Title	Computations for Coxeter arrangements and Solomon's descent algebra II: Groups of rank five and six
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Publication Date	2013-01-08
Publication Information	Bishop, Marcus, Douglass, J. Matthew, Pfeiffer, Götz, & Röhrle, Gerhard. (2013). Computations for Coxeter arrangements and Solomons descent algebra II: Groups of rank five and six. Journal of Algebra, 377(Supplement C), 320-332. doi: https://doi.org/10.1016/j.jalgebra.2012.11.047
Publisher	Elsevier
Link to publisher's version	https://doi.org/10.1016/j.jalgebra.2012.11.047
Item record	http://hdl.handle.net/10379/7004
DOI	http://dx.doi.org/10.1016/j.jalgebra.2012.11.047

COMPUTATIONS FOR COXETER ARRANGEMENTS AND SOLOMON'S DESCENT ALGEBRA II: GROUPS OF RANK FIVE AND SIX

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Abstract. In recent papers we have refined a conjecture of Lehrer and Solomon expressing the character of a finite Coxeter group W acting on the graded components of its Orlik-Solomon algebra as a sum of characters induced from linear characters of centralizers of elements of W. The refined conjecture relates the character above to a decomposition of the regular character of W related to Solomon's descent algebra of W. The refined conjecture has been proved for symmetric and dihedral groups, as well as for finite Coxeter groups of rank three and four. In this paper, we prove the conjecture for finite Coxeter groups of rank five and six. The techniques developed and implemented in this paper provide previously unknown decompositions of the regular and Orlik-Solomon characters of the groups considered.

1. Introduction

Let (W, S) be a finite Coxeter system. In previous articles [2–4] we proposed a conjecture relating the character ω of the Orlik-Solomon algebra of W to the regular character ρ of W. Based on a conjecture of Lehrer and Solomon [9], in this paper we prove the conjecture for the Coxeter groups of type B_5, B_6, D_5, D_6, and E_6. These computations, together with the remarks about reducible Coxeter groups following Theorem 2.3 of [2] and the proof of the conjecture for groups of type A in [3], prove the conjecture for all finite Coxeter groups of rank five and six. Our result is stated for these groups as the following theorem.

Theorem 1. *Suppose that W is a finite Coxeter group of rank five or six and that $\mathcal{R}$ is a set of conjugacy class representatives of W. Then for each $w \in \mathcal{R}$ there exists a linear character φ_w of $C_W(w)$ such that*

$$\rho = \sum_{w\in\mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \varphi_w \quad \textit{and} \quad \omega = \epsilon \sum_{w\in\mathcal{R}} \operatorname{Ind}_{C_W(w)}^W (\alpha_w \varphi_w)$$

where ϵ is the sign character of W and for $z \in C_W(w)$, $\alpha_w(z)$ denotes the determinant of the restriction of z to the 1-eigenspace of w in the complex reflection representation of W.

2010 *Mathematics Subject Classification.* 20F55, 20C15, 20C40, 52C35.

Key words and phrases. Coxeter group, descent algebra, Orlik-Solomon algebra.

Let $A(W)$ be the Orlik-Solomon algebra of W. The strategy for proving Theorem 1 is to decompose the $\mathbb{C}W$-modules $\mathbb{C}W$ and $A(W)$ into direct sums and prove a refinement of Theorem 1 for each summand. This method is somewhat stronger than directly proving Theorem 1, because it requires the solution to be compatible with the direct sum decompositions of $\mathbb{C}W$ and $A(W)$. This method also has the advantage that it splits the problem into smaller problems and provides additional insight into how the representations of W on $\mathbb{C}W$ and on $A(W)$ are related.

The decomposition of $\mathbb{C}W$ comes from idempotents e_λ in the descent algebra $\Sigma(W)$ constructed in [1]. These idempotents are indexed by subsets of S up to conjugacy in W. A class of conjugate subsets of S is called a *shape* of W. Denote the set of shapes of W by Λ. In [1] it is shown how to construct a quasi-idempotent e_L for any $L \subseteq S$ and then e_λ is the sum of the quasi-idempotents e_L where L runs over the subsets in the shape λ. It is also shown that $\{e_\lambda \mid \lambda \in \Lambda\}$ is a complete set of primitive orthogonal idempotents of $\Sigma(W)$. Since $\Sigma(W)$ is a subalgebra of $\mathbb{C}W$ and $1 = \sum_{\lambda\in\Lambda} e_\lambda$, we conclude that $\mathbb{C}W = \bigoplus_{\lambda\in\Lambda} e_\lambda\mathbb{C}W$ as a $\mathbb{C}W$-module. Denoting the character of $e_\lambda\mathbb{C}W$ by ρ_λ we have

$$\rho = \sum_{\lambda\in\Lambda} \rho_\lambda. \tag{1.1}$$

The corresponding decomposition of $A(W)$ comes from Brieskorn's Lemma. Let T be the set of reflections in the complex reflection representation V of W. Recall that the Orlik-Solomon algebra $A(W)$ may be defined as the quotient of the exterior algebra with generators $\{e_t \mid t \in T\}$ by the ideal generated by elements of the form $\sum_{i=1}^{k} (-1)^i e_{t_1} e_{t_2} \cdots \widehat{e_{t_i}} \cdots e_{t_k}$ for all sets $\{t_1, t_2, \ldots, t_k\} \subseteq T$ of linearly dependent reflections. Here, we say that a set of reflections is linearly dependent if the linear forms defining their reflecting hyperplanes are linearly dependent in the dual of V. For $t \in T$ we denote the image of the generator e_t in $A(W)$ by a_t. Thus, an arbitrary element of $A(W)$ can be expressed as a linear combination of monomials $a_{t_1} a_{t_2} \cdots a_{t_k}$ with $t_1, t_2, \ldots, t_k \in T$. The algebra $A(W)$ is a right $\mathbb{C}W$-module with $(a_{t_1} a_{t_2} \cdots a_{t_k}).w = a_{w^{-1}t_1w} a_{w^{-1}t_2w} \cdots a_{w^{-1}t_kw}$ for $w \in W$.

