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ON THE ABSENCE OF A NORMAL NONABELIAN SYLOW SUBGROUP

MARK W. BISSLER, JACOB LAUBACHER, AND COREY F. LYONS

ABSTRACT. Let G be a finite solvable group. We show that G does not have a normal nonabelian Sylow p -subgroup when its prime character degree graph $\Delta(G)$ satisfies a technical hypothesis.

1. INTRODUCTION AND PRELIMINARIES

In this paper we fix G to be a finite solvable group. Set $\mathbf{F}(G)$ to be the Fitting subgroup of G , write $\text{Irr}(G)$ for the set of irreducible characters of G , and let $\text{cd}(G)$ denote the set of irreducible character degrees of G . Denote $\rho(G)$ to be the set of primes that divide degrees in $\text{cd}(G)$. The prime character degree graph of G , written $\Delta(G)$, is the graph whose vertex set is $\rho(G)$. Two vertices p and q of $\Delta(G)$ are adjacent if there exists $d \in \text{cd}(G)$ such that pq divides d . In this paper, we concern ourselves solely with graphs $\Delta(G)$ that are connected with $\rho(G) \geq 5$.

We fix the following notation for an arbitrary vertex $p \in \rho(G)$:

$$\pi := \{q \in \rho(G) : q \text{ is adjacent to } p \text{ in } \Delta(G)\}$$

and

$$\rho := \{q \in \rho(G) : q \text{ is not adjacent to } p \text{ in } \Delta(G)\}.$$

Observe that ρ induces a complete subgraph in $\Delta(G)$ due to Pálffy's condition from [8], and that $\rho(G)$ is a disjoint union: $\{p\} \cup \pi \cup \rho$. We let π^* and ρ^* denote arbitrary nonempty subsets of π and ρ , respectively. Finally, let $\pi^* \cup \rho^*$ be an arbitrary vertex set which induces a complete subgraph in $\Delta(G)$. Let β be a subset of $\rho(G)$ that contains $\pi^* \cup \rho^*$ such that β also induces a complete subgraph in $\Delta(G)$. Set \mathcal{B} to be the union of all such β 's satisfying these properties for $\pi^* \cup \rho^*$. Consider $\tau := \mathcal{B} \setminus (\pi^* \cup \rho^*)$, and denote τ^* to be a subset of τ . Notice that τ could be empty, depending on the initial set $\pi^* \cup \rho^*$.

In this note we build upon the work done in [2] and [6] which concerns a normal nonabelian Sylow p -subgroup of a finite solvable group. We use tools from [2] which classify vertices as admissible. Recall the definition and result directly related to admissible vertices below.

Definition 1.1. ([2]) A vertex q of a graph Γ is **admissible** if:

- (i) the subgraph of Γ obtained by removing q and all edges incident to q does not occur as the prime character degree graph of any solvable group, and
- (ii) none of the subgraphs of Γ obtained by removing one or more of the edges incident to q occur as the prime character degree graph of any solvable group.

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Lemma 1.2 (Lemma 2.1 from [2]). *Let q be an admissible vertex of $\Delta(G)$. For every proper normal subgroup N of G , suppose that $\Delta(N)$ is a proper subgraph of $\Delta(G)$. Then $O^q(G) = G$.*

Most notably, our results pertain to groups G and graphs $\Delta(G)$ which do not satisfy Lemmas 2.3 and 2.4 from [2]. They also conclude in those lemmas that there is no normal nonabelian Sylow p -subgroup. This is a typical approach en route to showing that a graph cannot occur as the prime character degree graph of any solvable group.

Next we recall several lemmas due to Lewis which pertain to groups having a normal nonabelian Sylow subgroup.

