ABOUT THE DISTANCE BETWEEN RANDOM WALKERS ON SOME GRAPHS

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Abstract
We consider two or more simple symmetric walks on $\mathbb{Z}^d$ and the 2-dimensional comb lattice, and investigate the properties of the distance among the walkers.

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1 Introduction

Almost a hundred years ago, Pólya [11] in 1921 proved that on $\mathbb{Z}^1$ and $\mathbb{Z}^2$ simple random walks are recurrent and two independent walkers meet infinitely often with probability one, but on $\mathbb{Z}^d$, for $d \geq 3$, simple random walks are transient and two independent walkers meet only finitely often with probability one. Nowadays meeting at the same place at the same time is called a collision, so we will use this term, to avoid any confusion. On $\mathbb{Z}$ two walkers not only collide infinitely many times, but they collide even in the origin infinitely many times with probability one. In their landmark paper Dvoretzky and Erdős [8] in 1950 recall the celebrated Pólya result and, among their final

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remarks, they mention without proof that on $\mathbb{Z}$ three independent random walkers collide (all three together) infinitely often with probability one. A short elegant proof was given for this statement in Barlow, Peres and Sousi [1] in 2012. However four walkers in $\mathbb{Z}$, three walkers in $\mathbb{Z}^2$ and two walkers in $\mathbb{Z}^d$, with $d \geq 3$ will only collide finitely many times with probability 1. Khrishnapur and Peres [10] in 2004 studied this problem on the comb lattice. They proved that even though the comb is recurrent, two independent random walkers on the comb lattice collide only finitely often with probability 1.

Our question is that in case the walkers collide only finitely often, then how does their distance grow as a function of time. We want to establish upper class results for the distance of two or more walkers and lower class results for the distance of four or more walkers on $\mathbb{Z}$. Similarly, for the distance in $\mathbb{Z}^2$, we give upper class results for the distance of two or more walkers and lower class results for the distance of three or more walkers. For $\mathbb{Z}^d$, if $d \geq 3$, we get upper and lower class results for the distance of two or more walkers. Finally, we will investigate the distance of two or more walkers on the comb lattice.

We start with some definitions. Let $\mathbf{G}$ be a connected graph with vertex set $\mathbf{V}(\mathbf{G})$. Two neighboring connected vertices v and w form an edge of $\mathbf{G}$. A random walk $S(n)$ on $\mathbf{G}$ is defined with the following one step transition probabilities

$$p(u,v) := P(S(n+1) = v | S(n) = u) = \frac{1}{deg(u)} \tag{1.1}$$

for neighboring vertices u and v in $\mathbf{V}(\mathbf{G})$, where $deg(u)$ is the number of neighbours of u, otherwise $p(u,v) = 0$. We define the *graph distance*, which we will simply call *distance*, of u and v in $\mathbf{V}(\mathbf{G})$ as the minimal number of steps the walker needs to arrive from u to v. Formally,

$$d(u,v) := \min\{k > 0 : \ P(S(n+k) = v | S(n) = u) > 0\}. \tag{1.2}$$

Or, equivalently, the distance $d(u,v)$ is the length of the shortest path from vertex u to vertex v in $\mathbf{V}(\mathbf{G})$. In $\mathbb{Z}^d$ we will use Euclidean distance. In $\mathbb{Z}$ however these two distances are the same.

2 Preliminary results

In this section we list some known important results which we will need later on. Put $\mathbf{W}(t) = (W_1(t), W_2(t), ..., W_d(t))$, where $W_1(t), W_2(t), ..., W_d(t)$ are independent standard Wiener processes. Then the $\mathbb{R}^d$ valued process $\mathbf{W}(t)$ is called the standard d-dimensional Wiener process. Let $\mathbf{S}(n)$ be the location of a walker in $\mathbb{Z}^d$ after n-steps, where the simple symmetric walk $\mathbf{S}(n) = \mathbf{X}(1) + \mathbf{X}(2) + ... + \mathbf{X}(n)$, $n = 1, 2, ...$ and $\mathbf{X}(1), \mathbf{X}(2), ..., \mathbf{X}(n)$ are i.i.d. random vectors with $\mathbf{S}(0) = 0$,

$$P(\mathbf{X}(1) = e_i) = P(\mathbf{X}(1) = -e_i) = \frac{1}{2d}, \ i = 1, 2, ..., d, \tag{2.1}$$

where $e_1, e_2, ..., e_d$ are the orthogonal unit-vectors in $\mathbb{Z}^d$.

In Section 3 we investigate the distance on $\mathbb{Z}^d$, and show that the upper bound is a consequence of the law of the iterated logarithm (LIL), and that the lower bound can be established using a result of Dvoretzky and Erdős [8]. For the multidimensional LIL we refer, e.g., to Révész [12], Theorem 19.1.

Theorem A *For the d-dimensional standard Wiener process and the simple symmetric random walk we have for any $d \geq 1$*

$$\limsup_{t\to\infty} \frac{\|\mathbf{W}(t)\|}{\sqrt{t \log\log t}} = \sqrt{2} \quad a.s. \tag{2.2}$$

and

$$\limsup_{n\to\infty} \frac{\|\mathbf{S}(n)\|}{\sqrt{n \log\log n}} = \sqrt{\frac{2}{d}} \quad a.s. \tag{2.3}$$

We use the following definition (cf. Révész [12], page 36): The function $g(t)$ belongs to the lower-lower class (LLC) of the random process $\{Y(t),\, t \geq 0\}$ if for almost all $\omega \in \Omega$ there exists a $t_0 = t_0(\omega)$ such that $Y(t) > g(t)$ if $t > t_0$.

Theorem B Dvoretzky-Erdős [8] *Let $a(t)$ be a nonincreasing nonnegative function. Then, for the d-dimensional random walk $\mathbf{S}(n)$ and the standard Wiener process $\mathbf{W}(t)$*

$$t^{1/2}a(t) \in \mathrm{LLC}(\|\mathbf{W}(t)\|) \qquad (d \geq 3) \tag{2.4}$$

$$n^{1/2}a(n) \in \mathrm{LLC}(\|\mathbf{S}(n)\|) \qquad (d \geq 3) \tag{2.5}$$

if and only if

$$\sum_{n=1}^{\infty} (a(2^n))^{d-2} < \infty.$$

Here and throughout $\|\cdot\|$ denotes the Euclidean distance in d dimensions.

Remark 2.1 *Note that the same results hold when $a(n)$ is replaced by $c\,a(n)$ with any positive constant c.*

We will need a special case of the multidimensional invariance principle, explicitly stated in Révész [12], Theorem 18.2. (There are more precise results in the literature, but we don't need them here.)

Theorem C *On a rich enough probability space one can define a standard d-dimensional Wiener process $\{\mathbf{W}(t),\, t > 0\}$ and a simple random walk $\{\mathbf{S}(n),\, n = 0, 1, 2...\}$ on $\mathbb{Z}^d$ such that* for $d \geq 1$

$$\|\mathbf{S}(n) - \mathbf{W}(n/d)\| = O(n^{1/4}(\log\log n)^{3/4}) \quad \text{a.s.} \quad \textit{as } n \to \infty.$$

We will use the following result from linear algebra.

Lemma D *Let $\mathbb{S}$ be a vector space with basis $\{w_i,\ i = 1, 2, ..., n\}$. Define two subspaces $\mathbb{U}$ and $\mathbb{V}$ in $\mathbb{S}$ by*

$$\mathbb{U} = Span\{w_1 - w_i,\ i = 2, 3, ..., n\}$$
$$\mathbb{V} = Span\{\sum_{j=1}^{i}(w_j - w_{i+1}),\ i = 1, 2, ..., n-1\}. \tag{2.6}$$

Then the two subspaces defined above are the same.

The results presented so far are needed for proving our theorems about the d-dimensional random walk in Section 3. In Section 4 we consider the case of simple random walk on the 2-dimensional comb.

