Minimizing the mean projections of finite ρ-separable packings [*†]

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Abstract

A packing of translates of a convex body in the d-dimensional Euclidean space $\mathbb{E}^d$ is said to be totally separable if any two packing elements can be separated by a hyperplane of $\mathbb{E}^d$ disjoint from the interior of every packing element. We call the packing $\mathcal{P}$ of translates of a centrally symmetric convex body $\mathbf{C}$ in $\mathbb{E}^d$ a ρ-separable packing for given $\rho \geq 1$ if in every ball concentric to a packing element of $\mathcal{P}$ having radius ρ (measured in the norm generated by $\mathbf{C}$) the corresponding sub-packing of $\mathcal{P}$ is totally separable. The main result of this paper is the following theorem. Consider the convex hull $\mathbf{Q}$ of n non-overlapping translates of an arbitrary centrally symmetric convex body $\mathbf{C}$ forming a ρ-separable packing in $\mathbb{E}^d$ with n being sufficiently large for given $\rho \geq 1$. If $\mathbf{Q}$ has minimal mean i-dimensional projection for given i with $1 \leq i < d$, then $\mathbf{Q}$ is approximately a d-dimensional ball. This extends a theorem of K. Böröczky Jr. [Monatsh. Math. **118** (1994), 41–54] from translative packings to ρ-separable translative packings for $\rho \geq 1$.

1 Introduction

We denote the d-dimensional Euclidean space by $\mathbb{E}^d$. Let $\mathbf{B}^d$ denote the unit ball centered at the origin $\mathbf{o}$ in $\mathbb{E}^d$. A *d-dimensional convex body* $\mathbf{C}$ is a compact convex subset of $\mathbb{E}^d$ with non-empty interior $\operatorname{int}\mathbf{C}$. (If $d = 2$, then $\mathbf{C}$ is said to be a *convex domain.*) If $\mathbf{C} = -\mathbf{C}$, where $-\mathbf{C} = \{-x : x \in \mathbf{C}\}$, then $\mathbf{C}$ is said to be $\mathbf{o}$-*symmetric* and a translate $\mathbf{c} + \mathbf{C}$ of $\mathbf{C}$ is called centrally symmetric with center $\mathbf{c}$.

The starting point as well as the main motivation for writing this paper is the following elegant theorem of Böröczky Jr. [8]: Consider the convex hull $\mathbf{Q}$ of n non-overlapping translates of an arbitrary convex body $\mathbf{C}$ in $\mathbb{E}^d$ with n being sufficiently large. If $\mathbf{Q}$ has minimal mean i-dimensional projection for given i with $1 \leq i < d$, then $\mathbf{Q}$ is approximately a d-dimensional ball. In this paper, our main goal is to prove an extension of this theorem to ρ-separable translative packings of convex bodies in $\mathbb{E}^d$. Next, we define the concept of ρ-separable translative packings and then state our main result.

A packing of translates of a convex domain $\mathbf{C}$ in $\mathbb{E}^2$ is said to be *totally separable* if any two packing elements can be separated by a line of $\mathbb{E}^2$ disjoint from the interior of every packing element. This notion was introduced by G. Fejes Tóth and L. Fejes Tóth [9] . We can define a totally separable packing of translates of a d-dimensional convex body $\mathbf{C}$ in a similar way by requiring any two packing elements to be separated by a hyperplane in $\mathbb{E}^d$ disjoint from the interior of every packing element [6, 7].

Definition 1. *Let $\mathbf{C}$ be an $\mathbf{o}$-symmetric convex body of $\mathbb{E}^d$. Furthermore, let $\|\cdot\|_{\mathbf{C}}$ denote the norm generated by $\mathbf{C}$, i.e., let $\|\mathbf{x}\|_{\mathbf{C}} := \inf\{\lambda \mid \mathbf{x} \in \lambda\mathbf{C}\}$ for any $\mathbf{x} \in \mathbb{E}^d$. Now, let $\rho \geq 1$. We say that the packing*

$$\mathcal{P}_{\text{sep}} := \{\mathbf{c}_i + \mathbf{C} \mid i \in I \text{ with } \|\mathbf{c}_j - \mathbf{c}_k\|_{\mathbf{C}} \geq 2 \text{ for all } j \neq k \in I\}$$

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of (finitely or infinitely many) non-overlapping translates of $\mathbf{C}$ *with centers* $\{\mathbf{c}_i \mid i \in I\}$ *is a* ρ-separable packing *in* $\mathbb{E}^d$ *if for each* $i \in I$ *the finite packing* $\{\mathbf{c}_j + \mathbf{C} \mid \mathbf{c}_j + \mathbf{C} \subseteq \mathbf{c}_i + \rho\mathbf{C}\}$ *is a totally separable packing (in* $\mathbf{c}_i + \rho\mathbf{C}$*). Finally, let* $\delta_{\rm sep}(\rho, \mathbf{C})$ *denote the largest density of all* ρ*-separable translative packings of* $\mathbf{C}$ *in* $\mathbb{E}^d$*, i.e., let*

