The University of Queensland
Faculty of Business, Economics & Law
School Of Economics

INDIVIDUAL MANIPULATION: ANALYTICAL RESULTS OF FOUR VOTING RULES

By
THI THAO NGUYEN

Supervisors:
PROFESSOR ANDREW McLENNAN and DOCTOR SHINO TAKAYAMA

Estimated Word Count: 17,000

An ECON7930 submitted to the School of Economics, The University of Queensland, in partial fulfilment of the requirements for the degree of Master of Economics.

November 2016

Statement of Originality

I declare that the work presented in this Thesis is, to the best of my knowledge and belief, original and my own work, except as acknowledged in the text, and that material has not been submitted, either in whole or in part, for a degree at this or any other university.

Thi Thao Nguyen

16 November 2016

Acknowledgement

First and foremost, I would like to express my deep gratitude to my supervisors, Professor Andrew McLennan and Doctor Shino Takayama. Thank you so much for encouraging to me to be self-confident of conducting the research. Working with you has taught me one of the most valuable lessons in my life: challenge my ways of thinking to find the most convincing arguments.

I'm grateful to Toan Le Vo Khanh, Ingrid Riener and Nhan Phan Buu who took time to speak to me, and to read the draft for comment. Your company made my journey much more meaningful and enjoyable.

I owe special thanks to Doctor Priscilla Man who (even might not recognize an old student) had been generous in giving me advice on drawing pictures in Latex. Her reference had significantly improved the readability of the thesis.

To my parents and brother, thank you for always being on my side and reminding me to take care of myself. I love you and miss you so much.

Abstract

Aggregating individual preferences into collective choices is a central issue of multiagent systems and voting is the most general method used. A voting rule (social choice function) is a function that maps each preference profile to an element in the set of the alternatives. The famous Gibbard-Satterthwaite theorem asserts that all voting rules with three or more candidates are manipulable in the sense that at least one of the voters can improve the chances of getting a more favorable outcome by voting strategically (Gibbard, 1973, Satterthwaite, 1975). It is envisaged that manipulation is difficult or even infeasible for a voting rule as the existence of such situations may lead to a socially undesirable candidate being elected. This thesis analytically measures the revised volumes that represent the likelihood of individual manipulation under Plurality, Anti-plurality, Borda Count and Single Transferable Vote in the context of three-candidate elections, with the Impartial Anonymous Culture condition when preference profiles are normalized. The results show that Anti-plurality is most susceptible to individual manipulation while the most resistant rule to such a situation is Single Transferable Vote.

Keywords: Voting, Manipulation, Gibbard-Satterthwaite theorem, Single Transferable Vote

JEL Classification: D71, D72

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Research focus

Voting has long been used as a tool for collective decision making, with Athenian democracy known to have existed at least as far back as the 6^{th} century BC (Menton and Singh, 2012). For just as long, attempts have been made by political parties and the public to influence election outcomes. For example, when voters feel voting according to their personal preferences induces an undesirable outcome, they vote strategically to sway the outcome to achieve another viable option. A vote for a candidate that has no realistic chance of winning the election might be considered as "wasted". If a voter obtains a more favorable outcome by means of strategic voting, she[1] is said to be *manipulating* the voting rule. This outcome might not be socially preferred and could lead to unpopular and unintended changes in policy, the allocation of resources and even the political direction of a society.

Two recent voting outcomes have highlighted the unpredictability and complexity of a voting system. The UK's referendum and its subsequent decision of exiting the European Union, or the so-called Brexit was considered to have far-reaching consequences for all areas of British society and the economy for years to come[2] and the most recent

[1]For simplicity's sake, we use "she" to refer to a voter, and "he" to refer to a candidate.

[2]Source:`https://www.theguardian.com/politics/2016/jul/22/one-month-on-what-is-the-impact-of-the-brexit-vote-so-far`, retrieved 13 August 2016

election of Donald Trump for the US presidency is believed to have slid the country into predictable chaos. The New York Times stated that "rarely in the history of the American presidency has the exercise of choosing people to fill jobs had such a far-reaching impact on the nature and priorities of an incoming administration."[3]. People are still perplexed at these unexpected outcomes and thus, it is important to comprehend the voting system as well as the way people materialize their hopes in votes. This thesis attempts to demystify one of the many complex aspects of elections.

Manipulation is fraught, not only in human elections but also in the world of artificial intelligence and computer science because virtual elections have become a standard tool in preference aggregation. Voting was first used in artificial intelligence to solve planning problems, allowing agents to vote on the next step of the plan (Ephrati and Rosenschein, 1991). The web metasearch engine is another application of virtual elections where the engine treats other search engines as voters and the web pages as candidates (Dwork *et al.*, 2001). Voting mechanisms are also used in the intersection of human societies and the worlds of computer science to, say, build recommender systems for collaborative filtering (Pennock *et al.*, 2000). Since voting is commonly used and the consequences of voting manipulation are dire, it is crucial that a voting rule should be able to resist manipulation.

Extensive literature has addressed various facets of manipulation, and yet each result seems only to deepen the mystery[4]. Different approaches have been adopted to study manipulation of voting rules, ranging from the axiomatic to the concrete; from statistical, analytic and computational approaches to topological ones (Saari, 1995). "Suffice it to say for now that this conundrum has kept mathematicians, statisticians, political scientists, and economists busy for two centuries - to no avail"[5]. The development of

[3]Source:`http://www.nytimes.com/2016/11/13/us/politics/donald-trump-administration-appointments.html?ribbon-ad-idx=1&rref=homepage&module=Ribbon&version=origin&region=Header&action=click&contentCollection=Home%20Page&pgtype=article`, retrieved 13 November 2016

[4]Saari (1995) claims that it is possible to make every single candidate win an election if a "well-designed" voting rule is applied.

[5]Szpiro (2010)

mathematical field of game theory in 1940s and the use of Nash equilibrium to analyze strategic voting have changed the voting theory field. One of the most cited results in social choice theory, Arrow's impossibility theorem, inspired further significant results including the Gibbard-Satterthwaite theorem (Gibbard, 1973, Satterthwaite, 1975).

Rigorous study of voting manipulation dates back to the early 1970s when the philosopher Allan Gibbard and the economist Mark Satterthwaite independently established the theorem bearing their names (Taylor, 2005, Wallis, 2014). The theorem essentially states that all non-dictatorship[6] voting systems with more than two candidates are manipulable in the sense that at least one of the voters can improve the chances of getting a more favorable outcome by voting strategically. While the influence of game theory was implied in Arrow's impossibility theorem, the Gibbard-Satterthwaite theorem proofs explicitly treat voting rules as game-theoretic mechanisms, which provides the field of voting theory with a set of tools to examine a whole range of possible scenarios (Menton and Singh, 2012).

To circumvent the negative result stated in the Gibbard-Satterthwaite theorem, research has been conducted to check if finding a beneficial manipulation is computationally infeasible. The computational complexity approach has achieved some success, showing how hard it is to manipulate certain rules in the worst-case sense. However, computational difficulty does not preclude the existence of an efficient algorithm that can find a successful manipulation, and thus a worst-case analysis may be insufficient to guarantee the resistance to manipulation. This thesis departs from the computational approach, attempting to analytically measure the individual manipulability of four voting rules: Plurality, Anti-plurality, Borda Count and Single Transferable Vote (STV).

Voting manipulation is often classified according to the number of electors who attempt to manipulate the outcome, the *manipulators*. If only one voter manipulates the election outcome, it is called *individual manipulation*. If there is a group of people who

[6]Nondictatorial requires that no single voter should have the power to determine the outcome. In the situation where one voter determines the result of an election, voting would be a waste of time.

cooperate to manipulate the elections, it is called *coalitional manipulation.* Saari (1995) names the types of manipulation with one or a small group of voters *micromanipulation* while the type with a large group of voters is termed *macromanipulation.* To avoid ambiguity between the small and large group, we follow the *individual manipulation* and *coalitional manipulation* terminologies for the rest of the work.

The idea that an individual can change an election outcome might appear to be unrealistic[7] and what we attempt to measure is not the probability an individual can manipulate the outcome either. In fact, the probability that any one person can manipulate the election goes to zero in the limit when the number of voters goes to infinity. What does not go to zero is the ratio of the probability of manipulability under one voting rule to the probability of manipulability under another voting rule, which is what we are trying to measure. This measurement allows us to compare the four rules in terms of its vulnerability to individual manipulation.

1.2 Objectives

The thesis aims to obtain analytical results regarding the likelihood of successful individual manipulation in three-candidate elections under four voting rules: Plurality, Anti-plurality, Borda Count and STV, from which the relative probabilities of each scoring rule[8] with STV are computed. To achieve this aim, we first compute the volumes of regions representing possible individual manipulation Q by applying the decomposition method. After that, the volume Q is adjusted by its angle with possible manipulation to obtain the *revised volume*, which is interpreted as the likelihood of successful individual

[7]Saari (1990) argues that individual manipulation is easier to be justified compared to coalitional one because it only requires few voters to know or suspect what the actual outcome under sincere preferences is and decide to act strategically. Meanwhile, coalitional manipulation assumes wide-spread prior knowledge of who the real competitors are. Also, a coordinator is needed to ensure the correct number of voters casting the votes strategically. Often, manipulation means that a voter has to vote for someone other than her first preference. It is arguable that these groups of people should try to increase the number of sympathetic voters rather than voting strategically.

[8]Section 3.1.3 explains why Plurality, Anti-plurality and Borda Count belong to the scoring rule class.

manipulation.

The results enable us to compare and contrast voting rules, and find the one that is least susceptible to individual manipulation. Not all voting rules are equally manipulable, and profiles that offer strategic opportunities for one procedure do not necessarily offer strategic opportunities for another. However, manipulative behavior should not be the only decision rule in selecting a voting rule. Precautions are necessary, but extremes can be counterproductive. It is unrealistic to adopt a method that ensures sincere voting but severely distorts the voters' true intentions.

Our approach has two main advantages over the existing methods. Firstly, we do not have to worry about the effect of the electorate sizes on the results thanks to normalizing the preference profile. Furthermore, the angle of region Q with possible manipulation is taken into consideration, allowing the results to accurately reflect the likelihood of successful individual manipulation under each voting scheme.

1.3 Outline of thesis

The remainder of this thesis is structured as follows. Chapter 2 reviews the literature regarding manipulation, focusing on the four voting rules of interest. To understand these studies, we first introduce the assumptions most commonly used in the studies of voting manipulation. Then we discuss some desirable properties of voting rules and the measures of manipulability that have been proposed and studied. After that, each of the four is examined. There is also a focus on special properties of Borda Count and STV. We conclude with a discussion regarding each voting rule and a synthesis of results related to manipulation.

Chapter 3 is the theoretical core of the thesis, defining the individual manipulation as a linear programming problem and providing the methodology of computing such measure of manipulability. We redefine each voting rule in mathematical form and set up the basic framework from which the study of voting rule is converted into geometry.

After that, the regions that represent individual manipulation are defined for each rule with the description of how such manipulations are realized. Finally, the method for computing the volume of convex region and the revised volume is clearly delineated.

In presenting the results of each voting rule, Chapter 4 starts with vertices and the geometry of its faces, followed by the decomposition structure from which the volume is computed, and then the computation of the angle of region Q with possible manipulation. Because the computation of volume in high dimensional space is complex and error-prone, a standard Mathematica code is designed to generate artificial data for consistency check. The results are discussed with respect to the existing literature. Chapter 5 concludes and proposes possible extensions for future research.

Chapter 2

Literature Review

The famous Gibbard-Satterthwaite theorem asserts one of the main obstructions facing the social choice field: all nondictatorial voting rules with three or more candidates are manipulable. Since then, there has been extensive research attempting to measure the manipulability of a voting rule and to find the voting rule that motivates sincere voting while preserving as much desirable properties as possible. This review provides the theoretical framework for each rule, which can then be utilized analytically in the next chapter. We begin by examining the most commonly studied assumptions and the justification for continuing research with such assumptions. After that, the four voting rules: Plurality, Anti-plurality, Borda Count and STV are discussed in details.

2.1 Assumptions

Research on the probability of manipulation is often based on the assumptions that voters might have different preference rankings over the set of candidates. The most commonly-studied models are Impartial Culture (IC) and Impartial Anonymous Culture (IAC). Although these assumptions are often criticized for not reflecting realistic voting scenarios, an extensive amount of research based on these assumptions continues to be considered in the literature. Thus, before discussing the relevance of such studies, we first need to understand these assumptions and the significance of continuing using them in literature.

2.1.1 Impartial Culture

The IC assumption states that each possible preference ranking on the candidates is equally likely to represent the preferences of a randomly selected voter. Apart from assuming the preferences of any given voter are independent of those of the others, IC requires that there is a perfect balance of the expected ranking position of all candidates. In other words, no candidate has any advantage compared to other candidates in the preference rankings of a randomly selected voter, they are equally likely to be in any position in the preference ranking.

2.1.2 Impartial Anonymous Culture

The concept of IAC is based directly on the assumption that all voting situations are equally likely to be observed. In other words, voting situations follow a uniform distribution. IAC produces an expected balance of preferences on pairs of candidates. However, this balance only applies to voting situations with anonymous voters, not to a specific individual voter. A commonly used feature of voting rules is *anonymity* which states that the names of voters do not matter. With this property, the profile can be represented as *voting situation* where we list the numbers of voters for each possible rankings. Thus, we only need to consider voting situations, not profiles for voting rules which are anonymous.

There are valid arguments that support the use of IC and IAC in researching voting manipulation (Gehrlein and Lepelley, 2012). Firstly, we can compare different voting rules on the basis of relative impacts that manipulation can have on various voting situations. Secondly, by using such probability models to obtain closed-form representations, it is possible to isolate the effects of different parameters on the probabilities. Lastly, abandoning theoretical models to pursue empirical studies that are based only on the voting results from regular elections can lead to new problems regarding the validity of the results. For example, a minor change in a preference threshold parameter of the model may lead to very different rankings being obtained. In conclusion, it is clear that the classic assumptions for producing probability representations of voting paradoxes do have valid uses.

2.1.3 Poll assumption

Often, polls assist voters to ascertain if their most preferred candidate has any chance of winning the election or they need to vote strategically. The *poll assumption* states that voters whose most favorite candidate is not one of the two top candidates in a poll have incentives to adjust their preferences to vote for the one of the top two that they prefer. For this reason, those who do badly in polls may drop out before the election. The poll assumption applies primarily to the last poll taken before an election, but candidates frequently drop after earlier polls, knowing that their showing is not improving later.

The *poll assumption* is a rational justification for our research where we consider the possibility that a single voter can change the election outcome. Often an election is preceded by an informal count or poll. If a candidate does badly in polls, his supporters may change their votes because they understand that this candidate has no realistic chance of winning the election.

However, the opinion polls were largely proved to be wrong in the Brexit referendum and the American presidential election. Polls mistakenly predicted that Britain would vote to stay in the European Union in June and did not capture Republican loyalists who initially vowed not to vote for Mr. Trump, but changed their minds in the voting booth, leading to the biggest polling miss in the US presidential election in decades[1]. Thus, we need to bear in mind the possibility that polls fail to reach enough of those people who actually turn out to vote.

[1]Source:`http://www.nytimes.com/2016/11/09/business/media/media-trump-clinton.html?action=click&pgtype=Homepage&clickSource=story-heading&module=span-abc-region&region=span-abc-region&WT.nav=span-abc-region`, retrieved 12 Nov 2016.

2.1.4 Other assumptions

In this paper, the strategic behavior is considered decision-theoretic when individuals act optimally while assuming that others in the electorate will vote sincerely and every voter has a preference list that remains fixed throughout the voting process. We also assume that the manipulator has perfect knowledge regarding the remaining votes. This assumption is developed from the fact that if manipulation is hard in the most favorable condition for the manipulators, then it must be hard in more realistic settings.

2.2 Properties

2.2.1 Monotonicity

The monotonicity or “nonnegative responsiveness” property states that an individual cannot harm an alternative's chance of winning by ranking him higher. For example, if A is socially preferred to B, then an individual cannot harm A's aggregate position by ranking A above B in her ordering, with all other individual orderings held constant.

2.2.2 Independence of Irrelevant Alternatives

The independence of irrelevant alternatives (IIA) property implies that individual preferences for any pair of alternatives should not be influenced by other alternatives. These other alternatives can either be new alternatives introduced into the election or just the switching ranking of other existing candidates. If the election, for instance, determines that A is socially preferred to B, and suppose some electors change their preference lists. If no voter changes the relative positions of A and B (i.e. those who initially ranked A above B still do so, and those who preferred B to A continue to do so), then the voting rule should continue to select A over B.

