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VALUES OF HARMONIC WEAK MAASS FORMS ON HECKE ORBITS

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ABSTRACT. Let $q := e^{2\pi iz}$, where $z \in \mathbb{H}$. For an even integer k, let $f(z) := q^h \prod_{m=1}^{\infty}(1-q^m)^{c(m)}$ be a meromorphic modular form of weight k on $\Gamma_0(N)$. For a positive integer m, let T_m be the mth Hecke operator and D be a divisor of a modular curve with level N. Both subjects, the exponents $c(m)$ of a modular form and the distribution of the points in the support of $T_m.D$, have been widely investigated.

When the level N is one, Bruinier, Kohnen, and Ono obtained, in terms of the values of j-invariant function, identities between the exponents $c(m)$ of a modular form and the points in the support of $T_m.D$. In this paper, we extend this result to general $\Gamma_0(N)$ in terms of values of harmonic weak Maass forms of weight 0. By the distribution of Hecke points, this applies to obtain an asymptotic behaviour of convolutions of sums of divisors of an integer and sums of exponents of a modular form.

1. Introduction

Let $\mathbb{H}$ be the complex upper half plane. For a positive integer N, let $Y_0(N)$ be the modular curve of level N defined by $\Gamma_0(N)\backslash\mathbb{H}$, and $X_0(N)$ denote the compactification of $Y_0(N)$ by adjoining the cusps. Let $J_0(N)$ be the jacobian of a modular curve $X_0(N)$. We denote by $Div(C)$ the divisor group of a curve C. If f is a function on C and $D = \sum_{P\in C} n_P P$ is a divisor of C, we define

$$f(D) := \sum n_P f(P).$$

The mth normalized Hecke operator T_m acts on $Div(Y_0(N))$, and it is denoted by $T_n.D$ for $D \in Div(Y_0(N))$. We call $T_m.D$ *the mth Hecke orbit of* D. Especially, when D is a divisor corresponding to $i \in \mathbb{H}$, a point in the support of $T_m.D$ is called *a Hecke point*. Hecke points have been investigated from several perspectives such as their distribution on the fundamental domain for $\Gamma_0(N)$ [13, 14, 17, 16] and the rank of a subgroup of $J_0(N)$ generated by Hecke points [21], and so on. Let $q := e^{2\pi iz}$, where $z \in \mathbb{H}$. For an even integer k, let $f(z) := q^h \prod_{m=1}^{\infty}(1-q^m)^{c(m)}$ be a meromorphic modular form of weight k on $\Gamma_0(N)$. The exponents $c(m)$ of a modular form were investigated in various works (for examples, see [4, 5, 23]). For example, Borcherds [5] proved that if f has a Heegner divisor, then the mth exponent $c(m)$ is the m^2th coefficient of a fixed modular form of half integral weight. Bruinier, Kohnen, and Ono [9] obtained a connection between these exponents of a modular form and the points in the support of $T_n.D$.

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For the modular invariant j, let $J(z) := j(z) - 744$. For positive integers k and m, let $\sigma_k(m) := \sum_{d|m} d^k$, and $\sigma_f(m) := \sum_{d|m} c(d)$. Bruinier, Kohnen, and Ono [9] proved the following identities between values $J(T_m.D_f)$ and sum of exponents in the product expansion of f:

$$\sum_{d|m} c(d)d = 2k\sigma_1(m) + J(T_m.D_f)$$

for every positive integer m, where D_f denotes the divisor of f on $X_0(N)$. In other words, the value $J(T_m.D_f)$ can be expressed as the sum of the following values:

(1) a multiple of the divisor function $\sigma_1(m)$,
(2) the convolution of $\sigma_1(m)$ (sum of divisors) and $\sigma_f(m)$ (sum of exponents).

They applied this result to prove the modularity of the generating series for $\sigma_f(m)$ and to obtain several p-adic properties of $J(T_m.D_f)$ and exponents of a meromorphic modular form f. Based on the argument in [9], the result was extended to several cases such as $\Gamma_0(N)$ with genus zero by Ahlgren [2], Jacobi forms by Choie and Kohnen [12], and higher levels by the first author [11].

For general positive integers N, the first author studied in [11] the generalization of [9] to a harmonic weak Maass form $J_{N,1}$ of weight 0 defined as a Poincaré series (instead of a weakly holomorphic modular form of weight 0). It was proved in [11] that the value $J_{N,1}(T_m.D_f)$ can be expressed as the sum of the following values:

(1) a linear combination of the divisor functions $\sigma_1(nm)$ for $n|N$,
(2) the convolution of $\sigma_1(m)$ (sum of divisors) and $\sigma_f(m)$ (sum of exponents),
(3) the regularized Petersson inner product $R_{f,N}(m)$ of a meromorphic modular form and a cusp form.

In this paper, we show that $R_{f,N}(m)$, the value of the regularized Petersson inner product in identities [11], is zero, and so we give explicit identities between values $J_{N,1}(T_m.D_f)$ and sums of exponents in the product expansion of f. As an application, we obtain an asymptotic behavior for the convolution of $\sigma_1(m)$ (sum of divisors) and $\sigma_f(m)$ (sum of exponents) as $m \to \infty$.

Recently, Bringmann, Kane, Löbrich, Ono, and Rolen [7] showed that for any fixed N the generating series for $J_{N,1}(T_m.D_f)$ is basically modular. Moreover, their result implies that there is a cusp form such that, for each m, $R_{f,N}(m)$, the value of regularized Petersson inner product, is given by the mth coefficient of a fixed cusp form.

Let $\mathcal{F}_1$ denote the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ given by

$$\mathcal{F}_1 := \left\{ z \in \mathbb{H} \;\middle|\; |z| > 1,\ -\frac{1}{2} \le \mathrm{Re}(z) < \frac{1}{2} \right\} \cup \left\{ z \in \mathbb{H} \;\middle|\; |z| = 1,\ \mathrm{Re}(z) \le 0 \right\}$$

and

$$\mathcal{F}_N := \bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash \Gamma_0(N)} \gamma \mathcal{F}_1.$$

Here we choose coset representatives for $\mathrm{SL}_2(\mathbb{Z}) \backslash \Gamma_0(N)$ such that

$$\mathcal{F}_N \subset \left\{ z \in \mathbb{H} \;\middle|\; |\mathrm{Re}(z)| \le \frac{1}{2} \right\}.$$

Then, $\mathcal{F}_N$ is a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$. Let $\mathcal{C}_N$ be the set of inequivalent cusps of $\Gamma_0(N)$. Let k be an even integer and f be a meromorphic modular form of weight k on

$\Gamma_0(N)$. For $\tau \in \mathbb{H}\cup\{i\infty\}\cup\mathbb{Q}$, let Q_τ be the image of τ under the canonical map from $\mathbb{H}\cup\{i\infty\}\cup\mathbb{Q}$ to $X_0(N)$. For $\tau \in \mathbb{H}\cup\{i\infty\}\cup\mathbb{Q}$, we denote by $\nu_\tau^{(N)}(f)$ the order of zero of f at Q_τ on $X_0(N)$. Let us note

$$g(T_m.D_f) = \sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) g(T_m.\tau).$$

