^e_N(S,P)+S\hat{D}^e_N(R)) = \bigoplus_{i\geqq o}\{\hat{D}^e_N(S,P)_i/\bigcap_{r\geqq o}(\mathfrak{n}^r\,\hat{D}^e_N(S,P)_i+S\hat{D}^e_N(R)_i)\}.$$

Since $\hat{D}^e_N(S,P)$ is a finite module and $S\hat{D}_N(R)_i$ is a submodule of $\hat{D}_N(S,P)_i$, $S\hat{D}^e_N(R)_i$ is a closed set by Lemma 1.1. Hence we have

$$\hat{D}^e_N(S,R)= \bigoplus_{i\geqq o}\{\hat{D}^e_N(S,P)_i/S\hat{D}^e_N(R)_i\}$$

$$= \hat{D}^e_N(S,P)/S\hat{D}^e_N(R)$$

COROLLARY 2. *Notations and assumptions being as in Prop. 1.9. If $\hat{D}_N(R,P)=o$, then $\hat{D}_N(S,P) = \hat{D}_N(S,R)$.*

PROOF. We can obtain this proof in a similar way as Cor. 2 of Prop. 5 in [2].

Let S^* be the completion of S with respect to the $\mathfrak{n}$-adic topology and let us denote $\mathfrak{n}^*$ the extended ideal $\mathfrak{n}S^*$. Then it is well known that the topology of S^* as the limit space of S coincides with the $\mathfrak{n}^*$-adic topology and S^* is a Hausdorff $\mathfrak{n}^*$-adic ring. Under the condition (1), we can uniquely extend the homomorphism g to a ring homomorphism g^* of R^* into S^*.

PROPOSITION 1.10. *With notations as above, it holds the following equality*

$$\hat{D}_N(S^*, S)=o.$$

PROOF. Let $\{\hat{\delta}^i\}_{i\in N_o}$ be an R-derivation of rank N from S into a Hausdorff $\mathfrak{n}^*$-adic S^*-algebra E. Then $\{\hat{\delta}^i\}_{i\in N_o}$ can be uniquely extended to an R-derivation of rank N from S^* into E and we shall denote this extention by the same letter $\{\hat{\delta}^i\}_{i\in N_o}$. Since $\hat{\delta}^i(\mathfrak{n}^{*k})\subset\mathfrak{n}^{*k-i}E$ for $k\geqq i$, $\hat{\delta}^i$ is continuous. From this we have $\hat{D}_N(S^*, S)=o$ otherwise $\{\hat{d}^{\,i}_{S^*,S}\}_{i\in N_o}$ will give a non-trivial S-derivation of rank N from S^* into a Hausdorff $\mathfrak{n}^*$-adic S^*-algebra $\hat{D}_N(S^*,S)$.

PROPOSITION 1.11. *Let S be an R-algebra satisfying the condition* (1) *and assume that* $(S,\mathfrak{n})$ *is a Zariski ring. If there exists an algebra* $D_N^e(S,R)$ *of finite R-differential of rank N in S, then we have*

$$\widehat{D}_N^e(S^*, R^*)=\widehat{D}_N^e(S^*, R)=S^*\otimes_S\widehat{D}_N^e(S,R)=S^*\otimes_S D_N^e(S,R).$$

PROOF. Since, by our assumption, $(S^*,\mathfrak{n}^*)$ is a Zariski ring, $\widehat{D}_N^e(S^*, R^*)=D_N^e(S^*, R^*)$ by virtue of Prop. 1.8. By 2.2 in [1], Cor. 2 of Prop. 1.9 and Prop. 1.10, we have

$$\widehat{D}_N^e(S^*, R^*)=\widehat{D}_N^e(S^*,R)=S^*\otimes_S\widehat{D}_N^e(S,R)=S^*\otimes_S D_N^e(S,R).$$

COROLLARY. *Let S and R be local rings such that R is contained in S. Then if there exists an algebra $D_N^e(S,R)$ of finite R-differential of rank N in S, we have*

$$\widehat{D}_N^e(S^*, R^*)=\widehat{D}_N^e(S^*, R)=S^*\otimes_S\widehat{D}_N^e(S,R)=S^*\otimes_S D_N^e(S,R).$$

