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ON THE STRUCTURE OF AXIAL ALGEBRAS

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ABSTRACT. Axial algebras are a recently introduced class of non-associative algebra motivated by applications to groups and vertex-operator algebras. We develop the structure theory of axial algebras focussing on two major topics: (1) radical and simplicity; and (2) sum decompositions.

1. INTRODUCTION

Axial algebras are a new class of non-associative algebra introduced by Hall, Rehren and Shpectorov [7]. They axiomatise some key properties of vertex operator algebras (VOAs). VOAs were first introduced by physicists but particularly became of interest to mathematicians with Frenkel, Lepowsky and Meurman's [4] construction of the moonshine VOA $V^\natural$ whose automorphism group is the Monster M, the largest sporadic finite simple group. The rigorous theory of VOAs was developed by Borcherds [1] and it was instrumental in his proof of the monstrous moonshine conjecture.

An *axial algebra* is a commutative non-associative algebra A generated by a set of axes X. These axes are idempotents whose adjoint action decomposes the algebra as a direct sum of eigenspaces and the multiplication of eigenvectors satisfies a certain fusion law. Jordan and Matsuo algebras are examples of axial algebras with one of the simplest (and strongest) fusion laws. A slight relaxation of this fusion law adds the Griess algebra for the Monster M and other interesting examples.

Such axial algebras are of interest because the fusion law is $\mathbb{Z}_2$-graded and this leads to a naturally associated automorphism group. In this paper, we use a more general concept of a T-grading of a fusion law $\mathcal{F} = (\mathcal{F}, \star)$ by an abelian group T. For an axis $a \in X$, this induces a T-grading $A = \bigoplus_{t \in T} A_t$ on the algebra and there are natural automorphisms $\tau_a(\chi)$, for $\chi \in T^*$, associated to a, which we call *Miyamoto automorphisms*. The group generated by the set of all such $\tau_a(\chi)$, for $a \in X$ and $\chi \in T^*$, is called the *Miyamoto group* and it is a subgroup of the automorphism group of A.

We define an equivalence relation on sets of axes in an axial algebra. A set X of axes is closed if it is closed under the action of the Miyamoto group G defined by X; that is, $\bar{X} = X$, where $\bar{X} := X^G$. Two sets X and Y are equivalent if their closures $\bar{X}$ and $\bar{Y}$ are equal. We say that a property of an axial algebra is *stable* if it is invariant under equivalence of axes. In this paper, we introduce several new properties of axial algebras and we show that they, and some existing well-known properties, are stable. Firstly, we show that generation of axial algebras is stable.

That is, equivalent sets of axes generate the same algebra. The Miyamoto group of an axial algebra is also stable.

We introduce the *radical* $R(A, X)$ of an axial algebra A with axes X as the largest ideal of A not containing any axes from X. We show that this concept is well-defined and that the radical is stable, too.

This gives us a way to partition ideals of A into two classes, so that we may consider separately those which are contained in the radical and those which contain an axis. We introduce the *projection graph* on the set of axes X and show how this determines which axes are contained in a proper ideal.

A *Frobenius form* on an axial algebra is a non-zero (symmetric) bilinear form $(\cdot, \cdot)$ which associates with the algebra product. That is, $(a, bc) = (ab, c)$ for all $a, b, c \in A$. All currently known axial algebras admit such a form. Comparing our notion of the radical with that of the form, we have the following.

Theorem. *Let A be a primitive axial algebra with a Frobenius form. Then the radical $A^{\perp}$ of the Frobenius form coincides with the radical $R(A, X)$ of A if and only if $(a, a) \neq 0$ for all $a \in X$.*

In particular, when an axial algebra has a Frobenius form, we can use the above theorem as an easy way to find the radical $R(A, X)$. We also give an application of the above theorem to show that all the Norton-Sakuma algebras apart from 2B are simple.

In the second half of the paper, we discuss pairwise annihilating sum decompositions of axial algebras. That is, when A is a (vector space) sum of a set $\{A_i : i \in I\}$ of (not necessarily axial) subalgebras such that $A_i A_j = 0$ for $i \neq j$. We denote this by $A = \square_{i \in I} A_i$. We show that when the A_i are pairwise annihilating, the condition of A being the sum of A_i can be weakened to being generated by the A_i. We write $\langle\!\langle Y \rangle\!\rangle$ for the subalgebra generated by the set Y; hence the weaker condition is $A = \langle\!\langle A_i : i \in I \rangle\!\rangle$.

It is clear from our definition of the radical that the annihilator $\mathrm{Ann}(A) \subseteq R(A, X)$. As we will see, if A has a decomposition $A = \square_{i \in I} A_i$ and $\mathrm{Ann}(A) = 0$, then A is the direct sum of the A_i, which we denote $\boxplus_{i \in I} A_i$.

Theorem. *Suppose that $A = \square_{i \in I} A_i$ is a primitive axial algebra generated by the set of axes X. Let $X_i := X \cap A_i$ be the set of axes which are contained in A_i. Then, the X_i partition X and $A = \square_{i \in I} B_i$ where $B_i = \langle\!\langle X_i \rangle\!\rangle$.*

Moreover, this decomposition is invariant under arbitrary changes of axes (not just equivalence). That is, if $A = \langle\!\langle X \rangle\!\rangle = \langle\!\langle Y \rangle\!\rangle$ for two sets of primitive axes X and Y, then $\langle\!\langle X_i \rangle\!\rangle = \langle\!\langle Y_i \rangle\!\rangle$ for all $i \in I$.

This suggests the following definition.

Definition. The *non-annihilating graph* $\Delta(X)$ of an axial algebra A with generating axes X is the graph with vertex set X and, for $a \neq b$, an edge $a \sim b$ if and only if $ab \neq 0$.

It is clear that if $A = \square_{i \in I} A_i$, where the A_i are axial algebras, then the corresponding X_i are unions of connected components of $\Delta(X)$. It is natural to ask: is it not true that the finest pairwise annihilating sum decomposition of an axial algebra arises when each X_i is a single connected component of Δ? In particular, we make the following conjecture.

Conjecture. *The finest pairwise annihilating sum decomposition of an axial algebra A of Monster type arises when each X_i is just a single connected component of Δ.*

We show that the Miyamoto groups do indeed respect this decomposition of the axes.

Theorem. *Let A be a T-graded axial algebra with $0 \in \mathcal{F}_{1_T}$ and the components of $\Delta(X)$ be X_i for $i \in I$. Then, $G(X)$ is a central product of the $G(X_i)$.*

However, for the algebra, the picture is more complicated and we give a partial result in this direction. In order to do so, we introduce a new concept. A subspace $I \leq A$ is a *quasi-ideal* if I is invariant under multiplication with the axes X. We show that a quasi-ideal is stable and $G(X)$-invariant. The *spine* of A is defined as the quasi-ideal $Q(A, X)$ generated by the axes X. The algebra A is called *slender* if $A = Q(A, X)$. We prove that the spine of an axial algebra is stable. The conjecture above holds subject to a technical condition.

Theorem. *Let A be an axial algebra with a Seress fusion law and $A_i = \langle\langle X_i \rangle\rangle$ be the axial subalgebra generated by the connected component X_i of Δ. If all but possibly one A_i are slender, then $A = \square_{i \in I} A_i$.*

The definition of a Seress fusion law will be given later, but we note here that the Monster fusion law is Seress. An axial algebra A is *m-closed* if A is spanned by products in the axes of length at most m. Note that the spine is spanned by products in the axes of the form $x_1(x_2(\ldots (x_{k-1}x_k)\ldots))$. So, in particular, every 3-closed algebra is slender. Hence, we should expect the above result to apply to a large class of Seress axial algebras.

The paper is organized as follows. In Section 2, we recall the definition of axial algebras and review some basic properties. We discuss automorphisms and the Miyamoto group in Section 3. Here, we also introduce equivalence of sets of axes, stability and show that generation of axial algebras and the Miyamoto group are stable. Section 4 introduces the radical $R(A, X)$ and we show that it is stable. We also introduce the projection graph and use it to prove results about ideals. The Frobenius form is introduced and we prove some important properties. The main theorem in this section is that the radical of the form coincides with the radical of the algebra. In Section 5, we discuss pairwise annihilating sum decompositions of axial algebras and show when they are direct and axial. Finally, in Section 6, we introduce the non-annihilating graph Δ and our conjecture on connected components of Δ. We introduce quasi-ideals and the spine $Q(A, X)$ and show they are both stable. Finally, we prove results about the decomposition with respect to Δ.

We would like to thank Jonathan I. Hall and the referee for useful comments.

2. Axial algebras

2.1. **Fusion laws.** Throughout the paper $\mathbb{F}$ is an arbitrary field.

Definition 2.1. A *fusion law*[1] over $\mathbb{F}$ is a finite set $\mathcal{F}$ of elements of $\mathbb{F}$ together with a symmetric map $\star\colon \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$. We call a single instance $(\lambda, \mu) \mapsto \lambda \star \mu$ a *fusion rule*.

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[1]In previous papers on axial algebras [7, 8, 9, 11, 14] and before for Majorana algebras [13], this has been called fusion rules which led to singular/plural problems.

Since the values of $\star$ can be arranged in a symmetric square table, similar to a multiplication table, we sometimes call a fusion law a fusion table. We will often abuse notation and just write $\mathcal{F}$ for the fusion law $(\mathcal{F}, \star)$.

