The Radiation Efficiency of a Small Loop Antenna

For cave radio applications, we are not normally interested in the radiation from a loop antenna because the distance over which we are working is small (relative to a wavelength) and so only near-field effects need to be considered. However, it is still interesting to consider the radiation field, and to express the radiation efficiency in terms of the specific aperture where, just as for near-field operation, the number of turns on the antenna does not affect the result. ***David Gibson*** *explains radiation resistance, summarises the equations and shows how the skin effect in the wire and the proximity effect between turns of the antenna winding can be taken into account.*

The main purpose of this note was originally to prepare the ground for any reader who wished to consult papers on the proximity effect, e.g. [Smith, 1972a; b], that discuss it alongside radiation efficiency. One difficulty with those papers is in understanding the different notation in use, which this note hoped to clarify. However, I have now decided that, for cave radio purposes, a detailed study of the proximity effect is probably not required.

Introduction and Background: Maxwell's Equations

The equations of electrostatics and magnetostatics allow us to describe the fields and forces that arise from electric charge and current. They tell us, for example, that the magnetic field of an induction loop falls off with an inverse cube law, and that the magnetic field from a short wire falls off with a square law. The equations also tell us that, for a *long* wire, the magnetic field falls off with an inverse linear law. Thus, there is clearly much we can do with these laws, but the one thing they do not predict *at all* is radiation.

In this respect, the equations are similar to Newton's laws of motion, which do not, *in any way*, predict relativity. And, just as Einstein built on Newton's work, showing that very rapid motion gave rise to hitherto unexpected effects, so Maxwell built on the work of Ampère, Faraday, Gauss, Ørsted and others, showing that a rapid time variation of current gave rise to another hitherto unexpected effect.

Maxwell's equations demonstrate that the quasi-static near-fields are accompanied by far-fields for E and H that fall off in an inverse linear fashion, giving rise to a square law for a radiation of power.

For cave radio applications, we are not normally interested in the radiation from a loop antenna because it is so small, although Maxwell's equations are necessary to explain the 'optimum frequency' derivation for a cave radio system, (see [Gibson, 2010] §2.2.4). Radiation remains of interest, because it can be expressed in the same terminology (specific aperture) that we use for near-field systems.

Radiation Resistance

If we imagine the antenna as its Thévenin equivalent circuit, it is clear that the radiated power must be represented, from the point of view of the power source, by a resistance in which the power is dissipated. We call this the *radiation resistance.* Clearly, it is not a physical or 'ohmic' resistance but, as far as the power source is concerned, it exists; and it allows us to easily model the system. Radiation resistance also plays a part in receiving antennas, but that is a difficult concept, which I will avoid discussing here.

Given Maxwell's equations, the radiation resistance of a simple antenna is straightforward to calculate. We write down the fields produced by the current elements and we integrate those over a spherical surface that encloses the source at a sufficiently large distance that the near-field effects are played out inside it. This allows us to calculate the power flux across the surface of the sphere. Since we know the power P and the source current I the radiation resistance is simply $R = P / I^2$.

This procedure is covered in many textbooks. It can be shown that the radiated power of a small (relative to a wavelength) circular loop is[1]

$$P_r = m_d{}^2 \frac{Z_0 k_0{}^4}{6\pi} \qquad (1)$$

If the number of turns is N and the area of the loop is A then, assuming the current is uniform along the total length of the wire ($Nbk_0 << 1$), we can write the radiation resistance directly from (1) as

$$R_r = \frac{P_r}{I^2} = (NA)^2 \frac{Z_0 k_0{}^4}{6\pi} \qquad (2)$$

1 Symbols are listed at the end of this article.

but you may see this quoted in different forms. We can note that...

- The radiation resistance is proportional to the square of the number of turns. This immediately tells us that R_r is not a 'normal' resistance, which would be proportional to N, not to N^2.
- R_r is proportional to the inverse fourth power of frequency, so at the low frequencies we use for cave radio, it is practically zero.
- For a source of zero size ($A = 0$) the radiation resistance is zero. You may find, on the Internet, that R_r for an 'infinitesimal' dipole is frequently given as 0.3 Ω. This is ***not*** true!

