WAVE PROPAGATION IN A 3-D OPTICAL WAVEGUIDE

OLEG ALEXANDROV

School of Mathematics, University of Minnesota
206 Church Str SE, Minneapolis, MN 55455, USA
aoleg@math.umn.edu

GIULIO CIRAOLO

Dipartimento di Matematica U. Dini, University of Firenze
viale Morgagni 67/A, 50134 Firenze, Italy
ciraolo@math.unifi.it

In this paper we study the problem of electromagnetic wave propagation in a 3-D optical fiber. The goal is to obtain a solution for the time-harmonic field caused by a source in a cylindrically symmetric waveguide. The geometry of the problem, corresponding to an open waveguide, makes the problem challenging. To solve it, we construct a transform theory which is a nontrivial generalization of a method for solving a 2-D version of this problem given by Magnanini and Santosa.[3]

The extension to 3-D is made complicated by the fact that the resulting eigenvalue problem defining the transform kernel is singular both at the origin and at infinity. The singularities require the investigation of the behavior of the solutions of the eigenvalue problem. Moreover, the derivation of the transform formulas needed to solve the wave propagation problem involves nontrivial calculations.

The paper provides a complete description on how to construct the solution to the wave propagation problem in a 3-D optical waveguide with cylindrical symmetry. A follow-up article will study the particular cases of a step-index fiber and of a coaxial waveguide. In those cases we will obtain concrete formulas for the field and numerical examples.

Keywords: Wave propagation, Optical waveguides, Helmholtz equation, Green's function, Spectral representation.

1. Introduction

In this paper we study the electromagnetic wave propagation in a cylindrical optical fiber.[a] As model equation, we use the *Helmholtz equation*

$$\Delta u + k^2 n(x,y,z)^2 u = f(x,y,z), \quad (x,y,z) \in \mathbb{R}^3, \tag{1.1}$$

also called the *time-harmonic wave equation.* The number k is called the *wavenumber*, and the function f represents a source of energy. We require that the index of refraction $n(x,y,z)$ have the form:

$$n(x,y,z) = \begin{cases} n_{co}(\sqrt{x^2+y^2}), & \text{if } \sqrt{x^2+y^2} < R, \\ n_{cl}, & \text{if } \sqrt{x^2+y^2} \geq R, \end{cases}$$

where R is the radius of the waveguide, and $n_{co}(\cdot)$ is an arbitrary, bounded and integrable function with positive values.

The main result of this paper is the construction of a representation formula for a solution u of (1.1) satisfying suitable radiation conditions. To find such a formula is equivalent to finding a Green's function for problem (1.1), as then the field generated by the source f can be obtained as a superposition of fields generated by point sources. Our results generalize a similar formula obtained by Magnanini and Santosa[3] in the two-dimensional case.

In [3] it is shown that the energy of the electromagnetic field is divided into two parts: a part that propagates without loss inside the waveguide as a finite number of distinct guided modes, while the other part either decays exponentially along the fiber or is radiated outside. Our case reveals a new feature: for special choices of the parameters, new kinds of guided modes appear which, rather than decaying exponentially outside the fiber, vanish as a power of the distance from the fiber's axis.

As in,[3] we use the technique of separation of variables with a number of important differences specified below. In our case the relevant symmetries of the problem suggest that we separate variables in the cylindrical coordinates r, ϑ, and z. The equation in the angular variable ϑ is simple to solve for and will only introduce a Fourier series in ϑ. The separation in z can be solved as in.[3] More complicated is the study of the r-coordinate.

In the 2-D case, the authors of [3] start with a domain of the form $(x,z) \in [-t,t] \times (-\infty,\infty)$. They impose boundary conditions at $x = \pm t$, and then obtain the desired Green's function by setting $t \to \infty$. They use the fact that the separation of variables yields a differential equation in x whose solutions, in the region where the index $n(x)$ is constant, are sines and cosines whose zeros are uniformly distributed. This assumption turns out to play a fundamental role in calculating Green's function.

In our case we obtain a differential equation in the variable r which is somewhat related to Bessel's equation and, in particular, is singular at $r = 0$. A further

[a] We will use the terms optical waveguide and optical fiber interchangeably.

difficulty is that, unlike the 2-D case, the distribution of the zeroes of Bessel's functions is more complicated. The singularity and the non-uniform distribution of zeros make the method used in [3] inapplicable.

Our approach is to use the theory of singular self-adjoint eigenvalue problems for second order differential equations as presented in [2] and. [8] Applying this theory to our problem and doing the explicit calculations took some effort, chiefly for the reason that our self-adjoint eigenvalue problem has, just like in the 2-D case, a coefficient which is, with some restrictions, a general function, so we cannot obtain solutions for our equation in terms of concrete functions.

In spite of the differences we just mentioned with the work in [3], we wish to remark that qualitatively, the 2-D case and the 3-D case behave similarly. In both cases one ultimately needs to study the behavior of a differential operator (in the x variable in the 2-D case and in the r variable in the 3-D case). Both operators exhibit a *limit point* behavior (see [2] and [8]), as $|x| \to \infty$ and respectively $r \to \infty$. And their spectra have the same structure. They both have a discrete part and a continuous part.

The paper is organized as follows. In Section 2 we derive the second order self-adjoint eigenvalue problem associated with the Helmholtz equation (1.1). In Section 3 we will prove a set of technical lemmas aimed at studying the behavior of the solutions of this eigenvalue problem. In Section 4 we will classify the solutions of the eigenvalue problem which are "well-behaved" (in a sense to be specified there) as $r \to 0$ and $r \to \infty$. The motivation is that the electromagnetic field in the fiber will have a representation in terms of the "well-behaved" solutions of this eigenvalue problem. In Section 5 we summarize the theory of self-adjoint eigenvalue problems as exposed in [2] and. [8]

The functions defined in Section 5 are calculated in Section 6. In Section 7 the transform defined in Section 5 is computed. The obtained transform is used in Section 8 to find the Green's function for the Helmholtz equation (1.1), and in turn, to find the desired electromagnetic field in the fiber given the source.

2. The Eigenvalue Problem

A typical optical fiber is a cylindrical dielectric waveguide, made of silica glass or plastic. Its central region is called *core*, surrounded by *cladding*, which has a slightly lower index of refraction. The cladding is surrounded by a protective *jacket*. Most of the electromagnetic radiation propagates without loss as a set of guided modes along the fiber axis. The electromagnetic field intensity of the guided modes in the cladding decays exponentially along the radial direction. This is why, the radius of the cladding, which is typically several times larger than the radius of the core, can be considered infinite.

A typical optical fiber is *weakly guided*, which means that the difference in the indexes of refraction of the core and cladding is very small. In these conditions the electromagnetic field in the fiber is essentially *transverse* with each of the transverse components approximately satisfying the Helmholtz equation (1.1). This is the so-called *weakly guided approximation.* For a derivation of (1.1) from Maxwell's equations for a weakly guided fiber see,[7] chapters 30 and 32.

Because of the cylindrical geometry, it will be convenient to use the cylindrical coordinate system (r, ϑ, z), with z being the axial direction. Then, the index of refraction will depend on the r variable only, $n = n(r)$. In the new coordinates (1.1) becomes

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + k^2 n(r)^2 u = f(r, \vartheta, z). \tag{2.1}$$

This is a linear partial differential equation. The solution of this equation will be determined as soon as we find its *Green's function.* In order to obtain the latter, consider the homogeneous version of (2.1),

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + k^2 n(r)^2 u = 0.$$

Look for a solution in separated variables, $u(r, \vartheta, z) = Z(z)\,\Theta(\vartheta)\,v(r)$. It is quickly found that we must have

$$u(r, \vartheta, z) = e^{i\beta kz} e^{im\vartheta} v(r), \tag{2.2}$$

with $\beta \in \mathbb{C}$, $m \in \mathbb{Z}$, and $v(r)$ satisfying the differential equation

$$v'' + \frac{1}{r}v' + \left\{k^2 n(r)^2 - \beta^2 - \frac{m^2}{r^2}\right\}v = 0$$

(the derivative here, and in the rest of this paper, will always be with respect to the r variable).

Let $R > 0$ be the radius of the fiber core. Then $n(r) = n_{cl}$ for $r \geq R$. Denote

$$d^2 = k^2(n_0^2 - n_{cl}^2), \quad l = k^2(n_0^2 - \beta^2), \quad q(r) = k^2[n_0^2 - n(r)^2]. \tag{2.3}$$

Then this equation becomes

$$v'' + \frac{1}{r}v' + \left\{l - q(r) - \frac{m^2}{r^2}\right\}v = 0. \tag{2.4}$$

We will view (2.4) as an eigenvalue problem in $l \in \mathbb{C}$. The variable r is in $(0, \infty)$, the number m is an integer, the function q is bounded, measurable, real-valued and non-negative, with $q(r) = d^2 > 0$ for $r \geq R > 0$. It will be convenient to make a variable change. Denote $w(r) = \sqrt{r}v(r)$. We get the equation

$$w'' + \left\{l - q(r) - \frac{m^2 - 1/4}{r^2}\right\}w = 0, \quad r \in (0, \infty). \tag{2.5}$$

This will be our self-adjoint eigenvalue problem.

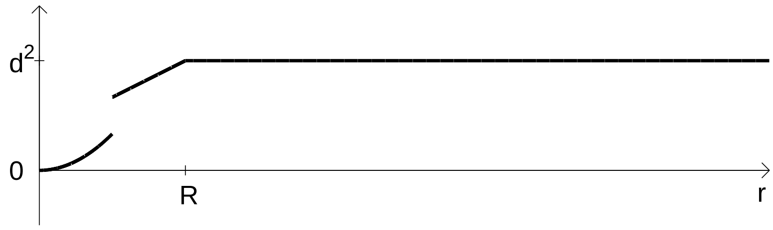

Fig. 1. The function q(r).

3. A Study of the Solutions of the Eigenvalue Problem

Before going further, we will need some information about the behavior of the solutions of the differential equation (2.5) as functions of r and l and m. This will be the subject of the next four lemmas.

Lemma 3.1 *There exists a solution* $j_m(r,l)$ $(r > 0,\ l \in \mathbb{C},\ m \in \mathbb{Z})$ *of* (2.5) *such that*

$$\lim_{r\to 0} \frac{j_m(r,l)}{r^{|m|+1/2}} = 1, \quad \lim_{r\to 0} \frac{j'_m(r,l)}{(|m|+1/2)\, r^{|m|-1/2}} = 1. \tag{3.1}$$

The functions $j_m(r,l)$ *and* $j'_m(r,l)$ *are analytic in* l *as* r *is fixed. There exists another solution* $y_m(r,l)$ *of* (2.5) *such that*

$$\lim_{r\to 0} \frac{y_m(r,l)}{r^{-|m|+1/2}} = 1, \quad \lim_{r\to 0} \frac{y'_m(r,l)}{(-|m|+1/2)\, r^{-|m|-1/2}} = 1, \text{ if } |m| \geq 1, \tag{3.2a}$$

and

$$\lim_{r\to 0} \frac{y_m(r,l)}{\sqrt{r}\ln r} = 1, \quad \lim_{r\to 0} \frac{y'_m(r,l)}{\ln r/(2\sqrt{r})} = 1, \text{ if } m = 0. \tag{3.2b}$$

Proof. We will assume $m \geq 0$, and then set $j_{-m} = j_m$ and $y_{-m} = y_m$. Make the variable change $w = r^{m+1/2}\sigma$ in (2.5). We obtain

$$\sigma'' + \frac{2m+1}{r}\sigma' + \{l - q(r)\}\,\sigma = 0.$$

Denote $k = 2m+1$, $k \geq 1$. Multiply this equation by r^k. We get

$$(r^k\sigma')' = r^k(q(r) - l)\,\sigma. \tag{3.3}$$

To prove this lemma we need to find two solutions $\sigma(r,l)$ and $\tau(r,l)$ of (3.3) such that

$$\lim_{r\to 0} \sigma(r,l) = 1, \quad \lim_{r\to 0} \sigma'(r,l) = 0, \tag{3.4}$$

$$\lim_{r\to 0} r^{2m}\tau(r,l) = 1, \quad \lim_{r\to 0} r^{2m+1}\tau'(r,l) = -2m, \quad \text{if } m \geq 1, \tag{3.5a}$$

and

$$\lim_{r\to 0} \frac{\tau(r,l)}{\ln r} = 1, \quad \lim_{r\to 0} r\tau'(r,l) = 1, \quad \text{if } m = 0. \tag{3.5b}$$

Also, $\sigma(r,l)$ must be analytic in l for r fixed. The idea to finding σ and τ is to rewrite (3.3) as an integral equation.

