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Economics Letters xxx (xxxx) xxx

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

On the consistency and the decisiveness of the double-minded decision-maker[☆]

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ARTICLE INFO

Article history:
Received 22 July 2019
Received in revised form 27 August 2019
Accepted 28 August 2019
Available online xxxx

JEL classification:
C00
D00
D01

MSC:
91B55
37E05

Keywords:
Completeness
Transitivity
Uni-preference
Bi-preference
Connectedness
Mixture-set

ABSTRACT

Under a continuity hypothesis on bi-preferences defined on a topologically-connected choice-set, as formalized by Giarlotta–Greco (2013), this letter reports that a mildly consistent, double-minded decision-maker is single-minded in the sense of being fully consistent and decisive. The results generalize recent work of Giarlotta–Watson (2019), and thereby provide a far-reaching generalization of a result of Schmeidler (1971) that has received considerable recent attention, and extend to mixture-sets of Herstein–Milnor (1953). They give another perspective on the authors' work on the Eilenberg–Sonnenschein (ES) research program embracing both the topological and algebraic registers in choice theory.

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1. Introduction

In a pioneering paper that draws on a rich literature in mathematical psychology, Giarlotta and Greco (2013) proposed the analysis of what can be seen as a double-minded decision-maker by introducing what they refer to as an NaP-preference structure. After referring to the classical theory based on a binary relation that is "simultaneously transitive and complete", Giarlotta and Watson (2017) write:

> This [classical] approach to preference modeling has many advantages, [but also] some serious drawbacks, since the two main assumptions of transitivity and completeness are hardly satisfied in most real-world applications. A NaP-preference (necessary and possible preference) is a pair of nested reflexive relations on a set such that the smaller is transitive, the larger is complete, and the two components jointly satisfy natural forms of mixed completeness and transitive coherence.[2]

Giarlotta and Watson (2019) refer to the two preferences as *rigid* and *soft*, and see the latter as a *transitively coherent enlargement of the other.*[3] Leaving aside linguistic dichotomies such as soft–hard, flexible-rigid, hesitant-firm, the point is that they consider

☆ The authors should like to acknowledge the inspiration received from a serendipitous meeting with Professor Alfio Giarlotta on July 1 at the 19th SAET Meetings in Ischia (Italy), and also a presentation of Professor Claudio Meo at the same conference. In addition to them, the authors are grateful to Max Amarante, Yorgos Gerasimou, Ani Ghosh, Debraj Ray and Eddie Schlee for stimulating conversation and correspondence. These results were finalized, and reported by the second author, at *Positivity X*, a conference held at the University of Pretoria July 8–12, 2019. He should like to thank the organizers for their hospitality. Needless to say, all errors of reading and interpretation are solely the authors'.

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1 We use the *certified random order* in order to list the authors; see Ray-Robson (2018, American Economic Review).

2 The authors introduce the concept of a *normalized* NaP-preferences, one whose smaller component is a partial order, and show that they are well-graded in the sense of Doignon and Falmagne (1997) on a finite set. A genesis of the concept can be traced to Giarlotta (2014), also see Giarlotta (2019).

3 All references to Giarlotta and Watson (2019) in this paper are to the preliminary draft circulated under the title "The interplay between two rationality tenets: extending Schmeidler's theorem to bi-preferences". The authors are grateful to Alfio Giarlotta for his unhesitating generosity in sharing this draft with us.

https://doi.org/10.1016/j.econlet.2019.108657
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a decision-maker who can be single-minded on some choices and double-minded regarding others. She is not uniformly decisive or uniformly hesitant, never always assured and never always skeptical.

