PATHOLOGICAL ABELIAN GROUPS: A FRIENDLY EXAMPLE

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Abstract. We show that the group of bounded sequences of elements of $\mathbb{Z}[\sqrt{2}]$ is an example of an abelian group with several well known, and not so well known, pathological properties. It appears to be simpler than all previously known examples for some of these properties, and at least simpler to describe for others.

1. History and introduction

In the first edition of his famous book *Infinite Abelian Groups* in 1954 [Kap54], Kaplansky proposed three "test problems" for abelian groups. The motivation was that if these did not have positive answers for some particular class of groups, then we could pretty much give up on a satisfactory structure theorem for that class. At the time, all three problems were open for general abelian groups, although they were all answered during the next decade.

The first problem asked whether, if two abelian groups G and H are each isomorphic to a direct summand of the other, then we must have $G \cong H$. In 1961 Sąsiada [Sąs61] provided a counterexample.

The second asked whether, if G and H are abelian groups with $G \oplus G$ and $H \oplus H$ isomorphic, then we must have $G \cong H$. In 1957 Jónsson [Jón57] provided a counterexample.

Subsequently, all manner of pathological phenomena concerning direct sum decompositions of abelian groups have been discovered, in many cases involving quite intricate constructions. One of the most well known is Corner's discovery [Cor64] of an abelian group G such that $G \cong G \oplus G \oplus G$, but $G \ncong G \oplus G$. This example gives another solution to Kaplansky's first two problems, since if we take $H = G \oplus G$, then G and H provide counterexamples for both problems.

In a later edition of his book, Kaplansky wrote "In this strange part of the subject anything that can conceivably happen actually does happen."

However, Kaplansky's third test problem asked whether, if $G \oplus \mathbb{Z}$ is isomorphic to $H \oplus \mathbb{Z}$, then we must have $G \cong H$. This was answered

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independently by Cohn [Coh56] and Walker [Wal56] in 1956, and this time the answer is "yes".

Perhaps this indicates that, although the bad behaviour of general abelian groups can be almost limitless, they can sometimes be restrained by the company of well-behaved groups like $\mathbb{Z}$.

On the internet site MathOverflow, Martin Brandenburg [Bra15] asked in 2015 whether there is an abelian group A such that $A \cong A \oplus \mathbb{Z}^2$, but $A \not\cong A \oplus \mathbb{Z}$. In fact the same question had been answered in 1985 by Eklof and Shelah [ES87], with a characteristically ingenious and intricate construction. They credit Sabbagh with asking the question.

The purpose of this paper is to give a simpler example of such a group, which also provides the simplest example that I know of Corner's phenomenon (and hence provides examples that answer Kaplansky's first two test problems, maybe not simpler to verify than existing examples, but easier to describe).

The group in question is the group of bounded sequences of elements of $\mathbb{Z}[\sqrt{2}]$.

2. The main theorem

We will frequently be dealing with sequences of numbers, or of functions, and we will use the compact notation $\underline{x}$ for a sequence $(x_0, x_1, \dots)$. By a finite sequence we will mean one with only finitely many nonzero terms.

For this section and the next, A will be the additive group of bounded (in the real norm) sequences $\underline{a}$ of elements of $\mathbb{Z}[\sqrt{2}]$. Our main theorem is

Theorem 2.1. *As abelian groups, $A \cong A \oplus \mathbb{Z}^2$, but $A \not\cong A \oplus \mathbb{Z}$.*

Clearly $A \oplus \mathbb{Z}[\sqrt{2}] \cong A$ via the map $(\underline{a}, b) \mapsto (b, a_0, a_1, \dots)$. Since $\mathbb{Z}[\sqrt{2}] \cong \mathbb{Z} \oplus \mathbb{Z}$ as abelian groups, $A \cong A \oplus \mathbb{Z} \oplus \mathbb{Z}$. We shall prove that $A \not\cong A \oplus \mathbb{Z}$.

