St. Norbert College
Digital Commons @ St. Norbert College

Faculty Creative and Scholarly Works

Classifying Character Degree Graphs With Six Verteces

Mark W. Bissler
Kent State University

Jacob Laubacher
St. Norbert College

Mark L. Lewis
Kent State University

Follow this and additional works at: https://digitalcommons.snc.edu/faculty_staff_works

CLASSIFYING CHARACTER DEGREE GRAPHS WITH 6 VERTICES

MARK W. BISSLER, JACOB LAUBACHER, AND MARK L. LEWIS

ABSTRACT. We investigate prime character degree graphs of solvable groups that have six vertices. There are one hundred twelve non-isomorphic connected graphs with six vertices, of which all except nine are classified in this paper. We also completely classify the disconnected graphs with six vertices.

1. INTRODUCTION

Throughout this paper, G will be a finite solvable group. We will write $\mathrm{Irr}(G)$ for the set of irreducible characters of G, and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Denote $\rho(G)$ to be the set of primes that divide degrees in $\mathrm{cd}(G)$ for the character degrees of G. The prime character degree graph of G, written $\Delta(G)$, is the graph whose vertex set is $\rho(G)$. Two vertices p and q of $\rho(G)$ are adjacent in $\Delta(G)$ if there exists $a \in \mathrm{cd}(G)$ where pq divides a. This type of graph has been studied in a variety of places (see [9], [10], [12], [14], [16], [20]).

In [16], Pálfy showed that if G is a solvable group for every three vertices in $\rho(G)$, there is some edge in $\Delta(G)$ incident to two of these vertices. With this in mind, we say that a graph Γ satisfies Pálfy's condition if for every three vertices there is some edge incident to two of them. Note that all graphs with three or fewer vertices that satisfy Pálfy's condition occur as $\Delta(G)$ for some solvable group G, as done in [8]. The graphs with four vertices that satisfy Pálfy's condition all occur except for the disconnected graph with two connected components each of size two and the connected graph with diameter three. These were shown not to occur in [17] and [20], respectively. Lastly, character degree graphs with five vertices were classified in [13], with the exception of one graph.

In [11], the third author settled the conjecture that prime character degree graphs have diameter at most two by constructing the graph in Figure 1. Later in [19] (see also [18]), it was shown that this is the unique graph with six vertices and diameter three that occurs as $\Delta(G)$.

Lately, families of graphs have also been studied. In [5], a family of graphs was classified as unable to occur as $\Delta(G)$, and this family was generalized in [3]. These families contain graphs with arbitrarily many vertices, however they do indeed classify several graphs with six vertices along the way, of which we will make use.
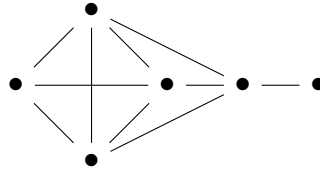
FIGURE 1. Diameter three graph which occurs as $\Delta(G)$

In this paper, we look specifically at graphs with six vertices, and we attempt to determine whether each graph can or cannot occur as the prime character degree graph of a solvable group. In [6], it was shown that there are a total of one hundred twelve non-isomorphic connected graphs with six vertices. Of the one hundred twelve graphs, only thirty-five satisfy Pálfy's condition. We then proceed to ask the following two questions:

(1) Which of these graphs can occur as the prime character degree graph of some solvable group?

(2) Which of these graphs cannot occur as the prime character degree graph of any solvable group?

After applying known results from the literature and our own results, we are able to conclude that at least thirteen of the connected graphs with six vertices occur as the prime character degree graph for some solvable group, as well as two disconnected graphs. We are left with nine graphs still in consideration. Our results can be summed up into the following theorem:

Main Theorem. *The graphs with six vertices that arise as $\Delta(G)$ for some solvable group G are precisely those graphs in Figures 1, 3, and 4, and possibly those in Figure 7.*

This work was done while the first author, Mark W. Bissler, was a Ph.D. student at Kent State University under the supervision of Mark L. Lewis. The results in this paper appear in the first author's Ph.D. dissertation [2]. Further work has been conducted with Jacob Laubacher on graphs with six vertices, and has been added.

2. Preliminaries

We aim to keep this note as self-contained as possible. We first recall the landmark results from Pálfy.

Lemma 2.1 (Pálfy's condition from [16])**.** *Let G be a solvable group and let π be a set of primes contained in $\Delta(G)$. If $|\pi| = 3$, then there exists an irreducible character of G with degree divisible by at least two primes from π. (In other words, any three vertices of the prime character degree graph of a solvable group span at least one edge.)*

In fact, recently Pálfy's condition was generalized:

Theorem 2.2. ([1]) *Let G be a finite solvable group. Then the complement graph $\bar{\Delta}(G)$ does not contain any cycle of odd length.*

Finally, Pálfy also has a result which controls the sizes of the components of a disconnected graph. This will be vital when determining which disconnected graphs are possible.

Theorem 2.3 (Pálfy's inequality from [17])**.** *Let G be a solvable group and $\Delta(G)$ its prime character degree graph. Suppose that $\Delta(G)$ is disconnected with two components having size a and b, where $a \leq b$. Then $b \geq 2^a - 1$.*

Most of the following results are from [5].

Definition 2.4. ([5]) A vertex p of a graph Γ is **admissible** if:

(i) the subgraph of Γ obtained by removing p and all edges incident to p does not occur as the prime character degree graph of any solvable group, and

(ii) none of the subgraphs of Γ obtained by removing one or more of the edges incident to p occur as the prime character degree graph of any solvable group.