While the authors are fully cognizant of the recent literature in decision theory, they do not draw on the long tradition in the economics of social cost–benefit analysis and project appraisal that takes intergenerational equity into account, and that also has a bi-preferential thrust. Marglin (1963) sights Gerhard Colm for giving importance to the frame of reference in determining preference relations, connects his position to Rousseau's views on the distinction between the individual and general will, and articulates a "schizophrenic" viewpoint in opposition to what he calls the "authoritarian" and the "interdependent" viewpoints. He writes:

> Do not be put off by the label "schizophrenia". I believe it a reasonably accurate label, [but] am, however, prevented from committing myself to it fully because of a dilemma inherent in schizophrenia: given two preference maps existing side by side, how can we choose one as representing the "true", or if you will, the "higher" preferences of the individual? Consequently, what significance can be attached to such concepts as Pareto optimality? That is, if a particular allocation of resources is Pareto-optimal in terms of one set of preference maps but nonoptimal in terms of the other, what does "optimal" mean?

To be sure, welfare economics is not our primary concern here, but what is missing in this literature, and what the modern bi-preference literature supplies, is some sort of integration of the two preference maps.[4]

In a broad ranging discussion of the literature antecedent to their work, Giarlotta and Greco (2013) had already drawn attention to the relevance of two directions of the decision theory literature: Aumann's (1962) relaxation of the completeness postulate on $\succsim$, and Luce's (1956) relaxation[5] of the transitivity postulate on $\sim$, confining ourselves only to the pioneers.[6] They "combine these alternative approaches to preference modeling into one", and note that bi-preferences are "designed in a way that transitivity and completeness hold jointly but not singularly". In particular, they detail how bi-preferences can arise from Bewley's *Knightian* preferences and Lehrer–Tepper's *justifiable* preferences. They also arise from the *transitive core* of a uni-preference as in Nishimura (2018).[7]

However, even leaving all questions of substance and/or descriptive realism aside, the results of this letter can be introduced purely in the technical register: as a simultaneous generalization of several results in the uni-preference literature that has been brought into prominence as a consequence of Schmeidler's striking 1971 result: there he deduced decisiveness for a decision-maker from his or her consistency of preferences *provided* that they were continuous in a connected topology. This was a sharp deduction of a behavioral consequence from a merely technical requirement. It was dramatic enough that the author did not see any need for a framing in terms of an antecedent literature going back to Eilenberg (1941).

Eilenberg inaugurated the study of ordered topological spaces by considering an anti-symmetric relation on a choice-set endowed with a topological structure. His results were respectively extended and elaborated by Sonnenschein (1965, 1967) and Sen (1967, 1969) to apply to contexts especially relevant to the theory of economic choice where a choice space of uncountable cardinality was partitioned into indifference classes.[8] Sonnenschein deduced full consistency for a decisive decision-maker from a rather mild consistency assumption on his or her preferences if they were continuous in a connected topology. As such he has a result that is a dual precursor to that of Schmeidler's. Sonnenschein's work was quickly followed by Sen's deconstruction of the consistency postulate, and one that eliminated topological considerations entirely from the subject. In recent work, Khan and Uyanık (2019), Galaabaatar et al. (2019) and Uyanık and Khan (2019) have given a systematic and synthetic exposition of this literature concerning a single-minded decision-maker: a single preference on a single choice-set.

It is in their most recent work that Giarlotta–Watson connect the bi-preference literature to Schmeidler's result, and in Sections 2 and 3, we present a substantial generalization that considerably relaxes their transitivity assumptions, and obtains full transitivity of their soft relation *along* with its completeness.[9] Furthermore, our results connect their work not only to that of Schmeidler but also to the classic work of Eilenberg–Sonnenschein and Herstein–Milnor. In Section 4 we present possible implications of our results for further work. Section 5 is devoted to the proofs, the bi-preference results also allow a sharpening of the earlier results on the uni-preferences. As such, it has both substantive and technical content.