Menton and Singh (2012) claim that IIA and the possibility of strategic behavior are mutually exclusive, meaning that the presence of one in a voting rule implies the absence of the other. However, we will see that this is not always true, particularly in the case of Borda Count in Section 2.6.2.

2.3 Measures of Manipulability

Various indices of manipulability of voting rules have been introduced and studied. While some measures such as computational complexity and axiomatic approaches can be used to evaluate both individual and coalitional manipulation, others are only suitable for coalitional manipulation; one such example is identifying the coalition size needed for a manipulation to be successful.

2.3.1 Computational complexity

Many researchers have pursued the computational complexity direction pioneered by Bartholdi III and Orlin (1991). The idea is to use the potential amount of time required to compute the solution to a manipulation problem as the criterion. If the amount of time required is a polynomial function of the problem's size, then the problem is tractable or nondeterministic polynomial time (NP) and it is easy to find an effective manipulation. The hardest problems are classified as NP-complete and if a voting rule is NP-complete, it is computationally resistant to manipulation.

Although software agents follow the algorithms coded without emotion rather than behaving like human beings, they have all the patience and computing power necessary to perform a complicated analysis (Conitzer *et al.*, 2007). Also, they are not bound by moral obligation or social pressure to act honestly, which enlarges the space of viable votes or viable manipulation and hence, maximizes their chance of finding the optimal solution (Faliszewski and Procaccia, 2010). Therefore, if the software agents cannot find an effective manipulation, we can conclude that the voting rule is immune to strategic voting.

Using computational complexity as a shield against manipulation has advantages over other methods, including restricting the voters' preferences to be single-peaked and introducing randomness in the elections. Often, social planning authorities cannot be sure if voters' preferences are single-peaked or not. Furthermore, the introduction of randomness would create too much noise and even make an election manipulable when it was not under deterministic situations (Conitzer *et al.*, 2007).

However, the computational barrier approach has recently been criticized because it relies on NP-completeness as an indicator of computational complexity Walsh (2010). The issue is that NP-completeness is only a worst-case notion. The fact that a problem is NP-complete simply means that there exist some instances, not necessary all, that are difficult to manipulate. In other words, the difficulty in constructing an efficient algorithm that *always* find a beneficial manipulation does not guarantee the nonexistence of an algorithm that *often* finds a successful manipulation. To guarantee a voting rule to be immune to manipulation, we want to make all the possible instances hard to manipulate. Thus, the conclusion about the (non)existence of voting rules that are usually hard to manipulate using the computational complexity approach is yet to be drawn.

2.3.2 The axiomatic approach

Since computational complexity is a worst-case analysis which may not sufficiently guarantee the resistance to manipulation, some authors take the axiomatic approach, showing that voting rules satisfying certain axioms are usually manipulable by a trivial algorithm. Conitzer and Sandholm (2006) are the first to take this approach. They show that common voting rules satisfy weak monotonicity and allow the manipulators to make either of exactly two candidates win. In addition, they argue that it is impossible to design a rule that both satisfies these two properties and resists manipulation at the same time.

Without imposing restrictions on a voting rule, Friedgut *et al.* (2008) show that an entirely random manipulation may succeed with nonnegligible probability. Under the

IAC assumption with exactly three candidates and a voting rule that has an outcome independent of the names of the candidates, if a randomized algorithm succeeds with only negligible probability, then the voting rule must be very close to being dictatorial. Although the result only holds for a restricted number of candidates, it covers every reasonable voting rule.

Xia and Conitzer (2008) extend the results of Friedgut *et al.* (2008) and obtain similar conclusion that holds for any constant number of candidates. However, there are additional conditions that the rules need to satisfy including homogeneity, anonymity, non-imposition and canceling-out conditions. It has been shown that both the positional scoring rule and STV do satisfy these conditions; thus, there exists at least a voter such that a random manipulation for that voter will succeed with a nonnegligible probability. Isaksson *et al.* (2012) successfully extend the result to settings with an arbitrary number of voters and at least four candidates.

2.3.3 Other measures of manipulability

The vulnerability of a voting scheme can be measured by counting the opportunities for manipulation or the number of candidates who can be made to win by manipulation (Chamberlin, 1985). Both the number of voters who can individually manipulate the elections and the Nitzan-Kelly's (NK) index, which is the share of all manipulable voting situations, have also been used to compare the manipulability of voting rules (Aleskerov *et al.*, 2015).

Saari (1995) notes that for individual manipulation to be successful, both the manipulated and sincere profiles should be very close from two sides to the profile boundary separating these outcomes. Two factors deciding if the manipulation would be successful are the choice of the voting rule and the abundance of opportunities for a successful manipulation. Strategic voting is not always used because the outcome depends on the manipulator's prior knowledge or expectation about the sincere election outcome.

Reyhani *et al.* (2010) propose a new probabilistic measure of manipulability in the case of coalition manipulation, acknowledging the limitations of two existing probabilistic measures of manipulability. The logical possibility of manipulation fails to measure the effort needed to gather and coordinate a coalition manipulation. There exist situations where two voting rules have the same logical probabilities, but one requires much more effort than the other does. The other measure that counts the necessary number of manipulators becomes defective when certain situations are not manipulable by any coalition. In addition, between two voting rules which requires the same number of manipulators, one might be more susceptible to manipulation than the other in some certain circumstances. The new probabilistic measure is suggested to reflect both the size and the prevalence of the coalition manipulation. Reyhani *et al.* (2010) define this measure as the minimum number of random agents that a potential instigator has to interview to get enough agents for a successful coalitional manipulation. The underlying assumption is that although the opinion distribution is known, the potential instigator does not know which agent holds which opinion and thus, has to interview them one by one.

Diss (2015) introduces the notions of self-selectivity and stability to examine the manipulability of Borda, Copeland and Plurality rules with three candidates. A voting rule is self-selective at some profiles if, given those profiles, there are no alternative rules that beat the given voting rule if the given voting rule is used to choose among the rules in the set. In this context, manipulability is understood as the possibility that a given voting rule become non-selective after manipulation. In addition, a set of voting rules is weakly stable if it always contains at least one self-selective rule at any profile or voting situation. In this case, if none of the voting rules in the set is self-selective, the society is not able to vote on how to vote. Diss (2015) concludes that the probability of individual and coalitional manipulations tend to vanish significantly when the notions of self-selectivity and stability are introduced.

Arribillaga and Massó (2016) identify three necessary and sufficient conditions that provide a systematic way to compare non-constant and non-dictatorial generalized median voter schemes in terms of their vulnerability to manipulation. The sets of manipulable preferences for each voting rules are used as the criterion while the possibility of incomparable pairs of social choice functions is not precluded. The authors note that existing literature has not distinguished the situation when only one manipulable instance is possible from the case when many such instances exist. In other words, manipulability of a social choice function does not indicate the degree of its lack of strategy-proofness but merely states that the social choice function is manipulable. The approach of applying a median voter scheme to the universal domain turns out to be relevant if agents have additional preferences on top of single-peaked preferences. The author concludes whether or not a strategy-proof median voter scheme (on the domain of single-peaked preferences) becomes manipulable on the universal domain depends very much on the identity of the agent, the particular properties of the additional preference, and the median voter scheme under consideration.

In the following sections, we will go into details for each voting rule. We provide the definition and an example how the voting rule works, special properties, how an individual can manipulate the election outcome in each voting rules and synthesize the results related to manipulation from existing literature.

2.4 Plurality

Plurality is the most common form of the electoral system, used for local and national elections in 43 of the 193 countries of the United Nations[2]. Plurality voting is particularly prevalent in the United Kingdom and former British colonies, including the United States, Canada, and India.

[2]Source: `http://aceproject.org/epic-en/CDMap?question=ES`, retrieved 2 August 2016.

2.4.1 Definition

The Plurality[3] selects the candidate with the most votes. Each voter casts vote for one of the alternatives and the winner is the alternative with the most votes. This makes Plurality the simplest voting system for voters and vote counting officials although drawing the district boundary lines can be a contentious issue. Plurality ballots (or single-mark ballots) can be categorized into two forms: either a blank ballot where the name of a candidate can be hand written or a more structured ballot that lists all the candidates and allows a mark to be made next to the name of a single candidate. A structured ballot can also include space for a write-in candidate.

Example 1. Three candidates A, B, C running in an election with 100 votes under Plurality scheme. If A received 40 first-place votes and while B and C each got 30, then A would be elected under Plurality.

The problem with the Plurality method is that it might select an unpopular candidate for the majority of voters (Wallis, 2014). In the example above, all the supporters of B and C thought that these two candidates were better-qualified than A. The problem of selecting an unpopular candidate is magnified when the number of candidates increases. Voters often believe that the Plurality elects the wrong candidate.

In the countries with two major political parties, to overcome this dilemma, it is common for each party to nominate one candidate representing the party. For example, in the United States, a *primary election* is held to select the party nominee if there are two or more members of a party wish to run for an election. The Democratic and Republican nominees then compete in the national election. This method, however, will not solve the problems if there are several major parties.

[3] It is also known as *first-past-the-post* in a system based on single-member districts or *winner-takes-all* in a system based on multi-member districts

2.4.2 Manipulation

If voters vote sincerely, they should cast their vote only for their most preferred alternative. However, while we know that no voting rule is completely immune to strategic behavior, Plurality has been shown to be particularly susceptible, both in theory and in practice (Saari, 1990, Forsythe *et al.*, 1996). A voter opts to vote for one of the two candidates she predicts is most likely to win, even if her true preference is neither because a vote for any other candidate will likely to be wasted and have no impact on the final result. Also, with the number of strategies equal to the number of alternatives, players can compute their best strategy easily, enabling them to vote strategically in a straightforward way.

A striking example was the US presidential election in 2000. The presence of Ralph Nader on the ballot swung the election from Al Gore to George W. Bush. If Ralph Nander had not run, Al Gore would have won Florida and the national election because the majority of Nader voters would have chosen Gore over Bush. This creates the dilemma "people should afford to vote sincerely only when their votes do not matter"[4] (Dixit and Nalebuff, 2008). If the election would be won by Bush whether a voter votes or not, then she might vote sincerely with her heart. If her vote counts (breaking a tie), then voting for Nader is a missed opportunity. In other words, a voter should vote strategically if she is a *pivotal voter*. We would like people who genuinely prefer Ralph Nader to have a way to express that view without having to give up their vote in Bush versus Gore but there seems no way that Plurality can allow voters to do so.

Numerous studies have reached the same conclusion about the vulnerability of Plurality to manipulation. Saari (2001) argues that Plurality is further away from symmetry, and thus provide more strategic opportunities. Hemaspaandra and Hemaspaandra (2007) produce one of the most beautiful results regarding the complexity of manipulation, named "the dichotomy theorem" that classifies the complexity of manipulation

[4]In the context where American politicians are calling everyone to vote, it is better to say each vote contributes to the final outcome.

in scoring rules. They state that every scoring rule having two or more point values assigned to candidates other than the favorite, except Plurality and its various guises, is NP-complete to manipulate.

The relationship between the degree of manipulability and the size of the electorate as well as the number of candidates has been extensively studied for Plurality. Aleskerov *et al.* (2015) estimate the level of manipulability of 7 aggregation procedures: Plurality, Approval, Borda's, Black's, Hare's, Nanson's and Threshold under both IC and IAC assumptions. Using Nitzan-Kelly's (NK) index, the authors find that for the case of 3 candidates with up to 10000 voters, Plurality is the worst aggregation procedure concerning manipulability for both IC and IAC. Furthermore, for all voting rules including Plurality, the degree of manipulability decreases with the size of the electorates because as we consider individual manipulability, the weight of one's preferences and its influence are decreasing with the growing number of voters. This finding is consistent with Smith (1999) which states that the manipulability index rises monotonically with increasing number of electors, peaking when the number of voters is between 10 and 20. However, Huang and Chua (2000) find the vulnerability of Plurality to be increasing as the size of the electorate increases in multiples of 12. Furthermore, Smith (1999) also conclude that when the size of the electorate is fixed, manipulability indices rise monotonically with an increasing number of candidates.

Meir *et al.* (2010) apply the game-theoretic solution concepts to voting games, studying the conditions under which Plurality voters' strategic behavior converges to a decision from which no voter want to deviate (Nash equilibrium). They assume the scenarios where voters have no initial knowledge regarding the preferences of the others and cannot coordinate their actions. It is found that the convergence depends on the features of the game such as the tie-breaking scheme and on the assumption regarding agents' weights and strategies.

Conitzer *et al.* (2007) focus on the computational complexity of voting rules when the number of candidates is small, and there is uncertainty about the other's votes.

They study not only whether a voting rule is hard to manipulate but also the number of candidates needed for the hardness to occur for both constructive manipulation (making a given candidate win) and destructive manipulation (making a given candidate not win). The authors find that both constructively and destructively manipulating the outcome under Plurality are easy with any number of candidates.

While agreeing that Plurality is the most manipulable, Bassi (2015) notes some experimental findings in favor of Plurality. This voting system yields not only more optimal behavior in the sense that players are more likely to play the equilibrium strategy, but also a higher level of social efficiency. Furthermore, Plurality never elects a Condorcet loser although it reduces the frequency of electing the Condorcet alternative.

2.5 Anti-plurality

Anti-plurality is a member of the scoring voting family. Anti-plurality has not been used for national election anywhere, although it is often used on a personnel committee (Saari, 2001).

2.5.1 Definition

The Anti-plurality rule selects the candidate with the least last-place rankings. Saari (1995) introduces Anti-plurality as a kinder and gentler way of identifying the bottom-ranked candidate where voters are instructed to vote for all but one candidate. An alternative, equivalent approach that we follow in this thesis is to understand Anti-plurality as a procedure that asks each voter to vote *against* one candidate. The following example illustrates how this voting rule operates.

Example 2. Three candidates A, B, C running in an election with 100 votes under Anti-plurality scheme. If A received 40 last-place votes and while B and C each got 30 and 50, then B would be elected under Anti-plurality.

2.5.2 Manipulation

According to Saari (1995), individual manipulability depends on how many profiles are near the boundary. Thus, a smaller boundary allows fewer chances of manipulation. The more symmetric the procedure is, the minimal boundary areas are and thus, the minimal exposure of opportunities to be successfully manipulated is. Saari (2001) constructs the "procedure line" with normalized outcomes as endpoints. Each point on this line is the profile's election outcome for some positional methods. The profile allows each candidate to be the winner with an appropriate way to tally the ballots, alerting the danger of electing the wrong candidate if an inappropriate voting method is to be applied. It is proven that Anti-plurality rule is further away from symmetry, and hence, provide more strategic opportunities.

Inferring from studies of a more general class of voting rules, individual manipulation under Anti-plurality is relatively easy. Friedgut *et al.* (2008) find that for nondictatorial neutral [5] voting rules with three candidates, a random manipulation by a single random voter will succeed with nonnegligible probability. Hemaspaandra and Hemaspaandra (2007) claim that for Plurality voting's transparently disguised translations including Anti-plurality, weighted manipulation can be done in polynomial time.

Furthermore, the vulnerability of Anti-plurality is found to accelerate with the size of the electorate. Huang and Chua (2000) provide an analytical characterization of the vulnerability properties of four scoring rules: Plurality, Anti-plurality, Plurality with runoff [6] and Anti-plurality with runoff [7]. The authors also find that the non-sequential mechanisms (Plurality and Anti-plurality) are more susceptible to manipulability than their sequential counterparts.

Coalitional manipulation is often shown to be hard under Anti-plurality. Xia *et al.*

[5]A social choice function is neutral if it is invariant under changes made to the names of the alternatives.

[6]In the absence of a majority winner, Plurality with runoff selects the two top scorers from the first round for the runoff. The winner of the runoff is the social choice.

[7]Anti-plurality with runoff sequentially eliminates the candidate with the most last-place rankings, and the ultimate survivor of the process is the social choice.

(2010) show that coalitional manipulation remains NP-complete for scoring rules even when votes are unweighted. Conitzer *et al.* (2007) find that the constructive coalitional weighted manipulation starts becoming NP-complete for Anti-plurality when the number of candidates is three. Meanwhile, Reyhani *et al.* (2010) note that although Anti-plurality is less susceptible to manipulation compared to Plurality and Borda count under IC, it often requires smaller coalitions in the situations where it is manipulable, and finding such coalitions is relatively easy.