	1	0
1	1	
0		0

	1	0	η
1	1		η
0		0	η
η	η	η	1,0

	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	1,0	β
β	β	β	β	1,0,α

FIGURE 1. Fusion laws $\mathcal{A}$, $\mathcal{J}(\eta)$, and $\mathcal{M}(\alpha, \beta)$

In Figure 1, we see three examples of fusion laws that have appeared in the literature. In the tables, we abuse notation by neglecting to write the set symbols. We also leave the entry blank to mean the empty set.

In the first example, the fusion law $\mathcal{A}$ consists of just the elements 1 and 0 of $\mathbb{F}$. Hence this is defined over every field $\mathbb{F}$. In the second example, $\mathcal{J}(\eta) = \{1, 0, \eta\}$, where $\eta \in \mathbb{F}$ and $1 \neq \eta \neq 0$. So this can be defined for any field $\mathbb{F}$ except $\mathbb{F}_2$. Similarly, in the third example, $\mathcal{M}(\alpha, \beta) = \{1, 0, \alpha, \beta\}$, where $\alpha, \beta \in \mathbb{F}$, $\alpha, \beta \notin \{1, 0\}$, and $\alpha \neq \beta$. Hence, for this to make sense, the field $\mathbb{F}$ must have at least four elements.

Given a fusion law $\mathcal{F}$ and a subset $\mathcal{H} \subseteq \mathcal{F}$, $\mathcal{F}$ induces fusion rules on $\mathcal{H}$ by defining

$$\lambda \circ \mu := (\lambda \star \mu) \cap \mathcal{H} \qquad \text{for } \lambda, \mu \in \mathcal{H}.$$

We call such a fusion law on $\mathcal{H}$ a *restriction* of $\mathcal{F}$. We say that $\mathcal{H}$ is a *sublaw* if $\lambda \circ \mu = \lambda \star \mu$ for all $\lambda, \mu \in \mathcal{H}$, that is, if $\mathcal{H}$ is *closed* for $\star$. For example, $\mathcal{A}$ is a sublaw of both $\mathcal{J}(\eta)$ and $\mathcal{M}(\alpha, \beta)$. We see that $\mathcal{J}(\eta)$ is a restriction of $\mathcal{M}(\alpha, \beta)$ in two ways: when $\eta = \alpha$ and when $\eta = \beta$. However, it is a sublaw only when $\eta = \alpha$. We will see some motivation for this definition in the next section.

2.2. **Axes and axial algebras.** Let A be a commutative non-associative (that is, not necessarily associative) algebra over $\mathbb{F}$. Recall that we write $\langle\!\langle Y \rangle\!\rangle$ to denote the subalgebra of A generated by the set of elements Y to differentiate it from the subspace $\langle Y \rangle$ spanned by Y. The adjoint of $a \in A$, denoted by ad_a, is the linear endomorphism of A defined by $b \mapsto ab$ for $b \in A$. For $\lambda \in \mathbb{F}$, let $A_\lambda(a)$ denote the λ-eigenspace of ad_a. That is, $A_\lambda(a) = \{b \in A : ab = \lambda b\}$. Clearly, $A_\lambda(a) \neq 0$ if and only if λ is an eigenvalue of ad_a. For $\Lambda \subseteq \mathbb{F}$, we write $A_\Lambda(a) = \bigoplus_{\lambda \in \Lambda} A_\lambda(a)$.

Definition 2.2. For a fusion law $\mathcal{F}$, an element $a \in A$ is an $\mathcal{F}$-*axis* if the following hold:

(A1) a is an idempotent; that is, $a^2 = a$;

(A2) ad_a is semisimple and all eigenvalues of ad_a are in $\mathcal{F}$; that is, $A = A_{\mathcal{F}}(a)$;

(A3) the fusion law $\mathcal{F}$ controls products of eigenvectors: namely,

$$A_\lambda(a) A_\mu(a) \subseteq A_{\lambda \star \mu}(a) \qquad \text{for } \lambda, \mu \in \mathcal{F}.$$

Note that, a being an idempotent, 1 is an eigenvalue of ad_a. For this reason, we will always assume that $1 \in \mathcal{F}$. We also allow for the possibility that $A_\lambda(a)$ is 0 for some $\lambda \in \mathcal{F}$.

Definition 2.3. An $\mathcal{F}$-axis a is *primitive* if $A_1(a) = \langle a \rangle$.

If a is a primitive axis then $A_1(a)A_\lambda(a) = A_\lambda(a)$, for all $\lambda \neq 0$, and $A_1(a)A_0(a) = 0$. Therefore, for primitive axes, we only need to consider fusion rules $\mathcal{F}$ satisfying $1 \star \lambda = \{\lambda\}$ for $\lambda \neq 0$ and $1 \star 0 = \emptyset$, provided that $0 \in \mathcal{F}$. All three fusion laws in Figure 1 possess this property.

Definition 2.4. An $\mathcal{F}$-*axial algebra* is a pair $A = (A, X)$, where A is a commutative non-associative algebra generated by the set X of $\mathcal{F}$-axes. An axial algebra (X, A) is *primitive* if each axis in X is primitive.

We will usually abuse notation and just refer to A as being an axial algebra without making reference to $\mathcal{F}$ and X where they are clear. We will also often consider just primitive axial algebras and so we will often skip this adjective.

A subalgebra $B \leq A$ is an *axial subalgebra* if there exists $Y \subseteq X$ such that $B = \langle\langle Y \rangle\rangle$. That is, if there exists $Y \subseteq X$ such that (B, Y) is itself an axial algebra. Let $\mathcal{H} \subseteq \mathcal{F}$ be a sublaw. Since $\mathcal{H}$ is closed, $A_\mathcal{H}(a)$ is a subalgebra of A, for all $a \in X$. However, it will not usually be an axial subalgebra as generation by axes is not guaranteed. A restriction of a fusion law does not in general lead to a subalgebra.

It is easy to show that in the class of primitive axial algebras, associative axial algebras are the same as $\mathcal{A}$-axial algebras [8, Corollary 2.9]. Furthermore, these are exactly the direct sum algebras $\mathbb{F} \oplus \ldots \oplus \mathbb{F}$. So we call $\mathcal{A}$ the *associative fusion law*.

Given any 3-transposition group (G, D), one can define a Matsuo algebra which has basis given by the elements of D and multiplication depending on the order of the product of the involutions. These are examples of $\mathcal{J}(\eta)$-axial algebras. For more details see the text before Example 4.12.

Every idempotent in a Jordan algebra satisfies the fusion law $\mathcal{J}(\frac{1}{2})$. This is known as the *Peirce decomposition*. Hence Jordan algebras generated by primitive idempotents are examples of $\mathcal{J}(\frac{1}{2})$-axial algebras. For this reason, we call $\mathcal{J}(\eta)$ the fusion law of *Jordan type* η.

Finally, the $196,884$-dimensional real Griess algebra [5], whose automorphism group is the sporadic simple Monster group M, is an example of an axial algebra with fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$. This fact was first discussed by Miyamoto in the context of VOAs in [16] and was checked directly in the Griess algebra by Ivanov in [13, Lemma 8.5.1]. In this paper, we call an axial algebra with this fusion law an axial algebra of *Monster type*, although this terminology is not yet stable and has also been used for an axial algebra with fusion law $\mathcal{M}(\alpha, \beta)$.

Historically, the axiomatics for axial algebras appeared as a generalisation of the axioms for Ivanov's *Majorana algebras* [13, Section 8.6], which in our terms are axial algebras of Monster type over $\mathbb{R}$, satisfying certain additional properties. In turn, Majorana algebras were defined to generalise properties of the Griess algebra and the moonshine VOA.

3. Automorphisms

3.1. **Axis subgroup.** The fusion laws $\mathcal{F}$ which particularly interest us are those where the axes lead to automorphisms of the algebra. Let us extend the operation $\star$ to arbitrary subsets Λ and M of the fusion table $\mathcal{F}$ via $\Lambda \star \mathrm{M} := \cup_{\lambda \in \Lambda, \mu \in \mathrm{M}} \lambda \star \mu$.

Definition 3.1. Suppose T is an abelian group. A T-*grading* of a fusion table $\mathcal{F}$ is a partition $\mathcal{F} = \cup_{t \in T} \mathcal{F}_t$ of $\mathcal{F}$ satisfying $\mathcal{F}_s \star \mathcal{F}_t \subseteq \mathcal{F}_{st}$ for all $s, t \in T$.

Note that we allow the possibility that some part $\mathcal{F}_t$ is the empty set. Suppose that $\mathcal{F}$ is T-graded. Let A be an $\mathcal{F}$-axial algebra and $a \in X$ an axis. For $t \in T$, we set $A_t(a) = A_{\mathcal{F}_t}(a) = \bigoplus_{\lambda \in \mathcal{F}_t} A_\lambda(a)$. Clearly, we have $A = \bigoplus_{t \in T} A_t(a)$. It follows from the above definition that $A_s(a)A_t(a) \subseteq A_{st}(a)$, that is, we have a T-grading of the algebra A for each axis a. Note that, since $\mathcal{F}_t$ may be empty, A_t may be trivial for some $t \in T$.

