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On Seifert fibered spaces embedding in 4-space, bounding definite manifolds and quasi-alternating Montesinos links

Committee:

Cameron Gordon, Supervisor

Robert Gompf

Tye Lidman

John Luecke

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by

Ahmad Issa Khalid Issa,

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On Seifert fibered spaces embedding in 4-space, bounding definite manifolds and quasi-alternating Montesinos links

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Ahmad Issa Khalid Issa, Ph.D.
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This dissertation is concerned with the question of which Seifert fibered spaces smoothly embed in the 4-sphere and the related question of which Seifert fibered spaces bound both a positive definite and a negative definite smooth 4-manifold. Using Donaldson's diagonalization theorem we derive strong obstructions in both of these settings. We construct new embeddings of Seifert fibered spaces in S^4 out of old ones, giving many new examples of Seifert fibered spaces which embed in S^4. Our results allow us to classify precisely when a Seifert fibered space over an orientable base surface smoothly embeds in S^4 provided $e > k/2$, where e is the normalized central weight and k is the number of singular fibers. Based on these results and an analysis of the Neumann-Siebenmann invariant $\overline{\mu}$, we make some conjectures concerning Seifert fibered spaces which embed in S^4. Finally, we classify the quasi-alternating Montesinos links, showing that a Montesinos link L is quasi-alternating if and only if its double branched cover is an L-space which bounds definite manifolds of both signs with torsion-free first homology.

Table of Contents

List of Figures

Chapter 1

Introduction

Let M be a connected, closed, oriented 3-manifold. It is a well-known fact that M bounds a smooth, oriented 4-manifold [Tho54]. However, one can ask more refined questions such as whether M bounds a smooth rational homology 4-ball, i.e. a 4-manifold W with $H_*(W;\mathbb{Q}) = H_*(B^4;\mathbb{Q})$, or whether it bounds a smooth negative definite 4-manifold. Answers to such questions, in addition to being interesting in their own right, have important implications for a number of important problems in 3- and 4-manifold topology and knot theory. As an example, suppose $K \subset S^3$ is a smoothly slice knot, that is, K bounds a smoothly embedded disk D in the 4-ball. The double branched cover of the 4-ball branched over D is a rational homology 4-ball with boundary $\Sigma_2(K)$, the double branched cover of S^3 branched over the knot K. Thus, one can obstruct a knot from being slice by showing that its double branched cover does not bound a rational homology 4-ball. One of the great triumphs of this strategy is the beautiful work of Lisca [Lis07a], where he shows that a 2-bridge knot is smoothly slice if and only if its double branched cover bounds a rational homology ball.

The obstruction used by Lisca is based on Donaldson's diagonalization theorem [Don87], which is one of the landmark results in smooth 4-manifold topology. Donaldson's theorem states that the intersection form of a smooth closed definite 4-manifold is diagonalizable, a fact which can be used to detect subtle topological information not accessible through classical methods. For example, it can be used in conjunction with work of Freedman and Quinn [Fre82a, FQ90] to show that there exist knots which are topologically slice, but not smoothly slice. This fact can be used to prove the existence of smooth 4-manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^4$.

This thesis is primarily concerned with applications of Donaldson's theorem to understanding Seifert fibered spaces and the types of 4-manifolds which they bound. We consider the problem of determining which Seifert fibered spaces smoothly embed in S^4. If a rational homology 3-sphere Y smoothly embeds in S^4, it splits S^4 into two rational homology 4-balls. Thus, obstructions to Y bounding a rational homology 4-ball are obstructions to embedding in S^4. When Y is a Seifert fibered space one can use Donaldson's theorem to analyse this problem. In fact, one can obtain stronger obstructions using Donaldson's theorem by exploiting the fact that Y must in fact bound two rational homology 4-balls and that they must glue to give S^4. Donald investigated this obstruction and used it to determine precisely which connected sums of lens spaces smoothly embed in S^4 [Don15]. In this thesis we analyse the case when Y is a Seifert fibered space. Our investigation naturally leads to a number of interesting results concerning which Seifert fibered spaces bound smooth definite 4-manifolds of both signs, which turns out to be related to the classification of quasi-alternating Montesinos links.

In Chapter 2, we use Donaldson's theorem to establish an inequality which gives strong restrictions on when the standard definite plumbing intersection lattice of a Seifert fibered space over S^2 can embed into a standard diagonal lattice, and give some applications. First, we answer a question of Neumann-Zagier on the relationship between Donaldson's theorem and Fintushel-Stern's R-invariant. As a corollary, we answer a question of Lidman-Tweedy concerning the non-vanishing of the Ozsváth-Szabó d-invariant for certain Seifert fibered integral homology spheres. We also characterise the Seifert fibered spaces Y with $b_1(Y) = 1$ which bound both positive and negative definite 4-manifolds. This in particular gives a short proof of the characterisation of Seifert fibered spaces which bound smooth rational homology $S^1 \times D^3$'s.