Each monomial $a_{t_1} a_{t_2} \cdots a_{t_k}$ determines a subspace of V, namely the intersection of the fixed point spaces of the reflections $t_1, t_2, \ldots, t_k$. If X is a subspace of V, then we denote by A_X the span of all monomials with fixed point space equal to X. Then taking A_λ to be the sum of the A_X for which X is the fixed point set of some conjugate of W_L for some $L \in \lambda$, we have a decomposition $A(W) = \bigoplus_{\lambda\in\Lambda} A_\lambda$. Denoting the character of A_λ by ω_λ we have

$$\omega = \sum_{\lambda\in\Lambda} \omega_\lambda.$$

Finally, we choose a set of conjugacy class representatives compatible with the decompositions above. A conjugacy class in W_L is called *cuspidal* if the fixed point set in the reflection representation of W_L of any of its elements is trivial. Now if we choose a fixed representative $L(\lambda)$ of each shape λ and let $\mathcal{C}_{L(\lambda)}$ be a set of representatives of the cuspidal classes in $W_{L(\lambda)}$, then by Theorem 3.2.12 of [5]

$$\mathcal{R} = \bigcup_{\lambda\in\Lambda} \mathcal{C}_{L(\lambda)} \quad \text{is a set of conjugacy class representatives of } W. \tag{1.2}$$

Suppose that $L \subseteq S$. The homogeneous component of $A(W_L)$ of highest degree is called the *top component* of $A(W_L)$. On the other hand, W_L is also a Coxeter group and thereby admits a system of quasi-idempotents as in [1], now denoted by e_J^L for $J \subseteq L$ to distinguish them from the quasi-idempotents e_J in $\mathbb{C}W$. Note that this notation does not agree with that used in [1, §7]. In analogy with $A(W_L)$ the homogeneous component $e_L^L\mathbb{C}W_L$ of $\mathbb{C}W_L$ is called the *top component* of $\mathbb{C}W_L$. We denote the characters of W_L afforded by the top components of $A(W_L)$ and $\mathbb{C}W_L$ by ω_L and ρ_L respectively.

Now the top components of $A(W_L)$ and $\mathbb{C}W_L$ are naturally W_L-stable subspaces of $A(W)$ and $\mathbb{C}W$ and it turns out that they are $N_W(W_L)$-stable by [3, Proposition 4.8]. Thus they afford characters $\widetilde{\omega_L}$ and $\widetilde{\rho_L}$ of $N_W(W_L)$ which are extensions of ω_L and ρ_L. Furthermore, if $L \in \lambda$ then by the same proposition,

$$\omega_\lambda = \operatorname{Ind}_{N_W(W_L)}^W \widetilde{\omega_L} \quad \text{and} \quad \rho_\lambda = \operatorname{Ind}_{N_W(W_L)}^W \widetilde{\rho_L}. \tag{1.3}$$

Consider the following refinement of Theorem 1. In the statement of this theorem we use the fact, proved in [8, Theorem 3.1], that if w is cuspidal in W_L, then $C_W(w) \subseteq N_W(W_L)$.

Theorem 2. *Suppose that (W, S) is a finite Coxeter system of rank five or six, that $L \subseteq S$, and that $\mathcal{C}_L$ is a set of representatives of the cuspidal conjugacy classes of W_L. Then for each $w \in \mathcal{C}_L$ there exists a linear character φ_w of $C_W(w)$ such that*

$$\widetilde{\rho_L} = \sum_{w\in\mathcal{C}_L} \operatorname{Ind}_{C_W(w)}^{N_W(W_L)} \varphi_w = \alpha_L \epsilon \widetilde{\omega_L}$$

where for $n \in N_W(W_L)$, $\alpha_L(n)$ denotes the determinant of the restriction of n to the subspace of fixed points of W_L in V.

To prove Theorem 1 we prove Theorem 2 for the representative $L(\lambda)$ of each shape λ. Then the characters φ_w that satisfy Theorem 2 with $L = L(\lambda)$ as λ varies over all shapes prove the first equality of Theorem 1 because

$$\begin{aligned}
\rho &= \sum_{\lambda\in\Lambda} \rho_\lambda && \text{by (1.1)}\\
&= \sum_{\lambda\in\Lambda} \operatorname{Ind}_{N_W(W_{L(\lambda)})}^W \widetilde{\rho_{L(\lambda)}} && \text{by (1.3)}\\
&= \sum_{\lambda\in\Lambda} \sum_{w\in\mathcal{C}_{L(\lambda)}} \operatorname{Ind}_{N_W(W_{L(\lambda)})}^W \operatorname{Ind}_{C_W(w)}^{N_W(W_{L(\lambda)})} \varphi_w && \text{by Theorem 2}\\
&= \sum_{w\in\mathcal{R}} \operatorname{Ind}_{C_W(w)}^W \varphi_w
\end{aligned}$$

where the last equality follows from transitivity of induction and (1.2). A similar argument proves the second equality in Theorem 1. We prove Theorem 2 in §3 and §4.

2. Implementation

As in [2], we have implemented the calculations for this article in the computer algebra system GAP [11] in conjunction with the CHEVIE [6] and the ZigZag [10] packages. In addition to our comments about the implementation in [2] we make the following remarks about the techniques new to this paper and improvements to old techniques.

2.1. **The Extension $\widetilde{\rho_L}$.** In this subsection we develop a formula for the character $\widetilde{\rho_L}$ of $N_W(W_L)$ for $L \subseteq S$. First we review the definitions of the constructions used in the process.