2. Notational and conceptual preliminaries

Let X be a set. A subset $\succsim$ of $X \times X$ denote a *binary relation* on X. We denote an element $(x, y) \in \succsim$ as $x \succsim y$. The *asymmetric part* $\succ$ of $\succsim$ is defined as $x \succ y$ if $x \succsim y$ and $y \not\succsim x$, and its *symmetric part* $\sim$ is defined as $x \sim y$ if $x \succsim y$ and $y \succsim x$. We call $x \bowtie y$ if $x \not\succsim y$ and $y \not\succsim x$. The inverse of $\succsim$ is defined as $x \precsim y$ if $y \succsim x$. Its asymmetric part $\prec$ is defined analogously and its symmetric part is $\sim$. We provide the descriptive adjectives pertaining to a relation in a tabular form for the reader's convenience in Table 1.[10]

Definition 1. A weak bi-preference on a set X is a pair $(\succsim^R, \succsim^S)$ of binary relations on X such that for all $x, y, z \in X$,

(i) $x \succsim^R y$ implies $x \succsim^S y$,
(ii) $\succsim^R$ is hemi-transitive,
(iii) $x \sim^i z \succsim^j y$, or $x \succsim^i z \sim^j y$ imply $x \succsim^S y$, where $i, j \in \{R, S\}$ and $i \neq j$.

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[4] Also see the subsequent follow-up of Marglin's work by Sen (1967), Anderson (1995) and their references; we are indebted to Hülya Eraslan. Moving to two other streams, Klaus and Meo (2019) use a bi-preference structure, *selfish* and *altruistic* preferences, in their recent investigation of housing markets, and one can discern such a structure also in Armstrong (1939), Chipman (1971) and Gorman (1971). A potentially fruitful connection to this literature in future work may represent a secondary contribution of the work reported here.

[5] This relaxation gave rise to an extensive literature on semiorders, a term due to Luce (1956); also see Wiener (1914) pioneering paper and the extended treatments of Fishburn and Monjardet (1992) and Luce (2000) for a more detailed references to this literature.

[6] Some writers may also want to include (Krantz, 1967); see also Krantz et al. (1971) for a comprehensive treatment.

[7] Nishimura and Ok (2018) take uni-preferences as primitive and define the *transitive core* as the largest weakly consistent relation contained in it; also see Giarlotta and Greco (2013), Giarlotta and Watson (2019) and Giarlotta (2019) for references to the work of Ghirardato, Gilboa, Maccheroni, Marinacci, Siniscalchi and others. The latter has a bibliography of 66 items.

[8] Eilenberg also tackled the problem of the representation of an anti-symmetric preference relation by a continuous function on a choice set, presumably independently taken up by Wold (1943–1944), and extended to the economic setting by Debreu (1954) and a rich stream of subsequent work. This issue of representation is not our concern here, and we hope to take it up in Uyanık and Khan (2019).

[9] See Giarlotta and Watson (2019), and also Footnote 3 above concerning it and the earlier draft.

[10] We draw the reader's attention to our label *hemi-transitive*: it originates in Rader (1963) and Sonnenschein (1965).

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Table 1
Properties of binary relations.

Reflexive	$x \succsim x \ \forall x \in X$
Complete	$x \succsim y$ or $y \succsim x \ \forall x, y \in X$
Transitive	$x \succsim y \succsim z \Rightarrow x \succsim z \ \forall x, y, z \in X$
Quasi-transitive	$x \succ y \succ z \Rightarrow x \succ z \ \forall x, y, z \in X$
Negatively transitive	$x \not\succsim y \not\succsim z \Rightarrow x \not\succsim z \ \forall x, y, z \in X$
Hemi-transitive	$x \succ y \sim z \Rightarrow x \succ z$ and $x \sim y \succ z \Rightarrow x \succ z \ \forall x, y, z \in X$

Definition 2. A weak bi-preference $(\succsim^R, \succsim^S)$ on a set X is strongly comonotonic if for all $x, y, z \in X$,

(i) $x \succ^S y$ implies $x \succ^R y$,
(ii) $x \succ^R y$ and $x \succ^S z \sim^S y$ imply $x \succ^S y$,
(iii) $x \succ^R y$ and $x \sim^S z \succ^S y$ imply $x \succ^S y$.

Strong comonotonicity strengthens the transitive coherence of the bi-preference and provides a two-way relationship between the asymmetric parts of the rigid, $\succsim^R$, and soft, $\succsim^S$, preferences.