Let T^* be the group of linear characters of T over $\mathbb{F}$, that is, the set of all homomorphisms from T to the multiplicative group of $\mathbb{F}$. For an axis a and $\chi \in T^*$, consider the linear map $\tau_a(\chi) \colon A \to A$ defined by

$$u \mapsto \chi(t)u \qquad \text{for } u \in A_t(a)$$

and extended linearly to A. Since A is T-graded, this map $\tau_a(\chi)$ is an automorphism of A and we call it a *Miyamoto automorphism* (see the footnote below). Furthermore, the map sending χ to $\tau_a(\chi)$ is a homomorphism from T^* to $\mathrm{Aut}(A)$.

Definition 3.2. We call the image T_a of the map $\chi \mapsto \tau_a(\chi)$, the *axis subgroup* of $\mathrm{Aut}(A)$ corresponding to a.

Usually, T_a is a copy of T^*, but occasionally, when some subspaces $A_t(a)$ are trivial, T_a can be isomorphic to a factor group of T^* over a non-trivial subgroup.

We will often consider fusion laws where T is the cyclic group C_2 of order two. If $\mathrm{char}(\mathbb{F}) = 2$, then $T^* = 1$ and we get no automorphisms. So, we will normally assume that $\mathrm{char}(\mathbb{F}) \neq 2$ when $T = C_2$. In this case, $T^* = \{\chi_1, \chi_{-1}\}$ where χ_1 is the trivial character. The automorphism $\tau_a(\chi_{-1})$ is (usually) non-trivial; we will denote it by τ_a and call it a *Miyamoto involution*[2]. Then $T_a = \langle \tau_a \rangle \cong C_2$.

Indeed, among our examples of fusion laws in Figure 1, the fusion tables $\mathcal{J}(\eta)$ and $\mathcal{M}(\alpha, \beta)$ are C_2-graded. Writing $C_2 = \{+, -\}$, the grading for $\mathcal{J}(\eta)$ is given by $\mathcal{J}(\eta)_+ = \{1, 0\}$ and $\mathcal{J}(\eta)_- = \{\eta\}$. For $\mathcal{M}(\alpha, \beta)$, the grading is given by $\mathcal{M}(\alpha, \beta)_+ = \{1, 0, \alpha\}$ and $\mathcal{M}(\alpha, \beta)_- = \{\beta\}$. Hence in these cases the axis subgroups are of order 2 (or 1 if $A_-(a) = 0$).

Recall that the Griess algebra A is an axial algebra with fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$. For an axis $a \in A$, the subgroup $T_a = \langle \tau_a \rangle$ has order two. It was Norton who first studied axes in the Griess algebra and his results were recorded by Conway in [2]. In particular, he shows that the involutions τ_a belong to the conjugacy class $2A$ in the Monster M. The fact that the mapping $a \mapsto \tau_a$ is a bijection is implicit in [2], a proof was sketched by Miyamoto in [16] and given by Höhn in [10].

Recall now that every axial algebra A comes with a set of generating axes X. In the following definition we slightly relax conditions on X by allowing it to be an arbitrary set of axes from A.

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[2]It was Miyamoto who first defined, in the context of VOAs, such an involution τ_a from a C_2-graded fusion law [16].

Definition 3.3. The *Miyamoto group* $G(X)$ of A with respect to the set of axes X is the subgroup of $\mathrm{Aut}(A)$ generated by the axis subgroups T_a, $a \in X$.

Since the $2A$ involutions generate the Monster, for the Griess algebra, we have $G(X) = M$ where X is the set of $2A$-axes.

3.2. **Closed sets of axes.** If a is an axis and $g \in \mathrm{Aut}(A)$, then a^g is again an axis. Indeed, it is easy to check that $A_\lambda(a^g) = A_\lambda(a)^g$ and, hence, for $\lambda, \mu \in \mathcal{F}$, we have $A_\lambda(a^g)A_\mu(a^g) = A_\lambda(a)^g A_\mu(a)^g = (A_\lambda(a)A_\mu(a))^g \subseteq A_{\lambda \star \mu}(a)^g = A_{\lambda \star \mu}(a^g)$.

Definition 3.4. A set of axes X is *closed* if $X^\tau = X$ for all $\tau \in T_a$ with $a \in X$. Equivalently, $X^{G(X)} = X$.

It is easy to see that the intersection of closed sets is again closed and so every X is contained in the unique smallest closed set $\bar{X}$ of axes. We call $\bar{X}$ the *closure* of X.

Lemma 3.5. *For a set of axes X, we have that $\bar{X} = X^{G(X)}$ and, furthermore, $G(\bar{X}) = G(X)$.*

Proof. Since $X \subseteq \bar{X}$, we have that $G(X) \leq G(\bar{X})$. Hence $X^{G(X)} \subseteq \bar{X}^{G(\bar{X})} = \bar{X}$. To show the reverse inclusion, it suffices to prove that $X^{G(X)}$ is closed.

Suppose that $b \in X^{G(X)}$. Then $b = a^g$ for some $a \in X$ and $g \in G(X)$. Note that $\tau_{a^g}(\chi) = \tau_a(\chi)^g$ and so $T_b = T_{a^g} = T_a^g$. Since $T_a \leq G(X)$ and $g \in G(X)$, we have that $T_b = T_a^g \leq G(X)^g = G(X)$. Hence, $G(X^{G(X)}) = G(X)$. Clearly, $X^{G(X)}$ is invariant under $G(X) = G(X^{G(X)})$. This means that $X^{G(X)}$ is closed, proving that $\bar{X} = X^{G(X)}$ and also $G(\bar{X}) = G(X^{G(X)}) = G(X)$. $\qquad \square$

Turning again to the example of the Griess algebra, it is well-known (see for example the tables in [17]) that the Monster M can be generated by three $2A$ involutions, say, τ_a, τ_b, and τ_c, for axes $a, b, c \in A$. Setting $X = \{a, b, c\}$, we have that $G(X) = \langle T_a, T_b, T_c \rangle = \langle \tau_a, \tau_b, \tau_c \rangle = M$. Hence $\bar{X} = X^{G(X)} = X^M$ is the set of all axes of A, since $\{\tau_a, \tau_b, \tau_c\}^M$ is clearly all of $2A$. (Recall that the map sending an axis to the corresponding $2A$ involution is bijective.) So here $\bar{X}$ (of size approximately 9.7×10^{19} [3]) is huge compared to the tiny X.

Definition 3.6. We say that sets X and Y of axes are *equivalent* (denoted $X \sim Y$) if $\bar{X} = \bar{Y}$.

Clearly, this is indeed an equivalence relation on sets of axes.

Definition 3.7. A property of an axial algebra is called *stable* if it is invariant under equivalence of axes.

In this paper, we will show that several properties of axial algebras are stable. Lemma 3.5 gives us the first of these.

Corollary 3.8. *The Miyamoto group of an axial algebra is stable.*

Since $\bar{X} = X^{G(X)}$ and, similarly, $\bar{Y} = Y^{G(Y)}$, we have the following.

Lemma 3.9. *Sets X and Y of axes are equivalent if and only if the following two conditions hold:*

1. *$G := G(X) = G(Y)$.*
2. *Every $x \in X$ is G-conjugate to some $y \in Y$ and, vice versa, every $y \in Y$ is G-conjugate to some $x \in X$.* $\qquad \square$

3.3. **Invariance.** Let $a \in X$ be an axis and W be a subspace of A invariant under the action of ad_a. Since ad_a is semisimple on A, it is also semisimple on W, and so $W = \bigoplus_{\lambda \in \mathcal{F}} W_\lambda(a)$, where $W_\lambda(a) = W \cap A_\lambda(a) = \{w \in W : aw = \lambda w\}$.

Let us note the following important property of axis subgroups T_a.

Lemma 3.10. *For an axis a, if a subspace $W \subseteq A$ is invariant under ad_a then W is invariant under every $\tau_a(\chi)$, $\chi \in T^*$. (That is, W is invariant under the whole T_a.)*

Proof. We have already observed that if W is invariant under ad_a then $W = \bigoplus_{\lambda \in \mathcal{F}} W_\lambda(a)$. Recall that $W_\lambda(a)$ is a subspace of $A_\lambda(a)$. Since $\tau = \tau_a(\chi)$ acts on $A_\lambda(a)$ as a scalar transformation, it leaves invariant every subspace of $A_\lambda(a)$. In particular, $W_\lambda(a)^\tau = W_\lambda(a)$ for every λ, and so $W^\tau = W$. $\square$

For example, ideals of A are invariant under ad_a for all axes a. Hence we have the following:

Corollary 3.11. *Every ideal I of A is $G(X)$-invariant for any set of axes X in A.*

Let us now prove the following important property. We denote by $\langle\!\langle X \rangle\!\rangle$ the subalgebra of A generated by the set of axes X.

Theorem 3.12. *Suppose that $X \sim Y$. Then $\langle\!\langle X \rangle\!\rangle = \langle\!\langle Y \rangle\!\rangle$. In particular, if X generates A then so does Y. Hence, generation of axial algebras is stable.*

Proof. Let $B = \langle\!\langle X \rangle\!\rangle$ and $C = \langle\!\langle Y \rangle\!\rangle$. Note that B is invariant under ad_a for every $a \in X$. Hence B is $G(X)$-invariant. Clearly, this means that $\bar{X} = X^{G(X)} \subseteq B$. Therefore, $Y \subseteq \bar{Y} = \bar{X} \subseteq B$, proving that $C \subseteq B$. Symmetrically, also $B \subseteq C$, and so $B = C$. $\square$

We note that the converse does not hold. That is, there exist sets of axes X and Y which are inequivalent, but which both generate the same axial algebra A. For example, there is an axial algebra of dimension 9 which is generated by a closed set of 6 axes (and has shape 3C2A and Miyamoto group S_4) [15, Table 40]. However, it is also generated (in fact, spanned by) a closed set of 9 axes. Since both sets are closed but of different sizes, they are clearly inequivalent.