$$\delta_{\rm sep}(\rho, \mathbf{C}) := \sup_{\mathcal{P}_{\rm sep}} \left(\limsup_{\lambda \to +\infty} \frac{\sum_{\mathbf{c}_i + \mathbf{C} \subset \mathbf{W}_\lambda^d} {\rm vol}_d(\mathbf{c}_i + \mathbf{C})}{{\rm vol}_d(\mathbf{W}_\lambda^d)} \right) ,$$

where $\mathbf{W}_\lambda^d$ *denotes the* d*-dimensional cube of edge length* 2λ *centered at* $\mathbf{o}$ *in* $\mathbb{E}^d$ *having edges parallel to the coordinate axes of* $\mathbb{E}^d$ *and* ${\rm vol}_d(\cdot)$ *refers to the* d*-dimensional volume of the corresponding set in* $\mathbb{E}^d$*.*

Remark 1. *Let* $\delta(\mathbf{C})$ *(resp.,* $\delta_{\rm sep}(\mathbf{C})$*) denote the supremum of the upper densities of all translative packings (resp., totally separable translative packings) of the* $\mathbf{o}$*-symmetric convex body* $\mathbf{C}$ *in* $\mathbb{E}^d$*. Clearly,* $\delta_{\rm sep}(\mathbf{C}) \leq \delta_{\rm sep}(\rho, \mathbf{C}) \leq \delta(\mathbf{C})$ *for all* $\rho \geq 1$*. Furthermore, if* $1 \leq \rho < 3$*, then any* ρ*-separable translative packing of* $\mathbf{C}$ *in* $\mathbb{E}^d$ *is simply a translative packing of* $\mathbf{C}$ *and therefore,* $\delta_{\rm sep}(\rho, \mathbf{C}) = \delta(\mathbf{C})$*.*

Recall that the mean i-dimensional projection $M_i(\mathbf{C})$ $(i = 1, 2, \ldots, d-1)$ of the convex body $\mathbf{C}$ in $\mathbb{E}^d$, can be expressed ([13]) with the help of mixed volume via the formula

$$M_i(\mathbf{C}) = \frac{\kappa_i}{\kappa_d} V(\overbrace{\mathbf{C}, \ldots, \mathbf{C}}^{i}, \overbrace{\mathbf{B}^d, \ldots, \mathbf{B}^d}^{d-i}),$$

where κ_d is the volume of $\mathbf{B}^d$ in $\mathbb{E}^d$. Note that $M_i(\mathbf{B}^d) = \kappa_i$, and the surface area of $\mathbf{C}$ is $S(\mathbf{C}) = \frac{d\kappa_d}{\kappa_{d-1}} M_{d-1}(\mathbf{C})$ and in particular, $S(\mathbf{B}^d) = d\kappa_d$. Set $M_d(\mathbf{C}) := {\rm vol}_d(\mathbf{C})$. Finally, let $R(\mathbf{C})$ (resp., $r(\mathbf{C})$) denote the circumradius (resp., inradius) of the convex body $\mathbf{C}$ in $\mathbb{E}^d$, which is the radius of the smallest (resp., largest) ball that contains (resp., is contained in) $\mathbf{C}$. Our main result is the following.

Theorem 1. *Let* $d \geq 2$*,* $1 \leq i \leq d-1$*,* $\rho \geq 1$*, and let* $\mathbf{Q}$ *be the convex hull of the* ρ*-separable packing of* n *translates of the* $\mathbf{o}$*-symmetric convex body* $\mathbf{C}$ *in* $\mathbb{E}^d$ *such that* $M_i(\mathbf{Q})$ *is minimal and* $n \geq \frac{4^d d^{4d}}{\delta_{\rm sep}(\rho, \mathbf{C})^{d-1}} \cdot \left(\rho \frac{R(\mathbf{C})}{r(\mathbf{C})}\right)^d$*. Then*

$$\frac{r(\mathbf{Q})}{R(\mathbf{Q})} \geq 1 - \frac{\omega}{n^{\frac{2}{d(d+3)}}}, \tag{1}$$

for $\omega = \lambda(d) \left(\frac{\rho R(\mathbf{C})}{r(\mathbf{C})}\right)^{\frac{2}{d+3}}$*, where* $\lambda(d)$ *depends only on the dimension* d*. In addition,*

$$M_i(\mathbf{Q}) = \left(1 + \frac{\sigma}{n^{\frac{1}{d}}}\right) M_i(\mathbf{B}^d) \left(\frac{{\rm vol}_d(\mathbf{C})}{\delta_{\rm sep}(\rho, \mathbf{C})\kappa_d}\right)^{\frac{i}{d}} \cdot n^{\frac{i}{d}},$$

where $-\frac{2.25 R(\mathbf{C})\rho d i}{r(\mathbf{C})\delta_{\rm sep}(\rho, \mathbf{C})} \leq \sigma \leq \frac{2.1 R(\mathbf{C})\rho i}{r(\mathbf{C})\delta_{\rm sep}(\rho, \mathbf{C})}$*.*

