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A GRAPH-THEORETIC ANALYSIS OF THE SEMANTIC PARADOXES

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Abstract. We introduce a framework for a graph-theoretic analysis of the semantic paradoxes. Similar frameworks have been recently developed for infinitary propositional languages by Cook [5, 6] and Rabern, Rabern, and Macauley [16]. Our focus, however, will be on the language of first-order arithmetic augmented with a primitive truth predicate. Using Leitgeb's [14] notion of semantic dependence, we assign reference graphs (rfgs) to the sentences of this language and define a notion of paradoxicality in terms of acceptable decorations of rfgs with truth values. It is shown that this notion of paradoxicality coincides with that of Kripke [13]. In order to track down the structural components of an rfg that are responsible for paradoxicality, we show that any decoration can be obtained in a three-stage process: first, the rfg is unfolded into a tree, second, the tree is decorated with truth values (yielding a dependence tree in the sense of Yablo [21]), and third, the decorated tree is re-collapsed onto the rfg. We show that paradoxicality enters the picture only at stage three. Due to this we can isolate two basic patterns necessary for paradoxicality. Moreover, we conjecture a solution to the characterization problem for dangerous rfgs that amounts to the claim that basically the Liar- and the Yablo graph are the only paradoxical rfgs. Furthermore, we develop signed rfgs that allow us to distinguish between 'positive' and 'negative' reference and obtain more fine-grained versions of our results for unsigned rfgs.

§1. Introduction. 'Why are some sentences paradoxical while others are not? Since Russell the universal answer has been: circularity, and more especially self-reference.' These are the opening lines of Stephen Yablo's article 'Paradox without self-reference' [22] that he concludes with the assertion that self-reference is neither necessary nor sufficient for liar-like paradoxes, drawing on the now famous example of an infinite sequence of sentences each of which says that all the sentences appearing later in the sequence are not true.

In 1970, about two decades before Yablo's discovery, Hans Herzberger [10] already argued that there are referential patterns other than circularity that should be counted as pathological. According to his approach, any sentence has a *domain*, the *set of objects it is about*. Herzberger concedes that 'the general notion of a domain is more readily indicated than explicated'. However, he gives the following rules of thumb. A sentence of the form 'A is (not) true' is about A; a sentence of the form 'All φs are (not) true' is about all the φs. Of course, some objects in the domain of

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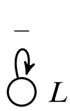
a sentence may be sentences themselves. Those sentences, too, have their own domain that may include sentences, and so forth. Let $D(\varphi)$ be the domain of the sentence φ and $D^2(\varphi)$ the union of the domains of all sentences in $D(\varphi)$. In this way, $D^k(\varphi)$ can be defined for all natural numbers k. (This hinges of course on the assumption that we have a definition of 'domain'.) Herzberger calls a sentences φ *groundless* iff for all k, $D^k(\varphi)$ is not empty. According to this picture, both the liar and Yablo's paradox are groundless; but while the liar is about itself, hence circular, no member of the Yablo sequence refers (directly or indirectly) to itself. Not all groundless sentences give rise to actual antinomies (the nonparadoxical truth-teller sentence is clearly groundless, for example), but they all suffer from 'vicious semantic regress', a form of 'semantic pathology' more general than merely involving a vicious circle, which, according to Herzberger, is responsible for the fact that groundless sentences 'lose their comprehensibility'. Thus, actual contradiction is 'but the extreme symptom of semantic pathology' ([10, pp. 149–150]).

Yablo did not answer the question 'Why are some sentences paradoxical while others are not?'; but the idea that each sentence has a domain invites the following crude answer:

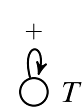
> *Some sentences are paradoxical because of their position in the reference graph of our language*, i.e., in the directed graph whose vertices are the sentences of the language, where two sentences ϕ, ψ are connected by an arc from φ to ψ iff φ is about (refers to, depends on) ψ.

Let us have a look at some informal examples. The paradigms of a self-referential statement are the liar and the truth-teller and it is plausible to represent their reference patterns by simple loops. In order to distinguish them, we might assign a '$-$' to the liar and a '$+$' to the truth-teller, indicating that the liar makes a negative statement about itself whereas the truth-teller makes a positive statement about itself.

L: (L) is false



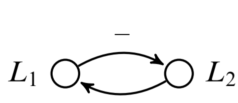
T: (T) is true



We can also consider pairs of sentences that, even if they are not directly self-referential, still exhibit some kind of circularity:

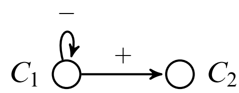
L_1: (L_2) is false

L_2: (L_1) is true



Similarily, for every natural number n, we can consider liar cycles of length n. A slightly different example is given by a version of Curry's paradox:

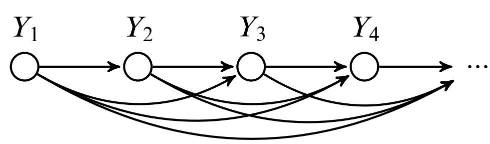
C_1: (C_1) is false or (C_2) is true $\qquad\qquad\qquad\qquad\qquad$ $C_2 : 1 + 1 = 3$



It is clear that self-reference or circularity is not a *sufficient* condition for paradox. But is self-reference or circularity a *necessary* condition for paradox? According to Yablo [22] that's not the case. Consider Yablo's paradox:

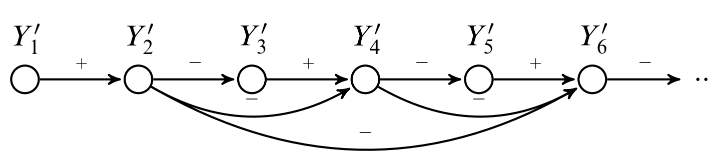
Y_1: (Y_n) is false for all $n > 1$,
Y_2: (Y_n) is false for all $n > 2$,
Y_3: (Y_n) is false for all $n > 3$,
Y_4: (Y_n) is false for all $n > 4$,
...

Informally still, we may represent the Yablo sequence by the following graph which does not contain any cycles.



In the above picture, every arc should be assigned a '−'. Variations on the Yablo sequence deliver new paradoxes whose reference graphs do not contain any loops.

Y_1': (Y_2') is true,
Y_2': (Y_3') is false and for all even $n > 2$, (Y_n') is false,
Y_3': (Y_4') is true,
Y_4': (Y_5') is false and for all even $n > 4$, (Y_n') are false,
Y_5': (Y_6') is true,
...



This raises the question of how many types of paradox there are and what reference patterns underlie them. What are the 'paradoxical nodes' of the reference graph? And can they be characterized in graph-theoretic terms? We shall call any approach to semantic paradoxes that is concerned with identifying paradoxical reference patterns a *Herzbergerian* or *reference-based* theory of semantic paradoxes. In order to develop such an account we have to (i) give a rigorous definition of the aboutness (reference, dependence) relation, (ii) give a rigorous definition of (potential) paradoxicality, and (iii) name a class of graph-theoretic properties that specify which nodes in the reference graph are (potentially) paradoxical.

1.1. Outline of the article. There have been several quite interesting approaches to characterize the notion of a paradoxical sentence for *infinitary propositional* languages using (unsigned) reference graphs (i.e., graphs that do not distinguish between 'positive' and 'negative' reference). We mention here in particular the work of Cook [5, 6] and, more recently, Rabern, Rabern, and Macauley [16].[1] Our goal is to extend their work in several ways: first, to develop reference graphs (rfgs) for first-order languages; second, to develop new tools for investigating paradoxical reference patterns; third, to develop the notion of a signed rfg. For definiteness, our study will focus on the language of arithmetic augmented with a primitive truth predicate. However, nothing essential hinges on the details of this language and we could have chosen any other interpreted first-order language that contains names and predicates applying to its own expressions as well. Choosing the language of arithmetic is convenient in so far as (i) it contains (via coding) a theory of its own syntax, and (ii) it is commonly used in the literature on formal theories of truth.

Propositional languages offer a straightforward way of defining rfgs in terms of the syntactic constituents of a sentence: ϕ refers to ψ iff ϕ contains a name of ψ within the scope of the truth predicate. This method no longer yields satisfactory results when we move to first-order languages. Instead, we will utilize Leitgeb's [14] notion of semantic dependence for that purpose: we assign rfgs to sentences in such a way that every node depends on the set of all its out-neighbours. An rfg can be decorated with truth values. A decoration is acceptable iff it assigns to each node (sentence) a value that is identical with its truth value relative to the Leitgeb valuation scheme V_L (given truth values for its out-neighbours). V_L piggybacks on the notion of dependence and can be treated in the general framework of Kripke [13]. We call a sentence *referentially paradoxical* iff it has no rfg that admits an acceptable decoration, and show that this is exactly the case if it has no truth value in any Kripke fixed point of V_L. The notion of referential paradoxicality bears some resemblance to Yablo's [21] notion of dependence-paradoxicality. A major difference, however, is that Yablo does not provide a concept of a reference graph, but only that of a dependence tree, which corresponds rather to a decorated unfolding of an rfg than to the rfg itself.

In Section 3 we will introduce a game-theoretic approach as a new tool of relating the reference pattern of a sentence to its paradoxicality. The major motivation for this is that we want track down the structural components of an rfg that are responsible for paradoxicality. We show that any decoration can be obtained in a three-stage process: first, the rfg is unfolded into a tree, second, the tree is decorated (yielding a dependence tree in the sense of Yablo [21]), and third, the decorated tree is re-collapsed onto the rfg. The concept of the *verification game* allows us to comprehend this three-stage

[1]Further notable work in this area includes Bolander [3], Dyrkolbotn and Walicki [8], Gaifman [9], and Walicki [19].

process as a game of perfect information between two players being played on the rfg.

In Section 4 we show that paradoxicality enters the picture only at stage three. Due to this we can isolate two basic patterns necessary for paradoxicality: the cycle and the so-called double path. In order to achieve these results (and many of those of Section 5) we develop the dichotomy between the *core* and the *periphery* of an rfg. We show that a reference pattern is only dangerous if it is located in the core. We also conjecture a solution to the characterization problem for dangerous rfgs that amounts to the claim that basically the Liar- and the Yablo graph are the only paradoxical rfgs.

In Section 5 we provide a more refined notion of reference that allows to distinguish, e.g., between the reference graph of the liar and that of the truth-teller. To this end we label the arcs of a large class of rfgs by '+' or by '−' depending on the syntactic structure of the nodes. Refined versions of our previous results are obtained: cycle as well as a double paths are dangerous only if they contain an odd number of negative arcs.

For reasons of space, the following interesting questions cannot be treated in the present article: are the methods and tools developed here applicable to Rabern et al.'s and Cook's framework? Can their notion of paradoxicality be interpreted in terms of a Kripke fixed point construction? Is there an analogon to our verification game that can be played on their graphs?, etc. In future work we will show that the answer to all of these questions is affirmative. In some sense, the frameworks of Rabern et al. and Cook can be embedded into ours. Key to this is the observation that their reference graphs can be defined in terms of a notion of dependence that corresponds to the Weak Kleene scheme (for infinitary propositional languages) in a similar sense as Leitgeb's notion of dependence corresponds to V_L.

1.2. Technical preliminaries. A *directed graph* consists of a set $V(G)$, the *vertices* (or *nodes*) of G, and of a set $A(G)$ of ordered pairs of vertices, called *arcs* of G. Throughout this article, we will use the shorter 'graph' instead of 'directed graph', except for Appendix A. There we will also discuss about undirected graphs and therefore call directed graphs explicitly 'digraphs'. If $x, y \in V(G)$ we denote an arc from x to y by (x, y); we call x its *tail* and y its *head*. For any vertex x, we call y an *out-neighbour* of x iff $(x, y) \in A(G)$ and an *in-neighbour* iff $(y, x) \in A(G)$. A graph H is a *subgraph* of G iff $V(H) \subseteq V(G)$ and $A(H) \subseteq A(G)$. In this case we also say that G *contains* H and write $H \subseteq G$.

A nonempty graph P (i.e., a graph with at least one vertex) is called a *path* (from a to b, of length $n-1$) iff there is an enumeration $(v_0, v_1, \dots, v_n)$ of $V(P)$ such that for all $0 \leq i, j \leq n$ $(v_i, v_j) \in A(P)$ iff $j = i + 1$ with $a = v_0$ and $b = v_n$. Note that a graph with one vertex and no arcs is a path of length 0. We call such a path *trivial*. A graph C is called a *cycle* (of length $n+1$) iff there is a (possibly trivial) path P of length n from a to b such that $V(C) = V(P)$ and $A(C) = A(P) \cup \{(b, a)\}$. A cycle of length 1 is called a *loop*.

For any graph G, call an infinite sequence of vertices $(v_0, v_1, \dots)$ of $V(G)$ an *infinite walk* in G iff for all $i \in \omega (v_i, v_{i+1}) \in A(G)$. Analogously we can define a finite walk. Note that a path must be itself a graph while a walk need not. Hence, one and the same vertex may occur more than once in a walk, while the sequence enumerating the vertices of a path P contains every vertex of P only once. A graph G is called *well-founded* iff there is no infinite walk in G.

We call a graph H a *subdivision* of G iff H is the result of replacing each $(x, y) \in A(G)$ by some path from x to y (possibly of length 1). A graph H is an *induced subgraph* of a graph G iff H is a subgraph of G and each arc of G between two vertices of H is also an arc of H.

A graph G is an *accessible pointed graph* iff G has a distinguished node, called its *root*, and every node of G is accessible from its root. Here, 'y is accessible from x' means that y can be reached in a *finite walk* starting from x, i.e., there is a sequence $(z_1, \dots, z_n)$ of nodes such that $z_1 = x$, $z_n = y$ and for each $1 \leq i < n$ there is an arc of G from x_i to x_{i+1}.

A tree T is a set of finite sequences (over some domain of objects) that is closed under initial segments, i.e., such that for all $s \in T$, if $t \subseteq s$ then $t \in T$. We will refer to the elements of the tree (the sequences) as *positions* of the tree. If s is a position of T, the set of T-*children* of s is the set $\{b \mid s \circ b \in T\}$, where $s \circ b$ denotes the result of extending the sequence s with the object a. If a is the last element of s, we will sometimes (by slight abuse of language) refer to the set $\{b \mid s \circ b \in T\}$ as the T-children of a. A position s of T is a *root* of T iff s has length 1. If s is a root, say $s = (a)$, we will (by slight abuse of language) also refer to the object a as a root of T. In this article, we will mostly be concerned with trees that have a unique root. A *branch* of a tree T is a subset of T that is linearly ordered by $\subseteq$ and closed under initial segments.

The object of our study is the first-order language of Peano arithmetic augmented with a primitive unary predicate symbol T. We denote the set of its sentences by $\mathcal{L}_T$. We fix some coding of $\mathcal{L}_T$ into ω; for technical simplicity, we assume it is a bijection, but nothing substantial hinges on that. We will frequently identify sentences with their codes. The language of $\mathcal{L}_T$ contains a name for each sentence φ—i.e., the numeral of (the code of) φ—that we shall denote by $\ulcorner\varphi\urcorner$. If φ is a sentence and $S \subseteq \mathcal{L}_T$ (i.e., $S \subseteq \omega$), we let $Val_S(\varphi)$ denote the truth value of φ in the classical interpretation $(\mathbb{N}, S)$, where $\mathbb{N}$ is the standard model of arithmetic and S is the extension (interpretation) of the truth predicate T. We also write $(\mathbb{N}, S) \models \phi$ to indicate that ϕ is true in the model $(\mathbb{N}, S)$. Throughout this article, $\models$ always refers to the classical satisfaction relation.

§2. Reference graphs for first-order languages. In the introduction, we have assigned rfgs to several paradoxical sentences relying merely on our intuitions about their referential relations.[2] In order to give a rigorous graph-theoretic analysis of the semantic paradoxes we need, of course, a systematic way of assigning an rfg to every sentence of our language.

[2]These intuitions, of course, shall prove to be accurate: According to the formal definition given in Section 2.5, each of the graphs depicted in the introduction is the canonical rfg (of the $\mathcal{L}_T$-formalization) of the corresponding sentence.

In order to do so, we need a precise definition of what we may call, following Herzberger [10], the *domain* of a sentence, i.e., the set of sentences that a given sentence is about (refers to, depends on). Propositional languages offer a simple way of doing that: ϕ refers to ψ iff there is a name α of ψ such that 'α is true' is a subformula of ϕ. A straightforward way to generalize this is to regard a quantified sentence as referring to all the sentences that its instances are referring to (amounting roughly to a Weak Kleene reference relation). We find this unsatisfactory, however. For example, we would like to say that a sentence of the form $\forall x(\phi(x) \to Tx)$ refers to all and only the objects satisfying the predicate $\phi(x)$. Under the current suggestion, the sentence would instead refer to every sentence. One reason why this is unfortunate is that in first-order languages, liar-like sentences often involve (restricted) quantifiers, e.g., $\exists x(\mathit{diag}(x) \wedge \neg Tx)$, where $\mathit{diag}(x)$ indicates that x is obtained from some diagonalization operation, or 'There is one and only one sentence written on the blackboard of room 104 and that sentence is false' (to take a natural language example). Clearly, we do not want to say that such sentences refer to all sentences of the language. Therefore, our proposal is to identify domains of sentences with dependence sets in the sense of Leitgeb [14], which seem more well-suited for that purpose.[3] Rfgs can then be defined in a straightforward way: the set of out-neighbours of a node ϕ must constitute a dependence set for ϕ. However, some sentences don't have a canonical dependence set, and accordingly many sentence will be assigned a *set* of rfgs. Based on Leitgeb's notion of semantic dependence, we will introduce a valuation scheme (called the Leitgeb valuation scheme, V_L) in terms of which we will define the notion of an acceptable decoration of an rfg. Then, generalizing on Rabern et al. [16], we will say that a sentence ϕ is paradoxical if and only if *there is no rfg* of ϕ that admits an acceptable decoration (where the quantifier in the definition takes care of the fact that ϕ might have several rfgs). As we will see later, our notion of paradoxicality coincides (extensionally) with that of Kripke [13] with respect to the Leitgeb valuation scheme. Therefore, we will start by reviewing Kripke's theory of truth. (This section also introduces two notions that are not found in [13]: that of a standard valuation scheme and that of a hypodoxical sentence.) Moreover, our notion of an rfg bears a close relationship to Yablo's notion of a dependence tree, which was introduced in Yablo [21]. Namely, dependence trees can be viewed as unfoldings of decorated rfgs. Hence, Yablo's work will be briefly reviewed in Section 2.2.

2.1. Kripke (1975). We assume that the reader is familiar with Kripke's fixed-point theory of truth and use this section only to fix some terminology. A *partial model* is an ordered pair $S = (S^+, S^-)$ such that $S^+, S^- \subseteq \mathcal{L}_T$ (encoded as a set of natural numbers) and $S^+ \cap S^- = \emptyset$. Thus, in this article, a partial model is always a consistent partial model. A *valuation*

[3]Admittedly, Leitgeb's notion of dependence is more well-suited for that purpose, but not perfect either. While it accords with many intuitions that we have about dependence, it violates a few. (More on that below.) The question how to fix these issues must be left open for future research.

scheme is a function V mapping pairs consisting of partial models and sentences to the set $\{0, 1, \frac{1}{2}\}$. A valuation scheme is *monotonic* iff for all partial models $(S^+, S^-) \subseteq (P^+, P^-)$[4] we have: whenever $V(S^+, S^-)(\phi) = v \in \{0, 1\}$ then $V(P^+, P^-)(\phi) = v$. Examples of such schemes include the Weak and Strong Kleene schemes, V_{WK} and V_{SK}. Let us call V *classically sound* iff for all partial models (S^+, S^-) and all sentences φ the following holds: if $V(S^+, S^-)(\varphi) \in \{0, 1\}$, then $V(S^+, S^-)(\varphi) = Val_{S^+}(\varphi)$. That is, a valuation scheme is classically sound if every sentence that receives a definite truth value under some partial model (S^+, S^-) receives the same truth value in its classical *close-off* or *closure*, $(\mathbb{N}, S^+)$. Let us call a valuation scheme V *standard* iff it is (i) monotonic, (ii) classically sound, and (iii) whenever $V_{WK}(S^+, S^-)(\varphi) = v \in \{0, 1\}$ then $V(S^+, S^-)(\varphi) = v$. The last condition is a nontriviality condition, implying that any standard valuation scheme is at least as strong as the Weak Kleene valuation scheme. The notion of a standard valuation scheme is adapted from Herzberger [11] and will play an important role later on. Obviously, the Weak and Strong Kleene valuation schemes are standard. We will later show (Section 4.2) that there is a maximal standard valuation scheme.

Let V be some standard valuation scheme. Given a partial model (S^+, S^-), the *Kripke-jump* $\mathcal{J}_V(S^+, S^-)$ is the partial model defined by

$$(\{\varphi \mid V(S^+, S^-)(\varphi) = 1\}, \{\varphi \mid V(S^+, S^-)(\varphi) = 0\}).$$

A partial model (S^+, S^-) is *sound* iff $(S^+, S^-) \subseteq \mathcal{J}_V(S^+, S^-)$. For any sound (X^+, X^-) there is a fixed point containing it, which can be obtained from (X^+, X^-) by iterating the jump operator $\mathcal{J}_V$. Kripke calls a sentence ϕ *grounded* (with respect to V) iff it is contained in the least fixed point of $\mathcal{J}_V$, and *ungrounded* otherwise. A sentence is called *Kripke-paradoxical* (w.r.t. V) iff there is no fixed point (of $\mathcal{J}_V$) in which it receives a definite truth value. Moreover, adapting a notion by Rabern et al. [16], we say that a sentence ϕ is *Kripke-hypodoxical* (w.r.t. V) iff there are two fixed points of $\mathcal{J}_V$ such that φ has one definite truth value in the first and a different definite truth value in the second. Kripke calls a fixed point *intrinsic* iff no sentence has a definite truth value in it conflicting with its truth value in any other fixed point. A sentence has an *intrinsic truth value* iff it has a definite truth value in some intrinsic fixed point. The set of all sentences having an intrinsic truth value forms a fixed point, the largest intrinsic fixed point.

PROPOSITION 2.1 (Kripke). *A sentence is Kripke-paradoxical or Kripke-hypo-doxical iff it receives no definite truth value in the largest intrinsic fixed point.*

2.2. Yablo (1982). Yablo [21] complements Kripke's 'bottom-up' approach by a 'top-down one': instead of checking whether a sentence φ is grounded by iterating the Kripke-jump operator and then checking whether it is in this fixed point, we start with φ and work our way down, applying an operation that could be called a reverse Kripke-jump. Let us make this a bit more precise.

[4]That is, if $S^+ \subseteq P^+$ and $S^- \subseteq P^-$.