Classifying vertices as admissible is shown to be quite powerful. One of the key results is below, of which we will use often.

Lemma 2.5. ([5]) *Let G be a solvable group, and suppose p is an admissible vertex of $\Delta(G)$. For every proper normal subgroup H of G, suppose that $\Delta(H)$ is a proper subgraph of $\Delta(G)$. Then $O^p(G) = G$.*

En route to showing that a graph does not occur as $\Delta(G)$ for any solvable group G, one of the techniques is to first show that G has no normal nonabelian Sylow subgroups.

Lemma 2.6. ([5]) *Let Γ be a graph satisfying Pálfy's condition. Let q be a vertex of Γ, and denote π to be the set of vertices of Γ adjacent to q, and ρ to be the set of vertices not adjacent to q. Assume that π is the disjoint union of nonempty sets π_1 and π_2, and assume that no vertex in π_1 is adjacent in Γ to any vertex in π_2. Let v be a vertex in π_2 adjacent to an admissible vertex s in ρ. Furthermore, assume there exists another vertex w in ρ that is not adjacent to v.*

Let G be a solvable group such that $\Delta(G) = \Gamma$, and assume that for every proper normal subgroup H of G, $\Delta(H)$ is a proper subgraph of $\Delta(G)$. Then a Sylow q-subgroup of G for the prime associated to q is not normal.

We now mention the refinement of admissible which again has consequences involving no normal nonabelian Sylow p-subgroup.

Definition 2.7. ([5]) A vertex p of a graph Γ is **strongly admissible** if:

(i) p is admissible, and

(ii) none of the subgraphs of Γ obtained by removing p, the edges incident to p, and one or more of the edges between two adjacent vertices of p occurs as $\Delta(G)$ for some solvable group G.

Lemma 2.8. ([5]) *Let G be a solvable group, and assume that p is a prime whose vertex is a strongly admissible vertex of $\Delta(G)$. For every proper normal subgroup H of G, suppose that $\Delta(G/H)$ is a proper subgraph of $\Delta(G)$. Then a Sylow p-subgroup of G is not normal.*

We next mention the result in [4], which is our final method in showing a group G has no normal nonabelian Sylow p-subgroup. First, however, we must set some necessary notation. We start by fixing a vertex p. Let π be the subset of vertices which are adjacent to p, and let ρ be the subset of vertices that are not adjacent to p. We will let π^* and ρ^* denote nonempty subsets of π and ρ, respectively. Next, for any subset $\pi^* \cup \rho^*$ which induces a complete subgraph of $\Delta(G)$, let β denote a subset of vertices which contains $\pi^* \cup \rho^*$ and also induces a complete subgraph of $\Delta(G)$. Setting $\mathcal{B}$ as the union of all such β's, we then denote $\tau = \mathcal{B} \setminus (\pi^* \cup \rho^*)$.

Hypothesis 2.9. ([4]) Concerning $\Delta(G)$, we assume the following:
  (i) for every vertex in ρ, there exists a nonadjacent vertex in π,
 (ii) for every vertex in π, there exists a nonadjacent vertex in ρ,
(iii) all the vertices in π are admissible. Moreover, no proper connected subgraph with vertex set $\{p\} \cup \pi^* \cup \rho$ occurs as the prime character degree graph of any solvable group,
 (iv) for each vertex set $\pi^* \cup \rho^*$ which induces a complete subgraph in $\Delta(G)$, all the vertices in the corresponding set τ are admissible. Moreover, no proper connected subgraph with vertex set $\rho(G) \setminus \tau^*$ occurs as the prime character degree graph of any solvable group, and
  (v) if a disconnected subgraph with vertex set $\rho(G)$ does not occur, then it must specifically violate Pálfy's inequality from [17]. Finally, if a disconnected subgraph with vertex set $\rho(G)$ does occur, then the sizes of the connected components must be $n > 1$ and $2^n - 1$.

Theorem 2.10. ([4]) *Assume Hypothesis 2.9. Then G has no normal nonabelian Sylow p-subgroup.*

The penultimate result we provide in this section will be the tool we use to show a graph cannot occur for any solvable group G. Note this is Lemma 2.6 of [5].

Lemma 2.11. ([5]) *Let Γ be a graph satisfying Pálfy's condition with $n \geq 5$ vertices. Also, assume there exist distinct vertices a and b of Γ such that a is adjacent to an admissible vertex c, b is not adjacent to c, and a is not adjacent to an admissible vertex d.*

Let G be a solvable group and suppose for all proper normal subgroups N of G we have that $\Delta(N)$ and $\Delta(G/N)$ are proper subgraphs of Γ. Let F be the Fitting subgroup of G and suppose that F in minimal normal in G. Then Γ is not the prime character degree graph of any solvable group.

Finally, we recall the main result from [5]. This constructs a family of graphs which cannot occur as $\Delta(G)$ for any solvable group G. This will be useful since the family contains graphs with six vertices.

Theorem 2.12. ([5]) *Let Γ be a graph satisfying Pálfy's condition with $k \geq 5$ vertices. Assume that there exists two vertices p_1 and p_2 in Γ such that*

(i) both p_1 and p_2 are of degree two,
(ii) p_1 is adjacent to p_2, and
(iii) p_1 and p_2 share no common neighbor.
Then Γ is not the prime character degree graph of any solvable group.