Remark 2. *It is worth restating Theorem 1 as follows: Consider the convex hull* $\mathbf{Q}$ *of* n *non-overlapping translates of an arbitrary* $\mathbf{o}$*-symmetric convex body* $\mathbf{C}$ *forming a* ρ*-separable packing in* $\mathbb{E}^d$ *with* n *being sufficiently large. If* $\mathbf{Q}$ *has minimal mean* i*-dimensional projection for given* i *with* $1 \leq i < d$*, then* $\mathbf{Q}$ *is approximately a* d*-dimensional ball.*

Remark 3. *The nature of the analogue question on minimizing* $M_d(\mathbf{Q}) = {\rm vol}_d(\mathbf{Q})$ *is very different. Namely, recall that Betke and Henk [4] proved L. Fejes Tóth's sausage conjecture for* $d \geq 42$ *according to which the smallest volume of the convex hull of* n *non-overlapping unit balls in* $\mathbb{E}^d$ *is obtained when the* n *unit balls form a sausage, that is, a linear packing (see also [2] and [3]). As linear packings of unit balls are* ρ*-separable therefore the above theorem of Betke and Henk applies to* ρ*-separable packings of unit balls in* $\mathbb{E}^d$ *for all* $\rho \geq 1$ *and* $d \geq 42$*. On the other hand, the problem of minimizing the volume of the convex hull of* n *unit balls forming a* ρ*-separable packing in* $\mathbb{E}^d$ *remains an interesting open problem for* $\rho \geq 1$ *and* $2 \leq d < 42$*. Last but not least, the problem of minimizing* $M_d(\mathbf{Q})$ *for* $\mathbf{o}$*-symmetric convex bodies* $\mathbf{C}$ *different from a ball in* $\mathbb{E}^d$ *seems to be wide open for* $\rho \geq 1$ *and* $d \geq 2$*.*

Remark 4. *Let $d \geq 2$, $1 \leq i \leq d-1$, $n > 1$, and let $\mathbf{C}$ be a given $\mathbf{o}$-symmetric convex body in $\mathbb{E}^d$. Furthermore, let $\mathbf{Q}$ be the convex hull of the totally separable packing of n translates of $\mathbf{C}$ in $\mathbb{E}^d$ such that $M_i(\mathbf{Q})$ is minimal. Then it is natural to ask for the limit shape of $\mathbf{Q}$ as $n \to +\infty$, that is, to ask for an analogue of Theorem 1 within the family of totally separable translative packings of $\mathbf{C}$ in $\mathbb{E}^d$. This would require some new ideas besides the ones used in the following proof of Theorem 1.*

In the rest of the paper by adopting the method of Böröczky Jr. [8] and making the necessary modifications, we give a proof of Theorem 1.

2 Basic properties of finite ρ-separable translative packings

The following statement is the ρ-separable analogue of the Lemma in [5] (see also Theorem 3.1 in [2]).

Lemma 1. *Let $\{\mathbf{c}_i + \mathbf{C} \mid 1 \leq i \leq n\}$ be an arbitrary ρ-separable packing of n translates of the $\mathbf{o}$-symmetric convex body $\mathbf{C}$ in $\mathbb{E}^d$ with $\rho \geq 1$, $n \geq 1$, and $d \geq 2$. Then*

$$\frac{n\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d\left(\cup_{i=1}^n \mathbf{c}_i + 2\rho\mathbf{C}\right)} \leq \delta_{\mathrm{sep}}(\rho, \mathbf{C}) \ .$$

Proof. We use the method of the proof of the Lemma in [5] (resp., Theorem 3.1 in [2]) with proper modifications. The details are as follows. Assume that the claim is not true. Then there is an $\epsilon > 0$ such that

$$\mathrm{vol}_d\left(\cup_{i=1}^n \mathbf{c}_i + 2\rho\mathbf{C}\right) = \frac{n\mathrm{vol}_d(\mathbf{C})}{\delta_{\mathrm{sep}}(\rho, \mathbf{C})} - \epsilon \tag{2}$$

Let $C_n = \{\mathbf{c}_i \mid i = 1, \ldots, n\}$ and let Λ be a packing lattice of $C_n + 2\rho\mathbf{C} = \cup_{i=1}^n \mathbf{c}_i + 2\rho\mathbf{C}$ such that $C_n + 2\rho\mathbf{C}$ is contained in a fundamental parallelotope of Λ say, in $\mathbf{P}$, which is symmetric about the origin. Recall that for each $\lambda > 0$, $\mathbf{W}^d_\lambda$ denotes the d-dimensional cube of edge length 2λ centered at the origin $\mathbf{o}$ in $\mathbb{E}^d$ having edges parallel to the coordinate axes of $\mathbb{E}^d$. Clearly, there is a constant $\mu > 0$ depending on $\mathbf{P}$ only, such that for each $\lambda > 0$ there is a subset L_λ of Λ with

$$\mathbf{W}^d_\lambda \subseteq L_\lambda + \mathbf{P} \text{ and } L_\lambda + 2\mathbf{P} \subseteq \mathbf{W}^d_{\lambda+\mu} \ . \tag{3}$$