A *fact* is an ordered pair (ϕ, v) consisting of a sentence ϕ and a truth value v that can be either 0 or 1. If $\mathcal{F}$ is a set of facts, we let $\mathcal{F}^+ := \{\varphi \mid (\varphi, 1) \in \mathcal{F}\}$ and $\mathcal{F}^- := \{\varphi \mid (\varphi, 0) \in \mathcal{F}\}$. If $\mathcal{F}^+ \cap \mathcal{F}^- = \emptyset$, we say that $\mathcal{F}$ is *consistent*. A consistent set of facts $\mathcal{F}$ can be viewed as a partial model $(\mathcal{F}^+, \mathcal{F}^-)$ encoded as a single set. Conversely, every partial model determines a unique set of facts (in the obvious way). *We will frequently identify consistent sets of facts and partial models.* We say, for instance, that $\mathcal{F}$ is a *sound* set of facts, meaning that $\mathcal{F}$ considered as the partial model $(\mathcal{F}^+, \mathcal{F}^-)$ is sound in the sense of Kripke. If $V(\mathcal{F}^+, \mathcal{F}^-)(\phi) = v \in \{0, 1\}$, we say that (ϕ, v) is *jump-entailed* by $\mathcal{F}$ (because ϕ will have value v in the Kripke-jump of $(\mathcal{F}^+, \mathcal{F}^-)$).

A *fact-dependence tree* (relative to V) is a tree T such that (i) every position s of T is a sequence of facts and (ii) for every position s of T, if (ϕ, v) is the last element of s, then (ϕ, v) is jump-entailed (w.r.t. V) by the set of T-children of s. A fact-dependence tree for (ϕ, v) is a dependence tree with (ϕ, v) as its root.[5]

A reverse Kripke-jump consists in the transition from a fact to a set of facts that jump-entails it. Hence, starting with a fact, one can successively generate a fact-dependence tree for it by iterated applications of reverse Kripke-jumps. Of course, since in general there is more than one sufficiency set for a fact, reverse Kripke-jumps are not unique, but there is rather a family of reverse Kripke-jumps, one among whose members we have to choose. Consequently, a fact may have infinitely many fact-dependence trees.

THEOREM 2.2 (Yablo). *A sentence φ is grounded (in the sense of Kripke) iff either $(\varphi, 0)$ or $(\varphi, 1)$ has a well-founded fact-dependence tree.*

According to Yablo [21], the paradoxicality of a sentence lies in the fact that 'when we unravel and chase down the sentences truth *or* falsity conditions, we are led to something absurd. And absurd here can only mean one of two things: either we are led to call a true (false) sentence false (true) (as when, for example, we choose to deny that Epimenides was really a Cretan), or we are led to maintain of a sentence that it is both true and false (as when we concede Epimenides nationality and elect to wrestle with the resulting self-dependency of his utterance).' In other words, we reach paradox iff we are led to assign truth values to sentences that are *unfaithful to the facts* or *inconsistent*.

Yablo tries to capture this intuition by the following definition. A fact (φ, v) is called *unfaithful* iff φ receives the truth value $1 - v$ in the least fixed point, while two facts of the form $(\psi, 0)$, $(\psi, 1)$ are called *opposite*. Yablo defines φ to be *dependence-paradoxical* iff every fact-dependence tree of $(\varphi, 0)$ as well as every fact-dependence tree for $(\varphi, 1)$ contains either an unfaithful fact or contains two opposite facts. He proves the following result:

[5]Actually, Yablo calls such trees simply 'dependence trees', but since we are going to introduce dependence trees later on whose nodes are simply sentences, instead of facts, we call them 'fact-dependence trees'.

THEOREM 2.3 (Yablo). *A sentence is dependence-paradoxical iff it is Kripke-paradoxical.*

We will work with a modified definition of unfaithfulness which, in our view, squares better with the picture invoked in the above quote. There might be facts that are compatible with the least fixed point (and thus faithful according to Yablo's definition) but which nevertheless lead inevitably to other facts which are not, when we 'unravel and chase down their truth *or* falsity conditions'. In some sense, such a fact would be faithful but not hereditarily faithful.[6]

DEFINITION 2.4. A fact-dependence tree T is *faithful* iff for all leaves (φ, v) of T, $v = 1$ iff $(\mathbb{N}, \emptyset) \models \varphi$, where a node of T is a leaf iff it has no children in T (that is the case when it is jump-entailed by the empty set). A fact (φ, v) is *faithful* iff there is a faithful fact-dependence tree with root (φ, v). A set of facts is *faithful* iff each of its members is a faithful fact.

Theorem 2.3 remains valid even with our version of 'unfaithful'. Our Theorem 3.9 can be regarded as a reformulation of 2.3. In Section 2.5, we will introduce the notion of a dependence tree, i.e., a tree whose nodes are sentences. To that end, we will focus on valuation schemes such that, whenever we decorate a dependence tree with truth values, the resulting tree will be a fact-dependence tree. This will be the case whenever V is *symmetric* (cf. Section 2.6).

2.3. Leitgeb (2005). Yablo's notion of fact-dependence as well as Kripke's notion of groundedness are somewhat parasitic on the notion of truth (i.e., on the Kripkean fixed point models for truth). In [14], Leitgeb gives a definition of groundedness that doesn't depend on the notion of (grounded) truth. He defines a relation of semantic dependence between sentences and sets of sentences as follows:

DEFINITION 2.5. A sentence φ *depends on* $\Phi \subseteq \mathcal{L}_T$ iff for all $\Psi \subseteq \mathcal{L}_T$: $Val_\Psi(\varphi) = Val_{\Phi \cap \Psi}(\varphi)$.

Thus, a sentence ϕ depends on a set of sentences Φ (encoded as a subset of ω) iff all sentences that are relevant for the evaluation of ϕ are among the Φs. Note that every sentence depends on ω. Leitgeb's notion of dependence has some neat properties: (1) Every T-free sentence depends on $\emptyset$. (2) A sentence of the form $T\ulcorner\phi\urcorner$ depends on $\{\phi\}$. (3) ϕ depends on Φ iff $\neg\phi$ depends on Φ. (4) If ϕ, ψ depend on Φ, then so do all truth-functional compositions of ϕ, ψ. (5) If for every n, $\phi(\overline{n})$ depends on Φ, then so do $\exists x\phi, \forall x\phi$. (6) A sentence of the form $\forall x(\phi(x) \to Tx)$, where $\phi(x)$ is T-free, depends on the extension of $\phi(x)$ in the standard model $\mathbb{N}$. (7) Dependence is closed under arithmetical equivalence (where two sentences of $\mathcal{L}_T$ are

[6] $(\exists x Tx, 0)$ is an example for such a fact with respect to the scheme V_L (which will be introduced in Section 2.4). This will become apparent in Section 4, where we will also discuss some deeper implications of the notion of faithfulness which rely crucially on our modification.

arithmetically equivalent iff they get the same truth value in every classical ω-model).

The operator $\mathbb{D}(\Phi) = \{\psi | \psi \text{ depends on } \Phi\}$ is monotonic: If $\Phi \subseteq \Psi$, then $\mathbb{D}(\Phi) \subseteq \mathbb{D}(\Psi)$. Thus, if φ depends on Φ, then φ also depends on any superset of Φ. Let us call a set of sentences Φ $\mathbb{D}$-*sound* iff $\mathbb{D}(\Phi) \supseteq \Phi$. This notion must not be confused with Kripke's notion of soundness (which is soundness with respect to the Kripke-jump). Given any $\mathbb{D}$-sound set of sentences S, we iterate the operator $\mathbb{D}$ as follows: $\mathbb{D}_0(S) = S$ and $\mathbb{D}_\alpha(S) = \mathbb{D}(\bigcup_{\beta<\alpha} \mathbb{D}_\beta(S))$ for any ordinal $\alpha > 0$. By the monotonicity of $\mathbb{D}$, this process reaches a fixed point $\mathbb{D}_{lf}(S) = \bigcup_{\alpha \in On} \mathbb{D}_\alpha(S)$. We call $\mathbb{D}_{lf}(S)$ the set of sentences *grounded in S*. A sentence is *grounded* (simpliciter) iff it is grounded in the empty set, and *ungrounded* otherwise.

By the monotonicity of the dependence operator, most sentences have infinitely many dependence sets. However, for certain sentences it is possible to single out a canonical dependence set. Following Leitgeb, we say that a sentence φ depends *essentially* on a set Φ iff φ depends on Φ and there is no proper subset $\Psi \subset \Phi$ such that φ also depends on Ψ. This set Φ, if it exists, is unique, since any sentence depending on Φ and on Ψ also depends on $\Phi \cap \Psi$. Most of the sentences usually considered in the literature on truth or the semantic paradoxes actually have essential dependence. However, there are many sentences that haven't. For example, consider (a formalization of) the following version of the Yablo sequence which we may call the *nested* Yablo sequence:

Y_n^*: There is an $m > n$ such that for all $k > m$, (Y_k^*) is false.

The reader may verify that each Y_n^* lacks essential dependence: For all $m > n$, Y_n^* depends on $\{Y_m^*, Y_{m+1}^*, Y_{m+2}^*, \dots\}$ but does not depend on the intersection of these sets, the empty set; hence there is no least set on which Y_n^* depends.

2.4. The Leitgeb valuation scheme V_L. Based on his notion of dependence, Leitgeb inductively defines the following extension for the truth predicate. Let $\Gamma_0 = \emptyset$ and $\Gamma_\alpha = \{\varphi \in \mathbb{D}_\alpha(\emptyset) \,|\, (\mathbb{N}, \bigcup_{\beta<\alpha} \Gamma_\beta) \models \phi\}$. Finally, let Γ_{lf} be the least fixed point of the Γ_α-hierarchy. Then the classical model $(\mathbb{N}, \Gamma_{lf})$ validates the T-biconditionals for all grounded sentences.[7] Leitgeb's theory can be related to Kripke's theory of truth in the following way.

DEFINITION 2.6. The *Leitgeb valuation scheme*, V_L, is given by the following clause:

$$V_L(S^+, S^-)(\varphi) = \begin{cases} 1, & \text{if } \varphi \text{ depends on } S^+ \cup S^- \text{ and } (\mathbb{N}, S^+) \models \phi, \\ 0, & \text{if } \varphi \text{ depends on } S^+ \cup S^- \text{ and } (\mathbb{N}, S^+) \not\models \phi, \\ \frac{1}{2}, & \text{if } \varphi \text{ does not depend on } S^+ \cup S^-. \end{cases}$$

Obviously, V_L is a monotonic valuation scheme. In fact, V_L is a standard valuation scheme in the sense of Section 2.1. Therefore, we are justified in

[7]Some philosophical aspects of Leitgeb's truth theory are discussed in Meadows [15], to which we refer the interested reader. An axiomatic truth theory based on Leitgeb's truth theory can be found in Schindler [17].

using the notion of Kripke-paradoxicality with respect to V_L, and similarly for the other notions introduced by Kripke.

Proposition 2.7. *The extension of the truth predicate in the minimal fixed point of $\mathcal{J}_L$ is identical to Leitgeb's fixed point model Γ_{lf}.*

A proof of this proposition can be found in Schindler [18, Proposition 5.2.17]. As a consequence, a sentence is grounded (in the sense of Leitgeb) iff it is grounded (in the sense of Kripke) with respect to V_L. Thus, we have embedded Leitgeb's theory into Kripke's framework.

2.5. Reference graphs for $\mathcal{L}_T$. We can finally turn to our definition of an rfg.

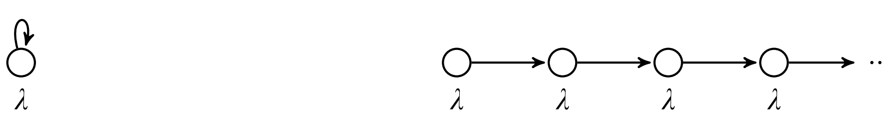
Definition 2.8. A *reference graph G of a sentence ϕ* is a directed graph with distinguished node ϕ (its root) such that

1. $V(G) \subseteq \mathcal{L}_T$,
2. every vertex of G is accessible from ϕ, and
3. every $\psi \in V(G)$ depends on the set of its out-neighbours.

A graph is a reference graph (simpliciter) if it is the rfg of some sentence. Since most sentences have infinitely many dependence sets, most sentences have infinitely many rfgs. Some sentences ϕ, however, have a *canonical rfg*, i.e., an rfg of ϕ that is contained as a subgraph in every rfg of ϕ. A notion closely related to that of a reference graph is that of a dependence tree.

Definition 2.9. A *dependence tree* is a tree T such that (i) every position s of T is a sequence of $\mathcal{L}_T$-sentences and (ii) for every position s of T, if ϕ is the last element of s, then ϕ depends on the set of T-children of s. A dependence tree for ϕ is a dependence tree with ϕ as its root.

Note that any rfg is an accessible pointed graph (as defined in the introduction). Therefore, any rfg of ϕ can be *unfolded* into a dependence tree for ϕ as follows. The unfolding of the graph G is the tree consisting of all finite walks in G starting from its root. The following illustration depicts the canonical rfg of the liar (left) and the dependence tree that is its unfolding (right):



Definition 2.10. A function $d : V(G) \to \{0,1\}$ is a *decoration* of G. A decoration d is *acceptable* iff for all vertices ψ of G:

$$V_L(d^+_\psi, d^-_\psi)(\psi) = d(\psi),$$

where $d^+_\psi = \{\chi \in out(\psi) | d(\chi) = 1\}$, $d^-_\psi = \{\chi \in out(\psi) | d(\chi) = 0\}$, and $out(\psi)$ is the set of all out-neighbours of ψ in G.

Acceptable decorations give us partial models validating the T-biconditionals of all sentences in $V(G)$ as follows:

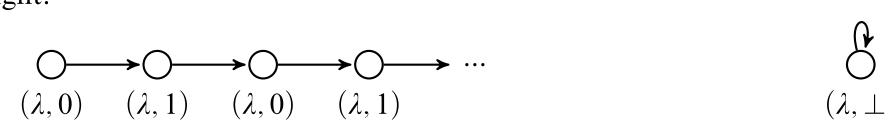
Theorem 2.11. *Let d be an acceptable decoration of a reference graph G, and let $S_d^+ := \{\phi \in V(G) \mid d(\phi) = 1\}$ and $S_d^- := \{\phi \in V(G) \mid d(\phi) = 0\}$. Then for all $\phi \in V(G)$: $V_L(S_d^+, S_d^-)(\phi) = V_L(S_d^+, S_d^+)(T\ulcorner\phi\urcorner) \in \{0, 1\}$.*

Decorated rfgs are closely related to what we earlier called *fact-dependence trees* (compare Section 2.2). Clearly, every acceptably decorated rfg (which is essentially a graph whose nodes are facts) can be unfolded into a fact-dependence tree. Conversely, every fact-dependence tree T can be *collapsed* into an rfg G. The vertices of G are those sentences ϕ such that, for some truth value v, the fact (ϕ, v) is a node of T. There is an arc from ϕ to ψ iff there is an arc in T from a fact containing ϕ to a fact containing ψ. Every fact-dependence tree induces a multidecoration on the rfg it collapses to.

Definition 2.12. The *multidecoration* of an rfg G induced by the fact-dependence tree T (where G is the rfg that T collapses into) is the function $D_T : V(G) \to \{0, 1, \perp\}$ such that

$$D_T(\phi) = \begin{cases} 1, & \text{if for every fact } (\phi, v) \text{ occurring in } T, v = 1, \\ 0, & \text{if for every fact } (\phi, v) \text{ occurring in } T, v = 0, \\ \perp, & \text{if } (\phi, 1) \text{ and } (\phi, 0) \text{ occur in } T. \end{cases}$$

Clearly, a multidecoration of G is a decoration iff no node gets assigned $\perp$ iff the fact-dependence tree T does not contain opposite facts. For example, the fact-dependence tree on the left induces the multidecorated rfg on the right:



The following notions are adapted from Rabern et al. [16] to our framework.

Definition 2.13. A sentence is *r-paradoxical* (‘r’ for ‘referentially’) iff it has no rfg that admits an acceptable decoration; it is *r-hypodoxical* iff it has an rfg that admits a verifying acceptable decoration and a falsifying acceptable decoration, where a decoration d of G is *verifying* iff d assigns 1 to the root of G and *falsifying* iff d assigns 0 to the root of G. A graph is *dangerous* iff it is isomorphic to an rfg of some r-paradoxical sentence.

The problem of stating necessary and sufficient condition for dangerousness is known as *the characterization problem*. Its solution is one of the most important goals of a graph-theoretic analysis of the paradoxes, and a mathematically challenging one.

2.6. Symmetry. While this article is mostly concerned with rfgs as defined in the previous section, i.e., graphs based on the Leitgeb valuation scheme V_L, it is possible to transfer our results to other valuation schemes provided they share certain properties with V_L. Given a standard valuation scheme V, let us say that a sentence ϕ *V-depends* on S iff there is a partition (S^+, S^-) of S such that $V(S^+, S^-)(\phi) \in \{0, 1\}$. Using the terminology of Yablo [21], ϕ V-depends on S iff there is a definite truth value v and a partition (S^+, S^-) of S such that (S^+, S^-) jump-entails the fact (ϕ, v) (cf. Section

2.2). Now, let us call a valuation scheme V *symmetric* iff for all sentences ϕ and sets of sentences S the following holds: if ϕ V-depends on S, then every partition (S^+, S^-) of S induces a definite truth value on ϕ (under V). Notice, however, that it is not required that every partition induces the same definite truth value on ϕ.

PROPOSITION 2.14. *Let V be a standard valuation scheme. Then V is symmetric iff V satisfies the following equation*:

$$V(S^+, S^-)(\varphi) = \begin{cases} 1, & \textit{if } \varphi \ V\textit{-depends on } S^+ \cup S^- \textit{ and } (\mathbb{N}, S^+) \models \phi, \\ 0, & \textit{if } \varphi \ V\textit{-depends on } S^+ \cup S^- \textit{ and } (\mathbb{N}, S^+) \not\models \phi, \\ \frac{1}{2}, & \textit{if } \varphi \textit{ does not } V\textit{-depend on } S^+ \cup S^-. \end{cases}$$

PROOF. $\Rightarrow$: Let V be symmetric and assume that $V(S^+, S^-)(\phi) = v \in \{0, 1\}$. Then by definition, ϕ V-depends on $S^+ \cup S^-$, and by classical soundness of V (cf. Section 2.1), v must coincide with the truth value that ϕ receives in the classical closure $(\mathbb{N}, S^+)$. On the other hand, if $V(S^+, S^-)(\phi) = \frac{1}{2}$, then ϕ cannot V-depend on $S^+ \cup S^-$, because otherwise the symmetry of V would imply that ϕ has a definite truth value in (S^+, S^-).

$\Leftarrow$: Suppose V satisfies the above equation and that ϕ V-depends on S. Let (S^+, S^-) be any partition of S. Clearly, $(\mathbb{N}, S^+) \models \phi$ or $(\mathbb{N}, S^+) \not\models \phi$. In either case, V assigns a definite truth value to ϕ. Hence, V is symmetric. $\dashv$

PROPOSITION 2.15. 1. *ϕ V_L-depends on S iff ϕ depends on S.*
2. *V_L is symmetric.*

PROOF. *Ad* (1): If ϕ V_L-depends on S then there is a partition (S^+, S^-) such that ϕ receives a definite truth value in the partial model (S^+, S^-). The definition of V_L therefore implies that ϕ must depend on $S^+ \cup S^-$. Conversely, if ϕ depends on $S = S^+ \cup S^-$, then ϕ will receive a definite truth value in the partial model (S^+, S^-). Hence ϕ V_L-depends on S.

Ad (2): Follows from (1) and the previous proposition. $\dashv$

LEMMA 2.16. *Let V be a symmetric standard valuation scheme. If φ V-depends on Φ, then for all partial models (S^+, S^-) with $S^+ \cup S^- \supseteq \Phi$: $V(S^+, S^-)(\varphi) = V(S^+ \cap \Phi, S^- \cap \Phi)(\varphi)$.*

PROOF. Since $S^+ \cup S^- \supseteq \Phi$, we have $\Phi = (S^+ \cap \Phi) \cup (S^- \cap \Phi)$. Since V is symmetric and φ V-depends on Φ we have $V((S^+ \cap \Phi)\,(S^- \cap \Phi))(\phi) = v$, where $v \in \{0, 1\}$. By monotonicity, $V(S^+, S^-)(\varphi) = v$. $\dashv$

Another symmetric valuation scheme, apart from V_L, is the Weak Kleene valuation scheme. This follows from the strong compositionality of the Weak Kleene scheme. In contrast, the Strong Kleene scheme is *not* symmetric. For instance, consider the sentence $\lambda \vee T\ulcorner 1 = 1 \urcorner$. This sentence V_{SK}-depends on the set $\{1 = 1\}$, but not every partition of that set induces a definite truth value on the sentence (under the Strong Kleene scheme). For instance, $\lambda \vee T\ulcorner 1 = 1 \urcorner$ receives the definite value 1 in the partial model $(\{1 = 1\}, \emptyset)$ while it receives the nondefinite value $\frac{1}{2}$ in the partial model $(\emptyset, \{1 = 1\})$.

We will now give a list of the properties of V_L that we will use in the proofs of our theorems. *If V is an arbitrary valuation scheme fulfilling the following* *conditions then all our theorems and proofs (except those in Section 5.3) remain valid if V_L is replaced by V and dependence is replaced by V-dependence*:

1. V is standard (i.e., monotonic, classically sound, at least as strong as Weak Kleene);
2. V is symmetric;
3. V-dependence is *weakly compositional* in the following sense:
   (a) Every arithmetical sentence V-depends on $\emptyset$,
   (b) ϕ V-depends on Φ iff $\neg\phi$ V-depends on Φ,
   (c) If ϕ and ψ V-depend on Φ then $\phi \wedge \psi, \phi \vee \psi$ V-depend on Φ,
   (d) Let $\phi(x)$ be a formula with exactly x free. If $\phi(\bar{n}/x)$ V-depends on Φ for all numerals $\bar{n}$ then $\forall x\phi, \exists x\phi$ V-depend on Φ.

An important example of a valuation scheme that satisfies all of the above conditions is the Weak Kleene scheme.

In the remainder of this article (unless otherwise stated), *dependence* will always mean *V_L-dependence*, *Kripke-paradoxical* will always mean *Kripke-paradoxical with respect to V_L* (and similarly for *Kripke-hypodoxical* and so on).

§3. **Kripke-games on reference graphs.** The main goal of this section is to investigate connections between the structure of an rfg and the set of decorations that are acceptable on it. In particular, we want to show that whenever an rfg lacks certain patterns, then it admits an acceptable decoration. To this end we show that any decoration can be obtained in a three-stage process. First, the rfg is unfolded into a dependence tree, second, the tree is decorated, yielding a fact-dependence tree, and third, the decorated tree is re-collapsed onto the rfg, inducing a multidecoration on it. In Section 4 we will show that paradoxicality enters the picture only at the third stage and that it is quite transparent how the structure of an rfg is responsible for the fact that the induced multidecoration is not an acceptable decoration. Due to this we can isolate two basic patterns necessary for paradoxicality.