§2. Characterizations of regular local rings. (equal characteristic case)

LEMMA 2.1. *Let R be a local ring and let $\mathfrak{m}$ be its maximal ideal generated by a minimal set $(u_1,\cdots,u_t)$ of generators. Let $S=R[X_1,\cdots,X_s]$ be a polynomial ring in indeterminates $X_1,\cdots,X_s$ over R and let us set $\mathfrak{a}=\mathfrak{m}S+(X_1,\cdots,X_s)$. Then we have the following equality*

$$\dim_{R/\mathfrak{m}}\mathfrak{a}/\mathfrak{a}^2=t+s.$$

(3.1 *in* [1])

LEMMA 2.2. *Let R be a ring and let $S=R[X_1,\cdots,X_s]$ be a polynomial ring in indeterminates $X_1,\cdots,X_s$ over R. Let $z_1,\cdots,z_s$ be elements of S such that $S=R[z_1,\cdots,z_s]$. Then $z_1,\cdots,z_s$ are algebraic independent over R.*

(3.2 *in* [1])

PROPOSITION 2.3. *Let R be a local ring (but not necessarily noetherian ring) and let $\mathfrak{m}$ be its maximal ideal. Let P be a subring of R and let k be the quotient field of $P/\mathfrak{m}\cap P$ and let us set $\mathfrak{n}=(\mathfrak{m}\cap P)R$. Assume that the residue field of R is a separable extension of k. Then, for each $i\in N$, we have a exact sequence*

$$o\longrightarrow\mathfrak{m}/(\mathfrak{n}+\mathfrak{m}^2)\xrightarrow{\widetilde{d}^i}\widehat{D}_N(R,P)_i/\widehat{I}(\mathfrak{m})_i\longrightarrow\widehat{D}_N(R/\mathfrak{m},k)_i\longrightarrow o$$

where $\widetilde{d}^i$ is induced by universal derivation of rank N in R. If there exists an algebra $D_N^e(R,P)$ of finite P-differential of rank N in R and if $\mathfrak{m}/\mathfrak{m}^2$ and $D_N^e(R/\mathfrak{m},k)_i$ are finitely generated for all $i\in N$, $\widehat{D}_N$ and $\widehat{I}_N$ can be replaced by $\widehat{D}_N^e$ and $\widehat{I}_N^e$ in the sequence.

PROOF. The assertion can be proved in a similar way as 3.3 in [1]. Therefore we omit the proof.

COROLLARY. *Let R be a local ring and let $\mathfrak{m}$ be its maximal ideal. Let P be the ring contained in R which is either a field or else a discrete valuation ring such that the prime element u of P is contained in $\mathfrak{m}^2$. Assume that the residue field of R is a separable extension of the residue field k of P. Then, for each $i\in N$, we have a exact sequence*

$$o \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \widehat{D}_N(R,P)_i/\widehat{I}_N(\mathfrak{m})_i \longrightarrow \widehat{D}_N(R/\mathfrak{m}, k)_i \longrightarrow o.$$

PROOF. When P is a field, the assertion is obvious. Hence we assume that P is a discrete valuation ring. Let us set $S=R/\mathfrak{m}^2$. From the assumption that $u \in \mathfrak{m}^2$, $k=P/uP$ is considered as a subfield of S. Since S is the local ring with the maximal ideal $\mathfrak{n}=\mathfrak{m}/\mathfrak{m}^2$, the residue field of S is a separable extension of k and $(\mathfrak{n} \cap k)S=o$, for each $i \in N$ we get the following exact sequence

$$o \longrightarrow \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \widehat{D}_N(S,k)_i/\widehat{I}_N(\mathfrak{n})_i \longrightarrow \widehat{D}_N(S/\mathfrak{n},k)_i \longrightarrow o.$$

By definition we have $S/\mathfrak{n}=R/\mathfrak{m}$ and $\mathfrak{n}=\mathfrak{n}/\mathfrak{n}^2$. Hence it is sufficient to prove that

$$\widehat{D}_N(S,k)_i/\widehat{I}_N(\mathfrak{n})_i \cong \widehat{D}_N(R,P)_i/\widehat{I}_N(\mathfrak{m})_i$$

But this is obvious.