If $J \subseteq L$ then the *parabolic transversal* of W_J in W_L is the set X_J^L of elements $w \in W_L$ satisfying $\ell(sw) > \ell(w)$ for all $s \in J$, where ℓ is the usual length function of W with respect to S. Then X_J^L can be calculated directly from the definition or by using the `ParabolicTransversal` function supplied by the ZigZag package.

In order to use the formula for $\widetilde{\rho_L}$ below, we need to be able to decompose an element of $N_W(W_L)$ into the product of an element of W_L and an element of the normalizer complement N_L of W_L. Recall that N_L consists of certain elements of the parabolic transversal X_L^S of W_L in W. Therefore, the decomposition of an element of $N_W(W_L)$ into a product nw with $n \in N_L$ and $w \in W_L$ is a special case of the more general decomposition of an element of W into a product of a coset representative in X_L^S by an element of W_L. In ZigZag this decomposition is implemented as the `ParabolicCoordinates` function.

The quasi-idempotents e_J^L are defined in [1] by means of the matrix $M = (m_{KJ})$, whose rows and columns are indexed by the subsets of S and whose (K, J)-entry is

$$m_{KJ} = \begin{cases} |\{x \in X_K \mid J^x \subseteq S\}| & \text{if } K \supseteq J \\ 0 & \text{otherwise.} \end{cases}$$

The matrix M can be calculated directly from the definition or by calling the method `Mu` supplied by the ZigZag package. Then putting $N = M^{-1} = (n_{KJ})$ we have $e_J^L = \sum_{K \subseteq L} n_{JK} x_K^L$ where x_K^L is the sum in $\mathbb{C}W_L$ of the elements of X_K^L.

In the following discussion let $w \in W_L$, $n \in N_L$, and $x \in \mathbb{C}W_L$. Observe that $N_W(W_L)$ acts on $\mathbb{C}W_L$ on the right by $x.(wn) = n^{-1}xwn$. Using this action we define the map

$$\gamma(wn, x) : \mathbb{C}W_L \to \mathbb{C}W_L \qquad \text{by} \qquad \gamma(wn, x)(v) = (xv).(wn) = n^{-1}xvwn$$

for $v \in \mathbb{C}W_L$.

The idempotent $e_L^L \in \mathbb{C}W_L$ determines a $\mathbb{C}N_W(W_L)$-stable decomposition of the group algebra of W_L, $\mathbb{C}W_L = e_L^L\mathbb{C}W_L \oplus (1 - e_L^L)\mathbb{C}W_L$. Calculating the trace of the action of $\gamma(wn, e_L^L)$ with respect to a basis of $\mathbb{C}W_L$ adapted to this decomposition, we find that $\widetilde{\rho_L}(wn) = \mathsf{Tr}\left(\gamma\left(wn, e_L^L\right)\right)$ since $\gamma\left(wn, e_L^L\right)$ sends $\left(1 - e_L^L\right)\mathbb{C}W_L$ to $e_L^L\mathbb{C}W_L$. Using the linearity of γ in its second argument we can further refine this to

$$\widetilde{\rho_L}(wn) = \sum_{y \in W_L} a_y \mathsf{Tr}\left(\gamma\left(wn, y\right)\right)$$

where the numbers a_y are such that $e_L^L = \sum_{y \in W_L} a_y y$.

For fixed $n \in N_L$ we define a right action $\cdot_n$ of W_L on W_L by

$$y \cdot_n z = nz^{-1}n^{-1}yz$$

for $y, z \in W_L$. Then the stabilizer $Z_n(y)$ of $y \in W_L$ under this action is $C_{W_L}\left(n^{-1}y\right)$. We denote the orbit of y by $\mathcal{O}_n(y) = \left\{nz^{-1}n^{-1}yz \mid z \in W_L/C_{W_L}\left(n^{-1}y\right)\right\}$. Now

$$\begin{aligned}
\mathsf{Tr}\left(\gamma\left(wn, y\right)\right) &= \left|\left\{z \in W_L \mid w^{-1} = y \cdot_n z\right\}\right| \\
&= \begin{cases} 0 & \text{if } w^{-1} \notin \mathcal{O}_n(y) \\ |Z_n(y)| & \text{if } w^{-1} \in \mathcal{O}_n(y) \end{cases} \\
&= \begin{cases} 0 & \text{if } y \notin \mathcal{O}_n\left(w^{-1}\right) \\ |C_{W_L}(wn)| & \text{if } y \in \mathcal{O}_n\left(w^{-1}\right), \end{cases}
\end{aligned}$$

where in the last equality we have used the fact that $w^{-1} \in \mathcal{O}_n(y)$ if and only if $y \in \mathcal{O}_n\left(w^{-1}\right)$, and if so, then the value of $\mathsf{Tr}\left(\gamma\left(wn, y\right)\right)$ is $\left|Z_n\left(w^{-1}\right)\right| = \left|C_{W_L}\left(n^{-1}w^{-1}\right)\right| = |C_{W_L}(wn)|$ by the calculation above. In conclusion, we obtain the following formula.

$$\begin{aligned}
\widetilde{\rho_L}(wn) &= \sum_{y \in \mathcal{O}_n(w^{-1})} a_y |C_{W_L}(wn)| \\
&= |C_{W_L}(wn)| \sum_{y \in \mathcal{O}_n(w^{-1})} \sum_{\substack{J \subseteq L \\ \mathcal{D}(y) \subseteq L \setminus J}} n_{LJ} \\
&= |C_{W_L}(wn)| \sum_{J \subseteq L} n_{LJ} \left|\mathcal{O}_n\left(w^{-1}\right) \cap X_J^L\right|.
\end{aligned}$$

Here we have used the descent set $\mathcal{D}(y) = \{s \in L \mid \ell(sy) < \ell(y)\}$ to derive the formula $\sum_{\mathcal{D}(y) \subseteq L \setminus J} n_{LJ}$ for a_y.