In the present section we will develop a tool that allows us to better control this decoration process and to comprehend its three stages in a single one: a multidecoration D of G can be thought of as being obtained as the result of a game between two players (the verification game), being played on G. More precisely, the fact-dependence tree that induces D can be identified with a strategy of the second player. These strategies, although as combinatorial objects slightly more complex than fact-dependence trees, tend to be more easily accessible for the human mind than the latter: they allow us to formulate many of our proofs in a more intuitive way. The verification game is parasitic on another game (the grounding game) which we discuss first.[8]

[8] In Welch [20] an extensive collection of games for truth can be found, each of which allows the characterization of the T-predicate (suggested by a particular formal theory of truth) in terms of a player's strategies. For our purposes, however, a mere characterization of the T-predicate's extension is not enough: we need a transparent reconstruction of the valuation process leading to this extension in the rules of the game.

3.1. The grounding game. For each sentence φ and set of sentences Φ we will define a (possibly) infinite game of perfect information, the *grounding game* $\mathcal{G}_G(\varphi, \Phi)$ between two players $(\exists)$ and $(\forall)$, such that $(\exists)$ has a winning strategy in $\mathcal{G}_G(\varphi, \Phi)$ iff φ is grounded in Φ.[9] Below we will see that unfoldings of rfgs for φ can be identified with strategies of player $(\exists)$ in the game $\mathcal{G}_G(\varphi, \emptyset)$. The rules of $\mathcal{G}_G(\varphi, \Phi)$ are the following.

1. The players $(\exists)$, $(\forall)$ move alternately. $(\forall)$ must move first and choose φ as his first move, φ_1. If $\varphi \in \Phi$, he cannot move.
2. As her n-th move $(\exists)$ must choose some set Φ_n on which φ_n depends.
3. If $n > 1$, as his n-th move $(\forall)$ must choose some sentence $\varphi_n \in \Phi_{n-1} \setminus \Phi$.

The winning conditions for $\mathcal{G}_G(\varphi, \Phi)$ are

- $(\exists)$ wins a run of the game if $(\forall)$ cannot move.
- $(\forall)$ wins a run of the game if it goes on forever.[10]

$$\phi_1 \xrightarrow{\text{depends on}} \Phi_1 \xrightarrow{\text{contains}} \phi_2 \xrightarrow{\text{depends on}} \Phi_2 \xrightarrow{\text{contains}} \cdots$$

We have a special interest in cases where the set parameter Φ denotes the empty set; we then omit the parameter and write $\mathcal{G}_G(\varphi)$.

3.1.1. Strategies.. Call any (possibly empty) finite sequence of legal moves in $\mathcal{G}_G(\varphi, \Phi)$ a *position* of $\mathcal{G}_G(\varphi, \Phi)$. Any position is either an $(\exists)$-position, i.e., a position in which $(\forall)$ is to move next, or an $(\forall)$-position, a position in which $(\exists)$ is to move next. The set of all $\mathcal{G}_G(\varphi, \Phi)$-positions forms a tree which we denote by $P_G(\varphi, \Phi)$.

DEFINITION 3.1. A *strategy for* $(\exists)$ *in* $\mathcal{G}_G(\varphi, \Phi)$ is a set $\sigma \subseteq P_G(\phi, \Phi)$ such that (i) the empty sequence is an element of σ; (ii) for all $(\exists)$-positions $p \in \sigma$: if $q \in P_G(\phi, \Phi)$ is a successor of p, then $q \in \sigma$; (iii) for all $(\forall)$-positions $p \in \sigma$: if there is a $q \in P_G(\phi, \Phi)$ that is a successor of p, then σ contains exactly one such q; (iv) nothing else is in σ.

A strategy for $(\exists)$ is a subtree of the tree of all legal positions, and each branch is a possible run of the game. If σ is a strategy in $\mathcal{G}_G(\phi, \Phi)$, we say (by slight abuse of language) that ϕ is the *root* of σ. Note that a strategy for $(\exists)$ might contain the empty sequence as its only element. We call this strategy the *trivial strategy*. Note that the trivial strategy is available to $(\exists)$ iff $(\forall)$ is not able to make a first move. This is never the case in the parameter-free games $\mathcal{G}_G(\phi)$ because here $(\forall)$ can always make a move.

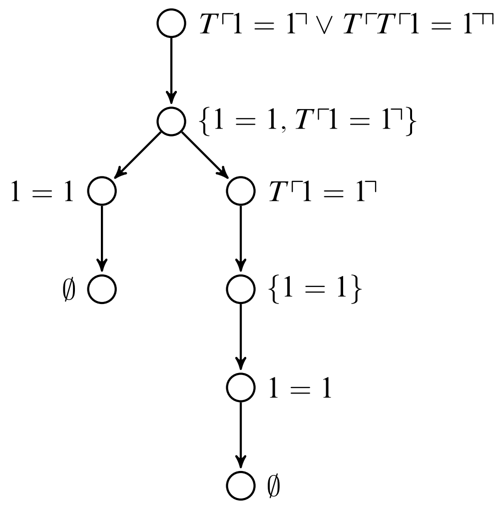
The definition of a $(\forall)$-strategy is obtained from Definition 3.1 by replacing every occurrence of '$(\exists)$' with '$(\forall)$' and vice versa. We call an $(\exists)$-strategy σ *homogenous* iff for each sentence ψ the following holds: if $(\exists)$ plays Ψ as a response to ψ in some σ-position, then $(\exists)$ plays Ψ as response to ψ in every σ-position. *We are only interested in homogenous strategies and therefore, in the remainder of the article,* strategy *will always mean* homogenous strategy!

[9] A version of the grounding game is described in Aczel's article on inductive definitions [1]. We thank Jönne Kriener for this hint.

[10] Note that $(\exists)$ can always make a move, since every sentence depends on some set, e.g., ω.

A strategy σ is a *winning strategy* for $(\exists)$ in $\mathcal{G}_G(\varphi, \Phi)$ iff she wins every run of $\mathcal{G}_G(\varphi, \Phi)$ that is *compatible* with σ, i.e., every run of $\mathcal{G}_G(\varphi, \Phi)$ that is a branch of the tree σ. Informally, this means that she wins every run of the game as long as she keeps to the strategy σ, regardless of the moves of her opponent $(\forall)$. Analogously, a *winning strategy* for $(\forall)$ is defined.

Let ϕ be the sentence $T\ulcorner 1 = 1\urcorner \vee T\ulcorner T\ulcorner 1 = 1\urcorner\urcorner$. The following is a winning strategy for $(\exists)$ in the game $\mathcal{G}_G(\phi)$.



The grounding game derives its name from the fact that a sentence φ is grounded in S (that is, $\phi \in \mathbb{D}_{lf}(S)$) iff $(\exists)$ has a winning strategy in the game $\mathcal{G}_G(\varphi, S)$.

THEOREM 3.2. *Let S be a $\mathbb{D}$-sound set. φ is grounded in S iff $(\exists)$ has a winning strategy in the game $\mathcal{G}_G(\varphi, S)$.*

PROOF. $\Rightarrow$: By induction on the rank_D of the sentences φ that are grounded in S, where $\text{rank}_D(\phi)$ is defined as the least ordinal α such that $\varphi \in \mathbb{D}_\alpha(S)$. Let $\text{rank}_D(\varphi)= 0$. Thus $\phi \in \mathbb{D}_0(S) = S$. Then the trivial strategy is a winning strategy for $(\exists)$. Now let $\text{rank}_D(\varphi) = \alpha$ for some ordinal $\alpha > 0$. Then φ depends on some $\Phi \subseteq \mathbb{D}_{lf}(S)$ whose members have strictly lower rank_D than φ. If $\Phi \subseteq S$ then again $(\exists)$ can choose S as her first move in $\mathcal{G}_G(\varphi, S)$ and this is a winning strategy for her. Otherwise $\Phi \neq \emptyset$, and by induction hypothesis $(\exists)$ has a winning strategy in $\mathcal{G}_G(\psi, S)$ for all $\psi \in \Phi$. Thus she plays Φ as her first move and whichever $\psi \in \Phi$ player $(\forall)$ chooses next, $(\exists)$ simply plays her winning strategy in $\mathcal{G}_G(\psi, S)$. This is a winning strategy for her in $\mathcal{G}_G(\varphi, S)$. (Observe that this is indeed a strategy, since $(\exists)$ can always choose the same Φ on various recurrences of ϕ.)

$\Leftarrow$: By induction on the strategy-rank of a sentence,

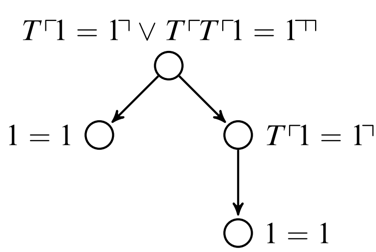
$$\text{rank}_G(\varphi) = \min\{\text{rank}(\sigma) \mid \sigma \text{ is a winning-strategy for } (\exists) \text{ in } \mathcal{G}_G(\varphi, S)\},$$

Here, $\text{rank}(\sigma) = \sup\{\text{rank}(\tau)+1) \mid \tau$ is the $(\exists)$-substrategy of σ in $\mathcal{G}_G(\psi, S)$, ψ is a possible response for $(\forall)$ to $(\exists)$'s first move in $\sigma\}$. Notice that any winning strategy for $(\exists)$ must be well-founded (as a tree), thus $\text{rank}(\sigma)$ is well-defined. Suppose that $(\exists)$ has a winning strategy σ in $\mathcal{G}_G(\varphi, S)$.

If $\text{rank}_G(\varphi) = 0$ then ϕ must depend on some subset of S and is therefore grounded in S. Now let $\text{rank}_G(\varphi) = \alpha$ for some ordinal $\alpha > 0$. Without loss of generality we may assume that $\text{rank}(\sigma) = \alpha$. Then $\text{rank}_G(\psi) < \alpha$ for all $\psi \in \Psi$, where Ψ is the first move of $(\exists)$ in σ. Thus by induction hypothesis all $\psi \in \Psi$ are grounded in S. Since φ depends on $\Psi \subseteq \mathbb{D}_{lf}(S)$, it follows that φ is grounded in S.

[Observe that we could have easily proved that $\text{rank}_D(\varphi)$=$\text{rank}_G(\varphi)$ for all sentences φ that are grounded in S.] $\dashv$

3.1.2. Strategies, dependence trees, and rfgs.. Our three notions of an rfg, a dependence tree, and an $(\exists)$-strategy are closely related. Say that a dependence tree T is *homogenous* iff for all positions s, t of T, if the last elements of s, t are identical, then the set of T-children of s, t are the same. Any homogenous dependence tree T for ϕ induces, or can be interpreted as, an $(\exists)$-strategy in $G_{\mathcal{G}}(\phi)$: $(\exists)$ chooses as her reply to a given $(\forall)$-move ψ simply the set of all the T-children of ψ. Second, given any $(\exists)$-strategy σ in $G_{\mathcal{G}}(\phi)$, a homogenous dependence tree T_σ for ϕ can be constructed by simply deleting all moves of $(\exists)$ (namely, the dependence sets Φ_n) from σ. Hence there is a canonical bijection between homogenous dependence trees and $(\exists)$-strategies. For example, the following dependence tree is obtained from the strategy depicted earlier.



Third, any dependence tree can be collapsed into an rfg and any rfg can be unfolded into a homogenous dependence tree. Finally, every $(\exists)$-strategy σ can be collapsed to an rfg $\Gamma(\sigma)$ whose unfolding is the dependence tree T_σ. The set of vertices of $\Gamma(\sigma)$ consists of the sentences occurring in σ; two vertices ψ, χ are joined by an arc from ψ to χ iff there is a run of the game (played according to σ) in which $(\forall)$ chooses ψ, χ consecutively. As a consequence, *we can interpret the grounding game for a sentence ϕ as being played on an rfg of ϕ*, namely the rfg that $(\exists)$'s chosen strategy collapses to.

Notice that a winning strategy for $(\exists)$ is a well-founded tree. We therefore obtain that a strategy σ for $(\exists)$ in $\mathcal{G}_G(\varphi)$ is a winning strategy for $(\exists)$ iff the rfg $\Gamma(\sigma)$ is well-founded. Combining this with Theorem 3.2, we obtain

Corollary 3.3. *A sentence φ is grounded iff φ has a well-founded reference graph.*

3.2. The verification game. The *verification game* $\mathcal{G}_T(\varphi, v, \mathcal{F})$ is quite similar to the grounding game $\mathcal{G}_G(\varphi, \Phi)$, but this time the players are not dealing merely with sentences φ and sets of sentences Φ, but with facts (ϕ, v) and *consistent* sets of facts $\mathcal{F}$. (Recall our conventions in Section 2.2

regarding facts and partial models.) A second difference to the grounding game is that a run of the verification game can end in a draw. Let us say that a partial model (Φ^+, Φ^-) is *compatible* with a set of facts $\mathcal{F}$ iff $\Phi^+ \cap \mathcal{F}^- = \emptyset$ and $\Phi^- \cap \mathcal{F}^+ = \emptyset$.

To every position of the game $\mathcal{G}_T(\varphi, v, \mathcal{F})$ a *mode* is associated, the mode that a run of the game assumes in this position. This mode is either the *verification mode* (=1) or the *falsification mode* (=0). The rules of $\mathcal{G}_T(\varphi, v, \mathcal{F})$ are

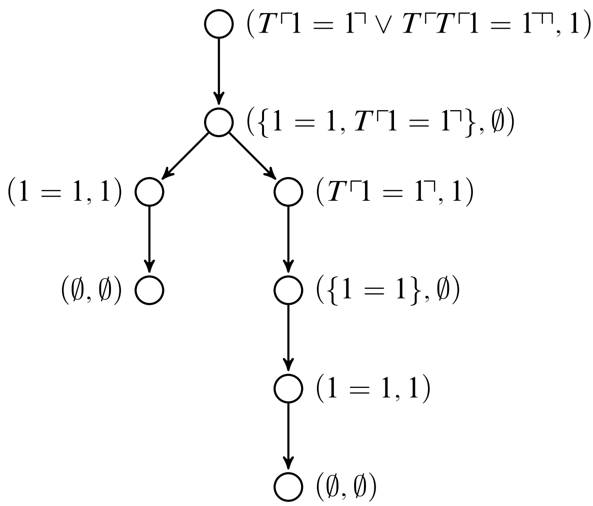
1. The game $\mathcal{G}_T(\varphi, 1, \mathcal{F})$ starts in the verification mode, $\mathcal{G}_T(\varphi, 0, \mathcal{F})$ starts in the falsification mode.
2. $(\forall)$ must move first and choose φ as his first move, φ_1. If $\varphi \in \mathcal{F}^+ \cup \mathcal{F}^-$ he cannot move.
3. As her n-th move, $(\exists)$ must choose some partial model (Φ_n^+, Φ_n^-) compatible with $\mathcal{F}$ such that φ_n depends on $\Phi_n^+ \cup \Phi_n^-$ and $(\mathbb{N}, \Phi_n^+) \models \varphi_n$ if the game is in verification mode, and $(\mathbb{N}, \Phi_n^+) \not\models \varphi_n$ if the game is in falsification mode.
4. If $n > 1$, as his n-th move $(\forall)$ must choose some sentence $\varphi_n \in (\Phi_{n-1}^+ \setminus \mathcal{F}^+) \cup (\Phi_{n-1}^- \setminus \mathcal{F}^-)$. If $\varphi_n \in \Phi_{n-1}^+$ then play continues in the verification mode. If $\varphi_n \in \Phi_{n-1}^-$ then play continues in the falsification mode.

The *winning conditions* for $\mathcal{G}_T(\varphi, v, \mathcal{F})$ are

- If a run of the game goes on forever it is declared a draw.
- If a player cannot move according to the rules 3 or 4 then the other player wins this run of the game.
- If $(\forall)$ cannot move according to rule 2 then he loses the game iff $\varphi \in \mathcal{F}^+$ and $v = 1$ or if $\varphi \in \mathcal{F}^-$ and $v = 0$. In the other cases he wins.

In order to relate our two notions of Kripke- and r-paradoxicality (cf. Theorem 3.9 and Proposition 3.10), special attention is paid to cases where the set parameter $\mathcal{F}$ denotes the empty set; we then write $\mathcal{G}_T(\varphi, v)$. Since we want to keep track of the mode of game, we represent the possible positions as follows: $((\phi_1, v_1), (\Phi_1^+, \Phi_1^-), \dots, (\phi_n, v_n), (\Phi_n^+, \Phi_n^-))$, where v_i is either 1 or 0 according as to whether the game is in verification or falsification mode after $(\forall)$'s i-th move. (Thus a position is an alternating sequence of facts and partial models.) We denote the set of legal positions of $\mathcal{G}_T(\phi, v, \mathcal{F})$ by $P_T(\phi, v, \mathcal{F})$. The definition of an $(\exists)$-strategy in $\mathcal{G}_T(\phi, v, \mathcal{F})$ is obtained from Definition 3.1 by replacing every occurrence of '$\mathcal{G}_G(\phi, \Phi)$' by '$\mathcal{G}_T(\phi, v, \mathcal{F})$' and every occurrence of '$P_G(\phi, \Phi)$' by '$P_T(\phi, v, \mathcal{F})$'. Just as before, the definition of a $(\forall)$-strategy is obtained by switching the roles of $(\exists)$ and $(\forall)$. An $(\exists)$-strategy in the verification game is *homogenous* iff for all sentences ψ the following holds: if $(\exists)$ plays a partition (Ψ^+, Ψ^-) of Ψ in response to ψ in some σ-position, then $(\exists)$ plays *some* partition of Ψ in response to ψ in every σ-position. Again, *our interest is only with homogenous strategies and for the remainder of the article,* strategy *will always mean* homogenous strategy!

For illustrative purposes, let us return to our earlier example. Let ϕ be again the sentence $T\ulcorner 1 = 1\urcorner \vee T\ulcorner T\ulcorner 1 = 1\urcorner\urcorner$. Then the following is a winning strategy for $(\exists)$ in the game $\mathcal{G}_T(\phi, 1)$.



One reason we are interested in the verification game is that it allows us to investigate the relation between rfgs and acceptable decorations of rfgs. First, observe that (sets of) facts can be seen as decorations of (sets of) sentences with truth values.

Definition 3.4. Let $\|(\phi, v)\| := \phi$ and $\|\mathcal{F}\| := \|(\mathcal{F}^+, \mathcal{F}^-)\| := \mathcal{F}^+ \cup \mathcal{F}^-$. We call (ϕ, v) a *decoration* of ϕ. We call $\mathcal{F}$ a *decoration* of Φ iff $\|\mathcal{F}\| = \Phi$. If σ is an $(\exists)$-strategy in the verification game, let $\|\sigma\| := \{\|p\| \mid p \in \sigma\}$, where $\|p\|$ is the sequence that results from p by applying the operator $\|\cdot\|$ to each component of p. We say that σ is *a decoration* of $\|\sigma\|$.

Clearly, if σ is a nonlosing $(\exists)$-strategy in the verification game (i.e., a strategy such that $(\exists)$ never loses as long as she plays according to it), then $\|\sigma\|$ is an $(\exists)$-strategy in the grounding game. Thus, a nonlosing strategy in the verification game can be seen as the result of decorating a strategy in the grounding game with truth values. Of course, there are many ways of decorating a grounding strategy. This changes under the stipulation that winning strategies must be mapped to winning strategies:

Theorem 3.5. *Let $\mathcal{F}$ be a consistent set of facts. Then player $(\exists)$ has a nontrivial winning strategy σ in $\mathcal{G}_G(\varphi, \|\mathcal{F}\|)$ iff $(\exists)$ has a nontrivial winning strategy σ' in either $\mathcal{G}_T(\varphi, 1, \mathcal{F})$ or in $\mathcal{G}_T(\varphi, 0, \mathcal{F})$. Moreover, σ' is the unique decoration of σ that is a winning strategy for $(\exists)$ in either $\mathcal{G}_T(\varphi, 1, \mathcal{F})$ or $\mathcal{G}_T(\varphi, 0, \mathcal{F})$.*

Proof. $\Rightarrow$: Let σ be a nontrivial winning strategy for $(\exists)$ in $\mathcal{G}_G(\varphi, \Phi)$, where $\Phi = \|\mathcal{F}\|$. As with Theorem 3.2, the proof is by induction on the strategy-rank of a sentence, $\mathrm{rank}_G(\phi)$. Let Ψ be $(\exists)$'s σ-response to φ. Then φ depends on Ψ and by induction hypothesis $(\exists)$ has either a winning strategy σ'_ψ in $\mathcal{G}_T(\psi, 1, \mathcal{F})$ or in $\mathcal{G}_T(\psi, 0, \mathcal{F})$, for all $\psi \in \Psi$. Let Ψ^+ be the set of all members of Ψ such that the first is the case and Ψ^- be the set of all members of Ψ such that the second alternative holds. If $Val_{\Psi^+}(\varphi)=1$, then playing (Ψ^+, Ψ^-) as her first move in $\mathcal{G}_T(\varphi, 1, \mathcal{F})$ followed by σ'_ψ as a response to $(\forall)$'s move ψ is a winning strategy for $(\exists)$ in $\mathcal{G}_T(\varphi, 1, \mathcal{F})$. If $Val_{\Psi^+}(\varphi)=0$, then playing (Ψ^+, Ψ^-) as her first move in $\mathcal{G}_T(\varphi, 0, \mathcal{F})$ followed by σ'_ψ as a response to $(\forall)$'s move ψ is a winning strategy for $(\exists)$ in $\mathcal{G}_T(\varphi, 0, \mathcal{F})$. By induction one proves that the strategy σ' thus defined is a decoration of the strategy σ and in fact the only decoration of σ that is a winning strategy for $(\exists)$ in either $\mathcal{G}_T(\psi, 1, \mathcal{F})$ or in $\mathcal{G}_T(\psi, 0, \mathcal{F})$.

$\Leftarrow$: Suppose w.l.o.g. that $(\exists)$ has a winning strategy σ in $\mathcal{G}_T(\phi, 1, \mathcal{F})$. Then σ is a well-founded tree (for otherwise $(\forall)$ could draw the game). Hence $\|\sigma\|$ is well-founded tree and thus a winning strategy for $(\exists)$ in $\mathcal{G}_G(\phi, \|\mathcal{F}\|)$. ⊣

Corollary 3.6. *Every well-founded rfg admits a unique acceptable decoration.*

The verification game derives its name from the property that a sentence φ is true in the fixed point of $\mathcal{J}_L$ generated by $\mathcal{F}$ iff $(\exists)$ has a winning strategy in $\mathcal{G}_T(\varphi, 1, \mathcal{F})$ and that φ is false in this fixed point iff $(\exists)$ has a winning strategy in $\mathcal{G}_T(\varphi, 0, \mathcal{F})$.