The 2-dimensional comb lattice $\mathbb{C}^2$ is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the x-axis. In this context the x-axis is usually called the backbone of the comb and the vertical lines are called teeth. A formal way of describing a simple random walk $\mathbf{C}(n)$ on the above 2-dimensional comb lattice $\mathbb{C}^2$ can be formulated via its transition probabilities as follows: for any integers x and y define

$$P(\mathbf{C}(n+1) = (x, y \pm 1) \mid \mathbf{C}(n) = (x, y)) = \frac{1}{2}, \quad \text{if } y \neq 0, \tag{2.7}$$

$$P(\mathbf{C}(n+1) = (x \pm 1, 0) \mid \mathbf{C}(n) = (x, 0)) = P(\mathbf{C}(n+1) = (x, \pm 1) \mid \mathbf{C}(n) = (x, 0)) = \frac{1}{4}. \tag{2.8}$$

A compact way of describing the just introduced transition probabilities for this simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ is via (1.1).

As far as we know, the first discussion of random walk on the comb was given by Weiss and Havlin [14]. Bertacchi and Zucca [3] obtained the following space-time asymptotic estimates for the n-step transition probabilities $p^{\mathbb{C}^2}(u, v, n)$, where $u = (x_1, y_1)$ and $v = (x_2, y_2)$ are two vertices on the comb. For any u and v fixed vertices on $\mathbb{C}^2$, with $deg(\cdot)$ as in (1.1),

$$p^{\mathbb{C}^2}(u, v, n) \sim \frac{2^{\frac{1}{4}-1} deg(v)}{\Gamma(\frac{1}{4}) n^{\frac{3}{4}}}, \qquad \text{as } n \to \infty \tag{2.9}$$

whenever $n + d(u, v)$ is even, and $p^{\mathbb{C}^2}(u, v, n) = 0$ if $n + d(u, v)$ is odd, and $\sim$ stands for asymptotic equality.

Here we recall our construction of two dimensional comb walk from [5], which we used there to prove Theorem K below, as we will need some parts of this construction later on. Consider a sample space large enough to contain two independent simple symmetric random walks $S_1(n)$ and $S_2(n)$, $n = 1, 2, \ldots$, on the integer lattice on the line, and an i.i.d. sequence of geometric random variables $G_i, i = 1, 2, \ldots$, with

$$P(G_1 = k) = \frac{1}{2^{k+1}}, \quad k = 0, 1, 2, \ldots$$

which is independent from the two random walks as well. We may then construct a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$ as follows. Let $\rho_2(N)$ be the time of the N-th return to zero of the second random walk $S_2(\cdot)$, i.e., $\rho(0) = 0$, and

$$\rho_2(N) := \min\{j > \rho_2(N-1) : S_2(j) = 0\}.$$

Put $T_N = G_1 + G_2 + \ldots G_N$, $N = 1, 2, \ldots$. For $n = 0, \ldots, T_1$, let $C_1(n) = S_1(n)$ and $C_2(n) = 0$. For $n = T_1 + 1, \ldots, T_1 + \rho_2(1)$, let $C_1(n) = C_1(T_1)$, $C_2(n) = S_2(n - T_1)$. In general, for $T_N + \rho_2(N) < n \leq T_{N+1} + \rho_2(N)$, let

$$C_1(n) = S_1(n - \rho_2(N)),$$

$$C_2(n) = 0,$$

and, for $T_{N+1} + \rho_2(N) < n \leq T_{N+1} + \rho_2(N+1)$, let

$$C_1(n) = C_1(T_{N+1} + \rho_2(N)) = S_1(T_{N+1}),$$

$$C_2(n) = S_2(n - T_{N+1}).$$

Then it can be seen in terms of these definitions for $C_1(n)$ and $C_2(n)$ that $\mathbf{C}(n) = (C_1(n), C_2(n))$ is a simple random walk on the 2-dimensional comb lattice $\mathbb{C}^2$. Define the local time of $S_i(n)$, $i = 1, 2$, at zero by

$$\xi_i(0, n) := \sum_{j=1}^{n} I\{S_i(j) = 0\}$$

and denote the number of horizontal and vertical steps of $\mathbf{C}(n)$ by H_n and V_n respectively, with $H_n + V_n = n$. Clearly H_n is the sum of $\xi_2(0, V_n)$ i.i.d. geometric random variables G_i as described above, where the last geometric random variable may be truncated. Moreover $\mathbf{C}(n) = (C_1(n), C_2(n)) = (S_1(H_n), S_2(V_n))$.

We will also need the following increment result of Csörgő and Révész ([7], page 115) for a random walk.

Theorem E *Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with mean zero and variance one, satisfying the following condition: there exists a t_0 such that $E(e^{tX_1})$ is finite for $|t| < t_0$. Let $0 < a_N$ be a non-decreasing sequence of integers such that N/a_N is also non-decreasing and $N/a_N \to \infty$. Then, for $S(n) = X_1 + X_2 + \cdots + X_n$, as $N \to \infty$, we have almost surely that*

$$\max_{0 \leq N - a_N} \max_{k \leq a_N} |S(n+k) - S(n)| = O(a_N^{1/2} (\log(N/a_N) + \log\log N)^{1/2}).$$

The following theorem is a version of Hoeffding's inequality, which is stated explicitly in [13].

Theorem F *Let G_i be i.i.d. random variables with the common geometric distribution $P(G_i = k) = 2^{-k-1}$, $\quad k = 0, 1, 2, \ldots$. Then*

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}(G_i-1)\right|>\lambda\right)\le 2\exp(-\lambda^2/8n)$$

for $0<\lambda<na$, *with some* $a>0$.

Consequence 2.1 *For any small* $0<\delta<a$ *and* n *big enough*

$$P\left(\sum_{i=1}^{n}G_i\ge(1+\delta)n\right)\le\frac{1}{n^2}.$$

Let $S(n)$ be a simple symmetric random walk on the line with local time $\xi(0,n)$ at zero. We recall from Csáki and Földes [6] the following result.

Theorem G *Suppose that* $x_n\to\infty$, $\dfrac{x_n}{n^{1/2}}\to 0$ *as* $n\to\infty$. *Then, for any* $0<\epsilon\le 1$,

$$P(\xi(0,n)\ge x_n n^{1/2})\le c\exp\left(-\frac{(1-\epsilon)x_n^2}{2}\right)$$

for some constant c *if* n *is large enough.*

Remark 2.2 *In what follows, we will disregard parity issues. Namely we will use results for the* $n-$*step transition probability for arbitrary* n,u *and* v, *when these results are only proved for even* n,u *and* v. *When these values are not even, and the corresponding probability is not zero, then their asymptotic behavior is the same as for even values (see Bertacchi and Zucca* [4], Sections 3, 4 and 10*).*

For completeness, we present the following trivial lemma, which is likely well-known.

Lemma H *Let* u *and* v *be two distinct vertices on* $\mathbb{C}^2$. *For any* n *such that* $n+d(u,v)$ *is even*

$$deg(u)p^{\mathbb{C}^2}(u,v,n)=deg(v)p^{\mathbb{C}^2}(v,u,n), \tag{2.10}$$

where $deg(\cdot)$ *is as in* (1.1).

Proof. Denote the set of all n-steps paths connecting vertex u to vertex v through the vertices $z_1,z_2,...,z_{n-1}$ by $\lambda_i:=(u,z_1^i,z_2^i,...z_{n-1}^i,v)$, and the set of all paths of length n from u to v by $\Lambda(u,v,n)$ and, for any vertex z, let $q(z)=\frac{1}{deg(z)}$. Then

$$p^{\mathbb{C}^2}(u,v,n)=\sum_{\lambda_i\in\Lambda(u,v,n)}q(u)q(z_1^i)q(z_2^i)...q(z_{n-1}^i),$$

$$p^{\mathbb{C}^2}(v,u,n)=\sum_{\lambda_i^*\in\Lambda(v,u,n)}q(v)q(z_{n-1}^i)q(z_{n-2}^i)...q(z_1^i),$$

where λ_i^* is the reversed path of λ_i. Hence, the above two probabilities are equal if $q(u) = q(v)$, and differ in a factor 2 if u and v are having different degrees. $\square$

In (2.9) the two vertices are fixed. We will need a more general estimate for the n-step transition probabilities $p^{\mathbb{C}^2}((0,0),(0,k),n)$. Define

$$\kappa = k/n \ \text{ and } \ \phi(\kappa) = \log((1-\kappa)^{\kappa-1}(1+\kappa)^{-\kappa-1}).$$

We recall the second statement from Theorem 5.5 of Bertacchi and Zucca [4].