2.6 Borda Count

Borda Count is a classic voting system which was named after the French mathematician and engineer Jean-Charles de Borda who devised the system in the 18^{th} century. Borda Count was proposed to cure one of the major problems in Plurality where the candidate with the most votes might be the unpopular one for the majority of voters. Borda Count, often described as a consensus-based voting system, is believed to better reflect the wishes of the electorate when more than two candidates are running for the election.

Borda Count has a rich and varied history of real-world use (Menton and Singh, 2012). Nauru uses Borda Count to elect members of the Parliament, and Kiribati uses Borda Count for selecting presidential election candidates. Borda Count has also been used to elect two ethnic minority members of the National Assembly of Slovenia and throughout the world by various private organizations and competitions (Davies *et al.*, 2011).

2.6.1 Definition

The winner under Borda Count is the candidate with the most points where a specific point is given for each ranking that voters rank over candidates. The following example demonstrates the process of finding Borda Count winner.

Example 3. Consider three candidates election with 14 votes and the ranking as follow:

6	6	2
A	B	C
B	A	A
C	C	B

If 1, $\frac{1}{2}$ and 0 points are given to the first, second and third ranked candidates, respectively, then the total points for three candidates A, B, C will be 10, 9, 2, respectively; hence, A is elected under Borda Count.

There exist different versions of Borda Count depending on the flexibility of preferences listed on the ballots. The simplest way is to allow voters to rank as many candidates as they wish, leaving all unranked candidate the minimum number of points. In some places like Nauru, voters are obliged to rank all candidates.

2.6.2 Properties

Borda Count does not always satisfy the IIA property, even in a small electorate (Wallis, 2014, Mas-Colell *et al.*, 1995). The reason is that the rank of a candidate depends on the placement of all other candidates.

Continuing with the example above but assume that the two voters who had preferences $C \succ A \succ B$ decided that B was a better candidate, the profile would be:

6	6	2
A	B	B
B	A	A
C	C	C

In this case, B would be the winner with 11 points compared to A with 10 points and C with 0 points. No voter changed their preference ordering of A and B but the electorate's order of those two candidates has changed.

2.6.3 Manipulation

Borda Count used to be criticized when it was first introduced such that Jean-Charles de Borda has famously dismissed criticism of his system's vulnerability to manipulation by saying "My scheme is only intended for honest men".

Computational complexity and experimental approaches suggest that Borda Count is particularly vulnerable to individual manipulation (Bartholdi III *et al.*, 1989). Smith (1999) finds that Borda Count is especially manipulable if the manipulator has complete knowledge of the others. Betzler *et al.* (2011) and Davies *et al.* (2011) show that Borda Count manipulation is NP-complete even in the case of three votes and two additional manipulators. Kim and Roush (1996) prove that Borda Count becomes coalitionally manipulable with probability 1 when the number of voters tends to infinity. Bassi (2015) experimentally studies behavior of electors under three voting systems: Plurality, Approval and Borda Count, finding that voters depart from their sincere strategy most under Borda Count, without playing the best response strategy.

Other measures of manipulability also suggest that Borda Count is especially manipulable. Chamberlin (1985) uses data generated through Monte Carlo simulation to study four voting rules: Borda Count, Coombs, Hare, and Plurality of four candidates under different electorate sizes. The author finds that Borda Count has the highest percentage election in which manipulation is logically possible. In other words, Borda Count is most frequently manipulable compared to the other three voting rules, with a large number of manipulable cases. Although Borda Count turns out to be less manipulable than Plurality when the number of candidates for whom manipulation is possible is used as the criterion, it is still the most manipulable voting rule among the four when combining the logical possibility and coalition size standards.

Favardin *et al.* (2002) characterize the conditions under which manipulation can occur with Borda Count and Copeland in three-candidate elections, deriving a closed-form representation of manipulability. The authors find that, under the IAC assumption, the Borda Count rule is more vulnerable to individual manipulations compared to Copeland though there also exists situation at which Copeland is manipulable but not Borda Count. Tie-breaking rules have a significant impact on the results, with the probability of a tied election tending to zero when the number of voters becomes large for Borda Count, but not for Copeland. If the tie-breaking rule changes from alphabetic order to random, the results for Borda Count would modify marginally while the vulnerability to strategic manipulation of Copeland could reduce significantly. Although Copeland suffers from no-show or abstention paradox, whereas the Borda Count (and all scoring rules) are immune, this does not change the conclusion. The reason is that voting situations which may lead to a no-show paradox have already been counted as manipulable situations.

Gehrlein and Lepelley (2004) claim that the theoretical probabilities of voting paradoxes cannot be considered completely negligible. With the assumption of IAC, the Borda Count rule is vulnerable to strategic manipulation by coalitions of voters in 50% of the preference profiles. However, the authors also note that these probabilities, computed under the classic assumptions, should not be interpreted as an estimate of what might happen in actual voting situations. Regarding the comparison of alternative voting rules, the vulnerability of Borda Count to coalitional manipulation in three-candidate elections with the IAC assumption is about 4.5 times higher than that of Plurality rule elimination. Although the relative results offer useful insights in the choice of voting rules, one needs to be cautious in examining if the results remain valid when relaxing the IAC assumptions.

With regards to the number of candidates needed to make manipulation infeasible, Conitzer *et al.* (2007) find that the constructive coalitional weighted manipulation starts becoming NP-complete when the number of candidates is three for Borda Count. In addition, constructive individual unweighted manipulation under uncertainty is also NP-complete even with three candidates. However, destructive manipulation can be done in polynomial time with any number of candidates.

Although many studies have concluded that Borda Count is one of the most vulnerable voting rules for both individual and coalitional manipulation, Pritchard and Slinko (2006) find Borda Count the optimal scoring rules, with the maximized asymptotic average threshold coalition size when the number of candidates equal to 3 or 4. Furthermore, Borda Count turns out to be one of the least vulnerable when the possibility of individual reactions or counter-coalitions of homogeneous voters is considered. Favardin and Lepelley (2006) consider manipulability when taking into account the possibility of reactions or counter-threats under IAC. They find that the possibility of reactions or counter-threats significantly reduces the manipulability measures of Borda Count.

2.7 Single Transferable Vote

The STV has had an unusual place in the echelons of electoral systems, being one of the oldest, yet rarest, proportional representation systems in use (Farrell and Katz, 2014). STV was part of the by-laws of the Society for Literary and Scientific Improvement of Birmingham in 1819[8], and was used in the Adelaide City Council election in 1984[9]. Research evidence suggests the STV began to be used again in the first part of the 20^{th} century, possibly because it was relatively difficult to implement this method before the advent of computers and technology. STV was adopted for electing the Tasmania lower house in 1907 and for electing the Australian Senate in 1949. Britain introduced STV to Ireland in 1920 and Malta in 1921. These two countries continue to use STV in their presidential elections after gaining independence (Farrell and Katz, 2014).

Although STV is used in local elections or at the organizational level in various places including New Zealand, the International Olympic Committee and American Political Science Association (Walsh, 2010), it has only been studied extensively in Australia, Ireland, and Malta, where it is used for national parliamentary elections. The fact that all three countries are former British colonies is not a coincidence: Britain had a tendency to experiment STV on its colonies (Farrell *et al.*, 2016). The retention of STV as Ireland's electoral system has been a debate on some occasions and the decision to retain STV has been quite deliberate. Most recently in the midst of the 2011 economic crisis, all Irish political parties included proposals for electoral reform in their manifestos, trying

[8] Source:`https://upload.wikimedia.org/wikipedia/en/2/2c/Thomas_wright_hill_laws_1819.pdf`. retrieved 15 August 2016

[9] Barber (2000)

to move voters away from STV, which is perceived to be a “voter-friendly” electoral system. The Constitutional Convention discussed the matter and recommended keeping a modified version of the existing STV, showing a trend towards a personalized electoral system.

2.7.1 Definition

According to the Oxford Advanced American Dictionary, Oxford University Press, STV is defined as[10]:

> An electoral system of proportional representation in which a person’s vote can be transferred to a second or further competing candidate (according to the voter’s stated order of preference) if the candidate of the first choice is eliminated during a succession of counts or has more votes than are needed for election.

STV is an iterative procedure. At the beginning of each iteration, the number of first-place votes is counted. The candidates with total first place votes equaling or exceeding the quota wins the election, and any excess votes are redistributed proportionally to the second preferences. If no candidate is reaching the quota, then the candidate with the least first votes will be eliminated and his votes are distributed to the next preferences. Quota can be re-evaluated for each iteration, which is called “dynamic STV” (Wallis, 2014). However, the process of re-evaluation is not significant because the re-evaluation neither increase nor decrease the quota more than one(Aleskerov and Karpov, 2013).

In the STV system, voters supply preference profiles of approved candidates. Voters could omit candidates at the bottom of preference tables, or leave some candidates unranked, but we only consider the full preference profiles in this thesis. Ranking a complete set of candidates is not difficult or demanding when we examine the electorates with a small number of candidates.

[10]Source:`https://en.oxforddictionaries.com/definition/single_transferable_vote`, retrieved 5 July 2016

In general, a quota is calculated depending on the number of voters and the number of seats to be filled. The Hare quota used to be applied for STV with the idea of taking the integer part of the ratio between the number of votes and the number of seats $\lfloor \frac{v}{k} \rfloor$, where v denotes the number of valid votes and k denotes the number of vacancies to be filled. In 1881, the Droop quota was proposed, which is defined as $\lfloor \frac{v}{k+1} \rfloor + 1$, the smallest number to ensure that no more candidates can reach the quota than the number of vacancies to be filled. Although the Droop quota can create a situation where majority coalition achieves fewer seats than a minority, it remains the most common quota used for STV (Tideman, 1995).

The choice of quota could have significant impacts on the election outcomes (Tideman, 1995). Take for example the situation where we need to select three candidates from a pool of four, two of which are from the Democrats party and the other two are from the Republican party. There are 100 voters in total with the following voting profile:

24	23	32	21
D	E	R	S
E	D	S	R
R	S	D	E
S	R	E	D

If the Hare quota, which is $\lfloor \frac{100}{3} \rfloor = 33$, is applied, no candidate reaches the quota in the first iteration. The candidate with the least first place vote, S, will be eliminated, resulting in the election of D, E and R. However, if the Droop quota, which is $\lfloor \frac{100}{3+1} \rfloor + 1 = 26$ is used, then R wins in the first round. His excess votes (6) will be distributed to the second preferences S, resulting in the election of S. Therefore, R, S and D will be elected using the Droop quota.

Although STV can be used to select a single seat or multiple candidates, from now on we will only consider choosing a single position, which is the focus of this thesis.

2.7.2 Properties

The complex, shifting behavior of STV with multiple candidates makes STV fail to possess some desirable properties, notably the IIA and monotonicity.

Independence of Irrelevant Alternatives

The violation of this property under STV can be best explained in the following example.

Example 4. Consider the situation where we need to elect one position from three candidates with 9 voters, with preference profile:

3	2	2	2
A	B	C	C
B	C	A	B
C	A	B	A

The Droop quota equals to $\lfloor \frac{9}{1+1} \rfloor + 1 = 5$. No candidate is winning in the first round and B is eliminated. His votes are distributed to the second preference resulting in the election of C.

Now, assume that one voter who has preference ranking $A \succ B \succ C$ changes her mind, placing B above A in the ballot. The new preference profile will be:

2	1	2	2	2
A	B	B	C	C
B	A	C	A	B
C	C	A	B	A

In this case, A is the candidate with the fewest first-place votes and thus eliminated. A's votes are distributed to B, resulting in the election of B. Thus, although no voter changes their ordering of B and C, the election outcome changes.

Negative Responsiveness

Monotonicity means that if one or more voters change their ranked preferences by placing one candidate higher on preference ranking, then the overall preference list should either be changed in favor of that candidate or else be unchanged. In other words, a candidate cannot be made less popular overall by having more supporters and this is a desirable property of a voting scheme. Unfortunately, this property of "nonnegative responsiveness" does not apply to STV. It has been shown that having more voters ranking a candidate as their first preferences could hurt his chance of winning the election under STV (Tideman, 1995). The following two examples illustrate this point.

Example 5. Assuming that one out of three candidates is to be elected from the following preference profiles:

3	4	3	1
A	B	C	C
B	C	A	B
C	A	B	A

Without any changes, no candidate has the Droop quota of 6. A will be eliminated, and B wins the election. However, if the last voter with preference $C \succ B \succ A$ happens to ranks B over C, then in the first round, B gets 5 first place votes when A and C ties. There is a positive probability that A remains and C is eliminated. In that case, the votes from C will be transferred to A, resulting in A winning the election.

Example 6. [11] Consider the profile:

1	4	6	5
A	A	B	C
B	C	C	A
C	B	A	B

No candidate meets the Droop quota of 9 in the first round. A and C are both eliminated, B wins the election. Now, if the first voter with preference $A \succ B \succ C$ were to promote B to first place, resulting in the profile:

[11] adapted fromWallis (2014)

1	4	6	5
B	A	B	C
A	C	C	A
C	B	A	B

A would be eliminated, and C beats B by 9-7.

Hence, a voter giving more support to a candidate under STV can actually lower his chance of winning the election. Wallis (2014) proves that under STV, examples that do not involve a tie will leave a probability on the outcome in the second round. Otherwise, it would involve more than one voter having changed their preference rankings. Stensholt (2004) notes that the nonmonotonicity property creates instability in the outcomes because a voter cannot be sure if ranking her favorite candidate first is the best support she could make. To keep the weaknesses of election method at a minimum level without losing too many of its strengths, Stensholt (2004) proposes using "tax cut algorithm[12]" on the reversed ballot ranking. The author claims that using tax cut, STV will avoid eliminations, which is often believed to lead to nonmonotonicity.

2.7.3 Manipulation

Before discussing manipulation under STV, it is necessary to distinguish manipulation from split-ticket. STV often leads to split-ticket where a voter's ranking is not structured by party. Voters can give a second preference to a candidate nominated by a second party rather than one nominated by the same party. Split-ticket voting is not strategic voting because STV allows voters to express their preferences over candidates rather than parties. The election of a candidate is based largely on their campaign for personal votes, with the party campaign being secondary (Marsh and Plescia, 2016, Marsh, 2007). The following example illustrates how individual manipulation works under STV.

[12] Stensholt (2004) defines tax cut tally as "an iterative algorithm that ranks the candidates by letting each voter transfer the surplus part of the vote from one candidate to the next". This algorithm takes into account the voting power of a voter in a specific round under STV.

Example 7. [13] Assume that we need to select 1 position from 4 candidates in an election with 7 voters. A candidate needs to obtain $\lfloor \frac{7}{1+1} \rfloor + 1 = 4$ to win the election. We start with the sincere preference profile:

2	1	2	1	2
B	B	D	C	C
A	D	C	A	D
D	A	B	B	A
C	C	A	D	B

In this sincere preference profile, A has no first-place votes so A will be eliminated first, resulting in profile:

2	1	2	1	2
B	B	D	C	C
D	D	C	B	D
C	C	B	D	B

Next, D has the smallest number of first place votes and is eliminated, leaving:

2	1	2	1	2
B	B	C	C	C
C	C	B	B	B

To this iteration, C gains 5 first place votes and wins the election.

Now suppose that one voter, the manipulator, changes the ballot from $B \succ A \succ D \succ C$ to $D \succ A \succ B \succ C$. The insincere profile is:

1	1	1	2	1	2
B	D	B	D	C	C
A	A	D	C	A	D
D	B	A	B	B	A
C	C	C	A	D	B

Again A is eliminated first, leaving:

[13] adapted from (Wallis, 2014, p.37)

1	1	1	2	1	2
B	D	B	D	C	C
D	B	D	C	B	D
C	C	C	B	D	B

In the next round, B has the fewest first-place votes and thus, is eliminated:

1	1	1	2	1	2
D	D	D	D	C	C
C	C	C	C	D	D

D is elected with 4 first place votes, which makes the manipulator better off. We say that the election has been manipulated successfully. By the following, we will highlight literature regarding whether or not it is easy for an individual to manipulate the election outcome under STV.

Despite exhibiting both inconsistency if the agenda changes in the sense that an introduction of a new candidate would change the relative ranking of former candidates and nonmonotonicity, existing literature has given enormous support to STV. It is believed that not only does STV keep the number of wasted votes to a minimum and equally take into account the opinions of each voter, but it also avoids the incentive to distort real preferences, encouraging voters to honestly rank candidates in the way they wish (Tideman, 1995).