Lemma 1.3 (Lemma 4.2 from [6]). *Let G be a solvable group with a normal nonabelian Sylow p -subgroup P . Let H be a p -complement for G . Assume that H acts faithfully on P , that $\rho \subseteq \rho(H)$, and that the Fitting subgroup of H is a π -group. Then $\rho(H) = \rho$.*

Lemma 1.4 (Lemma 4.3 from [6]). *Let G be a solvable group with a normal nonabelian Sylow p -subgroup P . Let H be a p -complement for G . Suppose that P' is minimal normal in G , and $\rho(H) = \rho$ is not empty. Then there is a nonempty subset π^* of π so that $\Delta(P'H)$ has two connected components: ρ and π^* . Furthermore, if $|\rho| = n$, then $|\pi^*| \geq 2^n - 1$.*

Lemma 1.5 (Lemma 3 from [5]). *Let G be a solvable group and let $p \in \rho(G)$. If P is a normal Sylow p -subgroup, then $\rho(G/P') = \rho(G) \setminus \{p\}$.*

To conclude this section, we recall two powerful results concerning when the prime character degree graph is disconnected.

Theorem 1.6 (Theorem 5.5 from [4]). *Let G be a solvable group and suppose that $\Delta(G)$ has two connected components. Then there is precisely one prime p so that the Sylow p -subgroup of the Fitting subgroup of G is not central in G .*

Theorem 1.7 (Pálffy's inequality from [9]). *Let G be a solvable group and $\Delta(G)$ its prime character degree graph. Suppose that $\Delta(G)$ is disconnected with two components having size a and b , where $a \leq b$. Then $b \geq 2^a - 1$.*

2. MAIN RESULTS

In this section we prove that G has no normal nonabelian Sylow p -subgroup. Hypothesis 2.1 below is strictly about the prime character degree graph $\Delta(G)$, whereas Hypothesis 2.2 is about the group G itself. Further, we always assume $|G|$ is minimal.

Hypothesis 2.1. Concerning $\Delta(G)$, we assume the following:

- (i) for every vertex in ρ , there exists a nonadjacent vertex in π ,
- (ii) for every vertex in π , there exists a nonadjacent vertex in ρ ,
- (iii) all the vertices in π are admissible. Moreover, no proper connected subgraph with vertex set $\{p\} \cup \pi^* \cup \rho$ occurs as the prime character degree graph of any solvable group,
- (iv) for each vertex set $\pi^* \cup \rho^*$ which induces a complete subgraph in $\Delta(G)$, all the vertices in the corresponding set τ are admissible. Moreover, no proper connected subgraph with vertex set $\rho(G) \setminus \tau^*$ occurs as the prime character degree graph of any solvable group, and
- (v) if a disconnected subgraph with vertex set $\rho(G)$ does not occur, then it must specifically violate Pálffy's inequality from [9]. Finally, if a disconnected subgraph with vertex set $\rho(G)$ does occur, then the sizes of the connected components must be $n > 1$ and $2^n - 1$.

Observe the following consequences of Hypothesis 2.1. For (i) and (ii), since $\Delta(G)$ is assumed to be connected, it is required that $|\pi| \geq 2$ and $|\rho| \geq 2$. As another consequence of (ii), all vertices in π are adjacent to each other. Otherwise, if there existed two nonadjacent vertices in π , then the complement graph would have an odd cycle. This is prohibited by the main theorem from [1]. Concerning (iii), one gleans that no proper connected subgraph with vertex set $\rho(G)$ occurs as the prime character degree graph of any solvable group (taking $\pi^* = \pi$). Depending on the initial subset $\pi^* \cup \rho^*$, (iv) may require that some of the vertices in ρ are admissible. Finally, we know from (v) that if a disconnected subgraph with vertex set $\rho(G)$ does indeed occur as the prime character degree graph of a solvable group, then it must be represented by Example 2.4 from [6].

Hypothesis 2.2. Concerning the group G , we assume P is a normal nonabelian Sylow p -subgroup of G , and we let H be a p -complement for G . We set $F := \mathbf{F}(H)$, which is necessarily nontrivial.

Concerning F , it is known there exists a degree in $\text{cd}(G)$ that is divisible by all the prime divisors of $|F|$. This corresponds to the vertices in $\pi(|F|)$ inducing a complete subgraph in $\Delta(G)$. Next, assuming G has a normal nonabelian Sylow p -subgroup, we can now see the full force of Hypothesis 2.1(v). As stated above, this hypothesis forces the disconnected graph (should it occur) to be represented by Example 2.4 from [4]. In that case, all normal Sylow subgroups are necessarily abelian. Therefore, assuming N is a group such that $\rho(N) = \rho(G)$, we in fact get $\Delta(N) = \Delta(G)$ by Hypothesis 2.1(iii). Since $|G|$ is minimal we conclude that $N = G$.