3. Constructions

Be begin this section by considering the disconnected graphs with six vertices. Due to Pálfy's condition, we can have at most two components. Moreover, each component must be a complete graph. Considering graphs with six vertices, therefore, yields three possibilities: graphs with component sizes one and five, two and four, and three and three. We know the graph with component sizes three and three cannot occur (see Figure 2). This is due to Pálfy's inequality by taking $a = b = 3$ and observing that $3 \geq 7$, a contradiction.
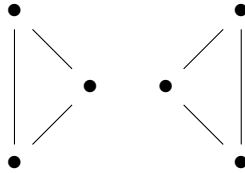

FIGURE 2. Disconnected graph which does not occur as $\Delta(G)$

We provide the construction of the two possible disconnected graphs, displayed in Figure 3. We mimic the process for the construction of these from [13]. First, we consider the construction of the graph with component sizes one and five. For this, consider the field of order 2^{32} acted on by its full multiplication group and then its Galois group. This group has character degree set $\{1, 2, 4, 8, 16, 32, 3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537\}$. Thus, this gives a graph with two connected components: $\{2\}$ and $\{3, 5, 17, 257, 65537\}$.

Next, we consider the graph with component sizes two and four. Consider the field of order 2^{35} acted on by its full multiplication group and then its Galois group. This group has character degree set $\{1, 5, 7, 35, 31 \cdot 71 \cdot 127 \cdot 122921\}$. Thus, this gives a graph with two connected components: $\{5, 7\}$ and $\{31, 71, 127, 122921\}$.

We also construct the graphs that can occur as prime character degree graphs via direct products. For groups H and K we have that $\rho(H \times K) = \rho(H) \cup \rho(K)$. We require also that $\rho(H)$ and $\rho(K)$ be disjoint sets; that is they must have different primes. Since there are infinitely many prime sets that occur for groups having
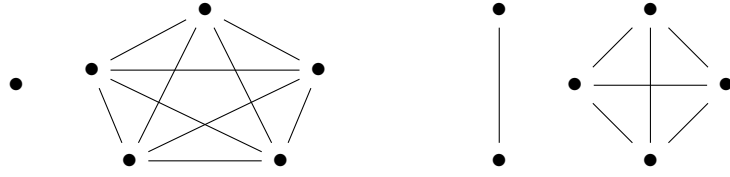
FIGURE 3. Disconnected graphs which occur as $\Delta(G)$

these graphs, it is not a problem to find such groups H and K. Next, there is an edge between vertices p and q in $\Delta(H \times K)$ if we have any of the following cases:

(1) $p, q \in \rho(H)$ and there is an edge between p and q in $\Delta(H)$,
(2) $p, q \in \rho(K)$ and there is an edge between p and q in $\Delta(K)$,
(3) $p \in \rho(H)$ and $q \in \rho(K)$, or
(4) $p \in \rho(K)$ and $q \in \rho(H)$.

Using this construction, we are able to produce the twelve graphs in Figure 4 that occur as $\Delta(G)$ for some solvable group G.
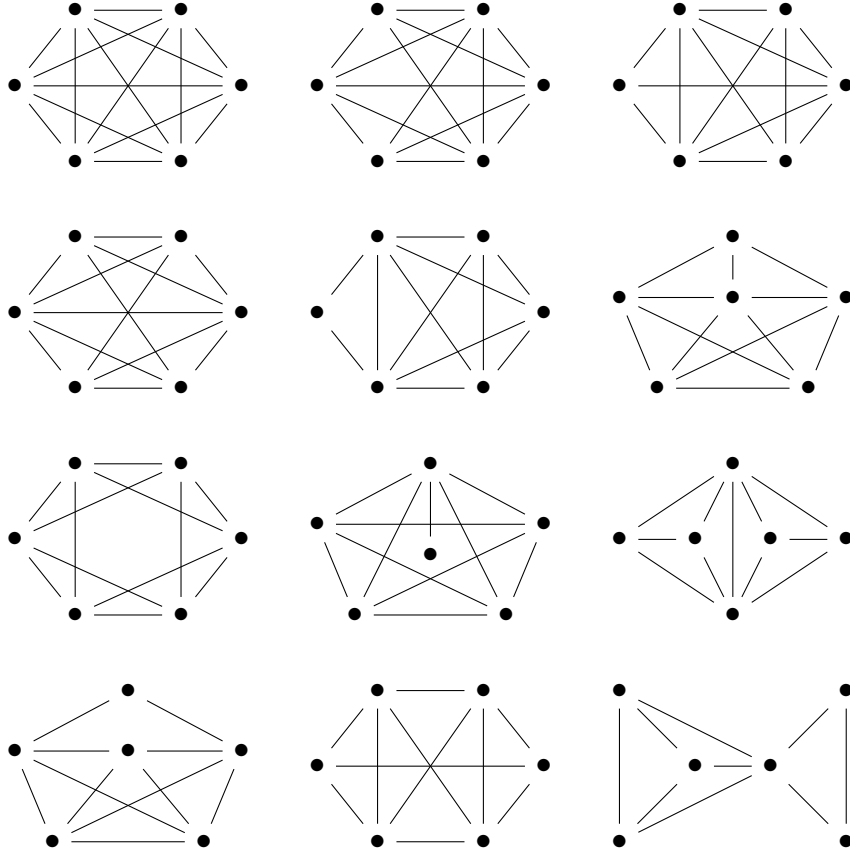
FIGURE 4. Direct products which occur as $\Delta(G)$

4. Reductions

We start by noting the number of connected graphs with six vertices is one hundred twelve by [6]. After applying Pálfy's condition to the one hundred twelve graphs, we reduce this number to thirty-five possible graphs. We use the results of [18] when dealing with a graph or subgraph that arises with diameter three. These results from [18] (see also her dissertation [19]) conclude that the graph in Figure 1 is the unique six-vertex graph with diameter three that occurs as $\Delta(G)$. Using this fact we can decrease the number of graphs from thirty-five graphs down to twenty-seven.