The manipulability of STV has often been studied using the computational complexity approach. STV is one of the very first rules shown to be hard-to-manipulate (NP-complete). Bartholdi III and Orlin (1991) study STV with both the number of candidates and the number of voters unbounded using the computational effort approach. They demonstrate that it is NP-complete under STV to find an "Effective Preference" or a "Preferred Outcome" where a manipulator can have their more preferred candidate elected compared to voting honestly. According to Bartholdi III and Orlin (1991), despite the fact all voting schemes with more than two candidates are manipulable, for some like STV it is infeasible to do so.

The relationship between manipulability and the number of voters as well as the number of candidates has been studied for STV. Walsh (2010) studies different distributions of votes. It is shown that with the simplest IC condition, the probability that an agent can manipulate the outcome decreases as the number of agents or number of candidates gets larger. In other words, the individual manipulability is only significant when the number of voters and the number of candidates are small. Subsequently, the study is extended to consider the case when votes are correlated and drawn from an urn model to find that the cost of computing an individual manipulation increases exponentially with the number of candidates. However, if the number of candidates is fixed at some different values up to 64, finding an individual manipulation or proving its nonexistence is relatively easy. Furthermore, for coalitional manipulation, the chance to successfully manipulate the election increases with the size of coalition.

Conitzer *et al.* (2007) find STV to be NP-complete when the number of candidates is equal to or greater than three, for both constructive and destructive manipulations. The authors first establish the results for constructive coalitional weighted manipulation in the complete information setting. They then prove that hardness of coalitional manipulation with complete information implies hardness of individual unweighted manipulation in the incomplete information setting. In other words, a constructive individual unweighted manipulation under uncertainty is NP-complete for STV even with three candidates.

Furthermore, although STV has an undesirable property of being negative responsive or nonmonotonic, it is also NP-complete to recognize nonmonotonicity in STV elections. Even better than that, this undesirable property of negative responsiveness can be considered as a factor that contributes to the hardness to manipulate of STV because to help a candidate win, a manipulator might have to rank that candidate later (not the first) in the preference order (Chamberlin, 1985). The difficult-to-manipulate under STV compared to other voting schemes is quite robust to some restrictive assumptions such as allowing for voters to learn about the preferences of others. This difficulty holds for all variants of STV which differ mostly in the way excess votes are reallocated to the next preferences (Chamberlin, 1985, Bartholdi III and Orlin, 1991).

However, the computational complexity approach has been challenged. NP-hardness only bounds for the worst-case complexity (Walsh, 2010). Using a real world sample analysis, Walsh (2010) concludes that although STV is computationally difficult to manipulate, manipulation can be relatively easy in practice. Theoretical papers including Conitzer and Sandholm (2006), Friedgut *et al.* (2008) show manipulation is often easy in practice despite NP-hardness. The results raise the concern if the computational complexity is really the shield against manipulation for the STV rule.

In conclusion, current literature suggests that scoring rules including Plurality, Anti-plurality and Borda Count are particularly susceptible to manipulation while STV is found hard to manipulate, removing the need for strategic voting and ensuring the elected candidates to reflect the diverse views of voters. We can now compute the exact manipulability of four voting rules and see if our results are consistent with those using different approaches.

Chapter 3

Model

To keep this chapter relatively self-contained, we start with the Preliminaries section, introducing the notations and redefining necessary concepts in mathematical terms. The regions where an individual can manipulate election outcome under each of four voting rules can then be defined. The chapter concludes with the general method used to compute the volume of convex regions in high dimensional space, with appropriate adjustment to our regions of interest.

3.1 Preliminaries

The formal definitions are first introduced in general forms for any finite number of alternatives and then delineated to four voting rules in a specific three-candidate election setting. The ideas are taken from Mas-Colell *et al.* (1995) and Taylor (2005) with adaptation to the new set of notations.

3.1.1 Basic setting

We consider the situation where a set K of voters (agents) need to make a collective choice from a set X of m candidates (alternatives). A voter k expresses a strict, complete and transitive preference ranking over candidates. This linear preference relation over the set of candidates, often called *ballot*, reflects some aspect of a voter's opinion about the desirability of different candidates. However, a ballot is just a voter's ordinal preference ordering which neither reflects the intensity between candidates nor describes how much a voter likes a particular candidate. The set of preference relations $\succeq$ on X is denoted $\mathscr{R}$. We also designate the subset of $\mathscr{R}$ that consists of strict preference relations (no two distinct alternatives are indifferent for $\succeq$) as $\mathscr{P}$.

Definition 1. Given any subset $\mathscr{A} \subset \mathscr{R}^K$, a *social choice function* $f : \mathscr{A} \to X$ defined on $\mathscr{A}$ assigns a chosen element $f(\succeq_1, \cdots, \succeq_K) \in X$ to every profile of individual preferences in $\mathscr{A}$.

A *social choice function* (resolute voting rule or simply, voting rule) differs from a *social choice correspondence* (non-resolute voting rule) in the sense that a social choice correspondence outputs a subset of the candidates, allowing for the tie situations; while for social choice function, the outcome is a single element of X[1]. In this thesis, we only focus on social choice functions, and for the sake of simplicity, we often call them *voting rules*.

Definition 2. The social choice function $f : \mathscr{A} \to X$ defined on $\mathscr{A} \subset \mathscr{R}^K$ is *monotonic* if for any two profiles $(\succeq_1, \cdots, \succeq_K) \in \mathscr{A}, (\succeq_1^{'}, \cdots, \succeq_K^{'}) \in \mathscr{A}$ with the property that the chosen alternative $x = f(\succeq_1, \cdots, \succeq_K)$ maintains its position from $(\succeq_1, \cdots, \succeq_K)$ to $(\succeq_1^{'}, \cdots, \succeq_K^{'})$, we have that $f(\succeq_1^{'}, \cdots, \succeq_K^{'}) = x$.

The social choice function is *monotonic* if an alternative can only be deselected by having some agents lowering their rankings for that candidate. As discussed in Chapter 2, our three scoring rules are monotonic while STV is not.

Definition 3. The social choice function $f : \mathscr{A} \to X$ defined on $\mathscr{A} \subset \mathscr{R}^K$ is *dictatorial* if there exists an agent $h \in K$, *the dictator*, such that for every profile $(\succeq_1, \cdots, \succeq_K) \in \mathscr{A}, f(\succeq_1, \cdots, \succeq_K) \in \{x \in X : x \succeq_h y \text{ for every } y \in X\}$.

The dictatorial social choice function always selects the dictator's most preferred alternative. Obviously, there is no point wasting time to vote if a dictatorial social choice

[1]In practice, elections involve various tie-breaking rules but here we disregard such complications.

function is in use because we just need the dictator to determine the outcome.

3.1.2 The Gibbard-Satterthwaite Theorem

We assume that each agent k has private information about her own types, which is θ_k that affects her decision. All the possible preference relations over X of agent k make up the set $\mathscr{R}_k = \{\succ_k : \succ_k = \succ_k (\theta_k) \quad \text{for some} \quad \theta_k \in \Theta_k\}$ with Θ_k denotes the set of all possible types of agent k. The Gibbard-Satterthwaite theorem[2] is stated as:

Theorem 1. (*The Gibbard-Satterthwaite Theorem*) Suppose that X is finite and contains at least three elements, that $\mathscr{R}_k = \mathscr{P}$ for all k, and that $f(\Theta) = X$. Then the social choice function $f(.)$ is truthfully implementable in dominant strategies if and only if it is dictatorial.

In this context, a social choice function is constructed as a game form where players are voters and player's strategies are the set of all possible orderings of preferences over the set of alternatives. A social choice function is said to be immune to manipulation if each voter has a strategy that will be at least as good as any other no matter what any other voter does, called the *dominant strategy*. Otherwise, a voter might have the incentive to depart from her actual preference to obtain a more favorable outcome.

The Gibbard-Satterthwaite theorem is restricted to social choice function and the finiteness of the set X. However, it has been shown that the result applies even without these requirements (Mas-Colell *et al.*, 1995). As our four voting rules are social choice functions, and we are considering the case of three-candidate elections which well satisfy the original Gibbard-Satterthwaite Theorem; we do not delineate in the expansion of the theorem.

[2]The proof of this theorem can be found in Mas-Colell *et al.* (1995) or Benoıt (2000).

3.1.3 Definitions of four voting rules

Scoring rules

Plurality, Anti-plurality and Borda Count belong to the class of scoring rule which assigns scores to each candidate using a weight vector $\vec{\alpha} = \langle \alpha_1, \cdots, \alpha_m \rangle$. A candidate receives α_1 points if ranked first, α_2 points if ranked second, etc. for every preference ranking in a ballot. The candidate with the highest total number of points wins the election.

Plurality is the most common scoring rule with $\vec{\alpha} = \langle 1, 0, \cdots, 0 \rangle$; Anti-plurality is the scoring voting rule with the score vector $\vec{\alpha} = \langle 1, \cdots, 1, 0 \rangle$; and Borda Count is the scoring rule which uses the weight vector $\vec{\alpha} = \langle 1, \frac{m-2}{m-1}, \frac{m-3}{m-1}, \cdots, 0 \rangle$.

Single Transferable Vote

Our final voting rule is essentially a "runoff system" which is based on the idea of iteratively eliminating the least preferred alternatives. We repeatedly use a single procedure to the same set of voters but with fewer and fewer alternatives remaining until a single candidate is left.

STV proceeds through a series of $m - 1$ rounds. The candidate with the lowest plurality score is eliminated in each round and each of the votes for that candidate is transferred to the next remaining candidate in the order given in the ballot. The winner is the last remaining candidate.

Four voting rules in three-candidate elections

We wish to consider elections with three candidates, denoting A, B, and C. There are 6 possible preference rankings (indexed j) from which a voter can choose. Let a_j be the number of voters who have j preference ranking. Table 3.1 represents six possible rankings for the three-candidate elections.

a_1	a_2	a_3	a_4	a_5	a_6
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Table 3.1: Preference ranking in three-candidate elections

A *profile* is a 6-tuple $P = (a_1, a_2, a_3, a_4, a_5, a_6)$ of non-negative integers such that $\sum_{j=1}^{6} a_j = a$ provides information about the number of voters in each preference ranking. Table 3.2 summarizes the information needed for candidate A to be elected in each of three scoring rules.

Table 3.2: How a candidate wins in three-candidate scoring rules

Voting rule	A beats B	A beats C
Plurality	$a_1 + a_2 > a_3 + a_4$	$a_1 + a_2 > a_5 + a_6$
Anti-plurality	$a_4 + a_6 < a_2 + a_5$	$a_4 + a_6 < a_1 + a_3$
Borda count	$a_1 + a_2 + \frac{1}{2}(a_3 + a_5) > a_3 + a_4 + \frac{1}{2}(a_1 + a_6)$	$a_1 + a_2 + \frac{1}{2}(a_3 + a_5) > a_5 + a_6 + \frac{1}{2}(a_2 + a_4)$

For A to win under STV, we need A to go into the the second round (the runoff). Without loss of generality, we can assume that B beats C in the first round and loses to A in the runoff. This means that C has fewest first-place vote, i.e. $a_1 + a_2 > a_5 + a_6, a_3 + a_4 > a_5 + a_6$; and after the first-place votes for C are redistributed to A and B, the total number of votes for A is greater than that of B, i.e. $a_1 + a_2 + a_5 > a_3 + a_4 + a_6$.

3.2 Geometrization

3.2.1 Normalized profiles

We divide the number of votes in each preference ranking by the total number of agents a to obtain a normalized profile $p = (p_1, p_2, p_3, p_4, p_5, p_6)$ where $p_j = a_j/a$ represents the fractions of voters who prefer the j preference ranking. Table 3.3 updates the possible rankings under a normalized profile case for a three-candidate election.

p_1	p_2	p_3	p_4	p_5	p_6
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Table 3.3: Possible rankings under normalized profiles

Because $p_j \geq 0$ and $\sum_1^6 p_j = 1$, the normalized profiles are in a 5 regular simplex in $\mathbb{R}^6$. The entire region Δ_5 in $\mathbb{R}^6$ is defined as:

$$\Delta_5 = \{p : p_j \geq 0, \sum_{j=1}^{6} p_j = 1\}.$$

We are considering individual manipulations with large electorates; hence, the contribution of one voter is very tiny. For a voter to successfully change the election outcome only by changing her own preference, the sincere profiles and the strategic profiles must be close to the boundary. Without loss of generality, we need to consider the situation where A and B tie and a voter who most prefers C can manipulate the outcome. Under the AIC assumption[3], the probability that a voter who most prefers C can manipulate the election by changing her preference ranking is measured by the volume of the region defined by a set of equalities and inequalities. In the following sections, we sequentially define the regions Q's that represent individual manipulation of four voting rules: Plurality, Anti-plurality, Borda Count and STV. We also identify vector of changes $\vec{d}$ of possible manipulations for each voting rule. The degree of successful individual manipulation under each voting rule will be the multiplication of Q with $sin\omega$, where ω is the angle of region Q with the vector of changes $\vec{d}$.

[3]This assumption is often criticized for being unrealistic. For example, the voting situation of (0,0,1) should not be considered as likely as (0.4,0.5,0.1). However, we have provided valid arguments for continuing research with this assumption in Chapter 2.

3.2.2 Plurality

An individual can manipulate the election outcome under Plurality when we have two leading candidates tie. Individuals who most prefer the third candidate can switch their votes to either one of two leading candidates and thus, determine the winner. The individual manipulability is measured as the volume of region Q^{plu} such that:

$$Q^{plu} = \{p \in \Delta_5; p_1 + p_2 = p_3 + p_4 > p_5 + p_6\}.$$

As can be seen from Table 3.3, the first possible manipulation over Q^{plu} is for individuals who have sincere preferences $C \succ A \succ B$ to report $A \succ C \succ B$. This insincere ballot, causing an increase in p_2 simultaneously with a decrease in p_5, will lead to the election of A. We have the first vector of changes $\vec{d_{1plu}} = \langle 0, 1, 0, 0, -1, 0\rangle$.

The second possible manipulation over Q^{plu} is for individuals who have sincere preferences $C \succ B \succ A$ to report $B \succ C \succ A$. This insincere, causing an increase in p_4 simultaneously with a decrease in p_6, will lead to the election of B. We have the second vector of changes $\vec{d_{2plu}} = \langle 0, 0, 0, 1, 0, -1\rangle$.

3.2.3 Anti-plurality

In Anti-plurality voting, each voter votes against a single candidate and the candidate with the fewest last-place votes wins the election. Anti-plurality in three-candidate elections is the scoring voting rule with the score vector $\vec{\alpha} = \langle 1, 1, 0\rangle$. The total number of votes against A, B, C are $p_4 + p_6, p_2 + p_5, p_1 + p_3$ respectively. When A and B tie, i.e: $p_4 + p_6 = p_2 + p_5$; and C is the least preferred candidate, i.e: $p_1 + p_3 > p_2 + p_5$, a voter who ranks C last can manipulate the outcome by belying the preference as it is different from the sincere one. Thus, the situation when an individual can manipulate

Anti-plurality is defined as:

$$Q^{ant} = \{p \in \Delta_5; p_4 + p_6 = p_2 + p_5 < p_1 + p_3\}.$$

Specifically, a voter currently in p_1 can report $A \succ C \succ B$, resulting in A winning the election. This possible manipulation can be represented by vector of changes $\vec{d_{1ant}} = \langle -1, 1, 0, 0, 0, 0\rangle$. The second possible manipulation is for those who have the sincere preferences $B \succ A \succ C$ to report $B \succ C \succ A$, resulting in B winning the election. This manipulation is characterized by vector of changes $\vec{d_{2ant}} = \langle 0, 0, -1, 1, 0, 0\rangle$. It is worth noticing that individuals who constitute p_1 and p_3 already have their preferred candidates on the top of ranking preferences, they do not have incentive to change their preference rankings.

3.2.4 Borda Count

We consider Borda Count in three-candidate elections with the most commonly studied scoring vector $\vec{\alpha} = \langle 1, \frac{1}{2}, 0\rangle$. The total score for A, B and C are $p_1 + p_2 + \frac{1}{2}(p_3 + p_5)$, $p_3 + p_4 + \frac{1}{2}(p_1 + p_6)$, and $p_5 + p_6 + \frac{1}{2}(p_2 + p_4)$ respectively. The region of individual manipulation under Borda Count is defined by:

$$Q^{bor} = \{p \in \Delta_5; p_1+p_2+\frac{1}{2}(p_3+p_5) = p_3+p_4+\frac{1}{2}(p_1+p_6); p_1+p_2+\frac{1}{2}(p_3+p_5) > p_5+p_6+\frac{1}{2}(p_2+p_4)\}.$$

There are four possible manipulations under Borda Count. Firstly, a voter with ranking $C \succ A \succ B$ in p_5 can report $A \succ C \succ B$ or $A \succ B \succ C$. Because we are considering scoring vector $\alpha = \langle 1, \frac{1}{2}, 0\rangle$, changing to $A \succ B \succ C$ will add the same score to both A and B and thus will not change the outcome. Only the insincere preference $A \succ C \succ B$ (instead of the true $C \succ A \succ B$) induces a better outcome for the manipulator where A wins the election instead of B. This possible manipulation is characterized by the vector of changes $\vec{d_{1bor}} = \langle 0, 1, 0, 0, -1, 0\rangle$.