Lemma 2.3. *Assume Hypotheses 2.1 and 2.2. Then $C_H(P) = 1$.*

Proof. Set $C := C_H(P)$. We follow the argument on page 259 in [6] in a generalized form to show $C = 1$.

First we observe that C is normal in G , and that $PC = P \times C$. Therefore, any prime in $\rho(C)$ must be adjacent to p in $\Delta(G)$. So we know that $\rho(C) \subseteq \pi$ and that r divides $|H : C|$ for every $r \in \rho$. Since we know that H/C acts faithfully on P , we see that the primes in $\pi(|H : C|)$ must lie in $\rho(G/C)$. Thus, $\rho \subseteq \rho(G/C)$. We then conclude that $\rho(G/C)$ must have one of the following vertex sets: (a) $\{p\} \cup \rho$, (b) $\{p\} \cup \pi^* \cup \rho$, or (c) $\{p\} \cup \pi \cup \rho$. Here π^* is taken as a nonempty proper subset of π .

Suppose (a). Note that none of the primes in π divide $|H : C|$. We fix a nonlinear character $\gamma \in \text{Irr}(P)$ and then γ extends to $\gamma \times 1_C$ in $P \times C$. For $\chi \in \text{Irr}(G|\gamma \times 1_C)$ notice that p divides $\chi(1)$ and $\chi(1)/\gamma(1)$ divides $|H : C|$. Since none of the primes in ρ divide $\chi(1)$ and the only primes that could possibly divide $|H : C|$ are the primes in ρ , we are forced to conclude that $\chi_{PC} = \gamma \times 1_C$, which implies $\chi_P = \gamma$. Using Gallagher's Theorem (Corollary 6.17 from [3]) we have that $\chi(1)d \in \text{Irr}(G)$ for every degree $d \in \text{cd}(H)$. Thus, $\rho(H) \subseteq \pi$. This means that H has a normal abelian Hall ρ -subgroup, and hence $O^q(G) < G$ for some $q \in \pi$. This is a contradiction since every vertex in π is admissible.

Suppose (b). By Hypothesis 2.1(iii), we know that any connected subgraph with vertex set $\{p\} \cup \pi^* \cup \rho$ does not occur as the prime character degree graph of any solvable group. If $\Delta(G/C)$ is a disconnected graph that does occur, then by Theorem 5.5 from [4] we conclude that G/C has a central Sylow q -subgroup for some $q \in \pi \setminus \pi^*$. This implies $O^q(G) < G$, which is a contradiction since every vertex in π is admissible.

We are left with (c), in which case $\rho(G/C) = \{p\} \cup \pi \cup \rho = \rho(G)$. By Hypothesis 2.1(v), we see that $|G/C| = |G|$, and hence $C = 1$. \square

Notice that Lemma 2.3 implies that H acts faithfully on P . With this in hand, we now investigate F , the Fitting subgroup of $H \cong G/P$.

Claim 2.4. *Assume Hypotheses 2.1 and 2.2. Then F is not a ρ -subgroup.*

Proof. For the sake of contradiction, we assume that F is a ρ -subgroup. Let $\gamma \in \text{Irr}(P)$ be a nonlinear character and let $\chi \in \text{Irr}(G|\gamma)$. We see that p divides both $\gamma(1)$ and $\chi(1)$, and that r does not divide $\chi(1)$ or $\chi(1)/\gamma(1)$ for all $r \in \rho$. Since this is true for all characters $\chi \in \text{Irr}(G|\gamma)$, we apply Theorem 12.9 of [7] to get that $G/P \cong H$ has an abelian Hall ρ -subgroup.