The definition of $\delta_{\mathrm{sep}}(\rho, \mathbf{C})$ implies that for each $\lambda > 0$ there exists a ρ-separable packing of $m(\lambda)$ translates of $\mathbf{C}$ in $\mathbb{E}^d$ with centers at the points of $C(\lambda)$ such that

$$C(\lambda) + \mathbf{C} \subset \mathbf{W}^d_\lambda$$

and

$$\lim_{\lambda \to +\infty} \frac{m(\lambda)\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{W}^d_\lambda)} = \delta_{\mathrm{sep}}(\rho, \mathbf{C}) \ .$$

As $\lim_{\lambda\to+\infty} \frac{\mathrm{vol}_d(\mathbf{W}^d_{\lambda+\mu})}{\mathrm{vol}_d(\mathbf{W}^d_\lambda)} = 1$ therefore there exist $\xi > 0$ and a ρ-separable packing of $m(\xi)$ translates of $\mathbf{C}$ in $\mathbb{E}^d$ with centers at the points of $C(\xi)$ and with $C(\xi) + \mathbf{C} \subset \mathbf{W}^d_\xi$ such that

$$\frac{\mathrm{vol}_d(\mathbf{P})\delta_{\mathrm{sep}}(\rho, \mathbf{C})}{\mathrm{vol}_d(\mathbf{P}) + \epsilon} < \frac{m(\xi)\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{W}^d_{\xi+\mu})} \text{ and } \frac{n\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{P}) + \epsilon} < \frac{n\mathrm{vol}_d(\mathbf{C})\mathrm{card}(L_\xi)}{\mathrm{vol}_d(\mathbf{W}^d_{\xi+\mu})} \ , \tag{4}$$

where $\mathrm{card}(\cdot)$ refers to the cardinality of the given set. Now, for each $\mathbf{x} \in \mathbf{P}$ we define a ρ-separable packing of $\overline{m}(\mathbf{x})$ translates of $\mathbf{C}$ in $\mathbb{E}^d$ with centers at the points of

$$\overline{C}(\mathbf{x}) := \{\mathbf{x} + L_\xi + C_n\} \cup \{\mathbf{y} \in C(\xi) \mid \mathbf{y} \notin \mathbf{x} + L_\xi + C_n + \mathrm{int}(2\rho\mathbf{C})\} \ .$$

Clearly, (3) implies that $\overline{C}(\mathbf{x}) + \mathbf{C} \subset \mathbf{W}^d_{\xi+\mu}$. Now, in order to evaluate $\int_{\mathbf{x}\in\mathbf{P}} \overline{m}(\mathbf{x})d\mathbf{x}$, we introduce the function $\chi_\mathbf{y}$ for each $\mathbf{y} \in C(\xi)$ defined as follows: $\chi_\mathbf{y}(\mathbf{x}) = 1$ if $\mathbf{y} \notin \mathbf{x} + L_\xi + C_n + \mathrm{int}(2\rho\mathbf{C})$ and $\chi_\mathbf{y}(\mathbf{x}) = 0$

for any other $\mathbf{x} \in \mathbf{P}$. Based on the origin symmetric $\mathbf{P}$ it is easy to see that for any $\mathbf{y} \in C(\xi)$ one has $\int_{\mathbf{x}\in\mathbf{P}} \chi_{\mathbf{y}}(\mathbf{x})d\mathbf{x} = \mathrm{vol}_d(\mathbf{P}) - \mathrm{vol}_d(C_n + 2\rho\mathbf{C})$. Thus, it follows in a straightforward way that

$$\int_{\mathbf{x}\in\mathbf{P}} \overline{m}(\mathbf{x})d\mathbf{x} = \int_{\mathbf{x}\in\mathbf{P}} \big(n\mathrm{card}(L_\xi) + \sum_{\mathbf{y}\in C(\xi)} \chi_{\mathbf{y}}(\mathbf{x})\big)d\mathbf{x} = n\mathrm{vol}_d(\mathbf{P})\mathrm{card}(L_\xi) + m(\xi)\big(\mathrm{vol}_d(\mathbf{P}) - \mathrm{vol}_d(C_n + 2\rho\mathbf{C})\big)\ .$$

Hence, there is a point $\mathbf{p} \in \mathbf{P}$ with

$$\overline{m}(\mathbf{p}) \geq m(\xi)\left(1 - \frac{\mathrm{vol}_d(C_n + 2\rho\mathbf{C})}{\mathrm{vol}_d(\mathbf{P})}\right) + n\mathrm{card}(L_\xi)$$

and so

$$\frac{\overline{m}(\mathbf{p})\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{W}^d_{\xi+\mu})} \geq \frac{m(\xi)\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{W}^d_{\xi+\mu})}\left(1 - \frac{\mathrm{vol}_d(C_n + 2\rho\mathbf{C})}{\mathrm{vol}_d(\mathbf{P})}\right) + \frac{n\mathrm{vol}_d(\mathbf{C})\mathrm{card}(L_\xi)}{\mathrm{vol}_d(\mathbf{W}^d_{\xi+\mu})}\ . \tag{5}$$