Theorem 3.7. *Let $\mathcal{F}$ be a consistent and sound set of facts. Then φ has the definite truth value v in the Kripke fixed point generated by $(\mathcal{F}^+, \mathcal{F}^-)$ iff $(\exists)$ has a winning strategy in $\mathcal{G}_T(\varphi, v, \mathcal{F})$.*

Proof. $\Rightarrow$: Suppose φ has the definite truth value v in the fixed point of $\mathcal{J}_L$ generated by $\mathcal{F}$. (Since $\mathcal{F}$ is consistent and sound, such a fixed point exists.) Hence φ is grounded in $\mathcal{F}^+ \cup \mathcal{F}^-$ and by Theorem 3.2, $(\exists)$ has a winning strategy σ in $\mathcal{G}_G(\varphi, \mathcal{F}^+ \cup \mathcal{F}^-)$. Then the strategy σ' as defined in the proof of Theorem 3.5 is a winning strategy for $(\exists)$ in $\mathcal{G}_T(\varphi, v, \mathcal{F})$.

$\Leftarrow$: Suppose $(\exists)$ has a winning strategy σ' in $\mathcal{G}_T(\varphi, v, \mathcal{F})$. Then by Theorem 3.5 $(\exists)$ has a winning strategy σ in $\mathcal{G}_G(\varphi,\ \mathcal{F}^+ \cup \mathcal{F}^-)$. By Theorem 3.2, φ is grounded in $\mathcal{F}^+ \cup \mathcal{F}^-$. Since $\mathcal{F}$ is consistent and sound, ϕ must have a definite truth value in the fixed point generated by $\mathcal{F}$. Therefore, if ϕ had value $1 - v$ in this fixed point, $(\exists)$ would have a winning strategy in $\mathcal{G}_T(\varphi, 1 - v, \mathcal{F})$ by the first part of this theorem. But by Theorem 3.5, this contradicts the assumption that $(\exists)$ has a winning strategy in $\mathcal{G}_T(\varphi, v, \mathcal{F})$. ⊣

3.3. Paradoxicality and verification strategies. Now let us investigate how acceptable decorations of rfgs are related to strategies in the verification game. In order to do so, we cannot focus solely on winning strategies but need a somewhat more liberal criterion for a good $(\exists)$-strategy σ in the verification-game:

Definition 3.8. A verification-strategy σ for $(\exists)$ is *faithful* iff $(\exists)$ never loses a game whenever she plays σ. We call σ *consistent* iff no sentence occurring in σ (played by $(\forall)$) occurs in both the verification- and the falsification-mode (i.e., if σ contains no opposite facts). A multidecoration is faithful (consistent) iff it is induced by faithful (consistent) verification-strategy.

Just as an $(\exists)$-strategy in the grounding game corresponds to a dependence tree, a faithful $(\exists)$-strategy σ in the verification game corresponds to a faithful fact-dependence tree. That is, given a faithful $(\exists)$-strategy in the verification game, one obtains a homogenous[11] faithful fact-dependence tree by deleting all $(\exists)$-moves from σ (i.e., the partial models (Φ_n^+, Φ_n^-)). Conversely, every homogenous faithful fact-dependence tree T induces a faithful verification strategy: $(\exists)$ chooses as her reply to a given $(\forall)$-move (ψ, v) simply the set of its T-children. Since, as we have noted in Section 2.5, every fact-dependence tree can be collapsed into an rfg, we can view the verification game as being played on an rfg as well. The following theorem can be seen as a reformulation of Theorem 2.3 of Yablo [21] within our framework. (Recall our discussion of this theorem and the notion of faithfulness in Section 2.2.)

Theorem 3.9. *A sentence ϕ has the truth value v in some Kripke fixed point (Φ^+, Φ^-) iff $(\exists)$ has a faithful consistent strategy σ in $\mathcal{G}_T(\phi, v)$. Moreover $(\Phi^+, \Phi^-) \supseteq (\mathcal{F}_\sigma^+, \mathcal{F}_\sigma^-)$, where $\mathcal{F}_\sigma$ is the set of all facts occurring in σ.*

Proof. $\Rightarrow$: Suppose φ has the truth value v in some fixed point (Φ^+, Φ^-). Then for each $\psi \in \Phi^+ \cup \Phi^-$ there is some consistent $(\Psi^+, \Psi^-) \subseteq (\Phi^+, \Phi^-)$ such that ψ depends on $\Psi^+ \cup \Psi^-$ and ψ has a definite truth value in (Ψ^+, Ψ^-). (This is so because ψ can only be in the fixed point if it is made true/false in some partial model that is a submodel of the fixed point, and ψ must depend on the union of the extension and anti-extension of that model, by definition of V_L.) Thus, using the axiom of choice, we can build up a strategy σ for player $(\exists)$ in $\mathcal{G}_T(\varphi, v)$ layer by layer. σ is faithful by construction and consistent because (Φ^+, Φ^-), being a fixed point, is consistent by definition.

$\Leftarrow$: Let σ be a consistent, faithful strategy for player $(\exists)$ in $\mathcal{G}_T(\varphi, v)$. Let $\mathcal{F}_\sigma$ be the set of all facts occurring in σ. Since σ is consistent, the pair $(\mathcal{F}_\sigma^+, \mathcal{F}_\sigma^-)$ is a partial model. Because σ is a faithful strategy, for each $(\psi, v') \in \mathcal{F}_\sigma$ there is a set of facts $\mathcal{E} \subseteq \mathcal{F}_\sigma$ such that $V_L(\mathcal{E}^+, \mathcal{E}^-)(\psi) = v'$. Hence by monotonicity of V_L, $(\mathcal{F}_\sigma^+, \mathcal{F}_\sigma^-)$ is sound. Hence there is some fixed point (Φ^+, Φ^-) extending it and φ has the truth value v in (Φ^+, Φ^-). $\dashv$

Hence, a sentence φ is Kripke-paradoxical iff every strategy for $(\exists)$ in $\mathcal{G}_T(\varphi, 1)$ or in $\mathcal{G}_T(\varphi, 0)$ is either unfaithful or inconsistent. On the other hand, φ is Kripke-hypodoxical iff there are faithful and consistent strategies for $(\exists)$ in $\mathcal{G}_T(\varphi, 1)$ and in $\mathcal{G}_T(\varphi, 0)$.

Proposition 3.10. *Let G be an rfg of ϕ and d be a multidecoration of G. Then d is an verifying acceptable decoration of G iff it is induced by a faithful consistent $(\exists)$-strategy in $\mathcal{G}_T(\phi, 1)$, and d is an falsifying acceptable decoration of G iff it is induced by a faithful consistent $(\exists)$-strategy in $\mathcal{G}_T(\phi, 0)$.*

Corollary 3.11. *A sentence is Kripke-paradoxical iff it is r-paradoxical.*

[11] A fact-dependence tree T is homogenous iff for all $s, t \in T$: if the first components (the sentences) of the last elements of s, t are the same then the T-children of s, t are partitions of the same set of sentences.

Given Proposition 3.10, we say that an $(\exists)$-strategy in the verification game is *acceptable* iff it is both faithful and consistent. Kripke [13] writes: 'The largest intrinsic fixed point is the unique 'largest' interpretation of $T(x)$ which is consistent with our intuitive idea of truth and makes no arbitrary choices in truth assignments.' The following theorem formulates this 'nonarbitrariness' in terms of decorations:

COROLLARY 3.12. *A sentence is in the largest intrinsic fixed point of $\mathcal{J}_{V_L}$ iff it has an rfg that admits either a verifying acceptable decoration or a falsifying acceptable decoration* (*but not both*). *As a consequence, a sentence is Kripke-hypodoxical iff it is r-hypodoxical.*

PROOF. $\Rightarrow$: Let $v \in \{0,1\}$ be the truth value assigned to φ by the largest intrinsic fixed point. By Theorem 3.9 there is a faithful and consistent $(\exists)$-strategy in $\mathcal{G}_T(\varphi, v)$. Suppose there is also some faithful and consistent $(\exists)$-strategy in $\mathcal{G}_T(\varphi, 1-v)$. By the right-to-left direction of Theorem 3.9 this implies that φ has the definite truth value v in one fixed point and $1-v$ in some other fixed point. But this contradicts the assumption that φ is intrinsic.

$\Leftarrow$: Suppose φ has no definite truth value in the largest intrinsic fixed point. By Proposition 2.1, this means that φ is either paradoxical or hypodoxical. By Theorem 3.9, in the first case there is no faithful and consistent strategy in either $\mathcal{G}_T(\varphi, 0)$ or in $\mathcal{G}_T(\varphi, 1)$. In the second case, there is a faithful and consistent strategy both in $\mathcal{G}_T(\varphi, 0)$ and in $\mathcal{G}_T(\varphi, 1)$. ⊣

§4. **Core and periphery of reference graphs.** We have seen that ϕ is Kripke-paradoxical iff any $(\exists)$-strategy in the verification game is either unfaithful or inconsistent (i.e., unacceptable). In this section we will show that $(\exists)$ can actually always choose a faithful strategy. This can be understood as some kind of normalization result for verification strategies and it is a crucial step towards our goal of a graph-theoretical understanding of paradoxicality because inconsistency, unlike unfaithfulness, can be related to structural properties (cf. Section 4.5) of an rfg. Unfaithfulness, on the other hand, cannot be related to structural properties—those that are preserved under graph isomorphism. The above mentioned result (Corollary 4.12) follows from Lemma 4.11, to which we will refer as the *Fundamental Lemma*—it is indeed the foundation for almost all the results in the rest of this article. The Fundamental Lemma also sheds light on the key concept of this section: the separation of an rfg into periphery and core. Roughly speaking, the periphery of an rfg consists of those parts whose unfolding can be decorated in a *unique* way by truth values such that the resulting verification strategy is faithful while in the core there are various such decorations. Intuitively, the periphery is the 'sphere of influence' of the 'atoms' (or sinks) of an rfg G (the nodes with no out-neighbours). If we imagine that the verification game is being played on G then the periphery of G is the part where $(\exists)$ has one and only one move that is nonlosing while she always has at least two nonlosing moves on the core. That she has indeed always at least one move that is nonlosing is an important consequence of the Fundamental Lemma. We will show that the phenomena of paradoxicality and hypodoxicality can be tied down to the structure of the core of an rfg. Moreover, we will give a characterization of the periphery in terms of Cantini's valuation scheme, V_{FV}.

4.1. The concept of core and periphery.

DEFINITION 4.1. Let G be an rfg and ψ be a vertex of G. We denote by G_ψ the subgraph of G induced by the vertices of G accessible from ψ.

1. Call ψ *bivalent in G* iff G_ψ has
   (a) a faithful multidecoration d with $d(\psi) = \perp$ or
   (b) a faithful multidecoration d_0 with $d_0(\psi) = 0$ and a faithful multidecoration with $d_1(\psi) = 1$.
2. Call ψ *univalent in G* iff G_ψ has no faithful multidecoration d with $d(\psi) = \perp$, and it either has a faithful multidecoration d_0 with $d_0(\psi) = 0$ or a faithful multidecoration d_1 with $d_1(\psi) = 1$, but not both. In the first case we call ψ 0-*univalent in G*, in the second case we call ψ 1-*univalent in G*.
3. The *core* of an rfg is the set of its bivalent vertices.
4. The *periphery* of an rfg is the set of its univalent vertices.

It is obvious that any grounded sentence lies in the periphery, while the liar λ lies in the core of every rfg in which it occurs. On the other hand, $\lambda \vee \neg\lambda$ always lies in the periphery, as one cannot make the sentence false. This is still the case if we consider the Weak Kleene scheme. $\lambda \vee \neg\lambda$ is paradoxical relative to V_{WK} (because it essentially V_{WK}-depends on $\{\lambda\}$) but, since one cannot make it false, lies in the periphery of any rfg in which it occurs. Similarly, $\exists x Tx$ is a sentence that is paradoxical relative to V_L but lies in the periphery of any rfg in which it occurs. There is one multidecoration that makes this sentence false, but it is unfaithful. (That $\exists x Tx$ is paradoxical relative to V_L relies on the fact that it depends essentially on the set of all sentences.)

The sentences in the core of an rfg could be described as 'ambiguous' while the sentences in the periphery are 'unambiguous'. So we can expect to tie down the phenomena of paradoxicality and hypodoxicality to the core of an rfg, ignoring its periphery. Now, we have already noted that some paradoxical sentences like $\exists x Tx$ are in the periphery rather than in the core. However, we will see later (cf. Theorem 4.30) that the reference patterns that make some sentence paradoxical lie within the core of its rfgs (e.g., in the rfg of $\exists x Tx$, that sentence has an arc pointing to itself, but this loop is not what makes it paradoxical: the problem is that the rfg contains the self-referential liar in its core). Notice that it not clear from the above definition that any vertex of an rfg belongs either to its core or to its periphery. However, in Section 4.3 we will show that every rfg can indeed be decomposed into core and periphery. A simple but useful observation about the periphery is the following:

Proposition 4.2. *Let G be an rfg.*

1. *No vertex of G belongs to both the core and the periphery of G.*
2. *If G admits a faithful multidecoration, then every vertex of G belongs either to its core or to its periphery.*
3. *Any faithful multidecoration of G is consistent on its periphery.*
4. *Any two faithful multidecorations of G coincide on its periphery.*

Proposition 4.3. *The well-founded part of an rfg is contained in its periphery.*

Proof. By Theorem 3.5. ⊣

The converse of this proposition fails. Here is an example of a nonwell-founded rfg all of whose vertices lie in its periphery: For each $n \in \omega$ let $\phi_n \leftrightarrow \neg T\ulcorner\phi_{n+1}\urcorner \wedge \neg T\ulcorner 1 = 1\urcorner$. Consider the canonical rfg of ϕ_0. Cleary there is one and only one faithful multidecoration of this graph, assigning 1 to $1 = 1$ and 0 to each ϕ_n. Later on, we will give an exact characterization of the sentences that are in the periphery of an rfg (Corollary 4.17). In order to formulate this result (and some other of our main results), we need to introduce an operator that maps every standard valuation scheme to an expansion of that scheme.

4.2. The saturated closure of V_L. Let V be a standard valuation scheme. We define

$$\hat{V}(\Phi^+,\Phi^-)(\phi) = \begin{cases} 1, & \text{if } \nexists(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-) \text{ s.t. } \Psi^+ \cap \Psi^- = \emptyset \wedge \\ & V(\Psi^+,\Psi^-)(\phi) = 0, \\ 0, & \text{if } \nexists(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-) \text{ s.t. } \Psi^+ \cap \Psi^- = \emptyset \wedge \\ & V(\Psi^+,\Psi^-)(\phi) = 1, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

We call $\hat{V}$ the *saturated closure* of V. Observe that $\hat{V}$ is well-defined since any (Φ^+,Φ^-) has an extension (Ψ^+,Ψ^-) such that $V(\Psi^+,\Psi^-)(\phi) \in \{0,1\}$ (namely, the classical close-off).

Lemma 4.4. *Let V be any standard valuation scheme. Then*:

1. *$\hat{V}$ is monotonic.*
2. *$V \leq \hat{V}$, i.e., $\hat{V}$ is at least as strong as V.*
3. *$\hat{V}$ is classically sound.*

Proof. *Ad* (1): Let $(\Phi^+,\Phi^-) \subseteq (\Psi^+,\Psi^-)$ and let $\hat{V}(\Phi^+,\Phi^-)(\varphi) = v \in \{0,1\}$. This means that $V(S^+,S^-)(\varphi) \in \{v, \frac{1}{2}\}$ for all $(S^+,S^-) \supseteq (\Phi^+,\Phi^-)$. Hence $V(S^+,S^-)(\varphi) \in \{v,\frac{1}{2}\}$ for all $(S^+,S^-) \supseteq (\Psi^+,\Psi^-)$. Thus $\hat{V}(\Psi^+,\Psi^-)(\varphi) = v$.

Ad (2): Let $v \in \{0,1\}$ and $V(\Phi^+,\Phi^-)(\phi) = v$. Since V is monotonic there is no $(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-)$ with $V(\Psi^+,\Psi^-)(\phi) = 1 - v$. Hence $\hat{V}(\Phi^+,\Phi^-)(\phi) = v$.

Ad (3): Assume w.l.o.g. that $\hat{V}(S^+,S^-)(\varphi) = 1$ and that $Val_{S^+}(\varphi) = 0$. Then $V(S^+,S^- \cup (\omega \setminus S^+))(\varphi) = 0$ since $V_{WK} \leq V$ and $V_{WK}(S^+,S^- \cup (\omega \setminus S^+))(\varphi) = 0$ (since V_{WK} is classically sound). But $(S^+,S^-) \subseteq (S^+,S^- \cup (\omega \setminus S^+))$. This is a contradiction to $\hat{V}(S^+,S^-)(\varphi) = 1$. ⊣

COROLLARY 4.5. *If V is a standard valuation scheme then so is $\hat{V}$.*

PROOF. Lemma 4.4(2) implies that $V_{WK} \leq \hat{V}$. By 4.4(1), $\hat{V}$ is monotonic and by 4.4(3), $\hat{V}$ is classically sound. ⊣

Now consider the supervaluation scheme V_{FV} given by Cantini in [4]:

$$V_{FV}(\Phi^+,\Phi^-)(\phi) = \begin{cases} 1, & \text{if } \forall(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-)(\Psi^+ \cap \Psi^- = \emptyset \Rightarrow \\ & (\mathbb{N},\Psi^+) \models \phi), \\ 0, & \text{if } \forall(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-)(\Psi^+ \cap \Psi^- = \emptyset \Rightarrow \\ & (\mathbb{N},\Psi^+) \not\models \phi), \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

We will call V_{FV} the *Cantini valuation scheme*. Obviously, it is a standard valuation scheme. We will now show that the Cantini valuation scheme is maximal among the standard schemes.

THEOREM 4.6. *Let V be any standard valuation scheme. Then $\hat{V} = V_{FV}$. In particular, $\hat{V}_L = V_{FV}$.*

PROOF. Suppose w.l.o.g. that $V_{FV}(\Phi^+,\Phi^-)(\phi) = 1$. Thus $(\mathbb{N},\Psi^+) \models \phi$ for all consistent $(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-)$. We want to show that $\hat{V}(\Phi^+,\Phi^-)(\phi) = 1$. Assume $\hat{V}(\Phi^+,\Phi^-)(\phi) \neq 1$. Then there is some consistent $(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-)$ with $V(\Psi^+,\Psi^-)(\phi) = 0$. By classical soundness, $(\mathbb{N},\Psi^+) \not\models \phi$. But this contradicts our above assumption.

Now suppose w.l.o.g. that $\hat{V}(\Phi^+,\Phi^-)(\phi) = 1$. Hence for all consistent $(S^+,S^-) \supseteq (\Phi^+,\Phi^-)$: $V(S^+,S^-)(\phi) \neq 0$. In particular, this holds for all consistent "classical" models, where (S^+,S^-) is classical if $S^+ \cup S^- = \omega$. Let $(\Psi^+,\Psi^-) \supseteq (\Phi^+,\Phi^-)$ be an arbitrary consistent model and consider $(\Psi^+,\Psi^- \cup (\omega \setminus \Psi^+))$, which is consistent and classical. Since V is at least as strong as V_{WK} this means that $V(\Psi^+,\Psi^- \cup (\omega \setminus \Psi^+))(\phi) = 1$ (because V_{WK} assigns a definite truth value in all classical models). Since V is classically sound we get $(\mathbb{N},\Psi^+) \models \phi$. Hence $V_{FV}(\Phi^+,\Phi^-)(\phi) = 1$. ⊣

COROLLARY 4.7. *V_{FV} is the maximal standard valuation scheme. Hence any monotonic valuation scheme $V > V_{FV}$ is classically unsound.*

The following game can be looked at as a more abstract version of the verification game *in the absence of a well-behaved (i.e., symmetric) dependence relation.* (Note that $\hat{V}_L$ is not symmetric.) In particular, there is no grounding game for $\hat{V}_L$. A second difference is that we keep track of the mode only indirectly: it is encoded in the truth-value component of $(\forall)$'s moves.

For each fact (φ, v) and each set of facts $\mathcal{F}$, define the *$\hat{V}_L$-verification game*, $\hat{\mathcal{G}}_T(\varphi, v, \mathcal{F})$, between two players $(\exists)$ and $(\forall)$ as follows:

1. $(\forall)$ must move first and choose (φ, v) as his first move (φ_1, v_1). If $\phi \in \mathcal{F}^+ \cup \mathcal{F}^-$ he cannot move.
2. As her n-th move $(\exists)$ must choose some partial model (Φ_n^+,Φ_n^-) compatible with $\mathcal{F}$ such that $\hat{V}_L(\Phi_n^+,\Phi_n^-)(\varphi_n) = v_n$.
3. If $n > 1$, as his n-th move $(\forall)$ must choose some fact (φ_n, v_n) such that $\varphi_n \in \Phi_{n-1}^+$ if $v_n = 1$ and $\varphi_n \in \Phi_{n-1}^-$ if $v_n = 0$.

The *winning conditions* for $\hat{\mathcal{G}}_T(\varphi, v, \mathcal{F})$ are

- If a run of the game goes on forever it is declared a draw.

- If a player cannot move according to the rules 2 or 3 then the other player wins this run of the game.
- If $(\forall)$ cannot move according to rule 1 then he loses the game iff $\varphi \in \mathcal{F}^+$ and $v = 1$ or if $\varphi \in \mathcal{F}^-$ and $v = 0$. In the other cases he wins.

To prepare the Fundamental Lemma we need the following lemmata.

PROPOSITION 4.8. *Let $\mathcal{F}$ be a consistent and sound set of facts. Then $(\exists)$ has a winning strategy either in $\hat{\mathcal{G}}_T(\varphi, 1, \mathcal{F})$ or in $\hat{\mathcal{G}}_T(\varphi, 0, \mathcal{F})$ (but not in both) iff φ is in the least fixed point of $\mathcal{J}_{\hat{V}_L}$ generated by $\mathcal{F}$.*

PROOF. $\Rightarrow$: By induction on the strategy-rank of φ.
$\Leftarrow$: By induction on the inductive rank of φ. $\dashv$

COROLLARY 4.9. *If there is some well-founded faithful fact-dependence tree for $\hat{V}_L$ with root (ϕ, v) then there is no well-founded fact-dependence tree for $\hat{V}_L$ with root $(\phi, 1 - v)$.*

LEMMA 4.10. *Let $\mathcal{F}$ be a consistent and sound set of facts and ϕ a sentence. Then there is either no winning strategy for $(\exists)$ in $\hat{\mathcal{G}}_T(\phi, v, \mathcal{F})$ or no faithful strategy for $(\exists)$ in $\mathcal{G}_T(\phi, 1 - v, \mathcal{F})$.*

PROOF. Assume ϕ is a sentence such that there is a winning $(\exists)$-strategy τ in $\hat{\mathcal{G}}_T(\phi, v, \mathcal{F})$ and a faithful $(\exists)$-strategy σ in $\mathcal{G}_T(\phi, 1 - v, \mathcal{F})$. By induction on the strategy-rank α of τ we show that this leads to a contradiction. The claim is trivial if τ is the trivial strategy. So assume that τ is nontrivial.