Secondly, a voter who is currently in group p_1 with sincere ranking $A \succ B \succ C$ can enhance the chance of winning for A by reporting $A \succ C \succ B$. The vector of changes $\vec{d_{2bor}} = \langle -1, 1, 0, 0, 0, 0 \rangle$ represents this manipulation opportunity.

Thirdly, an individual who has the true preference $C \succ B \succ A$ can report $B \succ C \succ A$, resulting in B winning the election. This manipulation is characterized by the vector of changes $\vec{d_{3bor}} = \langle 0, 0, 0, 1, 0, -1 \rangle$.

Finally, a voter with sincere preference $B \succ A \succ C$ can report $B \succ C \succ A$ on the ballot, resulting in the election of B. This situation is characterized by the vector of changes $\vec{d_{4bor}} = \langle 0, 0, -1, 1, 0, 0 \rangle$.

3.2.5 Single Transferable Vote

With STV, the way to manipulate is to change who gets into the runoff. Assume that C would lose a runoff to A, someone who does not like A can manipulate by placing B at the top of the ballot, even though C is their most preferred candidate, if doing so creates a runoff between A and B that B would win. Because we consider the situation where an individual can manipulate the outcome by letting B or C go into the runoff, this essentially means that B and C tie in the first round. The region of interest Q^{stv} is defined by:

$$Q^{stv} = \{p \in \Delta_5; p_1 + p_2 + p_3 > p_4 + p_5 + p_6; p_1 + p_2 + p_5 < p_3 + p_4 + p_6; p_3 + p_4 = p_5 + p_6\}.$$

There is only one possible manipulation under STV. We know that if C goes into the runoff, then A would win the election. Therefore, individual who likes B more than A, having preference $C \succ B \succ A$, has incentive to report $B \succ C \succ A$ to help B go into the runoff. This changes is characterized by the vector $\vec{d_{stv}} = \langle 0, 0, 0, 1, 0, -1 \rangle$.

3.3 Volume of Convex Region

The volume of a convex region is analytically measured using the decomposition method and double checked by simulated data. To perform the decomposition method, we first determine the coordinates of vertices that define each convex region Q. After that, we discuss how to decompose Q to reduce its dimension so that we can use the available techniques for computing its volume. The concept of "Singular Value Decomposition" can be applied to double check the 3D volume computation. As far as we know, the interpretation of "Singular Value Decomposition" as the volume in high dimensional space is new. It has an advantage that allows us to compute subvolume even in the case the original region collapses into smaller dimensional space. Appendix A provides an introduction into Singular Value Decomposition. This section concludes by introducing the method of computing the distance from a point to a subspace that is structured by some hyperplanes, which is an integral part of computing the convex volumes.

3.3.1 Decomposition

One method to measure the volume of a convex region is to break it into smaller pieces, whose volumes can be computed more readily, and then add up the results. In our case, the constraint sets are all convex, and it is well known that the intersection of convex sets is convex, thus the decomposition method is applicable. For each region, we start by selecting one of its vertices to act as the apex. The volume of region Q can be decomposed into a collection of pyramids sharing the same apex with different faces of the region as their bases. The volume of the apex with the base that contains the apex will be zero, so we only need to consider the faces that do not contain the selected vertex. When choosing a vertex to act as the apex, we pick the one that belongs to as many faces as possible. The fewer faces that contain the apex, the less volume computation involved. Figure 3.1 [4] illustrates this decomposition method in 3D.

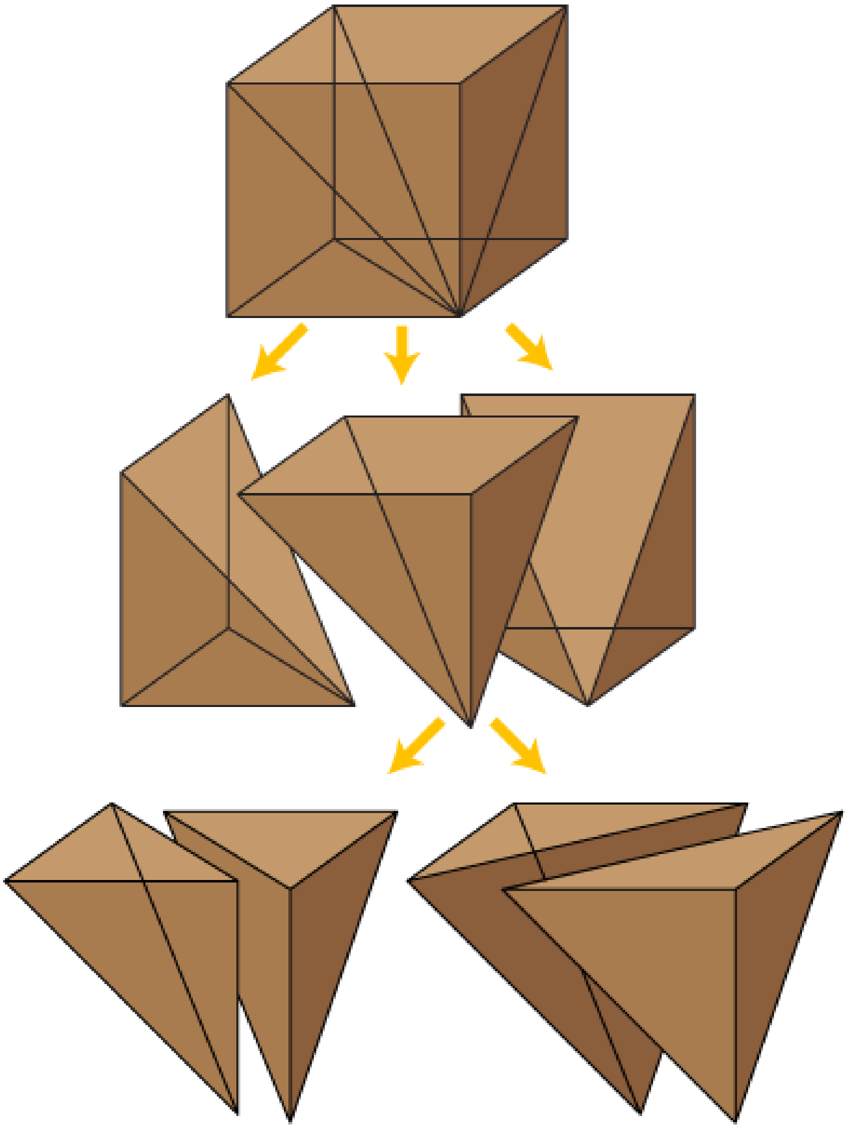


Figure 3.1: Partitioning the cube into three square pyramids and then, each square pyramid is partitioned into triangular pyramid in one of two ways.

3.3.2 Finding coordinates of vertices

We need to define the region of interest by identifying its vertices. The vertices in 6-dimensional spaces have to be the intersection of at least 6 hyperplanes which corre-

[4] Source: `http://mathoverflow.net/questions/190686/what-is-the-number-of-equitriangulations-of-the-n-cube`, retrieved 31 August 2016.

sponding to 6 equalities. In our studies, the regions of interests are always defined by a set of 2 equalities. At a vertex, the defining inequalities that are satisfied with equality determine the vertex. Therefore, we look at all subsets of the set of defining inequalities that might determine a vertex. If a subset is linearly independent and the point it determines satisfies all the other inequalities, then it constitutes a vertex of region Q.

Take for example the region Q^{plu}. Out of 7 inequalities ($p_i \geq 0$ and $p_1+p_2 > p_5+p_6$), we select a combination of four inequalities. There are $\binom{7}{4} = \frac{7!}{4!(7-4)!} = 35$ such combinations but only those that are linearly independent with the existing 2 equalities can create vertices. The solutions of linear system consisting of these four additional equalities, together with: $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$ and $p_1 + p_2 = p_3 + p_4$ are the coordinates of the vertices of 4-dimensional region Q^{plu}.

3.3.3 Projection

The volume of a pyramid in dimension n can be measured by $\frac{1}{n}Vh$ where V is the $(n-1)$ dimensional volume of the base, and h is the height from the apex to the base. Because our regions of interest are defined by 2 equalities in a 6-dimensional space, we only need to restrict our attention to volumes in 4D with 3D bases. The bases are faces of a convex object so are themselves convex, which allows us to compute their volumes by breaking them into 3D pyramids and adding up the results. This section explains the general approach to computing 3D volume bases that applies to vertices with any numbers of coordinates.

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors created from four vertices in a 3-dimensional space. We can think of this as a 3D pyramid over a 2D face and hence, we can apply exactly the same method detailed above. To compute the area of the triangle formed by vector $\vec{a}$ and $\vec{b}$, let $\vec{e_1}$ be the unit vector in the direction of $\vec{a}$, and $\vec{e_2}$ be the unit vector in the direction of $\vec{b} - (\vec{b}.\vec{e_1})\vec{e_1}$. Then, $\vec{e_1}$ and $\vec{e_2}$ are orthogonal unit vectors that span the plane containing vector $\vec{a}$ and $\vec{b}$. We can compute the area of the triangle by using $\vec{a}.\vec{e_1}$ as the base and $\vec{b}.\vec{e_2}$ as the height. The area of the triangle will be $\frac{1}{2}(\vec{a}.\vec{e_1})(\vec{b}.\vec{e_2})$.

To measure the volume of a pyramid over this triangle and vector $\vec{c}$, let $\vec{e_3}$ be the unit vector in the direction of $\vec{c} - (\vec{c}.\vec{e_1})\vec{e_1} - (\vec{c}.\vec{e_2})\vec{e_2}$. This determines an orthonormal basis for the three-space containing the pyramid. The process of finding orthonormal basis is often refered to as "Gram-Schmidt process". The height of the apex is $\vec{c}.\vec{e_3}$. Then, we can compute the volume of the pyramid by using the area of the triangle as the base and $\vec{c}.\vec{e_3}$ as the height. The volume of the pyramid will be $\frac{1}{3}\frac{1}{2}(\vec{a}.\vec{e_1})(\vec{b}.\vec{e_2})(\vec{c}.\vec{e_3}) = \frac{1}{6}(\vec{a}.\vec{e_1})(\vec{b}.\vec{e_2})(\vec{c}.\vec{e_3})$.

3.3.4 Distance from a point to a facet

Apart from computing the bases, we also need to determine the distance from the apex to the bases. If the base is defined by just one hyperplane, we simply apply the standard distance formula from one point to a hyperplane given in Appendix B. However, in our analysis, these bases are not simple hyperplanes with one parametric equation. They are the intersection of many hyperplanes. In 3D, we can imagine this as an analogy of the distance from a point to a line, which is illustrated in Figure 3.2.

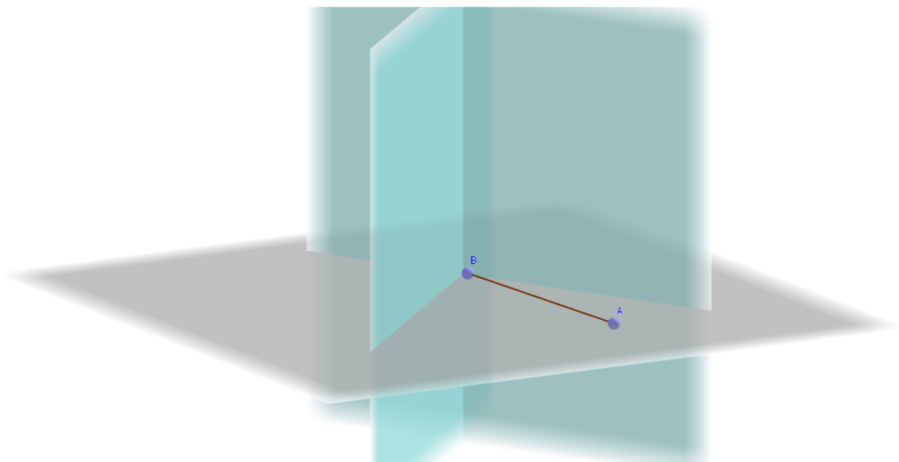


Figure 3.2: Distance from point A to the line that is the intersection of two planes, we find the parametric equation of a third plane that passes through A and takes the position vector of the line as its normal vector. Point B is the intersection of 3 planes. The distance from A to the line is the length AB.

We apply similar approach when our base is the intersection of k hyperplanes. We start by forming a matrix that contains the normal vectors of each hyperplane corresponding to the equalities. Subsequently, we find the Null Space of that matrix [5]. Each vector in this Null Space will be a normal vector of one additional hyperplane that determines the projection B of the apex A. The parametric equation of remaining hyperplanes that determine the point B is ready to be constructed, using the normal vector and the coordinates of a point it passes through (the apex A). These new hyperplanes, together with the original k hyperplanes will form a system of 6 equalities and the solutions of this system are the coordinates of B. The distance from A to k hyperplanes is equal to the length of AB.

3.3.5 Angle of possible manipulation with the region Q

Let ω be the angle of Q with possible manipulation, which is represented by a vector of changes, $\vec{d}$. Successful individual manipulation will be determined by $Qsin\omega$ as represented in Figure 3.3. Denote φ as the angle between the vector of changes $\vec{d}$ with the normal vector $\vec{n}$ of the hyperplane that determines the possible manipulation. For example in the case of Plurality, the hyperplane is $p_1 + p_2 = p_3 + p_4$ and thus, the normal vector $\vec{n_{plu}} = \langle 1, 1, -1, -1, 0, 0 \rangle$. From Figure 3.3, we can see that $sin\omega = |cos\varphi|$. Therefore, it is possible to directly compute $Q|cos\varphi|$ as the degree of successful manipulation where:

$$cos\varphi = \frac{\vec{d}.\vec{n}}{\|\vec{d}\|\|\vec{n}\|}.$$

[5]Null Space of matrix C is $N(C) = \{\vec{x} \in R^n | C\vec{x} = \vec{0}\}$

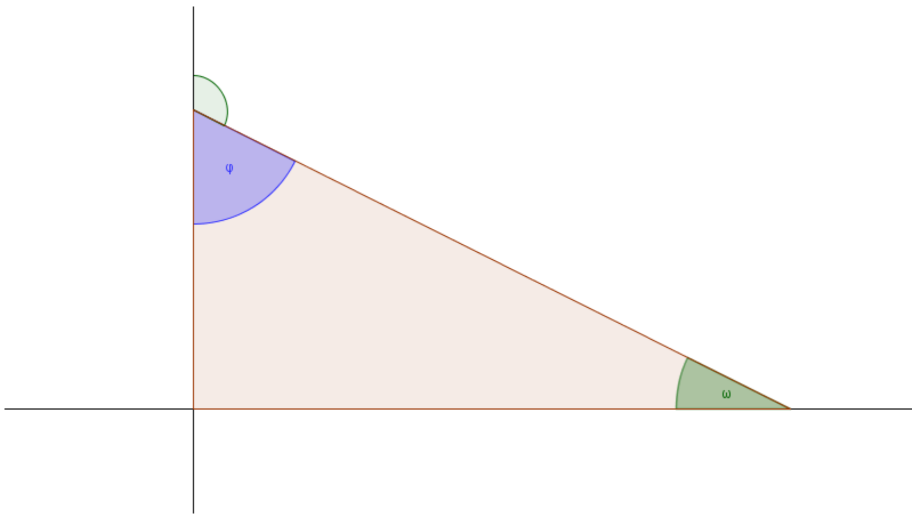


Figure 3.3: $Qsin\omega$ or $Q|cos\varphi|$ represents the degree of successful manipulation.

Chapter 4

Results and Discussion

In this chapter, the main results of four voting rules are presented, followed by a consistency check using simulated data. The chapter closes by discussing the implication of the results with respect to the existing literature.

4.1 Results

We first provide the general picture of the region Q for each voting rule, including its vertices and the shapes of its faces[1]. To compute the volume, we specify a decomposition structure for each region. The volume results are presented with more details under Plurality just to give a clearer view of our approach. The same procedure applies to other voting rules.