Now, (2) implies in a straightforward way that

$$\frac{\mathrm{vol}_d(\mathbf{P})\delta_{\mathrm{sep}}(\rho,\mathbf{C})}{\mathrm{vol}_d(\mathbf{P}) + \epsilon}\left(1 - \frac{\mathrm{vol}_d(C_n + 2\rho\mathbf{C})}{\mathrm{vol}_d(\mathbf{P})}\right) + \frac{n\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{P}) + \epsilon} = \delta_{\mathrm{sep}}(\rho,\mathbf{C}) \tag{6}$$

Thus, (4), (5), and (6) yield that

$$\frac{\overline{m}(\mathbf{p})\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{W}^d_{\xi+\mu})} > \delta_{\mathrm{sep}}(\rho,\mathbf{C})\ .$$

As $\overline{C}(\mathbf{p}) + \mathbf{C} \subset \mathbf{W}^d_{\xi+\mu}$ this contradicts the definition of $\delta_{\mathrm{sep}}(\rho,\mathbf{C})$, finishing the proof of Lemma 1. □

Definition 2. *Let $d \geq 2$, $\rho \geq 1$, and let $\mathbf{K}$ (resp., $\mathbf{C}$) be a convex body (resp., an $\mathbf{o}$-symmetric convex body) in $\mathbb{E}^d$. Then let $\nu_{\mathbf{C}}(\rho,\mathbf{K})$ denote the largest n with the property that there exists a ρ-separable packing $\{\mathbf{c}_i + \mathbf{C} \mid 1 \leq i \leq n\}$ such that $\{\mathbf{c}_i \mid 1 \leq i \leq n\} \subset \mathbf{K}$.*

Lemma 2. *Let $d \geq 2$, $\rho \geq 1$, and let $\mathbf{K}$ (resp., $\mathbf{C}$) be a convex body (resp., an $\mathbf{o}$-symmetric convex body) in $\mathbb{E}^d$. Then*

$$\left(1 + \frac{2\rho R(\mathbf{C})}{r(\mathbf{K})}\right)^{-d} \frac{\mathrm{vol}_d(\mathbf{C})\nu_{\mathbf{C}}(\rho,\mathbf{K})}{\delta_{\mathrm{sep}}(\rho,\mathbf{C})} \leq \mathrm{vol}_d(\mathbf{K}) \leq \frac{\mathrm{vol}_d(\mathbf{C})\nu_{\mathbf{C}}(\rho,\mathbf{K})}{\delta_{\mathrm{sep}}(\rho,\mathbf{C})}.$$

Proof. Observe that Lemma 1 and the containments $\mathbf{K} + 2\rho\,\mathbf{C} \subseteq \left(1 + \frac{2\rho R(\mathbf{C})}{r(\mathbf{K})}\right)\mathbf{K}$ yield the lower bound immediately.

We prove the upper bound. Let $0 < \varepsilon < \delta_{\mathrm{sep}}(\rho,\mathbf{C})$. By the definition of $\delta_{\mathrm{sep}}(\rho,\mathbf{C})$, if λ is sufficiently large, then there is a ρ-separable packing $\{\mathbf{c}_i + \mathbf{C} \mid 1 \leq i \leq n\}$ such that $C_n := \{\mathbf{c}_i \mid 1 \leq i \leq n\} \subset \mathbf{W}^d_\lambda$ and

$$\frac{n\mathrm{vol}_d(\mathbf{C})}{\mathrm{vol}_d(\mathbf{W}^d_\lambda)} \geq \delta_{\mathrm{sep}}(\rho,\mathbf{C}) - \varepsilon. \tag{7}$$

Sublemma 1. *If $\mathbf{X}$ and $\mathbf{Y}$ are convex bodies in $\mathbb{E}^d$ and $\mathbf{C}$ is an $\mathbf{o}$-symmetric convex body in $\mathbb{E}^d$, then*

$$\nu_{\mathbf{C}}(\rho,\mathbf{Y}) \geq \frac{\mathrm{vol}_d(\mathbf{Y})\nu_{\mathbf{C}}(\rho,\mathbf{X})}{\mathrm{vol}_d(\mathbf{X} - \mathbf{Y})}. \tag{8}$$

Proof. Indeed, consider any finite point set $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbf{X}$. Observe that the following are equivalent for a positive integer k:

- k is the maximum number a point of $\mathbf{X} - \mathbf{Y}$ is covered by the sets $\mathbf{x}_i - \mathbf{Y}$, $\mathbf{x}_i \in X$,
- k is the maximum number such that $\mathrm{card}((\mathbf{z} + \mathbf{Y}) \cap X) = k$ for some point $\mathbf{z} \in \mathbf{X} - \mathbf{Y}$.