Moreover, for a divisor $D = \sum n_z Q_z$ of $X_0(N)$, we can give a more explicit expression of $T_m.D$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ with positive determinant, we define the action of γ for $z \in \mathbb{H}$ by

$$\gamma z := \frac{az+b}{cz+d}.$$

For a positive integer m prime to N, let

$$T(m) := \{\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mid a,b,d \in \mathbb{Z},\ a>0,\ ad=m,\ \text{and } 0\le b<d\}.$$

Then, we have

$$T_m.D = \sum n_z \sum_{\gamma\in T(m)} Q_{\gamma z}.$$

Next, we define the Ramanujan theta-operator by

$$\theta(f)(z) := \frac{1}{2\pi i}\frac{d}{dz} f(z).$$

Let

$$f_\theta(z) := \frac{\theta f(z)}{f(z)} - \frac{k}{12}E_2(z),$$

where E_2 is the usual normalized Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$.

Let $N>1$ and I_v be the usual modified Bessel functions as in [1]. For a positive integer n, we define the Poincaré series of weight 0 and index n by

$$F_{N,n}(z,s) := \sum_{\gamma\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)} \pi\left|n\mathrm{Im}(\gamma z)\right|^{1/2} I_{s-\frac{1}{2}}(|2\pi n \mathrm{Im}(\gamma z)|)e(-n\mathrm{Re}(\gamma z)),$$

where $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ and $e(z) := e^{2\pi i z}$. Let $j_{N,n}(z)$ be the continuation of $F_{N,n}(z,s)$ as $s\to 1$ from the right. Then, the function $j_{N,n}$ is a harmonic weak Maass form of weight 0 on $\Gamma_0(N)$ (see [11, Section 2] for details). Let $J_{N,n}(z) := j_{N,n}(z) - \beta_{N,n}$, where $\beta_{N,n}$ is the constant term of the Fourier expansion of $j_{N,n}$ at the cusp $i\infty$.

For square-free N, let $D(N)$ be the number of divisors of N, and $\{d_1, d_2, \ldots, d_{D(N)-1}, N\}$ be the set of distinct divisors of N such that $d_{i_1} < d_{i_2}$ if $i_1 < i_2$. Let A_N be the $(D(N)-1)\times(D(N)-1)$ matrix whose ij-entry a_{ij} is defined by

$$a_{ij} = \left(1 - \frac{\gcd(d_i,d_j)^2}{d_j}\right).$$

Let $A_{f,j}$ be a matrix obtained from A_N by replacing the jth column of A_N with a column matrix whose ith component is $\nu_{1/d_i}^{(N)}(f) - \frac{k}{12}$. With this notation, we state our main theorem.

Theorem 1.1. *Let k be an even integer and $N > 1$ be a positive integer. Suppose that*

$$f(z) = q^{h_\infty} \prod_{n=1}^{\infty} (1-q^n)^{c(n)} \tag{1.1}$$

is a meromorphic modular form of weight k on $\Gamma_0(N)$. Then

$$-\sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) - \sum_{m=1}^{\infty} \left(\sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) \right) q^m = f_\theta(z) - \mathcal{E}_2(z),$$

where $\mathcal{E}_2$ is a modular form in the Eisenstein space of weight 2 on $\Gamma_0(N)$. Moreover, if N is square free, then, for every positive integer m prime to N,

$$-J_{N,1}(T_m.D_f) = \sum_{d|m} dc(d) + 24 \left(\sum_{1 \le j \le D(N)-1} \frac{\det(A_{f,j})}{\det(A_N)} + \frac{k}{12} \right) \sigma_1(m). \tag{1.2}$$

Remark 1.2. *The modular form $\mathcal{E}_2$ in Theorem 1.1 is determined by the order of zero or pole of f at each cusp. In many cases, a modular form $\mathcal{E}_2$ can be expressed as a sum of explicit modular forms. For example, if N is square free, then*

$$\mathcal{E}_2(z) = \sum_{1 \le j \le D(N)-1} \frac{\det(A_{f,j})}{\det(A_N)} (E_2(z) - d_j E_2(d_j z)).$$

Let $D := \sum_{z \in S} n_z Q_z$ be a divisor of $Y_0(N)$, where S is a finite set in $\mathcal{F}_N$. For a positive real number $r \ge 1$, we define a divisor $D_{>r}$ by

$$D_{>r} = \sum_{\substack{z \in S \\ \mathrm{Im}(\tilde{z}) > r}} n_z Q_{\tilde{z}}.$$

Here, $\tilde{z}$ is a complex number in $\mathcal{F}_N$, which is equivalent to z under the action of $\Gamma_0(N)$. By the argument of Duke [15] and equidistribution of Hecke points ([17], [13] and [14]), Theorem 1.1 implies the following theorem.

Theorem 1.3. *Let k, N, and f be given as in Theorem 1.1. Assume that N is square free. Let m be a positive integer prime to N, and h_f denote the sum of the orders of zero or pole of f at Q_τ on $Y_0(N)$. Then*

$$\lim_{m \to \infty} \frac{1}{\sigma_1(m)} \left(24 \left(\sum_{1 \le j \le D(N)-1} \frac{\det(A_{f,j})}{\det(A_N)} + \frac{k}{12} \right) \sigma_1(m) - \sum_{d|m} dc(d) - e\left(-\left(T_m.D_f\right)_{>1} \right) \right)$$
$$= \frac{3h_f}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \lim_{\epsilon \to 0} \int_{\mathcal{F}_N(\epsilon)} J_{N,1}(z) \frac{dxdy}{y^2},$$

where $c(n)$ are complex numbers determined by (1.1). Here, $\mathcal{F}_N(\epsilon)$ is defined by $\mathcal{F}_N - \cup_{\tau \in \mathcal{C}_N} B_\tau(\epsilon)$, where $B_\tau(\epsilon)$ is given in (3.1).

Recently, Ali and Mani [3] proved an upper bound for exponents $c(m)$ in the product expansion of f. The sum $\sum_{d|m} dc(d)$ looks like a kind of convolution of $\sigma_1(m)$ (a sum of divisors) and $\sigma_f(m)$

(a sum of exponents of f). The above inequality means that, as $m \to \infty$, this convolution has a similar asymptotic behavior as that of the sum of divisors of m except its main term.

The remainder of the paper is organized as follows. In Section 2, we introduce some preliminaries for meromorphic 1-forms on $X_0(N)$. In Section 3, we provide some basic facts on regularized Petersson inner product, and prove that f_θ is orthogonal to every cusp form of weight 2 on $\Gamma_0(N)$ with respect to regularized Petersson inner product if f is a meromorphic modular form on $\Gamma_0(N)$. In Section 4, we recall some results related to the distribution of Hecke points for $\Gamma_0(N)$. In Section 5, we prove our main theorems: Theorems 1.1 and 1.3.