Let (Φ^+, Φ^-) be $(\exists)$'s τ-response to ϕ. Then $\hat{V}_L(\Phi^+, \Phi^-)(\phi) = v$. Let (Ψ^+, Ψ^-) be $(\exists)$'s σ-response to ϕ. Then $V_L(\Psi^+, \Psi^-)(\phi) = 1-v$. Moreover, by rule 3 of the verification game for V_L, (Ψ^+, Ψ^-) is compatible with $\mathcal{F}$.

Let $\alpha = 0$. This means that $(\forall)$ cannot move anymore and accordingly $(\Phi^+, \Phi^-) = (\mathcal{F}^+, \mathcal{F}^-)$. Hence $\hat{V}_L(\mathcal{F}^+, \mathcal{F}^-)(\phi) = v$. Since $\mathcal{F}$ is compatible with (Ψ^+, Ψ^-), this is a contradiction. (First, $V_L(\Psi^+, \Psi^-)(\phi) = 1-v$. Since $\mathcal{F}$ is compatible with (Ψ^+, Ψ^-), $(\Psi^+ \cup \mathcal{F}^+, \Psi^- \cup \mathcal{F}^-)$ is a partial model that extends $(\mathcal{F}^+, \mathcal{F}^-)$. But by monotonicity of V_L, $V_L(\Psi^+ \cup \mathcal{F}^+, \Psi^- \cup \mathcal{F}^-)(\phi) = 1 - v$. This contradicts $\hat{V}_L(\mathcal{F}^+, \mathcal{F}^-)(\phi) = v$, by definition of $\hat{V}_L$.)

Now let $\alpha > 0$ and suppose the claim holds for all $\beta < \alpha$. Then

$$(*) \quad \Phi^+ \cap \Psi^- = \emptyset \text{ and } \Phi^- \cap \Psi^+ = \emptyset.$$

To prove $(*)$, assume w.l.o.g. that there is a $\psi \in \Psi^- \cap \Phi^+$. Let τ' be the substrategy of τ determined by the root ψ. Then τ' is a winning strategy for $(\exists)$ in $\hat{\mathcal{G}}_T(\psi, 1, \mathcal{F})$ (because $\psi \in \Phi^+$). Since the rank of τ' is less than α, by I.H. there is no faithful strategy for $(\exists)$ in $\mathcal{G}_T(\psi, 0, \mathcal{F})$. But this is contradicting the fact that σ is a faithful strategy in $\mathcal{G}_T(\psi, 0, \mathcal{F})$. (The latter is the case because σ is a faithful strategy in $\mathcal{G}_T(\phi, 1 - v, \mathcal{F})$ and $\psi \in \Psi^-$.)

Let $\chi^+ = \Psi^+ \cup \Phi^+$ and $\chi^- = \Psi^- \cup \Phi^-$. By $(*)$, (χ^+, χ^-) is a partial model. Since $\chi^+ \cap \chi^- = \emptyset$, and $(\chi^+, \chi^-) \supseteq (\Psi^+, \Psi^-)$, we obtain that $V_L(\chi^+, \chi^-)(\phi) = 1 - v$. Since also $(\chi^+, \chi^-) \supseteq (\Phi^+, \Phi^-)$, this contradicts $\hat{V}_L(\Phi^+, \Phi^-)(\phi) = v$, by definition of $\hat{V}_L$. $\dashv$

4.3. The Fundamental Lemma. For the following Fundamental Lemma, it will be convenient to compare $\hat{\mathcal{G}}_T$-strategies to $\mathcal{G}_G$-strategies. For any

$\hat{\mathcal{G}}_T$-strategy σ, let $\|\sigma\|$ be the result of applying the operator $\|\cdot\|$ to every component of every position of σ (cf. Section 3.2). Thus $\|\sigma\|$ is a tree just of same type as a strategy in the grounding-game, and we shall write, given an $(\exists)$-strategy τ in the latter, $\|\sigma\| \preceq \tau$ iff $\|\sigma\|$ is a subtree of τ with the same root as τ. Recall that where σ is a strategy, $\Gamma(\sigma)$ is the rfg that σ collapses into. Finally, recall that a set of facts is *faithful* iff each of its members is a faithful fact, where a fact (φ, v) is *faithful* iff there is a faithful fact-dependence tree with root (φ, v), iff $(\exists)$ has a faithful strategy in $\mathcal{G}_T(\varphi, v)$. We can now formulate the Fundamental Lemma:

LEMMA 4.11. *Let ϕ be a sentence and Φ any $\mathbb{D}$-sound set of sentences. Let σ be an $(\exists)$-strategy in $\mathcal{G}_G(\phi, \Phi)$ and let $\mathcal{F}$ be a faithful, consistent and sound set of facts with $\|\mathcal{F}\| = \Phi$.*

1. *There is a truth value v and a nonlosing $(\exists)$-strategy σ^* in $\mathcal{G}_T(\phi, v, \mathcal{F})$ with $\|\sigma^*\| = \sigma$.*
2. *The following statements are equivalent*:
   (a) *there are faithful decorations σ^* and $\bar{\sigma}^*$ of σ such that σ^* is a strategy in $\mathcal{G}_T(\varphi, 0, \mathcal{F})$ and $\bar{\sigma}^*$ is a strategy in $\mathcal{G}_T(\varphi, 1, \mathcal{F})$,*
   (b) *there is neither a winning strategy τ for $(\exists)$ in $\hat{\mathcal{G}}_T(\varphi, 0, \mathcal{F})$ nor in $\hat{\mathcal{G}}_T(\varphi, 1, \mathcal{F})$ such that $\|\tau\| \preceq \sigma$.*

We will give a proof of this Fundamental Lemma in Section 4.4. First let us note some of its important consequences:

COROLLARY 4.12. *Every rfg G admits a faithful multidecoration.*

This is the corollary we referred to as a 'normalization result' in the introduction of Section 4. In terms of Yablo's article that we quoted in Section 2.2: We—if we identify with $(\exists)$—can indeed always choose to maintain that Epimenides was really a Cretan and thus elect (structural) inconsistency over (nonstructural) unfaithfulness. But convincing as this may sound, the result is far from trivial—we will see this when we prove the Fundamental Lemma. The technical problem consists, roughly speaking, in rfgs generally being nonwell-founded, so that we cannot use inductive methods (running bottom-up) to define a faithful decoration on them. We have to find such a decoration running top-down (a rather co-inductive technique) and must take care that we never hit the ground where it is unfaithful. Further, the nontriviality of Lemma 4.11 is underlined by the fact that it doesn't hold for nonsymmetric valuation schemes, e.g., the Strong Kleene valuation. We refer the reader to Section 4.6 for further discussion.

COROLLARY 4.13. *Every rfg has a decomposition of its vertices into periphery and core.*

PROOF. By 4.2 and 4.12. ⊣

The following corollaries describe how paradoxicality and hypodoxicality can be related to the core:

COROLLARY 4.14. *Every faithful multidecoration of an rfg G that is consistent on the core of G is consistent, i.e., is a decoration of G.*

PROOF. By 4.2(3). ⊣

Corollary 4.15. *If ϕ is r-paradoxical or r-hypodoxical then every reference graph of ϕ has a nonempty core.*

Proof. By 4.2(3), 4.2(4), and 3.12. ⊣

The following notions are introduced to formulate further consequences of the Fundamental Lemma. They will also play a role in Section 5. Call a sentence ϕ *absolutely bivalent* or a *core-sentence* iff for every rfg G it is the case that ϕ occurs in the core of G whenever it occurs in G. Call ϕ *absolutely univalent* or a *periphery-sentence* iff for every rfg G it is the case that ϕ occurs in the periphery of G whenever it occurs in G. Our earlier notions of bivalence and univalence (Section 4.1) were relativized to a given graph G. We will now show that if a sentence ϕ is bivalent (univalent) in some rfg G, then ϕ is absolutely bivalent (univalent), i.e., ϕ is in the core (periphery) of *every* graph in which it occurs. This justifies us (in subsequent sections) to talk of univalent (bivalent) sentences simpliciter (omitting 'absolutely').

Corollary 4.16. *The periphery of any rfg G consists exactly of those vertices ϕ of G such that $(\exists)$ has a winning strategy either in $\hat{\mathcal{G}}_T(\varphi, 1)$ or in $\hat{\mathcal{G}}_T(\varphi, 0)$.*

Proof. Let G be a rfg and ϕ be a vertex of G. If ϕ is in the periphery of G, then by Lemma 4.11(2) applied to the graph G_ϕ, $(\exists)$ has a winning strategy in $\hat{\mathcal{G}}_T(\phi, 1)$ or in $\hat{\mathcal{G}}_T(\phi, 0)$. Now suppose w.l.o.g. that $(\exists)$ has a winning strategy in $\hat{\mathcal{G}}_T(\phi, 1)$. Then by Lemma 4.10 there is no faithful strategy for $(\exists)$ in $\mathcal{G}_T(\phi, 0)$. Hence ϕ cannot be bivalent and is therefore not in the core of G. ⊣

Corollary 4.17. *The periphery of any rfg G consists exactly of those sentences (i.e., nodes) of G that are Cantini-grounded. The 1-univalent sentences of G are those sentences of G that are true in the least Cantini fixed point and the 0-univalent sentences of G are those sentences of G that are false in the least Cantini fixed point.*

Proof. By Proposition 4.8 and the previous corollary, the periphery consists exactly of those sentences of G that are in the least fixed point of $\mathcal{J}_{\hat{V}_L}$. By Theorem 4.6, the least fixed point of $\mathcal{J}_{\hat{V}_L}$ consists exactly of the Cantini-grounded sentences. ⊣

Corollary 4.18.
1. *Every sentence is either absolutely univalent or absolutely bivalent.*
2. *Every absolutely univalent sentence is either absolutely 1-univalent or absolutely 0-univalent.*

Corollary 4.19. *A rfg G has an empty core iff for all $\phi \in V(G)$, ϕ is Cantini-grounded.*

4.4. Proof of the Fundamental Lemma. We now turn to the proof of the Fundamental Lemma. That is: Let ϕ be a sentence and Φ any $\mathbb{D}$-sound set of sentences. Let σ be an $(\exists)$-strategy in $\mathcal{G}_G(\phi, \Phi)$ and let $\mathcal{F}$ be a faithful, consistent and sound set of facts with $\|\mathcal{F}\| = \Phi$.

1. There is a truth value v and a nonlosing $(\exists)$-strategy σ^* in $\mathcal{G}_T(\phi, v, \mathcal{F})$ with $\|\sigma^*\| = \sigma$.

2. The following statements are equivalent:
   (a) there are faithful decorations σ^* and $\bar{\sigma}^*$ of σ such that σ^* is a strategy in $\mathcal{G}_T(\varphi, 0, \mathcal{F})$ and $\bar{\sigma}^*$ is a strategy in $\mathcal{G}_T(\varphi, 1, \mathcal{F})$,
   (b) there is neither a winning strategy τ for $(\exists)$ in $\hat{\mathcal{G}}_T(\varphi, 0, \mathcal{F})$ nor in $\hat{\mathcal{G}}_T(\varphi, 1, \mathcal{F})$ such that $\|\tau\| \preceq \sigma$.

PROOF. Let σ an $(\exists)$-strategy in $\mathcal{G}_G(\phi, \Phi)$. For ease of exposition, we will only show the case where $\Phi = \emptyset$. Clearly, $\emptyset$ is a $\mathbb{D}$-sound set as required by the lemma. Note that since $\Phi = \emptyset$, we must have $\mathcal{F} = \emptyset$ as well. Obviously, $\mathcal{F}$ is then a faithful, consistent and sound set of facts. Let T_σ be the dependence tree determined by σ. We will define a truth value assignment $v : T_\sigma \to \{0, 1\}$ by recursion on the length of $s \in T_\sigma$. The resulting pair (T_σ, v) will be a faithful fact-dependence tree for V_L with root ϕ. This will prove part (1) of the claim (because every fact-dependence tree corresponds to a unique verification strategy).

First, we need to introduce some notation. For any position (sequence) $s \in T_\sigma$, let $\pi(s)$ be the last element of s. Since T_σ is a dependence tree, $\pi(s)$ is a sentence. Let $\sigma(s)$ be the subtree of σ with $\pi(s)$ as its root. For any $s \in T_\sigma$ let $\hat{\sigma}(s)$ be a well-founded fact-dependence tree for $\hat{V}_L$ $(= V_{FV})$ with root $\pi(s)$ such that $\|\hat{\sigma}(s)\| \preceq \sigma(s)$, if such a fact-dependence tree exists, and let $\hat{\sigma}(s)$ be undefined otherwise. Let us say that $\hat{\sigma}(s)$ *is defined for 0* if its root is $(\pi(s), 0)$, and that $\hat{\sigma}(s)$ *is defined for* 1 if its root is $(\pi(s), 1)$. Note that by Lemma 4.9, if $\hat{\sigma}(s)$ is defined at all, it is either defined for 0 or 1, but not both, because then $\pi(s)$ is in the least fixed point of $\mathcal{J}_{FV}$. Observe that since all well-founded fact-dependence trees are compatible, this assignment of the values 0, 1 or 'undefined' does not depend on the choice of a fact-dependence tree.

We will define the decoration v of T_σ in such a way that for all positions $s \in T_\sigma$, $v(s)$ 'agrees' with $\hat{\sigma}(s)$ whenever the latter is defined. Here, the notion of agreement is spelled out in condition C_s below. The condition A_s below will ensure that (T_σ, v) is indeed a fact-dependence tree for V_L.

For any $s \in T_\sigma$, let C_s, A_s be the following statements:

C_s: for all $\psi \in out_\sigma(s)$: if $\hat{\sigma}(s \circ \psi)$ is defined for w, then $v(s \circ \psi) = w$,
A_s: $V_L(out^+_{\sigma,v}(s), out^-_{\sigma,v}(s))(\pi(s)) = v(s)$.

For $n \geq 0$, let C_{n+1}, A_{n+1} be the following statements:

C_{n+1}: for all $s \in T_\sigma \restriction n$, C_s holds,
A_{n+1}: for all $s \in T_\sigma \restriction n$, A_s holds.

Here, $T_\sigma \restriction n$ is the set of all positions of T_σ of length $\leq n$; $s \circ \psi$ is the result of extending the position s by the element ψ; $out_\sigma(s)$ is the set of sentences ψ such that $s \circ \psi \in T_\sigma$ and $out^+_{\sigma,v}(s)$ (resp. $out^-_{\sigma,v}(s)$) is the set of sentences ψ such that $(s \circ \psi \in T_\sigma)$ and ψ has the value 1 (0) according to v. The statements C_{n+1} and A_{n+1} can be read as 'v restricted to first $n+1$ levels of T_σ is *compatible* with $\hat{\sigma}$' and 'v restricted to first $n+1$ levels of T_σ is *acceptable*' (compare A_n with the definition of an acceptable decoration of an rfg). Note that the definition of A_s presupposes that v is already defined for all $s \circ \psi$, where ψ is an out-neighbour of $\pi(s)$ in T_σ. We will define v

by recursion on n, where in the n-th step of this procedure we suppose that v is already defined for all $s \in T_\sigma \upharpoonright n$ and expand the domain of v to all $s \in T_\sigma \upharpoonright n + 1$. Simultaneously we check, at the end of the n-th step, that C_{n+1} and A_{n+1} hold.

Let $n = 0$. Then $s = ()$. Let $v(s) = u$ if $\hat{\sigma}(s)$ is defined for u, and otherwise 0. Then C_1 and A_1 hold trivially. Note that we could define a second valuation v' by setting $v'(s) = u$ if $\hat{\sigma}(s)$ is defined for u, and otherwise 1. Observe that $v(s) \neq v'(s)$ iff $\hat{\sigma}(s)$ is undefined. This will be important for the proof of claim (2). However, we will only focus on the construction of v, as the construction for v' is completely analogous.

Now let $n > 0$ and $s \in T_\sigma \upharpoonright n$. By induction hypothesis $v(s)$ is already defined. We shall define $v(s \circ \psi)$ for all $\psi \in out_\sigma(s)$ in such a way that C_s, A_s hold. From this C_{n+1}, A_{n+1} will follow.

Let Ψ^+ be the set of all $\psi \in out_\sigma(s)$ such that $\hat{\sigma}(s \circ \psi)$ is defined for 1, let Ψ^- be the set of all $\psi \in out_\sigma(s)$ such that $\hat{\sigma}(s \circ \psi)$ is defined for 0, and let $\Psi^\perp$ be the set of all $\psi \in out_\sigma(s)$ such that $\hat{\sigma}(s \circ \psi)$ is undefined.

Case 1. $\hat{\sigma}(s)$ is defined for $w' \in \{0, 1\}$. Then $\hat{V}_L(\Psi^+, \Psi^-)(\pi(s)) = w'$. Define $v(s \circ \psi) = 1$, if $\psi \in \Psi^+$ and $v(s \circ \psi) = 0$, if $\psi \in \Psi^- \cup \Psi^\perp$. This yields C_s.

Since $\pi(s)$ depends on $out_\sigma(s)$, $V_L(\Psi^+, \Psi^- \cup \Psi^\perp)(\pi(s)) = w \in \{0, 1\}$. Since $\hat{V}_L$ is monotonic and at least as strong as V_L (Lemma 4.4) we get $w = w'$. On the other hand, by induction hypothesis C_n we have $v(s) = w = w'$. The monotonicity of V_L implies A_s.

Case 2. $\hat{\sigma}(s)$ is undefined. Let us prove the following claim:

($*$) There is some $(\Phi^+, \Phi^-) \supseteq (\Psi^+, \Psi^-)$ with $\Phi^+ \cup \Phi^- = out_\sigma(s)$ and $V_L(\Phi^+, \Phi^-)(\pi(s)) = v(s)$.

Proof. Suppose w.l.o.g. $v(s) = 0$. Assume that there is no such $(\Phi^+, \Phi^-) \supseteq (\Psi^+, \Psi^-)$ with $\Phi^+ \cup \Phi^- = out_\sigma(s)$ and $V_L(\Phi^+, \Phi^-)(\pi(s)) = 0$.

Since $\pi(s)$ depends on $out_\sigma(s)$, for all partial models (Φ^+, Φ^-) with $\Phi^+ \cup \Phi^- \supseteq out_\sigma(s)$ we obtain $V_L(\Phi^+, \Phi^-)(\pi(s)) = V_L(\Phi^+ \cap out_\sigma(s), \Phi^- \cap out_\sigma(s))(\pi(s))$, by Lemma 2.16. (Observe that in order to apply Lemma 2.16 it is essential that V_L is symmetric. We will give a detailed discussion of the role of symmetry in finding faithful decorations in Section 4.6.)

Hence by assumption there is no partial model $(\Phi^+, \Phi^-) \supseteq (\Psi^+, \Psi^-)$ with $V_L(\Phi^+, \Phi^-)(\pi(s)) = 0$. But this means by definition of $\hat{V}_L$ that $\hat{V}_L(\Psi^+, \Psi^-)(\pi(s)) = 1$. But $\hat{\sigma}(s \circ \psi)$ is defined for all ψ such that $\psi \in \Psi^+ \cup \Psi^-$. Hence for each $\psi \in \Psi^+ \cup \Psi^-$ there is a $v^*_\psi : \sigma(s \circ \psi) \to \{0, 1\}$ such that $(\sigma(s \circ \psi), v^*_\psi)$ is a fact-dependence tree for $\hat{V}_L$ with root ψ. Let $v^* = (\bigcup_{\psi \in \Psi^+ \cup \Psi^-} v^*_\psi) \cup (\pi(s), 1)$, and let $\tau(s)$ be the set of all $t \in \sigma(s)$ such that $t = (\pi(s))$ or $t(2) \in \Psi^+ \cup \Psi^-$, where $t(2)$ is the second element of t. Then $(\tau(s), v^*)$ is a well-founded fact-dependence tree for $\hat{V}_L$ with root $\pi(s)$, and $\|(\tau(s), v^*)\| \preceq \sigma(s)$. Hence $\hat{\sigma}(s)$ is defined for 1. Contradiction. ⊣

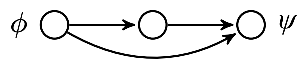
Define $v(s \circ \psi) = 1$, if $\psi \in \Phi^+$ and $v(s \circ \psi) = 0$, if $\psi \in \Phi^-$. Then ($*$) immediately yields C_s and A_s. So $v(s \circ \psi)$ is defined for all $\psi \in out_\sigma(s)$.

Since we have chosen s arbitrarily and any $t \in T_\sigma \restriction n+1$ is of the form $s \circ \psi$, $v(t)$ is defined for all $t \in T_\sigma \restriction n+1$. Hence C_{n+1} and A_{n+1} follow.

In this process of recursive definition we finally get a function $v : T_\sigma \to \{0,1\}$ and by induction C_n, A_n hold for every n. Clearly, from $\forall n \in \omega : A_n$ it follows that (T_σ, v) is a fact-dependence tree for V_L with root φ. Moreover (T_σ, v) is a faithful fact-dependence tree: our construction process provides an $(\exists)$-answer to any $(\forall)$-move whatsoever. This proves claim (1) of the theorem. Analogously, (T_σ, v') is also is a faithful fact-dependence tree for V_L with root φ.

Now for claim (2). By our definition of the valuations v and v' we obtain $v \neq v'$ iff there is no well-founded fact-dependence tree τ^* for $\hat{V}_L$ such that $\|\tau^*\| \preceq \sigma$. This proves $(b) \Rightarrow (a)$. Now suppose (a). Assume that there is winning strategy τ for $(\exists)$ in $\hat{\mathcal{G}}_T(\varphi, 0)$ such that $\|\tau\| \preceq \sigma$. Then, by Lemma 4.10, there is no faithful strategy for $(\exists)$ in $\mathcal{G}_T(\phi, 1)$. But this contradicts (a). Analogously, the assumption that there is a winning strategy τ for $(\exists)$ in $\hat{\mathcal{G}}_T(\varphi, 1)$ such that $\|\tau\| \preceq \sigma$ leads to a contradiction. ⊣

4.5. Paradoxicality and a graph's structural properties. Aside from its rather philosophical meaning hinted at above, Corollary 4.12 allows us (together with Theorem 3.9) to identify certain structural properties that all rfgs of a sentence share as necessary condition for its paradoxicality: Since there is always a faithful multidecoration of any rfg G_ϕ, the paradoxicality of ϕ must be due to the fact that all of the faithful multidecorations of all the rfgs of ϕ are inconsistent. But the property of lacking a faithful consistent multidecoration can be related to a graph's structural properties rather easily. In order to state these results, we need to introduce the following graph-theoretic notions. A graph G is a *tree* iff it there is some $r \in V(G)$ such that for every $x \in V(G)$ there is a unique walk from r to x. We call r the root of G. Note that the root of a tree is uniquely determined, so we can conceive any tree in a canonical way as an accessible pointed graph. A *double path* is a graph consisting of two paths originating both from the same vertex and rejoining in a different vertex, not touching each other in between. More precisely, a graph D is called a double path (from a to b) iff there are nontrivial paths P_1, P_2 from a to b such that $V(P_1) \cap V(P_2) = \{a, b\}$ and $V(D) = V(P_1) \cup V(P_2)$ and $A(D) = A(P_1) \cup A(P_2)$.