Definition 3. A weak bi-preference $(\succsim^R, \succsim^S)$ on X is non-trivial if $x \succ^S y$ for some $x, y \in X$.

Let $\succsim$ be a binary relation on a set X. For any $x \in X$, let $A_{\succsim}(x) = \{y \in X | y \succsim x\}$ denote the *upper section* of $\succsim$ at x and $A_{\precsim}(x) = \{y \in X | y \precsim x\}$ its *lower section* at x. A relation $\succsim$ on a topological space X has *closed (open) sections* if it has closed (open) upper and lower sections. Moreover, $\succsim$ is *continuous* if it has closed sections and its asymmetric part $\succ$ has open sections.

Definition 4. A weak bi-preference $(\succsim^R, \succsim^S)$ on a topological space is bi-continuous if $\succsim^R$ has closed sections and $\succ^S$ has open sections.

A set $\mathcal{S}$ is said to be a *mixture set* if for any $x, y \in \mathcal{S}$ and for any μ we can associate another element,[11] which we write as $x\mu y$, which is again in $\mathcal{S}$, and where for all $x, y \in \mathcal{S}$ and all λ, μ, (S1) $x1y = x$, (S2) $x\mu y = y(1-\mu)x$, (S3) $(x\mu y)\lambda y = x(\lambda\mu)y$.

Definition 5. A weak bi-preference $(\succsim^R, \succsim^S)$ on a mixture set $\mathcal{S}$ is scalarly bi-continuous if for all $x, y, z \in \mathcal{S}$,

(i) $\{\lambda \mid x\lambda y \succsim^R z\}$ and $\{\lambda \mid x\lambda y \precsim^R z\}$ are closed,
(ii) $\{\lambda \mid x\lambda y \succ^S z\}$ and $\{\lambda \mid x\lambda y \prec^S z\}$ are open.

Part (i) is equivalent to *mixture-continuity* of $\succsim^R$, and its part (ii) is a property slightly stronger than $\succsim^S$ being *Archimedean*[12] which requires that for all $x, y, z \in \mathcal{S}$ with $x \succ^S y$, there exist $\lambda, \delta \in (0, 1)$ such that $x\lambda z \succ^S y$ and $x \succ^S y\delta z$.

3. The results

This section presents two results on completeness and transitivity of weak bi-preferences. The first result shows that under a continuity hypothesis, a decision-maker with a non-trivial, strongly comonotonic and mildly consistent weak bi-preference defined on a topologically-connected choice-set, is decisive *and* consistent.

Theorem 1. *Let $(\succsim^R, \succsim^S)$ be a non-trivial, bi-continuous and strongly comonotonic weak bi-preference on a connected topological space. Then $\succsim^S$ is complete, transitive and continuous.*

Theorem 1 generalizes the main result of Giarlotta and Watson (2019) by showing that a weak consistency assumption on $(\succsim^R, \succsim^S)$ is sufficient to obtain both completeness *and* transitivity of $\succsim^S$. It is in this sense that the double minded-decision maker is single-minded. A weak bi-preference is a *bi-preference* if (ii) and (iii) in Definition 1 are replaced with their stronger versions (ii′) $\succsim^R$ is reflexive and transitive, and (iii′) $x \succsim^R z \succsim^S y$, or $x \succsim^S z \succsim^R y$ imply $x \succsim^S y$. The reader should note that our notion of weak bi-preference drops the reflexivity and weakens the joint transitivity of the bi-preferences assumed in Giarlotta and Watson (2019). Theorem 1 also generalizes Khan and Uyanık (2019, Proposition 1) result on uni-preferences to bi-preferences for connected spaces.

The second result suitably extends Theorem 1 to mixture sets.