In Chapter 3, we complete the classification of quasi-alternating Montesinos links. We show that the quasi-alternating Montesinos links are precisely those identified independently by Qazaqzeh-Chbili-Qublan and Champanerkar-Ording. A consequence of our proof is that a Montesinos link L is quasi-alternating if and only if its double branched cover is an L-space, and bounds both a positive definite and a negative definite 4-manifold with torsion-free first homology. This also leads to a classification of the Seifert fibered spaces which are formal L-spaces. The key to these classification results is a refinement of the inequality from Chapter 2 for Seifert fibered space bounding definite manifolds with torsion-free first homology.

In Chapter 4, using an obstruction based on Donaldson's theorem, we derive strong restrictions on when a Seifert fibered space $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ over an orientable base surface F can smoothly embed in S^4. As is often the case, an analysis of the lattice theoretic obstructions suggests interesting topological constructions. In the present case this leads us to show that if Y embeds in S^4 then the Seifert fibered spaces $F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_i}{q_i}, \frac{p_i}{q_i})$, where $1 \leq i \leq k$, also embeds in S^4. This allows us to classify precisely when Y smoothly embeds provided $e > k/2$, where e is the normalized central weight and k is the number of singular fibers. Based on these results and an analysis of the Neumann-Siebenmann invariant $\overline{\mu}$, we make some conjectures concerning Seifert fibered spaces which embed in S^4. Finally, we also provide some applications to doubly slice Montesinos links, including a way to construct new doubly slice links out of old ones and a classification of the smoothly doubly slice odd pretzel knots up to mutation.

Chapter 2

Seifert fibered spaces bounding definite manifolds[1]

2.1 Introduction

Donaldson's diagonalization theorem [Don87] has led to many great successes in understanding several important questions in low dimensional topology, and in knot theory in particular. For example, Donaldson's theorem can often be used to answer questions concerning sliceness, unknotting number, 3-manifolds bounding rational homology balls, and surgery questions. In these cases, one typically uses Donaldson's theorem to obstruct a certain 3-manifold from bounding a certain type of smooth negative definite 4-manifold, with the obstruction taking the form of the existence of a certain map of intersection lattices. However, understanding this obstruction for large families of examples is often highly non-trivial, and can require combinatorial ingenuity.

One appealing application of Donaldson's theorem is to prove the well-known fact that the Poincaré homology sphere $P = S^2(2; 2, \frac{3}{2}, \frac{5}{4})$ does not bound a smooth integral homology 4-ball. This fact can, of course, be proved

[1]This chapter consists of joint work with Duncan McCoy, and is based on our preprint *On Seifert fibered spaces bounding definite manifolds*, https://arxiv.org/abs/1807.10310, 2018.

in many other ways, for example by using Rokhlin's theorem, Fintushel-Stern's R-invariant, or by using the d-invariant coming from Heegaard Floer homology. Assuming that P is oriented to bound the positive E_8 plumbing, the proof by Donaldson's theorem is as follows. If P were the boundary of a smooth integral homology 4-ball W, then we could form a closed positive definite manifold by gluing $-W$ to the positive E_8 plumbing. Donaldson's theorem would then imply that the E_8 intersection form is diagonalizable, which is, of course, untrue. In fact, as the E_8 intersection form does not embed into any positive definite diagonal lattice, this argument shows that P does not bound any smooth negative definite 4-manifold. The purpose of this chapter is to generalize this argument to other Seifert fibered spaces. We prove the following theorem.

Theorem 2.1.1. *Let $Y = S^2(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space over S^2 in standard form, that is, with $e \geq 0$, $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \dots, k\}$ and $\varepsilon(Y) \geq 0$. Suppose that Y bounds a smooth 4-manifold W such that $\sigma(W) = -b_2(W)$ and the inclusion induced map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ is injective. Then there is a partition of $\{1, 2, \dots, k\}$ into at most e classes such that for each class C,*

$$\sum_{i \in C} \frac{q_i}{p_i} \leq 1.$$

We note that the condition that $\varepsilon(Y) := e - \sum_{i=1}^{k} \frac{q_i}{p_i} \geq 0$ in Theorem 2.1.1 guarantees that Y is oriented to bound a positive (semi-)definite plumbing 4-manifold. When Y is a rational homology sphere the map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ is automatically injective so in this case we are simply obstructing the existence of a negative definite manifold bounding Y. Although we do not discuss the details in this chapter, one can easily obtain analogous results for Seifert fibered spaces over any orientable base surface. In our notation, the Poincaré homology sphere oriented to bound the positive E_8 plumbing is $P = S^2(2; 2, \frac{3}{2}, \frac{5}{4})$, see Figure 2.2. The reader can easily verify that Theorem 2.1.1 obstructs P from bounding a negative definite manifold. When $k < 3$, the Seifert fibered spaces are lens spaces which are well known to bound both positive and negative definite smooth 4-manifolds. Finally, we note that the converse to Theorem 2.1.1 is not true. The integer homology sphere $S^2(1; 3, 5, \frac{13}{6})$ passes the obstruction, but does not bound a negative definite manifold as it bounds a positive definite plumbing whose intersection form does not embed in a diagonal lattice.

We give two applications of Theorem 2.1.1. First, we prove the following theorem.