Recall from Section 3.2 that the Monster can be generated by three $2A$ involutions. This means that there exist three axes a, b, and c in the Griess algebra A such that $M = \langle \tau_a, \tau_b, \tau_c \rangle$. Setting $B = \langle\!\langle a, b, c \rangle\!\rangle$, we see that B is invariant under M. Recalling from Section 3.1 that there is a bijection between axes and $2A$ involutions, we see that all axes are conjugate under M. This shows that B contains all axes from A, that is, $B = A$, since A is generated by axes. We have shown that $A = \langle\!\langle a, b, c \rangle\!\rangle$, which means that, despite its large dimension, A can be generated by just three axes.

4. Ideals, the radical and the Frobenius form

Throughout this section, suppose that A is an axial algebra with fusion law $\mathcal{F}$ over a field $\mathbb{F}$ and let X be the set of primitive axes which generate A.

4.1. **The radical.** Recall that if W is a subspace invariant under the action of ad_a for an axis a then $W = \bigoplus_{\lambda \in \mathcal{F}} W_\lambda(a)$, where $W_\lambda(a) = W \cap A_\lambda(a)$.

Lemma 4.1. *Let $a \in X$ be a primitive axis and W be a subspace of A invariant under the action of ad_a. Then, $a \in W$ if and only if $W_1(a) = W \cap A_1(a)$ is not 0.*

Proof. Since a is primitive, $A_1(a) = \langle a \rangle$ is 1-dimensional. In particular, we have the dichotomy: either $a \in W$ and $W_1(a) = A_1(a) = \langle a \rangle$, or $W_1(a) = 0$, and so $W \subseteq A_{\mathcal{F} \setminus \{1\}}(a)$. $\qquad \square$

In particular, the above lemma holds for ideals. We begin by considering those ideals which do not contain any axes from X.

Lemma 4.2. *Let Y be a set of primitive axes. There is a unique largest ideal that contains no axes from Y.*

Proof. By Lemma 4.1, an ideal, which is clearly invariant under the action of ad_a for all $a \in Y$, contains no axes from Y if and only if it is contained in $\cap_{a \in Y} A_{\mathcal{F} \setminus \{1\}}(a)$. Clearly, the sum of all such ideals is again contained in this intersection and so it does not contain any axes from Y. Hence there is indeed a unique largest ideal containing no axes from Y. $\qquad \square$

In light of this lemma, we make the following definition.

Definition 4.3. The *radical* $R(A, X)$ of A with respect to the generating set of primitive axes X is the unique largest ideal of A containing no axes from X.

Because the concept of the radical requires the axes to be primitive, we will assume primitivity whenever we talk about the radical. Abusing notation, we will drop either A or X where it is clear from context. The radical $R(A, X)$ of an axial algebra A is defined with respect to a given generating set of axes X. What if we take a different generating set?

Theorem 4.4. *If $X \sim Y$ are two equivalent sets of primitive axes, then $R(A, X) = R(A, Y)$. That is, the radical of an axial algebra is stable.*

Proof. We show that $R(A, X) = R(A, \bar{X})$. Clearly, the ideal $R(\bar{X})$ does not contain any axis from X, and so $R(\bar{X}) \subseteq R(X)$. Conversely, by Corollary 3.11, every ideal of A is invariant under $G(X)$. Since $R(X)$ contains no axis from X, it follows that $R(X)$ contains no axis from $X^{G(X)} = \bar{X}$. So $R(X) \subseteq R(\bar{X})$ and therefore $R(\bar{X}) = R(X)$.

If Y is equivalent to X, by Theorem 3.12, Y also generates A and so $R(Y)$ is defined. Furthermore, $R(Y) = R(\bar{Y}) = R(\bar{X}) = R(X)$. $\qquad \square$

This shows that our notion of the radical behaves well under the natural changes of generating sets of axes.

4.2. **Frobenius form.** Sometimes an $\mathcal{F}$-axial algebra also admits a bilinear form which behaves well with respect to the multiplication in the algebra.

Definition 4.5. A *Frobenius*[3] form on an $\mathcal{F}$-axial algebra A is a (non-zero) bilinear form $(\cdot, \cdot) \colon A \times A \to \mathbb{F}$ which associates with the algebra product. That is,

$$(a, bc) = (ab, c) \qquad \text{for all } a, b, c \in A.$$

Note that we do not place any restriction on the value of (a, a) for axes $a \in X$. This differs from definitions given in previous papers [7, 8, 11]. However, several key facts still hold. A Frobenius form is still necessarily symmetric [7, Proposition 3.5]. We also have the following important property:

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[3]The term Frobenius form is borrowed from Frobenius algebras which are associative algebras with such a bilinear form, called there a Frobenius form.

Lemma 4.6. *For an axis a, the direct sum decomposition $A = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$ is orthogonal with respect to every Frobenius form $(\cdot, \cdot)$ on A.*

Proof. Suppose $u \in A_\lambda(a)$ and $v \in A_\mu(a)$ for $\lambda \neq \mu$. Then $\lambda(u, v) = (\lambda u, v) = (au, v) = (ua, v) = (u, av) = (u, \mu v) = \mu(u, v)$. Since $\lambda \neq \mu$, we conclude that $(u, v) = 0$. $\square$

Let a be a primitive axis. Then we may decompose $u \in A$ with respect to a as $u = \sum_{\lambda \in \mathcal{F}} u_\lambda$, where $u_\lambda \in A_\lambda(a)$. We call u_λ the *projection* of u onto $A_\lambda(a)$. Focusing on the projection u_1, as a is primitive, $u_1 = \phi_a(u)a$ for some $\phi_a(u)$ in $\mathbb{F}$. It is easy to see that ϕ_a is linear in u.

Lemma 4.7. *Let $(\cdot, \cdot)$ be a Frobenius form on a primitive axial algebra A. Then, $(a, u) = \phi_a(u)(a, a)$ for any axis $a \in X$ and $u \in A$.*

Proof. We decompose $u = \sum_{\lambda \in \mathcal{F}} u_\lambda$ with respect to a, where $u_\lambda \in A_\lambda(a)$. Now, by Lemma 4.6, $(a, u) = (a, \sum_{\lambda \in \mathcal{F}} u_\lambda) = (a, u_1) = \phi_a(u)(a, a)$. $\square$

Let us now explore the connection of the Frobenius form to the radical of A. We write $A^\perp$ for the radical of the Frobenius form; that is,

$$A^\perp = \{u \in A : (u, v) = 0 \text{ for all } v \in A\}$$

Lemma 4.8. *The radical $A^\perp$ is an ideal of A.*

Proof. If $u \in A^\perp$ and $v, w \in A$, then $(uv, w) = (u, vw) = 0$ and so $uv \in A^\perp$. Since $(\cdot, \cdot)$ is also bilinear, $A^\perp$ is an ideal. $\square$

It follows from Lemma 4.6 that a primitive axis a is contained in $A^\perp$ if and only if $(a, a) = 0$. Therefore, $A^\perp$ contains no axes from the generating set X if and only if $(a, a) \neq 0$ for all $a \in X$. The following is a generalisation of Proposition 2.7 in [8].

Theorem 4.9. *Let $A = (A, X)$ be a primitive axial algebra with a Frobenius form. Then, the radical $A^\perp$ of the Frobenius form coincides with the radical $R(A, X)$ of A if and only if $(a, a) \neq 0$ for all $a \in X$.*

Proof. Let $R = R(A, X)$. If $A^\perp = R$ then $A^\perp$ contains no axes from X, and so, by Lemma 4.7, we have that $(a, a) \neq 0$ for all $a \in X$.

Conversely, suppose that $(a, a) \neq 0$ for all $a \in X$. Then $A^\perp$ contains no axes from X. Hence $A^\perp \subseteq R$. It remains to show that $R \subseteq A^\perp$, that is, that R is orthogonal to the entire A. Since X generates A, the algebra A is linearly spanned by all (non-associative) products w of the axes from X. Hence we just need to show that R is orthogonal to each product w. We prove this property by induction on the length of the product w.

If the length of w is one then $w = a$ is an axis from X. Since $a \notin R$, we have that $R \subseteq A_{\mathcal{F} \setminus \{1\}}(a)$, which by Lemma 4.6 means that R is orthogonal to w, as claimed. Now suppose that the length of w is at least two. Then $w = w_1 w_2$ for products w_1 and w_2 of shorter length. By the inductive assumption, we know that R is orthogonal to both w_1 and w_2. Therefore, $(w, R) = (w_1 w_2, R) = (w_1, w_2 R) = 0$, as R is an ideal. So, $R \subseteq A^\perp$ and hence $R = A^\perp$. $\square$

It is often additionally required that the Frobenius form satisfy $(a, a) = 1$ for each axis a. In view of Lemma 4.7, we call the Frobenius form satisfying $(a, a) = 1$

for all generating axes a the *projection form*. We will see later that the projection form, when it exists, is unique.

The existence of a projection form is included in the axioms of Majorana algebras by Ivanov [13]. He further requires the projection form to be positive-definite. (Recall that Majorana algebras are defined over $\mathbb{F} = \mathbb{R}$.) In particular, we have the following.