2. Residues of a meromorphic 1-form on $X_0(N)$

Let f be a meromorphic modular form of weight 2 on $\Gamma_0(N)$. Assume that t is a cusp of $\Gamma_0(N)$. Let $\sigma_t \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix such that $\sigma_t(i\infty) = t$, and $\Gamma_0(N)_t$ denote the stabilizer of the cusp t in $\Gamma_0(N)$. We define a positive integer α_t by

$$\sigma_t^{-1}\Gamma_0(N)_t\sigma_t = \left\{\pm\left(\begin{smallmatrix} 1 & \ell\alpha_t \\ 0 & 1\end{smallmatrix}\right) \;:\; \ell \in \mathbb{Z}\right\},$$

and we call α_t the width of $\Gamma_0(N)$ at the cusp t. The Fourier expansion of f at the cusp t is given by

$$(f|_2\sigma_t)(z) = \sum a_t(n)q^{n/\alpha_t},$$

where $|_k$ denotes the usual weight k slash operator. If a cusp t is equivalent to $i\infty$, the Fourier coefficients $a_t(n)$ of f at the cusp t are simply denoted by $a(n)$.

For $\tau \in \mathbb{H} \cup \{i\infty\} \cup \mathbb{Q}$, let Q_τ be the image of τ under the canonical map from $\mathbb{H} \cup \{i\infty\} \cup \mathbb{Q}$ to $X_0(N)$. Then, fdz can be considered as a meromorphic 1-form on $X_0(N)$. Thus, we denote by $\mathrm{Res}_{Q_\tau} fdz$ the residue of f at Q_τ on $X_0(N)$. Let $\mathrm{Res}_\tau f$ be the residue of f at τ on $\mathbb{H}$. The description of $\mathrm{Res}_{Q_\tau} fdz$ is given in terms of $\mathrm{Res}_\tau f$. For $\tau \in \mathbb{H}$, let e_τ be the order of the isotropy subgroup of $\Gamma_0(N)$ at τ. Then, we have

$$\mathrm{Res}_{Q_\tau} f\, dz = \begin{cases} \frac{1}{e_\tau}\mathrm{Res}_\tau f, & \text{if } \tau \in \mathbb{H}, \\ \frac{1}{2\pi i}\alpha_\tau a_\tau(0), & \text{if } \tau \in \mathcal{C}_N. \end{cases} \tag{2.1}$$

Let us note that if k is an even integer and f is a meromorphic modular form of weight k on $\Gamma_0(N)$, then f_θ is a meromorphic modular form of weight 2 on $\Gamma_0(N)$. The residue of f_θ at each point on $X_0(N)$ is determined by the order of its zero or pole of f at that point. Let $\mathrm{ord}_\tau(f)$ be the order of the zero or pole of f at τ on $\mathbb{H}$. Since we have

$$(cz+d)^{-2}E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) + \frac{12}{2\pi i}\cdot\frac{c}{cz+d}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we obtain

$$(f_\theta|_2\sigma_t)(z) = \frac{\theta(f|_k\sigma_t)(z)}{(f|_k\sigma_t)(z)} - \frac{k}{12}E_2(z)$$

for a cusp t. Thus, we have

$$\mathrm{Res}_{Q_\tau} f_\theta dz = \begin{cases} \frac{1}{2\pi i}\nu_\tau^{(N)}(f) & \text{if } \tau \in \mathbb{H}, \\ \frac{\alpha_\tau}{2\pi i}\left(\nu_\tau^{(N)}(f) - \frac{k}{12}\right) & \text{if } \tau \in \mathcal{C}_N. \end{cases} \tag{2.2}$$

3. Regularized Petersson inner product

Petersson defined an inner product of two cusp forms with the same weight. The Petersson inner product was extended by Borcherds [6] to the case in which one of the two forms is a weakly holomorphic modular form. In this section, following [6] and [11], we define regularized Petersson inner product of a cusp form and a meromorphic modular form with the same weight. We prove that if f is a meromorphic modular form on $\Gamma_0(N)$, then the regularized Petersson inner product of f_θ with any cusp form of weight 2 on $\Gamma_0(N)$ is zero.

Let k be an even integer and f be a meromorphic modular form of weight k on $\Gamma_0(N)$. Let $\mathrm{Sing}(f)$ be the set of singular points of f on $\mathcal{F}_N$. For a positive real number ε, an ε-disk $B_\tau(\varepsilon)$ at τ is defined by

$$B_\tau(\varepsilon) := \begin{cases} \{z \in \mathbb{H} \ : \ |z-\tau| < \varepsilon\}, & \text{if } \tau \in \mathbb{H}, \\ \{z \in \mathcal{F}_N \ : \ \mathrm{Im}(\sigma_\tau z) > 1/\varepsilon\}, & \text{if } \tau \in \{i\infty\} \cup \mathbb{Q}. \end{cases} \tag{3.1}$$

Let $\mathcal{F}_N(f,\varepsilon)$ be a punctured fundamental domain for $\Gamma_0(N)$ defined by

$$\mathcal{F}_N(f,\varepsilon) := \mathcal{F}_N - \bigcup_{\tau \in \mathrm{Sing}(f) \cup \mathcal{C}_N} B_\tau(\varepsilon).$$

Let g be a cusp form of weight k on $\Gamma_0(N)$. The regularized Petersson inner product $(f,g)_{reg}$ of f and g is defined by

$$(f,g)_{reg} := \lim_{\varepsilon \to 0} \int_{\mathcal{F}_N(f,\varepsilon)} f(z)\overline{g(z)} \frac{dxdy}{y^{k-2}}.$$

Then, we have the following proposition.

Proposition 3.1. *Let k be an even integer, and f be a meromorphic modular form of weight k on $\Gamma_0(N)$. Then, for every cusp form g of weight 2 on $\Gamma_0(N)$,*

$$(f_\theta, g)_{reg} = 0.$$

Proof. Let $\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$ be the unique normalized cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. Let

$$F(z) := \frac{f(z)^{12}}{\Delta(z)^k}.$$

Then, we have

$$d((\log_e |F(z)|^2)\overline{g(z)}d\overline{z}) = \frac{\partial_z F(z)\overline{F(z)}}{F(z)\overline{F(z)}}\overline{g(z)}dzd\overline{z} = \frac{\partial_z F(z)}{F(z)}\overline{g(z)}(-2i)dxdy.$$

Let us note that Δ has no zeros and no poles on $\mathbb{H}$. Therefore, according to [9, Theorem 1], we have

$$\frac{\theta(\Delta)}{\Delta} = E_2.$$

The function $\partial_z F(z)/F(z)$ is given as

$$\frac{\partial_z F(z)}{F(z)} = 12\frac{\partial_z f(z)}{f(z)} - k\frac{\partial_z \Delta(z)}{\Delta(z)} = 12\frac{\partial_z f(z)}{f(z)} - k(2\pi i)E_2(z) = (24\pi i)f_\theta(z).$$

Thus, we have

$$d((\log_e |F(z)|^2)\overline{g(z)}d\overline{z}) = (48\pi) f_\theta(z)\overline{g(z)}dxdy. \tag{3.2}$$