Let $q_\infty = \sup_{r\in[0,\infty)} q(r)$. If $\omega : [0,\infty) \to \mathbb{C}$ is a function, bounded and integrable on every compact subset of $[0,\infty)$, and $p \geq 0$ is an integer, then one has

$$\left| \int_0^r s^p \omega(s)\, ds \right| \leq \sup_{t\in[0,r]} |\omega(t)| \int_0^r s^p\, ds = \sup_{t\in[0,r]} |\omega(t)|\, \frac{r^{p+1}}{p+1}. \tag{3.6}$$

Consider the operator

$$T\omega(r) = \int_0^r t^{-k} \int_0^t s^k (q(s)-l)\omega(s)\, ds\, dt, \tag{3.7}$$

defined for complex-valued functions ω which are bounded and integrable on every compact subset of $[0,\infty)$. By applying (3.6) to the inner-most integral in (3.7) we deduce

$$\sup_{t\in[0,r]} |T\omega(t)| \leq (q_\infty + |l|) \sup_{t\in[0,r]} |\omega(t)| \frac{r^2}{2\,(k+1)}.$$

In particular, for any $r > 0$, T is a bounded linear operator from the space $\mathcal{C}([0,r])$ of continuous, complex-valued functions defined on $[0,r]$ onto itself. By using the same reasoning, one can show by induction that for any integer $n \geq 0$

$$|T^n\omega(r)| \leq (q_\infty + |l|)^n \sup_{t\in[0,r]} |\omega(t)| \frac{r^{2n}}{\{2\cdot 4 \cdots 2n\}\,\{(k+1)\cdot(k+3)\cdots(k+2n-1)\}}. \tag{3.8}$$

Using (3.6) it is easy to check that if function σ satisfies (3.4), then (3.3) is equivalent to

$$\sigma'(r,\lambda) = \frac{1}{r^k} \int_0^r s^k (q(s)-\lambda)\,\sigma(s,\lambda)\, ds, \tag{3.9}$$

which in turn is the same as

$$\sigma = 1 + T\sigma. \tag{3.10}$$

We will find σ by applying the method of successive approximations. Let $\sigma_0 \equiv 1$, and $\sigma_{n+1} = \sigma_0 + T\sigma_n$, $n \geq 0$. Then,

$$\sigma_n = \sigma_0 + T\sigma_0 + \cdots + T^n\sigma_0, \quad n \geq 0.$$

By using (3.8) we obtain that the series

$$\sigma = \sigma_0 + T\sigma_0 + \cdots + T^n\sigma_0 + \ldots, \quad \sigma_0 \equiv 1, \tag{3.11}$$

is uniformly convergent for r and l in compact sets. Since σ_n is continuous in r and analytic in l, the same property will hold for σ. Using the fact that the operator T

is continuous on $\mathcal{C}([0,r])$ for any $r > 0$, it follows from the above series that (3.10) holds, and therefore (3.3) holds.

Now prove the existence of τ satisfying (3.3) with the boundary conditions (3.5a) or (3.5b). That will be done in several steps to be described below. For the rest of the lemma, $\sigma = \sigma(r,l)$ will be the solution to (3.3) found above. We are not looking for any properties of τ with respect to l. Therefore, we will fix $l \in \mathbb{C}$, and will consider τ to be a function of r only. At step 1 we will prove that if τ is a solution to (3.3) which is linearly independent of σ, then τ is unbounded as $r \to 0$. At step 2, we will show that any solution τ to (3.3) has the property

$$|\tau(r)| \le A + Br^{-k}, 0 < r \le r_0, \tag{3.12}$$

for some $A > 0$, $B > 0, r_0 > 0$. At step 3 we will prove that any solution τ of (3.3) linearly independent of σ satisfies

$$\tau'(r) = O(r^{-k}) \text{ as } r \to 0. \tag{3.13}$$

That will imply (3.5a) or (3.5b) depending on whether $m \ge 1$ or $m = 0$, that is, $k \ge 3$ or $k = 1$.

Step 1. Show that any solution τ to (3.3) which is linearly independent of σ is unbounded as $r \to 0$. Assume that τ is such a solution and that it is bounded as $r \to 0$. From (3.3) we get

$$r^k \tau'(r) = c_1 + \int_0^r s^k (q(s) - l)\tau(r)\, ds$$

for some constant c_1. We cannot have $c_1 \neq 0$, because then $\tau'(r) = O(r^{-k})$ with $k \ge 1$, so τ cannot be bounded as $r \to 0$. Then, if $c_1 = 0$ we can divide by r^k and integrate again, to obtain

$$\tau = c_2 + T\tau,$$

with T the operator defined by (3.7) and c_2 another constant. Since the solution σ satisfies $\sigma = 1 + T\sigma$, we deduce that $\varphi = \tau - c_2\sigma$ will satisfy

$$\varphi = T\varphi.$$

Then $\varphi = T^n\varphi$ for any $n \ge 0$. By using (3.8) we get

$$|\varphi(r)| \le (q_\infty + |l|)^n \sup_{t\in[0,r]} |\varphi(t)| \frac{r^{2n}}{\{2 \cdot 4 \cdots 2n\} \{(k+1) \cdot (k+3) \cdots (k+2n-1)\}},$$

for any $r > 0$ and $n \ge 0$, which implies $\varphi = 0$, that is, $\tau = c_2\sigma$. This is a contradiction with the assumption that τ is linearly independent of σ.

Step 2. Show that for any τ solution of (3.3) inequality (3.12) holds. Let us again write (3.3) as an integral equation. This time we cannot integrate from $r = 0$, since we expect an unbounded solution as $r \to 0$. Let $r_0 > 0$ be a fixed number. It is easy to show that τ will satisfy (3.3) if and only if

$$\tau = \tau_0 + S\tau, \tag{3.14}$$

where

$$\tau_0(r) = c_0 + d_0 \int_{r_0}^{r} t^{-k}\, dt,$$

with $c_0 = \tau(r_0)$, $d_0 = r_0^k \tau'(r_0)$, and the operator S is defined on functions θ integrable on compact subsets of $(0, \infty)$ and is given by the formula

$$S\theta(r) = \int_{r_0}^{r} t^{-k} \int_{r_0}^{t} s^k (q(s) - l)\theta(s)\, ds\, dt.$$

Use once again the method of successive approximations to express τ as the sum of an infinite series. Let $c_0 \in \mathbb{R}$, $d_0 \in \mathbb{R}$, and define $\tau_0(r) = c_0 + d_0 \int_{r_0}^{r} t^{-k}\, dt$, $\tau_{n+1} = \tau_0 + S\tau_n$, $n \geq 0$. We have

$$\tau_n = \tau_0 + S\tau_0 + \cdots + S^n \tau_0, \quad n \geq 0.$$

Notice that if ω is an integrable function on $(0, r_0)$, then for $0 < r \leq r_0$

$$\left| \int_{r_0}^{r} s^{-k} \omega(s)\, ds \right| \leq \int_{r}^{r_0} s^{-k} |\omega(s)|\, ds \leq r^{-k} \int_{r}^{r_0} |\omega(s)|\, ds. \tag{3.15}$$

By applying (3.15) we get the estimate $|\tau_0(r)| \leq |c_0| + |d_0| r_0 r^{-k}$ for $0 < r \leq r_0$. Then,

$$\left| \int_{r_0}^{t} s^k (q(s) - l)\tau_0(s)\, ds \right| \leq \int_{t}^{r_0} (q_\infty + |l|)(r_0^k |c_0| + |d_0| r_0)\, ds = C(q_\infty + |l|)(r_0 - t),$$

where $C = r_0^k |c_0| + |d_0| r_0$. Apply (3.15) to estimate $S\tau_0$.

$$|S\tau_0(r)| \leq r^{-k} \int_{r}^{r_0} C(q_\infty + |l|)(r_0 - t)\, dt = C(q_\infty + |l|) r^{-k} \frac{(r_0 - r)^2}{2!}.$$

With the same reasoning, one obtains by induction

$$|S^n \tau_0(r)| \leq C(q_\infty + |l|)^n r^{-k} \frac{(r_0 - r)^{2n}}{(2n)!}, \quad 0 < r \leq r_0,\ n \geq 1.$$

As it was done for σ, one can prove that the series

$$\tau = \sum_{n=0}^{\infty} S^n \tau_0$$

is uniformly convergent for r in compact subsets of $(0, r_0]$ and that τ satisfies (3.14). One can then estimate τ.

$$|\tau(r)| \leq \sum_{n=0}^{\infty} |S^n \tau_0(r)| \leq |c_0| + Cr^{-k} e^{\sqrt{q_\infty + |l|}(r_0 - r)}, \quad 0 < r \leq r_0,$$

which implies (3.12).

Step 3. Show that any solution τ to (3.3) which is linearly independent of σ satisfies (3.13). From (3.12) we deduce that $r^k(q(r) - l)\tau(r)$ is bounded as $r \to 0$. Integrate (3.3) from 0 to r. We get

$$r^k \tau'(r) = c_3 + \int_0^r s^k (q(s) - l)\tau(s)\, ds, \tag{3.16}$$

with c_3 a constant.

Suppose that $c_3 = 0$. Then, from (3.12) and (3.16) it follows that $|r^k \tau'(r)| \leq Mr$ for $0 < r \leq r_0$ and some $M > 0$. But then, $|\tau'(r)| \leq M/r^{k-1}$ for $0 < r \leq r_0$. Note that k is an odd number, since $k = 2m + 1$ with $m \geq 0$ an integer. In particular $k \neq 2$, or $k - 1 \neq 1$. Since $\tau(r) = \tau(r_0) + \int_{r_0}^r \tau'(s)\, ds$, we obtain

$$|\tau(r)| \leq A_1 + \frac{B_1}{r^{k-2}},$$

for some $A_1 > 0$, $B_1 > 0$. This is the same as (3.12) but with $k-2$ instead of k. By repeating several times the same reasoning starting with (3.16) and the assumption $c_3 = 0$ we will be able to reduce the exponent of r in this inequality by 2 every time, until we get that τ must be bounded as $r \to 0$. As shown at step 1, then τ is linearly dependent of σ, which is a contradiction. Thus $c_3 \neq 0$. Then, (3.12) and (3.16) give us $\tau'(r) = O(r^{-k})$ as $r \to 0$, or (3.13).

From (3.13) and the equality $\tau(r) = \tau(r_0) + \int_{r_0}^r \tau'(s)\, ds$ it follows that as $r \to 0$, $\tau(r) = O(\ln r)$ if $k = 1$, and $\tau(r) = O(r^{-k+1})$ if $k > 1$. One can then get either (3.5a) or (3.5b) by conveniently multiplying τ by a non-zero constant. This proves the lemma. □

The following lemma will study the properties of $j_m(r, l)$ for l real.

Lemma 3.2 *Let $l = \lambda \in \mathbb{R}$. Then $j_m(r, \lambda)$ and $j'_m(r, \lambda)$ are real. If $\lambda \leq 0$, then $j_m(r, \lambda) \geq r^{|m|+1/2}$ and $j'_m(r, \lambda) > 0$.*

Proof. We will consider only the case $m \geq 0$, since $j_m(r, \lambda)$ is an even function of m. As λ is real, we have $q(r) - \lambda$ is real. With the notation from the proof of Lemma 3.1, $j_m(r, \lambda) = r^{m+1/2}\sigma(r, \lambda)$, with $\sigma(r, \lambda)$ given by the series (3.11). By using this series, and definition (3.7) of the operator T we deduce that $\sigma(r, \lambda)$ is real, and consequently, $j_m(r, \lambda)$ is real.

If $\lambda \leq 0$, then $q(r) - \lambda \geq 0$. In this case, the operator T will map non-negative functions into non-negative functions. We conclude from the construction (3.11) of σ that $\sigma(r, \lambda) \geq 1$, and so, $j_m(r, \lambda) \geq r^{m+1/2}$.

To show $j'_m(r,\lambda) > 0$ it suffices to prove $\sigma'(r,\lambda) \geq 0$, which is an immediate consequence of (3.9). $\square$

The next lemma will study the properties of $j_m(r,l)$ for large absolute value of m and $l = \lambda$ a real number.

Lemma 3.3 *Let $r > 0$ be a fixed number and $\Lambda \subset [0,\infty)$ be a bounded set. If $|m|$ is sufficiently large, then for all $\lambda \in \Lambda$*

$$j_m(r,\lambda) > 0, \quad j'_m(r,\lambda) > 0.$$

Proof. We can assume $m \geq 0$. Denote $k = 2m+1 \geq 1$. We have that $j_m(r,\lambda) = r^{m+1/2}\sigma(r,\lambda)$, with $\sigma(r,\lambda)$ defined by the series (3.11). Here we will prefer to use the notation $\sigma_m(r,\lambda)$ to emphasize its dependence on m. By using this series and estimate (3.8) it follows that

$$|1-\sigma_m(r,\lambda)| \leq \frac{C}{k+1}, \quad \lambda \in \Lambda,$$

with $C = C(\Lambda, q_\infty)$ a constant, where $q_\infty = \sup_{r\in[0,\infty)} q(r)$.

From (3.9) we deduce

$$|\sigma'_m(r,\lambda)| \leq \frac{D}{k+1}, \quad \lambda \in \Lambda,$$

with $D = D(\Lambda, q_\infty)$ another constant.