2.2. **The Extension $\widetilde{\omega_L}$.** In this subsection we discuss the calculation of $\widetilde{\omega_L}$ for $L \subseteq S$. As this calculation is almost identical to the calculation of ω, we begin with ω and discuss the minor modifications needed to calculate $\widetilde{\omega_L}$ at the end.

For computational purposes, rather than working with the set T of reflections in W, it is simpler to work with the positive roots of W. The positive roots are stored in CHEVIE as vectors in the `roots` component of a Coxeter group record, the first half being the positive roots and the second half being the negative of the first half. This means that whenever a calculation involving roots results in a negative root, we need to replace the negative root with its positive counterpart.

With this convention the generator a_t of $A(W)$ is denoted by a_r, where r is the positive root orthogonal to hyperplane fixed by t. To simplify the notation, we will denote a_r simply by r. This also reflects the way one implements $A(W)$ on a computer. Namely, the elements of $A(W)$ are represented by linear combinations of sequences $r_1 r_2 \cdots r_q$ of positive roots. We will also assume that any element $r_1 r_2 \cdots r_q$ satisfies $r_1 < r_2 < \cdots < r_q$, explicitly sorting the factors and inserting the appropriate sign ± 1 whenever the factors become unsorted. Here $<$ denotes a fixed total order on the positive roots, which can be simply be taken to be the order in which the roots appear in the `roots` component of the record for W.

Now since CHEVIE implements the element w of W as a permutation σ_w of the roots in V, it follows that if t is the reflection defined by the root r, then the conjugate $w^{-1}tw$ is the reflection defined by $r.\sigma_w$, which we simplify to $r.w$. Therefore, the action of W on $A(W)$ is given by $(r_1 r_2 \cdots r_q).w = (r_1.w)(r_2.w)\cdots(r_q.w)$.

We use the non-broken circuit basis $\mathcal{B}$ of $A(W)$ described in [2] to calculate its character ω. While this works exactly as in [2], we briefly describe some improvements to the algorithm that make the calculations in this paper possible. Let $n = |S|$ be the rank of W and recall that the non-broken circuit basis of $A(W)$ consists of elements of the form $r_1 r_2 \cdots r_n$ not containing certain sequences called *broken circuits* as subsequences. A broken circuit $r_{i_1} r_{i_2} \cdots r_{i_q}$ has the property that there exists a positive root r with $r > r_{i_q}$ for which $r_{i_1} r_{i_2} \cdots r_{i_q} r$ is dependent, so the defining relation for $A(W)$ implies that

$$(-1)^q \, r_{i_1} r_{i_2} \cdots r_{i_q} = \sum_{k=1}^{q} (-1)^k \, r_{i_1} r_{i_2} \cdots \widehat{r_{i_k}} \cdots r_{i_q} r. \tag{2.1}$$

Therefore, any element *not* in $\mathcal{B}$ can be expressed as a linear combination of lexicographically larger elements of $A(W)$ by applying (2.1) to a broken circuit subsequence. This observation is the rationale for the procedure for expressing an arbitrary element of $A(W)$ in terms of the non-broken circuit basis, but it also leads to a significant improvement in the calculation of ω.

Namely, to calculate the value of ω at an element $w \in W$, one in principle runs through all basis elements $b \in \mathcal{B}$, expressing $b.w$ as a linear combination of elements of $\mathcal{B}$ using (2.1) and storing the coefficients of the result into the rows of a matrix m. Then m represents the linear transformation w of $A(W)$ and $\omega(w)$ is the trace of m. We observe that if at any point in the calculation of $b.w$ we arrive at a monomial lexicographically larger than b, then this monomial cannot contribute to the trace of m. Such calculations can therefore be terminated. Furthermore, the matrix m itself exists only in concept. In practice we need only its diagonal entries. Therefore, we use the following algorithm.

COEFF *(Individual coefficient with respect to $\mathcal{B}$)* With respect to the non-broken circuit basis $\mathcal{B}$ of $A(W)$ this algorithm takes as input a monomial $a = r_1 r_2 \cdots r_n \in A(W)$ and a basis element $b \in \mathcal{B}$. It returns the coefficient of b when a is expressed with respect to $\mathcal{B}$.

```
if a > b then
    return 0
else if a ∈ ℬ then
    if a = b then
        return 1
    else
        return 0
else
    find a subsequence r_{i_1} r_{i_2} ··· r_{i_q} of a which is a broken circuit
    find a root r for which r_{i_1} r_{i_2} ··· r_{i_q} r is dependent
    return ∑_{j=1}^{m} (−1)^j COEFF (r_1 r_2 ··· \widehat{r_{i_j}} ··· r_n r, b)
```

Observe that in the last line of the algorithm, we have inserted r at the end of the first argument of **COEFF** for notational convenience. Moving the factor to its proper position will introduce a sign ± 1. Then to calculate $\omega(w)$ we simply calculate $\sum_{b\in\mathcal{B}}$ **COEFF** $(b.w, b)$.

Finally, to calculate the character $\widetilde{\omega_L}$ for $L \subseteq S$ we calculate the non-broken circuit basis of the top component of $A(W_L)$. Observe that an element $w \in N_W(W_L)$ is implemented as a permutation σ_w of the roots in V, so to apply w to an element $r_1 r_2 \cdots r_q$ of $A(W_L)$ each r_i must be replaced with its corresponding root in V. In CHEVIE this can be accomplished with the `rootInclusion` component of the W_L record. Then the permutation σ_w can be applied directly, followed by replacing each root with the corresponding root in the reflection representation of W_L using the `rootRestriction` component of the W_L record. With this modification, we proceed exactly as in the calculation of ω above.