In order to apply the Stokes theorem, we describe the boundary of $\mathcal{F}_N$. For a positive real number ε, we define

$$\gamma_\tau(\varepsilon) := \begin{cases} \{z \in \mathbb{H} \ : \ |z-\tau| = \varepsilon\} & \text{if } \tau \in \mathbb{H}, \\ \{z \in \mathcal{F}_N \ : \ \mathrm{Im}(\sigma_\tau z) = 1/\varepsilon\} & \text{if } \tau \in \{i\infty\} \cup \mathbb{Q}. \end{cases}$$

Assume that ε is sufficiently small. If $\partial^* \mathcal{F}_N(f,\varepsilon)$ denotes the closure of the set $\partial\mathcal{F}_N(f,\varepsilon) - \partial\mathcal{F}_N$ in $\mathbb{C}$, then

$$\partial^* \mathcal{F}_N(f,\varepsilon) = \bigcup_{\tau \in \mathrm{Sing}(f) \cup \mathcal{C}_N} \gamma_\tau(\varepsilon), \tag{3.3}$$

where ∂D denotes the boundary of D for a subset D of $\mathbb{C}$. From (3.2) and (3.3), the Stokes theorem implies

$$\begin{aligned} \int_{\mathcal{F}_N(f_\theta,\varepsilon)} f_\theta(z)\overline{g(z)}dxdy &= \int_{\partial^*\mathcal{F}_N(f_\theta,\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)\overline{g(z)}d\overline{z} \\ &= \sum_{\tau \in \mathrm{Sing}(f_\theta)\cup\mathcal{C}_N} \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)\overline{g(z)}d\overline{z}. \end{aligned}$$

For each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the absolute value $|(g|_2\gamma)(z)|$ exponentially decays as $\mathrm{Im}(z) \to \infty$, since g is a cusp form. Thus, if $\tau \in \mathcal{C}_N$, then $\lim_{\varepsilon\to 0} \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)\overline{g(z)}d\overline{z} = 0$.

To complete the proof, we assume that $\tau \in \mathrm{Sing}(f_\theta)$. Then

$$\begin{aligned} &\left| \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)\overline{g(z)}d\overline{z} \right| \\ &\leq \int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}|(\log_e |F(z)|^2)||\overline{g(z)}||d\overline{z}| \\ &\leq \max\{|(\log_e |F(z)|^2)| \ : \ z \in \gamma_\tau(\varepsilon)\} M_1 \int_{\gamma_\tau(\varepsilon)} |d\overline{z}| \ \text{ (some constant } M_1) \\ &\leq \max\{|(\log_e |F(z)|^2)| \ : \ z \in \gamma_\tau(\varepsilon)\} M_1 (2\pi\varepsilon). \end{aligned}$$

The function $F(z)$ can be expressed around τ as

$$F(z) = (z-\tau)^{12\nu_\tau^{(N)}(f)} F_0(z),$$

where $F_0(z)$ is a nowhere vanishing holomorphic function around τ. If ε is sufficiently small, then, for any $z \in \gamma_\tau(\varepsilon)$ we have

$$\begin{aligned} |(\log_e |F(z)|^2)| &\leq |(\log_e |(z-z_0)|^{24\nu_\tau^{(N)}(f)})| + |(\log_e |F_0(z)|^2)| \\ &\leq |(\log_e |(z-z_0)|^{24\nu_\tau^{(N)}(f)})| + M_2 \ \text{ (some fixed constant } M_2) \\ &= |24\nu_\tau^{(N)}(f) \log_e \varepsilon| + M_2. \end{aligned}$$

Thus, for sufficiently small ε, we have

$$\left|\int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)\overline{g(z)}d\overline{z}\right| \leq (|24\nu_\tau^{(N)}(f)\log_e \varepsilon| + M_2)M_1(2\pi\varepsilon).$$

This implies that, for $\tau \in \mathrm{Sing}(f_\theta)$,

$$\lim_{\varepsilon\to 0}\int_{\gamma_\tau(\varepsilon)} \frac{1}{48\pi}(\log_e |F(z)|^2)\overline{g(z)}d\overline{z} = 0.$$

Thus, we complete the proof. □

4. Equidistribution of Hecke points

Let $\{u_j\}_{j\geq 0}$ be an orthonormal basis of the residual and cuspidal spaces of $L^2(\Gamma_0(N)\backslash\mathbb{H})$, i.e., u_0 is a constant with the eigenvalue $\lambda_0 = 0$ and u_j is a Maass form for $\Gamma_0(N)$ with eigenvalue $\lambda_j = s_j(1-s_j)$ for $j \geq 1$. Further, assume that λ_j are ordered so that $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. For each cusp $t \in \mathbb{Q}\cup\{\infty\}$, let $E_t(z,s)$ be the Eisenstein series at t for $\mathrm{Re}(s) > 1$, which is given by

$$E_t(z,s) = \sum_{\gamma\in\Gamma_0(N)_t\backslash\Gamma_0(N)} (\mathrm{Im}(\sigma_t^{-1}\gamma z))^s.$$

Here, $\Gamma_0(N)_t \subset \Gamma_0(N)$ is the stability group of t. For the properties of $E_t(z,s)$, see [19, §15].

According to [19, Theorem 15.5], any $f \in L^2(\Gamma_0(N)\backslash\mathbb{H})$ has the spectral decomposition

$$f(z) = \sum_{j\geq 0} \langle f, u_j\rangle\, u_j(z) + \sum_{t\in\mathcal{C}_N} \frac{1}{4\pi}\int_{\mathbb{R}} \langle f, E_t(*, 1/2+ir)\rangle\, E_t(z, 1/2+ir)\, dr$$

(valid in L^2-sense) and converges absolutely and uniformly on compact sets if f and Δf are smooth and bounded.

We now follow the proof of [17, Theorem 3.1]. Let

$$\begin{aligned}
f_C &= \langle f, u_0\rangle = \text{ the projection of } f \text{ onto the constant subspace},\\
f_M(z) &= \sum_{j\geq 1} \langle f, u_j\rangle\, u_j(z),\\
f_E(z) &= \sum_{t\in\mathcal{C}_N} \frac{1}{4\pi}\int_{\mathbb{R}} \langle f, E_t(*, 1/2+ir)\rangle\, E_t(z, 1/2+ir)\, dr.
\end{aligned}$$

Note that

$$f_C = \langle f, u_0\rangle = \int_{\mathcal{F}_N} f(z)\, d\mu(z), \tag{4.1}$$

where $d\mu(z) := \frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]}\cdot\frac{dxdy}{y^2}$ is the normalized Haar measure; so, $\int_{\mathcal{F}_N} d\mu(z) = 1$.