4.1.1 Plurality

Vertices

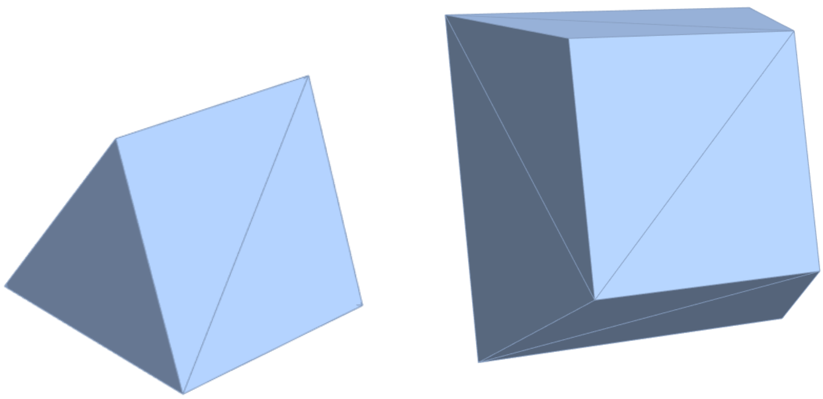
The 12 vertices that define the convex hull Q^{plu} in 4-dimensional space are:

[1]Obtaining an illustration of the whole region Q would be challenging because human being is still having difficulties visualizing objects in more than three dimensions. Although we have successfully formed the Plurality region, we do not yet know how to project 4D objects into 3D in general for other regions because of their complexity.

$$v_{13} = (\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0) \qquad v_{16}^3 = (\frac{1}{3}, 0, \frac{1}{3}, 0, 0, \frac{1}{3})$$
$$v_{14} = (\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0) \qquad v_{16}^4 = (\frac{1}{3}, 0, 0, \frac{1}{3}, 0, \frac{1}{3})$$
$$v_{23} = (0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0) \qquad v_{25}^3 = (0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0)$$
$$v_{24} = (0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0) \qquad v_{25}^4 = (0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0)$$
$$v_{15}^3 = (\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0) \qquad v_{26}^3 = (0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3})$$
$$v_{15}^4 = (\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0) \qquad v_{26}^4 = (0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}).$$

Faces

The region Q^{plu} has seven faces of Q^{plu} which are $P_1, P_2, P_3, P_4, P_5, P_6$ and H_2^{plu} (the bounding hyperplane corresponding to $p_1 + p_2 > p_5 + p_6$). As can be seen from Figure 4.1, faces P_1, P_2, P_3, and P_4 have prismatic forms, each containing six vertices while faces P_5 and P_6 are similar, each containing eight vertices. The face H_2^{plu} is a hypercube as illustrated in Figure 4.2.



(a) P_1, P_2, P_3, P_4 (b) P_5, P_6

Figure 4.1: Q^{plu} faces: $P_1, P_2, P_3, P_4, P_5, P_6$

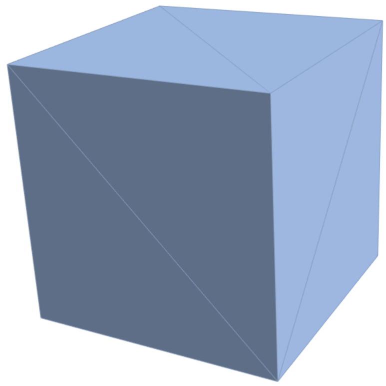


Figure 4.2: Q^{plu} faces: H_2^{plu}

Decomposition

Let H be the hyperplane corresponding to $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$; H_1^{plu} be the hyperplane corresponding to $p_1 + p_2 = p_3 + p_4$; and H_2^{plu} be the bounding hyperplane corresponding to $p_1 + p_2 > p_5 + p_6$. The vertices v_{24} is chosen the be the apex and the region Q^{plu} is then decomposed into a collection of pyramids having v_{24} as the apex and the faces that do not contain v_{24} as bases. There are three bases: P_4, P_2 and H_2^{plu}. Figure 4.3 shows the decomposition structure of Q^{plu}.

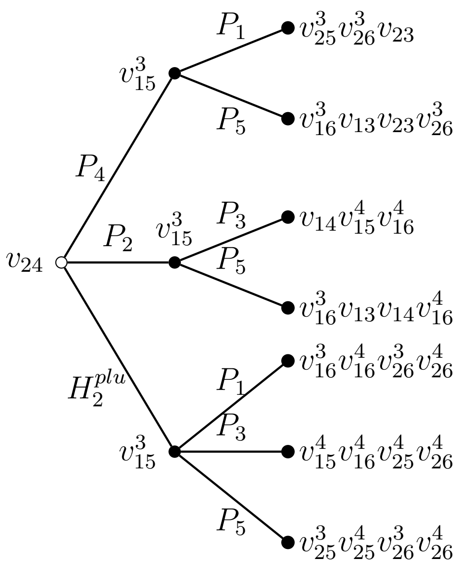


Figure 4.3: The left-most vertex in the diagram is the apex of 3 pyramids that form the region Q^{plu}. The 3-dimensional bases of these pyramids are divided into a number of pyramids with apexes given by the second column in the diagram. Vertices in each bases are listed on the right.

Volume

As can be seen from the decomposition structure Figure 4.3, we need to compute volumes of three pieces from the apex v_{24} to three bases: P_4, P_2 and H_2^{plu}.

Volume of the Base in P_4

There are 6 vertices in this base which are $v_{13}, v_{23}, v_{15}^3, v_{16}^3, v_{25}^3$ and v_{26}^3. We select the vertex v_{15}^3 for further decomposition. There are two faces that do not contain v_{15}^3 vertex which are P_1 and P_5. Of 6 vertices above, there are 3 belong to P_1 and 4 belong to P_5. The volume of the region formed by v_{15}^3 and P_1 is $\frac{\sqrt{14}}{324}$. The volume of the region formed by v_{15}^3 and v_{16}^3, v_{13}, v_{23} in P_5 is $\frac{\sqrt{14}}{216}$. The volume of the region formed by v_{15}^3 and $v_{16}^3, v_{26}^3, v_{23}$ in P_5 is $\frac{\sqrt{14}}{324}$. Thus, the volume of the base P_4 is equal to $\frac{7\sqrt{14}}{648}$.

Volume of the Base in P_2

There are 6 vertices in this base which are again decomposed using v_{15}^3 as the apex. The volume of the region formed by apex v_{15}^3 with P_3 equal to $\frac{\sqrt{14}}{324}$. The volume of the region formed by v_{15}^3 with P_5 equal to $\frac{\sqrt{14}}{216} + \frac{\sqrt{14}}{324}$. Thus, the volume of the base in P_2 is also $\frac{7\sqrt{14}}{648}$.

Volume of the Base in H_2^{plu}

There are 8 vertices of region Q^{plu} that satisfy the equation of hyperplane H_2^{plu}. To measure the volume of the convex hull in the base H_2^{plu}, we again apply the decomposition method, choosing v_{15}^3 as the apex. There are 3 faces that do not contain this v_{15}^3 apex, which are P_1, P_3 and P_5. There are four vertices in each of this hyperplane and they are linearly dependent. Thus, we can break it into two 3D pyramids and add up the results. The volume of the base in H_2^{plu} equals to $\frac{2\sqrt{2}}{27}$.

Distance from apex v_{24} to each base

We illustrate here the process of measuring distance from apex v_{24} to face P_2. Let matrix M consists of all the coefficients of three equalities that satisfy face P_2:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Null Space of M are found to contain three vectors:

$$\begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 2 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

By using three vectors of the Null Space as the normal vectors, we establish three additional hyperplanes that pass through the vertex v_{24}:

$$-p_1 - p_3 + 2p_6 = 0$$
$$-p_1 - p_3 + 2p_5 = 0$$
$$-p_3 + p_4 - \frac{1}{2} = 0.$$

These three new hyperplanes, together with the three old ones that define the face P_2 form the linear system of 6 equalities. Solving this linear system, we find the point $v^* = (\frac{5}{14}, 0, -\frac{1}{14}, \frac{3}{7}, \frac{1}{7}, \frac{1}{7})$. The distance from the apex v_{24} to the facet P_2 is the length $v_{24}v^*$, which equals to $\sqrt{\frac{3}{7}}$.

Using similar approach, the distance from apex v_{24} to face P_4 is found to be $\sqrt{\frac{3}{7}}$ while the distance to face H_2^{plu} is $\frac{1}{2\sqrt{3}}$.

In summary, the volume of the region that represent individual manipulation under Plurality is the sum of three pieces. Two pieces from apex v_{24} to face P_2 and P_4, each has the volume of $\frac{7}{432\sqrt{6}}$. The piece from apex v_{24} to face H_2^{plu} has the volume of $\frac{1}{54\sqrt{6}}$. Thus, the volume of interest is $\frac{11}{216\sqrt{6}}$.

Angle of Q^{plu} with possible manipulation and the revised volume

We find $cos\varphi = \frac{1}{2\sqrt{2}}$ for the first possible manipulation with $\vec{d_{1plu}} = \langle 0, 1, 0, 0, -1, 0\rangle$, and $cos\varphi = -\frac{1}{2\sqrt{2}}$ for the second possible manipulation with $\vec{d_{2plu}} = \langle 0, 0, 0, 1, 0, -1\rangle$. Therefore, the revised volume taking into account the possible manipulation and its angle under Plurality will be $\frac{11}{216\sqrt{6}} \times (\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}) = \frac{11}{432\sqrt{3}}$.

4.1.2 Anti-plurality

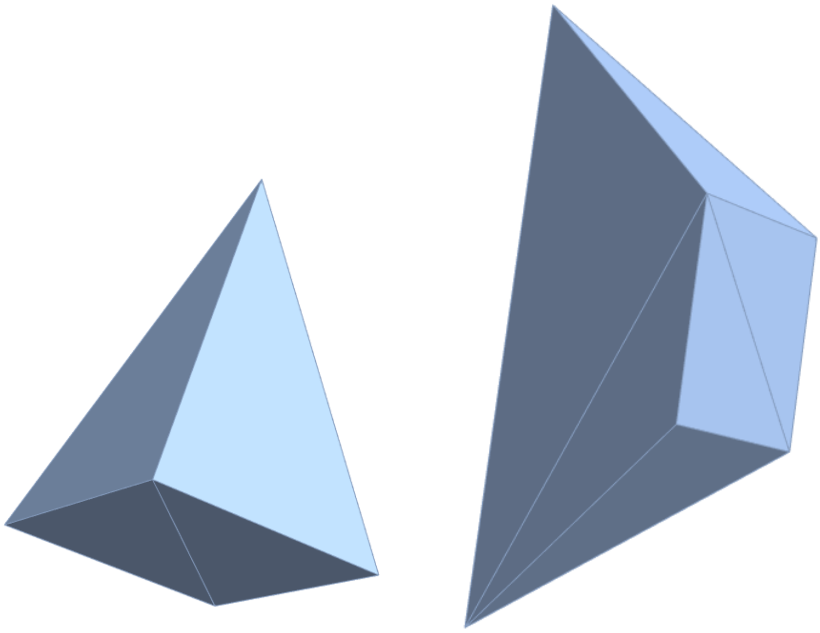
Vertices

We find 10 vertices that define the convex hull Q^{ant} in 4-dimensional space:

$$u_1 = (1, 0, 0, 0, 0, 0) \qquad v_{34}^2 = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$$
$$u_3 = (0, 0, 1, 0, 0, 0) \qquad v_{56}^1 = (\frac{1}{3}, 0, 0, 0, \frac{1}{3}, \frac{1}{3})$$
$$v_{56}^3 = (0, 0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}) \qquad v_{45}^1 = (\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0)$$
$$v_{45}^3 = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) \qquad v_{26}^1 = (\frac{1}{3}, \frac{1}{3}, 0, 0, 0, \frac{1}{3})$$
$$v_{36}^2 = (0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}) \qquad v_{24}^1 = (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, 0).$$

Faces

An illustration of Anti-plurality faces can be seen in Figure 4.4. P_1 and P_3 have similar shapes, each containing 5 vertices. P_2, P_4, P_5 and P_6 have similar shapes, each containing 6 vertices. H_2^{ant} is a hypercube looks like H_2^{plu} in Figure 4.2.



(a) P_1, P_3 (b) P_2, P_4, P_5, P_6

Figure 4.4: Q^{ant} faces

Decomposition

We choose u_1 as the apex. Two faces P_1 and $H_2^{ant} : p_2 + p_5 = p_1 + p_3$ do not contain u_1, thus, are considered as the bases. The decomposition is presented in Figure 4.5.

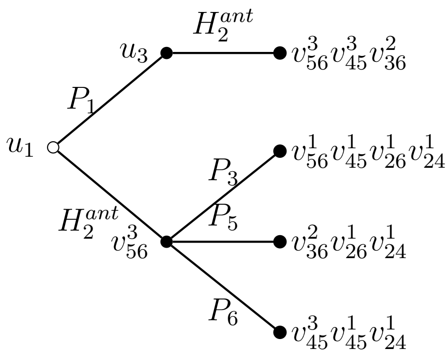


Figure 4.5: Q^{ant} decomposition structure

There are 5 vertices in P_1 which are $u_3, v_{56}^3, v_{45}^3, v_{36}^3$ and v_{34}^2. Within P_1, we again choose u_3 as the apex for further decomposition. There are only two faces that do not contain u_3 which are P_3 and H_2^{ant}. However, of the 5 vertices above, none belong to P_3 while 4 belongs to H_2^{ant}, which are $v_{56}^3, v_{45}^3, v_{36}^2$ and v_{34}^2.

In H_2^{ant}, there are 8 vertices that form a hypercube. v_{56}^3 is chosen as the apex for further decomposition with bases P_3, P_5 and P_6. As shown in Figure 4.5, base P_3 contains 4 vertices which are $v_{56}^1, v_{45}^1, v_{26}^1$ and v_{24}^1 while P_5 and P_6 each contains three vertices.

Volume

The volume of the base P_1 is $\frac{2\sqrt{5}}{81}$ with the distance from u_1 to P_1 equals to $\sqrt{\frac{6}{5}}$. The volume of base H_2^{ant} is $\frac{2\sqrt{2}}{27}$ with the distance from u_1 to H_2^{ant} equals to $\frac{1}{\sqrt{3}}$. Thus, the volume of the area interest under Anti-Plurality is $\frac{\sqrt{6}}{81}$.

Angle of Q_{ant} with possible manipulation and the revised volume

We find $cos\varphi = \frac{1}{2\sqrt{2}}$ where φ is the angle between the normal vector $\vec{n_{ant}} = \langle 0, 1, 0, -1, 1, -1\rangle$ and the first possible manipulation $\vec{d_{1ant}} = \langle -1, 1, 0, 0, 0, 0\rangle$. $cos\varphi = -\frac{1}{2\sqrt{2}}$ for the second possible manipulation with $\vec{d_{2ant}} = \langle 0, 0, -1, 1, 0, 0\rangle$. Therefore, the revised volume taking into account the possible manipulation and its angle under Anti-plurality is $\frac{\sqrt{6}}{81} \times (\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}) = \frac{\sqrt{3}}{81}$.