3. Proof of Theorem 2 when $L = S$

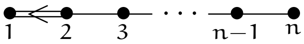
Observe that if $L = S$ then $\widetilde{\rho_S} = \rho_S$, $\widetilde{\omega_S} = \omega_S$, and $N_W(W_L) = W$. Observe also that $\alpha_S(w) = 1$ for all $w \in W$ since the space of fixed points of W is the zero subspace of V. Therefore, to verify Theorem 2 we need to find a character φ_w of $C_W(w)$ for each $w \in \mathcal{C}_S$ such that

$$\rho_S = \sum_{w\in\mathcal{C}_S} \operatorname{Ind}_{C_W(w)}^{W} \varphi_w = \epsilon\omega_S. \tag{3.1}$$

In this section we exhibit these characters for each irreducible Coxeter group W of rank five or six. Once the characters φ_w are specified, one verifies (3.1) by routine calculations, so we limit ourselves to displaying the characters $\operatorname{Ind}_{C_W(w)}^{W} \varphi_w$ (denoted simply by φ_w), ϵ, ρ_S, and ω_S only for the group $W = W(E_6)$.

Because each character φ_w is one-dimensional, it suffices to list its values on a generating set for the group $C_W(w)$. For the group $W(E_6)$ we have constructed generating sets for the groups $C_W(w)$ ad hoc. In type B generating sets for $C_W(w)$

are known, while in type D generating sets for $C_W(w)$ can be determined as described below. We use the notation for these generating sets from [2], which we now briefly review.

The cuspidal classes of $W(B_n)$ are indexed by partitions of n. We always display partitions in non-decreasing order without punctuation. With the labeling of the elements of S given by the diagram 

we define the following elements of $W(B_n)$, where we denote the elements of S by $1, 2, \dots, n$ rather than $s_1, s_2, \dots, s_n$ to improve legibility. If $\lambda = \lambda_1\lambda_2\cdots\lambda_k$ is a partition of n then for each $1 \leqslant i \leqslant k$ we define a negative λ_i-cycle

$$c_i = (j+1)\, j\, (j-1) \cdots 212 \cdots (j+\lambda_i) \qquad \text{where} \qquad j = \sum_{k=1}^{i-1} \lambda_k. \tag{3.2}$$

Then each c_i centralizes the element $w_\lambda = c_1c_2\cdots c_k$, which we take to be the representative of the cuspidal class labeled by λ. Whenever $\lambda_i = \lambda_{i+1}$ the element

$$x_i = \prod_{k=1}^{\lambda_i} (j+k)\,(j+k-1)\cdots(j+k-\lambda_i+1) \qquad \text{where} \qquad j = \sum_{k=1}^{i} \lambda_k \tag{3.3}$$

also centralizes w_λ. In fact, if $m(j) = \min\{k \mid \lambda_k = j\}$ then $C_W(w_\lambda)$ is generated by the elements x_i for all i satisfying $\lambda_i = \lambda_{i+1}$, together with the elements $c_{m(j)}$ for all j appearing as parts of λ. We remark that the elements defined in (3.2) and (3.3) coincide with the elements c_i and x_i defined in [2]. The character φ_{c_λ} of $C_W(w_\lambda)$ is denoted simply by φ_λ.

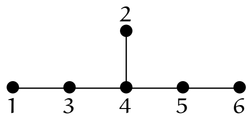
We view $W(D_n)$ as a reflection subgroup of $W(B_n)$ generated by the reflections $1' = 121$ and $2, 3, \dots, n$. Then $w_\lambda \in W(D_n)$ whenever λ has an even number of parts. In fact, such elements w_λ are representatives of the cuspidal classes of $W(D_n)$ and the centralizer $C_{W(D_n)}(w_\lambda)$ is the intersection of $W(D_n)$ with $C_{W(B_n)}(w_\lambda)$. We observe that the last factor $j+k-\lambda_i+1$ of (3.3) is at least λ_i+1 and that the other factors are greater than $j+k-\lambda_i+1$. This means that 1 is never a factor of x_i so that $x_i \in W(D_n)$. However, (3.2) shows that 1 occurs as a factor of c_i exactly once, making the replacement of 121 by $1'$ impossible. This shows that $c_i \notin W(D_n)$. Nevertheless, generators of $C_{W(D_n)}(w_\lambda)$ can often be found among products of an even number of the elements c_i.

In each of the following subsections we present the results of our calculations for the finite irreducible Coxeter groups of rank five and six. For each cuspidal class representative w we display a generating set of $C_W(w)$, where the generators are written as words in the Coxeter generators. At each generator, we display the value of the character φ_w. If ζ is an eigenvalue of w on V, we denote the determinant of the representation of $C_W(w)$ on the ζ-eigenspace of w in V by $\det|_\zeta$. If φ_w is a power of $\det|_\zeta$ for some ζ, then we also indicate this. By Springer's theory of regular elements [12], the centralizer $C_W(w)$ is a complex reflection group when w is a regular element. When this is the case, we identify $C_W(w)$ as such a group.

For $n \geqslant 1$ we denote the n^{th} root of unity $e^{2\pi i/n}$ by ζ_n, the cyclic group of size n by Z_n, and the symmetric group on n letters by S_n.

3.1. $W = W(E_6)$. We begin with $W(E_6)$ and present the calculations that lead to the proof of Theorem 2 for this group. For the other groups of rank five and six we present only the basic information described above.