These observations imply that for m large enough and $\lambda \in \Lambda$, $\sigma_m(r,\lambda)$ is close to 1, while $\sigma'_m(r,\lambda)$ is close to zero. Then, since

$$j_m(r,\lambda) = r^{m+1/2}\sigma_m(r,\lambda),$$

and

$$j'_m(r,\lambda) = r^{m-1/2}\{(m+1/2)\,\sigma_m(r,\lambda) + r\,\sigma'_m(r,\lambda)\},$$

it follows quickly that for m large enough both of these quantities are strictly positive. $\square$

Lemma 3.4 *For $l \in \{-\pi/2 < \arg(l-d^2) < 3\pi/2\}$, there exist two solutions $w_m(r,l)$ and $x_m(r,l)$, $r \geq R$, $m \in \mathbb{Z}$, of (2.5) which are analytic in l. If $l \in \mathbb{C}_+ = \{l : \mathfrak{Im}\, l > 0\}$, then as $r \to \infty$, $w_m(r,l)$ together with its derivative will decay exponentially, while $x_m(r,l)$ and its derivative will increase exponentially.*

Proof. Recall that $q(r) = d^2$ for $r \geq R$, therefore on $[R,\infty)$ (2.5) becomes

$$w'' + \left\{l - d^2 - \frac{m^2-1/4}{r^2}\right\}w = 0.$$

Set

$$w_m(r,l) = \sqrt{r}H_m^{(1)}(\sqrt{l-d^2}\,r), \quad x_m(r,l) = \sqrt{r}H_m^{(2)}(\sqrt{l-d^2}\,r), \tag{3.17}$$

where $H_m^{(1)}(\zeta)$ and $H_m^{(2)}(\zeta)$ are the Hankel functions of m-th order. They are analytic functions with the domain $\{-\pi < \arg\zeta < \pi\}$. Consider the function $\sqrt{z}$ defined on $\{-\pi/2 < \arg z < 3\pi/2\}$ with values in $\{-\pi/4 < \arg\zeta < 3\pi/4\}$. We obtain that $w_m(r,l)$ and $x_m(r,l)$ are analytic functions of $l \in \{-\pi/2 < \arg(l-d^2) < 3\pi/2\}$, which is the whole complex plane except those l for which $l - d^2$ has a zero real part and a non-positive imaginary part.

According to formulas (9.2.3) and (9.2.4) from,[1] we have the asymptotic expansions

$$H_m^{(1)}(\zeta) \sim \sqrt{2/(\pi\zeta)}\, e^{i(\zeta - m\pi/2 - \pi/4)}$$

and

$$H_m^{(2)}(\zeta) \sim \sqrt{2/(\pi\zeta)}\, e^{-i(\zeta - m\pi/2 - \pi/4)}$$

as $|\zeta| \to \infty$ and $|\arg\zeta| < \pi$. Then,

$$w_m(r,l) = O\big(e^{i\sqrt{l-d^2}\,r}\big), \quad x_m(r,l) = O\big(e^{-i\sqrt{l-d^2}\,r}\big) \text{ as } r \to \infty.$$

If $l \in \mathbb{C}_+$, then $\mathfrak{Im}\sqrt{l-d^2} > 0$ and thus, $i\sqrt{l-d^2}$ has a strictly negative real part. Therefore, as $r \to \infty$, $w_m(r,l)$ will decay exponentially, while $x_m(r,l)$ will increase exponentially. By formally differentiating the above equalities (for a rigorous justification one needs to use equalities (9.2.13) and (9.2.14) from [1]) we deduce that $w'_m(r,l)$ will decay exponentially, while $x'_m(r,l)$ will increase exponentially as $r \to \infty$. $\square$

4. Classification of the Solutions

The main purpose of this paper is to prove that under certain conditions, the solution to the Helmholtz equation (2.1) is a superposition of functions of the form (2.2). For each of the functions in the superposition, $v(r)$ will satisfy (2.4) with the notations of (2.3), and l will be a real variable, which we will denote by λ (thus we will reserve the notation l for the complex variable, and the notation λ for its restriction to the real axis). In addition the following properties will hold:

$$v(r) \text{ is bounded as } r \to 0, \tag{4.1a}$$

and

$$r \to \sqrt{r}v(r) \begin{cases} \text{is in } L^2(R,\infty), & \text{if } \lambda \le d^2, \\ \text{is bounded as } r \to \infty, & \text{if } \lambda > d^2. \end{cases} \tag{4.1b}$$

In this section we will study and classify the functions $v(r)$ with the properties (4.1a) and (4.1b).

A solution $v(r)$ of (2.4) has the form $v(r) = w(r)/\sqrt{r}$, with $w(r)$ satisfying (2.5). Lemma 3.1 shows how the solutions of (2.5) look like. It is clear that in order that $v(r)$ satisfy (4.1a), we need $w(r) = j_m(r,\lambda)$, or

$$v(r) = \frac{j_m(r,\lambda)}{\sqrt{r}}. \tag{4.2}$$

Condition (4.1b) is then satisfied if and only if

$$r \to j_m(r,\lambda) \begin{cases} \text{is in } L^2(R,\infty), & \text{if } \lambda \le d^2, \\ \text{is bounded as } r \to \infty, & \text{if } \lambda > d^2. \end{cases} \tag{4.3}$$

Next we will investigate for which λ condition (4.3) holds.

We will need to consider four cases: $\lambda \le 0$, $0 < \lambda < d^2$, $\lambda = d^2$ and $\lambda > d^2$. In each of these intervals $j_m(r,\lambda)$ and consequently $v(r)$, will have a different behavior.

Case 1. If $\lambda \le 0$, then according to Lemma 3.2, $j_m(r,\lambda) \ge r^{|m|+1/2}$, therefore $j_m(r,\lambda)$ will be not square integrable on (R,∞).

Case 2. Assume $0 < \lambda < d^2$. For $r \ge R$ the function $q(r)$ defined in (2.3) is constant and equal to d^2. Then (2.5) becomes

$$w'' + \left\{\lambda - d^2 - \frac{m^2 - 1/4}{r^2}\right\} w = 0, \quad r \in [R,\infty). \tag{4.4}$$

The solutions to this equation are

$$k_m(r,\lambda) = \sqrt{r} K_m(\sqrt{d^2 - \lambda}\, r), \quad \lambda < d^2, r \ge R, \tag{4.5}$$

and

$$\sqrt{r} I_m(\sqrt{d^2 - \lambda}\, r), \quad \lambda < d^2, r \ge R,$$

where K_m and I_m are the modified Bessel functions. Formulas (9.7.1) and (9.7.2) from [1] give us the expansions

$$\sqrt{s} K_m(s) \sim \sqrt{\pi/2}\, e^{-s} \text{ as } s \to \infty,$$

and

$$\sqrt{s} I_m(s) \sim \sqrt{\pi/2}\, e^{s} \text{ as } s \to \infty.$$

Thus, we have one solution decaying exponentially while another increasing exponentially. In order that $j_m(r,\lambda)$ be in $L^2(R,\infty)$ we need

$$j_m(r,\lambda) = C k_m(r,\lambda) \text{ for } r \ge R,$$

for some constant C. Since both $j_m(r,\lambda)$ and $C k_m(r,\lambda)$ satisfy the same second order differential equation, namely (4.4), then from the continuity of these functions and of their first derivatives at $r = R$ we find that

$$j_m(R,\lambda) - C k_m(R,\lambda) = 0,$$

and

$$j'_m(R,\lambda) - C k'_m(R,\lambda) = 0.$$

Thus, C must satisfy simultaneously two different conditions. This is possible if and only if

$$\frac{j'_m(R,l)}{j_m(R,l)} = \frac{k'_m(R,l)}{k_m(R,l)}, \quad \lambda < d^2. \tag{4.6}$$

Then we will have

$$j_m(r,\lambda) = \frac{j_m(R,\lambda)}{k_m(R,\lambda)} k_m(r,\lambda), \quad r \geq R. \tag{4.7}$$

It will be shown later that for each m, the set of λ such that (4.6) holds is finite. Notice that in this case $j_m(r,\lambda)$ will decay exponentially as $r \to \infty$. Its derivative has the same property, this follows from the asymptotic expansion

$$[\sqrt{s}K_m(s)]' \sim -\sqrt{\pi/2}\, e^{-s} \text{ as } s \to \infty \tag{4.8}$$

(according to the formulas (9.7.2) and (9.7.4) from.[1]) In conclusion, the only $\lambda \in (0, d^2)$ for which (4.3) holds, are those satisfying (4.6).

Case 3. Let now $\lambda = d^2$. Two linear independent solutions of (4.4) are in this case

$$r \to r^{1/2-|m|}, \quad m \in \mathbb{Z},$$

and

$$r \to \begin{cases} \sqrt{r}\ln r & \text{for } m = 0, \\ r^{1/2+|m|} & \text{for } m \neq 0. \end{cases}$$

The second solution is not bounded for any $m \in \mathbb{Z}$. With the same reasoning as before (4.6), one can show that $j_m(r,\lambda)$ will be proportional to the first of these two solutions if and only if

$$\frac{j_m'(R,l)}{j_m(R,l)} = -\frac{|m| - 1/2}{R}, \quad \lambda = d^2, \tag{4.9}$$

and then,

$$j_m(r,\lambda) = \frac{j_m(R,\lambda)}{R^{1/2-|m|}} r^{1/2-|m|}, \quad r \geq R. \tag{4.10}$$

The function $j_m(r,\lambda)$ will be in $L^2(R,\infty)$ if and only if $|m| \geq 2$.

A function of the form (2.2), with $v(r)$ given by (4.2), for which $0 < \lambda \leq d^2$ and either (4.6), or (4.9) (with $|m| \geq 2$) holds, is called a *guided mode.* Note that a guided mode decays in r either exponentially, or as $r^{-|m|}$ ($|m| \geq 2$).

Case 4. Let $\lambda > d^2$. Two solutions of (4.4) are then

$$a_m(r,\lambda) = \sqrt{r}J_m\big(\sqrt{\lambda - d^2}\, r\big), \quad b_m(r,\lambda) = \sqrt{r}Y_m\big(\sqrt{\lambda - d^2}\, r\big), \quad \lambda > d^2, \tag{4.11}$$

where J_m and Y_m are the Bessel functions of the first and second kind. Note the formulas

$$\sqrt{s}\, J_m(s) = \sqrt{\pi/2}\cos(s - m\pi/2 - \pi/4) + O(s^{-1/2}),$$
$$\sqrt{s}\, Y_m(s) = \sqrt{\pi/2}\sin(s - m\pi/2 - \pi/4) + O(s^{-1/2}),$$

as $s \to \infty$ (they are a particular case of formulas (9.2.1) and (9.2.2) from.[1]) We infer that $a_m(r,\lambda)$ and $b_m(r,\lambda)$ will be bounded as $r \to \infty$. By formally differentiating the above formulas it follows that $a_m'(r,\lambda)$ and $b_m'(r,\lambda)$ will also be bounded as $r \to \infty$.

The functions $a_m(r,\lambda)$ and $b_m(r,\lambda)$ are linearly independent, since J_m and Y_m are linearly independent. Then, for $r \geq R$, $j_m(r,\lambda)$ will be a linear combination of them. To find the coefficients of the linear combination, set

$$j_m(r,\lambda) = c_m(\lambda)a_m(r,\lambda) + d_m(\lambda)b_m(r,\lambda), \quad r \geq R.$$

By again using the continuity of these functions and their derivatives at $r = R$, we obtain a linear system which enables us to solve for c_m and d_m. Apply the equality

$$Y_m'(z)J_m(z) - J_m'(z)Y_m(z) = 2/(\pi z),$$

((9.1.16) from [1]) to find a value for the determinant of this linear system. We obtain

$$b_m'(R,\lambda)a_m(R,\lambda) - a_m'(R,\lambda)b_m(R,\lambda) = 2/\pi, \tag{4.12}$$

and therefore,

$$c_m(\lambda) = \frac{\pi}{2}\{b_m'(R,\lambda)j_m(R,\lambda) - j_m'(R,\lambda)b_m(R,\lambda)\}, \tag{4.13a}$$

$$d_m(\lambda) = -\frac{\pi}{2}\{a_m'(R,\lambda)j_m(R,\lambda) - j_m'(R,\lambda)a_m(R,\lambda)\}. \tag{4.13b}$$

It is easy to see that $j_m(r,\lambda)$ and its derivative will be bounded as $r \to \infty$, and so, condition (4.3) will be satisfied for all $\lambda > d^2$.

Let us look at the expression of (2.2) for $\lambda > d^2$ and with $v(r)$ given by (4.2). Notice that if $d^2 < \lambda < k^2 n_0^2$, then (2.2) will be oscillatory in z (recall that $k^2\beta^2 = k^2 n_0^2 - \lambda$). In this case we will say that (2.2) is a *radiation mode*. On the other hand, if $\lambda > k^2 n_0^2$, then β becomes imaginary. Depending on the sign of $\mathfrak{Im}\beta$ we will have exponential decay in one of the directions $z \to -\infty$, $z \to \infty$, and exponential growth in the other one. For $\lambda > k^2 n_0^2$, (2.2) will be called an *evanescent mode*.

5. The Theory of Eigenvalue Problems

Consider the eigenvalue problem

$$w'' + \{l - Q(r)\}w = 0, \quad r \in (0,\infty), \tag{5.1}$$

where $l \in \mathbb{C}$. Assume that Q is integrable over any compact subset of $(0,\infty)$ (in [2] and [8] the theory is developed only for continuous functions Q, but it is mentioned in a footnote on page 224 of [2] that it suffices for Q to be as we assume above). Let $0 < R < \infty$ be an arbitrary but fixed number. Let $\varphi(r,l)$ and $\theta(r,l)$ be the solutions of (5.1) with the boundary conditions

$$\begin{cases} \varphi(R,l) = 0, & \varphi'(R,l) = -1, \\ \theta(R,l) = 1, & \theta'(R,l) = 0. \end{cases} \tag{5.2}$$

Since (5.1) has an analytic dependence on the parameter l, the solutions $\theta(r,l)$ and $\varphi(r,l)$ will be analytic functions of l for r fixed.