PROPOSITION 2.4. *Let R be a local ring and let $\mathfrak{m}$ be its maximal ideal generated by a minimal set $(u_1,\cdots,u_t)$ of generators. Let P be a subring of R and let $N=\{1,2,\cdots,n\}$. Let $z_1,\cdots,z_s$ be elements of R such that $\widehat{D}_N(R/\mathfrak{m},P)$ is generated by $\{\widehat{d}^i_{R/\mathfrak{m},P}\bar{z}_1,\cdots,\widehat{d}^i_{R/\mathfrak{m},P}\bar{z}_s \mid i \in N\}$ where $\bar{z}_j=z_j+\mathfrak{m}$ and let us set*

$$\mathfrak{a}=\mathfrak{m} \oplus \bigoplus_{i \geqq 1} \widehat{D}^e_N(R,P)_i$$

Then we have

$$\text{(1)} \qquad dim_{R/\mathfrak{m}}\ \mathfrak{a}/\mathfrak{a}^2 \leqq t+n(t+s)$$

If the equality holds, then the sequence

$$o \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \widehat{D}^e_N(R,P)_i/\widehat{I}_N(\mathfrak{m})_i \longrightarrow \widehat{D}^e_N(R/\mathfrak{m},P)_i \longrightarrow o$$

is exact for each $i \in N$.

Conversely, if this sequence is exact for each $i \in N$ and if $\widehat{D}^e_N(R/\mathfrak{m},P)$ is a polynomial ring over $R/\mathfrak{m}$ in the indeterminates $\{\widehat{d}^i_{R/\mathfrak{m},P}\bar{z}_1,\cdots,\widehat{d}^i_{R/\mathfrak{m},P}\bar{z}_s \mid i \in N\}$, then the equality holds in (1).

PROOF. We can obtain this proof in a similar way as 3.4 in [1]. Therefore we omit the proof.

THEOREM 2.5. *Let R be a local ring of equal characteristic with maximal ideal $\mathfrak{m}$. Let k be a field contained in R such that $R/\mathfrak{m}$ is a finitely generated separable extension of dimension r over k. Assume that $\widehat{D}_N(R,k)$ is finite algebra and $N \neq \mathrm{N}$. Then $\widehat{D}_N(R,k)$ is $\mathfrak{m}$-adic free algebra if and only if R is regular.*

PROOF. Let $K=R/\mathfrak{m}$ and $N=\{1,2,\cdots,n\}$. Let $\alpha_1,\cdots,\alpha_r$ be elements of R such that their residue classes $\bar{\alpha}_1,\cdots,\bar{\alpha}_r$ modulo $\mathfrak{m}$ are separating transcendent base of K over k and let $(u_1,\cdots,u_r)$ be a minimal set of generators of $\mathfrak{m}$. Then $\widehat{D}_N(K,k)$ is generated by $\{\widehat{d}^i_{K,k}\bar{\alpha}_1\cdots, \widehat{d}^i_{K,k}\bar{\alpha}_r \mid i \in N\}$. Since K is separable over k, we have the following exact sequence for each $i \in N$.

$$o \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \widehat{D}_N(R,k)_i/\widehat{I}_N(\mathfrak{m})_i \longrightarrow \widehat{D}_N(K,k)_i \longrightarrow o$$

by Cor. of Prop. 2.3.

Now let us set $\mathfrak{a}=\mathfrak{m}\oplus\bigoplus_{i\geq 1}\widehat{D}_N(R,k)_i$. Then by Prop. 2.4 it holds the equality

$$\dim_K \mathfrak{a}/\mathfrak{a}^2 = t+n(t+r).$$

Since $\widehat{D}_N(R,k)$ is finitely generated by the assumption, we may view it as a polynomial ring over R in indeterminates $Y_1,\cdots,Y_s$. Therefore it is no loss of generality to assume that elements $Y_1,\cdots,Y_s$ are contained in $\mathfrak{a}$. Then we have