4.1.3 Borda Count

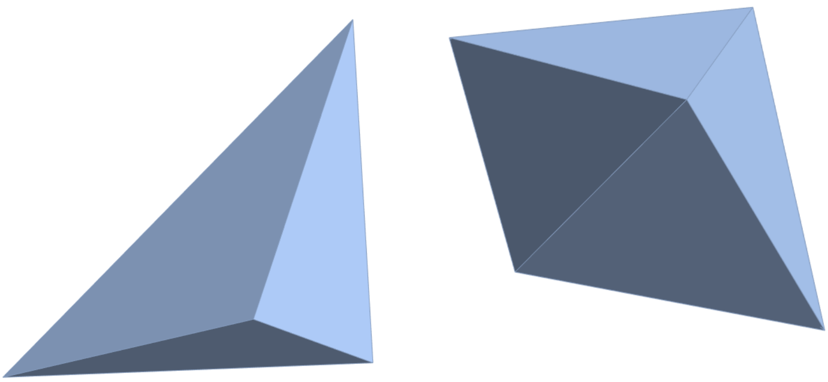
Vertices

We find 8 vertices that define the region of interest under Borda Count rule:

$$
\begin{aligned}
v_{35} &= (0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0) & v_{45}^1 &= (\frac{1}{3}, 0, 0, \frac{1}{3}, \frac{1}{3}, 0) \\
v_{24} &= (0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0) & v_3^2 &= (0, \frac{1}{3}, \frac{2}{3}, 0, 0, 0) \\
v_{13} &= (\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0) & v_4^1 &= (\frac{2}{3}, 0, 0, \frac{1}{3}, 0, 0) \\
v_{16} &= (\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}) & v_{36}^2 &= (0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}).
\end{aligned}
$$

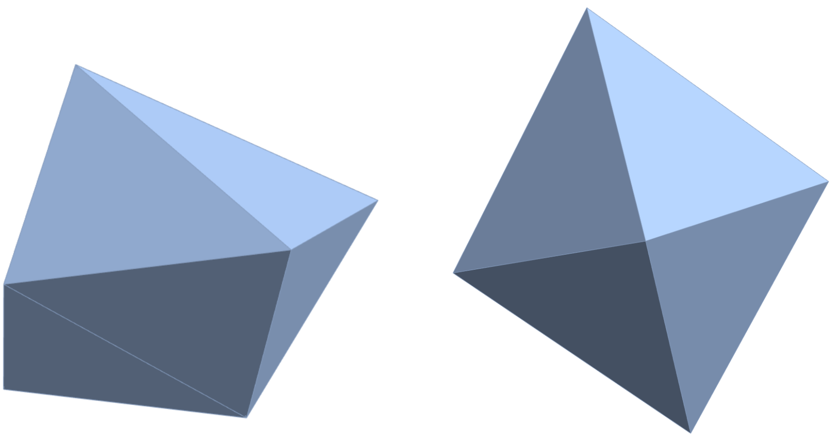
Faces

The region Q^{bor} also has 7 inequalities which define 7 faces. Figure 4.6 and Figure 4.7 illustrate how these faces look like. P_1 and P_3 have similar shapes, each containing 4 vertices. P_2 and P_4 have similar shapes, each containing 5 vertices. P_5 and P_6 have similar shapes, each containing 6 vertices. H_2^{bor} contains 5 vertices but its shape is different from P_2 and P_4.



(a) P_1, P_3 (b) P_2, P_4

Figure 4.6: Q^{bor} faces: P_1, P_2, P_3, P_4



(a) P_5, P_6 (b) H_2^{bor}

Figure 4.7: Q^{bor} faces P_5, P_6, H_2^{bor}

Decomposition

As being shown in Figure 4.8, v_{35} is chosen as the apex with bases P_3 and P_5. There are four vertices in P_3 and v_{24} is selected as the apex to compute the volume. There are 6 vertices in the base P_5 and again v_{24} is chosen as the apex. The volume in P_5 can be decomposed into volume from apex v_{24} to face P_2 and face P_4.

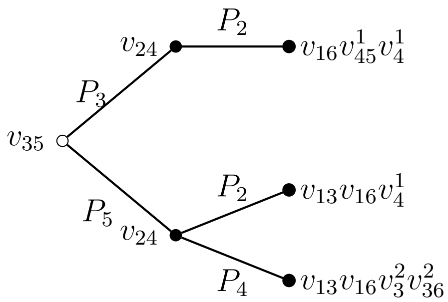


Figure 4.8: Q^{bor} decomposition structure

Volume

The base in P_3 formed by for vertices $v_{24}, v_{16}, v_{45}^1, v_4^1$ has the volume of $\frac{1}{12\sqrt{6}}$. The distance from v_{35} to P_3 is $\frac{1}{\sqrt{3}}$.

In P_5, the volume of the region formed by v_{24} and P_2, which includes v_{13}, v_{16}, v_4^1 is $\frac{1}{8\sqrt{6}}$. Because $v_{36}^2 - v_{13} = \frac{3}{2}(v_{16} - v_{13}) + (v_3^2 - v_{13})$, we can break the region formed by the apex v_{24} with 4 vertices $v_{13}, v_{16}, v_3^2, v_{36}^2$ into two pieces. The one formed by v_{24} and v_{13}, v_{16}, v_{36}^2 has the volume of $\frac{1}{8\sqrt{6}}$. The other formed by v_{24} and v_{13}, v_3^2, v_{36}^2 has the volume of $\frac{1}{12\sqrt{6}}$. Thus, the volume of base P_5 is $\frac{1}{3\sqrt{6}}$. The distance from v_{35} to P_5 equals to $\frac{1}{\sqrt{3}}$.

In conclusion, the volume of Q^{bor} is the sum of the volume created by v_{35} and P_3, and the volume formed by v_{35} and P_5, which equals to Q^{bor}, or $\frac{5}{144\sqrt{2}}$.

Angle of Q_{bor} with possible manipulation and the revised volume

With four possible vector of changes under Borda Count and the normal vector $\vec{n_{bor}} = \langle \frac{1}{2}, 1, -\frac{1}{2}, -1, \frac{1}{2}, -\frac{1}{2} \rangle$, we find that $cos\varphi = \frac{1}{2\sqrt{6}}$ for the first two vector of changes and

$cos\varphi = -\frac{1}{2\sqrt{6}}$ for vector of changes $\vec{d_{3bor}}$ and $\vec{d_{4bor}}$. Therefore, the revised volume under Borda Count is $\frac{5}{144\sqrt{2}} \times 4 \times \frac{1}{2\sqrt{6}} = \frac{5}{144\sqrt{3}}$.

4.1.4 Single Transferable Vote

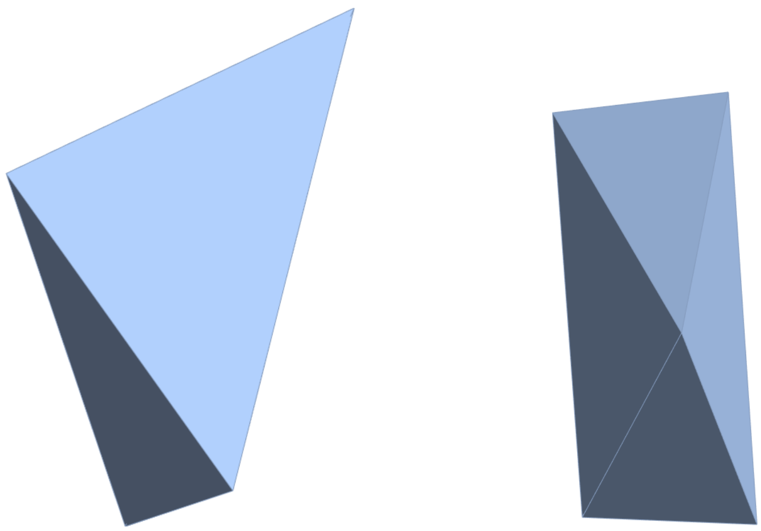
Vertices

The 6 vertices defining the region of interest under STV are listed below:

$$\begin{aligned}
v_{35} &= (0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0) & v_{36}^2 &= (0, \frac{1}{2}, \frac{1}{4}, 0, 0, \frac{1}{4}) \\
v_{36} &= (0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}) & v_{46}^1 &= (\frac{1}{2}, 0, 0, \frac{1}{4}, 0, \frac{1}{4}) \\
v_{46}^2 &= (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{4}) & v_{36}^1 &= (\frac{1}{2}, 0, \frac{1}{4}, 0, 0, \frac{1}{4}).
\end{aligned}$$

Faces

The region Q^{stv} has eight faces. As illustrated in Figure 4.9, P_1, P_2, P_4, H_2^{stv} each contains four vertices while P_5, H_3^{stv} each contains five vertices. P_3 only has two points, which makes it a line segment; and P_6 only has one point.



(a) P_1, P_2, P_4, H_2^{stv} (b) P_5, H_3^{stv}

Figure 4.9: Q^{stv} faces

Decomposition

We choose v_{35} as the apex for decomposition under STV as presented in 4.10. We then need to compute the volume in faces that do not contain v_{35}, which are P_3 and P_5. v_{36} is chosen as the apex for further decomposition in P_5.

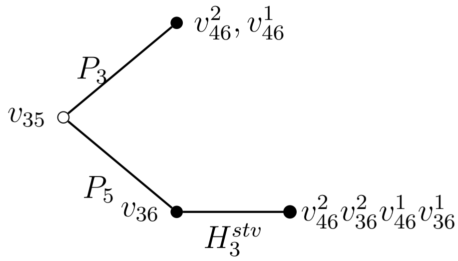


Figure 4.10: Q^{stv} decomposition structure

Volume

There are only two vertices in P_3, which are v_{46}^2 and v_{46}^1. These two vertices do not create a 3D volume. There are five vertices in P_5 that form a region with volume of $\frac{\sqrt{7}}{48\sqrt{2}}$. The distance from v_{35} to P_5 is $\sqrt{\frac{3}{7}}$. Thus, the volume of Q^{stv} is $\frac{1}{64\sqrt{6}}$.

Angle of Q_{stv} with possible manipulation and the revised volume

With the vector of changes $\vec{d_{stv}} = \langle 0, 0, 0, 1, 0, -1 \rangle$ and normal vector $\vec{n_{stv}} = \langle 0, 0, 1, 1, -1, -1 \rangle$, we found $cos\varphi = \frac{1}{\sqrt{2}}$. Therefore, the revised volume under STV equals to $\frac{1}{64\sqrt{6}} \times \frac{1}{\sqrt{2}} = \frac{1}{128\sqrt{3}}$.

4.2 Consistency check

In order to double check our complex analytical process, we generate a large number of points that satisfy all the equalities of each voting rule contained in a hypercube. We then filter these points into a subset that satisfies the inequalities in addition to the equalities. The ratio of the number of points in the subset to the number of points in the original set is the ratio of the volume formed by all constraints to the volume formed by equality constraints due to uniform randomization. We are certain that the dimensions of the two volumes are the same because applying inequalities on linear sets does not reduce their dimensions.

The volume of a hypercube in n dimensions, or an n cube is straightforward to compute[2]. With the edge length of r, the volume of an n cube is r^n. The intuition behind this formula is that to construct an n cube, we start with an $n-1$ cube and sweep it through space perpendicular to the hyperplane in which it lies. Figure 4.11 illustrates this process for the first three dimensions.

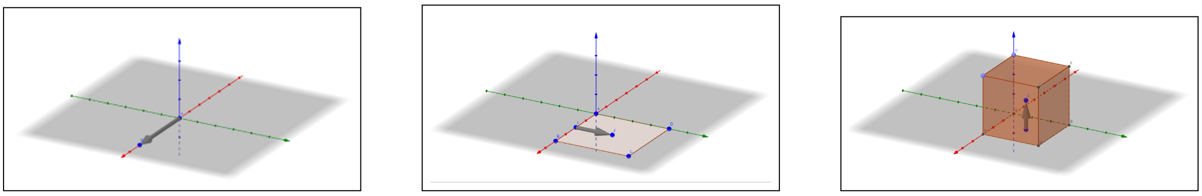


Figure 4.11: Constructing a 3-cube: A point at the origin sweeps out a line segment. The line segment sweeps out a square by moving perpendicularly to its length. The square sweeps out a cube by moving perpendicularly to its surface.

Without loss of generality, X is denoted as the matrix that contains all coefficients of the equalities in each voting rule. With all 4 voting rules in this paper, we have 2 equalities. We then find the Null Space of matrix X and its orthonormal basis. In our case, the orthonormal basis contains 4 vectors, each with 6 coordinates. The transpose of the orthonormal basis of X will give us 6 vectors, each with 4 coordinates, from which we can construct the 4-cube. To have the most efficient cube, we need to select the center of the cube that is the most symmetric. In our case, we choose the point $T = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. We also carefully select the edge length of the cube that is as small as possible but still sufficient for the 4-cube to cover all necessary points. The edge length is $2R$ with R chosen to be 0.8.

In the simulation, we first uniformly generate $m \times i$ points in a 4-cube, using i iter-

[2]Source:`http://www.physicsinsights.org/hypercubes_1.html`, retrieved 16 August 2016

ations, each of which generates m points. In this paper, we only run simulations with $m = 100,000$ and $i = 400$ but as is well-known, the greater these values, the more accurate the results. The number of points that satisfy the equalities is denoted as t. We then use Excel to filter the points that satisfy the inequalities in addition to equalities (denoted as n_a)[3]. The volume of the region that satisfies the set of equalities is:

$$V_e = \frac{(2R)^4 \times t}{m \times i}.$$

The ratio of the volume of the region that satisfies all inequalities and equalities (denoted as V_i), to the region that satisfies only the equalities (V_e) is:

$$\frac{V_i}{V_e} = \frac{n_a}{t}.$$

Thus, the volume of our area of interest is:

$$V_i = \frac{n_a}{t} . \frac{(2R)^4 \times t}{m \times i} = \frac{(2R)^4 \times n_a}{m \times i}.$$

Appendix C provides a typical code for the approximation of V_i using Mathematica.

4.3 Discussion

Table 4.1 shows that the decomposition method and the method of using simulated data deliver similar results with tiny errors so we are confident that the volumes have been precisely measured.

[3] Ideally, the design of an algorithm that can incorporate such inequalities to filter the points within Mathematica would significantly reduce the complexity of the process and the computational time.

Table 4.1: Summary of original volumes and consistency check

Voting Rule	Analytical	Simulated	Error
Plurality	$\frac{11}{216\sqrt{6}}$	$\frac{(2\times 0.8)^4\times 126950}{100000\times 400} \approx 0.020799488$	0.0000090
Anti-plurality	$\frac{\sqrt{6}}{81}$	$\frac{(2\times 0.8)^4\times 184614}{100000\times 400} \approx 0.03024715776$	0.0000065
Borda count	$\frac{5}{144\sqrt{2}}$	$\frac{(2\times 0.8)^4\times 149375}{100000\times 400} = 0.0244736$	-0.0000787
STV	$\frac{1}{64\sqrt{6}}$	$\frac{(2\times 0.8)^4\times 38864}{100000\times 400} \approx 0.00636747776$	-0.0000114

The results for four voting rules, taking into account possible manipulations and its angle with Q, are summarized in Table 4.2.

Table 4.2: Revised volumes

Voting Rule	Revised Volumes
Plurality	$\frac{11}{432\sqrt{3}}$
Anti-plurality	$\frac{\sqrt{3}}{81}$
Borda count	$\frac{5}{144\sqrt{3}}$
STV	$\frac{1}{128\sqrt{3}}$

As can be seen from Table 4.1 and Table 4.2, STV is far less susceptible to individual manipulation compared to three scoring rules and this is consistent with the existing literature. The resistance of STV to individual manipulation is attributed to the small defining region of possible manipulation and the existence of only one situation in which individual manipulation is possible under STV. As the ratios between the probabilities of individual manipulability under each scoring rule and under STV are computed, we can see that Plurality is approximately 3.3 times more vulnerable to individual manipulation than STV. Borda Count is even more vulnerable to individual manipulation, at about 4.4 times easier to be manipulated individually than STV. However, the most vulnerable rule to individual manipulation is Anti-plurality whose manipulability is about 4.7 times higher than that of STV.

It is worth noticing that although there are more situations where an individual can manipulate the election under Borda Count, the angles of such situations with the defining equality under Borda Count are small, which means that the manipulations have a low chance of success. Thus, the result suggesting that Borda Count is more susceptible to individual manipulation than Anti-plurality is not unexpected. Furthermore, our result is consistent with Smith (1999) who argues that Plurality is less manipulable than Borda Count.

Conitzer *et al.* (2007) suggest two methods to build the complexity in manipulation to protect the voting system. The first approach is to consider voters with different voting weights. Although unweighted voters are common, the introduction of weighted voters can fit the case of heterogenous agents better. For example, the weight can be interpreted as the size of the community that the voter represents (like a state), the size of the group (in parliament) or simply as an account of the case when a given agent has different decision-making power (shareholders in a company). The second approach entails giving up the complete information assumption by introducing probability distributions on the non-colluders' votes and examining if this changes the probability that a given voter could manipulate the election. It is found that incomplete information about how other people vote increases the difficulty of manipulation.

Chapter 5

Conclusion

5.1 Summary of findings

The ubiquity of voting systems has made research into voting theory increasingly important. This thesis is motivated by the lack of analytical results regarding individual manipulations of voting rules in the existing literature, and the recent unpredictable outcomes of the UK's Brexit and the US presidency, taking one step further to demystify a complex aspect of voting.

In this thesis, we have studied the individual manipulability of four voting rules: Plurality, Anti-plurality, Borda Count and STV in the context of three-candidate elections, with the IAC assumption when preference profiles are normalized. We have clearly described how an individual voter can manipulate the outcome under each rule and computed the volume of the regions representing such situations, taking the angle of the region with possible manipulation as the adjustment factor.

We find that STV is the least susceptible rule out of the four to individual manipulation and our result is consistent with other research using different approaches. Each of the three other scoring rules is more vulnerable to individual manipulation than is STV. Out of the four, Anti-plurality is the most susceptible rule to individual manipulation, with the likelihood of successful individual manipulation almost five times higher than that of STV.