Thus, $N\,\mathrm{vol}_d(\mathbf{Y}) \le \mathrm{card}((\mathbf{z}+\mathbf{Y})\cap X)\,\mathrm{vol}_d(\mathbf{X}-\mathbf{Y})$ for some $\mathbf{z}\in\mathbf{X}-\mathbf{Y}$. Hence, if $\{\mathbf{x}_i+\mathbf{C}\mid 1\le i\le N\}$ is an arbitrary ρ-separable packing with $X=\{\mathbf{x}_1,\dots,\mathbf{x}_N\}\subset\mathbf{X}$, then

$$\nu_{\mathbf{C}}(\rho,\mathbf{Y}) \ge \mathrm{card}((\mathbf{z}+\mathbf{Y})\cap X) \ge \frac{\mathrm{vol}_d(\mathbf{Y})N}{\mathrm{vol}_d(\mathbf{X}-\mathbf{Y})},$$

which implies (8). □

Applying (8) to $\mathbf{X}=\mathbf{W}^d_\lambda$ and $\mathbf{Y}=-\mathbf{K}$ and using (7), we obtain

$$\nu_{\mathbf{C}}(\rho,\mathbf{K}) \ge \frac{n\,\mathrm{vol}_d(\mathbf{K})}{\mathrm{vol}_d(\mathbf{W}^d_\lambda+\mathbf{K})} \ge \frac{\mathrm{vol}_d(\mathbf{K})}{\mathrm{vol}_d(\mathbf{W}^d_{\lambda+R(\mathbf{K})})}\cdot\frac{\mathrm{vol}_d(\mathbf{W}^d_\lambda)(\delta_{\mathrm{sep}}(\rho,\mathbf{C})-\varepsilon)}{\mathrm{vol}_d(\mathbf{C})},$$

which finishes the proof of Lemma 2. □

Definition 3. *Let $d\ge 2$, $n\ge 1$, $\rho\ge 1$, and let $\mathbf{C}$ be an $\mathbf{o}$-symmetric convex body in $\mathbb{E}^d$. Then let $R_{\mathbf{C}}(\rho,n)$ be the smallest radius $R>0$ with the property that $\nu_{\mathbf{C}}(\rho,R\,\mathbf{B}^d)\ge n$.*

Clearly, for any $\varepsilon>0$ we have $\nu_{\mathbf{C}}(\rho,(R_{\mathbf{C}}(\rho,n)-\varepsilon)\,\mathbf{B}^d)<n$, and thus, by Lemma 2 (for $\mathbf{K}=R_{\mathbf{C}}(\rho,n)\,\mathbf{B}^d$), we obtain

Corollary 1. *Let $d\ge 2$, $n\ge 1$, $\rho\ge 1$, and let $\mathbf{C}$ be an $\mathbf{o}$-symmetric convex body in $\mathbb{E}^d$. Then*

$$R_{\mathbf{C}}(\rho,n)^d \le \frac{\mathrm{vol}_d(\mathbf{C})n}{\delta_{\mathrm{sep}}(\rho,\mathbf{C})\kappa_d} \le (R_{\mathbf{C}}(\rho,n)+2\rho R(\mathbf{C}))^d\,. \tag{9}$$

Lemma 3. *Let $n\ge \frac{4^d\delta_{\mathrm{sep}}(\rho,\mathbf{C})\rho^d R(\mathbf{C})^d}{r(\mathbf{C})^d}$ and $i=1,2,\dots,d-1$. Then for $R=R_{\mathbf{C}}(\rho,n)$,*

$$M_i((R+\rho R(\mathbf{C}))\,\mathbf{B}^d) \le M_i(\mathbf{B}^d)\left(\frac{\mathrm{vol}_d(\mathbf{C})n}{\delta_{\mathrm{sep}}(\rho,\mathbf{C})\kappa_d}\right)^{\frac{i}{d}}\left(1+\frac{2\delta_{\mathrm{sep}}(\rho,\mathbf{C})^{\frac1d}\rho R(\mathbf{C})}{r(\mathbf{C})}\cdot\frac{1}{n^{\frac1d}}\right)^i.$$

Proof. Set $t=R+2\rho R(\mathbf{C})$. Then the first inequality in (9) yields that

$$R+\rho R(\mathbf{C}) \le \frac{t-\rho R(\mathbf{C})}{t-2\rho R(\mathbf{C})}\left(\frac{\mathrm{vol}_d(\mathbf{C})n}{\delta_{\mathrm{sep}}(\rho,\mathbf{C})\kappa_d}\right)^{\frac1d}.$$