Theorem I *If $\kappa \in [0, n^{-1/2-\epsilon}]$ for some $\epsilon > 0$, then as $n \to \infty$*

$$p^{\mathbb{C}^2}((0,2k),(0,0),2n) \sim \frac{\sqrt{2}e^{n\phi(\kappa)}}{\Gamma(1/4)n^{3/4}},$$

uniformly with respect to $\kappa \in [0, n^{-1/2-\epsilon}]$.

A simple calculation shows that $\phi(\kappa)$ is decreasing for $\kappa \geq 0$, hence $\phi(\kappa) \leq \phi(0) = 0$. Consequently $e^{n\phi(\kappa)} \leq 1$. Combining this with Lemma H, the following obtains.

Consequence 2.2 *For $k \leq n^{1/2-\epsilon}$ and some c>0,*

$$p^{\mathbb{C}^2}((0,0),(0,k),n) \leq \frac{c}{n^{3/4}}.$$

For any two vertices u and v on $\mathbb{C}^2$, we define the Green function associated with the random walk on the comb as

$$G(u,v|z) := \sum_{n=0}^{\infty} p^{\mathbb{C}^2}(u,v,n)z^n.$$

Bertacchi and Zucca in [4] show that for $u = (0,0)$, $G((0,0),(k,\ell)|z)$ can be given explicitly as follows:

$$G((0,0),(k,\ell)|z) = \begin{cases} \frac{1}{2}G(z)(F_1(z))^{|k|}(F_2(z))^{|\ell|} & \text{if } \ell \neq 0 \\ G(z)(F_1(z))^{|k|} & \text{if } \ell = 0, \end{cases}$$

where

$$G(z) = \frac{\sqrt{2}}{\sqrt{1-z^2+\sqrt{1-z^2}}},$$
$$F_1(z) = \frac{1+\sqrt{1-z^2}-\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}}{z},$$
$$F_2(z) = \frac{1-\sqrt{1-z^2}}{z}.$$

We will use this elegant result to get the asymptotic behavior of the probability $P(C_2(n) = 0)$. Selecting $\ell = 0$ and summing for all $k = 0, \pm 1, \pm 2, ...$, we easily obtain the generating function of $P(C_2(n) = 0)$. Namely, with the notation

$$H(z) := \sum_{n=0}^{\infty} P(C_2(n) = 0) z^n,$$

we have

$$H(z) = \sum_{k=-\infty}^{\infty} G((0,0)(k,0)|z) = G(z) + 2\sum_{k=1}^{\infty} G(z)(F_1(z))^k = G(z)\frac{1+F_1(z)}{1-F_1(z)}.$$

Now, just like as it is used in [2], we may apply the Hardy-Littlewood-Karamata theorem in the following form: If

$$F(z) = \sum a_n z^n \sim \frac{C}{(1-z)^{\alpha}} \quad \text{as} \quad z \to 1^- \quad \text{with} \quad \alpha \notin \{0, -1, -2, ...\},$$

and if $F(z)$ is analytic in some domain with the exception of $z = 1$, then

$$a_n \sim \frac{C}{\Gamma(\alpha)} n^{\alpha - 1} \quad \text{as} \quad n \to \infty.$$

An easy calculation yields the asymptotic behavior of $H(z)$. We only give a short indication of this calculation:

$$\begin{aligned}
G(z)\frac{(1+F_1(z))}{1-F_1(z)} &= \frac{\sqrt{2}}{\sqrt{1-z^2+\sqrt{1-z^2}}} \frac{z+1+\sqrt{1-z^2}-\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}}{z-1-\sqrt{1-z^2}+\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}} \\
&\sim \frac{\sqrt{2}}{(1-z^2)^{1/4}} \frac{2}{\sqrt{2}(1-z^2)^{1/4}} \\
&= \frac{2}{\sqrt{(1-z)(1+z)}} \\
&\sim \frac{\sqrt{2}}{\sqrt{1-z}}, \quad \text{as} \quad z \to 1^-.
\end{aligned}$$

Thus, we have $\alpha = 1/2$, implying the following result.

Consequence 2.3 *For the second coordinate $C_2(n)$ of the comb walk we have*

$$P(C_2(n) = 0) \sim \frac{\sqrt{2}}{\sqrt{\pi n}}, \quad \text{as } n \to \infty.$$

A further insight to the nature of the random walk on a comb was provided by Bertacchi [2], who established the following remarkable weak convergence result for the walk $\mathbf{C}(n) = (C_1(n), C_2(n))$ on the comb $\mathbb{C}^2$.

Theorem J *For the random walk* $\{\mathbf{C}(n) = (C_1(n), C_2(n)); n = 0, 1, 2, \ldots\}$ *on* $\mathbb{C}^2$,*we have*

$$\left(\frac{C_1(nt)}{n^{1/4}}, \frac{C_2(nt)}{n^{1/2}};\, t \geq 0\right) \xrightarrow{\text{Law}} (W_1(\eta_2(0,t)), W_2(t);\, t \geq 0), \quad n \to \infty, \tag{2.11}$$

where W_1, W_2 *are two independent standard Wiener processes (Brownian motions) and* $\eta_2(0,t)$ *is the local time process of* W_2 *at zero, and* $\xrightarrow{\text{Law}}$ *denotes weak convergence on* $C([0,\infty), \mathbb{R}^2)$ *endowed with the topology of uniform convergence on compact intervals.*

For the definition of $\eta_2(0,t)$ see e.g. Révész [12], page 107.

In our paper [5] we gave a joint strong approximation result for the two coordinates of this walk.

Theorem K *On an appropriate probability space for the random walk* $\{\mathbf{C}(n) = (C_1(n), C_2(n));$ $n = 0, 1, 2, \ldots\}$ *on* $\mathbb{C}^2$, *one can construct two independent standard Wiener processes* $\{W_1(t);\, t \geq 0\}$, $\{W_2(t);\, t \geq 0\}$ *so that, as* $n \to \infty$, *we have with any* $\varepsilon > 0$

$$n^{-1/4}|C_1(n) - W_1(\eta_2(0,n))| + n^{-1/2}|C_2(n) - W_2(n)| = O(n^{-1/8+\varepsilon}) \quad \text{a.s.},$$

where $\eta_2(0,\cdot)$ *is the local time process at zero of* $W_2(\cdot)$.

From the many consequences of this result, we will need the following two.

Corollary 2.1 *For the horizontal and vertical coordinates of* $\mathbf{C}(n) = (C_1(n), C_2(n))$ *we have*

$$\limsup_{n\to\infty} \frac{|C_1(n)|}{n^{1/4}(\log\log n)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.}, \tag{2.12}$$

$$\limsup_{n\to\infty} \frac{|C_2(n)|}{(2n\log\log n)^{1/2}} = 1 \quad \text{a.s.} \tag{2.13}$$

For the distribution of the hitting time of a simple random walk, we need the following result (cf., e.g., Feller [9], Ch. 3.7, Theorem 2 and Theorem 3).

Theorem L *Let* $\{S(i),\, i = 0, 1, 2, \ldots\}$ *be a simple symmetric random walk on the line with* $S(0) = 0$, *and define the hitting time*

$$\beta(r) = \min\{i > 0 :\, S(i) = r\}, \tag{2.14}$$

where r *is a positive integer. Then*

$$P(\beta(r) = N) = \frac{r}{N}\binom{N}{\frac{N+r}{2}} 2^{-N}, \quad N = r, r+1, \ldots \tag{2.15}$$

whith $N + r$ *even, and*

$$\lim_{r\to\infty} P(\beta(r) < ur^2) = \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{u}}^{\infty} e^{-s^2/2}\, ds, \quad u > 0. \tag{2.16}$$

3 Distance on $\mathbb{Z}^d$

Let $\{\mathbf{S}_i(\cdot), \quad i=1,2,...,K\}$, be K independent random walks on $\mathbb{Z}^d$, the paths of the K walkers. We consider the maximal distance between K walkers as follows.

$$D_K^{\mathbb{Z}^d}(n) := \max_{i\neq j,\ i,j\leq K} \|\mathbf{S}_i(n) - \mathbf{S}_j(n)\|,$$

where $\|\cdot\|$ denotes Euclidean distance.

Similarly, for $\{\mathbf{W}_i(\cdot), \quad i=1,2,...,K\}$, K independent standard d-dimensional Wiener processes, all starting from 0, let

$$D_K^{\mathbb{R}^d}(t) = \max_{i\neq j,\ i,j\leq K} \|\mathbf{W}_i(t) - \mathbf{W}_j(t)\|.$$

Concerning upper class results, we prove our next result from the law of the iterated logarithm (LIL).