Theorem 2.1.2. *Let $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a Seifert fibered integral homology sphere in standard form, that is, with $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \ldots, k\}$, $e > 0$ and with Y oriented to bound a smooth positive definite plumbing 4-manifold. If Y bounds a smooth negative definite 4-manifold, then $e = 1$.*

In the course of proving Theorem 2.1.2, we obtain a positive answer to the following question asked by Neumann-Zagier [NZ85].

Question: *Let Y be as in Theorem 2.1.2. If the intersection form of the plumbing of Y is diagonalizable over $\mathbb{Z}$, must e be equal to 1?*

The motivation for this question comes from the R-invariant. Fintushel-Stern [FS85] used gauge theory to define an invariant $R(Y)$ of Seifert fibered integral homology spheres with the property that if $R(Y) > 0$ then Y does not bound a smooth negative definite 4-manifold W with $H_1(W)$ having no 2-torsion. Fintushel-Stern originally gave an expression for $R(Y)$ as a trigonometric sum involving the Seifert invariants of Y. Neumann-Zagier [NZ85] proved that these sums could be simply evaluated in terms of the central weight e of the standard positive definite plumbing bounding Y, showing that $R(Y) = 2e - 3$. Thus, if $e > 1$ then the R-invariant shows that Y does not bound a smooth negative definite 4-manifold W with $H_1(W)$ having no 2-torsion. In this light, the positive answer to Neumann-Zagier's question implies that this result obtained from the R-invariant is also a consequence of Donaldson's theorem.

We are in fact able to prove a more general version of Theorem 2.1.2 which holds for all $|H_1(Y)| \in \{1, 2, 3, 5, 6, 7\}$, see Theorem 2.5.1 of Section 2.5. Some particular cases of Theorem 2.1.2 are known. In their original paper, Neumann-Zagier [NZ85] claimed to have proved the cases when $k = 3$, and when $k = 4$ and $e \neq 3$, but do not provide a proof, remarking that their proof was "clearly not the right proof". The special case when $e = k - 1$ follows from [LL11, Lemma 3.3].

Finally, we note that a positive answer to Neumann-Zagier's question is a special case of a more general conjecture made by Neumann [Neu89], stating that if an integral homology sphere Y is given as the boundary of a positive definite plumbing tree Γ and the intersection lattice of Γ is isomorphic to a diagonal lattice, then some vertex of Γ has weight 1. This general form of Neumann's conjecture for graph manifolds remains open.

Lidman-Tweedy [LT18, Remark 4.3] asked whether a Seifert fibered integral homology sphere with central weight different from 1 must have non-vanishing Heegaard-Floer d-invariant. As a corollary of Theorem 2.1.2, we answer their question positively.

Corollary 2.1.3. *Let Y be a Seifert fibered integral homology sphere, and let $e \in \mathbb{Z}$ be the central weight in the standard definite plumbing graph for Y. If $|e| \neq 1$, then $d(Y) \neq 0$.*

As a second application, we give a short proof of the following theorem which, in particular, gives a classification of the Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s.

Theorem 2.1.4. *Let Y be a Seifert fibered space over S^2 with $H_*(Y;\mathbb{Q}) \cong H_*(S^1 \times S^2;\mathbb{Q})$. The following are equivalent:*

1. *Y is of the form $S^2(k; \frac{p_1}{q_1}, \frac{p_1}{p_1-q_1}, \ldots, \frac{p_k}{q_k}, \frac{p_k}{p_k-q_k})$, where $k \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \ldots, k\}$.*

2. *$Y = \partial W$, where W is a smooth 4-manifold with $H_*(W;\mathbb{Q}) \cong H_*(S^1 \times D^3;\mathbb{Q})$.*

3. Y is the boundary of smooth 4-manifolds W_+ and W_- such that $\sigma(W_\pm) = \pm b_2(W_\pm)$ and each of the inclusion-induced maps $H_1(Y;\mathbb{Q}) \to H_1(W_\pm;\mathbb{Q})$ is injective.

Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s naturally arise in two contexts. First, a Seifert fibered space rational homology $S^1 \times S^2$ which embeds in S^4 necessarily bounds a rational homology $S^1 \times D^3$. Indeed, in this context Donald [Don15, Proof of Theorem 1.3] proved the implication (2) implies (1) of Theorem 2.1.4. Second, a smoothly slice 2-component Montesinos link has double branched cover a Seifert fibered space over S^2 bounding a rational homology $S^1 \times D^3$. Motivated by trying to determine the slice 2-component Montesinos links, Aceto [Ace15, Theorem 1.2] also classified Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s.

Much like the proofs by Donald and Aceto, our proof also proceeds by means of Donaldson's theorem. However, their proofs rely on the work of Lisca [Lis07b] which gives a detailed analysis on sums of linear lattices embedding in a full-rank lattice. We give a short proof of Theorem 2.1.4 circumventing the reliance on Lisca's work. We obtain the additional equivalent condition (3) in Theorem 2.1.4, since our method does not require the lattice embeddings to have full-rank.