Theorem 2. *Let $(\succsim^R, \succsim^S)$ be a non-trivial, scalarly bi-continuous and strongly comonotonic weak bi-preference on a mixture set. Then $\succsim^S$ is complete, transitive, mixture-continuous and Archimedean.*

Theorem 2 provides a two-fold generalization of Galaabaatar et al. (2019, Theorem 1): it replaces uni-preferences by bi-preferences, and drops the reflexivity assumption. It generalizes Dubra (2011), Karni and Safra (2015) and McCarthy and Mikkola (2018), and it provides a simple alternative proof of the portmanteau of all these results.[13] Note that the scalar bi-continuity assumption is weaker than bi-continuity property of a weak bi-relation defined on a convex subset of a topological vector space; see Uyanık and Khan (2019). Hence, Theorem 2 obtains completeness and transitivity of $\succsim^S$ under a weaker continuity assumption.

4. Concluding remarks

In this section we see possible implications of our two theorems for future work. The first priority is to study the two-way relationship, as emphasized in Khan and Uyanık (2019), between the topological structure of the choice space and the assumptions on weak bi-preferences. Second, we can ask what additional mathematical structures such as convexity (independence, betweenness, additivity or convex upper sections) and monotonicity would yield. Third, it is also interesting to see if the analysis of Sen's deconstruction of the transitivity postulate, as detailed in Khan and Uyanık (2019), and in particular Giarlotta's (2014) examination of the sequences of mixed preferences, applies to weak bi-preferences. Fourth, given that the Archimedean axiom is weaker than part (ii) of the scalar bi-continuity property, it is not known to us if the latter property can be replaced by the former in Theorem 2. Fifth, can the relaxation of the continuity postulate in Gerasimou (2013) and Uyanık and Khan (2019) yield anything of interest? Finally, it is surely worth examining the implications of our results for semiorders and interval orders, and more generally for directions in choice theory that have been recently investigated by Gerasimou (2018), Cerreia-Vioglio et al. (2018), Cerreia-Vioglio and Ok (2018) and Nishimura and Ok (2018).

5. Proofs of the results

First, we present a result due to Sen (1969, Theorem I) and Khan and Uyanık (2019, Proposition 2) on the deconstruction of the transitivity postulate.[14]

[11] In deference to Herstein and Milnor (1953), lower case Greek letters consistently denote real numbers in [0,1].

[12] The Archimedean property and part (ii) of Definition 5 are equivalent if $\succsim^S$ is mixture-continuous; see Galaabaatar et al. (2019, Proposition 1) for details.

[13] See Footnote 15 in Section 5 for details of this simplification that was entirely missed in Galaabaatar et al. (2019).

[14] Part (a) of Lemma 1 is observed and proved by Sonnenschein (1965, Theorem 3). Sen's re-statement highlights the topology-free nature of this result. See Section 3.4 of Khan and Uyanık (2019) for extensions and discussion.

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Lemma 1. *The following are true for any binary relation $\succsim$ on a set X.*

(a) *If $\succsim$ is complete and hemi-transitive, then $\sim$ is transitive,*
(b) *If $\succ$ is negatively transitive, then $\succsim$ is hemi-transitive and quasi-transitive,*
(c) *$\succsim$ is transitive if and only if it is hemi-transitive, quasi-transitive, and $\sim$ is transitive.*

We next turn to the proof of Theorem 1.

Proof of Theorem 1. First, consider the following claim.

Claim 1. (a) *$\succ^S$ is negatively transitive, and* (b) *$\succsim^S$ is complete.*

It follows from negative transitivity of $\succ^S$ and (b) of Lemma 1 that $\succsim^S$ is quasi-transitive and hemi-transitive. Completeness of $\succsim^S$, and (a) and (c) of Lemma 1 imply that $\succsim^S$ is transitive. It follows from $\succsim^S$ is complete and $\succ^S$ has open sections that $\succsim^S$ has closed sections. Therefore $\succsim^S$ is continuous. It remains to prove Claim 1 in order to complete the proof.