Corollary 4.10. *Every Majorana algebra has trivial radical; that is, every non-zero ideal contains one of the generating primitive axes.*

Proof. Indeed, since the Frobenius form is positive definite, we have that $(u, u) > 0$ for every $u \neq 0$. In particular, this is true for axes, and so, by Theorem 4.9, the radical of the algebra is the same as the radical of the Frobenius form, which is zero. □

For $\mathcal{J}(\eta)$-axial algebras, which are called *axial algebras of Jordan type* η, we do not need to assume the existence of a projection form. Every primitive axial algebra of Jordan type automatically admits a projection form [11]. Hence, we can state the following.

Corollary 4.11. *The radical of every primitive axial algebra of Jordan type coincides with the radical of its projection form.*

We wish to give an example, but first we must define the class of Matsuo algebras. For any group of 3-transpositions (G, D), we define the *Matsuo algebra* A with respect to (G, D) which has basis D and multiplication $\circ$ given by

$$a \circ b = \begin{cases} a & \text{if } a = b \\ 0 & \text{if } o(ab) = 2 \\ \frac{\eta}{2}(a + b - c) & \text{if } o(ab) = 3, \text{ where } c = a^b = b^a \end{cases}$$

(Clearly here the field should not be of characteristic two.) By [8, Theorem 1.5], all Matsuo algebras with $\eta \neq 0, 1$ are examples of axial algebras of Jordan type η. It can be seen that the projection Frobenius form for A is given by

$$(a, b) = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{if } o(ab) = 2 \\ \frac{\eta}{2} & \text{if } o(ab) = 3 \end{cases}$$

Using the basis D, the form has Gram matrix

$$F = I + \frac{\eta}{2} M$$

where I is the identity matrix and M is the adjacency matrix of the non-commuting graph on D. The form has a radical precisely when F is not of full rank. From the above equation for F, we see this occurs if and only if $\eta = -\frac{2}{\lambda}$ for a non-zero eigenvalue λ of M. Furthermore, the radical of the form coincides with the λ-eigenspace of F. For example, the valency κ of the non-commuting graph is an eigenvalue of M and the corresponding eigenspace is 1-dimensional spanned by the all-one vector. Hence, when $\eta = -\frac{2}{\kappa}$, the projection form has 1-dimensional radical.

In general, since M has finitely many eigenvalues, there are only finitely many values of η, for which the Frobenius form on the Matsuo algebra has a non-zero radical.

Here is a concrete example.

Example 4.12. The group $G = S_5$ with the conjugacy class D of transpositions is a 3-transposition group and so leads to a Matsuo algebra A. By calculation (for example, see [12]), M has eigenvalues $6, 1, -2$. The value $\lambda = -2$ leads to $\eta = 1$, so this may be discarded. When $\lambda = \kappa = 6$, $\eta = -\frac{1}{3}$ and the radical is spanned by the element $r := \sum_{a \in D} a$. When $\lambda = 1$, $\eta = -2$ and we have a 4-dimensional radical spanned by elements of the form

$$(i, j) + (i, k) + (i, l) - (m, j) - (m, k) - (m, l)$$

where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$.

4.3. **Ideals and the projection graph.** Having considered ideals which do not contain any axes, we now turn our attention to ideals I that do contain an axis a. What other axes does I contain? Suppose b is another primitive axis of A. Recall that $A = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(b)$ and so $a = \sum_{\lambda \in \mathcal{F}} a_\lambda$ for unique $a_\lambda \in A_\lambda(b)$. Since I is invariant under ad_b, we have $I = \bigoplus_{\lambda \in \mathcal{F}} I_\lambda(b)$ and hence $a_\lambda \in I$ for each $\lambda \in \mathcal{F}$. In particular, the projection a_1 is in I. Since b is primitive, $a_1 = \phi_b(a)b$ is a scalar multiple of b. Hence, if $a_1 \neq 0$ then $b \in I$. This motivates the following construction.

Definition 4.13. Let A be a primitive axial algebra. We define the *projection graph* Γ to be the directed graph with vertex set X and a directed edge from a to b if the projection a_1 of a onto b is non-zero. That is, if $\phi_b(a) \neq 0$.

Given a directed graph Γ, the *out set* $Out(\Gamma, Y)$ of a subset of vertices Y is the set of all the vertices v reachable from Y by a directed path from $x \in Y$ to v.

The following lemma follows from the discussion above.

Lemma 4.14. *Let A be a primitive axial algebra and Γ be its projection graph. If Y is a set of axes contained in an ideal I then $Out(\Gamma, Y)$ is also fully contained in I.*

Recall that a directed graph Γ is *strongly connected* if every vertex is reachable by a directed path from any other.

Corollary 4.15. *Let A be a primitive axial algebra with a strongly connected projection graph. Then every proper ideal of A is contained in the radical.*

Recall from Corollary 3.11 that every ideal is invariant under the Miyamoto group G. Hence, as a further improvement, when $X = \bar{X}$ is closed, we may quotient out by the action of G to form the quotient graph $\bar{\Gamma} := \Gamma/G$. It has as vertices orbits of axes with a directed edge from a^G to b^G if there exist axes $a' \in a^G$ and $b' \in b^G$ such that the projection a_1' of a' onto b' is non-zero. We call $\bar{\Gamma}$ the *orbit projection graph*.

Corollary 4.16. *Let A be a primitive axial algebra with a strongly connected orbit projection graph. Then every proper ideal of A is contained in the radical.*

We now consider the properties of the projection graph Γ when there is a Frobenius form.

Lemma 4.17. *Let A be a primitive axial algebra that admits a Frobenius form. Suppose that $(a, a) \neq 0 \neq (b, b)$ for $a, b \in X$. The following are equivalent:*
  1. *There is a directed edge $a \to b$ in Γ.*
  2. *There is a directed edge $a \leftarrow b$ in Γ.*

   3. $(a, b) \neq 0$.

Proof. By Lemma 4.7, $(a, b) = \phi_a(b)(a, a)$, where the projection $b_1 = \phi_a(b)a$. Since the form is symmetric, the result follows. □

In light of the above result, when A is a primitive axial algebra that admits a Frobenius form that is non-zero on the axes, we may consider Γ to be an undirected graph.

4.4. **Uniqueness of the Frobenius form.** The same concept of the projection graph is useful when we want to establish uniqueness of the Frobenius form.

Lemma 4.18. *A Frobenius form on a primitive axial algebra A is uniquely determined by its values (a, a) for $a \in X$.*

Proof. If two Frobenius forms have the same values of (a, a) for all $a \in X$ then their difference (which also associates with the algebra product) satisfies $(a, a) = 0$ for all $a \in X$. Hence it suffices to show that the latter condition forces the form to be zero.

Clearly, A is spanned by products of axes and so we just need to show that $(u, v) = 0$ for all u and v that are products of axes. We use induction on the length of the products of axes for v. If v has length one, it is itself an axis in X. By Lemma 4.7, $(u, v) = \phi_v(u)(v, v) = 0$. Suppose now that v has length at least two, which means that we may write $v = v_1 v_2$, where v_1 and v_2 are shorter products. Then $(u, v) = (u, v_1 v_2) = (u v_1, v_2)$. By induction, the latter value is zero. □

In particular, for the form to be non-zero, at least one value (a, a) must be non-zero. Clearly, we can scale the form so that (a, a) takes any non-zero value we like, say $(a, a) = 1$. By Lemma 4.7, $\phi_a(b)(a, a) = (b, a) = (a, b) = \phi_b(a)(b, b)$. If $\phi_b(a) \neq 0$, we can deduce $(b, b) = \frac{\phi_a(b)}{\phi_b(a)}(a, a)$; that is, the value of (b, b) can be determined from the value of (a, a).

Recall that in the projection graph Γ on X we have a directed edge from a to b exactly when $\phi_b(a) \neq 0$. Hence the known values on a subset $Y \subset X$ allow us to deduce all values on the out set $Out(\Gamma, Y)$. In particular, we have the following.

Proposition 4.19. *If the projection graph Γ of a primitive axial algebra A is strongly connected then the Frobenius form on A, if it exists, is unique up to scaling.*

The equation $(b, b) = \frac{\phi_a(b)}{\phi_b(a)}(a, a)$ means also that, for the Frobenius form to be a projection form (up to scaling), we must have $\frac{\phi_a(b)}{\phi_b(a)} = 1$ for every edge of Γ; that is, $\phi_a(b) = \phi_b(a)$. Note that this condition may not be satisfied. For example, recent work of Joshi on double axes in Matsuo algebras [14] unearthed examples of axial algebras with fusion law $\mathcal{M}(2\eta, \eta)$, where the unique (up to scaling) Frobenius form is not a projection form.

Let us now discuss when the Frobenius form on A is invariant under the Miyamoto group $G(X)$. Clearly, this requires that $(a^g, a^g) = (a, a)$ for all $a \in X$. It turns out that this condition is also sufficient.