Recall that in Section 3 we constructed twelve graphs via direct products (see Figure 3); if we remove these twelve graphs from the reduced list of twenty-seven, we are then left with fifteen graphs. In this section we will show that the graphs in Figure 5 cannot occur as $\Delta(G)$ for any solvable group G.
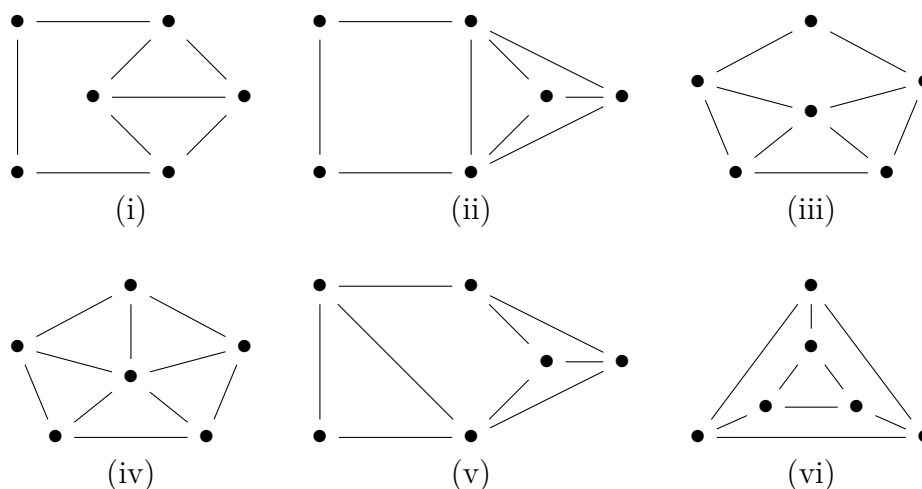

FIGURE 5. Connected six-vertex graphs investigated in this paper

Theorem 4.1. *Graphs (i) and (ii) of Figure 5 are not the prime character degree graph of a solvable group.*

Proof. Using the main result from [5] (see Theorem 2.12 above), we have that graphs (i) and (ii) from Figure 5 fall into the infinite family shown not to occur as the prime character degree graph for any solvable group. One observes that these are also classified in the families studied in [3]. ∎

Theorem 4.2. *Graphs (iii) and (iv) of Figure 5 are not the prime character degree graph of a solvable group.*

Proof. By the main result from [1] (see Theorem 2.2 above), any graph whose complement graph has an odd cycle can be eliminated. The complement of graphs

(iii) and (iv) contain a cycle of length five, thus they cannot be the prime character degree graph for a solvable group. ∎

Theorem 4.3. *Graph (vi) of Figure 5 is not the prime character degree graph of a solvable group.*

Proof. In [3], the authors show that the graph (vi) of Figure 5 cannot occur as $\Delta(G)$ for any solvable group G by showing that every vertex is admissible. This graph is also classified in an infinite family which cannot occur as the character degree graph for any solvable group. We mention also, that in [21], a result about regular character degree graphs was shown which also proves graph (vi) is not the prime character degree graph of a solvable group. ∎

4.1. **Eliminating graph (v) of Figure 5.** Due to the length of the proof, we dedicate this separate subsection to eliminating graph (v) of Figure 5. It is beneficial to label our vertices, as in Figure 6.
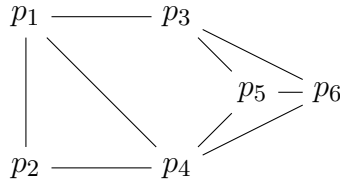

FIGURE 6. The graph in Theorem 4.4

Theorem 4.4. *There is no solvable group G such that $\Delta(G)$ is the graph found in Figure 6.*

Proof. Suppose G is a counterexample with $|G|$ minimal. We label the primes in $\rho(G)$ as the vertices in Figure 6. Our first goal will be to show that G does not have any normal nonabelian Sylow subgroups. We first show which vertices of Figure 6 are strongly admissible.

Claim: *The vertices p_1, p_4, p_5, and p_6 are strongly admissible vertices, and thus G does not have a corresponding normal Sylow subgroup for p_1, p_4, p_5 and p_6.*

Proof: We start by considering the vertex p_1. If we lose an edge between p_1 and p_3, then we arrive at a graph with diameter three, which is not the unique one, so we cannot lose this edge. If we lose an edge between p_1 and p_2, we violate Pálfy's condition with p_1, p_2, and p_6. If we lose the edge between p_1 and p_4, we have the graph (i) of Figure 5, which has been shown to not occur. Now let us assume that the vertex p_1 is removed from the graph. Let P_1 be a Sylow p_1-subgroup of G and assume that $P_1 \triangleleft G$. By Lemma 3 of [12] we know $\mathrm{cd}(G/P_1') = \mathrm{cd}(G) \setminus \{p_1\}$, so

the subgraph will have vertex set $\{p_2, p_3, p_4, p_5, p_6\}$. If no edges are lost, we arrive at a subgraph with five vertices that has diameter three, which we know is not possible by the main theorem of [10]. The only possible edge that could be lost in this subgraph is the edge between p_2 and p_4, and this subgraph would violate Pálfy's condition. Thus, p_1 is strongly admissible.