Proof of Claim 1. **(a)** Note that $\succ^S$ is negatively transitive if and only if $x \succ^S y$ implies $x \succ^S z$ or $z \succ^S y$ for all $x, y, z \in \mathcal{S}$. Pick $x, y \in X$ such that $x \succ^S y$. It follows from strong comonotonicity that $x \succ^R y$. We first show that

$$A_{\prec^S}(x) \cup A_{\succ^S}(y) = A_{\precsim^R}(x) \cup A_{\succsim^R}(y). \tag{1}$$

The forward inclusion immediately follows from strong comonotonicity. In order to prove the backward inclusion, pick $z \in X$ such that $x \succsim^R z$ or $z \succsim^R y$. Let $x \succsim^R z$. Then, either $x \sim^R z$ or $x \succ^R z$. Assume $x \sim^R z$. Then $x \succ^R y$ and hemi-transitivity of $\succsim^R$ imply that $z \succ^R y$. It follows from the definition of weak bi-preference and $z \sim^R x$ that $z \sim^S x$. Then $z \succ^R y$, $z \sim^S x \succ^S y$ and strong comonotonicity imply that $z \succ^S y$. Hence, $z \in A_{\succ^S}(y)$. Now assume $x \succ^R z$. Then the definition of weak bi-preference implies that either $x \succ^S z$ or $x \sim^S z$. If $x \succ^S z$, then $z \in A_{\prec^S}(x)$. Then let $x \sim^S z$. Then $z \sim^S x \succ^R y$ implies that $z \succsim^S y$. Hence either $z \succ^S y$ or $z \sim^S y$. If $z \sim^S y$, then $x \succ^S y \sim^S z$, $x \succ^R z$ and strong comonotonicity imply that $x \succ^S z$. This contradicts $x \sim^S z$. Hence, $z \succ^S y$ must hold. Therefore, $z \in A_{\succ^S}(y)$. The proof for $z \succsim^R y$ is analogous.

Note that $x \succ^S y$ implies that $A_{\prec^S}(x) \cup A_{\succ^S}(y) \neq \emptyset$. Then it follows from bi-continuity of the weak bi-preference, Eq. (1) and the connectedness of X that $X = A_{\prec^S}(x) \cup A_{\succ^S}(y)$. Hence, $x \succ^S y$ implies that for all $z \in X$, $x \succ^S z$ or $z \succ^S y$. Therefore, $\succ^S$ is negatively transitive.
(b) Assume there exists $x, y \in X$ such that $x \bowtie^S y$. We first show that

$$A_{\succ^S}(x) \cap A_{\succ^S}(y) = A_{\succsim^R}(x) \cap A_{\succsim^R}(y). \tag{2}$$

The forward inclusion immediately follows from strong monotonicity. In order to prove the backward inclusion, pick $z \succsim^R x$ and $z \succsim^R y$. Then the definition of weak bi-preference implies that $z \succsim^S x$ and $z \succsim^S y$. If $x \sim^S z$, it follows from $z \succ^R y$ or $z \sim^R y$, and the definition of weak bi-preference that $x \succsim^S y$. This furnishes us a contradiction with $x \bowtie^S y$. Hence $z \succ^S x$. An analogous argument implies $z \succ^S y$. Therefore $z \in A_{\succ^S}(x) \cap A_{\succ^S}(y)$.

It follows from the non-triviality assumption that there exist $\bar{x}, \bar{y} \in X$ such that $\bar{x} \succ^S \bar{y}$. Part (a) implies that $\succ^S$ is negatively transitive, therefore $\bar{x} \succ^S x$ or $x \succ^S \bar{y}$. Assume $\bar{x} \succ^S x$. Then negative transitivity of $\succ^S$ implies that $y \succ^S x$ or $\bar{x} \succ^S y$. Since $x \bowtie^S y$, therefore $\bar{x} \succ^S y$. Hence $\bar{x} \in A_{\succ^S}(x) \cap A_{\succ^S}(y)$. Note that the set $A_{\succ^S}(x) \cap A_{\succ^S}(y)$ does not contain x. Since $\succ^S$ has open sections and $\succsim^R$ has closed sections, therefore Eq. (2) implies that $A_{\succ^S}(x) \cap A_{\succ^S}(y)$ is both open and closed. This contradicts the connectedness of X. Analogously, $x \succ^S \bar{y}$ furnishes us a contradiction with the connectedness of X. Therefore, $\succsim^S$ is complete. □ □