Any solution to (5.1) linearly independent of φ can be represented, up to a constant multiple, in the form

$$\psi = \theta + M\varphi, \tag{5.3}$$

with $M \in \mathbb{C}$. Let $\tau \in \mathbb{R}$ and $0 < t < \infty$. Look for a solution of (5.1) of the form (5.3) to satisfy the boundary condition

$$\cos\tau\psi(t,l) + \sin\tau\psi'(t,l) = 0.$$

It is a direct calculation to check that we need

$$M = M(l) = -\frac{\theta(t,l)\cos\tau + \theta'(t,l)\sin\tau}{\varphi(t,l)\cos\tau + \varphi'(t,l)\sin\tau}. \tag{5.4}$$

Let $\mathbb{C}_+$ be the open upper complex-half-plane, $\mathbb{C}_+ = \{l : \mathfrak{Im}\, l > 0\}$. As shown in Chapter 2 of [8] the following results hold: as $t \to 0$, $M(l)$ converges uniformly on compact subsets of $\mathbb{C}_+$ to a function $M_0(l)$ analytic on $\mathbb{C}_+$. Moreover, the function

$$\psi_0(r,l) = \theta(r,l) + M_0(l)\,\varphi(r,l), \quad l \in \mathbb{C}_+, \tag{5.5}$$

is in $L^2(0,R)$, and one has

$$\int_0^R |\psi_0(r,l)|^2\,dr = \frac{\mathfrak{Im}\,M_0(l)}{\mathfrak{Im}\,l}. \tag{5.6}$$

As $t \to \infty$, $M(l)$ converges uniformly on compact subsets of $\mathbb{C}_+$ to an analytic function $M_\infty(l)$ on $\mathbb{C}_+$, and if

$$\psi_\infty(r,l) = \theta(r,l) + M_\infty(l)\,\varphi(r,l), \quad l \in \mathbb{C}_+, \tag{5.7}$$

then $\psi_\infty(r,l) \in L^2(R,\infty)$ and

$$\int_R^\infty |\psi_\infty(r,l)|^2\,dr = -\frac{\mathfrak{Im}\,M_\infty(l)}{\mathfrak{Im}\,l}. \tag{5.8}$$

Note that the obtained M_0, M_∞, ψ_0, ψ_∞ depend on the parameter $\tau \in \mathbb{R}$. Thus, possibly these quantities, and therefore the representation given below, in Theorem 5.1, will not be unique. This might be true in general, but in our concrete case, given by (2.5), these quantities will turn out to be unique, as we will see from Lemma 6.1. So then the transform we are looking for (which is calculated in Theorems 7.1 and 7.2) will be unique.

In Section 9.5 of [2] and Chapter 3 of [8] it is proved that for any $\lambda \in \mathbb{R}$ the following limits exist

$$\xi(\lambda) = \lim_{\delta\to 0^+}\int_0^\lambda -\mathfrak{Im}\frac{1}{M_0(s+i\delta) - M_\infty(s+i\delta)}\,ds, \tag{5.9a}$$

$$\eta(\lambda) = \lim_{\delta \to 0^+} \int_0^\lambda -\mathfrak{Im} \frac{M_0(s+i\delta)}{M_0(s+i\delta) - M_\infty(s+i\delta)} \, ds, \tag{5.9b}$$

$$\zeta(\lambda) = \lim_{\delta \to 0^+} \int_0^\lambda -\mathfrak{Im} \frac{M_0(s+i\delta) M_\infty(s+i\delta)}{M_0(s+i\delta) - M_\infty(s+i\delta)} \, ds. \tag{5.9c}$$

It is shown there that the functions ξ and ζ are non-decreasing, and that η is with bounded variation. In addition, for any $\lambda_0 < \lambda_1$ real numbers,

$$\{\eta(\lambda_1) - \eta(\lambda_0)\}^2 \leq \{\xi(\lambda_1) - \xi(\lambda_0)\}\{\zeta(\lambda_1) - \zeta(\lambda_0)\}, \tag{5.10}$$

as stated on page 252 of [2] (with a different notation). Then, equalities (3.1.8), (3.1.9), (3.1.10) from [8] give us an expansion formula for a function $g \in L^2(0, \infty)$ in terms of $\theta(r, l)$, $\varphi(r, l)$ and the functions ξ, η and ζ. The same result is proved in [2] at Theorem 5.2. In this reference the representation result is stated more rigorously, so we will prefer it over.[8] To state the result we need some notations.

Denote $\rho = (\xi, \eta, \zeta)$. For any vector $\Gamma = (\Gamma_1, \Gamma_2)$, where $\Gamma_1, \Gamma_2 : \mathbb{R} \to \mathbb{C}$, let

$$||\Gamma||^2 = \int_{-\infty}^{\infty} |\Gamma_1(\lambda)|^2 \, d\xi + 2\mathfrak{Re}\{\Gamma_1(\lambda) \, \bar{\Gamma}_2(\lambda)\} \, d\eta + |\Gamma_2(\lambda)|^2 \, d\zeta. \tag{5.11}$$

The fact that ξ and η are non-decreasing, together with (5.10), gives us that $||\Gamma||^2 \geq 0$. It is easy to check that $|| \cdot ||$ is a semi-norm. Denote by $L^2(\rho)$ the space of all $\Gamma = (\Gamma_1, \Gamma_2)$ such that $||\Gamma|| < \infty$. This is then the statement of Theorem 5.2 from.[2]

Theorem 5.1 *If $g \in L^2(0, \infty)$, the vector $\Gamma = (\Gamma_1, \Gamma_2)$, where*

$$\Gamma_1(\lambda) = \int_0^\infty \theta(r, \lambda) g(r) \, dr, \quad \Gamma_2(\lambda) = \int_0^\infty \varphi(r, \lambda) g(r) \, dr,$$

converges in $L^2(\rho)$, that is, there exists $\Gamma \in L^2(\rho)$ such that

$$||\Gamma - \Gamma^{cd}|| \to 0 \text{ as } c \to 0, \, d \to \infty,$$

where for $0 < c < d < \infty$

$$\Gamma_1^{cd}(\lambda) = \int_c^d \theta(r, \lambda) g(r) \, dr, \quad \Gamma_2^{cd}(\lambda) = \int_c^d \varphi(r, \lambda) g(r) \, dr. \tag{5.12}$$

The expansion

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \big\{\theta(r, \lambda) \Gamma_1(\lambda) \, d\xi(\lambda) + \theta(r, \lambda) \Gamma_2(\lambda) \, d\eta(\lambda) \\ + \varphi(r, \lambda) \Gamma_1(\lambda) \, d\eta(\lambda) + \varphi(r, \lambda) \Gamma_2(\lambda) \, d\zeta(\lambda)\big\}$$

holds, with the latter integral convergent in $L^2(0,\infty)$*, that is,* $g^{\sigma\tau} \to g$ *in* $L^2(0,\infty)$ *as* $\sigma \to -\infty$, $\tau \to \infty$*, where for* $-\infty < \tau < \sigma < \infty$

$$g^{\sigma\tau}(r) = \frac{1}{\pi}\int_{\sigma}^{\tau} \{\theta(r,\lambda)\Gamma_1(\lambda)\,d\xi(\lambda) + \theta(r,\lambda)\Gamma_2(\lambda)\,d\eta(\lambda) + \varphi(r,\lambda)\Gamma_1(\lambda)\,d\eta(\lambda) + \varphi(r,\lambda)\Gamma_2(\lambda)\,d\zeta(\lambda)\}. \quad (5.13)$$

We have the Parseval identity

$$\int_0^\infty |g(r)|^2\,dr = \frac{1}{\pi}||\Gamma||^2. \quad (5.14)$$

6. Computing the Measures

In the sections to follow we will apply the results of Section 5 to our particular eigenvalue equation, given by (2.5). Thus, for the function $Q(r)$ in (5.1) we will have the expression

$$Q(r) = q(r) - \frac{m^2 - 1/4}{r^2}, \quad r \in (0,\infty).$$

In this section we will calculate the measures given in (5.9a), (5.9b) and (5.9c).

Lemma 6.1 *Let* $M_0^m(l)$, $M_\infty^m(l), \psi_0^m(r,l)$, $\psi_\infty^m(r,l)$ $(l \in \mathbb{C}_+ = \{l : \mathfrak{Im}\, l > 0\})$ *be the quantities defined in Section* 5 *for equation* (2.5) *(we use the superscript* m *to emphasize their dependence on* $m \in \mathbb{Z}$*). Let* $j_m(r,l)$ $w_m(r,l)$ *be the solutions of* (2.5) *defined respectively in Lemma 3.1 and by* (3.17)*. Then,*

$$M_0^m(l) = -\frac{j_m'(R,l)}{j_m(R,l)}, \quad M_\infty^m(l) = -\frac{w_m'(R,l)}{w_m(R,l)}, \quad (6.1)$$

and

$$\psi_0^m(r,l) = \frac{j_m(r,l)}{j_m(R,l)}, \quad \psi_\infty^m(r,l) = \frac{w_m(r,l)}{w_m(R,l)}. \quad (6.2)$$

Proof. It is easy to note that $j_m(r,l)$ and $y_m(r,l)$ defined in Lemma 3.1 are linearly independent solutions of (2.5), thus any other solution will be a linear combination of these two. Then $\theta(r,l)$ and $\phi(r,l)$ defined in Section 5 can be represented as

$$\begin{cases} \theta(r,l) = \alpha(l)j_m(r,l) + \beta(l)y_m(r,l), \\ \varphi(r,l) = \gamma(l)j_m(r,l) + \delta(l)y_m(r,l), \end{cases} \quad (6.3)$$

for some coefficients α, β, γ, δ. Let

$$\Delta(l) = j_m'(R,l)y_m(R,l) - j_m(R,l)y_m'(R,l). \quad (6.4)$$

Being the Wronskian of two linearly independent solutions, $\Delta(l)$ is non-zero. Using the boundary conditions (5.2) it is easy to find that

$$\alpha = -\frac{y_m'(R,l)}{\Delta(l)},\ \beta = \frac{j_m'(R,l)}{\Delta(l)},\ \gamma = -\frac{y_m(R,l)}{\Delta(l)},\ \delta = \frac{j_m(R,l)}{\Delta(l)}. \tag{6.5}$$

From (6.3) and (5.4) it follows that

$$M(l) = -\frac{\{\alpha j_m(t,l) + \beta y_m(t,l)\}\cos\tau + \{\alpha j_m'(t,l) + \beta y_m'(t,l)\}\sin\tau}{\{\gamma j_m(t,l) + \delta y_m(t,l)\}\cos\tau + \{\gamma j_m'(t,l) + \delta y_m'(t,l)\}\sin\tau}.$$

For $t \to 0$ the formulas (3.1), (3.2a) and (3.2b) tell us that the term $y_m(t,l)$ will dominate $j_m(t,l)$ and $j_m'(t,l)$, while $y_m'(t,l)$ will dominate $y_m(t,l)$. Then,

$$M_0^m(l) = \lim_{t\to 0} M(l) = -\frac{\beta}{\delta}.$$

By applying (6.5) we obtain the first equality in (6.1). To get the first equality in (6.2) use definition (5.5) of $\psi_0^m(r,l)$ and equalities (6.3) to (6.5).

The quantities $M_\infty^m(l)$ and $\psi_\infty^m(r,l)$ are calculated in exactly the same way. One needs to express $\varphi(r,l)$ and $\theta(r,l)$ in terms of $w_m(r,l)$ and $x_m(r,l)$, then put $t \to \infty$ in (5.4) and use the properties of $w_m(r,l)$, $x_m(r,l)$, and their derivatives shown in Lemma 3.4. The lemma is proved. □

The next lemma will study the properties of the functions $M_0^m(l)$, $M_\infty^m(l)$, and $M_0^m(l) - M_\infty^m(l)$. Here we will set some notation and write some formulas which will be used in the lemma. By λ we will denote a real variable. Let $w_m(r,\lambda)$, $k_m(r,\lambda)$, $a_m(r,\lambda)$, and $b_m(r,\lambda)$ be the functions defined by (3.17), (4.5) and (4.11). The following equalities hold,

$$H_m^{(1)}(iz) = \frac{2}{i\pi}e^{-im/2}K_m(z), \quad -\pi < \arg z \le \pi/2$$

and

$$H_m^{(1)}(z) = J_m(z) + iY_m(z), \quad -\pi < \arg z < \pi, \tag{6.6}$$

(formulas (9.6.4) and (9.1.3) from.[1]) We deduce

$$w_m(r,\lambda) = \frac{2}{i\pi}e^{-im/2}k_m(r,\lambda), \quad \lambda < d^2, \tag{6.7a}$$

and

$$w_m(r,\lambda) = a_m(r,\lambda) + ib_m(r,\lambda) \quad \lambda > d^2. \tag{6.7b}$$

Lemma 6.2 *$M_0^m(l)$ is meromorphic across all the complex plane, while $M_\infty^m(l)$ ($l \in \mathbb{C}_+$) extends continuously to the real axis. $M_0^m(\lambda) - M_\infty^m(\lambda)$ is real or infinite for $\lambda < d^2$, and it has a finite number of zeros on the real axis, all in the interval $(0, d^2]$.*

Proof. As stated in Lemma 3.1, $j_m(R,l)$ and $j'_m(R,l)$ will be analytic functions of $l \in \mathbb{C}$. Then $M_0^m(l)$, being obtained as their ratio, will be a meromorphic function.

The function $w_m(r,l)$ was defined in Lemma 3.4. It was proved there that $w_m(r,l)$ is defined and analytic for all $l \in \{-\pi/2 < \arg(l-d^2) < 3\pi/2\}$. In particular $w_m(r,\lambda)$ is defined for all real $\lambda \neq d^2$. To prove that $M_\infty^m(l)$ extends continuously to the real axis, it suffices to show that its denominator, $w_m(r,\lambda)$, is not zero for $\lambda \in \mathbb{R}\backslash\{d^2\}$ and that $\lim_{l\to d^2} M_\infty^m(l)$ exists.