Now we look at the vertex p_4. If an edge is lost between p_2 and p_4, we violate Pálfy's condition with vertices p_2, p_3, and p_4. We have already shown we cannot lose the edge between p_1 and p_4, so we consider the edge between p_4 and p_5 or p_6. If the edge between p_4 and p_5 is removed, we then arrive at a graph with diameter three, which is not the unique diameter three graph. If we lose the edge between p_4 and p_6, we then also arrive at a diameter three graph that is not the unique diameter three graph with six vertices. Let P_4 be a Sylow p_4-subgroup of G and assume that $P_4 \triangleleft G$. By Lemma 3 of [12] we know $\mathrm{cd}(G/P_4') = \mathrm{cd}(G) \setminus \{p_1\}$, so the subgraph will have vertex set $\{p_1, p_2, p_3, p_5, p_6\}$. If we remove the vertex p_4 from the graph, the subgraph that loses no other edges has fives vertices and diameter three, which we know is not possible again, by the main theorem of [10]. The only edges in this subgraph that could be lost are the edges between p_1 and p_2 and the edge between p_5 and p_6. In either case we result in a subgraph that violates Pálfy's condition. Thus, p_4 is strongly admissible.

We next show p_5 is strongly admissible and note that a similar argument will show that p_6 is strongly admissible. If an edge is lost between p_3 and p_5, then we violate Pálfy's condition with p_3, p_5, and p_2. If an edge is lost between p_5 and p_6, we again violate Pálfy's condition with p_5, p_6, and p_2. The edge between p_4 and p_5 we have already shown cannot be lost, so we now consider when the vertex p_5 is removed from the graph. Let P_5 be a Sylow p_5-subgroup of G and assume that $P_5 \triangleleft G$. If p_5 is removed from the graph, then by Lemma 3 of [12], we know $\mathrm{cd}(G/P_5') = \mathrm{cd}(G) \setminus \{p_5\}$, so the subgraph will have vertex set $\{p_1, p_2, p_3, p_4, p_6\}$. If we remove the vertex p_5 from the graph, the resulting subgraph is the graph shown to not be possible in [13]. The only edges that this subgraph could lose are between two vertices that are adjacent to p_5. Thus, we have the following two cases: the edge between p_4 and p_5, and the edge between p_3 and p_6. If both edges are lost, we result in a subgraph that violates Pálfy's condition, and if either one is only lost, we arrive at a subgraph that has five vertices and diameter three, which is not possible by the main theorem of [10] or a graph that violates Pálfy's condition. Thus, p_5 is strongly admissible, and p_6 is also. By Lemma 2.8 we have that G does not have a normal nonabelian Sylow p_i-subgroup for $i \in \{1, 4, 5, 6\}$. $\square$

Next we show that G does not have a normal Sylow p_3-subgroup by showing that the vertex p_3 satisfies the hypotheses of Lemma 2.6.

Claim: The group G does not have a normal Sylow p_3-subgroup.

Proof: Consider p_3. Then we have that $\pi = \{p_1, p_5, p_6\}$ and $\rho = \{p_2, p_4\}$. We can write $\pi = \pi_1 \cup \pi_2$, a disjoint union where $\pi_1 = \{p_1\}$ and $\pi_2 = \{p_5, p_6\}$. We have shown that p_1 is strongly admissible previously, and we know that p_5 is adjacent to p_4, which is an admissible vertex in ρ. Finally, we note that $p_2 \in \rho$ and p_2 is not adjacent to p_5. Thus, the hypotheses of Lemma 2.6 are satisfied, and we have that G does not have a normal Sylow p_3-subgroup. $\square$

We next show that no subgraphs of the graph in Figure 6 with six vertices can occur.

Claim: *The graph in Figure 6 has no proper subgraphs with six vertices that occur as a prime character degree graph of a solvable group.*

Proof: Suppose H is a group where $\rho(H) = \rho(G)$, $\Delta(H)$ is a proper subgraph of $\Delta(G)$, and $|H| < |G|$. Applying Pálfy's condition to p_1, p_2, and p_6 in $\Delta(H)$, we see that p_1 and p_2 must be adjacent in $\Delta(H)$. Again, if we apply Pálfy's condition to p_2, p_3, and p_5 or p_6 in $\Delta(H)$, we see that there must be an edge between p_3 and p_5 and also between p_3 and p_6. Applying Pálfy's condition to p_2, p_5, and p_6, we see that there must be an edge between p_5 and p_6 in $\Delta(H)$. Again, applying Pálfy's condition to p_2, p_4, and p_3 in $\Delta(H)$, we see that there must be an edge between p_2 and p_4. Finally, we claim there must be an edge between p_1 and p_4 in $\Delta(H)$, since Pálfy's condition implies that there must be an edge between p_1 and p_4 or between p_4 and p_6. If there is an edge between p_4 and p_6, we arrive at a subgraph with diameter three, which was a graph that was excluded earlier. If we have an edge between p_1 and p_4, we conclude that this graph along with any possible edge added, can be ruled out by having diameter three, violating Pálfy's condition, or a graph already ruled out. Thus, $\Delta(H)$ does not occur as the prime character degree graph in any case. $\square$

Claim: *The group G does not have a normal nonabelian Sylow p_2-subgroup.*

Proof: We will show that the graph in Figure 6 satisfies all the conditions of Hypothesis 2.9 for $p = p_2$. Consequently, we have that $\pi = \{p_1, p_4\}$ and $\rho = \{p_3, p_5, p_6\}$.