The following notation is useful in the rest of this section. For any binary relation $\succsim$ on a mixture set $\mathcal{S}$ and any $x, y, z \in \mathcal{S}$, let

$$I_{\succsim}(x, y, z) = \{\lambda \mid x\lambda y \succsim z\} \ \text{ and } \ I_{\precsim}(x, y, z) = \{\lambda \mid z \succsim x\lambda y\}.$$

The sets $I_{\succ}(x, y, z)$ and $I_{\prec}(x, y, z)$ are analogously defined.

We now turn to the proof of Theorem 2. The construction of the proof is similar to that of Theorem 1, but there are subtle differences because of the lack of a topological structure on the choice set. Moreover, for the special case of uni-preferences, the proof we present here is distinct from, and considerably simpler than, the proof of Theorem 1 of Galaabaatar et al. (2019).[15] Their proof hinges crucially on the reflexivity assumption. Our new method-of-proof uses a construction similar to the proof of Theorem 1 and allows us to drop the reflexivity of $\succsim^R$.

Proof of Theorem 2. First consider the following claim.

Claim 2. (a) *$\succ^S$ is negatively transitive, and* (b) *$\succsim^S$ is complete.*

Analogous to the proof of Theorem 1, Claim 2 and Lemma 1 imply that $\succsim^S$ is transitive. Since $\succsim^S$ is complete, and the sets $I_{\succ^S}(x, y, z)$ and $I_{\prec^S}(x, y, z)$ are open for all $x, y, z \in \mathcal{S}$, therefore the sets $I_{\succsim^S}(x, y, z)$ and $I_{\precsim^S}(x, y, z)$ are closed. Hence $\succsim^S$ is mixture-continuous. Then Proposition 1 of Galaabaatar et al. (2019) implies that $\succsim^S$ is Archimedean. It remains to prove Claim 2 in order to complete the proof.

Proof of Claim 2. **(a)** Pick $x, y \in \mathcal{S}$ such that $x \succ^S y$, and pick $z \in X$. It follows from strong comonotonicity that $x \succ^R y$. We next show that

$$I_{\succ^S}(y, z, y) \cup I_{\prec^S}(y, z, x) = I_{\succsim^R}(y, z, y) \cup I_{\precsim^R}(y, z, x).$$

The forward inclusion immediately follows from the strong comonotonicity. In order to prove the backward inclusion, pick $\lambda \in I_{\succsim^R}(y, z, y) \cup I_{\precsim^R}(y, z, x)$. Then $y\lambda z \succsim^R y$ or $x \succsim^R y\lambda z$. Let $x \succsim^R y\lambda z$. Then, either $x \sim^R y\lambda z$ or $x \succ^R y\lambda z$. Assume $x \sim^R y\lambda z$. It follows from $x \succ^R y$ and hemi-transitivity of $\succsim^R$ that $y\lambda z \succ^R y$. Moreover, it follows from the definition of weak bi-preference and $y\lambda z \sim^R x$ that $y\lambda z \sim^S x$. Then $y\lambda z \succ^R y$, $y\lambda z \sim^S x \succ^S y$ and strong comonotonicity imply that $y\lambda z \succ^S y$. Hence, $\lambda \in I_{\succ^S}(y, z, y)$. Now assume $x \succ^R y\lambda z$. Then the definition of weak bi-preference implies that either $x \succ^S y\lambda z$ or $x \sim^S y\lambda z$. If $x \succ^S y\lambda z$, then $\lambda \in I_{\prec^S}(y, z, x)$. Then assume $x \sim^S y\lambda z$. Then $y\lambda z \sim^S x \succ^R y$ imply that $y\lambda z \succsim^S y$. Hence either $y\lambda z \succ^S y$ or $y\lambda z \sim^S y$. If $y\lambda z \sim^S y$, then $x \succ^S y \sim^S y\lambda z$, $x \succ^R y\lambda z$ and strong comonotonicity imply that $x \succ^S y\lambda z$. This contradicts $x \sim^S y\lambda z$. Hence, $y\lambda z \succ^S y$ must hold. Therefore, $\lambda \in I_{\succ^S}(y, z, y)$. The proof for $y\lambda z \succsim^R y$ is analogous.