Proposition 4.20. *The Frobenius form $(\cdot, \cdot)$ is invariant under the action of $G(X)$ if and only if $(a^g, a^g) = (a, a)$ for all $a \in X$ and $g \in G(X)$.*

Proof. We have already mentioned that if the form is $G(X)$-invariant then $(a^g, a^g) = (a, a)$ for all $a \in X$ and $g \in G(X)$. Conversely, suppose that $(a^g, a^g) = (a, a)$ for all

$a \in X$ and $g \in G(X)$. Fixing g, define a second form $(\cdot, \cdot)'$ by $(u, v)' := (u^g, v^g)$. It is straightforward to check that $(\cdot, \cdot)'$ is bilinear and, furthermore, Frobenius. Since $(a, a)' = (a^g, a^g) = (a, a)$ for each $a \in X$, we deduce from Proposition 4.18 that $(u, v)' = (u, v)$ for all $u, v \in A$. That is, $(u^g, v^g) = (u, v)$, proving that the form is $G(X)$-invariant. □

Finally, if we are only interested in $G(X)$-invariant Frobenius forms then the uniqueness of such form can be checked via the orbit projection graph.

Proposition 4.21. *Let A be a primitive axial algebra with a strongly connected orbit projection graph. Then a $G(X)$-invariant Frobenius form on A, if it exists, is unique up to scaling.*

4.5. **An application: Norton-Sakuma algebras.** The 2-generated primitive axial algebras of Monster type with a Frobenius form are well-known and have been completely classified. There are nine such algebras, known as *Norton-Sakuma* algebras. For the Griess algebra, the list was obtained by Norton [2] and known to be complete after the uniqueness proof for the Monster [6]. The general result was shown in the context of OZ-type VOAs by Sakuma [18] and was proved for axial algebras of Monster type with a Frobenius form by Hall, Rehren and Shpectorov in [7]. All the Norton-Sakuma algebras arise as subalgebras of the Griess algebra and their isomorphism type is determined by the conjugacy class of $\tau_a \tau_b$, where a and b are two axes which generate the algebra. For this reason, they are usually labelled 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A and 6A. For a full description of these see, for example, [13]. We note that the known Frobenius forms on Norton-Sakuma algebras are inherited from the Griess algebra and, as such, they are positive definite projection forms invariant under the respective Miyamoto groups.

Proposition 4.22. *All the Norton-Sakuma algebras, except 2B, are simple and have unique Frobenius form (up to scaling).*

Proof. It follows from the table on page 213 in [13] that $\phi_a(b) = 0$ if and only if a and b generate a 2B algebra; that is, $ab = 0$. In particular, for the algebras 2A, 3A, 3C, 4B, 5A, and 6A the projection graph is a complete (unoriented) graph. For the algebra 4A, the projection graph is the complete graph K_4 minus a matching; that is, a 4-cycle. Hence for all these algebras the projection graph is unoriented and connected and hence strongly connected.

On the one hand, Corollary 4.15 now implies that every proper ideal is contained in the radical, and the latter is trivial by Corollary 4.10. Hence the algebra is simple.

On the other hand, Proposition 4.19 tells us that the Frobenius form is unique up to scaling. □

Note that $2B \cong \mathbb{R} \oplus \mathbb{R}$ and so it is not simple and, furthermore, the values (a, a) and (b, b) for the primitive axes in this algebra can be chosen arbitrarily. So a Frobenius form on 2B is definitely not unique up to scaling.

5. SUM DECOMPOSITIONS

If our definition of radical is good then we can expect that axial algebras with a trivial radical are semisimple, that is, direct sums of simple axial algebras. Hence it is natural to discuss here decompositions of axial algebras.

5.1. **Sums of algebras.** Suppose A is a commutative non-associative algebra and $\{A_i : i \in I\}$, are subalgebras of A, for some indexing set I. Recall that we say A is the *sum* of the subalgebras A_i, written $A = \sum_{i \in I} A_i$, if it is a sum of the A_is as a vector space. In particular, every element $u \in A$ can be written as $u = \sum_{i \in I} u_i$, where $u_i \in A_i$ for all i. As usual, if the decomposition $u = \sum_{i \in I} u_i$ is unique for each $u \in A$, we call A the *direct sum* of the subalgebras A_i and write $A = \bigoplus_{i \in I} A_i$.

Lemma 5.1. *If A is generated by a set $\{A_i : i \in I\}$ of pairwise annihilating subalgebras, then $A = \sum_{i \in I} A_i$ is the sum of the subalgebras.*

Proof. We must show that $\sum_{i \in I} A_i$ is the whole of A. Taking two elements $u = \sum_{i \in I} u_i$ and $v = \sum_{i \in I} v_i$ of the subspace $\sum_{i \in I} A_i$, we see that $uv = (\sum_{i \in I} u_i)(\sum_{j \in I} v_j) = \sum_{i \in I} u_i v_i$, since all other pairwise products are zero. Hence $\sum_{i \in I} A_i$ is closed with respect to multiplication and so it is a subalgebra. Since it also contains all A_i, we conclude that $\sum_{i \in I} A_i$ coincides with $\langle\!\langle A_i : i \in I \rangle\!\rangle = A$. $\qquad\square$

In light of the above result, from now on, we will write $A = \square_{i \in I} A_i$ when A is a sum of pairwise annihilating subalgebras $\{A_i : i \in I\}$ and say it has a pairwise annihilating sum decomposition. In addition, if this sum is direct, we write $A = \boxplus_{i \in I} A_i$. In this case, A is isomorphic to the *external direct sum* defined as the Cartesian product $\prod_{i \in I} A_i$ taken with the entrywise operations.

Recall the following standard definition.

Definition 5.2. The *annihilator* of a commutative algebra A is

$$\text{Ann}(A) := \{u \in A : uA = 0\}.$$

Manifestly, $\text{Ann}(A)$ is an ideal. Returning to the axial algebra case, recall that the radical is the largest ideal $R(A, X)$ of A not containing any axes $x \in X$.

Lemma 5.3. *For an axial algebra A, $\text{Ann}(A) \subseteq R(A, X)$.*

Proof. Clearly, $\text{Ann}(A)$ does not contain any axes in X as $a \cdot a = a \neq 0$ for $a \in X$. $\qquad\square$

The next example shows that the annihilator does not necessarily equal the radical of an axial algebra.

Example 5.4. Recall the Matsuo algebra for the group S_5 from Example 4.12. If $\eta = -\frac{1}{3}$, then the radical is spanned by

$$\sum_{a \in D} a$$

which is easy to check is also in the annihilator. So for $\eta = -\frac{1}{2}$, $R(A, X) = \text{Ann}(A)$.

However, for $\eta = -2$, the radical is 4-dimensional and is spanned by elements of the form

$$(i, j) + (i, k) + (i, l) - (m, j) - (m, k) - (m, l)$$

where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$. However, a simple calculation shows that such a vector is not in the annihilator. Furthermore, these vectors span an irreducible submodule, so this implies the annihilator must be trivial. Hence, for $\eta = -2$, $0 = \text{Ann}(A) \subsetneqq R(A, X)$.

Proposition 5.5. *If $A = \square_{i \in I} A_i$, then*

1. $A_i \cap (\square_{j \neq i} A_j) \subseteq \text{Ann}(A_i)$

2. $\mathrm{Ann}(A) = \square_{i \in I} \mathrm{Ann}(A_i)$

Proof. Suppose that $u \in A_i \cap (\square_{j \neq i} A_j)$ and let $a \in A$. We may decompose $a = \sum_{j \in I} a_j$. We have $ua = \sum_{j \in I} ua_j = ua_i + \sum_{j \neq i} ua_j$. Since $u \in \square_{j \neq i} A_j$, we see that $ua_i = 0$. On the other hand, u is in A_i and hence $\sum_{j \neq i} ua_j = 0$ too. Therefore $uA = 0$ and $u \in \mathrm{Ann}(A) \cap A_i \subseteq \mathrm{Ann}(A_i)$.

Since $A_i A_j = 0$ for all $j \neq i$, $\mathrm{Ann}(A_i) \subseteq \mathrm{Ann}(A)$. Hence $\square_{i \in I} \mathrm{Ann}(A_i) \subseteq \mathrm{Ann}(A)$. Conversely, if $u = \sum_{j \in I} u_j \in \mathrm{Ann}(A)$ then, taking $v \in A_i$, we get that $0 = uv = (\sum_{j \in I} u_j)v = \sum_{j \in I} u_j v = u_i v$. So $u_i v = 0$ for all $v \in A_i$; that is, $u_i \in \mathrm{Ann}(A_i)$. Therefore, $\mathrm{Ann}(A) \subseteq \square_{i \in I} \mathrm{Ann}(A_i)$ and so we have equality. $\square$

Suppose we take two different decompositions of an element u and consider how these can differ.

Lemma 5.6. *Suppose $A = \square_{i \in I} A_i$ and $u \in A$. For any two decompositions $u = \sum_{i \in I} u_i = \sum_{i \in I} u_i'$ of u, the difference $d_i = u_i - u_i'$ lies in $\mathrm{Ann}(A_i)$ for each i.*

Proof. Clearly, $d_i \in A_i$. On the other hand, $d_i = \sum_{j \neq i} (u_j' - u_j) = -\sum_{j \neq i} d_j \in \square_{j \neq i} A_j$. So $d_i \in A_i \cap \square_{j \neq i} A_j$. By Proposition 5.5, $d_i \in \mathrm{Ann}(A_i)$. $\square$

In particular, the following is true.

Corollary 5.7. *If $A = \square_{i \in I} A_i$ and $\mathrm{Ann}(A) = 0$, then $A = \boxplus_{i \in I} A_i$.*

Recall that, for a primitive axial algebra A, $\mathrm{Ann}(A) \subseteq R(A, X)$ and hence the assumption that the annihilator is trivial is satisfied when the radical $R(A, X)$ is trivial.