Since p_1 is not adjacent to p_5 or p_6 and p_4 is not adjacent to p_3, we satisfy (i) and (ii) of Hypothesis 2.9.

We have already shown that p_1 and p_4 are admissible, and one can check that no connected subgraph with vertex set $\{p\} \cup \{p_1\} \cup \rho$ or $\{p\} \cup \{p_4\} \cup \rho$ occurs as the prime character degree graph for any solvable group G. Moreover, the previous claim verifies that no subgraph with vertex set $\{p\} \cup \pi \cup \rho$ occurs either, and so we satisfy (iii).

For the next condition, (iv), we start by investigating the possible subsets $\pi^* \cup \rho^*$, which are: (1) $\{p_1\} \cup \{p_3\}$, (2) $\{p_4\} \cup \{p_5, p_6\}$, (3) $\{p_4\} \cup \{p_5\}$, and (4) $\{p_4\} \cup \{p_6\}$.

Notice that (1) and (2) have their corresponding sets $\tau = \varnothing$, and so there is nothing to verify. For (3), we note that $\tau = \{p_6\}$. By a previous claim, we have verified that p_6 is admissible, and one can easily check that no connected subgraph with vertex set $\rho(G) \setminus \{p_6\} = \{p_1, p_2, p_3, p_4, p_5\}$ occurs as $\Delta(G)$ for any solvable group G. Observe that a symmetric argument will work for (4), and thus (iv) is satisfied.

The only possible disconnected graph would have components with vertex sets $\{p\} \cup \pi$ and ρ, but this is exactly the graph in Figure 2 which does not occur as the prime character degree graph of any solvable group, and in particular violates Pálfy's inequality. Thus, (v) is satisfied.

Since we satisfy all the conditions for Hypothesis 2.9, we apply Theorem 2.10 and have that G has no normal nonabelian Sylow p_2-subgroup. $\square$

We have shown that G has no normal nonabelian Sylow subgroups. Let F be the Fitting subgroup of G. We note that $\rho(G) = \pi(|G : F|)$, and thus, $\rho(G) = \rho(G/\Phi(G))$ where $\Phi(G)$ is the Frattini subgroup of G. Suppose M is a normal subgroup of G so that $\rho(G/M) = \rho(G)$. If $M > 1$, then the minimality of $|G|$ would imply that $\Delta(G/M)$ is a proper subgraph of $\Delta(G)$, which violates a previous claim. Thus, we see that if $\rho(G/M) = \rho(G)$, then $M = 1$. Now, for $\Phi(G)$, we have that $\rho(G/\Phi(G)) = \rho(G)$, so that $\Phi(G) = 1$. We may now apply Lemma III 4.4 of [7] to see that there is a subgroup H of G so that $G = HF$ and $H \cap F = 1$. We let E denote the Fitting subgroup of H.

Claim: *The Fitting subgroup F of G is a minimal normal subgroup.*

Proof: Suppose that there is a normal subgroup N of G so that $1 < N < F$. By Theorem III 4.5 of [7], there is a normal subgroup M of G so that $F = N \times M$. Since $N > 1$ and $M > 1$, we have $\rho(G/N) \subset \rho(G)$ and $\rho(G/M) \subset \rho(G)$. For any prime $p \in \rho(G) \setminus \rho(G/N)$, we know that G/N has a normal abelian Sylow p-subgroup. The class of finite groups with an abelian and normal Sylow p-subgroup is a formation, so p must lie in $\rho(G/M)$. Thus, $\rho(G) = \rho(G/N) \cup \rho(G/M)$.

If $p \in \rho(G) \setminus \rho(G/N)$, then p is not in $\rho(G/F) = \rho(H)$; so E must then contain the Sylow p-subgroup of H. Since $p \in \rho(G)$, it follows that p divides $|H|$, and thus p will divide $|E|$. Recall that $\mathrm{cd}(G)$ contains a degree divisible by all the prime divisors of $|EF : F| = |E|$. We conclude that $\rho(G) \setminus (\rho(G/M) \cap \rho(G/N))$ lies in a complete subgraph of $\Delta(G)$. Therefore, $\rho(G) \setminus (\rho(G/M) \cap \rho(G/N))$ lies in the subsets: (1) $\{p_1, p_2, p_4\}$, (2) $\{p_1, p_3\}$, (3) $\{p_3, p_5, p_6\}$, or (4) $\{p_4, p_5, p_6\}$.

Suppose that (1) occurs. This implies that $\{p_3, p_5, p_6\} \subseteq \rho(G/N) \cap \rho(G/M)$. Since $\rho(G/M) \cup \rho(G/N) = \rho(G)$, we know that $\rho(G/M)$ or $\rho(G/N)$ must therefore be one of the following: (i) $\{p_3, p_5, p_6, p_1, p_2\}$, (ii) $\{p_3, p_5, p_6, p_2, p_4\}$, or (iii) $\{p_3, p_5, p_6, p_1, p_4\}$. Assume (i), that is, we have $\rho(G/N) = \{p_1, p_2, p_3, p_5, p_6\}$. By Theorem 5.5 of [10], G/N has a central Sylow p_4-subgroup. However, this