Define the characters $\varphi_d = \varphi_{w_d}$ in the following table, where the conjugacy classes of W are labeled by their Carter diagrams d. Here the elements of S are labeled as in the Coxeter graph



and r denotes the reflection defined by the highest root of W.

d	w_d	Gen	φ_d	$C_W(w_d)$	Det
A_2^3	$12356r$	24542314	ζ_3	G_{25}	$\det\|_{\zeta_3}$
		13	ζ_3		
		56	ζ_3		
$E_6(a_2)$	$w_{E_6}^2$	w_{E_6}	1	G_5	
		234543	ζ_3		
A_5A_1	$13456r$	$w_{A_5A_1}$	ζ_3		
		2345432	-1		
		r	-1		
$E_6(a_1)$	$34w_{E_6}$	$w_{E_6(a_1)}$	ζ_9	Z_9	$\det\|_{\zeta_9}$
E_6	123456	w_{E_6}	-1	Z_{12}	$(\det\|_{\zeta_{12}})^6$

Finally, the values of the characters φ_d^W together with ρ_S and ω_S are shown in the following table.

W	$\emptyset$	A_1^4	A_1^2	A_2^3	A_2	A_2^2	$D_4(a_1)$	A_3A_1	A_4	$E_6(a_2)$	D_4	A_5A_1	$A_2A_1^2$
$\varphi_{A_2^3}$	80	16	·	−10	−4	2	8	·	·	−2	−2	−2	·
$\varphi_{E_6(a_2)}$	720	16	·	−18	−12	−6	8	·	·	−2	−2	−2	·
$\varphi_{A_5A_1}$	1440	32	·	−36	12	−3	·	·	·	−4	2	−1	·
$\varphi_{E_6(a_1)}$	5760	·	·	−72	·	·	·	·	·	·	·	·	·
φ_{E_6}	4320	96	·	108	·	·	−16	·	·	12	·	·	·
ϵ	1	1	1	1	1	1	1	1	1	1	1	1	1
ρ_S	12320	160	·	−28	−4	−7	·	·	·	4	−2	−5	·
ω_S	12320	160	·	−28	−4	−7	·	·	·	4	−2	−5	·

W	$E_6(a_1)$	E_6	A_1	A_1^3	$A_3A_1^2$	A_3	A_2A_1	$A_2^2A_1$	A_5	D_5	A_4A_1	$D_5(a_1)$
$\varphi_{A_2^3}$	-1	2	.	.	.	.	.	.	.	.	.	.
$\varphi_{E_6(a_2)}$	.	2	.	.	.	.	.	.	.	.	.	.
$\varphi_{A_1A_5}$	.	.	-120	-8	.	.	.	3	1	.	.	.
$\varphi_{E_6(a_1)}$	.	.	.	.	.	.	.	.	.	.	.	.
φ_{E_6}	.	-4	.	.	.	.	.	.	.	.	.	.
ϵ	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
ρ_S	-1	.	-120	-8	.	.	.	3	1	.	.	.
ω_S	-1	.	120	8	.	.	.	-3	-1	.	.	.

3.2. $W = W(B_5)$**.** The characters defined in the following table satisfy Theorem 2 for $W = W(B_5)$ when $L = S$.

λ	Gen	Word	φ_λ	$C_W(w_\lambda)$	Det
1^5	S		ϵ	W	$\det\|_{-1}$
1^32	c_1	1	-1		
	c_4	43212345	-1		
	x_1	2	-1		
	x_2	3	-1		
12^2	c_1	1	-1		
	c_2	2123	-1		
	x_2	4354	-1		
1^23	c_1	1	-1		
	c_3	3212345	ζ_6		
	x_1	2	-1		
23	c_1	12	-1		
	c_2	3212345	ζ_6		
14	c_1	1	-1		
	c_2	212345	-1		
5	c_1	12345	ζ_{10}	Z_{10}	$\det\|_{\zeta_{10}}$

3.3. $W = W(B_6)$. The characters defined in the following table satisfy Theorem 2 for $W = W(B_6)$ when $L = S$.

λ	Gen	Word	φ_λ	$C_W(w_\lambda)$	Det
1^6	S		ϵ	W	$\det\|_{-1}$
$1^4 2$	c_1	1	-1		
	c_5	5432123456	-1		
	x_1	2	-1		
	x_2	3	-1		
	x_3	4	-1		
$1^2 2^2$	c_1	1	-1		
	c_3	321234	-1		
	x_1	2	-1		
	x_3	5465	-1		
2^3	c_1	12	-1	$Z_4 \wr S_3$	
	x_1	3243	-1		
	x_2	5465	-1		
$1^3 3$	c_1	1	-1		
	c_4	432123456	ζ_6		
	x_1	2	-1		
	x_2	3	-1		
123	c_1	1	-1		
	c_2	2123	-1		
	c_3	432123456	ζ_6		
3^2	c_1	123	ζ_6	$Z_6 \wr S_2$	$\det\|_{\zeta_6}$
	x_1	432543654	-1		
$1^2 4$	c_1	1	-1		
	c_2	32123456	-1		
	x_1	2	-1		
24	c_1	12	-1		
	c_2	32123456	-1		
15	c_1	1	-1		
	c_2	2123456	ζ_{10}		
6	c_1	123456	ζ_6	Z_{12}	$(\det\|_{\zeta_{12}})^2$

3.4. $W = W(D_5)$. The characters defined in the following table satisfy Theorem 2 for $W = W(D_5)$ when $L = S$.

λ	Gen	Word	φ_λ	$C_W(w_\lambda)$	Det
$1^3 2$	$w_{1^3 2}$	$1'2321'3431'2345$	ζ_4		
	c_1	$1'$	-1		
	x_1	2	-1		
	x_2	3	-1		
23	w_{23}	$1'3w_{14}$	ζ_{12}		
14	w_{14}	$1'2345$	ζ_8	Z_8	$\det\vert_{\zeta_8}$

3.5. $W = W(D_6)$. The characters defined in the following table satisfy Theorem 2 for $W = W(D_6)$ when $L = S$.