$$\mathfrak{a}=\mathfrak{m}\widehat{D}_N(R,k)+(Y_1,\cdots,Y_s)$$

and from Lemma 2.1

$$\dim_K \mathfrak{a}/\mathfrak{a}^2 = t+s.$$

Thus we obtain $s=n(t+r)$. By Lemma 2.2, $\widehat{D}_N(R,k)$ is a polynomial ring over R in indeterminates $\{\widehat{d}^{\,i}_{R,k}\alpha_1,\cdots,\widehat{d}^{\,i}_{R,k}\alpha_r,\ \widehat{d}^{\,i}_{R,k}u_1,\cdots,\widehat{d}^{\,i}_{R,k}u_t \mid i\in N\}$. Next we shall show that $u_1,\cdots,u_t$ is a regular system of parameters of R. Let $F(X_1,\cdots,X_t)$ be a homogeneous polynomial of $R[X]$ with degree i such that $F(u_1,\cdots,u_t)\in\mathfrak{m}^{i+1}$. In $\widehat{D}_N(R,k)$, it holds that, for sufficiently large N such that $i\in N$,

$$\widehat{d}^{\,i}_{R,k}\,F(u_1,\cdots,u_t)=F(\widehat{d}^{\,1}_{R,k}u_1,\cdots,\ \widehat{d}^{\,1}_{R,k}u_t)+G \text{ with } G\in\mathfrak{m}\widehat{D}_N(R,k)$$

Since $\widehat{d}^{\,i}_{R,k}(\mathfrak{m}^{i+1})\subset\mathfrak{m}\widehat{D}_N(R,k)$, we have

$$F(\widehat{d}^{\,1}_{R,k}u_1,\cdots,\ \widehat{d}^{\,1}_{R,k}u_t)\in\mathfrak{m}\widehat{D}_N(R,k).$$

Hence all coefficients of F must be contained in $\mathfrak{m}$ because $\widehat{d}^{\,1}_{R,k}u_1,\ \cdots,\ \widehat{d}^{\,1}_{R,k}u_t$ are contained in $\widehat{D}_N(R,k)$.

Conversely, let R^* be the completion of R and let us denote by $\mathfrak{m}^*$ the extended ideal $\mathfrak{m}R^*$. Then $K=R^*/\mathfrak{m}^*$ and $R^*=K[[u_1,\cdots,u_t]]$. Hence we see that, by the same way as in the proof of Prop. 7 and Cor. 3 of Prop. 9 in [3], $\widehat{D}_N(R^*,K)$ and $\widehat{D}_N(K,k)$ can be expressed as polynomial rings over R^* in $n\cdot t$ indeterminates and over K in $n\cdot r$ indeterminates respectively. Moreover we have the following exact sequence for each $i\in N$

$$\mathfrak{m}^*/\mathfrak{m}^{*2}\longrightarrow \widehat{D}_N(R^*,k)_i/\widehat{I}_N(\mathfrak{m}^*)_i\longrightarrow \widehat{D}_N(R^*/\mathfrak{m}^*,\ k)_i\longrightarrow 0$$

By this fact, we see that $\widehat{D}_N(R^*,k)$ is generated by $\{\widehat{d}^{\,i}_{R^*k}\,\alpha_1,\cdots,\ \widehat{d}^{\,i}_{R^*k}\,\alpha_r,\ \widehat{d}^{\,i}_{R^*k}u_1,\cdots,\ \widehat{d}^{\,i}_{R^*k}u_t \mid i\in N\}$. Thus by Lemma 2.2, $\widehat{D}_N(R^*,k)$ is free R^*-algebra. By Prop. 1.11, $\widehat{D}_N(R^*,k)_i=R^*\otimes\widehat{D}_N(R,k)_i$ for all $i\in N$. Therefore above isomorphism is given by corresponding $\widehat{d}^{\,i}_{R^*k}\,\alpha_j,\ \widehat{d}^{\,i}_{R^*k}\,u_l$, to $1\otimes\widehat{d}^{\,i}_{R,k}\,\alpha_j,\ 1\otimes\widehat{d}^{\,i}_{R,k}u_l$ $(j=1,\cdots,s,\ l=1,\cdots,t)$ respectively.

The assertion follows from this.

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