would imply $O^{p_4}(G) < G$, a contradiction as we have shown p_4 is strongly admissible, which implies $O^{p_4}(G) = G$. A similar argument works for (ii). Finally, assume we have (iii), that is, $\rho(G/N) = \{p_3, p_5, p_6, p_1, p_4\}$. We know $\rho(G/N) \cup \rho(G/M) = \rho(G)$, and so we consider $\rho(G/M)$ in this case. We have that $\{p_3, p_5, p_6, p_2\} \subset \rho(G/M)$. We see that the only possible graph arising from the set of $\rho(G/M)$ must be disconnected and have two components. By Theorem 5.5 of [10], we have that G/M has either a central Sylow p_1-subgroup or a central Sylow p_4-subgroup. This implies $O^{p_1}(G) < G$ or $O^{p_4}(G) < G$, respectively. However, this is a contradiction as we showed p_1 and p_4 are strongly admissible, and hence $O^{p_1}(G) = G = O^{p_4}(G)$.

Now suppose (2) occurs. This implies that $\{p_2, p_4, p_5, p_6\} \subseteq \rho(G/N) \cap \rho(G/M)$. We consider the possible cases for $\rho(G/N)$ and $\rho(G/M)$. Now we know that $\rho(G/M)$ or $\rho(G/N)$ must contain p_3 or p_1 as $\rho(G) = \rho(G/N) \cup \rho(G/M)$. Without loss of generality, assume that $\rho(G/N) = \{p_1, p_2, p_4, p_5, p_6\}$ and $\rho(G/M) = \{p_2, p_3, p_4, p_5, p_6\}$. For $\Delta(G/M)$, we have seen that the only possible graph that could occur with this vertex set is the disconnected graph. Again by Theorem 5.5 of [10], G/M then has a central Sylow p_1-subgroup. However, this would imply $O^{p_1}(G) < G$, a contradiction as we have shown p_1 is strongly admissible, which implies $O^{p_1}(G) = G$. Thus, (2) cannot occur.

Suppose (3) occurs. We then have that E contains a Hall $\{p_3, p_5, p_6\}$-subgroup of H. Since $\mathrm{cd}(G)$ has a degree divisible by all the primes dividing $|E|$, we have that $|E|$ is divisible by no other primes and $|E|$ is tha Hall $\{p_3, p_5, p_6\}$-subgroup of H. Let $\chi \in \mathrm{Irr}(G)$ with $p_3 p_5 p_6$ dividing $\chi(1)$. Let θ be an irreducible constituent of χ_{FE}. Now, $\chi(1)/\theta(1)$ divides $|G : FE|$ and $\chi(1)$ is relatively prime to $|G : FE|$. We determine that $\chi_{FE} = \theta$. Since p_3, p_5 and p_6 divide $\theta(1)$ and the only possible prime divisors of $a \in \mathrm{cd}(G/FE)$ are p_1, p_2, or p_4, we conclude via Gallagher's theorem that $\mathrm{cd}(G/FE) = \{1\}$ and G/FE is abelian. We now have that $O^{p_4}(G) < G$, a contradiction. So, (3) cannot occur.

Suppose (4) occurs. Then we have that $\{p_1, p_2, p_3\} \subseteq \rho(G/N) \cap \rho(G/M)$. We know that $\rho(G/M)$ or $\rho(G/N)$ must be one of the following: (i) $\{p_1, p_2, p_3, p_4, p_5\}$, (ii) $\{p_1, p_2, p_3, p_4, p_6\}$, or (iii) $\{p_1, p_2, p_3, p_5, p_6\}$. Note that for each of these cases, the only graph that occurs with the respective vertex set is the disconnected graph. Applying Theorem 5.5 from [10] yields a central Sylow subgroup, which carries us to our contradiction using the admissibility of p_6, p_5, and p_4, respectively. Therefore (4) cannot occur. □

Now we apply Lemma 2.11 for the final contradiction to prove Theorem 4.4. We note that p_2 and p_3 are adjacent to admissible vertices p_4 and p_5, p_3 is not adjacent to p_4, and p_2 is not adjacent to p_5. Since we have shown F is minimal normal, and the graph we consider has greater than four vertices, Lemma 2.5 of [5] is satisfied (see Lemma 2.11 above), and thus the graph in Figure 6 is not the prime character degree graph for any solvable group G. ■

4.2. **Remaining graphs.** There are still nine remaining graphs with six vertices that have yet to be classified (see Figure 7).
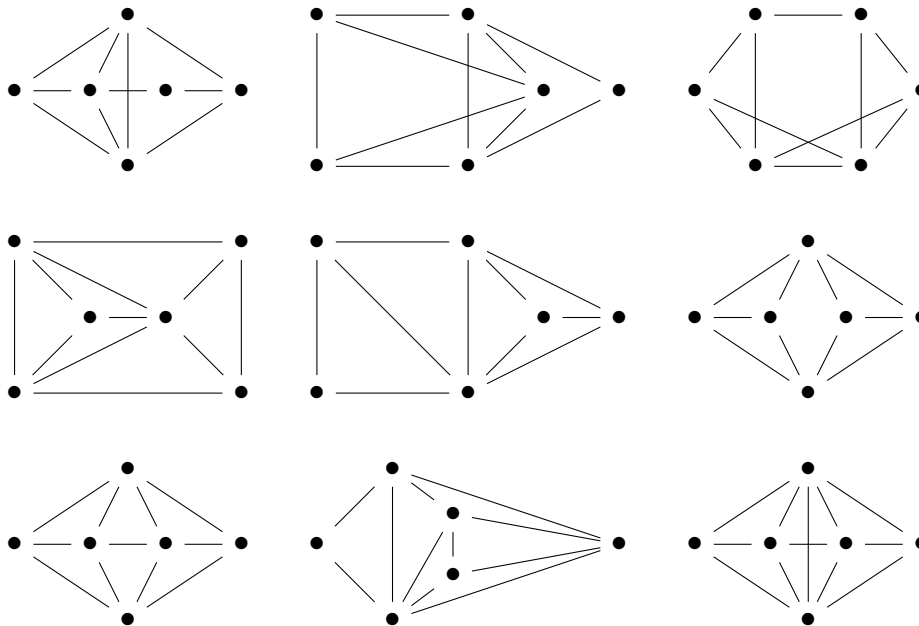