5.2. **Idempotents.** Axial algebras are generated by idempotents. So let us take a look at idempotents when our algebra has a pairwise annihilating sum decomposition.

Lemma 5.8. *Suppose $A = \square_{i \in I} A_i$ and $a \in A$ is an idempotent. Then a can be written as $a = \sum_{i \in I} a_i$, where every $a_i \in A_i$ is an idempotent.*

Proof. Consider first an arbitrary decomposition $a = \sum_{i \in I} a_i'$ and set $a_i = (a_i')^2$. Note that $a = a^2 = (\sum_{i \in I} a_i')^2 = \sum_{i \in I} (a_i')^2 = \sum_{i \in I} a_i$. By Lemma 5.6, $d_i = a_i - a_i' \in \mathrm{Ann}(A_i)$. Therefore, $a_i^2 = (a_i' + d_i)^2 = (a_i')^2 + 2a_i' d_i + d_i^2 = (a_i')^2 = a_i$. Hence each a_i is indeed an idempotent. $\square$

Recall that we call an axis a primitive when the 1-eigenspace of ad_a coincides with $\langle a \rangle$. Similarly, we call a non-zero idempotent $a \in A$ *primitive* when the 1-eigenspace of ad_a is 1-dimensional.

Lemma 5.9. *Let $A = \square_{i \in I} A_i$. Then*
1. *Every idempotent of A is contained in at most one A_i.*
2. *Every primitive idempotent is contained in exactly one A_i.*

Proof. First of all, note that a non-zero idempotent cannot lie in two summands. Indeed, if $a \in A_i$ and $a \in A_j$ with $i \neq j$ then $a \in A_i \cap A_j \subseteq \mathrm{Ann}(A_i)$. Hence $a = a^2 = 0$; a contradiction.

Write $a = \sum_{i \in I} a_i$, where every $a_i \in A_i$ is an idempotent. Note that $aa_i = (\sum_{j \in I} a_j)a_i = a_i a_i = a_i = 1a_i$. Hence all a_i are contained in the 1-eigenspace of ad_a. By primitivity, if two components, a_i and a_j, are non-zero then $a_i = \lambda a_j$ for some $\lambda \in \mathbb{F}^\times$. Then, $a_i \in A_i \cap A_j$ and so by the first part, $a_i = 0$, a contradiction. $\square$

Theorem 5.10. *Suppose that $A = \square_{i \in I} A_i$ is a primitive axial algebra with a set of axes X. Let $X_i := A_i \cap X$ be the set of axes in A_i and let $B_i = \langle\!\langle X_i \rangle\!\rangle$. Then the X_i partition X and $A = \square_{i \in I} B_i$ is the sum of pairwise annihilating axial subalgebras.*

Proof. By Lemma 5.9, every axis from X lies in one and only one set X_i; that is, the sets X_i form a partition of X.

Clearly, for $i \neq j$, we have $B_i B_j \subseteq A_i A_j = 0$. So we just need to show that the subalgebras B_i generate A. Since X generates A, the algebra is spanned by all products of axes. Hence it suffices to show that each product is contained in some B_i. Clearly, if all axes involved in a product are from the same part X_i then the product lies in B_i. Hence we just need to consider the case where the product w involves axes from two different parts X_i and X_j. In this case we will show that the product is zero by induction on the length of the product. Clearly the length of w is at least two, and so we have $w = w_1 w_2$, where w_1 and w_2 are shorter products. If, say, w_1 involves axes from two different parts then $w_1 = 0$ by induction and so $w = 0$. Hence we can assume that w_1 only contains axes from one part, say X_i. Similarly, we can assume that w_2 only contains axes from X_j. However, this means that $w_1 \in B_i$ and $w_2 \in B_j$, and so $w = w_1 w_2 \in B_i B_j = 0$. So indeed every product lies in some summand B_i and so the subalgebras B_i generate (in fact, span) A. $\square$

This means that if an axial algebra decomposes as a sum of pairwise annihilating subalgebras, it also decomposes as a sum of pairwise annihilating axial subalgebras. Furthermore, the summands come from partitions of the generating set X satisfying $X_i X_j = 0$ for all $i \neq j$.

Theorem 5.11. *Suppose that $A = \square_{i \in I} A_i$ is a primitive axial algebra such that $A = \langle\!\langle X \rangle\!\rangle = \langle\!\langle Y \rangle\!\rangle$ for two different generating sets of axes X and Y. Let $X_i = A_i \cap X$ and $Y_i = A_i \cap Y$ and define $B_i = \langle\!\langle X_i \rangle\!\rangle$ and $C_i = \langle\!\langle Y_i \rangle\!\rangle$ as the axial algebras generated by the X_i and Y_i respectively. Then, $B_i = C_i$ for all $i \in I$.*

Proof. By Theorem 5.10, A has a decomposition $A = \square_{i \in I} B_i$ and another decomposition $A = \square_{i \in I} A_i$. By Lemma 5.9, each axis $y \in Y$ is contained in a unique A_i and a unique B_i. However, since A_i is a subalgebra, $B_i = \langle\!\langle X_i \rangle\!\rangle \leq A_i$ for all $i \in I$. So, for each each $y \in Y$ there exists a unique $i \in I$ such that $y \in B_i \leq A_i$. Hence, $C_i = \langle\!\langle Y_i \rangle\!\rangle \leq B_i$ and by symmetry the result follows. $\square$

Corollary 5.12. *The decomposition of a primitive axial algebra into a sum of pairwise annihilating axial subalgebras is stable under arbitrary changes of axes.*

6. THE NON-ANNIHILATING GRAPH $\Delta(X)$

We can view the results above in a graph-theoretic way.

Definition 6.1. The *non-annihilating graph* $\Delta(X)$ has vertex set X and an edge $a \sim b$ between $a \neq b$ if $ab \neq 0$.

Such a graph was introduced for axial algebras of Jordan type in [9]. In the case of Matsuo algebras, $\Delta(X)$ is also the non-commuting graph of the transpositions X. We will compare the non-annihilating graph to the projection graph introduced earlier in Section 4. In order to make the comparison, we consider all the edges in the non-annihilating graph to be directed edges in both directions.

Lemma 6.2. *The projection graph is a subgraph of the non-annihilating graph.*

Proof. Let $a \not\sim b$ in the non-annihilating graph. Then $ab = 0$ and so the projection of a onto b and b onto a are both zero. $\square$

Lemma 6.3. *For primitive axial algebras of Monster type which admit a Frobenius form which is non-zero on each axis, the non-annihilating graph is the same as the projection graph.*

Proof. For these algebras, every subalgebra $\langle\langle a, b \rangle\rangle$ is a Norton-Sakuma algebra and, by inspection, only the subalgebra 2B has zero projections. $\square$

We note that the condition that the algebra admits a Frobenius form which is non-zero on each axis holds in all known examples.

Suppose that $A = \square_{i \in I} A_i$. Then by Theorem 5.10, we may partition X into a union of X_i and $A = \square_{i \in I} B_i$, where each B_i is an axial algebra generated by X_i. In particular, if $a \in X_i$ and $b \in X_j$, $i \neq j$, then $ab \in B_i B_j = 0$. This means that each X_i is a union of connected components of $\Delta(X)$. It seems natural to ask: is it not true that the finest annihilating sum decomposition of A arises when each X_i is just a single connected component of Δ? For the Monster fusion law, we conjecture that this is indeed the case:

Conjecture 6.4. *The finest pairwise annihilating sum decomposition of a primitive axial algebra A of Monster type arises when each X_i is just a single connected component of Δ.*

Equivalently, set X_i to be the ith connected component of $\Delta(X)$. Then certainly $ab = 0$ for $a \in X_i$, $b \in X_j$, whenever $i \neq j$. Define $A_i = \langle\langle X_i \rangle\rangle$. The above conjecture means that A decomposes as a sum of the A_i. The argument as in Theorem 5.10 above shows that the A_i generate A. What is missing is the claim that $A_i A_j = 0$ for $i \neq j$.

For axial algebras of Jordan type η (those with fusion law $\mathcal{J}(\eta)$), the above conjecture holds and is Theorem A in [9]. We note that axial algebras of Jordan type are 1-closed and their fusion law is Seress.

While we do not have any examples to the contrary, we cannot prove Conjecture 6.4 in full generality. We give a partial result, but before that we show that the groups behave well with respect to the finest sum decomposition.

Before we do so, we make an observation. So far we have completely ignored the fusion law $\mathcal{F}$ for A. However, if $\Delta(X)$ does have more than one component, then in particular there exists two axes $a, b \in X$ such that $ab = 0$. So at the very least we must have that $0 \in \mathcal{F}$.

6.1. **Miyamoto group.** Suppose that our axial algebra A is T-graded, so that it has a Miyamoto group G. Recall that $\mathcal{F}_t$ denotes the part of the grading partition corresponding to $t \in T$. For example, $\mathcal{F}_{1_T}$ is the part corresponding to the identity element $1_T \in T$. We always have that $1 \in \mathcal{F}_{1_T}$.

Lemma 6.5. *Let $a, b \in X$ such that $ab = 0$. Then $0 \in \mathcal{F}_{1_T}$ and $[T_a, T_b] = 1$.*

Proof. Suppose $0 \in \mathcal{F}_t$ for some $t \in T$. Then $0 * 0 \subseteq \mathcal{F}_{t^2}$. On the other hand, Note $b \in A_0(a)$ and $b^2 = b$. This means that $t^2 = t$, and so $t = 1_T$. We have shown that $0 \in \mathcal{F}_{1_T}$.