Finally, we note that Theorem 2.1.1 also plays a key role in Chapter 4, where we analyse which Seifert fibered spaces smoothly embed in S^4, and in particular, completely determine the Seifert fibered spaces $Y = S^2(e; r_1, \ldots, r_k)$

with $r_i \in \mathbb{Q}_{>1}$ for all i, $\varepsilon(Y) > 0$ and $e > k/2$ which smooothly embed in S^4.

In Section 2.2, we recall some standard facts and establish notation and conventions. In Section 2.3, we prove the key technical theorem used to prove Theorem 2.1.1. In Section 2.4, we analyse when gluing compact 4-manifolds with boundary results in a definite 4-manifold and give a proof of Theorem 2.1.1. In Section 2.5, we prove Theorem 2.5.1 answering Neumann-Zagier's question, as well as prove Corollary 2.1.3. Finally, in Section 2.6 we prove Theorem 2.1.4 determining the Seifert fibered spaces which bound rational homology $S^1 \times D^3$'s.

2.2 Preliminaries

In this section we briefly recall some standard facts about Seifert fibered spaces and intersection lattices, and establish notation and conventions. See [NR78] for a more indepth treatment on Seifert fibered spaces and plumbings.

Given $r \in \mathbb{Q}_{>1}$, there is a unique (negative) continued fraction expansion

$$r = [a_1, \ldots, a_h]^- := a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots \, a_{h-1} - \cfrac{1}{a_h}}},$$

where $h \geq 1$ and $a_i \geq 2$ are integers for all $i \in \{1, \ldots, h\}$. We associate to r the weighted linear graph (or linear chain) given in Figure 2.1. We call the vertex with weight labelled by a_i the ith vertex of the linear chain associated to r, so that the vertex labelled with weight a_1 is the first, or starting vertex of the linear chain.

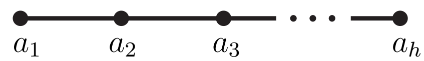

Figure 2.1: Weighted linear chain representing $r = [a_1, \ldots, a_h]^-$.

We denote by $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ the Seifert fibered space over S^2 given in Figure 2.2, where $e \in \mathbb{Z}$, and $\frac{p_i}{q_i} \in \mathbb{Q}$ is non-zero for all $i \in \{1, \ldots, k\}$. The generalised Euler invariant of Y is given by $\varepsilon(Y) = e - \sum_{i=1}^{k} \frac{q_i}{p_i}$. Every Seifert fibered space Y is (possibly orientation reversing) homeomorphic to one in standard form, i.e. such that $\varepsilon(Y) \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \ldots, k\}$. We henceforth assume that Y is in standard form. If $\varepsilon(Y) \neq 0$ then Y is a rational homology sphere with $|H_1(Y)| = |p_1 \cdots p_k \varepsilon(Y)|$, and if $\varepsilon(Y) = 0$ then Y is a rational homology $S^1 \times S^2$.

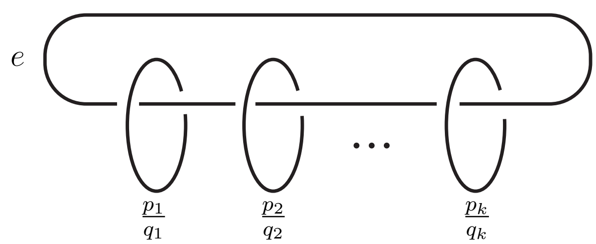

Figure 2.2: Surgery presentation for the Seifert fibered space $S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$.

For each $i \in \{1, \ldots, k\}$, we have the unique continued fraction expansion $\frac{p_i}{q_i} = [a_1^i, \ldots, a_{h_i}^i]^-$ where $h_i \geq 1$ and $a_j^i \geq 2$ are integers for all $j \in \{1, \ldots, h_i\}$. We associate to $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ the weighted star-shaped graph in Figure 2.3. The ith leg of the star-shaped graph is the weighted linear subgraph for p_i/q_i generated by the vertices labelled with weights $a_1^i, \ldots, a_{h_i}^i$. The degree k vertex labelled with weight e is called the central vertex.

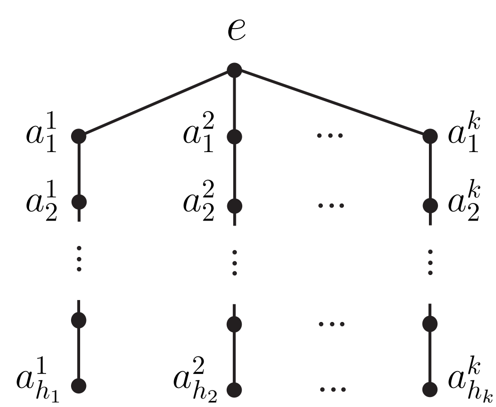

Figure 2.3: The weighted star-shaped plumbing graph Γ.