Let $\lambda_j(n)$ be the nth Fourier coefficient of u_j. By the Ramanujan conjecture, there exists $\theta \geq 0$ such that $|\lambda_j(n)| \leq cn^{\theta+\epsilon}$, for any $\epsilon > 0$. So, we get

$$\frac{1}{\sigma_1(n)}\|T_n f_M\|_2 \leq cn^{-\frac{1}{2}+\theta+\epsilon}\|f_M\|_2. \tag{4.2}$$

Note that the value of θ has been lowered to $\frac{7}{64}$ by Kim and Sarnak [20, Appendix 2].

In [22, §6, §7 and §8], an explicit change-of-basis formula between the Eisenstein series attached to cusps and newform Eisenstein series attached to pairs of primitive Dirichlet characters is described. The Eisenstein series attached to a Dirichlet character is an eigenfunction of Hecke operators T_n for $\gcd(n, N) = 1$, and the absolute values of the corresponding eigenvalues are bounded above by $\sigma_0(n)n^{-\frac{1}{2}}$. So, we get

$$\frac{1}{\sigma_1(n)}\|T_n f_E\|_2 \leq c n^{-\frac{1}{2}+\epsilon}\|f_E\|_2. \tag{4.3}$$

If we combine (4.1), (4.2), and (4.3), then we obtain the following theorem. For more general result, see [13].

Theorem 4.1. *Let $f \in L^2(\Gamma_0(N)\backslash\mathbb{H})$. For a positive integer n prime to N, we have*

$$\left\|\frac{1}{\sigma_1(n)}T_n f - \int_{\mathcal{F}_N} f(z)\, d\mu(z)\right\|_2 \leq c_\epsilon n^{-\frac{1}{2}+\theta+\epsilon}\|f\|_2$$

for any $\epsilon > 0$. The constant c_ϵ depends on ϵ.

The pointwise convergence can be derived from [14, Proposition 8.2]. Note that elliptic differential operators are differential operators that generalize the Laplace-Beltrami operator Δ. For an integer $m \geq 2$, assume that $f, \Delta^m f \in L^2(\Gamma_0(N)\backslash\mathbb{H})$. Then, by [14, Proposition 8.2], for a compact subset $\omega \subset \mathcal{F}_N$, there exist constants $C_1(\omega)$ and $C_2(\omega)$ such that, for any $z_0 \in \omega$

$$\begin{aligned}&\left|\frac{1}{\sigma_1(n)}T_n f(z_0) - \int_{\mathcal{F}_N} f(z)\, d\mu(z)\right| \\ &\quad\leq C_1(\omega)\left\|\frac{1}{\sigma_1(n)}T_n f - \int_{\mathcal{F}_N} f(z)\, d\mu(z)\right\|_2 + C_2(\omega)\left\|\frac{1}{\sigma_1(n)}T_n(\Delta^m f) - \int_{\mathcal{F}_N} (\Delta^m f)(z)\, d\mu(z)\right\|_2.\end{aligned}$$

So, we have the following corollary.

Corollary 4.2. *Assume that $f, \Delta^2 f \in L^2(\Gamma_0(N)\backslash\mathbb{H})$. Take a compact $\omega \subset \Gamma_0(N)\backslash\mathbb{H}$ and a positive number ϵ. Then, there exists a constant $C_{\omega,\epsilon}$ depending on ω and ϵ, such that, for a positive integer n prime to N, for any $z_0 \in \omega$,*

$$\left|\frac{1}{\sigma_1(n)}T_n f(z_0) - \int_{\mathcal{F}_N} f(z)\, d\mu(z)\right| \leq C_{\omega,\epsilon} n^{-\frac{1}{2}+\theta+\epsilon}\max\{\|f\|_2, \|\Delta^2 f\|_2\}.$$

5. Proofs

Let $M_k^{Eis}(\Gamma_0(N))$ be the space of modular forms orthogonal to all the cusp forms of weight k on $\Gamma_0(N)$, which is called the Eisenstein space of weight k on $\Gamma_0(N)$. In the following lemma, we prove that if N is square-free, then, for a positive integer n prime to N, the nth coefficient of a modular form in $M_2^{Eis}(\Gamma_0(N))$ is a multiple of $\sigma_1(n)$. Recall the notations $D(N)$, d_j, and A_N from Section 1. Now, we prove the following lemma related to properties for modular forms in an Eisenstein space.

Lemma 5.1. *Suppose that $\mathcal{E}_2(z) := \sum_{n=0}^\infty b(n)q^n$ is a modular form in $M_2^{Eis}(\Gamma_0(N))$, and that N is square free. Then, the following statements are true.*

(1) *There exists a constant* c *such that for every positive integer* n *prime to* N,

$$b(n) = c\sigma_1(n).$$

(2) *Assume that the constant term of* $\mathcal{E}_2(z)$ *at cusp* $1/d_i$ *is* c_{d_i}. *Let* A_j *be the matrix obtained from* A *by replacing the* j*th column of* A *with a column matrix whose* i*th component is* c_{d_i}. *Then*

$$\mathcal{E}_2(z) = \sum_{1\le j\le D(N)-1} \frac{\det(A_j)}{\det(A_N)}(E_2(z) - d_jE_2(d_jz)).$$

Proof. (1) We claim that there is a basis of $M_2^{Eis}(\Gamma_0(N))$ consisting of modular forms $E_2(z) - dE_2(dz)$, where $d \neq 1$ are the divisors of N. Assume that the claim is true. Then, $\mathcal{E}_2(z)$ can be expressed as a linear combination of $E_2(z) - d_jE_2(d_jz)$ having the form

$$\mathcal{E}_2(z) = \sum_{1\le j\le D(N)-1} a_j(E_2(z) - d_jE_2(d_jz)).$$

Recall that E_2 has the Fourier expansion of the form

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n. \tag{5.1}$$

Then, the nth coefficient of $\mathcal{E}_2(z)$ is given by

$$-24 \sum_{1\le j\le D(N)-1} a_j(\sigma_1(n) - d_j\sigma_1(n/d_j))$$

for $n > 0$, and a_j does not depend on n. Here, $\sigma_1(n/d) = 0$ if n is not divisible by d. Thus, we have the proof of the lemma.

Now, we prove the claim. Suppose that

$$\sum_{1\le j\le D(N)-1} a_j(E_2(z) - d_jE_2(d_jz)) = 0,$$

where a_j are complex numbers. We assume that complex numbers a_j are not all zero. Then, we have

$$\sum_{1\le j\le D(N)-1} a_jE_2(z) = \sum_{1\le j\le D(N)-1} a_jd_jE_2(d_jz).$$

Comparing the nth coefficients of the forms on both sides for n prime to N, we have

$$\sum_{1\le j\le D(N)-1} a_jE_2(z) = \sum_{1\le j\le D(N)-1} a_jd_jE_2(d_jz) = 0.$$

Take the smallest positive integer $d_{j_0}|N$ such that $a_{j_0} \neq 0$. Then, we have

$$-a_{j_0}d_{j_0}E_2(d_{j_0}z) = \sum_{1\le j\le D(N)-1} a_jd_jE_2(d_jz) - a_{j_0}d_{j_0}E_2(d_{j_0}z).$$

Comparing the d_{j_0}th coefficients of the forms on both sides, we have $a_{j_0} = 0$. This is a contradiction. Therefore, the modular forms $E_2(z) - dE_2(dz)$, $d|N$ and $d \neq 1$, are linearly independent.