If $\lambda < d^2$, then $w_m(r,\lambda) \neq 0$ because of (4.5) and (6.7a), since the Bessel function $K_m(s)$ takes real strictly positive values for $s > 0$ (as it follows from section (9.6.1) of.[1]) If $\lambda > d^2$, then $w_m(r,\lambda) \neq 0$ because of (4.11) and (6.7b), since for $s > 0$ the Bessel functions $J_m(s)$ and $Y_m(s)$ take real values and cannot be zero at the same time.

Show that $\lim_{l\to d^2} M_\infty^m(l)$ exists. If $m \geq 0$, then according to (9.1.8) and (9.1.9) from, [1] we have that for $z \in \mathbb{C}$, $z \to 0$

$$H_m^{(1)}(z) \sim -(1/\pi)(m-1)!(z/2)^{-m}, \text{ for } m \geq 1,$$

and

$$H_m^{(1)}(z) \sim (-2i/\pi)\ln z, \text{ for } m = 0.$$

We can extend these to $m < 0$, by using (6.6) together with

$$J_{-m}(z) = (-1)^m J_m(z), Y_{-m}(z) = (-1)^m Y_m(z), \quad m \in \mathbb{Z}$$

(formula (9.1.5) from.[1]) One can obtain the behavior of the derivative of $H_m^{(1)}(z)$ as $z \to 0$ by formally differentiating the above. Set $z = R\sqrt{l-d^2}$. We deduce that for all $m \in \mathbb{Z}$

$$\lim_{l\to d^2} M_\infty^m(l) = \frac{|m|-1/2}{R}. \tag{6.8}$$

Note that $M_0^m(\lambda)$ is real or infinite for $\lambda < d^2$, being the quotient of $-j'_m(R,\lambda)$ and $j_m(R,\lambda)$ both of which are real according to Lemma 3.1. We have that $M_\infty^m(\lambda)$ is real for $\lambda < d^2$, that follows from (6.7a) and by using again the properties of the function $K_m(s)$. Then, $M_0^m(\lambda) - M_\infty^m(\lambda)$ is real or infinite for $\lambda < d^2$.

Let us show the last part of this lemma, the fact that $M_0^m(\lambda) - M_\infty^m(\lambda)$ finite number of zeros on the real axis, all in the interval $(0, d^2]$. Let first prove that this function can have no zeros for $\lambda \leq 0$. From (6.7a) we have

$$M_0^m(\lambda) - M_\infty^m(\lambda) = -\frac{j'_m(R,\lambda)}{j_m(R,\lambda)} + \frac{k'_m(R,\lambda)}{k_m(R,\lambda)} = -\frac{D_m(R,\lambda)}{j_m(R,\lambda)k_m(R,\lambda)},$$

where

$$D_m(r,\lambda) = j'_m(r,\lambda)k_m(r,\lambda) - j_m(r,\lambda)k'_m(r,\lambda).$$

Assume for some λ, $M_0^m(\lambda) - M_\infty^m(\lambda) = 0$. Then $D_m(R,\lambda) = 0$. We will prove that is false. First note that $D_m(r,\lambda) = D_m(R,\lambda)$ for $r \geq R$. Indeed, $j_m(r,\lambda)$ and $k_m(r,\lambda)$ will satisfy (2.5) (for $j_m(r,\lambda)$ this follows by its definition, for $k_m(r,\lambda)$ it follows from the observation that according to (6.7a), $k_m(r,\lambda)$ is a constant times $w_m(r,\lambda)$ which, as defined by (3.17), is a solution of (2.5) for $r \geq R$). Then it is easy to check that

$$D'_m(r,\lambda) = j''_m(r,\lambda)k_m(r,\lambda) - j_m(r,\lambda)k''_m(r,\lambda) = 0,$$

so $D_m(r,\lambda)$ is constant in r. Second, note that for r large enough $D_m(r,\lambda) < 0$. Indeed, for $r > 0$, $j_m(r,\lambda) > 0$, $j'_m(r,\lambda) > 0$ by Lemma 3.2, and $k_m(r,\lambda) > 0$ by the properties of Bessel functions. Also, for r sufficiently large we have $k'_m(r,\lambda) < 0$, this is a consequence of (4.8). We infer that for r large enough $D_m(r,\lambda) < 0$. These two observations imply $D_m(R,\lambda) < 0$, therefore $D_m(R,\lambda)$ is non-zero, and $M_0^m(\lambda) - M_\infty^m(\lambda) \neq 0$.

Now let $0 < \lambda \leq d^2$. We showed that $M_0^m(l)$ is meromorphic on the whole complex plane. The function $w_m(r,l)$, as defined by (3.17), is analytic on $\{-\pi/2 < \arg(l-d^2) < 3\pi/2\}$, in particular it is analytic on the set of l such that $\mathfrak{Re}\, l < d^2$. Then $M_\infty^m(l) = -w'_m(R,l)/w_m(R,l)$ is meromorphic on the same region, and so is $M_0^m(l) - M_\infty^m(l)$. Therefore, it can have only a discrete number of zeros on the interval $(0,d^2)$.

Assuming that the number of zeros is infinite, their only possible accumulation point is $\lambda = d^2$. So, there would exist a sequence $\lambda_n < d^2$, $n \geq 1$, with $\lim \lambda_n = d^2$ and $M_0^m(\lambda_n) - M_\infty^m(\lambda_n) = 0$ for all $n \geq 1$. According to formulas (9.1.3), (9.1.10) and (9.1.11) from, [1] the Hankel function $H^{(1)}(z)$ will have the representation

$$H_m^{(1)}(z) = f_1(z) + f_2(z)\ln z,$$

with $f_1(z)$ and $f_2(z)$ meromorphic functions of $z \in \mathbb{C}$. Then, if we recall definition (3.17) of $w_m(r,l)$, one can calculate that we will have the representation

$$M_0^m(l) - M_\infty^m(l) = \frac{g_1(t) + g_2(t)\ln t}{g_3(t) + g_4(t)\ln t},$$

with g_1, g_2, g_3 and g_4 meromorphic functions on the whole complex plane, and $t = \sqrt{l - d^2} \in \mathbb{C}$. We infer

$$g_1(t_n) + g_2(t_n)\ln t_n = 0,$$

for all $n \geq 1$, where $t_n = \sqrt{\lambda_n - d^2}$. Two cases are possible. If $g_2(t_n)$ is zero for infinitely many n, this implies $g_1(t_n) = 0$ at the same points. Then the meromorphic functions $g_1(t)$ and $g_2(t)$ are identically zero, and so is $M_0^m(l) - M_\infty^m(l)$, obtaining a contradiction. Otherwise, if $g_2(t_n)$ is zero only for finitely many n, we can write

$$\ln t_n = -\frac{g_1(t_n)}{g_2(t_n)}, \quad n \geq n_0.$$

On the right-hand side we have a meromorphic function. As $n \to \infty$ we have $t_n \to 0$, thus, for some integer p, there follows

$$\ln|t_n| = O(|t_n|^p),$$

which is clearly impossible. This contradiction shows that $M_0^m(\lambda) - M_\infty^m(\lambda)$ can have only finitely many zeros on $[0, d^2)$. Another possible zero could be at $\lambda = d^2$.

Lastly, show that $M_0^m(\lambda) - M_\infty^m(\lambda)$ is never zero on (d^2, ∞). According to (6.1) and (6.7b),

$$M_\infty^m(\lambda) = -\frac{a'_m(R,\lambda) + ib'_m(R,\lambda)}{a_m(R,\lambda) + ib_m(R,\lambda)},$$

and so,

$$\begin{aligned} M_0^m(\lambda) - M_\infty^m(\lambda) &= -\frac{j'_m}{j_m} + \frac{a'_m + ib'_m}{a_m + ib_m} \\ &= -\frac{(j'_m a_m - a'_m j_m) + i(j'_m b_m - b'_m j_m)}{j_m(a_m + ib_m)} \end{aligned} \tag{6.9}$$

(the arguments (R, λ) were omitted for simplicity). If we assume that for some $\lambda > d^2$, $M_0^m(\lambda) - M_\infty^m(\lambda) = 0$, this implies

$$\begin{aligned} j'_m(R,\lambda)a_m(R,\lambda) - a'_m(R,\lambda)j_m(R,\lambda) = 0, \\ j'_m(R,\lambda)b_m(R,\lambda) - b'_m(R,\lambda)j_m(R,\lambda) = 0 \end{aligned}$$

(since these quantities are real, as it follows from Lemma 3.2 and (4.11)). The numbers $j_m(R,\lambda)$ and $j'_m(R,\lambda)$ cannot be both zero, since $j_m(r,\lambda)$ is a non-zero solution of the second order differential equation (2.5). In this case the vectors $(a_m(R,\lambda), a'_m(R,\lambda))$ and $(b_m(R,\lambda), b'_m(R,\lambda))$ must be linearly dependent. That cannot be since the functions $a_m(r,\lambda)$ and $b_m(r,\lambda)$ $(r \geq R)$ are also solutions of (2.5), and they are linearly independent. Thus $M_0^m(\lambda) - M_\infty^m(\lambda) \neq 0$. This finishes the lemma. □

Theorem 6.1 *Let ξ_m, η_m and ζ_m be the functions defined by* (5.9a), (5.9b) *and* (5.9c)*. There exists a non-decreasing function $\chi_m : \mathbb{R} \to \mathbb{R}$ such that the following measures are equal*

$$\begin{aligned} d\xi_m(\lambda) &= j_m(R,\lambda)^2 \, d\chi_m(\lambda), \\ d\eta_m(\lambda) &= -j'_m(R,\lambda)j_m(R,\lambda) \, d\chi_m(\lambda), \\ d\zeta_m(\lambda) &= j'_m(R,\lambda)^2 \, d\chi_m(\lambda). \end{aligned} \tag{6.10}$$

The function χ_m is identically zero for $\lambda \in (-\infty, 0]$, is piecewise constant for $\lambda \in (0, d^2)$ where it has a finite number of discontinuities, and is continuous for $\lambda \in (d^2, \infty)$.

Proof. Denote

$$M(l) = -\frac{1}{M_0^m(l) - M_\infty^m(l)}, \quad l \in \mathbb{C}_+.$$

According to (5.9a), for any $\lambda_0 < \lambda_1$ real numbers

$$\xi_m(\lambda_1) - \xi_m(\lambda_0) = \lim_{\delta \to 0^+} \int_{\lambda_0}^{\lambda_1} \mathfrak{Im}\, M(s + i\delta) \, ds.$$

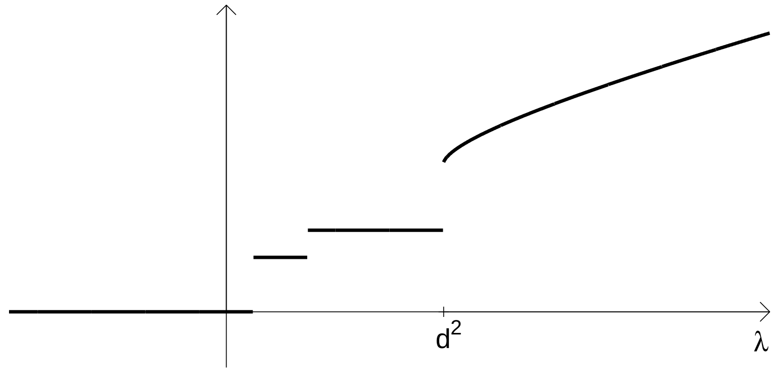

Fig. 2. The function $\chi_m(\lambda)$.

In particular, if $M(l)$ extends continuously to the interval $[\lambda_0, \lambda_1]$, then by using Lebesgue's theorem of dominant convergence it is easy to show that

$$\xi_m(\lambda_1) - \xi_m(\lambda_0) = \int_{\lambda_0}^{\lambda_1} \mathfrak{Im}\, M(s)\, ds. \tag{6.11}$$

The same kind of reasoning clearly holds for η_m and ζ_m.

As it follows from Lemma 6.2, if $\lambda \leq 0$ then $M_0^m(\lambda) - M_\infty^m(\lambda)$ is real or infinite, and non-zero. By applying (6.11) we find that for any $\lambda_1 < \lambda_2 < 0$, $\xi_m(\lambda_1) - \xi_m(\lambda_0) = 0$. From (5.9a) we have that $\xi_m(0) = 0$, and thus $\xi_m(\lambda) = 0$ for all $\lambda \leq 0$. In the same fashion one obtains that for $\lambda \leq 0$, $\eta_m(\lambda) = 0$ and $\zeta_m(\lambda) = 0$. Set $\chi_m(\lambda) = 0$ for $\lambda \leq 0$, and (6.10) will hold.

Let $\lambda_1^m < \lambda_2^m < \cdots < \lambda_{P_m}^m$ be the points in the interval $(0, d^2]$ where, according to Lemma 6.2, $M_0^m(\lambda) - M_\infty^m(\lambda) = 0$. We can use Lemma 6.2 and (6.11) to deduce that ξ_m, η_m and ζ_m are constant on each of the intervals making up $(0, d^2]\backslash\{\lambda_1^m, \lambda_2^m, \ldots, \lambda_{P_m}^m\}$. Set χ_m to be constant on each of these intervals. At each of the points $\lambda_1^m, \lambda_2^m, \ldots, \lambda_{P_m}^m$ the functions ξ_m, η_m and ζ_m could have a jump. To show (6.10) on $(0, d^2]$ we need to find a relationship between the jumps of these functions. We will return to this shortly.