Theorem 3.1 *For $K\geq 2$ and $d=1,2,\ldots,$ we have*

$$\limsup_{t\to\infty} \frac{D_K^{\mathbb{R}^d}(t)}{\sqrt{t\log\log t}} = 2 \quad a.s. \tag{3.1}$$

and

$$\limsup_{n\to\infty} \frac{D_K^{\mathbb{Z}^d}(n)}{\sqrt{n\log\log n}} = \frac{2}{\sqrt{d}} \quad a.s. \tag{3.2}$$

For the lower classes, we prove the following results.

Theorem 3.2 *For $d=1,2,\ldots$ and $K\geq 1+\frac{3}{d}$ we have*

$$\sqrt{t}a(t) \in \mathrm{LLC}(D_K^{\mathbb{R}^d}(t)) \tag{3.3}$$

and

$$\sqrt{n}a(n) \in \mathrm{LLC}(D_K^{\mathbb{Z}^d}(n)) \tag{3.4}$$

if and only if

$$\sum_{n=1}^{\infty} (a(2^n))^{Kd-d-2} < \infty. \tag{3.5}$$

The above theorems show that the behavior of distance of random walks and that of the distance of Wiener processes are very similar. This is not the case in two dimensions with two walkers. As it was mentioned in the Introduction, on $\mathbb{Z}^2$ two independent walkers will collide infinitely often almost surely. On the other hand, two independent standard Wiener processes won't collide infinitely often, their distance for $t > 0$ big enough, will be at least $t^{-(\log t)^{\epsilon}}$ for any $\epsilon > 0$, (see Révész [12], page 208, Remark 3).

Proof of Theorem 3.1. It suffices to prove (3.1) for Wiener processes. The random walk case (3.2) follows from the strong approximation in Theorem C.

Consider the case $K = 2$ first. Then

$$D_2^{\mathbb{R}^d}(t) = \|\mathbf{W}_1(t) - \mathbf{W}_2(t)\| = \sqrt{2}\|\mathbf{W}^*(t)\|,$$

where $\mathbf{W}^*$ is a standard d-dimensional Wiener process. Hence by Theorem A, (3.1) is true in this case. $D_K^{\mathbb{R}^d}(t)$ is the maximum of $\binom{K}{2}$ distances for each of which (3.1) holds. This implies the upper part of the conclusion. The lower part is immediate, namely

$$D_K^{\mathbb{R}^d}(t) \geq D_2^{\mathbb{R}^d}(t).$$

□

Proof of Theorem 3.2. We first prove the convergent part for $d = 1$. Define

$$W_i^*(t) := \frac{\sum_{j=1}^{i}(W_j(t) - W_{i+1}(t))}{\sqrt{i^2 + i}}, \ i = 1, 2, ..., K - 1. \tag{3.6}$$

It is an easy calculation to show that $W_i^*(t)$ $i = 1, 2, ..., K - 1$ are independent standard Wiener processes. Hence for the $K - 1$ dimensional standard Wiener process defined by

$$\mathbf{W}^*(t) := (W_1^*(t), W_2^*(t), ..., W_{K-1}^*(t)),$$

we can apply Theorem B with $d = K - 1$ to get that, if $\{a(n),\ n = 1, 2, ...\}$ satisfy (3.5) then for $t > t_0(\omega)$ we have

$$\|\mathbf{W}^*(t)\| \geq \sqrt{t}a(t).$$

This implies that for some $i \leq K - 1$

$$|W_i^*(t)| \geq \frac{\sqrt{t}(a(t))}{\sqrt{K - 1}},$$

which, in turn, implies that for the absolute value of one of the summands $W_j(t) - W_{i+1}(t),\ j = 1, 2, ..., i$ of $W_i^*(t)$, we have that

$$\left|\frac{W_j(t) - W_{i+1}(t)}{\sqrt{i^2 + i}}\right| \geq \frac{\sqrt{t}(a(t))}{i\sqrt{K - 1}},$$

implying that there exists a pair $1 \leq i < j \leq K$ such that

$$|W_i(t) - W_j(t)| \geq \frac{1}{\sqrt{K - 1}}\sqrt{t}a(t).$$

Thus, using Remark 2.1, we proved the convergent part of the theorem for $d = 1$.

Divergent part: Suppose that

$$\sum_{n=1}^{\infty} (a(2^n))^{K-3} = \infty. \tag{3.7}$$

Then, according to Theorem B again, there is random sequence $t_k \to \infty$ such that

$$\|\mathbf{W}^*(t_k)\| \leq \sqrt{t_k} a(t_k)$$

almost surely. Thus, for all $i = 1, 2, ..., K-1$,

$$|W_i^*(t_k)| \leq \sqrt{t_k} a(t_k)$$

almost surely as well. Now, applying Lemma D, we can conclude that each

$$\{W_i(t) - W_j(t),\ i < j,\ i = 1, 2, ..., K-1,\ j = 2, 3, ..., K\}$$

can be expressed as a linear combination of $W_i^*(t)$, $i = 1, 2, ..., K-1$. This in turn implies that for all $i < j,\ i = 1, 2, ..., K-1.\ j = 2, 3, ..., K$, we have almost surely for our random sequence $t_k \to \infty$ that, for some appropriate constant c_K we have

$$|W_i(t_k) - W_j(t_k)| \leq c_K \sqrt{t_k} a(t_k),$$

implying that, almost surely,

$$D_K^{\mathbb{R}}(t_k) \leq c_K \sqrt{t_k} a(t_k)$$

as well, proving Theorem 3.3 in case $d = 1$.

For $d > 1$, we perform the previous transformation for each coordinate separately. So we get d times $K-1$ independent Wiener processes. For these $d(K-1)$ independent Wiener processes we apply Theorem B again. Repeating the arguments in the case $d = 1$, we can finally obtain (3.3). Now (3.4) follows from the strong invariance in Theorem C. $\square$

Remark 3.1 *We can get the limiting distribution of $D_K^{\mathbb{Z}}(n)$ via the exact calculation for $D_K^{\mathbb{R}}(t)$. Let $W_i(t)/\sqrt{t} := N_i$ for $i = 1, 2, 3, ..., K$, and, conditioning on the largest of them, say N_1, we have*

$$P\left(\frac{D_K^{\mathbb{R}}(t)}{\sqrt{t}} < z\right) = K \int_{-\infty}^{\infty} P\left(\frac{D_K^{\mathbb{R}}(t)}{\sqrt{t}} \leq z,\ N_i < x,\ i = 2, 3, ..., K | N_1 = x\right) P(N_1 = x)\, dx$$

$$= K \int_{-\infty}^{\infty} P(x - z \leq N_i \leq x,\ i = 2, 3, ..., K) P(N_1 = x)\, dx$$

$$= K \int_{-\infty}^{\infty} (\Phi(x) - \Phi(x - z))^{K-1} \phi(x)\, dx = K \int_{-\infty}^{\infty} \left(\int_{x-z}^{x} \phi(u)\, du\right)^{K-1} \phi(x)\, dx.$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the distribution and density functions of the standard normal random variable. Consequently, we conclude the following result:

$$\lim_{n\to\infty} P\left(\frac{D_K^{\mathbb{Z}}(n)}{\sqrt{n}} < z\right) = K\int_{-\infty}^{\infty}\left(\int_{x-z}^{x}\phi(u)\,du\right)^{K-1}\phi(x)\,dx. \tag{3.8}$$

It might be of interest to get the d-dimensional analog of this result.

4 Distance on the comb

As it was mentioned in the Introduction, Krishnapur and Peres [10] introduced a fascinating class of graphs where simple random walks continue to be recurrent, but the respective paths of two independent random walks meet only finitely many times with probability 1. In particular, the 2-dimensional comb lattice has this property. So, for K independent walks $\{\mathbf{C}^{(i)}(n) = (C_1^{(i)}(n), C_2^{(i)}(n)) \quad i = 1, 2, ..., K\}$, we want to investigate

$$D_K^{\mathbb{C}^2}(n) = \max_{i\neq j,\ i,j\leq K} d(\mathbf{C}^{(i)}(n), \mathbf{C}^{(j)}(n)),$$

the maximal distance between the K walkers at time n, where $d(x, y)$ was defined in (1.2).

The upper class result is an easy consequence of our strong approximation in Theorem K.