Let us note

$$\dim_{\mathbb{C}} M_2^{Eis}(\Gamma_0(N)) = D(N) - 1.$$

since N is square free. Thus,

$$\{(E_2(z) - dE_2(dz) \ : \ d \mid N \text{ and } d \neq 1\}$$

is a basis of $M_2^{Eis}(\Gamma_0(N))$. This completes the proof of the claim.

(2) From the proof of (1), we may assume that

$$\mathcal{E}_2(z) = \sum_{1\leq j\leq D(N)-1} a_j(E_2(z) - d_jE_2(d_jz)).$$

Let us note that $E_2(z) - \frac{3}{\pi \mathrm{Im}(z)}$ is a non-holomorphic modular form of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$. By direct computation, there are $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\mu_j \in \mathbb{Z}$ such that

$$\begin{pmatrix} d_j & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d_i & 1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & \mu_j \\ 0 & d_j/\gcd(d_j,d_i) \end{pmatrix}\begin{pmatrix} \gcd(d_j,d_i) & 0 \\ 0 & 1 \end{pmatrix}$$

Thus,

$$(E_2(z) - d_jE_2(d_jz))|_2 \left(\begin{smallmatrix} 1 & 0 \\ d_i & 1 \end{smallmatrix}\right) = E_2(z) - \frac{\gcd(d_j,d_i)^2}{d_j} E_2\left(\frac{\gcd(d_j,d_i)^2}{d_j} z + \frac{\mu_j \gcd(d_j,d_i)}{d_j}\right).$$

This implies that a_j are the solution of the system

$$c_{d_i} = \sum_{1\leq j\leq D(N)-1} \left(1 - \frac{\gcd(d_j,d_i)^2}{d_j}\right) a_j$$

for $1 \leq i \leq D(N) - 1$. Thus, the Cramer's rule completes the proof. □

Now, we prove Theorem 1.1.

Proof. Note that

$$\{E_2(z) - dE_2(dz) \mid d \mid N, \ d \neq 1\}$$

forms a basis of $M_2^{Eis}(\Gamma_0(N))$ by the proof of Lemma 5.1. Therefore, we can take a modular form $\mathcal{E}_2 \in M_2^{Eis}(\Gamma_0(N))$ such that the constant term of $\mathcal{E}_2$ at each cusp except cusps equivalent to $i\infty$ is the same as that of f_θ. Suppose that $\mathcal{E}_2$ has the Fourier expansion of the form

$$\mathcal{E}_2(z) = \sum_{n=0}^{\infty} b(n)q^n.$$

Note that, by (2.1) and (2.2), we have

$$\frac{2\pi i}{e_\tau}\mathrm{Res}_\tau f_\theta = \nu_\tau^{(N)}(f)$$

for $\tau \in \mathcal{F}_N$. Thus, from [11, Lemma 3.1], we obtain

$$(f_\theta - \mathcal{E}_2, \xi_0(j_{N,m}))_{reg} = \beta_{N,m}(a_\theta(0) - b(0)) + a_\theta(m) - b(m) + \sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) j_{N,m}(\tau), \tag{5.2}$$

where $a_\theta(m)$ is the mth Fourier coefficient of f_θ and ξ_0 is a differential operator defined by

$$\xi_0(f)(z) := 2i\overline{\frac{\partial}{\partial \bar{z}} f(z)}.$$

By the same argument in the proof of [11, Lemma 3.1], we have

$$(f_\theta - \mathcal{E}_2, \xi_0(J_{N,m}))_{reg} = a_\theta(m) - b(m) + \sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau). \tag{5.3}$$

Note that $\xi_0(j_{N,m}) = \xi_0(J_{N,m})$ since $J_{N,m}(z) = j_{N,m}(z) - \beta_{N,m}$. Therefore, from (5.2) and (5.3), we have

$$a_\theta(0) - b(0) = \frac{1}{\beta_{N,m}} \left(\sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) - \sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) j_{N,m}(\tau) \right) = -\sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f). \tag{5.4}$$

Proposition 3.1 implies that

$$(f_\theta - \mathcal{E}_2, \xi_0(j_{N,m}))_{reg} = 0.$$

Therefore, from (5.3), we have

$$a_\theta(m) - b(m) = -\sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) \tag{5.5}$$

for every positive integer m. Thus, from (5.4) and (5.5), we obtain

$$f_\theta(z) - \mathcal{E}_2(z) = -\sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) - \sum_{m=1}^{\infty} \left(\sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) \right) q^m. \tag{5.6}$$

By (1.1) and the Fourier expansion of E_2 given in (5.1), f_θ has the Fourier expansion of the form

$$f_\theta(z) = h_\infty + \sum_{n=1}^{\infty} \sum_{d|n} dc(d) q^n - \frac{k}{12} + 2k \sum_{n=1}^{\infty} \sigma_1(n) q^n. \tag{5.7}$$

Let us note that the constant term of f_θ at cusp t is $\nu_t^{(N)}(f) - k/12$. Suppose that m is prime to N. Then, Lemma 5.1 implies that

$$b(m) = -24 \left(\sum_{1\le j\le D(N)-1} \frac{\det(A_{f,j})}{\det(A_N)} \right) \sigma_1(m). \tag{5.8}$$

Here, $A_{f,j}$ is a matrix obtained from A_N by replacing the jth column of A_N with a column matrix whose ith component is $\frac{\nu_{1/d_i}^{(N)}(f)}{\alpha_{1/d_i}} - \frac{k}{12}$. Let us note that if $\gcd(m,N) = 1$, then $J_{N,m} = J_{N,1}|T_m$. Therefore, by (5.5), (5.7), and (5.8), we have

$$\begin{aligned} -J_{N,1}(T_m.D_f) &= -J_{N,m}(D_f) = -\sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) J_{N,m}(\tau) = a_\theta(m) - b(m) \\ &= \sum_{d|m} dc(d) + 24 \left(\sum_{1\le j\le D(N)-1} \frac{\det(A_{f,j})}{\det(A_N)} + \frac{k}{12} \right) \sigma_1(m). \end{aligned}$$

□

To prove Theorem 1.3, we follow the argument of the proof of [15, Proposition 3]. We fix $\epsilon > 0$. Let $\psi_\epsilon : \mathbb{R}_{>0} \to \mathbb{R}$ be a C^∞ function with $0 \le \psi_\epsilon(y) \le 1$ for all $y \in \mathbb{R}_{>0}$ and

$$\psi_\epsilon(y) = \begin{cases} 0, & \text{if } y \le 1, \\ 1, & \text{if } y > 1+\epsilon. \end{cases}$$

For a positive integer n, consider the Poincaré series defined by

$$P_{n,\epsilon}(z) := \sum_{\gamma \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} \psi_\epsilon(\mathrm{Im}(\gamma z)) e(-n(\gamma z)). \tag{5.9}$$

From this, we obtain the following proposition.