Since we assumed that F was in fact a ρ -subgroup, we now get that F is the abelian Hall ρ -subgroup. Thus $O^q(G) < G$ for some $q \in \pi$, which is a contradiction since all the vertices in π are admissible. Therefore F is not a ρ -subgroup. \square

Claim 2.5. *Assume Hypotheses 2.1 and 2.2. Then F is not a $\pi^* \cup \rho^*$ -subgroup.*

Proof. For the sake of contradiction, we assume that F is a $\pi^* \cup \rho^*$ -subgroup. It is sufficient to suppose $\pi(|F|) = \pi^* \cup \rho^*$. Observe that in this scenario, Hypothesis 2.1(i) and (ii) force π^* and ρ^* to be proper subsets of π and ρ , respectively.

We fix the following notation:

$$\eta := \{s \in \rho(G) \setminus (\{p\} \cup \pi(|F|)) : s \text{ and } q \text{ are not adjacent in } \Delta(G) \text{ for some } q \in \pi(|F|)\}.$$

In fact, $\eta = \rho(G) \setminus (\{p\} \cup \mathcal{B})$ and η must contain at least one element from $\pi \setminus \pi^*$ and from $\rho \setminus \rho^*$. For otherwise this would contradict Hypothesis 2.1(i) and (ii) as $\pi^* \cup \rho^*$ forms a complete subgraph of $\Delta(G)$. Thus η is nonempty.

Let $\gamma \in \text{Irr}(PF)$ be such that q divides $\gamma(1)$ for all $q \in \pi^* \cup \rho^*$. For any character $\chi \in \text{Irr}(G|\gamma)$, we have that q divides $\chi(1)$ for all $q \in \pi^* \cup \rho^*$, and that r does not divide $\chi(1)$ or $\chi(1)/\gamma(1)$ for any $r \in \eta$. By Theorem 12.9 from [7], we see that $G/PF \cong H/F$ has an abelian Hall η -subgroup. Let $M = O_{\mathcal{B}}(H)$, and note that $F \subseteq M$. Set E/M to be the Fitting subgroup of H/M . Since H/M has an abelian Hall η -subgroup, we see that E/M is the abelian Hall η -subgroup.

Next we have that $\rho(PE)$ contains $\{p\} \cup \pi^* \cup \rho^* \cup \eta$. However this may be a proper subset of $\rho(G)$. Notice that by construction we have that $\rho(G) \setminus (\{p\} \cup \pi^* \cup \rho^* \cup \eta) = \tau$. Hence $\rho(PE)$ must be one of the following: (a) $\rho(G) \setminus \tau^*$ or (b) $\rho(G)$.

Suppose (a). Then by Hypothesis 2.1(iv) we see that no proper connected subgraph with vertex set $\rho(G) \setminus \tau^*$ occurs as the prime character degree graph of any solvable group. If $\Delta(PE)$ is a disconnected graph that occurs, then by Theorem 5.5 from [4] we observe that PE has a central Sylow q -subgroup for some $q \in \tau^*$. This implies $O^q(G) < G$, which is a contradiction since every vertex in τ is admissible.

We are left with (b). So $\rho(PE) = \rho(G)$, and by Hypothesis 2.1(v) we see that $PE = G$. Since E/M is abelian, and since $\eta \cap \pi \neq \emptyset$, we have that $O^s(G) < G$ for some $s \in \pi$. This is a contradiction since all the vertices in π are admissible. Therefore F is not a $\pi^* \cup \rho^*$ -subgroup. \square

Claim 2.6. *Assume Hypotheses 2.1 and 2.2. Then F is not a π -subgroup.*

Proof. For the sake of contradiction, we suppose F is a π -subgroup of H and hence $\rho \subseteq \rho(H)$. Furthermore, H acts faithfully on P as consequence of Lemma 2.3. We apply Lemma 4.2 from [6] which yields $\rho(H) = \rho$.

Next we will see that P' is minimal normal in G . Suppose that X is normal in G such that $1 \leq X \leq P'$ and P'/X is a chief factor for G . Our goal will be to show that $X = 1$.

We know $\rho(G/X) \supseteq \rho(G/P') = \rho(G) \setminus \{p\}$ (by Theorem 3 from [5]). However, $p \in \rho(G/X)$ since G/X has a normal nonabelian Sylow p -subgroup. It follows that $\rho(G) = \rho(G/X)$, and hence $|G/X| = |G|$ by Hypothesis 2.1(v). Therefore $X = 1$ and P' is minimal normal in G .