Example of a double path between φ and ψ.

The proof of the following useful lemma is straightforward.

LEMMA 4.20. *Let G be an apg. Then the following claims are equivalent*:

1. *G is a tree.*
2. *The map that collapses the unfolding of G onto G is a bijection.*
3. *G contains neither a cycle nor a double path.*

COROLLARY 4.21. *If a sentence φ has a reference graph that is a tree, then φ is not r-paradoxical.*

PROOF. Let G be a rfg of ϕ which is a tree. Let σ be an $(\exists)$-strategy in the grounding game such that $\Gamma(\sigma) = G$. By Theorem 4.12, there is a truth value v and an $(\exists)$-strategy σ^* in the game $\mathcal{G}_T(\phi, v)$ such that σ^* is a faithful decoration of σ. Let $\mathcal{F}$ be the set of all facts occurring in σ^*. Since G is a tree, no sentence ψ occurring in σ^* can occur in both contexts $(\psi, 1)$ and $(\psi, 0)$. Thus, σ is consistent. By Theorem 3.9, φ has the definite truth value v in some fixed point. ⊣

COROLLARY 4.22. *If a sentence is r-paradoxical, then each of its rfgs contains a directed cycle or a double path.*[12]

Now let us turn to the classification problem for dangerous graphs. First, we can formulate and prove in our framework the following result of Rabern et al. [16]:

THEOREM 4.23. *A finite rfg is dangerous iff it contains a directed cycle.*

PROOF. The left-to-right direction follows from Corollary 3.3. For the other direction, one first shows that a graph is dangerous iff some subgraph of it is dangerous. Rabern et al. [16, Lemma 2] have proved that for their own framework, and the proof can be adapted to our own framework as well. Now, suppose that G contains a directed cycle. Then, by the previous remark, we can simply assume that G is a directed cycle, let's say of length n. But then G is isomorphic to an rfg of a liar cycle of length n. ⊣

It is worth noticing that while the directed cycle is the reference pattern underlying the liar family, the double path is underlying any member of the Yablo sequence. However, it can be shown that if φ has an rfg with no cycles and only *finitely* many double paths, then φ is not r-paradoxical. Unlike cycles, double paths must come in flocks in order to make an rfg dangerous.

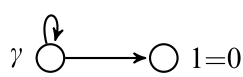
CONJECTURE 4.24. *A reference graph is dangerous iff it contains a subdivision of the liar-graph as a subgraph or the Yablo-graph as a* finitary minor.[13]

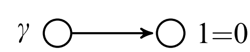
[12]Shortly before finishing this article it came to our attention that Jongeling et al. [12] have a theorem apparently mirroring the above one. (They call 'double reference' what we call 'double path'.) However, there are quite a few differences between their result and ours. First, they define 'double reference' only for a fragment of their (propositional) language, a restriction that probably could be dispensed with if they gave a rigorous definition of reference graph (which they don't). But the essential difference is that their language (the fact that it is a propositional language is not so important) contains only propositional variables but no propositional *constants* (unlike the language in Rabern et al. [16]). This is a substantial restriction which amounts to banning atomic *sentences* from the language entirely and which makes the proof of their version of the theorem much easier. In particular, it renders superfluous to prove something analogous to our Fundamental Lemma, since there is no such a thing as an unfaithful truth-value assignment in such a framework.

[13]For the notion of a minor in the context of digraphs consult the Appendix. Intuitively, the minor-relation is a more liberal form of the subgraph-relation that allows that connected sets of vertices can be contracted. By *finitary* we mean that any set of vertices that is contracted to one vertex must be finite.

This conjecture is motivated by an attempt to make the notion of a graph containing *many* double paths precise: An acyclic graph should contain the Yablo-graph as a finitary minor iff it deviates considerably from being a tree in the sense that it contains many double paths. This conjecture, if correct, implies that in some sense every r-paradoxical sentence is reducible either to the liar or the Yablo paradox.

4.6. Symmetry revisited. Now let us adopt again the more abstract point of view on valuation schemes already discussed in Section 2.6. At this point we are in a better position to fully appreciate the importance of our valuation schemes being symmetric for this kind of structural analysis of the paradoxes. Recall that for an arbitrary valuation scheme V a sentence ϕ V*-depends* on S iff there is a partition (S^+, S^-) of S such that $V(S^+, S^-)(\phi) \in \{0, 1\}$ and that V is called symmetric iff V-dependence of ϕ on S implies that $V(S^+, S^-)(\phi) \in \{0, 1\}$ for all partitions (S^+, S^-) of S. In the following we will have a look at two sentences that violate symmetry with respect to Strong Kleene valuation. First, consider a sentence γ such that $\gamma \leftrightarrow (\neg T\ulcorner\gamma\urcorner \vee T\ulcorner 1 = 0\urcorner)$ and the following two graphs:





The first graph is an rfg for γ under both V_{SK} and V_L; however, under V_{SK}, this rfg is not canonical, because the second graph is also an rfg for γ (observe that $V_{SK}(\{1 = 0\}, \emptyset)(\gamma) = 1$). So we have a well-founded but nevertheless dangerous V_{SK}-rfg. Hence, the theorems of the previous sections do not hold for V_{SK}. Let us have a closer look at what goes wrong here. The $(\exists)$-strategy σ that corresponds to the unfolding of the last rfg looks like this: $(\exists)$ responds to γ by $\{1 = 0\}$ and to $1 = 0$ by $\emptyset$, which is a winning strategy for her in the grounding game. The V_{SK} analogue to the Fundamental Lemma would yield a $v \in \{0, 1\}$ and a nonlosing strategy σ^* for $(\exists)$ in $\mathcal{G}_T(\gamma, v)$ that is a decoration of σ. Let's check whether this holds.

First recall that Proposition 2.14 fails for V_{SK} (indeed $V_{SK}(\emptyset, \{1 = 0\})(\gamma) = \frac{1}{2}$, but $(\mathbb{N}, \emptyset) \models \gamma$), so the rules for the V_{SK}-verifiction game must formulated analogous to the case of $\hat{V}_L$—otherwise $(\exists)$ could come up with a winning strategy in $\mathcal{G}_T(\gamma, 1)$ which would defy Theorem 3.9. Suppose the game starts in the verification mode. $(\exists)$ cannot play $(1 = 0, 0)$ since $V_{SK}(\emptyset, \{1 = 0\})(\gamma) = \frac{1}{2}$, so her only move is $(1 = 0, 1)$. But playing so loses the game because $(\exists)$ cannot respond to $(\forall)$'s move $1 = 0$. By the same reasoning, the only possible strategy for $(\exists)$ in $\mathcal{G}_T(\gamma, 0)$ is a losing strategy. So the Fundamental Lemma (and its corollaries) fail under V_{SK} due to its lack of symmetry. This means that the paradoxicality of γ cannot be captured by the structural property that all of its rfgs have a loop—it is rather reflected by the fact each of its rfgs has a loop *or no faithful multidecoration*. But this last property is not structural, i.e., not preserved under graph isomorphism. So symmetry is essential to make paradoxicality visible in the rfg of a sentence.

One may suspect that a way out of this dilemma would be to work with a modified notion of V-dependence. For instance, let ϕ V*-depend* on S iff for *every* partition (S^+, S^-) of S $V(S^+, S^-)(\phi) \in \{0, 1\}$. Wouldn't that make V symmetric with respect to this notion of dependence? In a certain way it would. However, now we have a different problem. Consider the sentence $\lambda \vee T\ulcorner 1 = 1 \urcorner$ (cf. Section 2.6). This sentence is in the least fixed point, since $V_{SK}(\{1 = 1\}, \emptyset)(\lambda \vee T\ulcorner 1 = 1\urcorner) = 1$, but under the new notion of dependence it has no loop-free rfg! So while in the first example with the original notion of dependence we failed to read off the paradoxicality of a sentence from some rfg, with the modified notion of dependence we fail to recognize a grounded sentence, both due to a lack of symmetry. However, this doesn't mean that nonsymmetric valuation schemes are simply unaccessible for some generalisation of our methods, just that such a generalisation is not that straightforward. This, however, will be left for future research.

4.7. The core-graph. We call the subgraph of an rfg G induced by its core the *core-graph* of G and the subgraph induced by its periphery its *periphery-graph*. The aim of this section is to prove a sharper version of the main result of Section 4.5: paradoxicality is due to the existence of a cycle or double path *in the core-graph*. Such a more precise localisation will be of great importance in Section 5, where we will introduce the distinction between positive and negative arcs of an rfg—its syntactic signature. The reason is that the syntactic signature, although its restriction to the periphery-graph may show some strange behaviour, its restriction to the core graph will give us valuable semantic information on the rfg.

Let us start with the following simple observation: Call a graph G *sinkless* iff G is nonempty and every vertex of G has an out-neighbour.

PROPOSITION 4.25. *The core-graph of any rfg is empty or sinkless.*

PROOF. Let C be the core-graph of G and let ψ be a vertex of C. Suppose ψ has no out-neighbour (in C). Then ψ depends on the periphery of G. Hence ψ is univalent. Contradiction. $\dashv$

In Section 3.2 we have defined the function Γ which collapses each $(\exists)$-strategy σ in the verification game (with an empty set parameter) to an rfg $G = \Gamma(\sigma)$, inducing a multidecoration on G. Since we have already seen that paradoxicality is related closely to the core, it looks promising to develop a tool that allows us to decorate the core-graph independently: if there is some obstruction to a consistent decoration in the structure of G, it should be located entirely in the core-graph and not arise from some interaction between core and periphery. To this end let us expand the collapsing operator Γ to games with an arbitrary parameter.

Let Φ be a set of sentences and let σ be an $(\exists)$-strategy in the grounding game $\mathcal{G}_G(\phi, \Phi)$. Define the rfg $\Gamma(\sigma)$ associated to σ as follows: The set of vertices of $\Gamma(\sigma)$ consists of the sentences occurring in σ; two vertices ψ, χ are joined by an arc from ψ to χ iff $(\phi, \dots, \psi, \Psi, \chi) \in \sigma$ for some Ψ, i.e., if there is a run of the game (played according to σ) in which $(\forall)$ chooses ψ, χ consecutively. Note that if $\Phi \neq \emptyset$, a strategy may have the form $\{()\}$, where $()$ denotes the empty sequence. This is exactly the case if $(\forall)$ cannot make a first move, i.e., if $\phi \in \Phi$. An analogous situation occurs in the verification game $\mathcal{G}_T(\phi, v, (\Phi^+, \Phi^-))$. But here the empty strategy is not necessarily a winning strategy: it is winning iff ϕ is in the component of (Φ^+, Φ^-) indicated by v, otherwise it is a losing strategy.

Definition 4.26. A set $B \subseteq V(G)$ is called a *basis* of G iff for each $x \in V(G)$ there is a unique $b \in B$ such that x is accessible from b.

In particular, two different elements of a basis are not accessible from one another. As a preparation for the following, define for any set of sentences Φ and any $\mathcal{G}_G(\cdot,\cdot)$-strategy τ a set of sequences $cut(\tau, \Phi) = \{cut(p, \Phi) \mid p \in \tau\}$, where $cut(p, \Phi) = p$ if no member of Φ occurs in p as a $(\forall)$-move, and otherwise let $cut(p, \Phi) = \emptyset$. So $cut(\tau, \Phi)$ can be thought of as cutting off every branch of the strategy tree τ at the first occurrence of some member of Φ. Cutting a strategy by Φ yields, under certain conditions, a strategy in the game relative to Φ:

Lemma 4.27. *Let σ be an $(\exists)$-strategy in $\mathcal{G}_G(\phi, \Phi)$ and let $\Psi \supseteq \Phi$ with $\phi \notin \Psi$. Then $cut(\sigma, \Psi)$ is an $(\exists)$-strategy in $\mathcal{G}_G(\phi, \Psi)$.*

Let $G^*[b]$ denote the graph induced by the set of all $\psi \in V(G)$ accessible from b. Let $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ be the least Cantini fixed point.

Lemma 4.28. *Let G be an rfg and A its core.*

1. *A has a basis B.*
2. *for each $\phi \in B$ there is a unique $(\exists)$-strategy σ_ϕ in $\mathcal{G}_G(\phi, \|\mathcal{C}\|)$ such that σ_ϕ corresponds to the unfolding of $A^*[\phi]$.*

Proof. *Ad* (1): For $x, y \in V(A)$ we write $x \leq y$ iff $x = y$ or there is a walk in A from x to y. We write $x \equiv y$ iff $x \leq y$ and $y \leq x$. Observe that $x \equiv y$ iff $x = y$ or there is a cycle $X \subseteq A$ such that $x \in V(X)$ and $y \in V(X)$.

The relation $\leq$ induces a partial ordering $\widetilde{\leq}$ on the set $\widetilde{V}(A)$ of $\equiv$-equivalence classes of $V(A)$. Observe that $\widetilde{<}$ is a well-founded relation, i.e., has no infinite descending chain. ($A \subseteq G$ and G has a root.) Let $\widetilde{B}$ the set of all elements of $\widetilde{V}(A)$ that are minimal with respect to $\widetilde{<}$. Then for all $b_0 \neq b_1 \in \widetilde{B}$: $b_0 \widetilde{\not\leq} b_1$ and $b_1 \widetilde{\not\leq} b_0$. On the other hand let $x \in \widetilde{V}(A)$. Then there is some $b \in \widetilde{B}$ with $b \widetilde{\leq} x$: For either $\{y \mid y \widetilde{<} x\} = \emptyset$. In this case $x \in \widetilde{B}$ and thus $b = x \widetilde{\leq} x$. Or there is some $\widetilde{<}$-minimal element b of $\{y \mid y \widetilde{<} x\}$ and $b \in \widetilde{B}$. Let $B \subseteq V(A)$ be a set of representatives of $\widetilde{B}$. It follows that B is a basis of A.

Ad (2): Let $\phi \in B \subseteq A$. Then there is a unique $(\exists)$-strategy τ_ϕ in $\mathcal{G}_G(\phi, \emptyset)$ that corresponds to the unfolding of $G^*[\phi]$. Define $\sigma_\phi = cut(\tau_\phi, \|\mathcal{C}\|)$. Then σ_ϕ is an $(\exists)$-strategy $\tau\phi$ in $\mathcal{G}_G(\phi, \|\mathcal{C}\|)$. Since by Corollary 4.17 $\|\mathcal{C}\|$ is identical with the periphery of G and G is decomposed into core and periphery by Corollary 4.13, σ_ϕ corresponds to the unfolding of $A^*[\phi]$. ⊣

Now let us transfer the key-concept of the unfolding of an rfg to the core-graph. A core-graph has not necessarily a root, but as shown above it always has a basis which just plays the role of generalized root: Let G be an rfg and B a basis of the core-graph C of G. Let $\Sigma_B(G) = \bigcup\{\sigma_\phi \mid \phi \in B\}$, where σ_ϕ is defined as in Lemma 4.28. We call $\Sigma_B(G)$ the *strategy-unfolding* of C relative to B. In game-theoretic terms, $\Sigma_B(G)$ can be thought of as a strategy

in a game that consists of the games $\mathcal{G}_G(\phi_b, \|\mathcal{C}\|)$ (for each $\phi_b \in B$) being played simultaneously, where $(\exists)$ wins the composite game iff she wins each component, and $(\forall)$ wins iff he wins one component. Due to lack of space we shall not work out this idea but simply conceive of $\Sigma_B(G)$ as an unfolding of a graph with potentially more than one root.
In order to prove our main result of this section we want to provide a consistent and faithful to the strategy unfolding of a core-graph with no cycles and double paths. The following concepts will be needed in the below theorem but the will also play a key role in Section 5. Call a set of facts $\mathcal{F}$ a *verifier* (*falsifier*) for ϕ iff $V_L(\mathcal{F})(\phi) = 1$ ($V_L(\mathcal{F})(\phi) = 0$). A verifier (falsifier) is called *faithful* iff for all $(\psi, v) \in \mathcal{F}$: if ψ is u-univalent then $v = u$.

LEMMA 4.29. *A univalent sentence has either a faithful verifier or a faithful falsifier, but not both.*

PROOF. W.l.o.g. let ϕ be 1-univalent. Then Lemma 4.11 yields a faithful strategy σ for $(\exists)$ in the game $\mathcal{G}_T(\phi, 1)$. Then $(\exists)$'s first σ-move is a faithful verifier $\mathcal{F}$ for ϕ. Now assume that there is also a faithful falsifier $\mathcal{F}'$ for ϕ. But then an application of Lemma 4.11 to each element of $\mathcal{F}'$ yields a faithful falsifying strategy for ϕ, in addition to the faithful verifying strategy provided by the first application. ⊣

The main result of this section is a sharper version of Corollary 4.22:

THEOREM 4.30. *If the core-graph of an rfg G contains no directed cycle and no double path, then G has an acceptable decoration.*

PROOF. Let G be an rfg such that the core-graph C of G contains neither a directed cycle nor a double path. Let ϕ be the root of G. Let σ be the strategy that is induced by the unfolding of G. Let B be a basis for C and $\tau = \Sigma_R(G)$. B is at most countable, since our language is countable.

Let $(\psi_n)_{n\in\kappa}$ be an enumeration of B, (so $\kappa \leq \omega$). Denote by τ_n the component of τ that has root ψ_n. Thus τ_n is a strategy in $\mathcal{G}_G(\psi_n, \|\mathcal{C}\|)$ by the definition of τ and Lemma 4.28(2). (Again, $\mathcal{C}$ denotes the least Cantini fixed point.) By recursion on n we will define sequences $(\tau_n^*)_{n<\kappa}$, $(v_n)_{n<\kappa}$, and $(\mathcal{F}_n)_{n<\kappa}$ such that

- $\mathcal{F}_n$ is a faithful, consistent and sound set of facts,
- τ_n^* is a consistent nonlosing $(\exists)$-strategy in $\mathcal{G}_T(\psi_n, v_n, \mathcal{F}_n)$,
- $\mathcal{F}_{\tau_n^*} \cap \mathcal{F}_n = \emptyset$, and
- τ_n^* is a decoration of $cut(\tau_n, \|\mathcal{F}_n\|)$.

Recall that $\mathcal{F}_{\tau_n^*}$ denotes the set of all facts (ψ, v) such that ψ is a $(\forall)$-moves that occurs in τ_n^* and v is the mode that the game assumes after ψ has been played. Let $n = 0$. Put $\mathcal{F}_0 = \mathcal{C}$. Then $\mathcal{F}_0$ is a faithful, consistent and sound set of facts. An application of Lemma 4.11(1) to the $\mathcal{G}_G(\psi_0, \mathcal{C}^+ \cup \mathcal{C}^-)$-strategy τ_0 yields a v and a nonlosing $\mathcal{G}_T(\psi_0, v, (\mathcal{C}^+, \mathcal{C}^-))$-strategy τ_0^*. But τ_0^* is also consistent: C_{ψ_0} is an apg and τ_0 is the strategy induced by its unfolding. Since C contains no directed cycle and no double path, Lemma 4.20 yields that the map that collapses τ_0 and thus τ_0^* to C_{ψ_0} is a bijection. Hence τ_0^* is consistent. Moreover, $\mathcal{F}_{\tau_0^*} \cap \mathcal{F}_0 = \emptyset$ (by definition of strategy), and since τ_0^* is a nonlosing strategy, it is a decoration of $\tau_0 = cut(\tau_0, \|\mathcal{F}_0\|)$.

Now let $n > 0$. Put $\mathcal{F}_n = \mathcal{F}_{n-1} \cup \mathcal{F}_{\tau_{n-1}^*}$. By induction hypothesis, $\mathcal{F}_{n-1}$ is a faithful consistent and sound set of facts. τ_{n-1}^* is a consistent nonlosing strategy. $\mathcal{F}_n$ is a faithful set of facts; hence, $F_{\tau_{n-1}^*}$ is a faithful consistent and sound set of facts. Since $\mathcal{F}_{\tau_{n-1}^*} \cap \mathcal{F}_{n-1} = \emptyset$, $\mathcal{F}_n$ is also a faithful, consistent and sound set of facts. Since τ_n is a $\mathcal{G}_G(\psi_n, \|\mathcal{F}_0\|)$-strategy, $\mathcal{F}_n \supseteq F_0$ and $\psi_n \notin \|\mathcal{F}_n\|$ (since B is a basis), we can apply Lemma 4.27 to obtain that $cut(\tau_n, \|\mathcal{F}_n\|)$ is a $\mathcal{G}_G(\psi_n, \|\mathcal{F}_n\|)$-strategy. Thus an application of Lemma 4.11(1) to $cut(\tau_n, \|\mathcal{F}_n\|)$ yields a v and a nonlosing $\mathcal{G}_T(\psi_0, v, \mathcal{F}_n)$-strategy τ_n^* which is a decoration of $cut(\tau_n, \|\mathcal{F}_n\|)$. By definition of a strategy, we obtain that $\mathcal{F}_{\tau_n^*} \cap \mathcal{F}_n = \emptyset$. The consistency of τ_n^* is proved completely analogously to the case $n = 0$.

Let $\mathcal{F}_\kappa = \bigcup_{n<\kappa} \mathcal{F}_n$. Since the sequence $(\mathcal{F}_n)_{n<\kappa}$ is monotonic and each $\mathcal{F}_n$ is faithful and consistent, $\mathcal{F}_\kappa$ is also faithful and consistent. Moreover,

$(*)$ for each $\psi \in V(G)$ there exists a unique u such that $(\psi, u) \in \mathcal{F}_\kappa$.

The uniqueness part of the claim follows from the consistency of $\mathcal{F}_\kappa$. Since $\tau = \Sigma_R(G)$ and $C = \Gamma(\Sigma_B(G))$ by definition, the existence part follows for bivalent ψ from the fact that for all n, τ_n^* is a decoration of $cut(\tau_n, \|\mathcal{F}_n\|)$ and $\tau = \bigcup_{n<\kappa} cut(\tau_n, \|\mathcal{F}_n\|)$ (by definition $\tau = \bigcup_{n<\kappa} \tau_n$ and if any χ does occur in τ_n but not in $cut(\tau_n, \|\mathcal{F}_n\|)$ then there is some $k < n$ such that χ occurs in τ_k). For univalent ψ it follows from the fact that $\mathcal{C} \subseteq F_\kappa$.