Now, since $0 \in \mathcal{F}_{1_T}$, $b \in A_0(a)$ is fixed by $\tau_a(\chi)$ for all $\chi \in T^*$. Therefore, $\tau_b(\chi')^{\tau_a(\chi)} = \tau_{b^{\tau_a(\chi)}}(\chi') = \tau_b(\chi')$ and hence $[T_a, T_b] = 1$. $\square$

Theorem 6.6. *Let A be a T-graded axial algebra and the components of $\Delta(X)$ be X_i for $i \in I$. Then, $G(X)$ is a central product of its subgroups $G(X_i)$.*

Proof. The Miyamoto group $G(X_i)$ is generated by all T_a with $a \in X_i$. By Lemma 6.5, $[T_a, T_b] = 1$ for all $a \in X_i$, $b \in X_j$, $i \neq j$. Hence, every element of $G(X_i)$ commutes with every element of $G(X_j)$. Since $G(X) = \langle G(X_i) : i \in I \rangle$, it is a central product of the $G(X_i)$. $\square$

So under the mild assumption that 0 is in the trivially graded part, the finest sum decomposition of the non-annihilating graph induces a central product of the corresponding Miyamoto groups.

6.2. **Quasi-ideals.** Before we consider the algebra decomposition, we first introduce a new concept.

Definition 6.7. Suppose A is an axial algebra generated by a set X of axes. A *quasi-ideal* in A with respect to the generating set X is a subspace $I \subseteq A$ such that $aI \subseteq I$ for all $a \in X$.

Clearly, every ideal is a quasi-ideal. The converse is not true as we will see from the example given at the very end of the paper.

The above definition of a quasi-ideal I depends on a particular set of generating axes. Suppose the fusion law $\mathcal{F}$ is T-graded. Since I is invariant under each ad_a, $a \in X$, Lemma 3.10 implies that I is invariant under the action of each T_a, and hence it is invariant under $G(X)$. Therefore, for every $b = a^g \in \bar{X}$, we have that $bI = a^g I^g \subseteq (aI)^g = I^g = I$ and so I is also a quasi-ideal with respect to the closure $\bar{X}$ of X. We have the following.

Proposition 6.8. *Let $I \subseteq A$ and X and Y be two sets of axes.*

1. *If I is a quasi-ideal with respect to X, then it is invariant under the action of $G(X)$.*
2. *If $X \sim Y$, then I is a quasi-ideal with respect to X if and only if it is a quasi-ideal with respect to Y. That is, being a quasi-ideal is stable.*

So the concept of quasi-ideals behaves well with respect to natural changes of generators. We now introduce an important example of a quasi-ideal.

Definition 6.9. The *spine* of an axial algebra A is the quasi-ideal $Q(A, X)$ generated by all axes X. If $Q(A, X) = A$, then we say that A is *slender*.

It is clear that the spine contains the axes X and is spanned by all products of the form $x_1(x_2(\ldots(x_{k-1}x_k)\ldots))$ where $x_i \in X$. Recall that A is m-closed if it is spanned by products of axes of length at most m.

Lemma 6.10. *If A is a 3-closed axial algebra, then A is slender.*

Proposition 6.11. *Let A be an axial algebra and $X \sim Y$ be two equivalent sets of axes. Then, $Q(A, X) = Q(A, Y)$. That is, the spine of an axial algebra is stable.*

Proof. It suffices to show equality of $Q(A, X)$ and $Q(A, \bar{X})$. Clearly, $Q(A, X) \subseteq Q(A, \bar{X})$. On the other hand, by Proposition 6.8, $Q(A, X)$ is invariant under $G(X)$, which means that $\bar{X} \subset Q(A, X)$. Hence $Q(A, \bar{X}) \subseteq Q(A, X)$, and hence we have equality. $\square$

6.3. **Algebras with Seress fusion laws.** As we noted before, $0 \in \mathcal{F}$. However, if A were to have annihilating sum decompositions, then this imposes further constraints on $\mathcal{F}$. If an axis a lies in the summand A_i then every A_j, $j \neq i$, is contained in the 0-eigenspace of ad_a, since $aA_j = 0$. In particular, as A_j is a subalgebra, $0 \in 0 \star 0$. In order to show our partial result, we will, in fact, require a lot more than this.

Definition 6.12. The fusion law $\mathcal{F}$ is *Seress* if $0 \in \mathcal{F}$ and for any $\lambda \in \mathcal{F}$ we have $0 \star \lambda \subseteq \{\lambda\}$.

Note that for 1, we already have that $1 \star \lambda \subseteq \{\lambda\}$. So, for Seress fusion laws, it follows that $1 \star 0 \subseteq \{1\} \cap \{0\} = \emptyset$. Also note that $0 \star 0 \subseteq \{0\}$ implies that $A_0(a)$ is a subalgebra for every axis a.

The following lemma was first given by Seress for the Monster fusion law (hence the name Seress for the property of the fusion law). Hall, Rehren and Shpectorov noticed that the same proof holds in a more general setting.

Lemma 6.13 (Seress Lemma)**.** [7, Proposition 3.9] *If $\mathcal{F}$ is Seress, then every axis a associates with $A_1(a) + A_0(a)$. That is, for $x \in A$ and $y \in A_1(a) + A_0(a)$, we have that*
$$a(xy) = (ax)y.$$
In other words, ad_a and ad_y commute.

Proof. Since the associativity identity is linear in y, we may consider $y \in A_1$ and $y \in A_0$ separately. Associativity is also linear in x, so, since we may decompose x with respect to $A = \bigoplus_{\lambda \in \mathcal{F}} A_\lambda(a)$, it suffices to check for $x \in A_\lambda$. As $\mathcal{F}$ is Seress, $1 \star \lambda, 0 \star \lambda \subseteq \{\lambda\}$ and so $xy \in A_\lambda$ for $y \in A_1$, or $y \in A_0$. Hence,
$$a(xy) = \lambda xy = (\lambda x)y = (ax)y. \qquad \square$$

Suppose A is generated by the set of axes $X = Y_1 \cup Y_2$, where for all $a \in Y_1$ and $b \in Y_2$ we have $ab = 0$. (We write $Y_1 Y_2 = 0$.) Let $A_i = \langle\!\langle Y_i \rangle\!\rangle$.

Theorem 6.14. *Let A be an axial algebra with $X = Y_1 \cup Y_2$ satisfying $Y_1 Y_2 = 0$. If the fusion law is Seress, then $Q(A_1, Y_1)$ annihilates A_2.*

Proof. First of all, note that, for $x \in Y_1$, since $A_0(x)$ is a subalgebra and $Y_2 \subseteq A_0(x)$, we have that $xA_2 = 0$. This means that $Y_1 \subseteq U$, where $U := \{u \in A_1 : uA_2 = 0\} = \mathrm{Ann}(A_2) \cap A_1$ is the annihilator of A_2 in A_1. Now, for $u \in U$ and $v \in A_2$, by Seress's Lemma, $(xu)v = x(uv) = x0 = 0$. So, $xu \in U$. Since this is true for all $x \in Y_1$, U is a quasi-ideal. Therefore, U is a quasi-ideal containing Y_1, implying that $Q(A_1, Y_1) \subseteq U$. $\qquad \square$

Corollary 6.15. *Suppose that A is an axial algebra such that $X = Y_1 \cup Y_2$ is a disjoint union of axes and the fusion law is Seress. If A_1 is slender, then $A = A_1 \square A_2$.*

We can now state our partial result for the conjecture about the non-annihilating graph.

Theorem 6.16. *Let A be an axial algebra with a Seress fusion law and let X_i be the components of $\Delta(X)$ with $A_i = \langle\!\langle X_i \rangle\!\rangle$. If all but possibly one A_i are slender, then $A = \square_{i \in I} A_i$.*

Proof. This follows from Corollary 6.15 using induction on $|I|$. $\qquad \square$

Recall that a 3-closed axial algebra is slender, so the above theorem holds when all but at most one A_i are 3-closed.

As noted above, an axial algebra A of Jordan type η is 1-closed and Seress. So, by Corollary 6.16, $A = \square_{i \in I} A_i$, where X_i are the connected components of $\Delta(A)$. This is part (2) of Theorem A in [9].

The Ising fusion law $\mathcal{M}(\alpha, \beta)$, of which the Monster fusion law is a special case, is also Seress. Most of the examples we know for $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ are 2-closed, while a few are 1- or 3-closed [15, Table 4]. So we should expect the above decomposition theorem to apply to a wide class of examples.

However, there exist examples of axial algebras with fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ that are not 3-closed. In [15], we found an 18-dimensional primitive axial algebra with Miyamoto group $S_3 \times S_3$, which is 4-closed, but not 3-closed. In fact, in this example, $17 = \dim(Q(A, X)) < \dim(A) = 18$. So this algebra is not slender. Note also that, since it is 4-closed and $Q(A, X) \subsetneq A$, there exists some product of axes of the form $(ab)(cd)$ that completes a basis for $Q(A, X)$ to a basis for A. Indeed this shows that $Q(A, X)$ is not an ideal and hence that not all quasi-ideals are ideals.

Finally, we note that none of the proofs in this last section required primitivity.

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