We first want to understand group homomorphisms $\varphi : A \to \mathbb{Z}[\sqrt{2}]$. We shall show in Proposition 2.5 that there are none apart from the obvious ones. This is analogous to a well known result of Specker [Spe50] stating that the only group homomorphisms from the group of all integer sequences (the Baer-Specker group) to $\mathbb{Z}$ are the obvious ones of the form $\underline{x} \mapsto \sum_i b_i x_i$ for a finite sequence $\underline{b}$ of integers. We will adapt one proof of Specker's result that combines ideas of Sąsiada and Łoś.

For $n \in \mathbb{N}$, let $A_n \cong \mathbb{Z}[\sqrt{2}]$ be the subgroup of A consisting of sequences whose terms are all zero, apart from possibly the nth term.

Lemma 2.2. *Let $\varphi : A \to \mathbb{Z}[\sqrt{2}]$ be a group homomorphism. Then $\varphi(A_n) = 0$ for all but finitely many n.*

Proof. (cf. Sąsiada [Sąs59].) If not, we can choose $n_0 < n_1 < \dots$ so that $\varphi(A_{n_k}) \neq 0$ for all k. The intersection of A_{n_k} with $\ker(\varphi)$ has rank at most one, so we can choose $x_k \in A_{n_k}$ so that its n_kth term is equal to $2^{m_k}(\sqrt{2}-1)^{l_k}$, where we inductively choose m_k so that 2^{m_k} does not divide $\varphi(x_{k-1})$ and subsequently choose l_k so that $\varphi(x_k) \neq 0$ and $|x_k| < 1$.

Consider the sequences in A whose n_kth term, for each k, is either x_k or 0, with all other terms zero. Since there are uncountably many such sequences, φ must agree on two of them. Taking the difference of these two, we get a nonzero sequence in $\ker(\varphi)$ whose first nonzero term, in the n_kth place for some k, is $\pm x_k$ and with all other terms divisible by a higher power of 2 than $\varphi(x_k)$. But this is a contradiction, since $\varphi(x_k) = \pm\varphi(y)$, where y is the sequence obtained by replacing the first non-zero term by zero. □

Let $A' < A$ be the subgroup $\{\underline{a} \in A \mid \sum_n a_n \text{ converges}\}$. We can adapt the previous proof, by replacing the condition $|x_k| < 1$ with the condition $|x_k| < 2^{-k}$, to get the following variant of the lemma.

Lemma 2.3. *Let $\varphi : A' \to \mathbb{Z}[\sqrt{2}]$ be a group homomorphism. Then $\varphi(A_n) = 0$ for all but finitely many n.*

Lemma 2.4. *Let $\varphi : A \to \mathbb{Z}[\sqrt{2}]$ be a group homomorphism such that $\varphi(A_n) = 0$ for all n. Then $\varphi = 0$.*

Proof. (cf. Łoś [Łoś54].) Suppose not. Choose $\underline{a} \in A$ with $\varphi(\underline{a}) \neq 0$. If $\underline{b} \in A'$ then the partial sums $b_0 + b_1 + \cdots + b_n$ are bounded, so we can define a homomorphism $\theta : A' \to A$ by

$$\theta(\underline{b}) = (b_0 a_0, (b_0 + b_1)a_1, (b_0 + b_1 + b_2)a_2, \dots).$$

If $\underline{e_k} \in A'$ is the sequence which is zero except that the kth term is 1, then $\theta(\underline{e_k})$ differs from $\underline{a}$ only in the first k terms, so $\varphi\theta\left(\underline{e_k}\right) = \varphi(\underline{a}) \neq 0$ for every k, contradicting Lemma 2.3. □

Proposition 2.5. *The only group homomorphisms $\varphi : A \to \mathbb{Z}[\sqrt{2}]$ are the maps*

$$\underline{a} \mapsto \sum_n \varphi_n(a_n),$$

for finite sequences $\underline{\varphi}$ of group homomorphisms $\varphi_n : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$.