Let Γ be either the weighted star-shaped graph for Y, or a disjoint union of weighted linear graphs. There is an oriented smooth 4-manifold X_Γ given by plumbing D^2-bundles over S^2 according to the weighted graph Γ. We denote by $|\Gamma|$ the number of vertices in Γ. Let $N = |\Gamma|$ and denote the vertices of Γ by $v_1, v_2, \dots, v_N$. The zero-sections of the D^2-bundles over S^2 corresponding to each of $v_1, \dots, v_N$ in the plumbing together form a natural spherical basis for $H_2(X_\Gamma)$. With respect to this basis, which we call the vertex basis, the intersection form of X_Γ is given by the weighted adjacency matrix Q_Γ with entries Q_{ij}, $1 \leq i, j \leq N$ given by

$$Q_{ij} = \begin{cases} \mathrm{w}(v_i), & \text{if } i = j \\ -1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge ,} \\ 0, & \text{otherwise} \end{cases}$$

where $\mathrm{w}(v_i)$ is the weight of vertex v_i. Denoting by Q_{X_Γ} the intersection form of X_Γ, we call $(H_2(X_\Gamma), Q_{X_\Gamma}) \cong (\mathbb{Z}^N, Q_\Gamma)$ the intersection lattice of X_Γ

(or of Γ). We denote the intersection pairing of two elements $x, y \in \mathbb{Z}^N$ by $x \cdot y = x^T Q_\Gamma y$. Now assume that Γ is the star-shaped plumbing for Y. If $\varepsilon(Y) > 0$ then X_Γ is a positive definite 4-manifold and Γ is the standard positive definite plumbing graph for Y. If $\varepsilon(Y) = 0$, then X_Γ is a positive semi-definite manifold.

Let $\iota : (\mathbb{Z}^N, Q_\Gamma) \to (\mathbb{Z}^m, \mathrm{Id})$, $m > 0$, be a map of lattices, i.e. a $\mathbb{Z}$-linear map preserving pairings, where $(\mathbb{Z}^m, \mathrm{Id})$ is the standard positive diagonal lattice. We call ι a lattice embedding if it is injective. We adopt the following standard abuse of notation. First, for each $i \in \{1, \ldots, N\}$, we identify the vertex v_i with the corresponding ith basis element of $(\mathbb{Z}^N, Q_\Gamma)$. Moreover, we shall identify an element $v \in (\mathbb{Z}^N, Q_\Gamma)$ with its image $\iota(v) \in (\mathbb{Z}^m, \mathrm{Id})$.

2.3 The embedding inequality

In this section, we prove Theorem 2.3.2 below, which is the key technical result of this chapter. In particular, it will be used in the next section to prove Theorem 2.1.1. We begin with some continued fraction identities which we will need.

Lemma 2.3.1. *Let $\{a_i\}_{i\geq 1}$ be a sequence of integers with $a_i \geq 2$ for all i, and let $p_k/q_k = [a_1, \ldots, a_k]^-$ for all $k \geq 1$. Then we have the following identities:*

(a) $q_n p_{n-1} - p_n q_{n-1} = 1$ *for all* $n \geq 2$.

(b) $[a_1, \ldots, a_n, x]^- = \frac{x p_n - p_{n-1}}{x q_n - q_{n-1}}$, *for all* $n \geq 2$ *and* $x \in \mathbb{R}$ *such that both sides are well defined.*

(c) $p_n = \det \begin{pmatrix} a_1 & -1 & 0 & 0 \\ -1 & a_2 & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & a_n \end{pmatrix}$ *and* $q_n = \det \begin{pmatrix} a_2 & -1 & 0 & 0 \\ -1 & a_3 & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & a_n \end{pmatrix}$ *for all* $n \geq 2$.

Proof. For $a \in \mathbb{R}$, let M_a denote the matrix $M_a = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$. If $q/r = [a_2, \ldots, a_n]^-$, then $\frac{p_n}{q_n} = a_1 - \frac{r}{q}$. In particular, we have

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = M_{a_1} \begin{pmatrix} q \\ r \end{pmatrix}$$

Thus, one can inductively show that

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = M_{a_1} \cdots M_{a_n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.3.1}$$

and furthermore that

$$\begin{pmatrix} p_n & -p_{n-1} \\ q_n & -q_{n-1} \end{pmatrix} = M_{a_1} \cdots M_{a_n}. \tag{2.3.2}$$

Identity (a) follows by taking determinants of (2.3.2) and observing that $\det M_a = 1$ for any a. Identity (b) follows from combining (2.3.1) and (2.3.2) to get

$$M_{a_1} \cdots M_{a_n} M_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_n & -p_{n-1} \\ q_n & -q_{n-1} \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} xp_n - p_{n-1} \\ xq_n - q_{n-1} \end{pmatrix}.$$

The identities in (c) can easily be proven by induction using the observation that

$$\det \begin{pmatrix} a_1 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & a_n \end{pmatrix} = a_1 \det \begin{pmatrix} a_2 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & a_n \end{pmatrix} - \det \begin{pmatrix} a_3 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & a_n \end{pmatrix}.$$

□

The following theorem is the key technical result of this chapter.