The remaining case, $\lambda > d^2$, is treated similarly. Equation (6.9) in the proof of Lemma 6.2 gives an expression for $M_0^m(\lambda) - M_\infty^m(\lambda)$ on this interval (with the notation from (4.11)). By using the fact that the quantities a_m, b_m and j_m together with their derivatives are real, we can calculate

$$\begin{aligned}\mathfrak{Im}\, M(\lambda) &= \mathfrak{Im}\left\{\frac{j_m(a_m + ib_m)}{(j_m' a_m - a_m' j_m) + i(j_m' b_m - b_m' j_m)}\right\}\\ &= \frac{j_m^2(b_m' a_m - a_m' b_m)}{(j_m' a_m - a_m' j_m)^2 + (j_m' b_m - b_m' j_m)^2}.\end{aligned}$$

Use (4.12) to simplify the numerator of this fraction. Apply (6.11). We get that for any $d^2 < \lambda_0 < \lambda_1$,

$$\xi_m(\lambda_1) - \xi_m(\lambda_0) = \frac{\pi}{2} \int_{\lambda_0}^{\lambda_1} \frac{j_m(R,\lambda)^2}{c_m(\lambda)^2 + d_m(\lambda)^2} \, d\lambda,$$

where $c_m(\lambda)$ and $d_m(\lambda)$ are defined by (4.13a) and (4.13b). So we have

$$d\xi_m(\lambda) = \frac{\pi}{2} \frac{j_m(R,\lambda)^2 \, d\lambda}{c_m(\lambda)^2 + d_m(\lambda)^2}.$$

We want (6.10) to hold. Define $\chi_m(\lambda)$ for $\lambda > d^2$ such that

$$d\chi_m(\lambda) = \frac{\pi}{2} \frac{d\lambda}{c_m(\lambda)^2 + d_m(\lambda)^2},$$

then the first of the three identities (6.10) is valid. It is easy to repeat the same calculation for η_m and ζ_m and show that for $\lambda > d^2$ the other two identities in (6.10) hold.

Now we will return to what is the longest part of the proof, the study of what happens at the points $\lambda \in (0, d^2]$ where $M_0^m(\lambda) - M_\infty^m(\lambda) = 0$. Let λ_0 be such a point. An immediate observation is that $M_0^m(\lambda_0)$ is finite and $j_m(R,\lambda_0) \neq 0$. Indeed, it was proved in Lemma 6.2 that $M_\infty^m(\lambda)$ is finite for λ real. So, $M_0^m(\lambda_0)$ which equals $M_\infty^m(\lambda_0)$ is finite. Then, since $M_0^m(\lambda_0) = -j_m'(R,\lambda_0)/j_m(R,\lambda_0)$ and because $j_m'(R,\lambda_0)$ cannot become zero simultaneously with $j_m(R,\lambda_0)$ ($j_m(r,l)$ is a solution of (2.5)), we deduce $j_m(R,\lambda_0) \neq 0$.

With this observation in hand, in order to prove that (6.10) holds at λ_0 one needs to show that

$$d\eta_m(\lambda_0) = M_0^m(\lambda_0) \, d\xi_m(\lambda_0), \quad d\zeta_m(\lambda_0) = M_0^m(\lambda_0)^2 \, d\xi_m(\lambda_0), \tag{6.12}$$

and then define $d\chi_m(\lambda_0) = d\eta_m(\lambda_0)/j_m(R,\lambda_0)^2$.

Let r_0 be the jump of ξ_m at λ_0. Recall, ξ_m was defined by (5.9a), so,

$$r_0 = \lim_{\varepsilon \to 0^+} \lim_{\delta \to 0^+} \int_{\lambda_0 - \varepsilon}^{\lambda_0 + \varepsilon} -\mathfrak{Im} \frac{1}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \, ds.$$

By using (5.9b), the first equality in (6.12) can be written as

$$\lim_{\varepsilon \to 0^+} \lim_{\delta \to 0^+} \int_{\lambda_0 - \varepsilon}^{\lambda_0 + \varepsilon} -\mathfrak{Im} \frac{M_0^m(s+i\delta)}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \, ds = M_0^m(\lambda_0) \, r_0.$$

$M_0^m(l)$ is analytic around λ_0. Therefore, in a neighborhood of λ_0

$$M_0^m(s+i\delta) = M_0^m(\lambda_0) + \{M_0^m(s) - M_0^m(\lambda_0)\} + i\delta H(s+i\delta),$$

with H a continuous function around λ_0. Substitute this above. By using the fact that $M_0^m(\lambda_0)$ is real and the definition of r_0, we get the equivalent equality

$$M_0^m(\lambda_0)\, r_0 + \lim_{\varepsilon\to 0^+}\lim_{\delta\to 0^+}\int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} -\mathfrak{Im}\frac{\{M_0^m(s)-M_0^m(\lambda_0)\}+i\delta H(s+i\delta)}{M_0^m(s+i\delta)-M_\infty^m(s+i\delta)}\,ds = M_0^m(\lambda_0)\, r_0, \quad (6.13)$$

so we have to prove that the limit of the integral on the left-hand side of (6.13) is zero. To do this, we will need additional information about $M_0^m - M_\infty^m$.

Let $l = s + i\delta$ ($\delta > 0$). Recall that formulas (5.6) and (5.8) hold, where ψ_0^m and ψ_∞^m are defined in (5.5) and (5.7) and an expression for them is given by (6.2). Add equalities (5.6) and (5.8). We obtain

$$\frac{1}{\delta}\mathfrak{Im}\{M_0^m(l)-M_\infty^m(l)\} = \int_0^R \left|\frac{j_m(r,l)}{j_m(R,l)}\right|^2 dr + \int_R^\infty \left|\frac{w_m(r,l)}{w_m(R,l)}\right|^2 dr.$$

This equality gives us two things. First, that $\mathfrak{Im}\{M_0^m(l) - M_\infty^m(l)\} > 0$, therefore,

$$-\mathfrak{Im}\frac{1}{M_0^m(l)-M_\infty^m(l)} > 0. \quad (6.14)$$

Second, for l close to λ_0, the quantity $\delta^{-1}\mathfrak{Im}\{M_0^m(l) - M_\infty^m(l)\}$ is bounded from below by a strictly positive number, say ω_∞^{-1}. Then $\delta^{-1}|M_0^m(l) - M_\infty^m(l)|$ is bounded below by the same number and therefore,

$$\frac{\delta}{|M_0^m(l)-M_\infty^m(l)|} \le \omega_\infty. \quad (6.15)$$

The integral in (6.13) can be written as

$$\int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} -\mathfrak{Im}\frac{M_0^m(s)-M_0^m(\lambda_0)}{M_0^m(s+i\delta)-M_\infty^m(s+i\delta)}\,ds + \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} -\mathfrak{Im}\frac{i\delta H(s+i\delta)}{M_0^m(s+i\delta)-M_\infty^m(s+i\delta)}\,ds. \quad (6.16)$$

Denote $c_\varepsilon = \sup_{|s-\lambda_0|\le\varepsilon} |M_0^m(s) - M_0^m(\lambda_0)|$. Since M_0^m is continuous in a neighborhood of λ_0, we will have $c_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us estimate the first integral from

(6.16).

$$\left| \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} -\mathfrak{Im} \frac{M_0^m(s) - M_0^m(\lambda_0)}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \right| ds \leq \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} \left| \mathfrak{Im} \frac{M_0^m(s) - M_0^m(\lambda_0)}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \right| ds$$
$$= \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} |M_0^m(s) - M_0^m(\lambda_0)| \left| \mathfrak{Im} \frac{1}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \right| ds$$
$$\leq c_\varepsilon \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} \left| \mathfrak{Im} \frac{1}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \right| ds$$
$$= c_\varepsilon \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} -\mathfrak{Im} \frac{1}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \, ds$$
$$= c_\varepsilon[\xi_m(\lambda_0+\varepsilon) - \xi_m(\lambda_0-\varepsilon)].$$

In deriving this we used the fact that $M_0^m(s) - M_0^m(\lambda_0)$ is real, together with (6.14) and (5.9a). Clearly as $\varepsilon \to 0$, the integral goes to zero.

Now estimate the second integral in the sum (6.16). Let H_∞ be an upper bound of $|H(s+i\delta)|$ for $l = s+i\delta$ in a neighborhood of λ_0. Inequality (6.15) implies that as $\varepsilon \to 0$,

$$\left| \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} -\mathfrak{Im} \frac{i\delta H(s+i\delta)}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \, ds \right| \leq \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} H_\infty \omega_\infty \, ds = 2\varepsilon H_\infty \omega_\infty \to 0.$$

This proves the first equality in (6.12).

Let us prove the second equality in (6.12). Recall that ζ_m is given by (5.9c). The above approach does not apply immediately, since unlike $M_0^m(l)$ (the numerator in (5.9b)), the function $M_0^m(l)M_\infty^m(l)$ (the numerator in (5.9c)) will not be analytic around λ_0 if $\lambda_0 = d^2$ (since as seen from Lemma 3.4, $w_m(r,l)$ is not defined in a neighborhood of $l = d^2$). The idea is then to use the equality

$$\frac{xy}{x-y} = \frac{x^2}{x-y} - x,$$

to write the jump of ζ_m at λ_0 as

$$\lim_{\varepsilon\to 0^+} \lim_{\delta\to 0^+} \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} -\mathfrak{Im} \frac{M_0^m(s+i\delta)^2}{M_0^m(s+i\delta) - M_\infty^m(s+i\delta)} \, ds + \lim_{\varepsilon\to 0^+} \lim_{\delta\to 0^+} \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} \mathfrak{Im}\, M_0^m(s+i\delta) \, ds.$$

The limit of the second integral in the sum will be zero, since as argued above, M_0^m will be finite at λ_0 (and therefore, around λ_0) and thus the quantity inside the integral is bounded. For the first integral in the sum we can proceed in the same way we calculated the jump of η_m. This finishes the proof of (6.10).

Finally, we need to justify the claim that χ_m is a non-decreasing function. From (6.10) we have

$$d\xi_m(\lambda) + d\zeta_m(\lambda) = \{j_m(R,\lambda)^2 + j_m'(R,\lambda)^2\}d\chi_m(\lambda).$$

The left-hand side of this is non-negative measure, since by theory ξ_m and ζ_m are non-decreasing. The number $j_m(R,\lambda)^2 + j_m'(R,\lambda)^2$ is strictly positive, as $j_m(R,\lambda)$ and $j_m'(R,\lambda)$ cannot be both zero, $j_m(r,\lambda)$ being a non-zero solution to the second order differential equation (2.5). Then we get that $d\chi_m(\lambda)$ is a non-negative measure, which shows that χ_m is non-decreasing. □

Corollary 6.1 *Let $\lambda \in (0, d^2]$ be a discontinuity point for χ_m. Then, if $\lambda < d^2$, the following hold:*

$$\frac{j_m'(R,\lambda)}{j_m(R,\lambda)} = \frac{k_m'(R,\lambda)}{k_m(R,\lambda)}, \tag{6.17a}$$

and

$$j_m(r,\lambda) = \frac{j_m(R,\lambda)}{k_m(R,\lambda)} k_m(r,\lambda), \quad r \geq R. \tag{6.17b}$$

While for $\lambda = d^2$,

$$\frac{j_m'(R,\lambda)}{j_m(R,\lambda)} = -\frac{|m| - 1/2}{R}, \tag{6.18a}$$

and

$$j_m(r,\lambda) = \frac{j_m(R,\lambda)}{R^{1/2-|m|}} r^{1/2-|m|}, \quad r \geq R. \tag{6.18b}$$

In particular, for $\lambda \in (0, d^2]$ a discontinuity point of χ_m, $j_m(r,\lambda)$ decays exponentially as $r \to \infty$ if $\lambda < d^2$, and $j_m(r,\lambda) \sim r^{1/2-|m|}$ as $r \to \infty$ if $\lambda = d^2$.

Proof. As it follows from the proof of Theorem 6.1, for such a λ we will have $M_0^m(\lambda_0) - M_\infty^m(\lambda_0) = 0$. With the help (6.1), (6.7a), and (6.8), we deduce (6.17a) and (6.18a). Notice that these equalities are exactly the conditions (4.6) and (4.9). Then (6.17b) and (6.18b) follow from (4.7) and (4.10). □

7. Computing the Transform

Denote by $L^2(\chi_m)$ the space of all functions $G : \mathbb{R} \to \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} |G(\lambda)|^2 \, d\chi_m(\lambda) < \infty,$$

where χ_m is the non-decreasing function defined in Theorem 6.1.