Proof. Clearly every such φ is a group homomorphism. By Lemmas 2.2 and 2.4 every group homomorphism is of this form. □

Proposition 2.6. *Every group endomorphism $\beta : A \to A$ is determined by the compositions $\beta_{mn} : A_n \to A \xrightarrow{\beta} A \to A_m$, where, for each m, all but finitely many β_{mn} are zero.*

Proof. Since A is a subgroup of a direct product of copies of $\mathbb{Z}[\sqrt{2}]$ in an obvious way, this follows immediately from Proposition 2.5. □

In other words, this means that if we think of sequences as infinite column vectors then we can represent β as an infinite matrix of homomorphisms $\beta_{mn} : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$, with finitely many nonzero entries in each row. In fact, every such matrix will describe a homomorphism from A to the group of all sequences of elements of $\mathbb{Z}[\sqrt{2}]$, but extra conditions are needed to ensure that the image of this homomorphism consists of bounded sequences.

Lemma 2.7. *Let $\vartheta : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{2}]$ be a group homomorphism that is not a $\mathbb{Z}[\sqrt{2}]$-module homomorphism. Then for any $\epsilon > 0$ and $N > 0$ there is some $x \in \mathbb{Z}[\sqrt{2}]$ with $|x| < \epsilon$ and $|\vartheta(x)| > N$.*

Proof. $\vartheta(\sqrt{2}) - \sqrt{2}\vartheta(1) \neq 0$, since ϑ is not a $\mathbb{Z}[\sqrt{2}]$-module homomorphism. Choose sequences $\underline{a}$ and $\underline{b}$ of nonzero integers such that $a_n + b_n\sqrt{2} \to 0$, so that necessarily $|b_n| \to \infty$. Then

$$\left|\vartheta(a_n + b_n\sqrt{2})\right| = \left|(a_n + b_n\sqrt{2})\vartheta(1) + b_n\left(\vartheta(\sqrt{2}) - \sqrt{2}\vartheta(1)\right)\right| \to \infty.$$

□

Lemma 2.8. *Let $\beta : A \to A$ be a group endomorphism, and define $\beta_{mn} : A_n \to A_m$ as in Proposition 2.6. For all but finitely many n, β_{mn} is a $\mathbb{Z}[\sqrt{2}]$-module homomorphism for all m.*

Proof. Suppose not. Since, for each m, $\beta_{mn} = 0$ for all but finitely many n, we can recursively choose $m_0 < m_1 < \dots$ and $n_0 < n_1 < \dots$ so that $\beta_{m_k n_k}$ is not a $\mathbb{Z}[\sqrt{2}]$-module homomorphism for any k, and such that $\beta_{m_k n_l} = 0$ for $k < l$. For if we have chosen $m_0, \dots, m_{k-1}$ and $n_0, \dots, n_{k-1}$ consistent with these properties, then we can choose n_k large enough that $\beta_{m n_k} = 0$ for all $m \leq m_{k-1}$ and such that there is some m_k for which $\beta_{m_k n_k}$ is not a $\mathbb{Z}[\sqrt{2}]$-module homomorphism.

By Lemma 2.7 we can recursively choose $x_{n_k} \in \mathbb{Z}[\sqrt{2}]$ such that, identifying each A_n with $\mathbb{Z}[\sqrt{2}]$ in the obvious way, $|x_{n_k}|$ is bounded, but

$$\left|\beta_{m_k n_k}(x_{n_k}) + \sum_{l<k} \beta_{m_k n_l}(x_{n_l})\right| \to \infty.$$

But then, if we set $x_n = 0$ for $n \notin \{n_0, n_1, \dots\}$, the sequence $\underline{x}$ is bounded, and so in A, but $\beta(\underline{x})$ is unbounded, contradicting the fact that $\beta(\underline{x}) \in A$. □

Proof of Theorem 2.1. As pointed out earlier, the fact that $A \cong A \oplus \mathbb{Z}^2$ is clear.