The hypotheses of Lemma 4.3 from [6] are satisfied, and therefore $\Delta(P'H)$ has disconnected components with vertex sets ρ and π^* such that

$$(1) \quad |\pi^*| \geq 2^{|\rho|} - 1.$$

In particular, notice that

$$(2) \quad |\rho| \leq |\pi^*|.$$

Hypothesis 2.1(v) gives two cases concerning the possibility of a disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$. We will investigate these below, but first notice that by our assumptions, the two connected components of a possibly disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$ are forced to have vertex sets ρ and $\{p\} \cup \pi$.

For Case 1, we consider if no disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$ occurs. By Hypothesis 2.1(v) we know that for component sizes a and b (with $a \leq b$), we must have the situation where $b < 2^a - 1$. This gives two subcases:

Case 1(a): $|\rho| = a$ and $|\{p\} \cup \pi| = b$. For this scenario we have

$$|\pi^*| \leq |\pi| < |\{p\} \cup \pi| = b < 2^a - 1 = 2^{|\rho|} - 1,$$

which contradicts (1).

Case 1(b): $|\{p\} \cup \pi| = a$ and $|\rho| = b$. Thus

$$|\rho| = b \geq a = |\{p\} \cup \pi| > |\pi| \geq |\pi^*|,$$

which contradicts (2).

For Case 2, we consider if a disconnected subgraph of $\Delta(G)$ with vertex set $\rho(G)$ occurs. By Hypothesis 2.1(v) we know the components must be of sizes $n > 1$ and $2^n - 1$, which gives two subcases:

Case 2(a): $|\rho| = n > 1$ and $|\{p\} \cup \pi| = 2^n - 1$. Observe that

$$|\pi^*| \leq |\pi| < |\{p\} \cup \pi| = 2^n - 1 = 2^{|\rho|} - 1,$$

which contradicts (1).

Case 2(b): $|\{p\} \cup \pi| = n > 1$ and $|\rho| = 2^n - 1$. Similarly notice

$$|\rho| = 2^n - 1 > n = |\{p\} \cup \pi| > |\pi| \geq |\pi^*|,$$

which contradicts (2). Therefore F is not a π -subgroup. \square

We have shown that no primes in $\rho(G) \setminus \{p\}$ will divide $|F|$. This leads to the following result:

Theorem 2.7. *Assume Hypothesis 2.1. Then G has no normal nonabelian Sylow p -subgroup.*

Proof. For the sake of contradiction, suppose Hypothesis 2.2. That is, suppose G has a normal nonabelian Sylow p -subgroup P , and let H be a p -complement for G with Fitting subgroup F .

By Claims 2.4, 2.5, and 2.6, we see that F must be trivial since no primes divide $|F|$. As H is solvable it must have a nontrivial Fitting subgroup, a contradiction. \square

Consider the following graphs with the appropriate vertex p labeled in Figure 1 for several examples and applications of Theorem 2.7.

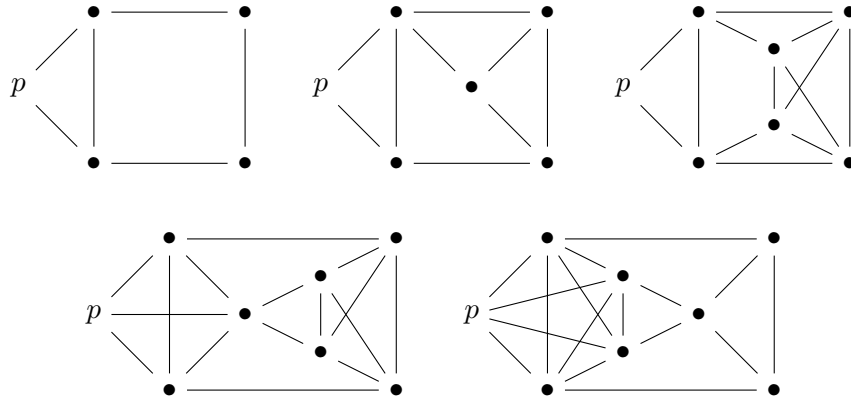


FIGURE 1. Graphs satisfying Hypothesis 2.1

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