Radiation Efficiency

Although textbooks give formulas for radiation resistance, they do not, generally, discuss radiation *efficiency*. This is because, for normal radio applications, R_r is high (e.g. ≈73 Ω for a half-wave dipole) and it swamps the low ohmic resistance R. But in our case, R_r is much lower than R.

The radiation *efficiency* is $R_r/(R+R_r)$. But if $R_r << R$, as it usually is for cave radio, then a simpler formula, R_r/R – the *radiation fraction* – will give a similar result.

It is useful to express R in terms of a figure of merit that I call specific aperture Φ [Gibson, 1999], which is defined by

$$m_d = NIA = \Phi\sqrt{P} \qquad (3)$$

from which the ohmic resistance is just

$$R = (\Phi/NA)^2 \qquad (4)$$

With the specific aperture Φ defined as

$$\Phi = \tfrac{1}{2} b \sqrt{M \frac{\sigma}{\rho}} \qquad (5)$$

we can substitute (4) in (2) to get

$$\frac{R_r}{R} = \frac{P_r}{P} = \Phi^2 \frac{Z_0 k_0{}^4}{6\pi} \qquad (6)$$

The salient point is that the radiation fraction (and the radiation efficiency) does not depend on the number of turns! The result can be used alongside formulas for Q-factor and bandwidth which, likewise, do not depend on N when expressed in terms of Φ but this note is too short to give examples and develop the concept further.

Skin Effect

In addition to assuming 'small' antennas, (6) also assumes a uniform distribution of current across the antenna wire. In practice, this is not the case, due to the skin effect, which causes it to fall off as $\exp(-r/\delta)$ where r is the distance from the surface of the wire and δ is the skin depth,

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{7}$$

There is the additional assumption that $\delta << a$. With this assumption, it can be shown that the total current is equivalent to a uniform current flowing in a skin of width δ. The cross-sectional area of the skin is $2\pi a\delta$ so R is increased by

$$\frac{\pi a^2}{2\pi a\delta} = \tfrac{1}{2} a \big/ \delta \tag{8}$$

and P_r/P drops by this factor and is now

$$\frac{R_r}{R} = \frac{P_r}{P} = \Phi^2 \frac{Z_0 k_0{}^4}{6\pi} \Big/ \left(\frac{1}{2}\frac{a}{\delta}\right) \tag{9}$$

Multi-Turn Antennas – The Proximity Effect

The proximity effect is similar in concept to the skin effect – that is, a redistribution of the current in the wires – but the proximity effect operates independently of the skin effect and has the effect of forcing the current to the outermost edges of a bundle of wires. We can consider two extreme cases – if the wires in the bundle are very close together then the current distribution is like that of a single larger wire and, if they are well spaced, they will behave like individual wires. In neither case is there any specific 'proximity' effect but, between the extremes, there is an additional effect. I described the proximity effect in [Gibson, 1995], referring to the classic Butterworth paper, to which [Smith, 1972a] also refers. In summary we can add a term to (6) to get

$$\frac{R_r}{R} = \frac{P_r}{P} = \Phi^2 \frac{Z_0 k_0{}^4}{6\pi} \Big/ \left(\frac{1}{2}\frac{a}{\delta}\left(\frac{R_p}{R_s}+1\right)\right) \tag{10}$$

where R_p is the resistance due to the proximity effect and R_s is due to the skin effect alone. There is the assumption that the current is uniform over the length of the wire; i.e. the length is much less than λ:

$$N k_0 b << 1 \tag{11}$$

The million-dollar question is, of course, "how do we calculate R_p / R_s?".

Smith's papers go into this in detail, but the analysis is complicated. Fortunately, *Figures 5 and 6*, and *Tables I and II* in [Smith, 1972a] provide some answers, but my current thinking is that, in reality, we do not need to worry about the proximity effect, *provided* that we space our conductors by c/a > 2 and preferably c/a > 4, where c is Smith's notation for the half-spacing. That is, the conductors should be four radiuses apart. This conclusion is also reached by [Paul, 2009]. This spacing has other advantageous effects such as reducing the mutual inductance [Gibson, 2019] and the self-capacitance.