Theorem 2.3.2. *Let $\iota : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^m, Id)$ be a map of lattices, where $m > 0$ and Γ is a disjoint union of weighted linear chains representing fractions $\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k} \in \mathbb{Q}_{>1}$. Suppose that there is a unit vector $w \in (\mathbb{Z}^m, Id)$ which pairs non-trivially with (the image of) the starting vertex of each linear chain. Then*

$$\sum_{i=1}^{k} \frac{q_i}{p_i} \leq 1.$$

Moreover, if we have equality then w has pairing ± 1 with the starting vertex of each linear chain. More precisely, if we have equality then for each linear chain either

(a) the unit vector w has pairing ± 1 with the first vertex of the chain and has trivial pairing with every other vertex in the chain, or

(b) the starting vertex of the chain has weight 2, $(w \cdot v_1, w \cdot v_2) = (\pm 1, \mp 1)$, where v_1, v_2 are the first two vertices of the chain, and w pairs trivially with every other vertex of the chain.

Proof. Let $\{e_1, \dots, e_m\}$ denote the orthonormal basis of coordinates vectors of $(\mathbb{Z}^m, \mathrm{Id})$. Since the unit vectors in $(\mathbb{Z}^m, \mathrm{Id})$ are precisely those vectors of the form $\pm e_i$ where $i \in \{1, \dots, m\}$, by a change of basis if necessary, we may assume that $w = e_1 \in (\mathbb{Z}^m, \mathrm{Id})$. Write $\iota : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^m, \mathrm{Id})$ as an integer matrix with respect to the vertex basis of $(\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$, and let M be the transpose of this matrix. Since ι preserves intersection pairings we have,

$u^T Q_\Gamma v = \iota(u)^T \iota(v) = (M^T u)^T (M^T v) = u^T M M^T v$ for all $u, v \in (\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$. Thus,

$$MM^T = Q_\Gamma = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix},$$

where for each $i \in \{1, \ldots, k\}$, A_i on the diagonal represents a block matrix of the form

$$A_i = \begin{pmatrix} a_1^i & -1 & 0 & 0 \\ -1 & a_2^i & -1 & 0 \\ 0 & -1 & \ddots & -1 \\ 0 & 0 & -1 & a_{h_i}^i \end{pmatrix}$$

where $[a_1^i, \ldots, a_{h_i}^i]^-$ is the continued fraction expansion for p_i/q_i. If a matrix A can be written as a product $M'M'^T$, then[2]

$$\det A \geq 0. \tag{2.3.3}$$

We will prove the theorem by applying (2.3.3) to a matrix of the form $A = M'M'^T$, where M' is a suitable modification of M.

We may write M in the form

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix}$$

where for all $i \in \{1, \ldots, k\}$, M_i is a matrix such that $M_i M_i^T = A_i$. By the assumption that e_1 pairs non-trivially with each of the starting vertices of the linear chains, we may assume that each matrix M_i is non-zero in its top left

[2]Let $v \neq 0$ be an eigenvector of A with eigenvalue λ. We have $v^T A v = \lambda \|v\|^2 = \|M'^T v\|^2 \geq 0$, thus $\lambda \geq 0$. Since $\det A$ is a product of eigenvalues, this implies that $\det A \geq 0$ as required.

entry. For each $i \in \{1, \ldots, k\}$, let $p_i/q_i = [a^i_1, \ldots, a^i_{h_i}]^-$ be the standard continued fraction expansion and choose M'_i to be the submatrix of M_i obtained by taking the first l_i rows, where l_i is chosen so that the first column w_i of M'_i takes one of the two forms as follows:

(Form 1) If $a^i_1 = 2$ and the first column of M_i is of the form $\pm\begin{pmatrix}1 & -1 & \cdots\end{pmatrix}^T$ then we may take $w_i = \pm\begin{pmatrix}1 & -1 & 0 & \cdots & 0 & v\end{pmatrix}^T$, where $v = 0$ only if $M'_i = M_i$.

(Form 2) Otherwise, we may take w_i to be of the form $w_i = \begin{pmatrix}u & 0 & \cdots & 0 & v\end{pmatrix}^T$, where $v = 0$ only if $M'_i = M_i$.

Let M' be the matrix

$$M' = \begin{pmatrix} 1\,0 \cdots 0 \\ M'_1 \\ \vdots \\ M'_k \end{pmatrix}.$$

Then the product $A = M'M'^T$ takes the form of the block matrix

$$M'M'^T = \begin{pmatrix} 1 & w_1^T & \cdots & w_k^T \\ w_1 & A'_1 & & 0 \\ \vdots & & \ddots & \\ w_k & 0 & & A'_k \end{pmatrix}.$$

Claim. $\det A$ *can be written in the form*

$$\det A = (P_1 \cdots P_k)(1 - \sum_{i=1}^{k} \frac{Q_i}{P_i}),$$

where $P_i = \det A'_i > 0$ *and* $Q_i = -\det\begin{pmatrix} 0 & w_i^T \\ w_i & A'_i \end{pmatrix}$ *is a quantity depending only on* A'_i *and* w_i.