Thus, by the second inequality in (9) and the condition that $n\ge\frac{4^d\delta_{\mathrm{sep}}(\rho,\mathbf{C})\rho^d R(\mathbf{C})^d}{r(\mathbf{C})^d}\ge\frac{4^d\delta_{\mathrm{sep}}(\rho,\mathbf{C})\rho^d R(\mathbf{C})^d\kappa_d}{\mathrm{vol}_d(\mathbf{C})}$, we obtain that

$$\frac{t-\rho R(\mathbf{C})}{t-2\rho R(\mathbf{C})} = 1+\left(\frac{t}{\rho R(\mathbf{C})}-2\right)^{-1} \le 1+\frac{2\delta_{\mathrm{sep}}(\rho,\mathbf{C})^{\frac1d}\rho R(\mathbf{C})\kappa_d^{\frac1d}}{\mathrm{vol}_d(\mathbf{C})^{\frac1d}}\cdot\frac{1}{n^{\frac1d}} \le 1+\frac{2\delta_{\mathrm{sep}}(\rho,\mathbf{C})^{\frac1d}\rho R(\mathbf{C})}{r(\mathbf{C})}\cdot\frac{1}{n^{\frac1d}}.$$

□

3 Proof of Theorem 1

In the proof that follows we are going to use the following special case of the Alexandrov-Fenchel inequality ([13]): if $\mathbf{K}$ is a convex body in $\mathbb{E}^d$ satisfying $M_i(\mathbf{K})\le M_i(r\,\mathbf{B}^d)$ for given $1\le i<d$ and $r>0$, then

$$M_j(\mathbf{K}) \le M_j(r\,\mathbf{B}^d) \tag{10}$$

holds for all j with $i<j\le d$. In particular, this statement for $j=d$ can be restated as follows: if $\mathbf{K}$ is a convex body in $\mathbb{E}^d$ satisfying $M_d(\mathbf{K})=M_d(r\,\mathbf{B}^d)$ for given $d\ge 2$ and $r>0$, then $M_i(\mathbf{K})\ge M_i(r\,\mathbf{B}^d)$ holds for all i with $1\le i<d$.

Let $d \geq 2$, $1 \leq i \leq d-1$, $\rho \geq 1$, and let $\mathbf{Q}$ be the convex hull of the ρ-separable packing of n translates of the $\mathbf{o}$-symmetric convex body $\mathbf{C}$ in $\mathbb{E}^d$ such that $M_i(\mathbf{Q})$ is minimal and

$$n \geq \frac{4^d d^{4d}}{\delta_{\rm sep}(\rho, \mathbf{C})^{d-1}} \cdot \left(\rho \frac{R(\mathbf{C})}{r(\mathbf{C})}\right)^d. \tag{11}$$

By the minimality of $M_i(\mathbf{Q})$ we have that

$$M_i(\mathbf{Q}) \leq M_i(R\,\mathbf{B}^d + \mathbf{C}) \leq M_i((R + \rho R(\mathbf{C}))\,\mathbf{B}^d) \tag{12}$$

with $R = R_{\mathbf{C}}(\rho, n)$. Note that (12) and Lemma 3 imply that

$$M_i(\mathbf{Q}) \leq \left(1 + \frac{2\delta_{\rm sep}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^i M_i(\mathbf{B}^d) \left(\frac{\mathrm{vol}_d(\mathbf{C})}{\delta_{\rm sep}(\rho, \mathbf{C})\kappa_d}\right)^{\frac{i}{d}} \cdot n^{\frac{i}{d}}.$$

We examine the function $x \mapsto (1+x)^i$, where, by (11), we have $x \leq x_0 = \frac{1}{2d^4}$. The convexity of this function implies that $(1+x)^i \leq 1 + i(1+x_0)^{i-1}x$. Thus, from the inequality $\left(1 + \frac{1}{2d^4}\right)^{d-1} \leq \frac{33}{32} < 1.05$, where $d \geq 2$, the upper bound for $M_i(\mathbf{Q})$ in Theorem 1 follows.

On the other hand, in order to prove the lower bound for $M_i(\mathbf{Q})$ in Theorem 1, we start with the observation that (10) (based on (12)), (11), and Lemma 3 yield that

$$S(\mathbf{Q}) \leq S((R + \rho R(\mathbf{C}))\,\mathbf{B}^d) \leq d\kappa_d \left(\frac{n\,\mathrm{vol}_d(\mathbf{C})}{\delta_{\rm sep}(\rho, \mathbf{C})\kappa_d}\right)^{\frac{d-1}{d}} \left(1 + \frac{2\delta_{\rm sep}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{d-1}. \tag{13}$$

Thus, (13) together with the inequalities $S(\mathbf{Q})r(\mathbf{Q}) \geq \mathrm{vol}_d(\mathbf{Q})$ (cf. [11]) and $\mathrm{vol}_d(\mathbf{Q}) \geq n\,\mathrm{vol}_d(\mathbf{C})$ yield

$$r(\mathbf{Q}) \geq \left(1 + \frac{2\delta_{\rm sep}(\rho, \mathbf{C})^{\frac{1}{d}} \rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{-(d-1)} \frac{\mathrm{vol}_d(\mathbf{C})^{\frac{1}{d}} \delta_{\rm sep}(\rho, \mathbf{C})^{\frac{d-1}{d}}}{d\kappa_d^{\frac{1}{d}}} \cdot n^{\frac{1}{d}}. \tag{14}$$