Proposition 5.2. *Let θ be given as in Section 4. Fix $n \in \mathbb{Z}_{\ge 1}$, $\epsilon > 0$, and $z_0 \in \mathbb{H}$. For any positive integer m prime to N and any $\epsilon' > 0$, we have*

$$\left| \frac{1}{\sigma_1(m)} \left\{ \sum_{\substack{ad=m, \\ b \pmod d}} J_{N,n}\left(\frac{az_0+b}{d}\right) - \sum_{\substack{ad=m, \\ b \pmod d}} P_{n,\epsilon}\left(\frac{az_0+b}{d}\right) \right\} - \lim_{\epsilon'' \to 0} \int_{\mathcal{F}_N(\epsilon'')} J_{N,n}(z)\, d\mu(z) \right|$$
$$\le C_{z_0,\epsilon'} m^{-\frac{1}{2}+\theta+\epsilon'} \max\{\|F_{n,\epsilon}\|_2, \|\Delta^2 F_{n,\epsilon}\|_2\},$$

where $F_{n,\epsilon} := J_{N,n} - P_{n,\epsilon}$ and $C_{z_0,\epsilon'}$ is the constant given as in Corollary 4.2.

Proof. For a positive integer n, let $P_{n,\epsilon}$ be the Poincaré series as in (5.9). From [11, Theorem 2.1], it follows that $F_{n,\epsilon} \in L^2(\Gamma_0(N)\backslash\mathbb{H})$ for a fixed $n \ge 1$.

Recall that for $\phi \in L^2(\Gamma_0(N)\backslash\mathbb{H})$ and $m \ge 1$ with $\gcd(m,N)=1$, the normalized Hecke operator T_m can be represented by

$$T_m\phi(z) = \sum_{\gamma \in T(m)} \phi(\gamma z).$$

By Corollary 4.2, we find that for any $\epsilon' > 0$ and m

$$\left| \frac{1}{\sigma_1(m)} (T_m F_{n,\epsilon})(z_0) - \int_{\mathcal{F}_N} F_{n,\epsilon}(z)\, d\mu(z) \right| \le C_{z_0,\epsilon'} m^{-\frac{1}{2}+\epsilon'+\theta} \max\{\|F_{n,\epsilon}\|_2, \|\Delta^2 F_{n,\epsilon}\|_2\}. \tag{5.10}$$

For $z_0 \in \mathbb{H}$, we have

$$\frac{1}{\sigma_1(m)} T_m F_{n,\epsilon}(z_0) = \frac{1}{\sigma_1(m)} \left\{ \sum_{\substack{ad=m, \\ b \pmod d}} J_{N,n}\left(\frac{az_0+b}{d}\right) - \sum_{\substack{ad=m, \\ b \pmod d}} P_{n,\epsilon}\left(\frac{az_0+b}{d}\right) \right\}. \tag{5.11}$$

Note that

$$\lim_{\epsilon'' \to 0} \int_{\mathcal{F}_N(\epsilon'')} P_{n,\epsilon}(z) \frac{dx\, dy}{y^2} = \int_{\Gamma_0(N)_\infty \backslash \mathbb{H}} \psi_\epsilon(y) e^{-2\pi i n z} \frac{dxdy}{y^2} = \int_0^\infty \psi_\epsilon(y) e^{2\pi n y} \frac{dy}{y^2} \cdot \int_0^1 e^{2\pi i n x}\, dx = 0$$

for every positive integer n. So, we get

$$\int_{\mathcal{F}_N} F_{n,\epsilon}(z)\, d\mu(z) = \lim_{\epsilon'' \to 0} \int_{\mathcal{F}_N(\epsilon'')} J_{N,n}(z)\, d\mu(z). \tag{5.12}$$

If we combine (5.10), (5.11), and (5.12), then we get the desired result. □

We define

$$Q_{1,\epsilon}(z) := \psi_\epsilon(\mathrm{Im}(\tilde{z}))e(-\tilde{z})$$

for $\epsilon > 0$. Then, we obtain the following proposition.

Proposition 5.3. *For any m with $\gcd(m, N) = 1$, we obtain*

$$\left|\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f) - Q_{1,\epsilon}(T_m.D_f)\right) - h_f \lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')} J_{N,1}(z)d\mu(z)\right| \tag{5.13}$$
$$\leq H_f C(f,\epsilon')m^{-\frac{1}{2}+\theta+\epsilon'} \max\{\|F_{1,\epsilon}\|_2, \|\Delta^2 F_{1,\epsilon}\|_2\},$$

where h_f denotes the sum of the orders of zero or pole of f at Q_τ on $Y_0(N)$.

Proof. Let $\epsilon > 0$ be fixed. Note that

$$J_{N,1}(T_m.D_f) = \sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) \sum_{\substack{ad=m \\ b \pmod d}} J_{N,1}\left(\frac{a\tau+b}{d}\right)$$

and

$$P_{1,\epsilon}(T_m.D_f) = \sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(N)}(f) \sum_{\substack{ad=m, \\ b \pmod d}} P_{1,\epsilon}\left(\frac{a\tau+b}{d}\right).$$

Therefore, by Proposition 5.2, for any m with $\gcd(m, N) = 1$, we have

$$\left|\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f) - P_{1,\epsilon}(T_m.D_f)\right) - h_f \lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')} J_{N,1}(z)d\mu(z)\right| \tag{5.14}$$
$$\leq H_f C(f,\epsilon')m^{-\frac{1}{2}+\theta+\epsilon'} \max\{\|F_{1,\epsilon}\|_2, \|\Delta^2 F_{1,\epsilon}\|_2\},$$

for any $\epsilon' > 0$, where $H_f := \sum_{\tau\in\mathcal{F}_N} |\nu_\tau^{(N)}(f)|$ and $C(f,\epsilon') := \max\{C_{\tau,\epsilon'} \mid \tau \in \mathcal{F}_N,\ \nu_\tau^{(N)}(f) \neq 0\}$.

Recall that $\tilde{z}$ is a unique complex number in $\mathcal{F}_N$ which is equivalent to z under the action of $\Gamma_0(N)$. If $\mathrm{Im}(\tilde{z}) > 1$, then for any $\gamma \in \Gamma_0(N)$, $\mathrm{Im}(\gamma\tilde{z}) \leq 1$ unless $\gamma \in \Gamma_0(N)_\infty$.

Suppose that $\mathrm{Im}(\tilde{z}) \leq 1$ and that there exists $\gamma \in \Gamma_0(N)$ such that $\mathrm{Im}(\gamma\tilde{z}) > 1$. Then, there exists $\ell \in \mathbb{Z}$ such that $-\frac{1}{2} < \mathrm{Re}(\gamma\tilde{z}) + \ell \leq \frac{1}{2}$, and so

$$\gamma\tilde{z} + \ell = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \gamma\tilde{z} \in \mathcal{F}_N.$$

Since $\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)\gamma \in \Gamma_0(N)$, we have $\gamma\tilde{z} + \ell = \tilde{z}$, so $\mathrm{Im}(\gamma\tilde{z}) = \mathrm{Im}(\tilde{z}) \leq 1$, which is a contradiction. Therefore, if $\mathrm{Im}(\tilde{z}) \leq 1$, then for any $\gamma \in \Gamma_0(N)$, we get $\mathrm{Im}(\gamma\tilde{z}) \leq 1$.