Proof. Since Γ represents a positive definite lattice, $P_i > 0$ for all i. By the multi-linearity of the determinant in both rows and columns we have

$$\det\begin{pmatrix} 1 & w_1^T & \dots & w_k^T \\ w_1 & A_1' & & 0 \\ \vdots & & \ddots & \\ w_k & 0 & & A_k' \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & A_1' & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & A_k' \end{pmatrix} + \sum_{1\le i,j\le k} \det B_{ij},$$

where B_{ij} is the matrix

$$B_{ij} = \begin{pmatrix} 0 & \cdots & w_j^T & \cdots & 0 \\ \vdots & A_1' & & & \\ w_i & & \ddots & 0 & \\ \vdots & & 0 & \ddots & \\ 0 & & & & A_k' \end{pmatrix}.$$

Since A_i' has full rank, w_i can be expressed as a rational linear combination of the columns of A_i', and hence $\det B_{ij} = 0$ for all $i \neq j$. For $i \in \{1, \dots, k\}$, by row and column operations, we can put B_{ii} into the form of a diagonal block matrix with diagonal blocks $\begin{pmatrix} 0 & w_i^T \\ w_i & A_i' \end{pmatrix}$, $A_1', \dots, A_{i-1}', A_{i+1}', \dots, A_k'$ without changing the determinant. Hence, $\det B_{ii}$ is the product of the determinants of these blocks, that is, $\det B_{ii} = -(P_1 \cdots P_k)\frac{Q_i}{P_i}$. □

Since $P_i > 0$ for all i, the previous claim combined with $\det A \geq 0$ (see (2.3.3)) shows that

$$\sum_{i=1}^{k} \frac{Q_i}{P_i} \leq 1. \tag{2.3.4}$$

So to prove the inequality in the theorem it suffices to show that $Q_i/P_i \geq q_i/p_i$ for each $i \in \{1, \dots, k\}$. To do this it suffices to consider some fixed $i \in \{1, \dots, k\}$. For convenience, let $P/Q = P_i/Q_i$ and $p/q = p_i/q_i =$

$[a_1, a_2, \ldots, a_h]^-$ where $a_j \geq 2$ for all $j \in \{1, \ldots, h\}$, and let $l = l_i$ be the number of rows of A_i'.

Consider the following identity obtained by adding the second row to the first row, and the second column to the first column:

$$\det \begin{pmatrix} 0 & -1 & 1 & \cdots & v \\ -1 & 2 & -1 & & 0 \\ 1 & -1 & a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & 0 & & -1 & a_l \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 & \cdots & v \\ 1 & 2 & -1 & & 0 \\ 0 & -1 & a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & 0 & & -1 & a_l \end{pmatrix}. \tag{2.3.5}$$

Recall that w_i takes one of two possible forms. By applying the above identity if w_i takes the form (Form 1), we see that regardless of the form that w_i takes, Q is equal to the determinant of a matrix of the following form

$$Q = -\det \begin{pmatrix} 0 & u & 0 & \cdots & v \\ u & a_1 & -1 & & 0 \\ 0 & -1 & a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & 0 & & -1 & a_l \end{pmatrix}, \tag{2.3.6}$$

where if $(u, v) = (\pm 1, \mp 1)$ then either $l > 2$ or $a_1 > 2$. If w_i takes the form (Form 1), we define $u \in \{\pm 1\}$ via Equation (2.3.6) by applying the identity in (2.3.5).

For $j \in \{1, \ldots, h\}$, let r_j/s_j denote the continued fraction $[a_1, \ldots, a_j]^-$. Note that $P = r_l$.

Claim.

$$Q = u^2 s_l + 2uv + v^2 r_{l-1}$$

Proof. Applying cofactor expansion along the first column and first row in (2.3.6) gives

$$Q = u^2C_1 + (-1)^{l+1}uvC_2 + (-1)^{l+1}uvC_3 + v^2C_4,$$

where $C_1 = \det \begin{pmatrix} a_2 & -1 & \cdots & 0 \\ -1 & a_3 & -1 & \\ \vdots & -1 & \ddots & -1 \\ 0 & & -1 & a_l \end{pmatrix}$, $C_2 = \det \begin{pmatrix} -1 & a_2 & -1 & \cdots & 0 \\ 0 & -1 & a_3 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ & & & -1 & a_{l-1} \\ 0 & & & & -1 \end{pmatrix}$,

$C_3 = \det \begin{pmatrix} -1 & 0 & & \cdots & 0 \\ a_2 & -1 & & & \\ -1 & a_3 & -1 & & \\ & -1 & \ddots & -1 & \\ 0 & & & a_{l-1} & -1 \end{pmatrix}$ and $C_4 = \det \begin{pmatrix} a_1 & -1 & \cdots & 0 \\ -1 & a_2 & -1 & \\ \vdots & -1 & \ddots & -1 \\ 0 & & -1 & a_{l-1} \end{pmatrix}$.

Using the continued fraction identities in Lemma 2.3.1, we see that $C_1 = s_l$ and $C_4 = r_{l-1}$. Finally, notice that C_2 (resp. C_3) is the determinant of an upper (resp. lower) triangular matrix with $l-1$ diagonal entries all of which are -1, hence $C_2 = C_3 = (-1)^{l-1}$. □

Claim. *We have* $\frac{Q}{P} \geq \frac{q}{p}$ *with equality only if* $u = \pm 1$ *and* $v = 0$.