Theorem 4.1 *For the distance of K walkers on the comb we have*

$$\limsup_{n\to\infty} \frac{D_K^{\mathbb{C}^2}(n)}{2\sqrt{n\log\log n}} = 1 \quad \text{a.s.} \tag{4.1}$$

Proof. First we prove the theorem for two walkers. Observe that for the second coordinates of our two walkers we have from Theorem K that

$$\begin{aligned} |C_2^{(1)}(n) - C_2^{(2)}(n)| &= |W^*(n) - W^{**}(n)| + O(n^{3/8+\epsilon}) \quad \text{a.s.}, \\ |C_2^{(1)}(n) + C_2^{(2)}(n)| &= |W^*(n) + W^{**}(n)| + O(n^{3/8+\epsilon}) \quad \text{a.s.}, \end{aligned} \tag{4.2}$$

where $W^*(n)$ and $W^{**}(n)$ are two independent standard Wiener processes. Then both $\dfrac{W^*(n) - W^{**}(n)}{\sqrt{2}}$ and $\dfrac{W^*(n) + W^{**}(n)}{\sqrt{2}}$ are standard Wiener processes again, for which the LIL holds. Combining this with (4.2), we get that

$$\limsup_{n\to\infty} \frac{|C_2^{(1)}(n) - C_2^{(2)}(n)|}{2\sqrt{n\log\log n}} = \limsup_{n\to\infty} \frac{|C_2^{(1)}(n) + C_2^{(2)}(n)|}{2\sqrt{n\log\log n}} = 1 \quad \text{a.s.}. \tag{4.3}$$

Applying now Corollary 2.1, (2.12) implies that $|C_1^{(1)}(n) - C_1^{(2)}(n)|$ can't have a significant contribution to $\limsup_n d(\mathbf{C}^{(1)}(n), \mathbf{C}^{(2)}(n))$. Thus the distance of the two walkers is essentially the difference or the sum of their second coordinates, depending on whether they are on the same tooth or not. Consequently, we get

$$\limsup_{n\to\infty} \frac{D_2^{\mathbb{C}^2}(n)}{2\sqrt{n\log\log n}} = 1 \quad \text{a.s.}\,, \tag{4.4}$$

so we have Theorem 4.1 for $K = 2$. From here on the proof for $K > 2$ is exactly the same as in Theorem 3.1; by definition, $D_K^{\mathbb{C}^2}(n)$ is the maximum of $\binom{K}{2}$ distances for each of which (4.4) holds. This implies the upper part of the theorem. The lower part is immediate, namely

$$D_K^{\mathbb{C}^2}(n) \geq D_2^{\mathbb{C}^2}(n).$$

□

We now turn to the lower class results.

Theorem 4.2 *For $K = 2$ and any $\epsilon > 0$*

$$P\left(D_2^{\mathbb{C}^2}(n) \leq (1+\epsilon)\frac{2^{9/4}}{3^{3/4}} n^{1/4}(\log\log n)^{3/4} \text{ i.o.}\right) = 1. \tag{4.5}$$

Proof. The main idea of the proof is the following. Consider the second coordinates of the two walkers. They behave like simple symmetric walks, except that sometimes, when horizontal steps occur, they don't move. But we know that in n steps the number of vertical steps is $n(1-o(1))$. If they actually would move like simple symmetric walks, then, as it was mentioned in the Introduction, they would meet infinitely often at the origin. This means that with probability one there would be infinitely many n_k when the second coordinates of the two walkers would be zero, and hence they both would be on the x-axis. At these occasions their distance can't be more than what Corollary 2.1 implies, i.e., thus we arrive to our conclusion as well.

Turning to the actual proof that the two walkers are on the x-axis at the same time infinitely often with probability 1, according to Consequence 2.3, for n big enough, we have

$$P(C_2(n) = 0) \geq \frac{1}{2\sqrt{n}}. \tag{4.6}$$

Consider now two independent walkers $\mathbf{C}^{(i)}(n) = (C_1^{(i)}(n), C_2^{(i)}(n)),\ i = 1, 2$ on the comb. Let U denote the number of collisions at zero of their second coordinates $C_2^{(1)}(n)$ and $C_2^{(2)}(n)$. Then by (4.6)

$$E(U) = E\left(\sum_{n=1}^{\infty} I\{C_2^{(1)}(n) = C_2^{(2)}(n) = 0\}\right) \geq \sum_{n=1}^{\infty}\left(\frac{1}{2\sqrt{n}}\right)^2 = +\infty.$$

As the number of collisions at zero of the second coordinates follows a geometric distribution, having infinite expectation implies that there is an infinite number of such collisions at zero. Thus, almost surely, there is a random sequence $n_k \to \infty$ such that $C_2^{(1)}(n_k)$ and $C_2^{(2)}(n_k)$ are simultaneously on the backbone (x-axis) of the comb. This, in turn, implies our theorem by (2.12) in Corollary 2.1. $\square$

As to the lower lower class (LLC) result, first we prove the following result for $K = 2$.

Theorem 4.3 *For every $\epsilon > 0$ for n big enough*

$$D_2^{\mathbb{C}^2}(n) > n^{1/4-\epsilon} \quad \text{a.s.}$$

Proof. Define the events

$$\begin{aligned} A_n &= \{D_2^{\mathbb{C}^2}(n) \le n^{1/4-\epsilon},\ C_1^{(1)}(n) \ne C_1^{(2)}(n)\}, \\ B_n &= \{D_2^{\mathbb{C}^2}(n) \le n^{1/4-\epsilon},\ C_1^{(1)}(n) = C_1^{(2)}(n)\}. \end{aligned} \tag{4.7}$$

Then

$$P(D_2^{\mathbb{C}^2}(n) \le n^{1/4-\epsilon}) = P(A_n) + P(B_n).$$

We show that

$$P(A_n\, i.o.) = P(B_n\, i.o.) = 0.$$

First we give an upper bound for $P(A_n)$. To this end, we need a couple of lemmas.

To begin with, consider only one walk $\mathbf{C}(n)$. Recall the construction of the comb walk in Section 2, where we defined $G_1, G_2, \dots$ to be i.i.d. geometric random variables with

$$P(G_1 = k) = \frac{1}{2^{k+1}}, \quad k = 0, 1, 2, \dots,$$

as the number of horizontal steps after each return to the backbone. Recall also that H_n and V_n are the number of horizontal and vertical steps, respectively, in the first n steps of $\mathbf{C}(\cdot)$. Then it is easy to see that

$$H_n \le \sum_{i=1}^{\xi_2(0,V_n)} G_i \le \sum_{i=1}^{\xi_2(0,n)} G_i,$$

where $\xi_2(0,\cdot)$ is the local time at zero of the simple symmetric walk $S_2(\cdot)$ of the vertical steps. Let

$$M(C_1, n) := \max_{0 \le k \le n} |C_1(k)|,$$

the absolute maximum of the horizontal coordinate of $\mathbf{C}(\cdot)$ in n steps.

Lemma 4.1 *For n big enough*

$$P(M(C_1, n) \ge n^{1/4} \log n) \le \frac{3}{n}.$$

Proof. First we give an estimate for the upper tail of H_n. Observe that, for n big enough, on account of Theorem G, we have with an appropriate constant $c > 0$ and arbitrary $\varepsilon \in (0, 1/2)$ that

$$P\left(\xi_2(0,n) \geq 2\sqrt{n \log n}\right) \leq \frac{c}{n^{2(1-\varepsilon)}} \leq \frac{1}{n}. \tag{4.8}$$

By (4.8) and applying Consequence 2.1, we get that

$$\begin{aligned} P\left(H_n \geq 3\sqrt{n \log n}\right) &\leq P\left(\sum_{i=1}^{\xi_2(0,n)} G_i \geq 3\sqrt{n \log n}\right) \\ &\leq P\left(\sum_{i=1}^{\xi_2(0,n)} G_i \geq 3\sqrt{n \log n},\, \xi_2(0,n) < 2\sqrt{n \log n}\right) + \frac{1}{n} \\ &\leq \frac{1}{n} + P\left(\sum_{i=1}^{2\sqrt{n \log n}} G_i \geq 3\sqrt{n \log n}\right) \leq \frac{2}{n} \end{aligned} \tag{4.9}$$

if n is big enough.