λ	Gen	Word	φ_λ	$C_W(w_\lambda)$	Det
1^6	S		ϵ	W	$\det\vert_{-1}$
$1^2 2^2$	$c_1 c_3$	$31'234$	ζ_4		
	x_1	2	-1		
	x_3	5465	-1		
$1^3 3$	$c_1 c_4$	$43w_{15}$	ζ_3		
	x_1	2	-1		
	x_2	232	-1		
3^2	w_{3^2}	$1'343w_{15}$	ζ_3	$G(6,2,2)$	$\det\vert_{\zeta_6}$
	c_1^2	$1'232$	ζ_3		
	x_1	432543654	-1		
24	w_{24}	$1'3w_{15}$	ζ_8^3		
	c_2^2	$(3w_{15})^2$	ζ_4		
15	w_{15}	$1'23456$	ζ_5	Z_{10}	$(\det\vert_{\zeta_{10}})^2$

4. Proof of Theorem 2 when L is a proper subset of S

Recall that the normalizer in W of W_L factors as the semidirect product of W_L and a normalizer complement [7]. When the semidirect product is a direct product, W_L is called *bulky*. It is shown in [4] that Theorem 2 holds if either W_L is bulky or the rank of W_L is two or less. Also, it is shown in [3] that Theorem 2 holds if W_L is a direct product of Coxeter groups of type A. Thus, to prove Theorem 1 it suffices to prove Theorem 2 for all pairs W, W_L where the rank of W is five or six and L is a proper subset of S for which the following hold.

(1) W_L is not bulky in W,
(2) W_L has rank at least three, and
(3) W_L is not a direct product of Coxeter groups of type A.

After consulting the table of bulky parabolic subgroups in [2], it remains to consider the pairs shown in Table 1.

W	W_L
B_5	A_2B_2
B_6	A_2B_2, A_2B_3, A_3B_2, $A_1^2B_2$
D_5	D_4
D_6	D_4, D_5
E_6	D_4

Table 1. List of pairs W, W_L to be considered for Theorem 2

We consider each such pair W, W_L separately in the following subsections. For each pair we indicate representatives of the cuspidal conjugacy classes of W_L, generators of the centralizers of these representatives, and linear characters of the centralizers that satisfy the conclusion of Theorem 2. Additionally, we also give the values of $\widetilde{\rho_L}$, $\widetilde{\omega_L}$, α_L, and ϵ for the pair $W(B_5), W(A_2B_2)$ and the pair $W(E_6), W(D_4)$.

In the following sections we use the symbol w_n to denote a representative of the n^{th} conjugacy class of a group in the list of conjugacy classes returned by the command `ConjugacyClasses` in GAP. We denote the longest element of W by w_0 and the longest element in W_L by w_L. As in §3 the symbols $1, 2, \ldots, n$ denote the elements of S.

4.1. $W = W(B_5)$. As an illustration, we provide somewhat more detail in the case where $W = W(B_5)$ and $L = \{1, 2, 4, 5\}$. The cuspidal conjugacy classes in W_L are represented by w_{13} and w_{15}. The centralizer of w_{15} is $\langle w_{15}\rangle \times \langle w_0\rangle = Z_{12} \times Z_2$. The centralizer of w_{13} is generated by $C_W(w_{15})$ and 1. We define the characters φ_{13} and φ_{15} by supplying their values at these generators shown in the following table.

L	Type	Characters
$\{1,2,4,5\}$	A_2B_2	$\varphi_{13} : (w_{15}, w_0, 1) \mapsto (\zeta_3, 1, -1)$
		$\varphi_{15} : (w_{15}, w_0) \mapsto (\zeta_6, 1)$

Then $\varphi_{13}^{N_W(W_L)} + \varphi_{15}^{N_W(W_L)} = \widetilde{\rho_L}$. This character is shown in Table 2 together with $\widetilde{\omega_L}$, α_L, and ϵ. The conjugacy classes of $N_W(W_L)$ are listed in the order determined by GAP where W is constructed using the command `W:=CoxeterGroup("B",5)` and $N_W(W_L)$ is constructed using the command

```
Normalizer(W,ReflectionSubgroup(W,[1,2,4,5])).
```

The classes are labeled by the orders of their elements and an additional letter to distinguish them from one another. We see that $\widetilde{\rho_L} = \alpha_L \epsilon \widetilde{\omega_L}$ so that Theorem 2 holds for the pair W, W_L. This completes the proof of Theorem 1 for W.

$N_W(W_L)$	1a	2a	3a	2b	2c	6a	2d	2e	6b	2f	2g	6c	2h	2i	6d
$\widetilde{\rho_L}$	6	$\cdot$	−3	6	$\cdot$	−3	−2	$\cdot$	1	−2	$\cdot$	1	−2	$\cdot$	1
$\widetilde{\omega_L}$	6	$\cdot$	−3	6	$\cdot$	−3	2	$\cdot$	−1	2	$\cdot$	−1	2	$\cdot$	−1
α_L	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	1	1	1
ϵ	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1	−1

$N_W(W_L)$	2j	2k	6e	4a	4b	12a	4c	4d	12b	2l	2m	6f	2n	2o	6g
$\widetilde{\rho_L}$	−2	$\cdot$	1	−2	$\cdot$	1	−2	$\cdot$	1	6	$\cdot$	−3	6	$\cdot$	−3
$\widetilde{\omega_L}$	2	$\cdot$	−1	−2	$\cdot$	1	−2	$\cdot$	1	6	$\cdot$	−3	6	$\cdot$	−3
α_L	−1	−1	−1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1
ϵ	1	−1	1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1

TABLE 2. Characters of $N_W(W_L)$ where $W = W(B_5)$ and $L = \{1, 2, 4, 5\}$

4.2. $W = W(B_6)$. The characters defined in the following table satisfy Theorem 2 for $W = W(B_6)$ and W_L as in Table 1. For notational convenience, set $M = \{1,2,3,4,5\}$ and r is the reflection corresponding to the highest long root of W.