Applying the assumption (11) and $\mathrm{vol}_d(\mathbf{C}) \geq \kappa_d r(\mathbf{C})^d$ to (14), we get that

$$r(\mathbf{Q}) \geq \left(1 + \frac{1}{2d^4}\right)^{-(d-1)} \frac{\delta_{\rm sep}(\rho, \mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})}{d} n^{\frac{1}{d}} \geq \frac{4d^3}{(1 + \frac{1}{2d^4})^{d-1}} R(\mathbf{C}) \geq 31 R(\mathbf{C}). \tag{15}$$

Let $\mathbf{P}$ denote the convex hull of the centers of the translates of $\mathbf{C}$ in $\mathbf{Q}$. Then, (15) implies

$$r(\mathbf{P}) \geq r(\mathbf{Q}) - R(\mathbf{C}) \geq \frac{30}{31} r(\mathbf{Q}) \geq \frac{8\delta_{\rm sep}(\rho, \mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})}{9d} \cdot n^{\frac{1}{d}}. \tag{16}$$

Hence, by (16) and Lemma 2,

$$\mathrm{vol}_d(\mathbf{Q}) \geq \mathrm{vol}_d(\mathbf{P}) \geq \left(1 + \frac{9d\rho R(\mathbf{C})}{4\delta_{\rm sep}(\rho, \mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{-d} \cdot \frac{n\,\mathrm{vol}_d(\mathbf{C})}{\delta_{\rm sep}(\rho, \mathbf{C})}, \tag{17}$$

which implies in a straightforward way that

$$\mathrm{vol}_d(\mathbf{Q}) \geq \left(1 + \frac{9d\rho R(\mathbf{C})}{4\delta_{\rm sep}(\rho, \mathbf{C}) r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{-d} \cdot \frac{n\,\mathrm{vol}_d(\mathbf{C})}{\delta_{\rm sep}(\rho, \mathbf{C})}. \tag{18}$$

Note that (10) (see the restated version for $j = d$) implies that $M_i(Q) \geq \left(\frac{\mathrm{vol}_d(Q)}{\kappa_d}\right)^{\frac{i}{d}} \kappa_i$. Then, replacing $\mathrm{vol}_d(Q)$ by the right-hand side of (18), and using the convexity of the function $x \mapsto (1+x)^{-i}$ for $x > -1$ yields the lower bound for $M_i(\mathbf{Q})$in Theorem 1.

Finally, we prove the statement about the spherical shape of $\mathbf{Q}$, that is, the inequality (1). As in [8], let

$$\theta(d) = \frac{1}{2^{\frac{d+3}{2}}\sqrt{2\pi}\sqrt{d}(d-1)(d+3)} \min\left\{ \frac{3}{\pi^2 d(d+2)2^d}, \frac{16}{(d\pi)^{\frac{d-1}{4}}} \right\}.$$

Using the inequality $\frac{\kappa_{d-1}}{\kappa_d} \geq \sqrt{\frac{d}{2\pi}}$ (cf. [1]) and (6) of [10], we obtain

$$\left(\frac{S(\mathbf{Q})}{S(\mathbf{B}^d)}\right)^d \left(\frac{\mathrm{vol}_d(\mathbf{B}^d)}{\mathrm{vol}_d(\mathbf{Q})}\right)^{d-1} - 1 \geq \theta(d) \cdot \left(1 - \frac{r(\mathbf{Q})}{R(\mathbf{Q})}\right)^{\frac{d+3}{2}}$$

(see also (5) of [8]). Substituting (13) and (17) into this inequality, we obtain

$$\left(1 + \frac{2\delta_{\mathrm{sep}}(\rho,\mathbf{C})^{\frac{1}{d}}\rho R(\mathbf{C})}{r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{d(d-1)} \left(1 + \frac{9d\rho R(\mathbf{C})}{4\delta_{\mathrm{sep}}(\rho,\mathbf{C})^{\frac{d-1}{d}} r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}}\right)^{d(d-1)} \geq \left(\frac{S(\mathbf{Q})}{S(\mathbf{B}^d)}\right)^d \left(\frac{\mathrm{vol}_d(\mathbf{B}^d)}{\mathrm{vol}_d(\mathbf{Q})}\right)^{d-1}.$$

By the assumptions $d \geq 2$ and (11), it follows that

$$4d^2(d-1)\frac{\rho R(\mathbf{C})}{\delta_{\mathrm{sep}}(\rho,\mathbf{C}) r(\mathbf{C})} \cdot \frac{1}{n^{\frac{1}{d}}} \geq \theta(d)\left(1 - \frac{r(\mathbf{Q})}{R(\mathbf{Q})}\right)^{\frac{d+3}{2}}. \qquad (19)$$

Note that by [12], $\frac{1}{\delta_{\mathrm{sep}}(\rho,\mathbf{C})} \leq \frac{2^{\frac{3d}{2}} \cdot \sqrt{\binom{\frac{d(d+1)}{2}}{d}}}{(d+1)^{\frac{d}{2}}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}+1\right)}$. This and (19) implies (1), finishing the proof of Theorem 1.

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