Why all the fuss about proximity effect in the past? Originally the problem was how to wind high-efficiency air-cored coils for RF applications in situations where the space for a winding was limited. One question was how many turns should be crammed into the space. Our applications are slightly different, and we usually have the luxury of designing an antenna from scratch. It is therefore better to design to avoid both the skin and proximity effects.

Non-Radiating Antennas

For most cave radio antennas (10) is not helpful, because there is very little radiation. But the net effect of (10) is to reduce Φ, whether there is radiation or not, to

$$\Phi \to \Phi \Big/ \sqrt{\frac{1}{2}\frac{a}{\delta}\left(\frac{R_p}{R_s}+1\right)} \tag{12}$$

Example: suppose we design a cave radio loop antenna. We space the turns to eliminate the proximity effect, and we calculate δ=0.3 mm with 2 mm dia. wire. (12) tells us that Φ will be reduced to 77% of the value we expected. So, using 2 mm wire to boost the mass of the antenna was possibly a counter-productive idea.

(8) seems to suggest that there is a skin depth *advantage* if $a < 2\delta$. This would be impossible and what it really indicates is that the formulation does not apply when $\delta << a$ is not true. Thus, it also indicates that $a < 2\delta$ is a good design rule.

Smith's Analysis

If you wish to study Smith's results, you will hopefully find this introductory note useful. Smith does not use the concept of a radiation fraction – he uses radiation *efficiency* – so his expressions are a little more complicated. He also uses a different terminology, using "20" for $Z_0/6\pi$, and n, β_0, R^s, R_0 where I use N, k_0, Z_s, R_s. Also, he does not simplify using Φ. But my (10) is equivalent to his (9).

Smith uses surface resistance (a.k.a. surface impedance) in his formulations for radiation efficiency, rather than skin depth. The *surface impedance* Z_s is

$$Z_s = \frac{1}{\delta\sigma} = \sqrt{\frac{\omega\mu}{2\sigma}} \tag{13}$$

Now the skin resistance per unit length is

$$\frac{R_s}{\ell} = \frac{1}{2\pi a\delta\sigma} \tag{14}$$

but $\ell = 2\pi bN$ and so we can write

$$R_s = Z_s \frac{bN}{a} \tag{15}$$

There is no point in trying to substitute that into (10) because Z_s offers no advantage to us over δ as a way to characterise the antenna – especially when we have already eliminated N.

Concluding Remarks

We have characterised the radiation efficiency of a loop antenna in terms of the specific aperture, allowing for skin depth in the wire and the proximity effect (10). However, we have asserted that a detailed study of proximity effect is not required – we just need to space the conductors by about two diameters. This principle applies to all antennas – even if non-radiating – and it should be remembered that in most cases, small loop antennas do not radiate any appreciable power. In such a situation (6), (9) and (10) are clearly useless, but the general principles remain intact, with (3) and (12) applying.

References

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Symbols Used in this Article

λ	Wavelength	m
σ	Electrical conductivity of the wire	S/m
ρ	Mass density of wire	kg
ω	Angular frequency	Hz
μ	Magnetic permeability of free space	H/m
δ	Skin depth	m
Φ	Specific aperture	$m^2/\sqrt{\Omega}$
A	Area of loop	m^2
a	Radius of wire	m
b	Radius of loop	m
c	Half-spacing between turns	m
I	Current	A
k_0	Wave number, $2\pi/\lambda$	m^{-1}
ℓ	Length	m
M	Mass of wire in an antenna	kg
m_d	Magnetic dipole moment, NIA	Am^2
N	Number of turns of wire	–
P P_r	Antenna power: dissipated / radiated	W
R	Ohmic resistance of antenna	Ω
R_p R_r R_s	Resistance: proximity, radiation, skin	Ω
Z_0 Z_s	Impedance: free space, surface	Ω