Thus, we have

$$P_{1,\epsilon}(z) = P_{1,\epsilon}(\tilde{z}) = \sum_{\gamma\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)} \psi_\epsilon(\mathrm{Im}(\gamma\tilde{z}))e(-\gamma\tilde{z}) = Q_{1,\epsilon}(z).$$

Therefore, from (5.14), we obtain the desired result. □

From Proposition 5.2 and Proposition 5.3, we obtain the following theorem. This gives the distribution of values of $J_{N,1}$ on Hecke orbits.

Theorem 5.4. *We have*

$$\lim_{m\to\infty}\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f)-e\left((T_m.D_f)_{>1}\right)\right) = \frac{3h_f}{\pi[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]}\lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')} J_{N,1}(z)\frac{dxdy}{y^2}.$$

Proof. Let $\epsilon>0$ be fixed. For any positive integer m which is prime to N, we have

$$\begin{aligned}&\left|\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f)-e\left((T_m.D_f)_{>1}\right)\right)-h_f\lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')}J_{N,1}(z)d\mu(z)\right|\\ &\le\left|\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f)-Q_{1,\epsilon}(T_m.D_f)\right)-h_f\lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')}J_{N,1}(z)d\mu(z)\right|\\ &+\frac{1}{\sigma_1(m)}\left|Q_{1,\epsilon}(T_m.D_f)-e\left((T_m.D_f)_{>1}\right)\right|.\end{aligned}\tag{5.15}$$

Note that

$$\begin{aligned}&|Q_{1,\epsilon}(T_m.D_f)-e(-(T_m.D_f))_{>1}|\\ &\le\sum_{\substack{\tau\in\mathcal{F}_N\\ \nu_\tau^{(N)}(f)\neq 0}}\left|\nu_\tau^{(N)}(f)\right|\sum_{\gamma\in T(m)}\begin{cases}|\psi_\epsilon(\mathrm{Im}(\widetilde{\gamma\tau})-1|\,|e(-\widetilde{\gamma\tau})|, & \text{if } 1<\mathrm{Im}(\widetilde{\gamma\tau})\le 1+\epsilon,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{5.16}$$

Now, we follow the proof of [15, Proposition 3]. Fix $0<\epsilon<\frac{1}{4}$ and consider the incomplete Eisenstein series

$$g_\epsilon(z):=\sum_{\gamma\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)}\phi_\epsilon(\mathrm{Im}\gamma z),$$

where $\phi_\epsilon:\mathbb{R}_{>0}\to\mathbb{R}$ is a smooth function supported in $(1-\epsilon,1+2\epsilon)$ with $0\le\phi_\epsilon(y)\le 1$ for all $y\in\mathbb{R}_{>0}$ and $\phi_\epsilon(y)=1$ for $1\le y\le 1+\epsilon$. By Corollary 4.2, we see that for any $\epsilon'>0$, $z_0\in\mathbb{H}$, and m with $\gcd(m,N)=1$,

$$\left|\frac{1}{\sigma_1(m)}T_m g_\epsilon(z_0)-\int_{\mathcal{F}_N}g_\epsilon(z)d\mu(z)\right|\le C_{z_0,\epsilon'}m^{-\frac{1}{2}+\epsilon'+\theta}\max\{\|g_\epsilon\|_2,\|\Delta^2 g_\epsilon\|_2\}.\tag{5.17}$$

Then, there exists a constant $D(f,\epsilon')$ such that

$$\begin{aligned}\frac{1}{\sigma_1(m)}|Q_{1,\epsilon}(T_m.D_f)-e(-(T_m.D_f)_{>1})|&\le\frac{e^{2\pi(1+\epsilon)}}{\sigma_1(m)}\sum_{\substack{\tau\in\mathcal{F}_N\\ \nu_\tau^{(N)}(f)\neq 0}}\left|\nu_\tau^{(N)}(f)\right|(T_m g_\epsilon)(\tau)\\ &\le C_f e^{2\pi(1+\epsilon)}\left(\int_{\mathcal{F}_N}g_\epsilon(z)\,d\mu(z)+D(f,\epsilon')m^{-\frac{1}{2}+\theta+\epsilon'}\max\{\|g_\epsilon\|_2,\|\Delta^2 g_\epsilon\|_2\}\right),\end{aligned}\tag{5.18}$$

where $C_f:=\#\left\{\tau\in\mathcal{F}_N\;\middle|\;\nu_\tau^{(N)}(f)\neq 0\right\}\times\max\left\{\left|\nu_\tau^{(N)}(f)\right|\;\middle|\;\tau\in\mathcal{F}_N\right\}$.

Therefore, from Proposition 5.3 and (5.15), we have

$$\begin{aligned}&\left|\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f)-e\left((T_m.D_f)_{>1}\right)\right)-h_f\lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')}J_{N,1}(z)d\mu(z)\right|\\&\leq\left(H_fC(f,\epsilon')\max\{\|F_{1,\epsilon}\|_2,\|\Delta^2F_{1,\epsilon}\|_2\}+C_fe^{2\pi(1+\epsilon)}D(f,\epsilon')\max\{\|g_\epsilon\|_2,\|\Delta^2g_\epsilon\|_2\}\right)m^{-\frac{1}{2}+\theta+\epsilon'}\\&\quad+C_fe^{2\pi(1+\epsilon)}\int_{\mathcal{F}_N}g_\epsilon(z)\,d\mu(z).\end{aligned}\tag{5.19}$$

For a fixed ϵ, taking $m\to\infty$, we get

$$\begin{aligned}&\lim_{m\to\infty}\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f)-e\left((T_m.D_f)_{>1}\right)\right)\\&\qquad=h_f\lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')}J_{N,1}(z)d\mu(z)+C_fe^{2\pi(1+\epsilon)}\int_{\mathcal{F}_N}g_\epsilon(z)\,d\mu(z).\end{aligned}\tag{5.20}$$

Note that (5.19) holds for any fixed $0<\epsilon<\frac{1}{4}$. Since

$$\int_{\mathcal{F}_N}g_\epsilon(z)d\mu(z)=\frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]}\int_0^\infty\phi_\epsilon(y)\,\frac{dy}{y^2}\to 0,$$

as $\epsilon\to 0$, we get

$$\lim_{m\to\infty}\frac{1}{\sigma_1(m)}\left(J_{N,1}(T_m.D_f)-e\left((T_m.D_f)_{>1}\right)\right)=h_f\lim_{\epsilon''\to 0}\int_{\mathcal{F}_N(\epsilon'')}J_{N,1}(z)d\mu(z).$$

□

Finally, Theorem 1.3 comes from Theorem 5.4 and (1.2).

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