Theorem 7.1 *Let $g \in L^2(0,\infty)$. The integral*

$$G_m(\lambda) = \int_0^\infty j_m(r,\lambda) g(r) \, dr \tag{7.1}$$

is convergent in $L^2(\chi_m)$*, in the sense that there exists* $G_m \in L^2(\chi_m)$ *such that* $G_m^{cd} \to G_m$ *in* $L^2(\chi_m)$ *as* $c \to 0$ *and* $d \to \infty$*, where*

$$G_m^{cd}(\lambda) = \int_c^d j_m(r,\lambda) g(r)\, dr, \quad 0 < c < d < \infty. \tag{7.2}$$

The equality

$$g(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} j_m(r,\lambda) G_m(\lambda)\, d\chi_m(\lambda) \tag{7.3}$$

holds, in the sense that $g^{\sigma\tau} \to g$ *in* $L^2(0,\infty)$ *as* $\tau \to -\infty$, $\sigma \to \infty$*, where*

$$g^{\tau\sigma}(r) = \frac{1}{\pi} \int_{\tau}^{\sigma} j_m(r,\lambda) G_m(\lambda)\, d\chi_m(\lambda), \quad -\infty < \sigma < \tau < \infty. \tag{7.4}$$

We have the Parseval identity

$$\int_0^{\infty} |g(r)|^2\, dr = \frac{1}{\pi} \int_{-\infty}^{\infty} |G_m(\lambda)|^2\, d\chi_m(\lambda). \tag{7.5}$$

Proof. We will apply Theorem 5.1. First note that if $\Gamma = (\Gamma_1, \Gamma_2)$ with Γ_1, Γ_2 : $\mathbb{R} \to \mathbb{C}$, then because of (6.10), the norm $||\cdot||$ as defined in (5.11) can be written in the form

$$||\Gamma||^2 = \int_{-\infty}^{\infty} |j_m(R,\lambda)\Gamma_1(\lambda) - j_m'(R,\lambda)\Gamma_2(\lambda)|^2\, d\chi_m(\lambda). \tag{7.6}$$

Recall the identities (6.3) to (6.5) for expressing $\theta(r,\lambda)$ and $\varphi(r,\lambda)$ in terms of $j_m(r,\lambda)$ and $y_m(r,\lambda)$. Note that from (6.4) and (6.5) we get the following equalities involving the coefficients α, β, γ, and δ

$$\alpha(\lambda) j_m(R,\lambda) - \gamma(\lambda) j_m'(R,\lambda) = 1, \tag{7.7}$$

and

$$\beta(\lambda) j_m(R,\lambda) - \delta(\lambda) j_m'(R,\lambda) = 0. \tag{7.8}$$

Then, for $0 < c < d < \infty$, the functions Γ_1^{cd} and Γ_2^{cd} defined by (5.12) become

$$\Gamma_1^{cd}(\lambda) = \int_c^d \big\{\alpha(\lambda) j_m(r,\lambda) + \beta(\lambda) y_m(r,\lambda)\big\} g(r)\, dr,$$

$$\Gamma_2^{cd}(\lambda) = \int_c^d \big\{\gamma(\lambda) j_m(r,\lambda) + \delta(\lambda) y_m(r,\lambda)\big\} g(r)\, dr.$$

We denoted $\Gamma^{cd} = (\Gamma_1^{cd}, \Gamma_2^{cd})$. Let

$$\tilde{\Gamma}_1^{cd}(\lambda) = \int_c^d \alpha(\lambda) j_m(r,\lambda) g(r)\, dr, \quad \tilde{\Gamma}_2^{cd}(\lambda) = \int_c^d \gamma(\lambda) j_m(r,\lambda) g(r)\, dr. \tag{7.9}$$

and set $\tilde{\Gamma}^{cd} = (\tilde{\Gamma}_1^{cd}, \tilde{\Gamma}_2^{cd})$. We have

$$||\Gamma^{cd} - \tilde{\Gamma}^{cd}|| = 0. \tag{7.10}$$

That follows from (7.6) and (7.8). Theorem 5.1 guarantees the existence of $\Gamma = (\Gamma_1, \Gamma_2) \in L^2(\rho)$ such that

$$||\Gamma - \Gamma^{cd}|| \to 0 \text{ as } c \to 0,\ d \to \infty.$$

Then, equality (7.10) says that we have $||\Gamma - \tilde{\Gamma}^{cd}|| \to 0$ as $c \to 0$, $d \to \infty$. But, according to (7.6),

$$||\Gamma - \tilde{\Gamma}^{cd}||^2 = \int_{-\infty}^{\infty} |\{j_m(R,\lambda)\Gamma_1(\lambda) - j_m'(R,\lambda)\Gamma_2(\lambda)\} - \{j_m(R,\lambda)\tilde{\Gamma}_1^{cd}(\lambda) - j_m'(R,\lambda)\tilde{\Gamma}_2^{cd}(\lambda)\}|^2\, d\chi_m. \tag{7.11}$$

Note that

$$j_m(R,\lambda)\tilde{\Gamma}_1^{cd}(\lambda) - j_m'(R,\lambda)\tilde{\Gamma}_2^{cd}(\lambda) = \int_c^d j_m(r,\lambda) g(r)\, dr.$$

That follows from (7.7) and (7.9). Thus, if we denote

$$G_m(\lambda) = j_m(R,\lambda)\Gamma_1(\lambda) - j_m'(R,\lambda)\Gamma_2(\lambda),\ \lambda \in \mathbb{R}, \tag{7.12}$$

we obtain from (7.11) that

$$\int_{-\infty}^{\infty} |G_m(\lambda) - G_m^{cd}(\lambda)|^2\, d\chi_m \to 0 \text{ as } c \to 0,\ d \to \infty$$

(G_m^{cd} was defined by (7.2)). This shows the first part of Theorem 7.1.

Next we need to show that representation (7.3) holds. It suffices to prove that $g^{\tau\sigma}$ as defined by (5.13) in Theorem 5.1 is the same as $g^{\tau\sigma}$ defined in (7.4). And they are. To check this one needs to start with $g^{\tau\sigma}$ as given in (5.13), substitute $\theta(r,\lambda)$ and $\varphi(r,\lambda)$ from (6.3), use (6.10) to express ξ_m, η_m and ζ_m in terms of χ_m, use the equalities (7.7) and (7.8), and finally use definition (7.12) for $G_m(\lambda)$.

Lastly, the Parseval identity (7.5) follows from (5.14), (7.6) and (7.12). The theorem is proved. □

Theorem 7.2 *Let $g \in L^2(0,\infty)$. Let χ_m be the non-decreasing function defined in Theorem 6.1. Let $0 < \lambda_1^m < \cdots < \lambda_{P_m}^m \le d^2$ ($P_m \ge 0$) be the points where χ_m is discontinuous. Let $r_1^m, \ldots, r_{P_m}^m$ be the corresponding jumps. Let $a_m(r,\lambda)$ and $b_m(r,\lambda)$ be the functions defined by (4.11), and $c_m(\lambda)$ and $d_m(\lambda)$ be defined by (4.13a) and (4.13b). Then,*

$$r_k^m = \pi \left\{ \int_0^\infty j_m(r,\lambda_k^m)^2 \, dr \right\}^{-1}, \; k = 1, \ldots, P_m, \tag{7.13}$$

$$d\chi_m(\lambda) = \frac{\pi}{2} \frac{d\lambda}{c_m(\lambda)^2 + d_m(\lambda)^2}, \quad \lambda \in (d^2, \infty). \tag{7.14}$$

We have the representation

$$g(r) = \frac{1}{\pi} \sum_{k=1}^{P_m} r_k^m j_m(r,\lambda_k^m) G_m(\lambda_k^m) + \frac{1}{2} \int_{d^2}^\infty \frac{j_m(r,\lambda) G_m(\lambda)}{c_m(\lambda)^2 + d_m(\lambda)^2} \, d\lambda. \tag{7.15}$$

Proof. In Theorem 6.1 we proved the existence of the function χ_m and along the way we found its continuous part, that is, equality (7.14). We need to find its discrete part, that is, the value of all the jumps of χ_m. Then (7.15) will follow by applying (7.3).

Let us notice the following observation. If $\lambda_1 \neq \lambda_2$, and $w_1(r)$ and $w_2(r)$ satisfy (2.5) with $\lambda = \lambda_1$, and $\lambda = \lambda_2$ respectively, then for any $0 < c < d < \infty$

$$\int_c^d w_1(r) w_2(r) \, dr = -(\lambda_1 - \lambda_2)^{-1} [w_1'(r) w_2(r) - w_1(r) w_2'(r)] \Big|_d^c. \tag{7.16}$$

Indeed, we can write

$$\int_c^d w_1''(r) w_2(r) \, dr = - \int_c^d \left\{ \lambda_1 - q(r) - \frac{m^2 - 1/4}{r^2} \right\} w_1(r) w_2(r) \, dr,$$

and

$$\int_c^d w_1(r) w_2''(r) \, dr = - \int_c^d w_1(r) \left\{ \lambda_2 - q(r) - \frac{m^2 - 1/4}{r^2} \right\} w_2(r) \, dr.$$

If we integrate by parts the left-hand sides of these two equalities, and then subtract from first the second, we get exactly (7.16).

Before we prove (7.13), recall the behavior of $j_m(r,\lambda)$ as $r \to \infty$. For $\lambda \le d^2$ a discontinuity point of χ_m it is described in Corollary 6.1, while for $\lambda > d^2$ see the discussion in Section 4.

Let $\lambda_0 \le d^2$ be one of the discontinuity points of χ_m. Let r_0 be the corresponding jump. We will consider two cases: when $j_m(r,\lambda_0)$ is square integrable, and when it is not. The first case splits into two subcases: we can have either $\lambda_0 < d^2$, or

$\lambda_0 = d^2$ with $|m| \geq 2$. The second case happens for $\lambda_0 = d^2$ and $|m| \in \{0, 1\}$. Let us start with the first case.

Denote $g(r) = j_m(r, \lambda_0)$. Since $g(r)$ is square integrable, we can apply Theorem 7.1 for this particular function. We will show that the corresponding $G_m(\lambda)$ as defined by (7.1) is such that

$$G_m(\lambda) = \begin{cases} \int_0^\infty j_m(r, \lambda_0)^2\, dr, & \text{if } \lambda = \lambda_0, \\ 0, & \text{if } \lambda \neq \lambda_0. \end{cases} \tag{7.17}$$

for all λ such that $d\chi_m(\lambda) \neq 0$. Then (7.13) will follow promptly; one needs to apply the Parseval identity (7.5) and notice that since r_0 is the jump of χ_m at λ_0, then $d\chi_m(\lambda_0) = r_0 \delta(\lambda - \lambda_0)$ with δ being Dirac's function.

Consider first the subcase $\lambda_0 < d^2$. That (7.17) is true for $\lambda = \lambda_0$ follows immediately from (7.1). Assume now $\lambda \neq \lambda_0$. Let us compute $G_m^{cd}(\lambda)$ for $0 < c < d < \infty$ as defined in (7.2). Use (7.16) with $w_1(r) = j_m(r, \lambda_0)$, $w_2(r) = j_m(r, \lambda)$. Put $c \to 0$ and $d \to \infty$. From (3.1) we deduce that

$$j_m'(c, \lambda_0) j_m(c, \lambda) - j_m(c, \lambda_0) j_m'(c, \lambda) \to 0 \text{ as } c \to 0.$$

Also, both $j_m'(d, \lambda_0) j_m(d, \lambda)$ and $j_m(d, \lambda_0) j_m'(d, \lambda)$ go to zero as $d \to \infty$ if $d\chi_m(\lambda) \neq 0$, since on one hand, $j_m(r, \lambda_0)$ and its derivative decrease exponentially, and on the other hand, $j_m'(r, \lambda)$ and its derivative either decrease exponentially for $\lambda < d^2$, or behave like a power of r for $\lambda = d^2$, or are bounded for $\lambda > d^2$. In any case we get that $G_m^{cd}(\lambda) \to 0$ as $c \to 0$ and $d \to \infty$, so $G_m(\lambda) = 0$.

The subcase $\lambda_0 = d^2$ with $|m| \geq 2$ follows in the same way. The statement that for $\lambda \neq \lambda_0$ and $d\chi_m(\lambda) \neq 0$ both $j_m'(d, \lambda_0) j_m(d, \lambda)$ and $j_m(d, \lambda_0) j_m'(d, \lambda)$ go to zero as $d \to \infty$ is argued in a little different way. We have that $j_m(r, \lambda_0)$ and its derivative decay as a negative power of r for $r \to \infty$, while $j_m(r, \lambda)$ and its derivative either decay exponentially for $\lambda < \lambda_0$, or stay bounded for $\lambda > \lambda_0$. But the conclusion is the same, $G_m^{cd}(\lambda) \to 0$ as $c \to 0$ and $d \to \infty$, and thus (7.17) holds in this case too.

Now consider the second case, $\lambda_0 = d^2$ and $|m| \in \{0, 1\}$, when, as we remarked above, $j_m(r, \lambda_0)$ is not square integrable. For $0 < c < d < \infty$ define

$$g(r) = \begin{cases} j_m(r, \lambda_0), & \text{if } c < r < d, \\ 0, & \text{otherwise.} \end{cases}$$

This function *will be* square integrable. Let $G_m(\lambda)$ be the corresponding transform of $g(r)$ as defined by (7.1). Apply the Parseval identity (7.5). We get

$$\int_c^d |j_m(r, \lambda_0)|^2\, dr = \frac{1}{\pi} \int_{-\infty}^{\infty} |G_m(\lambda)|^2\, d\chi_m(\lambda)$$
$$\geq \frac{1}{\pi} \int_{\{\lambda_0\}} |G_m(\lambda)|^2\, d\chi_m(\lambda) = \frac{1}{\pi} G_m(\lambda_0)^2 r_0.$$

From (7.1) we obtain that

$$G_m(\lambda_0) = \int_c^d |j_m(r, \lambda_0)|^2 \, dr.$$

Then we can write

$$\pi \left\{ \int_c^d |j_m(r, \lambda_0)|^2 \, dr \right\}^{-1} \geq r_0.$$

By putting $c \to 0$, $d \to \infty$ and noticing that $r_0 \geq 0$, being a jump of the non-decreasing function χ_m, we deduce $r_0 = 0$. □

With this theorem, Corollary 6.1 and the classification obtained in Section 4, we can characterize completely the points of discontinuity of χ_m.