FIGURE 7. Unknown six-vertex graphs

REFERENCES

[1] Zeinab Akhlaghi, Carlo Casolo, Silvio Dolfi, Khatoon Khedri, and Emanuele Pacifici. On the character degree graph of solvable groups. *Proc. Amer. Math. Soc.*, 146(4):1505–1513, 2018.

[2] Mark W. Bissler. *Character degree graphs of solvable groups.* OhioLINK Electronic Theses and Dissertations Center, Columbus, Ohio, 2017. Dissertation (Ph.D.)–Kent State University.

[3] Mark W. Bissler and Jacob Laubacher. Classifying families of character degree graphs of solvable groups. arXiv:1708.00119, 2017.

[4] Mark W. Bissler, Jacob Laubacher, and Corey F. Lyons. On the absence of a normal non-abelian Sylow subgroup. arXiv:1803.06691, 2018. Accepted, to appear in *Comm. Algebra.*

[5] Mark W. Bissler and Mark L. Lewis. A family of graphs that cannot occur as character degree graphs of solvable groups. arXiv:1707.03020, 2017.

[6] Dragos Cvetković and Milenko Petrić. A table of connected graphs on six vertices. *Discrete Math.*, 50:37–49, 1984.

[7] Bertram Huppert. *Endliche gruppen. I.* Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967.

[8] Bertram Huppert. Research in representation theory at Mainz (1984–1990), in *Representation theory of finite groups and finite-dimensional algebras*, ed. by G. O. Michler and C. M. Ringel (Bielefeld, May 15–17, 1991), Progr. Math. 95, Basel 1991, 17–36.

[9] I. Martin Isaacs. *Character theory of finite groups.* Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)].

[10] Mark L. Lewis. Solvable groups whose degree graphs have two connected components. *J. Group Theory*, 4(3):255–275, 2001.

[11] Mark L. Lewis. A solvable group whose character degree graph has diameter 3. *Proc. Amer. Math. Soc.*, 130(3):625–630, 2002.

[12] Mark L. Lewis. Solvable groups with character degree graphs having 5 vertices and diameter 3. *Comm. Algebra*, 30(11):5485–5503, 2002.

[13] Mark L. Lewis. Classifying character degree graphs with 5 vertices. In *Finite groups 2003*, pages 247–265. Walter de Gruyter, Berlin, 2004.

[14] Mark L. Lewis. An overview of graphs associated with character degrees and conjugacy class sizes in finite groups. *Rocky Mountain J. Math.*, 38(1):175–211, 2008.

[15] Olaf Manz and Thomas R. Wolf. *Representations of solvable groups*. Cambridge University Press: Cambridge, 1993.

[16] Péter Pál Pálfy. On the character degree graph of solvable groups. I. Three primes. *Period. Math. Hungar.*, 36(1):61–65, 1998.

[17] Péter Pál Pálfy. On the character degree graph of solvable groups. II. Disconnected graphs. *Studia Sci. Math. Hungar.*, 38:339–355, 2001.

[18] Catherine B. Sass. *Prime character degree graphs of solvable groups having diameter three*. OhioLINK Electronic Theses and Dissertations Center, Columbus, Ohio, 2014. Dissertation (Ph.D.)–Kent State University.

[19] Catherine B. Sass. Character degree graphs of solvable groups with diameter three. *J. Group Theory*, 19(6):1097–1127, 2016.

[20] Jiping Zhang. On a problem by Huppert. *Bejing Daxue Xuebao Ziran Kexue Ban*, 34(2-3):143–150, 1998.

[21] Claudio Paolo Morresi Zuccari. Regular character degree graphs. *J. Algebra*, 411:215–224, 2014.

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, U.S.A.
  E-mail address: `mbissle2@math.kent.edu`

Department of Mathematics, St. Norbert College, De Pere, Wisconsin 54115, U.S.A.
  E-mail address: `jacob.laubacher@snc.edu`

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, U.S.A.
  E-mail address: `lewis@math.kent.edu`

arXiv.org > math > arXiv:1709.09278v3

Mathematics > Group Theory

Classifying Character Degree Graphs With 6 Vertices

Mark W. Bissler, Jacob Laubacher, Mark L. Lewis

(Submitted on 26 Sep 2017 (v1), last revised 11 Aug 2018 (this version, v3))

We investigate prime character degree graphs of solvable groups that have six vertices. There are one hundred twelve non-isomorphic connected graphs with six vertices, of which all except nine are classified in this paper. We also completely classify the disconnected graphs with six vertices.

Download:

Current browse context:
math.GR

Change to browse by:

References & Citations

Bookmark

Submission history

Which authors of this paper are endorsers? | Disable MathJax (What is MathJax?)