Let

$$M(n) := \max_{0 \leq k \leq n} |S(k)|$$

be the absolute maximum of a simple symmetric random walk $S(\cdot)$ in n steps. Recall that $C_1(n) = S_1(H_n)$. Then the well-known large deviation result for the maximum (see e.g. Révész [12], p. 21) and (4.9) imply that

$$\begin{aligned} P(M(C_1, n) \geq n^{1/4} \log n) &= P(\max_{0 \leq i \leq n} |C_1(i)| \geq n^{1/4} \log n) \\ &\leq \frac{2}{n} + P(M(3\sqrt{n \log n}) \geq n^{1/4} \log n) \leq \frac{3}{n} \end{aligned} \tag{4.10}$$

if n is big enough. □

Lemma 4.2 *Let* $\mathbf{C}(n) = (C_1(n), C_2(n))$ *be a random walk on* $\mathbb{C}^2$*. There exists a constant* $c > 0$ *such that we have*

$$P(C_1(n) = x, C_2(n) = y) \leq \frac{c}{n^{3/4}} \quad \textit{for all } (x, y) \textit{ with } |y| \leq n^{1/2-\epsilon}.$$

Proof. In what follows, unimportant constants will be denoted by c, whose value might change from line to line. For simplicity, we work with even coordinates, and that, as we remarked earlier, does not restrict generality. Recall that H_n is the number of horizontal steps in the first n steps of the walk. We have

$$
\begin{aligned}
P(C_1(2n)=2r, C_2(2n)=2j) &= \sum_k P(C_1(2n)=2r, C_2(2n)=2j|H_{2n}=2k)P(H_{2n}=2k)\\
&= \sum_k P(S_1(2k)=2r)P(C_2(2n)=2j|H_{2n}=2k)P(H_{2n}=2k)\\
&\le \sum_k P(S_1(2k)=0)P(C_2(2n)=2j|H_{2n}=2k)P(H_{2n}=2k)\\
&= P(C_1(2n)=0, C_2(2n)=2j). \qquad (4.11)
\end{aligned}
$$

The second equality above follows from the fact that when the number of horizontal steps are fixed, then $C_1(\cdot)$ is a simple symmetric walk, denoted by $S_1(\cdot)$, which is independent of the second coordinate. The above inequality, on the other hand, is true, as

$$
\max_{-k\le r\le k} P(S_1(2k)=2r) = P(S_1(2k)=0).
$$

To finish the proof, observe that by Lemma H, for $j\ne 0$,

$$
P(C_1(2n)=0, C_2(2n)=2j) = p^{\mathbb{C}^2}((0,0),(0,2j),2n) = \frac{1}{2}p^{\mathbb{C}^2}((0,2j),(0,0),2n).
$$

Now our lemma follows from Consequence 2.2. □

Returning now to the proof of Theorem 4.3, we can give the following upper bound for $P(A_n)$. Define the event

$$
U_n := \{M(C_1^{(i)},n)\le n^{1/4}\log n,\ i=1,2\}.
$$

Then, by Lemma 4.1, we have for U_n^c, the complement of U_n, that

$$
P(U_n^c)\le \frac{6}{n}.
$$

Define now the set of pairs of points on the comb

$$
\begin{aligned}
Q(n,\epsilon) = \{(x_1,y_1),(x_2,y_2) : |x_i|\le n^{1/4}\log n,\ i=1,2,\ |x_1-x_2|\le n^{1/4-\epsilon},\\
|y_1|\le n^{1/4-\epsilon},\ |y_2|\le n^{1/4-\epsilon}\}.
\end{aligned}
$$

By Lemma 4.2

$$
\begin{aligned}
P(A_n) &\le P(|C_2^{(1)}(n)|\le n^{1/4-\epsilon},\ |C_2^{(2)}(n)|\le n^{1/4-\epsilon},\ |C_1^{(1)}(n)-C_1^{(2)}(n)|\le n^{1/4-\epsilon})\\
&\le P(|C_2^{(1)}(n)|\le n^{1/4-\epsilon},\ |C_2^{(2)}(n)|\le n^{1/4-\epsilon},\ |C_1^{(1)}(n)-C_1^{(2)}(n)|\le n^{1/4-\epsilon}, U_n)+\frac{6}{n}\\
&\le \sum_{Q(n,\epsilon)} P(\mathbf{C}^{(1)}(n)=(x_1,y_1), \mathbf{C}^{(2)}(n)=(x_2,y_2))+\frac{6}{n}\\
&\le 16(n^{1/4}\log n)(n^{1/4-\epsilon})^3\left(\frac{c}{n^{3/4}}\right)^2+\frac{6}{n}\le \frac{c\log n}{n^{1/2+3\epsilon}}, \qquad (4.12)
\end{aligned}
$$

implying that for the subsequence $n_k = k^\alpha$, with any $\alpha > 2$,

$$\sum_{k=1}^{\infty} P(A_{n_k}) < \infty.$$

This implies that, almost surely for $k \geq k_0(\omega)$, A_{n_k} does not occur, which in turn means that if the two walkers are on different teeth, then either

$$A_1(k) := \{|C_2^{(1)}(n_k)| \geq n_k^{1/4-\epsilon}\}, \quad \text{or } A_2(k) := \{|C_2^{(2)}(n_k)| \geq n_k^{1/4-\epsilon}\},$$
$$\text{or } A_3(k) := \{|C_1^1(n_k) - C_2^1(n_k)| \geq n_k^{1/4-\epsilon}\}$$

will occur. We want to show that we can select $\alpha > 2$ such that for any n, with $n_k \leq n \leq n_{k+1}$, if one of the events $\{A_i(k)\ i = 1, 2, 3\}$ occurs, then

$$D_2^{\mathbb{C}^2}(n) \geq n^{1/4-\epsilon}$$

will occur as well, as long as two walkers are on different teeth. Since $n_k = k^\alpha$, we have $n_{k+1} - n_k \sim \alpha k^{\alpha-1}$. So we have to show that in $\alpha k^{\alpha-1}$ steps the increments of the three processes in the events $\{A_i(k)\ i = 1, 2, 3\}$ are less than $n_k^{1/4-\epsilon}$. The first two of these three processes are simple symmetric walks, while the third one is the difference of two simple symmetric walks, but with a much smaller number of steps (as there are possible vertical excursions when the horizontal move pauses). By Theorem E the increment of these walks in $\alpha k^{\alpha-1}$ steps is almost surely less than

$$k^{\frac{\alpha-1}{2}} \log k,$$

while

$$n_k^{1/4-\epsilon} = k^{(1/4-\epsilon)\alpha}.$$

So we need to have

$$\frac{\alpha - 1}{2} < \alpha \left(\frac{1}{4} - \epsilon\right),$$

which is equivalent to $\alpha < \frac{2}{1+4\epsilon}$. On the other hand, for the convergence of $\sum_k P(A_{n_k})$ we need that $\alpha(1/2 + 3\epsilon) \geq 1$ should hold, which is equivalent to $\alpha > \frac{2}{1+6\epsilon}$. So, for any $\epsilon > 0$, we can find an appropriate

$$\frac{2}{1+6\epsilon} < \alpha < \frac{2}{1+4\epsilon},$$

and conclude by the Borel-Cantelli Lemma that

$$P(A_n\, i.o.) = 0. \tag{4.13}$$

To show that $P(B_n\, i.o.) = 0$, recall the definition of B_n in (4.7). Equivalently,

$$B_n = \{C_1^{(1)}(n) = C_1^{(2)}(n),\, |C_2^{(1)}(n) - C_2^{(2)}(n)| \leq n^{1/4-\varepsilon}\} = B_n^1 \cup B_n^2,$$

where

$$B_n^1 = \{C_1^{(1)}(n) = C_1^{(2)}(n),\ |C_2^{(1)}(n) - C_2^{(2)}(n)| \le n^{1/4-\varepsilon}, \max(|C_2^{(1)}(n)|, |C_2^{(2)}(n)|) \le n^{1/4}\},$$

$$B_n^2 = \{C_1^{(1)}(n) = C_1^{(2)}(n),\ |C_2^{(1)}(n) - C_2^{(2)}(n)| \le n^{1/4-\varepsilon}, \max(|C_2^{(1)}(n)|, |C_2^{(2)}(n)|) > n^{1/4}\}.$$