Define a truth value v_0 by stipulating that $v_0 = u$ iff (ϕ, u) occurs in $\mathcal{F}_\kappa$. Due to $(*)$, v is well defined. Now let us define τ^* as follows: Let (ψ, v) be an $(\forall)$-move in $G_T(\phi, v_0)$. Let Ψ the σ-answer to ψ. Let $(\mathcal{F}_\kappa^+ \cap \Psi, \mathcal{F}_\kappa^- \cap \Psi))$ be the τ^*-response to (ψ, v). We claim that τ^* is a faithful and consistent $(\exists)$-strategy in $G_T(\phi, v_0)$. Since F_κ is a consistent set of facts, τ^* is a consistent strategy if it is a strategy at all. Moreover, it is a faithful strategy, if it is a strategy at all: $(\exists)$ has a response to any $(\forall)$-move whatsoever. In order to establish that τ^* is indeed a strategy we have to show, for each $(\forall)$-move (ψ, v), that the following holds for the $(\exists)$-response $(\mathcal{F}_\kappa^+ \cap \Psi, \mathcal{F}_\kappa^- \cap \Psi)$:

(i) ψ depends on $\|(\mathcal{F}_\kappa^+ \cap \Psi, \mathcal{F}_\kappa^- \cap \Psi)\|$,
(ii) $(\mathcal{F}_\kappa^+ \cap \Psi, \mathcal{F}_\kappa^- \cap \Psi)$ is a verifier for ψ if $v = 1$ and a falsifier for ψ if $v = 0$.

Claim (i) is clear. In order to prove (ii), let n be the least number such that $(\psi, v) \in \mathcal{F}_n$.

Case 1. $n = 0$. Then ψ is univalent. Because of (i) $(\mathcal{F}_\kappa^+ \cap \Psi, \mathcal{F}_\kappa^- \cap \Psi)$ is either an verifier or a falsifier for ψ. If $v = 1$ it cannot be a falsifier because of Lemma 4.29 (since a faithful verifier exists in this case). So it must be a verifier. Likewise it follows that it is a falsifier if $v = 0$.

Case 2. $n > 0$. Then ψ is bivalent. Then τ_{n-1}^* is a $(\exists)$-strategy in $\mathcal{G}_T(\psi_n, v_n, \mathcal{F}_{n-1})$ which contains a response $\mathcal{E}$ of $(\exists)$ to (ψ, v) (because τ_{n-1}^* is a nonlosing strategy). But $\mathcal{E} = (\mathcal{F}_\kappa^+ \cap \Psi, \mathcal{F}_\kappa^- \cap \Psi))$. To see this observe that it follows from $(*)$ that $\|\mathcal{E}\| = \Psi = \|(\mathcal{F}_\kappa^+ \cap \Psi, \mathcal{F}_\kappa^- \cap \Psi)\|$. From this follows the claimed identity because $\mathcal{F}_\kappa^+$ is consistent. But this proves (ii), since $\mathcal{E}$ is a verifier of (ψ, v) if $v = 1$, and a falsifier if $v = 0$. ⊣

The main purpose of the above theorem is its application in Section 5. There we will provide a 'signed' version of it (Theorem 5.10). It is crucial to formulate and prove Theorem 5.10 in the 'core-graph version' rather than in the 'absolute version' because its proof involves Theorem 5.9 which has no 'absolute' counterpart (cf. the example discussed at length in Section 5.1).

However, Theorem 4.30 can also be applied to identify rfgs of nonparadoxical sentences that elude Corollary 4.22. Take the example given right after Proposition 4.3. The canonical rfg of ϕ_0 is nonwell-founded and contains infinitely many double paths. Its core graph (since it is empty!) contains none of them. This is a rather trivial example because here the core is empty, but it can easily be modified: take some rfg T_0 that is a nonwell-founded tree but has infinitely many sinks (nodes with no successor). Let T_1 arise from T_0 by replacing each sink of T_0 by some ϕ_n of the just discussed counterexample to Proposition 4.3. Then T_1 has a nonempty core iff T_0 has a nonempty core (which can easily be achieved) and T_1 is nonwell-founded (in an even more intricate way than T_0) and it has infinitely many double paths. But since the core-graph of T_1 is contained in the subgraph corresponding to T_0, it contains no double path and no cycle. Hence by Theorem 4.30, T_1 has an acceptable decoration.

Before proceeding to Section 5, let us recall that all our results in this article (except some in Section 5), in particular the above theorem, do not only hold for the Leitgeb valuation scheme V_L but also for any standard symmetric valuation schemes whose dependence relation is weakly compositional (cf. Section 2.6), such as the Weak Kleene scheme. Theorem 4.6 (together with the Corollaries 4.13 and 4.17) ensures that the core has some kind of 'transvaluational' absoluteness property: suppose G_ϕ^{WK} is some Weak-Kleene rfg of ϕ while G_ϕ^L is a Leitgeb rfg. Let ψ be any vertex occurring in both graphs (certainly ϕ is such a vertex). Then ψ is in the core of G_ϕ^{WK} iff it is in the core of G_ϕ^L, because by Theorem 4.6 the core nodes of G_ϕ^{WK} are the sentences of G_ϕ^{WK} that are not in the least Cantini fixed point—and the same holds true for G_ϕ^L. This means, in the light of Theorem 4.30, that the stronger valuation scheme V_L does not make V_{WK}-paradoxical sentences nonparadoxical by shifting sentences from the core to the periphery but rather by cutting sentences out from the core, and with them the dangerous reference patterns (cycle and double path) that lie in the core. For example, $\lambda \vee \neg\lambda$ is paradoxical relative to V_{WK} but not to V_L. The disjunction V_{WK}-depends on the liar, which lies in the core, but the disjunction V_L-depends on the empty set. In the next section we will see that not all cycles and double paths in the core are equally dangerous but that danger depends on the *sign* of its arcs.

§5. **Signed reference graphs.** The introduction of signed rfgs seems to us to be of great importance for a full understanding of paradoxicality. As we mentioned earlier, the canonical (unsigned) rfg of the liar is identical with the canonical (unsigned) rfg of the truth-teller. But while the liar is clearly paradoxical, the truth-teller is not. As Herzberger pointed out, both sentences suffer from some form of semantic regress. In our framework, this is captured by the fact that all of their (unsigned) rfgs contain a directed cycle as a subgraph. However, what distinguishes the liar from the truth-teller, and makes the former paradoxical, is that the liar makes a 'negative' statement about itself, claiming itself to be false, while the truth-teller makes a 'positive' statement about itself, claiming itself to be true. Accordingly, we should label the rfg of the liar with a '$+$' and the rfg of the truth-teller with a '$-$'. One obvious way to make the distinction between 'positive' and 'negative' precise is by syntactic analysis of the sentence: the liar is a T-negative sentence, while the truth teller is T-positive. One problem with this account is that it does not cover all rfgs. Another problem is that there seems to be no obvious method to link patterns of syntactic graph signatures to (the lack of) acceptable decorations. Our approach in this section is to introduce in addition to the notion of syntactic signature the notion of a *semantic signature*. A semantic signature is an arc labelling which is induced by an $(\exists)$-strategy in the verification game. Our main result states that for a large class of rfgs G there is a semantic signature (induced by a faithful verification strategy) which matches the syntactic signature *on the core-graph of G*. This provides us with a method to find conditions for the existence of acceptable decorations in terms of a graph's syntactic signature. This gives us more nuanced existence results than those in the previous sections.

Remark. This section contains some results that cannot automatically be transferred to other standard symmetric valuation schemes whose dependence relation is weakly compositional. The results that cannot be transferred are Lemma 5.8, Theorem 5.9, its three corollaries, and Theorem 5.15. However, it is not hard to see that Lemma 5.8, Theorem 5.9, and its three corollaries, do hold, e.g., for the Weak Kleene scheme.

5.1. Syntactic and semantic signature. A *signature* $\mathcal{S}$ on an rfg G is a map from the set of arcs of G to the *set of labels* $\{+,-,\perp\}$. Call $\mathcal{S}$ *coherent* if its set of labels is $\{+,-\}$, call it *positive* if its set of labels is $\{+\}$, *negative* if its set of labels is $\{-\}$. Call a sentence φ *T-positive* iff every occurrence of the truth predicate in φ is in the scope of an even number of negation signs. Call a sentence φ *T-negative* iff every occurrence of the truth predicate in φ is in the scope of an odd number of negation signs. (This definition assumes that the material conditional $\rightarrow$ is *not* among the primitive symbols but defined in terms of negation and disjunction.) Call an rfg G *pure* iff every vertex of G that contains the T-symbol is either T-positive or T-negative. Call G *mainly* pure iff every vertex in the core of G is either T-positive or T-negative. Call $(G,\mathcal{S})$ *mainly* coherent (negative, positive) iff $\mathcal{S}$ is coherent (negative, positive) on the core-graph of G. We will now describe a method for assigning labels to the arcs of rfgs based on the syntactic shape of their nodes, followed by a method to label arcs of an rfg according to certain semantic properties.

DEFINITION 5.1. Let G be an rfg.

1. Suppose G is pure. The *syntactic signature* of G is the signature that assigns to an arc a of G the label '$+$' if the tail of a is T-positive (and not arithmetical) and the label '$-$' if the tail of a is T-negative (and not arithmetical). (If the tail of a is arithmetical it is assigned the label '$\perp$'.)[14]
2. Let G be a mainly pure rfg. The *syntactic core-signature* of G is the signature that is defined on any arc of the core-graph of G just like the syntactic signature and that assigns any arc not in the core-graph the symbol '$\perp$'.

DEFINITION 5.2. We define a *semantically signed rfg* as follows. Let T be a fact-dependence tree and G the rfg that T collapses into. Let (ϕ, ψ) be an arc of G. We put a '$+$' on the arc if there is a sequence s and truth values v_ϕ, v_ψ such that (ϕ, v_ϕ) is the last element of s, $s \circ (\psi, v_\psi)$ is in T, and $v_\phi = v_\psi$. We put a '–' on the arc if there is a sequence s and truth values v_ϕ, v_ψ such that (ϕ, v_ϕ) is the last element of s, $s \circ (\psi, v_\psi)$ is in T, and $v_\phi \neq v_\psi$. We put a '$\perp$' on the arc if both of the former conditions are satisfied.

Alternatively, we may give the following equivalent definition. Let σ be a verification strategy and G be the rfg that σ collapses into. Let (ϕ, ψ) be an arc of G. We put a '$+$' on this arc if there is a run of the game in which $(\forall)$ choses ϕ, ψ consecutively without changing the mode of the game. We put a '–' on this arc if there is a run of the game in which $(\forall)$ choses ϕ, ψ consecutively with a change in the mode of the game. We put a '$\perp$' on the arc if both of the former conditions are satisfied.

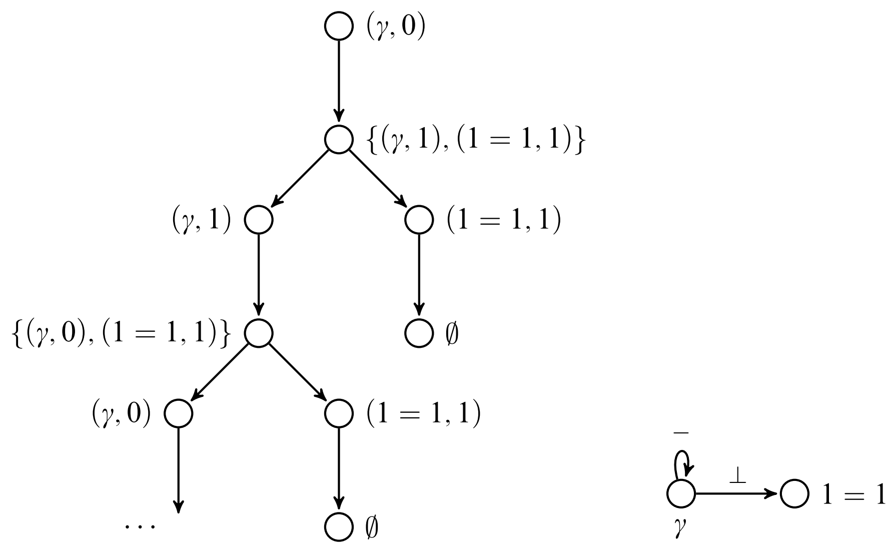
A pair $(G, \mathcal{S})$ is a *semantically signed rfg* iff $\mathcal{S}$ is a signature on G induced by some verification strategy on G. We call $(G, \mathcal{S})$ and $\mathcal{S}$ faithful iff the inducing strategy is faithful.

There are sentences φ such that there is no faithful $(\exists)$-strategy σ in $\mathcal{G}_T(\varphi, v)$ such that the induced semantically signed rfg is coherent. For example, let γ be a sentence such that $\gamma \leftrightarrow \neg T\ulcorner\gamma\urcorner \vee \neg T\ulcorner 1 = 1\urcorner$. Then γ has no faithful and coherent signed rfg. To see this, note that γ depends essentially on $\{\gamma, 1 = 1\}$. Consider the following schema which describes all possible $(\exists)$-responses to $(\forall)$-moves in $\mathcal{G}_T(\gamma, v)$, assuming that $(\exists)$ plays some strategy σ^* that is a decoration of the canonical strategy σ in $\mathcal{G}_G(\gamma)$:

1. $(\gamma, 0) \rightarrow \{(\gamma, 1), (1 = 1, 1)\}$,
2. $(\gamma, 1) \rightarrow \{(\gamma, 0), (1 = 1, 1)\}$,
3. $(\gamma, 1) \rightarrow \{(\gamma, 0), (1 = 1, 0)\}$,
4. $(\gamma, 1) \rightarrow \{(\gamma, 1), (1 = 1, 0)\}$.

Observe that σ^* is unfaithful whenever (3) or (4) occur in σ^*. So assume that neither (3) or (4) occur in σ^*. But then both (1) and (2) must occur in σ^* alternatingly: whenever $(\exists)$ plays (1) in one move she must play (2) in her next move and vice versa.

[14]This case is rather unusual, since any rfg that contains an arithmetical sentence with an out-neighbour is obviously redundant.



A faithful $(\exists)$-strategy for γ and the induced graph.

But this means that in the semantically signed rfg induced by σ^*, the arc leading from γ to $1 = 1$ must bear both labels '+' and '−'. Since any $(\exists)$-strategy in $\mathcal{G}_T(\gamma, v)$ has a decoration of σ as a substrategy it follows that there is no faithful and coherent semantically signed rfg for γ. Further observe that σ^* is inconsistent as soon as (1) or (2) occur in it. Hence by Theorem 3.9, γ is paradoxical. Observe, however, that if the semantically signed rfg of γ is induced by a faithful strategy, then the incoherent arcs are not in the core-graph. In the next subsection we will show that for a large class of rfgs there exists a semantic signature that is coherent on the core (Theorem 5.9). Note that every consistent strategy induces a coherent semantic signature.

The next theorems deal with some special cases where we obtain faithful and coherent semantic signatures.

THEOREM 5.3. 1. *Every nonr-paradoxical sentence has an rfg that has a faithful and coherent semantic signature.*
2. *Every well-founded rfg has a unique faithful and coherent semantic signature.*

PROOF. *Ad* 1: If ϕ is not paradoxical then there is a consistent verification strategy. *Ad* 2: Follows from Corollary 3.6. ⊣

THEOREM 5.4. *Let $(G, \mathcal{S})$ be a faithful and mainly coherent semantically signed rfg. Suppose further that the core-graph of $(G, \mathcal{S})$ has only cycles and double paths with an even number of negative arcs. Then every strategy that induces $(G, \mathcal{S})$ is acceptable.*

PROOF. Suppose there is some strategy σ that induces $(G, \mathcal{S})$ and σ is not acceptable. By Corollary 4.30 the core-graph of G must contain a cycle C or a double path D. Since σ is faithful by assumption, it must be inconsistent. Moreover, the node ψ where the inconsistency occurs must lie, in the first case, somewhere on C or, in the second case, at the end of D (at the point where both paths meet).

CASE 1. Let C consist of the nodes $\psi = \psi_0, \ldots, \psi_n = \psi$, with ψ being the point of inconsistency of C. W.l.o.g. let $((\psi_0, 0), \ldots, (\psi_n, 1))$ be a sequence corresponding to C in the unfolding of G. Since the truth value of ψ_0 is 0 and the truth value of ψ_n is 1, and since C only contains positive or negative arcs (because $(G, \mathcal{S})$ is mainly coherent), there must be an odd number of truth value changes on the walk from ψ_0 to ψ_n. But this implies that C consists of an odd number of negative arcs. This contradicts our assumptions.

CASE 2. Let D_0 and D_1 be the two paths that D consists of. Let $\psi_0, \ldots, \psi_n$ and $\varphi_0, \ldots, \varphi_m$ be the nodes of D_0 and D_1, respectively, where $\psi_0 = \varphi_0$ and $\psi = \psi_n = \varphi_m$. W.l.o.g. let $((\psi_0, 0), \ldots, (\psi_n, 0))$ and $((\varphi_0, 0), \ldots, (\varphi_m, 1))$ be sequences corresponding to D_0 and D_1, respectively, in the dependence tree of G. But this means that D_0 consists of an even number of negative arcs while D_1 consists of an odd number of arcs. Hence D consists of an odd number of arcs. This contradicts our assumption. $\dashv$

5.2. Matching syntactic and semantic signature. The following lemmata are needed to prove the main result of this section, Theorem 5.9, which asserts that the syntactic signature of any pure graph can be matched by a faithful semantic signature on its core.

LEMMA 5.5. 1. *Any arithmetical sentence is univalent.*
2. *$T(\phi)$ is bivalent iff ϕ is bivalent.*
3. *$\neg\phi$ is bivalent iff ϕ is bivalent.*
4. *If $\phi \wedge \psi$ is bivalent then one of the following holds*
   (a) *Both ϕ and ψ are bivalent,*
   (b) *One of the sentences ϕ, ψ is bivalent and the other is 1-univalent.*
5. *If $\phi \vee \psi$ is bivalent then one of the following holds*
   (a) *Both ϕ and ψ are bivalent,*
   (b) *One of the sentences ϕ, ψ is a bivalent and the other is 0-univalent.*
6. *If $\forall x\phi$ is bivalent then one of the following holds*
   (a) *For all $n \in \omega$, $\phi(\overline{n}/x)$ is bivalent,*
   (b) *There is some $n \in \omega$ such that $\phi(\overline{n}/x)$ is bivalent and for all $m \in \omega$ such that $\phi(\overline{m}/x)$ is not bivalent $\phi(\overline{m}/x)$ is 1-univalent.*
7. *If $\exists x\phi$ is bivalent then one of the following holds*
   (a) *For all $n \in \omega$, $\phi(\overline{n}/x)$ is bivalent,*
   (b) *There is some $n \in \omega$ such that $\phi(\overline{n}/x)$ is bivalent and for all $m \in \omega$ such that $\phi(\overline{m}/x)$ is not bivalent $\phi(\overline{m}/x)$ is 0-univalent.*

PROOF. By Corollary 4.17, a sentence ϕ is v-univalent iff ϕ has the truth value v in the least Cantini fixed point. ϕ is bivalent iff ϕ is not in the least Cantini fixed point. From this all the claims follow straightforwardly. $\dashv$

Call a nonempty set of facts $\mathcal{F}$ *mainly positive* (*mainly negative*) iff $\mathcal{F}$ assigns value 1 (0) to each bivalent sentence occurring in it. Recall that a set of facts $\mathcal{F}$ is a *verifier* (*falsifier*) for ϕ iff $V_L(\mathcal{F})(\phi) = 1$ ($V_L(\mathcal{F})(\phi) = 0$). A verifier (falsifier) is *faithful* iff for all $(\psi, v) \in \mathcal{F}$: if ψ is u-univalent then $v = u$.

LEMMA 5.6. 1. *Every 1-univalent sentence has a faithful and mainly positive verifier $\mathcal{V}^+$ and a faithful and mainly negative verifier $\mathcal{V}^-$.*

2. *Every* 0*-univalent sentence has a faithful and mainly negative falsifier* $\mathcal{F}^-$ *and a faithful and mainly positive falsifier* $\mathcal{F}^+$.

PROOF. Let ϕ be a 1-univalent sentence and let Φ be a set ϕ depends on. By Corollary 4.17, $\hat{V}_L(\mathcal{C}^+, \mathcal{C}^-)(\phi) = 1$, where $(\mathcal{C}^+, \mathcal{C}^-)$ is the least fixed point of $\hat{V}_L$ (i.e., the least Cantini fixed point). Let $\Psi^+ = \mathcal{C}^+ \cup (\Phi \setminus \mathcal{C}^-)$ and $\Psi^- = \mathcal{C}^-$. Then (Ψ^+, Ψ^-) assigns a positive value to each bivalent sentence occurring in it, and it assigns to each univalent sentence in it the right truth value. Moreover, ϕ depends on $\Psi^+ \cup \Psi^-$. By definition of $\hat{V}_L$, $V_L(\Psi^+, \Psi^-)(\phi) \neq 0$. Hence $V_L(\Psi^+, \Psi^-)(\phi) = 1$, i.e., (Ψ^+, Ψ^-) is a faithful mainly positive verifier for ϕ.

The remaining claims are proved in a completely analogous manner. $\dashv$

A formula is in *negation normal form* iff it is build from atomic and negated atomic formulae using $\wedge, \vee, \forall$, and $\exists$ only, without further use of $\neg$.

LEMMA 5.7. *Let ϕ be a bivalent sentence that is in negation normal form.*

1. *If ϕ is T-positive then it has a faithful mainly positive verifier $\mathcal{V}^+$ and a faithful mainly negative falsifier $\mathcal{F}^-$.*
2. *If ϕ is T-negative then it has a faithful mainly negative verifier $\mathcal{V}^-$ and a faithful mainly positive falsifier $\mathcal{F}^+$.*

PROOF. By induction on the syntactic complexity of ϕ.

1. If ϕ is an arithmetical sentence then the claim holds trivially since ϕ is univalent.

2.a) $\phi \equiv Tt$. Then ϕ depends on $\{t^{\mathbb{N}}\}$. Then $\mathcal{V}^+ = \{(t^{\mathbb{N}}, 1)\}$ and $\mathcal{F}^- = \{(t^{\mathbb{N}}, 0)\}$ are as desired.

2.b) $\phi \equiv \neg Tt$. Again, ϕ depends on $\{t^{\mathbb{N}}\}$. Then $\mathcal{V}^- = \{(t^{\mathbb{N}}, 0)\}$ and $\mathcal{F}^+ = \{(t^{\mathbb{N}}, 1)\}$ are as desired.