Corollary 7.1 *Let $\lambda \in \mathbb{R}$. Then λ is a discontinuity point of χ_m if and only if $j_m(\cdot, \lambda) \in L^2(0, \infty)$, and if and only if $0 < \lambda < d^2$ and* (6.17a) *holds, or $\lambda = d^2$ and* (6.18a) *holds.*

8. Finding Green's Function

In this section we will show that under certain conditions, the solution of the Helmholtz equation (1.1), which in the cylindrical coordinate system (r, ϑ, z) is written as (2.1), is unique. We will find a representation for it in terms of the source $f(r, \vartheta, z)$, the eigenfunctions $j_m(r, \lambda)$ of equation (2.5) satisfying Lemma 3.1 and the measure $d\chi_m(\lambda)$ defined in Theorem 6.1. Before proving this result we will need one lemma.

Lemma 8.1 *Let $u \in C^1(\mathbb{R}^3)$. Then u can be written as*

$$u(r, \vartheta, z) = \sum_{m \in \mathbb{Z}} e^{im\vartheta} u_m(r, z). \tag{8.1}$$

For each $m \in \mathbb{Z}$ and $z \in \mathbb{R}$, the function $r \to u_m(r, z)$ is in $C^1[0, \infty)$ and

$$\lim_{r \to 0} \left[j_m(r, \lambda) \frac{\partial \{\sqrt{r} u_m(r, z)\}}{\partial r} - \frac{\partial j_m(r, \lambda)}{\partial r} \{\sqrt{r} u_m(r, z)\} \right] = 0. \tag{8.2}$$

Proof. Equality (8.1) is nothing but the Fourier series of the function $\vartheta \to u(r, \vartheta, z)$. The smoothness of the obtained u_m follows from the formula for the Fourier coefficients,

$$u_m(r, z) = \frac{1}{2\pi} \int_0^{2\pi} u(r, t, z) e^{-imt} \, dt.$$

Let us prove (8.2). Write it as

$$\lim_{r \to 0} \left[\sqrt{r} j_m(r, \lambda) \, \frac{\partial u_m(r, z)}{\partial r} + u_m(r, z) \left(\frac{j_m(r, \lambda)}{2\sqrt{r}} - \sqrt{r} \frac{\partial j_m(r, \lambda)}{\partial r} \right) \right] = 0.$$

The first term in the sum clearly goes to zero as $r \to 0$. For the second term, by applying Lemma 3.1 to the expression in parentheses we get

$$\lim_{r\to 0}\left(\frac{j_m(r,\lambda)}{2\sqrt{r}} - \sqrt{r}\frac{\partial j_m(r,\lambda)}{\partial r}\right) = \lim_{r\to 0}\left(\frac{1}{2}r^{|m|} - \left\{|m| + \frac{1}{2}\right\}r^{|m|}\right) = -\lim_{r\to 0}|m|r^{|m|} = 0,$$

for all $m \in \mathbb{Z}$. □

We assume that the source f is continuous and with compact support. These will be the conditions on the solution u of (2.1) which will guarantee its uniqueness. First, we will impose the condition that $u \in C^1(\mathbb{R}^3)$. Second, suppose that for all $m \in \mathbb{Z}$, $z \in \mathbb{R}$ the following equality holds

$$\lim_{r\to\infty}\left[j_m(r,\lambda)\frac{\partial\{\sqrt{r}u_m(r,z)\}}{\partial r} - \frac{\partial j_m(r,\lambda)}{\partial r}\{\sqrt{r}u_m(r,z)\}\right] = 0, \tag{8.3}$$

with the functions $u_m(r,z)$ defined by (8.1).

Denote by $U_m(\lambda,z)$ the transform of the function $r \to \sqrt{r}u_m(r,z)$ given by (7.1),

$$U_m(\lambda,z) = \int_0^\infty j_m(\rho,\lambda)\sqrt{r}u_m(r,z)\,d\rho, \tag{8.4}$$

The fourth requirement is the *radiation condition*

$$\begin{cases} \lim\limits_{|z|\to\infty}\left[\dfrac{\partial U_m(\lambda,z)}{\partial|z|} - i\sqrt{k^2n_0^2-\lambda}\,U_m(\lambda,z)\right] = 0, & \text{for } \lambda \le k^2n_0^2 \text{ with } d\chi_m(\lambda)\neq 0, \\ \lim\limits_{|z|\to\infty} U_m(\lambda,z) = 0, & \text{for } \lambda > k^2n_0^2. \end{cases} \tag{8.5}$$

These conditions are physically motivated. The condition on f says that the source must be finite in size. Equation (8.3) signifies a fast decay of the electromagnetic field intensity as $r \to \infty$. The radiation condition (8.5) means that as $|z| \to \infty$, the electromagnetic field can be separated in two parts. The first part, for $\lambda \le k^2n_0^2$, is oscillatory, and then condition (8.5) is just a version of the the Sommerfeld radiation condition, which guarantees that field is *outgoing*, that is, it is being radiated away. The second part implies that the energy of the field is decaying.

Remark 8.1 *The study of a suitable radiation condition on u for this problem is very complicated. More details on these arguments can be found in* [6].

Theorem 8.1 *With the above assumptions, the solution of* (2.1) *can be represented as*

$$u(r,\vartheta,z) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} G(r,\rho;\vartheta,t;z,\zeta)f(\rho,t,\zeta)\,\rho\,dt\,d\rho\,d\zeta, \tag{8.6}$$

where

$$G(r,\rho;\vartheta,t;z,\zeta)=$$
$$\frac{1}{2\pi^2}\frac{1}{\sqrt{r\rho}}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{+\infty}\frac{e^{i|z-\zeta|\sqrt{k^2n_0^2-\lambda}}}{2i\sqrt{k^2n_0^2-\lambda}}e^{im(\vartheta-t)}\,j_m(\rho,\lambda)\,j_m(r,\lambda)\,d\chi_m(\lambda),$$
$$0<r,\rho;\ 0\le\vartheta,t\le 2\pi;\ z,\zeta\in\mathbb{R},\quad (8.7)$$

and χ_m *is the non-decreasing function defined in Theorem 6.1.*

Proof. The function $f(\rho,\vartheta,z)$ can be decomposed as

$$f(\rho,\vartheta,z)=\sum_{m\in\mathbb{Z}}e^{im\vartheta}f_m(\rho,z),\quad (8.8)$$

with

$$f_m(\rho,z)=\frac{1}{2\pi}\int_0^{2\pi}f(\rho,t,z)e^{-imt}\,dt.\quad (8.9)$$

Look for $u(\rho,\vartheta,z)$ in the form (8.1). By plugging (8.1) and (8.8) into (2.1) we deduce that for each m, $u_m(\rho,z)$ needs to satisfy the equation

$$\frac{\partial^2 u_m}{\partial z^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u_m}{\partial\rho}\right)+\left(k^2n(\rho)^2-\frac{m^2}{\rho^2}\right)u_m=f_m,\quad (8.10)$$

for all $m\in\mathbb{Z}$. Let $\lambda\in\mathbb{R}$ be such that $d\chi_m(\lambda)\neq 0$. Let $U_m(\lambda,z)$ be the transform of $u_m(\rho,z)$ given by (8.4), and let

$$F_m(\lambda,z)=\int_0^\infty j_m(\rho,\lambda)\sqrt{\rho}f_m(\rho,z)\,d\rho\quad (8.11)$$

be the transform of f_m. Multiply (8.10) on both sides by $\sqrt{\rho}j_m(\rho,\lambda)$ and integrate from 0 to ∞. We obtain

$$\frac{\partial^2 U_m}{\partial z^2}+\int_0^\infty j_m(\rho,\lambda)\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u_m(\rho,z)}{\partial\rho}\right)d\rho+$$
$$\int_0^\infty j_m(\rho,\lambda)\left(k^2n(\rho)^2-\frac{m^2}{\rho^2}\right)\sqrt{\rho}u_m(\rho,z)\,d\rho=F_m.$$

Use integration by parts twice for the first integral in the above equation. By applying Lemma 8.1 and equality (8.3) we get

$$\frac{\partial^2 U_m}{\partial z^2}+\int_0^\infty\left\{\frac{\partial^2 j_m(\rho,\lambda)}{\partial\rho}+\left(k^2n(\rho)^2-\frac{m^2-1/4}{\rho^2}\right)j_m(\rho,\lambda)\right\}\sqrt{\rho}u_m(\rho,\lambda)\,d\rho=F_m.$$

Recall that $j_m(\rho,\lambda)$ satisfies (2.5) with $q(r)$ given by (2.3). Then,

$$\frac{\partial^2 U_m}{\partial z^2} + (k^2 n_0^2 - \lambda) U_m = F_m. \tag{8.12}$$

The solution to (8.12) which satisfies (8.5) is easily found,

$$U_m(\lambda, z) = \int_{-\infty}^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_0^2 - \lambda}}}{2i\sqrt{k^2 n_0^2 - \lambda}} F_m(\lambda, \zeta)\, d\zeta,$$

or if we use (8.11),

$$U_m(\lambda, z) = \int_{-\infty}^{+\infty} \int_0^{\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_0^2 - \lambda}}}{2i\sqrt{k^2 n_0^2 - \lambda}} j_m(\rho,\lambda)\sqrt{\rho} f_m(\rho,\zeta)\, d\rho\, d\zeta.$$

$U_m(\lambda, z)$ was defined by (8.4). Using the inversion formula (7.3) given in Theorem 7.1 we can recover $u_m(r,z)$,

$$\sqrt{r} u_m(r,z) = \frac{1}{\pi} \int_{-\infty}^{\infty} j_m(r,\lambda) U_m(\lambda, z)\, d\chi_m(\lambda),$$

or

$$u_m(r,z) = \frac{1}{\pi\sqrt{r\rho}} \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} \int_0^{\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_0^2 - \lambda}}}{2i\sqrt{k^2 n_0^2 - \lambda}} j_m(\rho,\lambda) j_m(r,\lambda) \rho f_m(\rho,\zeta)\, d\rho\, d\zeta\, d\chi_m(\lambda).$$

Now, to get (8.6) with $G(r,\rho;\vartheta,t;z,\zeta)$ given by (8.7) we need to substitute $f_m(\rho,z)$ from (8.9), find $u(r,\vartheta,z)$ from (8.1) and interchange the order of integration so that the inner-most integral is the one with respect to λ. □

The theorem we just proved shows that the electromagnetic field generated by the source f can be decomposed in two parts: the guided part, which is a sum of guided modes decaying in r either exponentially or as a power of r, and a radiation part, which is obtained by summing in m and integrating in λ. For each $m \in \mathbb{Z}$ the set of guided modes is finite, as it was shown in Lemma 6.2. The next theorem will prove a stronger result.

Theorem 8.2 *The total number of guided modes (in all $m \in \mathbb{Z}$) is finite.*

Proof. We just need to show that for $|m|$ large enough there are no more guided modes. A mode $j_m(r,\lambda)$ is guided, if

$$M_0^m(\lambda) - M_\infty^m(\lambda) = 0.$$

All λ for which this equality happens are in $(0, d^2]$, as proved in Lemma 6.2. By using (6.1), (6.7a) and (6.8) we can write this equality as

$$\frac{j_m'(R,\lambda)}{j_m(R,\lambda)} = \frac{k_m'(R,\lambda)}{k_m(R,\lambda)}, \text{ if } \lambda < d^2,$$

or

$$\frac{j'_m(R,\lambda)}{j_m(R,\lambda)} = -\frac{|m| - 1/2}{R}, \text{ if } \lambda = d^2,$$

where $k(r, \lambda)$ is given by (4.5).

The left-hand side of these equalities is strictly positive for $|m|$ large, as it follows from Lemma 3.3. Thus, the second of these equalities is not possible. To show that the first one cannot happen, it suffices to prove that $k'_m(R, \lambda) < 0$ for $|m| \geq 1$, since we know that $k_m(r, \lambda) > 0$, for all $r > 0$.

The function $r \to k_m(r, \lambda)$ will satisfy the equation

$$k''_m(r,\lambda) = \left\{ d^2 - \lambda + \frac{m^2 - 1/4}{r^2} \right\} k_m(r,\lambda),$$

which implies that $k''_m(r, \lambda) > 0$ for $r > 0$, and so, $k'_m(r, \lambda)$ is an increasing function of r. From this and (4.8) it follows that $k'_m(r, \lambda) < 0$ for all $r > 0$. This finishes the proof of the theorem. □

9. Conclusion

In this paper we have constructed a framework for analyzing waveguide problems which is based on a transform theory. The construction of the transform was more difficult than, but the final form relatively similar to, the 2-D case.[3] The primary tool in obtaining the transform was the theory of self-adjoint singular eigenvalue problems.

This paper completes the study of the wave propagation in a infinite cylindrical waveguide. We obtained a Green's function valid for every choice of the index of refraction of the core with cylindrical symmetry. In particular, it is enough to solve (6.17a) for $\lambda \in (0, d^2]$ and the differential equation (2.5) in the core region in order to obtain the corresponding Green's function.

The obtained formula for Green's functions is very amenable to computation. In a future article we will calculate explicitly Green's function in the cases of a *step-index fiber* and a *coaxial waveguide* and will display numerical results.

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