We first show that $P(B_n^1\, i.o.) = 0$. $P(B_n^1)$ can be estimated similarly to $P(A_n)$ in (4.12). We obtain

$$P(B_n^1) \le \frac{c \log n}{n^{3/4}}.$$

Choosing $n_k = k^\alpha$ with

$$\frac{4}{3} < \alpha < \frac{2}{1+4\varepsilon},$$

we have $P(B_{n_k}^1\, i.o.) = 0$, i.e., there exists a k_0 such that $B_{n_k}^1$ does not occur almost surely if $k \ge k_0$. As we already proved that $P(A_n\, i.o.) = 0$, and we will prove that $P(B_n^2\, i.o.) = 0$, we may assume that for $k \ge k_0$ neither A_{n_k} nor $B_{n_k}^2$ occur. Consequently, it suffices to consider the case when

$$|C_2^{(1)}(n_k) - C_2^{(2)}(n_k)| > n_k^{1/4-\varepsilon},$$

since otherwise, either

$$C_1^{(1)}(n_k) \ne C_1^{(2)}(n_k),$$

in which case A_{n_k} does not occur, or

$$\max(|C_2^{(1)}(n_k)|, |C_2^{(2)}(n_k)|) > n_k^{1/4}, \qquad C_1^{(1)}(n_k) = C_1^{(2)}(n_k),$$

in which case $B_{n_k}^2$ does not occur. Now let $n_k \le n < n_{k+1}$. We have to show that if $C_1^{(1)}(n) = C_1^{(2)}(n)$ and $\max(|C_2^{(1)}(n)|, |C_2^{(2)}(n)|) \le n^{1/4}$, then

$$|C_2^{(1)}(n) - C_2^{(2)}(n)| > n^{1/4-\varepsilon}, \tag{4.14}$$

i.e., B_n^1 does not occur with probability 1 for large n. The increments of $C_2(\cdot)$ in (n_k, n_{k+1}) are almost surely less than

$$k^{\frac{\alpha-1}{2}} \log k < n_k^{1/4-\varepsilon},$$

so it can be seen that (4.14) holds, i.e., B_n^1 does not occur, so $P(B_n^1\, i.o.) = 0$.

To prove $P(B_n^2\ i.o.) = 0$, we need the following Lemma.

Lemma 4.3 *Let E_n and B_n be two sequences of events on the same probability space. Introduce the notations*

$$E_n^* := \bigcup_{j=n}^{\infty} E_j. \quad \text{and} \quad B_{n,m}^* = B_n \cap B_{n-1}^c \cap \ldots \cap B_m^c, \quad m < n, \quad \text{with} \quad B_{n,n}^* = B_n, \tag{4.15}$$

where B^c denotes the complement of B. Assume that

$$P(E_n \, i.o.) = 0, \tag{4.16}$$

and

$$P(E_n^*|B_{n,m}^*) \geq C > 0 \tag{4.17}$$

for large enough $m \leq n$ with some constant C. Then we also have

$$P(B_n \, i.o.) = 0.$$

Proof. It is known that $P(E_n \, i.o.) = 0$ is equivalent to $\lim_{n\to\infty} P(E_n^*) = 0$. $B_{n,m}^*$ means that n is the first index, when B_i occurs with $i \geq m$. Then by (4.17) we have that

$$P(E_n^* \cap B_{n,m}^*) = P(E_n^*|B_{n,m}^*)P(B_{n,m}^*) \geq CP(B_{n,m}^*).$$

Since $B_{k,m}^*$ are disjoint for different k, and $B_k \supset B_{k,m}^*$, we have

$$\begin{aligned} P(\bigcup_{k=m}^{\infty} E_k^* \cap B_k) &\geq P(\bigcup_{k=m}^{\infty} E_k^* \cap B_{k,m}^*) = \sum_{k=m}^{\infty} P(E_k^* \cap B_{k,m}^*) \\ &\geq C \sum_{k=m}^{\infty} P(B_{k,m}^*) = CP(\bigcup_{k=m}^{\infty} B_{k,m}^*) = CP(\bigcup_{k=m}^{\infty} B_k). \end{aligned} \tag{4.18}$$

By (4.16), $P(E_m^* \cap B_m \, i.o.) = 0$ as well, or equivalently,

$$\lim_{m\to\infty} P(\bigcup_{k=m}^{\infty} E_k^* \cap B_k) = 0.$$

Consequently, by (4.18),

$$\lim_{m\to\infty} P\left(\bigcup_{k=m}^{\infty} B_k\right) = 0,$$

hence $P(B_n \, i.o.) = 0$. $\square$

To complete the proof of Theorem 4.3, we have to prove $P(B_n^2 \; i.o.) = 0$. Recall the result of Krishnapur and Peres [10] that $P(\mathbf{C}^{(1)}(n) = \mathbf{C}^{(2)}(n) \; i.o.) = 0$. Similarly, it can be shown that for

$$E_n = \{C_1^{(1)}(n) = C_1^{(2)}(n), \, |C_2^{(1)}(n) - C_2^{(2)}(n)| \leq 1\}, \tag{4.19}$$

we have also $P(E_n \; i.o.) = 0$. To apply Lemma 4.3 with E_n as in (4.19) and B_n replaced by B_n^2, we have to prove that

$$P(E_n^*|B_{n,m}^{2*}) \geq c \tag{4.20}$$

with some constant $c > 0$, by showing that if $\mathbf{C}^{(1)}(n)$ and $\mathbf{C}^{(2)}(n)$ are as in B_n^2, then before returning to the backbone, with positive probability they either meet at some point, or are at distance 1. Now define

$$\tau_1 = \min\{k \geq 0 : |C_2^{(1)}(n+k) - C_2^{(2)}(n+k)| \leq 1\}, \tag{4.21}$$

$$\tau_2^{(j)} = \min\{k \geq 0 : C_2^{(j)}(n+k) = 0\}, \quad j = 1, 2, \quad \tau_2 = \min(\tau_2^{(1)}, \tau_2^{(2)}). \tag{4.22}$$

Since, under the condition $B_{n,m}^{2*}$, both $C_2^{(1)}(n)$ and $C_2^{(2)}(n)$ are either positive or negative, we have

$$P(E_n^*|B_{n,m}^{2*}) \geq P(\tau_1 < \tau_2|B_{n,m}^{2*}).$$

What we have to show is that this last probability can be bounded from below by a positive constant. This will be achieved by estimating the distributions of τ_1 and τ_2, and applying Theorem L in Section 2, since these distributions are equivalent in terms of $\beta(r)$ as in Theorem L.

Lemma 4.4

$$\lim_{m\to\infty} P(\tau_1 < n^{1/2-\varepsilon}|B_{n,m}^{2*}) = 1, \qquad \lim_{m\to\infty} P(\tau_2 > n^{1/2-\varepsilon/2}|B_{n,m}^{2*}) = 1.$$

Proof.

$$\begin{aligned} &P\,(\tau_1 < n^{1/2-\varepsilon}|B_{n,m}^{2*}) \\ &= \sum_{z_1,z_2} P(\tau_1 < n^{1/2-\varepsilon}|B_{n,m}^{2*},\, C_2^{(1)}(n) = z_1,\, C_2^{(2)}(n) = z_2) P(C_2^{(1)}(n) = z_1,\, C_2^{(2)}(n) = z_2|B_{n,m}^{2*}), \end{aligned}$$

and

$$\begin{aligned} &P\,(\tau_2 > n^{1/2-\varepsilon/2}|B_{n,m}^{2*}) = \\ &= \sum_{z_1,z_2} P(\tau_2 > n^{1/2-\varepsilon/2}\,|B_{n,m}^{2*},\, C_2^{(1)}(n) = z_1,\, C_2^{(2)}(n) = z_2) P(C_2^{(1)}(n) = z_1,\, C_2^{(2)}(n) = z_2|B_{n,m}^{2*}), \end{aligned}$$

where the summation $\sum_{z_1,z_2}$ stands for all permissible values of z_1, z_2, under the condition $B_{n,m}^{2*}$. In fact, under the condition $\{B_{n,m}^{2*},\, C_2^{(1)}(n) = z_1,\, C_2^{(2)}(n) = z_2\}$,

$$C_2^{(1)}(n+k),\, C_2^{(2)}(n+k), \quad k = 0, 1, \ldots, \tau_2,$$

are two independent simple random walks, starting at z_1 and z_2, respectively, and avoiding 0 before τ_2. Moreover, since under the above condition, both $C_2^{(1)}(n)$ and $C_2^{(2)}(n)$ are either positive or negative,