3. Let $\phi \equiv \psi_1 \wedge \psi_2$. By Lemma 5.5, one of the following alternatives holds

3.1. Both ψ_1 and ψ_2 are bivalent. Let us first deal with the case where ϕ is T-positive. By induction hypothesis (I.H.) there are $\mathcal{V}_1^+$, $\mathcal{F}_1^-$ for ψ_1 and $\mathcal{V}_2^+$, $\mathcal{F}_2^-$ for ψ_2. Let $\mathcal{V}^+ = \mathcal{V}_1^+ \cup \mathcal{V}_2^+$ and $\mathcal{F}^- = \mathcal{F}_1^- \cup \mathcal{F}_2^-$. Since ψ_1 depends on $\|\mathcal{V}_1^+\|$ and ψ_2 depends on $\|\mathcal{V}_2^+$ (by I.H.), ϕ depends on $\|\mathcal{V}_1^+ \cup \mathcal{V}_2^+$ since V-dependence is weakly compositional (cf. Section 2.4). Hence $\mathcal{V}^+$ is a verifier for ϕ. Similarly for $\mathcal{F}^-$.

Now for the case that ϕ is T-negative. By I.H. there are $\mathcal{V}_1^-$, $\mathcal{F}_1^+$ for ψ_1 and $\mathcal{V}_2^-$, $\mathcal{F}_2^+$ for ψ_2. We let $\mathcal{V}^- = \mathcal{V}_1^- \cup \mathcal{V}_2^-$ and $\mathcal{F}^+ = \mathcal{F}_1^+ \cup \mathcal{F}_2^+$.

3.2. W.l.o.g. let ψ_1 be bivalent and let ψ_2 be 1-univalent. Again, we start with the case where ϕ is T-positive. By I.H., there are $\mathcal{V}_1^+$ and $\mathcal{F}_1^-$ for ψ_1. Since ψ_2 is 1-univalent, by Lemma 5.6 there are $\mathcal{V}_2^+$, and $\mathcal{F}_2^-$ for ψ_2. Again, let $\mathcal{V}^+ = \mathcal{V}_1^+ \cup \mathcal{V}_2^+$ and $\mathcal{F}^- = \mathcal{F}_1^- \cup \mathcal{V}_2^-$.

Now let ϕ be T-negative. By I.H., there are $\mathcal{V}^+{-}_1$ and $\mathcal{F}_1^+$ for ψ_1. Since ψ_2 is 1-univalent, by Lemma 5.6 there are $\mathcal{V}_2^-$, and $\mathcal{F}_2^+$ for ψ_2. Again, let $\mathcal{V}^- = \mathcal{V}_1^- \cup \mathcal{V}_2^-$ and $\mathcal{F}^+ = \mathcal{F}_1^+ \cup \mathcal{V}_2^+$.

4. $\phi \equiv \psi_1 \vee \psi_2$. By Lemma 5.5, one of the following alternatives holds Both ψ_1 and ψ_2 are bivalent, or w.l.o.g. ψ_1 is bivalent and ψ_2 is 0-univalent. Both cases are analogous to Case 3.

5. $\phi \equiv \forall x\psi$. We only treat the T-positive case, the T-negative is symmetrical. By Lemma 5.5, one of the following alternatives holds

5.1. For all $n \in \omega$, $\phi_x(n)$ is bivalent. By I.H., for each n there are $\mathcal{V}_n^+, \mathcal{F}_n^-$ $(\mathcal{V}_n^-, \mathcal{F}_n^+)$ for $\phi(\overline{n}/x)$. Analogously to 3.1 set $\mathcal{V}^+ = \bigcup_{n\in\omega} \mathcal{V}_n^+$ and $\mathcal{F}^- = \bigcup_{n\in\omega} \mathcal{F}_n^-$.

5.2. There is some $n \in \omega$ such that $\phi(\overline{n}/x)$ is bivalent and for all $m \in \omega$ such that $\phi(\overline{m}/x)$ is not bivalent, $\phi(\overline{m}/x)$ is 1-univalent. Completely analogous we define $\mathcal{V}^+ = (\bigcup_{n\in B} \mathcal{V}_n^+) \cup (\bigcup_{n\in U} \mathcal{V}_n^+)$ and $\mathcal{F}^- = (\bigcup_{n\in B} \mathcal{F}_n^-) \cup (\bigcup_{n\in U} \mathcal{V}_n^-)$, where B is the set of all $n \in \omega$ such that $\phi(\overline{n}/x)$ is bivalent while U is the set of all $n \in \omega$ such that $\phi(\overline{n}/x)$ is univalent. ⊣

The following result shows that Lemma 5.7 holds for all bivalent sentences, not just those in negation normal form. The result relies on particular properties of V_L that do not follow automatically from its being a standard symmetric valuation scheme with a weakly compositional dependence relation. Hence, the main theorem of this section can only be transferred to other valuation schemes if they allow for a corresponding normal form theorem.

LEMMA 5.8. *For every $\phi \in \mathcal{L}_T$ there is a $\phi' \in \mathcal{L}_T$ such that*

1. *ϕ' is in negation normal form;*
2. *$V_L(S^+; S^-)(\phi) = V_L(S^+, S^-)(\phi')$, for all S^+, S^-; and*
3. *ϕ, ϕ' depend on the same sets of sentences.*

Hence, Lemma 5.7 holds for all bivalent sentences.

PROOF. Let ϕ be given. Using some standard transformation rules we can obtain a negation normal formula ϕ' that is classically equivalent to ϕ and has the same atomic formulae as ϕ. This implies that (2) and (3) hold. ⊣

THEOREM 5.9. *Let G be a mainly pure graph. Then G admits a faithful semantic signature that matches its syntactic signature on the core-graph of G.*

PROOF. Let G be a mainly pure rfg of some sentence ϕ and let σ be the strategy induced by its unfolding. We recursively define a strategy σ^* for $(\exists)$ in the game $\mathcal{G}_T(\phi, v)$, where $v = 1$ if ϕ is 1-univalent or bivalent, and $v = 0$ if 0-univalent. The semantic signature induced by σ^* will be as desired. For any node ψ of G, let $out_G(\psi)$ be the set that of out-neighbours of ψ in G. Let ψ_n be $(\forall)$'s n-th move and v_n be the mode of the game of this move. We define

CASE 1. ψ_n is T-positive.

If ψ_n is some sentence in the core of G, let $(\exists)$'s answer be

$$\mathcal{F}_n = \begin{cases} \mathcal{V}^+(\psi_n) \cap out_G(\psi_n), & \text{if } v_n = 1, \\ \mathcal{F}^-(\psi_n) \cap out_G(\psi_n), & \text{if } v_n = 0. \end{cases}$$

where $\mathcal{V}^+(\psi_n)$ $(\mathcal{F}^-(\psi_n))$ is the faithful and mainly positive verifier (the faithful and mainly negative falsifier) given by Lemma 5.7.

Now suppose ψ_n is in the periphery of G and 1-univalent. If $n = 1$ then $v_n = 1$ by assumption. If $n > 1$ then also $v_n = 1$, since by induction hypothesis, $\mathcal{F}_n$ is a faithful set of facts. Let $(\exists)$'s answer be

$\mathcal{F}_n = \mathcal{V}^+(\psi_n) \cap out_G(\psi_n)$.

This is a legal move and a faithful set of facts, where the existence of the faithful and mainly positive verifier $\mathcal{V}^+(\psi_n)$ is given by Lemma 5.6.

Suppose ψ_n is in the periphery of G and 0-univalent. Then $v_n = 0$ by the same argument as in the previous case. Let $(\exists)$'s answer be

$\mathcal{F}_n = \mathcal{F}^-(\psi_n) \cap out_G(\psi_n)$.

Again, this is a legal move and a faithful set of facts, where the existence of the faithful and mainly negative falsifier $\mathcal{F}^-(\psi_n)$ is given by Lemma 5.6.

CASE 2. ψ_n is T-negative.

If ψ_n is some sentence in the core of G, let $(\exists)$'s answer be

$$\mathcal{F}_n = \begin{cases} \mathcal{V}^-(\psi_n) \cap out_G(\psi_n), & \text{if } v_n = 1, \\ \mathcal{F}^+(\psi_n) \cap out_G(\psi_n), & \text{if } v_n = 0. \end{cases}$$

where $\mathcal{F}^+(\psi_n)$ $(\mathcal{V}^-(\psi_n))$ is the faithful and mainly positive falsifier (the faithful and mainly negative verifier) given by Lemma 5.7.

Now suppose ψ_n in the periphery of G and 1-univalent. Then $v_n = 1$. Let $(\exists)$'s answer be $\mathcal{F}_n = \mathcal{V}^-(\psi_n) \cap out_G(\psi_n)$. If ψ_n is 0-univalent then $v_n = 1$. Let $(\exists)$'s answer be $\mathcal{F}_n = \mathcal{F}^+(\psi_n) \cap out_G(\psi_n)$.

Then σ^* a is faithful strategy and a decoration of σ. Let $\mathcal{S}$ be the signature induced by σ^* on G. Then by construction $\mathcal{S}$ matches the syntactic signature of G on its core-graph. (Observe that by collapsing σ^* to G, no $\perp$ will arise because in the syntactic signature the label of an arc depends only on its tail). ⊣

The following theorem is the 'signed' variant of Corollary 4.30.

COROLLARY 5.10. *Let G be a pure rfg. If the core-graph of G has only cycles and double paths with an even number of negative arcs then G has an acceptable decoration.*

PROOF. This follows from Theorems 5.4 and 5.9. ⊣

COROLLARY 5.11. *Every mainly positive rfg has an acceptable decoration.*

COROLLARY 5.12. *Every mainly negative rfg whose core-graph has only cycles and double paths with an even number of negative arcs has an acceptable decoration.*

5.3. F-systems. A special case of rfgs with a *syntactic* signature that has attracted some attention in the literature is what Rabern et al. [16] call F-systems (F standing for *False*): each sentence in such a system claims that all the sentences it is referring to are false. The interest in F-systems is understandable, since many of the classical paradoxes like the Liar or Yablo's paradox are of this form. F-systems are the subject of Cook [5], where he establishes a correspondence between the acceptable decorations of an F-system and the *kernels* of its graph: this is interesting because graph kernels have been investigated quite systematically by graph theorists and any results, e.g., on necessary or sufficient conditions for the existence of a kernel are straightforwardly translated into results on conditions for the (non)paradoxicality of F-systems via Cook's correspondence. Our own definition of an F-system is a generalized version of Rabern et al.'s version of Cook's framework: We drop the requirement that the graphs must be *sink free*, which allows us, for example, to analyse also Curry-like paradoxes within this framework and thus to apply the theory of graph kernels to them.

DEFINITION 5.13. Let G be a canonical rfg. We call G an *F-system* iff for all vertices ψ of G the following holds

1. If ψ is a not a sink of G then $\psi \equiv \forall x(\phi \rightarrow \neg Tx)$, where ϕ is a T-free formula.
2. If ψ is a sink of G then ψ is a true arithmetical sentence.

Recall that a sentence of the form $\forall x(\phi \rightarrow \neg Tx)$, where ϕ is T-free, depends on the set $\{n \mid \mathbb{N} \models \phi(n)\}$, and in fact essentially so (cf. Section 2.3). This will be used frequently in the proof of the theorem below. However, this means that the results in the present section cannot automatically be transferred to other valuation schemes, since the property in question does not follow from weak compositionality alone. A particularly odd case is given by the Weak Kleene scheme—it is easy to see that there are no (nontrivial) F-systems for the Weak Kleene scheme, where a nontrivial F-system is one that contains at least one node that is not a sink. The reason is that under this scheme, any sentence of the form $\forall x(\phi \rightarrow \neg Tx)$ will V_{WK}-depend on the set of all sentences because of the strong compositionality of Weak Kleene. The theorem below will therefore hold trivially.

DEFINITION 5.14. Let G be a graph. $K \subseteq G$ is called a *kernel* of G iff

1. All elements of K are independent in G, i.e., G has no arc a such that both the head of a and the tail of a are elements of K.
2. For all $x \in V(G) \setminus K$ there is a $y \in K$ such that there is a G-arc from x to y.

Cook's correspondence theorem carries over straightforwardly:

THEOREM 5.15. *Let G be an F-system. Then there is a bijection between the set of the kernels of G and the set of the acceptable decorations of G.*

PROOF. Let K be a kernel of G. Define $d_K : V(G) \rightarrow \{0,1\}$ by $d_K(\psi) = 1$ if $\psi \in K$ and $d_K(\psi) = 0$ if $\psi \notin K$. We have to show that d_K is acceptable, i.e., that for all vertices ψ of G the following holds

$$V_L(\Psi^+_d, \Psi^-_d)(\psi) = d(\psi),$$

where $\Psi^+_d = \{\chi \in \Psi \mid d_K(\chi) = 1\}$, $\Psi^-_d = \{\chi \in \Psi \mid d_K(\chi) = 0\}$ and Ψ is the set of all out-neighbours of ψ in G.

Suppose that $d_K(\psi) = 1$. Then $\psi \in K$ and thus (since K is a kernel) for all $\chi \in \Psi : d_K(\chi) = 0$. Hence $V_L(\Psi^+_d, \Psi^-_d)(\psi) = V_L(\emptyset, \Psi)(\psi) = Val_\emptyset(\forall x(\phi \rightarrow \neg Tx)) = 1$.

Now suppose that $d_K(\psi) = 0$. Then $\psi \notin K$. Hence there is some $\chi \in K$ with $\chi \in \Psi$. Thus $d_K(\chi) = 1$ and $\Psi^+_d \neq \emptyset$. Since G is canonical, χ cannot be arithmetical (i.e., is not a sink of G), hence $\psi \equiv \forall x(\phi \rightarrow \neg Tx)$. Therefore $V_L(\Psi^+_d, \Psi^-_d)(\psi) = Val_{\Psi^+_d}(\forall x(\phi \rightarrow \neg Tx))$. But ψ depends essentially on $\{n \in \omega \mid \mathbb{N} \models \phi(n)\}$ and by assumption also essentially on Ψ. This means $\Psi = \{n \in \omega \mid \mathbb{N} \models \phi(n)\}$. Hence $(\mathbb{N}, \Psi^+_d) \models \phi(\chi)$ and $(\mathbb{N}, \Psi^+_d) \models T(\chi)$. Hence $Val_{\Psi^+_d}(\forall x(\phi \rightarrow \neg Tx)) = 0 = d_K(\psi)$. Thus, d_K is an acceptable decoration of G.

Clearly, the map h is injective. To show that h is also surjective, let d be any acceptable decoration of G. Let $K = \{\psi \in V(G) \mid d(\psi) = 1\}$. We have to show that K is a kernel. Let $\psi, \chi \in V(G)$. If χ is an out-neighbour of ψ then $\psi \equiv \forall x(\phi \to \neg Tx)$. Suppose that $d(\chi) = 1$. Then $d(\psi) = 0$ by the same argument as above. Hence any two vertices of K are independent.

Now suppose that $\psi \in V(G) \setminus K$. Then $d(\psi) = 0$. Since d is acceptable, ψ cannot be arithmetical. Hence ψ is not a sink and $\psi \equiv \forall x(\phi \to \neg Tx)$. Let Ψ be the set of all out-neighbours of ψ. Since d is an acceptable decoration and $\Psi = \{n \in \omega \mid \mathbb{N} \models \phi(n)\}$, there is some $\chi \in \Psi$ such that $d(\chi) = 1$, hence $\chi \in K$. ⊣

§**6. Appendix: The concept of finitary minor.** The purpose of this appendix is to define and motivate the concept of a *minor* used in the conjecture in Section 4.5. In order to do so, let us first have a look at the notion of a minor for *undirected* graphs and try to grasp the basic intuition behind it. Then, in a second step, we will define a concept of minor for *directed* graphs in such a way that this basic intuition carries over. Intuitively speaking, an undirected graph G contains an undirected graph H as minor iff H is isomorphic to a graph H' that results from G by some kind of simplification operations which allow to (i) delete arbitrary edges of G and (ii) contract connected subgraphs of G to single nodes. We call an undirected graph G *connected* iff for each $x, y \in V(G)$ there is a path in G with end vertices x and y. In the following let G and H be undirected and simple (i.e., loop-free) graphs. We denote by $E(G)$ the set of the edges of G. The crucial concept in the definition of the minor relation between undirected graphs is that of an inflation (cf. [7]).

Definition 6.1. An undirected graph G is an *inflation* of an undirected graph H if $V(G)$ admits a partition $\{V_x \mid x \in V(H)\}$ such that the following conditions hold

1. for all $x \neq y \in V(H)$, $(x, y) \in E(H)$ iff there is some $e \in E(G)$ connecting a vertex in V_x and a vertex in V_y.
2. for all $x \in V(H)$, $G[V_x]$ is connected.

The notion of an inflation gives rise to a graph homomorphism $f : G \to H$ *contracting* the inflation G to H. Let $f : V(G) \to V(H)$ defined by $f(x) = y$ iff $x \in V_y$ (where V_y is the component associated to $y \in V(H)$ in the partition of $V(G)$). Then f is a *graph homomorphism* between simple undirected graphs i.e., a surjective map such that (i) $(x, y) \in E(G)$ implies that $(f(x), f(y)) \in E(H)$ or $f(x) = f(y)$ and (ii) $(x', y') \in E(H)$ implies that there are $x, y \in V(G)$ such that $x' = f(x)$ and $y' = f(y)$. Moreover, due to condition (2) in the definition of inflation, f has the following property.

Definition 6.2. A graph homomorphism $f : G \to H$ between simple undirected graphs has the *path lifting property* iff for each path p' joining x' and y' in H there are $x, y \in V(G)$ and a path p joining x and y in G such that $f(x) = x'$ and $f(y) = y'$ and $f(z) \in V(p')$ for all $z \in V(p)$.

Loosely speaking, the path lifting property ensures that the image H of G under the contracting map can be used as a road map for the 'real world' G: whenever a road map shows a road p' leading from x' to y' you want to be sure that there is a real road p that corresponds to p'.

Definition 6.3. An undirected graph G is a *minor* of H iff there is some $H' \subseteq H$ such that H' is an inflation of G.

So every minor of H is a 'road map' of some subgraph of H. In this sense, the minor relation can be seen as a more liberal form of the subgraph relation, allowing in addition the subgraph to be shrinked down. Or, in other words: that a graph G is contained in H as a minor means that G is 'hidden' in H and can be made visible by drawing a suitable road map of a region of H.

In the theory of undirected graphs the concept of minor is certainly one of the most fruitful among graph-theoretic concepts. There is an abundance of so-called excluded-minor characterizations of various graph-theoretic properties (cf. Diestel [7, Chapter 4].) However, switching from undirected to directed graphs, there is some problem in the minor-definition to be dealt with: the concept of connectedness has more than one counterpart when it comes to digraphs (e.g., weak connectedness, strong connectedness). But having isolated the path lifting property as the crucial point, it should be clear how to choose the right notion of connectedness.

Definition 6.4. Call a digraph G a *finitary inflation* of a digraph H if $V(G)$ admits a partition $\{V_x \mid x \in V(H)\}$ such that the following conditions hold

1. for all $x, y \in V(H)$:
   (a) if $x \neq y$ then $(x, y) \in A(H)$ iff there is some $a \in A(G)$ with tail in V_x and head in V_y,
   (b) if $x = y$ then $(x, y) \notin A(H)$.
2. for all $x \in A(H)$ and all $y, z \in V_x$: if y has an in-neighbour in $V(G) \setminus V_x$ and z has an out-neighbour in $V(G) \setminus V_x$ then there is a path $P \subseteq G[V_x]$ from y to z.
3. for all $x \in V(H)$, V_x is finite.

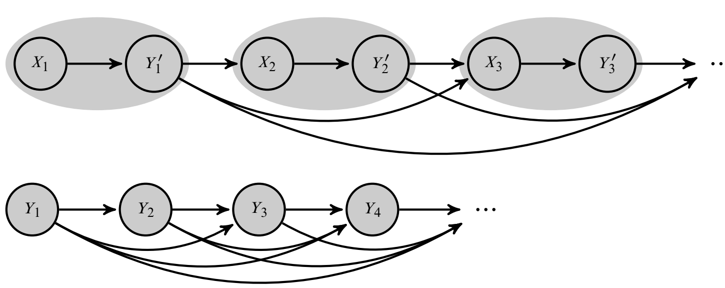
Definition 6.5. A digraph G is a *finitary minor* of H (we also say H contains G as a finitary minor) iff there is some $H' \subseteq H$ such that H' is a finitary inflation of G.

In analogy to the undirected case we get a map $f : V(G) \to V(H)$ defined by $f(x) = y$ iff $x \in V_y$ (where V_y is a the component associated to $y \in V(H)$ of the partition of $V(G)$). Then f is a digraph homomorphism mapping digraphs to *loop-free digraphs*, i.e., a surjective map such that (i) $(x, y) \in A(G)$ implies that $(f(x), f(y)) \in A(H)$ or $f(x) = f(y)$ and (ii) $(x', y') \in A(H)$ implies that there are $x, y \in V(G)$ such that $x' = f(x)$ and $y' = f(y)$.

Observe that condition 1(b) ensures that no digraph G can be an inflation of a digraph H that contains a loop. This implies in particular that the 'inner arcs' of a component V_x, i.e., the arcs of $G[V_x]$ are not contracted to a loop by applying the contraction homomorphism f, but simply disappear. Thus f is not a *danger preserving operation* in the sense of Rabern et al., i.e., an operation that maps dangerous graphs to dangerous graphs. The reason for this clause is a simple technical one: allowing 'inner arcs' to be contracted to a loop would lead to new artificial loops as a product of the operation of contraction. Since the Yablo graph does not contain any loops, it cannot be the result of a nontrivial contraction and thus our conjecture would turn out trivially false. The fact that, given our definition of inflation (contraction), even existing cycles of a component $G[V_x]$ are eliminated, should not pose any problem for our purposes. If a graph contains a loop it is dangerous anyway; the unsolved part of the characterization problem concerns only acyclic graphs and only for these has the operation of contraction to turn out as danger preserving. The second condition in the above definition ensures that f satisfies the following property:

Definition 6.6. A graph homomorphism $f : G \to H$ from digraphs to loop-free digraphs has the *path lifting property* iff for each path p' fom x' to y' in H there are $x, y \in V(G)$ and a path p from x to y in G such that $f(x) = x'$ and $f(y) = y'$ and $f(z) \in V(p')$ for all $z \in V(p)$.

The third condition is added because we want, as mentioned above, a contraction to be a danger preserving operation on acyclic graphs. In the following illustration, the second graph is a (finitary) contraction of the first one.



Hence the second graph is a finitary minor of the first. As shown by Rabern et al. the first graph is dangerous while it does *not* contain a subdivision of the Yablo graph. This illustrates that the minor concept is more liberal than the concept of subdivision, as it captures more dangerous graphs.

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