Note that $x \succ^S y$ implies that $1 \in I_{\prec^S}(y, z, x)$. Then it follows from scalar bi-continuity of $(\succsim^R, \succsim^S)$ that $I_{\succ^S}(y, z, y) \cup I_{\prec^S}(y, z, x)$ is non-empty, open and closed subset of the connected set $[0, 1]$. Then $I_{\succ^S}(y, z, y) \cup I_{\prec^S}(y, z, x) = [0, 1]$. It follows from $0 \in I_{\succ^S}(y, z, y) \cup I_{\prec^S}(y, z, x)$ that $z \succ^S y$ or $x \succ^S z$. Therefore, $\succ^S$ is negatively transitive.
(b) Assume there exists $u, v \in \mathcal{S}$ such that $u \bowtie^S v$. It follows from non-triviality that $x \succ^S y$ for some $x, y \in \mathcal{S}$. Then, part (a) implies that $x \succ^S u$ or $u \succ^S y$. Let $x \succ^S u$. Then, part (a) implies that $x \succ^S v$ or $v \succ^S u$. Since $u \bowtie^S v$, therefore $x \succ^S v$. Hence, $x \succ^S u$ and $x \succ^S v$. Next, we show that

$$I_{\succ^S}(x, u, v) \cap I_{\succ^S}(x, u, u) = I_{\succsim^R}(x, u, v) \cap I_{\succsim^R}(x, u, u).$$

15 Note that Galaabaatar–Khan–Uyanık use both mixture-continuity and Archimedean assumptions. As noted in Footnote 12, it is important for the reader to realize that under the mixture-continuity assumption, Archimedean property and part (ii) of scalar bi-continuity assumption are equivalent, hence the generalization.

Please cite this article as: M. Uyanık and M.A. Khan, On the consistency and the decisiveness of the double-minded decision-maker. Economics Letters (2019) 108657, https://doi.org/10.1016/j.econlet.2019.108657.

ECOLET: 108657

One of the inclusion relationship directly follows from strong comonotonicity. In order to prove the other direction, pick $\lambda \in I_{\succsim^R}(x,u,v) \cap I_{\succsim^R}(x,u,u)$. Then $x\lambda u \succsim^R v, u$. The definition of weak bi-preference implies that $x\lambda u \succsim^S v, u$. Assume $x\lambda u \sim^S v$. Then it follows from the definition of weak bi-preference and $x\lambda u \succsim^R u$ that $v \succsim^S u$. This furnishes us a contradiction with $u \bowtie^S v$. Hence, $x\lambda u \succ^S v$, i.e., $\lambda \in I_{\succ^S}(x,u,v)$. Now assume $x\lambda u \sim^S u$. Then an analogous argument implies that $u \succsim^S v$. This furnishes us a contradiction with $u \bowtie^S v$. Hence, $x\lambda u \succ^S v$, i.e., $\lambda \in I_{\succ^S}(x,u,u)$.

It follows from S1 and $u \bowtie^S v$ that $1 \in I_{\succ^S}(x,u,v) \cap I_{\succ^S}(x,u,u)$ and $0 \notin I_{\succ^S}(x,u,v) \cap I_{\succ^S}(x,u,u)$. Scalar bi-continuity of $(\succsim^R, \succsim^S)$ implies $I_{\succsim^R}(x,u,v) \cap I_{\succsim^R}(x,u,u)$ is closed and $I_{\succ^S}(x,u,v) \cap I_{\succ^S}(x,u,u)$ is open. Therefore, we obtain a non-empty proper subset of [0,1] which is both open and closed. This contradicts the connectedness of [0,1].

The proof is analogous for $u \succ^S y$. Therefore, $\succsim^S$ is complete. □ □

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