Suppose $A \cong A \oplus \mathbb{Z}$. Then there is a monomorphism $\beta : A \to A$ such that $A/\beta(A) \cong \mathbb{Z}$. By Proposition 2.6 and Lemma 2.8, β is described by an infinite matrix (β_{mn}) with only finitely many columns containing any entries that are not $\mathbb{Z}[\sqrt{2}]$-module homomorphism. So for sufficiently large n, if $A[n] \leq A$ is the subgroup of sequences whose first n terms are zero, then the restriction of β to $A[n]$ is a $\mathbb{Z}[\sqrt{2}]$-module homomorphism, and so $A/\beta(A[n])$ is a $\mathbb{Z}[\sqrt{2}]$-module, and so must have even rank, since $(A/\beta(A[n])) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space over $\mathbb{Q}(\sqrt{2})$. However, as an abelian group $A/\beta(A[n]) \cong \mathbb{Z}^{2n+1}$. □

3. Corollaries

For A the group considered in Section 2, let $B = A \oplus \mathbb{Z}$, so $A \not\cong B$ by Theorem 2.1. Using the fact that $A \cong A \oplus A$, it is easy to see that B provides an example of Corner's phenomenon.

Theorem 3.1. *$B \cong B \oplus B \oplus B$, but $B \not\cong B \oplus B$.*

Proof. The map $(\underline{a}, \underline{b}) \mapsto (a_0, b_0, a_1, b_1, \ldots)$ is an isomorphism of abelian groups $A \oplus A \to A$, so

$$B \oplus B \cong A \oplus \mathbb{Z} \oplus A \oplus \mathbb{Z} \cong A \oplus A \cong A \not\cong B.$$

However,

$$B \oplus B \oplus B \cong B \oplus A \cong A \oplus \mathbb{Z} \oplus A \cong A \oplus \mathbb{Z} = B.$$

□

Consequently, A and B also answer Kaplansky's first two test problems: each is a direct summand of the other, and $A \oplus A \cong B \oplus B$.

4. Variants

Now let $n > 1$ and let A be the group of bounded sequences of elements of $\mathbb{Z}[\sqrt[n]{2}]$.

A straightforward adaptation of the proof of our main theorem shows that $A \cong A \oplus \mathbb{Z}^n$ but $A \not\cong A \oplus \mathbb{Z}^m$ for $0 < m < n$. Eklof and Shelah [ES87] also give such an example.

Also, Theorem 3.1 generalizes to show that $B = A \oplus \mathbb{Z}$ is isomorphic to the direct sum of $n + 1$ copies of itself, but not to the direct sum of $m + 1$ copies if $0 < m < n$.

References

[Bra15] Martin Brandenburg, *A is isomorphic to $A \oplus \mathbb{Z}^2$, but not to $A \oplus \mathbb{Z}$*, MathOverflow, https://mathoverflow.net/q/218113 (version: 2015-12-02).

[Coh56] P. M. Cohn, *The complement of a finitely generated direct summand of an abelian group*, Proc. Amer. Math. Soc. **7** (1956), 520–521.

[Cor64] A. L. S. Corner, *On a conjecture of Pierce concerning direct decompositions of Abelian groups*, Proc. Colloq. Abelian Groups (Tihany, 1963), Akadémiai Kiadó, Budapest, 1964, pp. 43–48.

[ES87] Paul C. Eklof and Saharon Shelah, *On groups A such that $A \oplus \mathbf{Z}^n \cong A$*, Abelian group theory (Oberwolfach, 1985), Gordon and Breach, New York, 1987, pp. 149–163.

[Jón57] Bjarni Jónsson, *On direct decompositions of torsion-free abelian groups*, Math. Scand. **5** (1957), 230–235.

[Kap54] Irving Kaplansky, *Infinite abelian groups*, University of Michigan Press, Ann Arbor, 1954.

[Łoś54] Jerzy Łoś, *On the complete direct sum of countable abelian groups*, Publ. Math. Debrecen **3** (1954), 269–272 (1955).

[Sąs59] E. Sąsiada, *Proof that every countable and reduced torsion-free Abelian group is slender*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 143–144.

[Sąs61] ______, *Negative solution of I. Kaplansky's first test problem for Abelian groups and a problem of K. Borsuk concerning cohomology groups*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **9** (1961), 331–334.

[Spe50] Ernst Specker, *Additive Gruppen von Folgen ganzer Zahlen*, Portugaliae Math. **9** (1950), 131–140.

[Wal56] Elbert A. Walker, *Cancellation in direct sums of groups*, Proc. Amer. Math. Soc. **7** (1956), 898–902.

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