Our geometric approach to computing volume has two main advantages. Firstly, we take into account the angle of the individual manipulability with possible manipulation, accurately reflecting the likelihood of successful individual manipulation under each voting scheme. Secondly, the geometric approach allows us to obtain the general results that hold true for any number of voters. This strengthens our conclusion that STV is the least susceptible rule to individual manipulation.

5.2 Suggestions for future research

There are two possible extensions of our research. Firstly, for other paradoxical voting situations, the IAC assumption tends to increase the opportunities of such paradoxes happening, which draws us to the conclusion that if paradoxes are rare under IAC, we can say they are not likely to be a real threat. However, the IAC assumption actually reduces the chance of successful individual manipulations. Instead of exaggerating the likelihood of manipulation, it understates the probability that such events might be observed. The reason is because manipulation tends to happen when we have closely tie situations. Meanwhile, IAC spreads the probability of each voting situation evenly, giving too much weight on profiles that are actually very unlikely to occur, resulting in zero probability limit of observing individual manipulation. Therefore, it is worth considering conducting manipulation research with a more realistic set of assumptions, which also takes into account domino effects that one voter could have on others.

Although our results are robust to any number of voters, the question remains open for numbers of candidates greater than or equal to four. Our decomposition method would become unnecessarily complex with about 24 dimensions in the case of just four candidates. Therefore, new solutions needed for the general problem with m candidates, and to show that for all voting rules that satisfy symmetry and unanimity, STV is the least susceptible rule against individual and coalitional manipulations with any small number of candidates.

Appendix A

Singular Value Decomposition

Singular Value Decomposition is one of the most beautiful and useful results from linear algebra which states that any transformation can be expressed as a rotation followed by a scaling followed by another rotation. Singular Value Decomposition has the same idea with diagonalization but in a more general sense that applies not only to square matrices but also the rectangulars. Recall that an $n \times n$ matrix A is *diagonalizable* if we can decompose A into:

$$A = PDP^{-1}$$

where P is an invertible matrix and D is a diagonal matrix.

The secret to constructing matrix P is to let the columns of P be the eigenvectors of A. If D is a diagonal matrix with eigenvalues on the diagonal, $AP = PD$. Because P is invertible if and only if the columns of P are linearly independent, A is diagonalizable if and only if A has n linearly independent eigenvectors. We can present P and D in the following forms:

$$P = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

To get the Singular Value Decomposition, we take the advantage of the fact that for any matrix A, A^TA is symmetric and thus, their eigenvectors form an orthonormal basis.

Consider the eigenvectors x_i and corresponding eigenvalues λ_i. Let $\sigma_i = \sqrt{\lambda_i}$ and $r_i = \frac{Ax_i}{\sigma_i}$. Three matrices are constructed from these values. The first one is the diagonal matrix, or *scale matrix*, Σ that has σ_i values on the diagonal (filled by 0 if do not have enough σs). The other two are the rotation matrix U with r_i as columns and the matrix V with x_i as columns. It is proven that $U\Sigma V^T = A$. We also know that multiplying eigenvalues of a square matrix gives us its determinant, and the determinant has an interpretation as the volume of hypercube created by those vectors. It is analogous that matrix Σ, the 2^{nd} matrix of our Singular Value Decomposition can be used to compute the volume of the region formed by vectors of matrix A. This conclusion is utilized to double check all the volume results computed in this paper, in addition to the projection method.

Appendix B

Distance from one point to a hyperplane

Given the parametric equation of a hyperplane $ap_1 + bp_2 + cp_3 + dp_4 + ep_5 + fp_6 + g = 0$ and a point $p = (p_1^*, p_2^*, p_3^*, p_4^*, p_5^*, p_6^*)$, the normal vector to the plane is given by:

$$\vec{n} = \langle a, b, c, d, e, f \rangle$$

A vector from the plane to the point is given by:

$$\vec{w} = -\langle p_1 - p_1^*, p_2 - p_2^*, p_3 - p_3^*, p_4 - p_4^*, p_5 - p_5^*, p_6 - p_6^* \rangle$$

Projecting $\vec{w}$ onto $\vec{n}$ gives the distance d from the point to the hyperplane:

$$\begin{aligned}
d &= \frac{|\vec{n}.\vec{w}|}{\|\vec{n}\|} \\
&= \frac{|a(p_1 - p_1^*) + b(p_2 - p_2^*) + c(p_3 - p_3^*) + d(p_4 - p_4^*) + e(p_5 - p_5^*) + f(p_6 - p_6^*)|}{\sqrt{a^2 + b^2 + c^2 + d^2 + e^2 + f^2}} \\
&= \frac{|ap_1 + bp_2 + cp_3 + dp_4 + ep_5 + fp_6 - ap_1^* - bp_2^* - cp_3^* - dp_4^* - ep_5^* - fp_6^*|}{\sqrt{a^2 + b^2 + c^2 + d^2 + e^2 + f^2}} \\
&= \frac{|-g - ap_1^* - bp_2^* - cp_3^* - dp_4^* - ep_5^* - fp_6^*|}{\sqrt{a^2 + b^2 + c^2 + d^2 + e^2 + f^2}} \\
&= \frac{ap_1^* + bp_2^* + cp_3^* + dp_4^* + ep_5^* + fp_6^* + g}{\sqrt{a^2 + b^2 + c^2 + d^2 + e^2 + f^2}}
\end{aligned}$$

Appendix C

Mathematica Code

C.1 Faces

In our case, all the vertices have 6 coordinates but the faces that form the bases for volume computation are in 3-dimensional space. To visualize the faces, we bring those vertices to 3 coordinates points and draw a convex hull formed by them. The following code is for P_1 face under Anti-plurality. To obtain the results for other faces, we only need to change the vertex coordinates.

$$\text{ant1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{pmatrix};$$

```
a1=With[{vectors=#-First@ant1&/@ant1},
N@DeleteCases[Orthogonalize[vectors],
{0,0,0,0,0,0}].Transpose[vectors]//Transpose];
ConvexHullMesh[a1]
```

C.2 Distance

The algorithm to find the distance is designed in accordance with discussion in Section 3.3.4. The following is the code for distance from apex v_{35} to face P_3 under Borda count.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix};$$

```
n=NullSpace[A]//MatrixForm
```

$$m = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -4 & 3 & 0 & 1 & 0 & 0 \end{pmatrix};$$

```
LinearSolve[m,{1,0,0,0,1/2,0}]
{1/18,0,0,2/9,5/9,1/6}
Norm[{0,0,1/2,0,1/2,0}-LinearSolve[m,{1,0,0,0,1/2,0}]]
1/Sqrt[3]
```

C.3 Volume

To compute the volume of a pyramid, we first define three vectors v_1, v_2, v_3 from the coordinates of 4 vertices. Subsequently, unit vectors that form the orthogonal basis e_1, e_2, e_3 are computed. Finally, volume is computed using the formula discussed in Section 3.3.3.

```
v1={1/2,0,0,1/4,0,1/4}-{0,1/2,0,1/4,0,1/4};
```

```
v2={1/2,0,1/4,0,0,1/4}-{0,1/2,0,1/4,0,1/4};
v3={0,0,1/2,0,0,1/2}-{0,1/2,0,1/4,0,1/4};
e1=v1/Norm[v1];
e2=(v2-(v2.e1) e1)/Norm[(v2-(v2.e1) e1)]//Simplify;
e3=(v3-(v3.e1) e1-(v3.e2) e2)/Norm[(v3-(v3.e1) e1-(v3.e2) e2)]//Simplify;
V=1/6 (v3.e3) (v1.e1) (v2.e2)//Simplify
Sqrt[7/2]/96
```

C.4 Simulation

The following is the code for consistency check as discussed in Section 4.2 under STV.

```
f[{p1_,p2_,p3_,p4_,p5_,p6_}]:=
If[p1>=0&&p2>=0&&p3>=0&&p4>=0&&p5>=0&&p6>=0,
{p1,p2,p3,p4,p5,p6}]
```

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix};$$

```
n=NullSpace[X];
Q=Orthogonalize[n]\[Transpose]//N;
m=100000;
xp={{1/6},{1/6},{1/6},{1/6},{1/6},{1/6}};
cs=RandomVariate[UniformDistribution[{-1,1}],
{m,4}]\[Transpose];
XN=Q.cs 0.8;
xps=Table[{1/6,1/6,1/6,1/6,1/6,1/6},{i,1,m}]\[Transpose];
G=XN+xps;
For[i=1,i<401,i++,
```

```
cs=RandomVariate[UniformDistribution[{-1,1}],
{m,4}]\[Transpose];
XN=Q.cs 0.8;G=XN+xps;Gt=G\[Transpose];
a[i]=DeleteCases[f/@Gt,Null]]
Export["datastv.xls",Flatten[Array[a,{400}],1],"Table"]
datastv.xls
SystemOpen["datastv.xls"]
```

Bibliography

Aleskerov, F., Ivanov, A., Karabekyan, D., and Yakuba, V. (2015). Manipulability of aggregation procedures in impartial anonymous culture. *Procedia Computer Science*, 55:1250–1257.

Aleskerov, F. and Karpov, A. (2013). A new single transferable vote method and its axiomatic justification. *Social Choice and Welfare*, 40(3):771–786.

Arribillaga, R. P. and Massó, J. (2016). Comparing generalized median voter schemes according to their manipulability. *Theoretical Economics*, 11(2):547–586.

Barber, K. L. (2000). *A right to representation: proportional election systems for the twenty-first century*. Ohio State University Press.

Bartholdi III, J. J. and Orlin, J. B. (1991). Single transferable vote resists strategic voting. *Social Choice and Welfare*, 8(4):341–354.

Bartholdi III, J. J., Tovey, C. A., and Trick, M. A. (1989). The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6(3):227–241.

Bassi, A. (2015). Voting systems and strategic manipulation: An experimental study. *Journal of Theoretical Politics*, 27(1):58–85.

Benoıt, J.-P. (2000). The Gibbard–Satterthwaite theorem: a simple proof. *Economics Letters*, 69(3):319–322.

Betzler, N., Niedermeier, R., and Woeginger, G. J. (2011). Unweighted coalitional manipulation under the Borda rule is NP-hard. In *IJCAI*, volume 11, pages 55–60. Citeseer.

Chamberlin, J. R. (1985). An investigation into the relative manipulability of four voting systems. *Behavioral Science*, 30(4):195–203.

Conitzer, V. and Sandholm, T. (2006). Nonexistence of voting rules that are usually hard to manipulate. In *AAAI*, volume 6, pages 627–634.

Conitzer, V., Sandholm, T., and Lang, J. (2007). When are elections with few candidates hard to manipulate? *Journal of the ACM (JACM)*, 54(3):14.

Davies, J., Katsirelos, G., Narodytska, N., and Walsh, T. (2011). Complexity of and algorithms for Borda manipulation. In *AAAI*, volume 11, pages 657–662.

Diss, M. (2015). Strategic manipulability of self-selective social choice rules. *Annals of Operations Research*, 229(1):347–376.

Dixit, A. K. and Nalebuff, B. (2008). *The art of strategy: a game theorist's guide to success in business & life*. WW Norton & Company.

Dwork, C., Kumar, R., Naor, M., and Sivakumar, D. (2001). Rank aggregation methods for the web. In *Proceedings of the 10th international conference on World Wide Web*, pages 613–622. ACM.

Ephrati, E. and Rosenschein, J. S. (1991). The Clarke tax as a consensus mechanism among automated agents. In *AAAI*, volume 91, pages 173–178.

Faliszewski, P. and Procaccia, A. D. (2010). AI's war on manipulation: Are we winning? *AI Magazine*, 31(4):53–64.

Farrell, D. M. and Katz, R. S. (2014). Assessing the proportionality of the single transferable vote. *Representation*, 50(1):13–26.

Farrell, D. M., Suiter, J., and Harris, C. (2016). The challenge of reforming a "voter-friendly" electoral system: the debates over Ireland's single transferable vote. *Irish Political Studies*, pages 1–18.

Favardin, P. and Lepelley, D. (2006). Some further results on the manipulability of social choice rules. *Social Choice and Welfare*, 26(3):485–509.

Favardin, P., Lepelley, D., and Serais, J. (2002). Borda rule, Copeland method and strategic manipulation. *Review of Economic Design*, 7(2):213–228.

Forsythe, R., Rietz, T., Myerson, R., and Weber, R. (1996). An experimental study of voting rules and polls in three-candidate elections. *International Journal of Game Theory*, 25(3):355–383.

Friedgut, E., Kalai, G., Nisan, N., et al. (2008). *Elections can be manipulated often*. Citeseer.

Gehrlein, W. V. and Lepelley, D. (2004). Probability calculations in voting theory: an overview of recent results. *Reasoned Choices: Essays in Honor of Academy Professor Hannu Nurmi*.

Gehrlein, W. V. and Lepelley, D. (2012). The value of research based on simple assumptions about voters' preferences. In *Electoral Systems*, pages 173–199. Springer.

Gibbard, A. (1973). Manipulation of voting schemes: a general result. *Econometrica: journal of the Econometric Society*, pages 587–601.

Hemaspaandra, E. and Hemaspaandra, L. A. (2007). Dichotomy for voting systems. *Journal of Computer and System Sciences*, 73(1):73–83.

Huang, H. and Chua, V. C. (2000). Analytical representation of probabilities under the IAC condition. *Social Choice and Welfare*, 17(1):143–155.

Isaksson, M., Kindler, G., and Mossel, E. (2012). The geometry of manipulation—a quantitative proof of the Gibbard-Satterthwaite theorem. *Combinatorica*, 32(2):221–250.

Kim, K. H. and Roush, F. W. (1996). Statistical manipulability of social choice functions. *Group Decision and Negotiation*, 5(3):263–282.

Marsh, M. (2007). Candidates or parties? Objects of electoral choice in Ireland. *Party Politics*, 13(4):500–527.

Marsh, M. and Plescia, C. (2016). Split-Ticket voting in an STV system: Choice in a non-strategic context. *Irish Political Studies*, 31(2):163–181.

Mas-Colell, A., Whinston, M. D., Green, J. R., et al. (1995). *Microeconomic theory*, volume 1. Oxford university press New York.

Meir, R., Polukarov, M., Rosenschein, J. S., and Jennings, N. R. (2010). Convergence to equilibria in plurality voting. In *Proc. of 24th Conference on Artificial Intelligence (AAAI-10)*, pages 823–828.

Menton, C. and Singh, P. (2012). Manipulation can be hard in tractable voting systems even for constant-sized coalitions. *Computer Science Review*, 6(2):71–87.

Pennock, D. M., Horvitz, E., Giles, C. L., et al. (2000). Social choice theory and recommender systems: Analysis of the axiomatic foundations of collaborative filtering. In *AAAI/IAAI*, pages 729–734.

Pritchard, G. and Slinko, A. (2006). On the average minimum size of a manipulating coalition. *Social Choice and Welfare*, 27(2):263–277.

Reyhani, R., Pritchard, G., and Wilson, M. C. (2010). A new measure of manipulability of voting rules.

Saari, D. (2001). *Chaotic elections!: A mathematician looks at voting.* American Mathematical Soc.

Saari, D. G. (1990). Susceptibility to manipulation. *Public Choice*, 64(1):21–41.

Saari, D. G. (1995). Geometry of voting, volume 3 of studies in economic theory.

Satterthwaite, M. A. (1975). Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of economic theory*, 10(2):187–217.

Smith, D. A. (1999). Manipulability measures of common social choice functions. *Social Choice and Welfare*, 16(4):639–661.

Stensholt, E. (2004). Single transferable votes with tax cuts. *SIAM review*, 46(3):417–442.

Szpiro, G. G. (2010). *Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present.* Princeton University Press.

Taylor, A. D. (2005). *Social choice and the mathematics of manipulation.* Cambridge University Press.

Tideman, N. (1995). The single transferable vote. *The Journal of Economic Perspectives*, 9(1):27–38.

Wallis, W. D. (2014). *The mathematics of elections and voting.* Springer.

Walsh, T. (2010). An empirical study of the manipulability of single transferable voting. In *ECAI*, volume 10, pages 257–262.

Xia, L. and Conitzer, V. (2008). A sufficient condition for voting rules to be frequently manipulable. In *Proceedings of the 9th ACM Conference on Electronic Commerce*, pages 99–108. ACM.

Xia, L., Conitzer, V., and Procaccia, A. D. (2010). A scheduling approach to coalitional manipulation. In *Proceedings of the 11th ACM conference on Electronic commerce*, pages 275–284. ACM.