$$|C_2^{(1)}(n+k) - C_2^{(2)}(n+k)|$$

for $0 \le k \le \tau_1$ behaves also as a simple random walk with even number of steps, starting at $|z_1 - z_2|$, τ_1 being the first hitting time of zero or one, depending on the parity of $|z_1 - z_2|$. The distribution of τ_1 is equivalent to that of the first hitting time of $|z_1 - z_2|$ or $|z_1 - z_2| - 1$ of a simple random walk, starting from 0 and considering even number of steps. Denoting by $S(\cdot)$ a simple random walk on the line, it can be seen that

$$P(\tau_1 < n^{1/2-\varepsilon} | B^{2*}_{n,m}) \ge P(\tau_1 < n^{1/2-\varepsilon} | C_1^{(1)}(n) = C_1^{(2)}(n), |C_2^{(1)}(n) - C_2^{(2)}(n)| = 2[n^{1/4-\varepsilon}])$$

$$= P(\min\{k \ge 0 : S(2k) = 0\} < n^{1/2-\varepsilon} | S(0) = 2[n^{1/4-\varepsilon}]) = P(\beta(2[n^{1/4-\varepsilon}]) < 2n^{1/2-\varepsilon}),$$

since $2\min\{k \ge 0 : S(2k) = 0\}$ under the condition $S(0) = 2[n^{1/4-\varepsilon}]$ has the same distribution as $\beta(2[n^{1/4-\varepsilon}])$ in Theorem L.

Concerning τ_2, suppose that $|C_2^{(i)}(n)| < |C_2^{(j)}(n)|$. Then it suffices to consider the time when the random walk $C_2^{(i)}(n+k)$, $k \ge 0$, reaches 0 , otherwise the two random walks will meet before τ_2, i.e., $\tau_1 < \tau_2$. Under the condition $B^{2*}_{n,m}$, we have

$$P(\tau_2 > n^{1/2-\varepsilon/2} | B^{2*}_{n,m}) \ge P(\min\{k \ge 0 : \; S(k) = 0\} > n^{1/2-\varepsilon/2} | S(0) = [n^{1/4}])$$

$$= P(\beta([n^{1/4}]) > n^{1/2-\varepsilon/2}),$$

where $\beta(\cdot)$ is defined in (2.14) and $[\cdot]$ denotes integral part. Now Lemma 4.4 follows from the limiting distributions of hitting times in Theorem L of Section 2. $\square$

To complete the proof of Theorem 4.3, it follows from Lemma 4.4 that

$$P(\tau_1 < \tau_2 | B^{2*}_{n,m}) \ge P(\tau_1 < n^{1/2-\varepsilon}, \, \tau_2 > n^{1/2-\varepsilon/2} | B^{2*}_{n,m})$$

$$\ge P(\tau_1 < n^{1/2-\varepsilon} | B^{2*}_{n,m}) - P(\tau_2 < n^{1/2-\varepsilon/2} | B^{2*}_{n,m}) \ge c > 0,$$

where the first term tends to 1, the second term tends to zero, as $n \to \infty$, so the difference is greater than a positive constant c. This means that the two random walks will meet after time n with positive probability, i.e., (4.20) holds. Hence, using Lemma 4.3, we have $P(B_n^2 \, i.o.) = 0$. This completes the proof of Theorem 4.3. $\square$

Concerning the lower classes for more than 2 walkers, we have the following result.

Theorem 4.4 *Let $a(n)$ be a nonincreasing nonnegative function. Then, for $K \ge 3$*

$$\sqrt{n}a(n) \in \mathrm{LLC}(D_K^{\mathbb{C}^2}(n)) \tag{4.23}$$

if and only if

$$\sum_{n=1}^{\infty} (a(2^n))^{K-2} < \infty. \tag{4.24}$$

Proof. First assume that

$$\sum_{n=1}^{\infty}(a(2^n))^{K-2}=\infty.$$

Then, by the Dvoretzky-Erdős Theorem (Theorem B in Section 2) and the strong approximation in Theorem K, we have

$$\max_{1\le j\le K}|C_2^{(j)}(n)|\le\sqrt{\sum_{j=1}^{K}(C_2^{(j)}(n))^2}\le n^{1/2}a(n)/3$$

infinitely often with probability 1, since we have also

$$\sum_{n=1}^{\infty}(a(2^n)/3)^{K-2}=\infty.$$

By (2.12) of Corollary 2.1 for the horizontal distance, we have

$$|C_1^{(i)}(n)-C_1^{(j)}(n)|\le n^{1/2}a(n)/3$$

for all $1\le i,j\le K$ and all large enough n, with probability 1. Then

$$d(\mathbf{C}^{(i)}(n),\,\mathbf{C}^{(j)}(n))\le|C_2^{(i)}(n)|+|C_2^{(j)}(n)|+|C_1^{(i)}(n)-C_1^{(j)}(n)|,$$

and, consequently, we have

$$D_K^{\mathbb{C}^2}(n)\le 2\max_{1\le j\le K}|C_2^{(j)}(n)|+\max_{1\le i,j\le K}|C_1^{(i)}(n)-C_1^{(j)}(n)|\le n^{1/2}a(n)$$

infinitely often with probability 1. This verifies the first part of Theorem 4.4.

To show the other part, i.e., assuming that

$$\sum_{n=1}^{\infty}(a(2^n))^{K-2}<\infty,$$

we have to prove

$$D_K^{\mathbb{C}^2}(n)>n^{1/2}a(n) \tag{4.25}$$

for all large enough n with probability 1. The idea is similar to the proof of Theorem 4.3 concerning the event B_n^2 in the case when the 2 random walks are on the same tooth at time n. In fact, we show that one of the random walks has to be high on some tooth by the Dvoretzky-Erdős Theorem B, and no other random walks can be close to this one on the same tooth. We note that the constants in the following proof are not too important, one could also choose different suitable constants.

Assume that we have K independent random walks $\mathbf{C}^{(i)}(\cdot)$, $i=1,\ldots,K$ on the comb. By Theorem B and Theorem K, we have

$$\max_{1\le j\le K}(|C_2^{(j)}(n)|)>5n^{1/2}a(n).$$

If there is no other random walk on the same tooth than the one taking the above maximum of $|C_2^{(j)}(n)|$, then obviously (4.25) holds. So we can consider the case when two random walks are on the same tooth at time n, and one of them is higher than $5n^{1/2}a(n)$. For fixed $1\le i,j\le K$, define the event

$$B_n^{i,j}=\{C_1^{(i)}(n)=C_1^{(j)}(n)\,,\,|C_2^{(i)}(n)-C_2^{(j)}(n)|\le n^{1/2}a(n),\,\max(|C_2^{(i)}(n)|,|C_2^{(j)}(n)|)>5n^{1/2}a(n)\}.$$

We show that for fixed i,j, $P(B_n^{i,j}\ i.o.)=0$, by applying Lemma 4.3, with E_n as in (4.19). Note that $B_n^{i,j}$ implies that $\min(|C_2^{(i)}(n)|\,,\,|C_2^{(j)}(n)|)>3n^{1/2}a(n)$.

Define τ_1 and τ_2 as in (4.21) and (4.22), with the obvious modification that $(1),(2)$ should be $(i),(j)$. Here again, we can consider τ_2 to be the time when the lower one of $|C_2^{(i)}(n)|$ and $|C_2^{(j)}(n)|$ reaches zero. Then, similarly to the proof of Theorem 4.3, using Theorem L,

$$P(\tau_1<\tau_2|B_{n,m}^{i,j*})\ge P(\tau_1<2na^2(n)|B_{n,m}^{i,j*})-P(\tau_2<9na^2(n)/2|B_{n,m}^{i,j*})$$

$$\ge P(\beta([2n^{1/2}a(n)])<4na^2(n))-P(\beta([3n^{1/2}a(n)])<9na^2(n)/2)>c$$

with some positive constant c. Hence, by Lemma 4.3, $P(B_n^{i,j}\ i.o.)=0$ for all $1\le i,j\le K$. This means that there is at least one distance $|C_2^{(i)}(n)-C_2^{(j)}(n)|$ larger than $n^{1/2}a(n)$, for all large n with probability 1, so (4.25) follows. This completes the proof of Theorem 4.4. □

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