L	Type	Characters
$\{1,2,4,5\}$	A_2B_2	$\varphi_{13} : (w_{15}, w_0, w_M, 1) \mapsto (\zeta_3, 1, 1, -1)$ $\varphi_{15} : (w_{15}, w_0, w_M) \mapsto (\zeta_6, 1, 1)$
$\{1,2,3,5,6\}$	A_3B_2	$\varphi_{12} : (1, 2, 3, 56, w_0) \mapsto (-1, -1, -1, \zeta_3, -1)$ $\varphi_{24} : (w_{24}, 1, w_0) \mapsto \big(\zeta_3^2, -1, -1\big)$ $\varphi_{30} : (123, 56, w_0) \mapsto \big(-\zeta_3, \zeta_3^2, -1\big)$
$\{1,2,4,5,6\}$	A_2B_3	$\varphi_{23} : (1, 2, 456, w_0) \mapsto (-1, -1, \zeta_4, -1)$ $\varphi_{25} : (12, 456, w_0) \mapsto (-1, \zeta_4, -1)$
$\{1,2,4,6\}$	$A_1^2B_2$	$\varphi_{12} : (1, 2, 4, 6, 5465, r) \mapsto (-1, -1, -1, -1, 1, 1)$ $\varphi_{20} : (12, 4, 6, 5465, r) \mapsto (-1, -1, -1, 1, 1)$

4.3. $W = W(D_5)$. The characters defined in the following table satisfy Theorem 2 for $W = W(D_5)$ and W_L as in Table 1.

L	Type	Characters
$\{1', 2, 3, 4\}$	D_4	$\varphi_3 = \epsilon$ $\varphi_9 : (x_1, 1'w_0) \mapsto (-1, \zeta_4)$ $\varphi_{11} : (w_{11}, w_0) \mapsto (\zeta_3, 1)$

4.4. $W = W(D_6)$. The characters defined in the following table satisfy Theorem 2 for $W = W(D_6)$ and W_L as in Table 1. For notational convenience, set $M = \{1', 2, 3, 4, 5\}$ and $x_1 = 3243$.

L	Type	Characters
$\{1', 2, 3, 4\}$	D_4	$\varphi_3 : (1', 2, 3, 4, 6, w_M) \mapsto (-1, -1, -1, -1, 1, 1)$ $\varphi_9 : (6, x_1, 2w_M) \mapsto (1, -1, \zeta_4)$ $\varphi_{11} : (w_{11}, 6, w_M) \mapsto (\zeta_3, 1, 1)$
$\{1', 2, 3, 4, 5\}$	D_5	$\varphi_7 : (w_7, w_0, 1', 2, 3) \mapsto (\zeta_4, -1, -1, -1, -1)$ $\varphi_{15} : (w_{15}, w_0) \mapsto (\zeta_{12}, -1)$ $\varphi_{17} : (w_{17}, w_0) \mapsto (\zeta_8, -1)$

4.5. $W = W(E_6)$. Let $L = \{2, 3, 4, 5\}$. The cuspidal conjugacy classes in W_L are represented by w_3, w_9, and w_{11}. The class containing w_{11} is also labeled by the partition 13 in the notation used in type D_4. It is convenient to take $y_{13} = 2354$ as a representative of this class instead of w_{11}. The centralizer of y_{13} is generated by y_{13}, w_M, w_N where $M = \{2, 3, 4, 5, 6\}$ and $N = \{1, 2, 3, 4, 5\}$. Then W_M and W_N both are of type D_5. Notice that conjugation by w_M exchanges 2 with 3 while conjugation by w_N exchanges 2 with 5. Set $x_1 = 4354$. Define the following characters.

L	Type	Characters
$\{2, 3, 4, 5\}$	D_4	$\varphi_3 : (3, 4, w_M, w_0) \mapsto (-1, -1, 1, 1)$
		$\varphi_9 : (x_1, 2w_M, 243w_0) \mapsto (-1, -\zeta_4, \zeta_4)$
		$\varphi_{11} : (y_{13}, w_M, w_N) \mapsto (\zeta_3, 1, 1)$

In this case, $N_W(W_L) \cong W(F_4)$. Then using the notation from [5, Table C.3] for the irreducible characters of $W(F_4)$ (which is identical to the notation used in CHEVIE), we have

$$\begin{aligned}
\varphi_3^{N_W(W_L)} &= \chi''_{(1,12)} \\
\varphi_9^{N_W(W_L)} &= \chi'_{(6,6)} + \chi''_{(6,6)} \\
\varphi_{11}^{N_W(W_L)} &= \chi''_{(2,4)} + \chi_{(9,2)} + \chi''_{(9,6)} + \chi_{(12,4)} \\
\widetilde{\rho_L} &= \chi''_{(1,12)} + \chi''_{(2,4)} + \chi_{(9,2)} + \chi''_{(9,6)} + \chi'_{(6,6)} + \chi''_{(6,6)} + \chi_{(12,4)} \\
\widetilde{\omega_L} &= \chi'_{(1,12)} + \chi'_{(2,16)} + \chi_{(9,10)} + \chi'_{(9,6)} + \chi'_{(6,6)} + \chi''_{(6,6)} + \chi_{(12,4)}.
\end{aligned}$$

Now since $\alpha_L \epsilon = \chi_{(1,24)}$ is the sign character of $W(F_4)$, the calculations above together with [5, Table C.3] show that the characters $\varphi_3, \varphi_9, \varphi_{11}$ satisfy Theorem 2 for the pair W, W_L.

Acknowledgments: The authors would like to acknowledge support from the DFG-priority program SPP1489 "Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory".

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