Proof. Recall that if $v = 0$ then $r_l/s_l = p/q$. Thus if $v = 0$, $Q/P = u^2q/p$. Since $u \neq 0$, we clearly have $Q/P \geq q/p$ with equality only if $u^2 = 1$, as required. Thus assume that $v \neq 0$. In this case, if $l = h$, or equivalently, $p/q = r_l/s_l$ then $P = p$ and $Q = u^2s_l + 2uv + v^2r_{l-1} > s_l = q$ and thus $Q/P > q/p$. Hence, we assume that $p/q \neq r_l/s_l$ and, in particular, that $p/q = [a_1, \ldots, a_l, x]^-$ where $x = [a_{l+1}, \ldots, a_h]^- > 1$. Thus, by Lemma 2.3.1 we have

$$\frac{p}{q} = \frac{xr_l - r_{l-1}}{xs_l - s_{l-1}}.$$

Note that since u and v are both non-zero we have that

$$Q = (s_l - 1)u^2 + (u+v)^2 + (r_{l-1} - 1)v^2 \geq s_l + r_{l-1} - \varepsilon,$$

where we take $\varepsilon = 2$ if $u = -v \in \{\pm 1\}$ and $\varepsilon = 1$ otherwise. Note that if $r_{l-1} = 2$, then $\varepsilon = 1$, as $l = a_1 = 2$ implies we cannot have $u = -v \in \{\pm 1\}$ by the condition stated immediately following Equation (2.3.6). In either case we always have

$$r_{l-1} - \varepsilon \geq 1.$$

Thus we obtain

$$\begin{aligned}
\frac{Q}{P} - \frac{q}{p} &\geq \frac{s_l + r_{l-1} - \varepsilon}{r_l} - \frac{xs_l - s_{l-1}}{xr_l - r_{l-1}} \\
&= \frac{(r_{l-1} - \varepsilon)(xr_l - r_{l-1}) + r_l s_{l-1} - s_l r_{l-1}}{r_l(xr_l - r_{l-1})} \\
&= \frac{(r_{l-1} - \varepsilon)(xr_l - r_{l-1}) - 1}{r_l(xr_l - r_{l-1})} \\
&\geq \frac{(xr_l - r_{l-1}) - 1}{r_l(xr_l - r_{l-1})} \\
&> 0,
\end{aligned} \tag{2.3.7}$$

where we used the identity $r_l s_{l-1} - s_l r_{l-1} = -1$ from Lemma 2.3.1 to obtain the third line, $r_{l-1} - \varepsilon \geq 1$ to obtain the fourth line, and finally that $p = xr_l - r_{l-1} > 1$. This gives the desired inequality, proving the claim. □

The claim together with (2.3.4) proves that $\sum_{i=1}^{k} \frac{q_i}{p_i} \leq 1$ with equality only if $w = e_1$ has pairing ± 1 with each starting vertex. Moreover, equality implies that we equality in the above claim, which depending on whether w_i takes form (Form 1) or (Form 2), implies either (a) or (b) holds in the statement of the theorem.

□

2.4 Definite 4-manifolds and the Seifert fibered space inequality

Now we consider when gluing two 4-manifolds can result in a closed definite 4-manifold.

Proposition 2.4.1. *Let U_1 and U_2 be connected 4-manifolds with $\partial U_1 = -\partial U_2 = Y$. Then the closed 4-manifold $X = U_1 \cup_Y U_2$ is positive definite if and only if*

(a) the inclusion-induced map $(i_1)_* \oplus (i_2)_* \colon H_1(Y;\mathbb{Q}) \to H_1(U_1;\mathbb{Q}) \oplus H_1(U_2;\mathbb{Q})$ is injective and

(b) for $i = 1, 2$, U_i is positive semi-definite, or equivalently has signature

$$\sigma(U_i) = b_2(U_i) + b_1(U_i) - b_3(U_i) - b_2(Y).$$

Proof. In this proof all homology groups will be taken with rational coefficients. First, for $i = 1, 2$, consider the following segment of the long exact sequence in homology of the pair (U_i, Y):

$$0 \to H_3(U_i) \to H_3(U_i, Y) \to H_2(Y) \to H_2(U_i) \xrightarrow{Q} H_2(U_i, Y), \qquad (2.4.1)$$

where Q can be represented by the intersection form matrix with respect to suitable bases. Hence, the null space of Q is precisely the image of $H_2(Y) \to H_2(U_i)$. By exactness and Lefschetz duality, the rank of the map $H_2(Y) \to H_2(U_i)$ is $b_2(Y) - b_1(U_i) + b_3(U_i)$. This gives an upper bound on the signature of U_i:

$$\sigma(U_i) \leq b_2(U_i) + b_1(U_i) - b_3(U_i) - b_2(Y), \tag{2.4.2}$$

with equality if and only if U_i is positive semi-definite. Now consider the segment of the Mayer-Vietoris sequence

$$\begin{aligned} 0 \to & H_3(U_1) \oplus H_3(U_2) \to H_3(X) \to H_2(Y) \to H_2(U_1) \oplus H_2(U_2) \to \\ \to & H_2(X) \to H_1(Y) \to H_1(U_1) \oplus H_1(U_2) \to H_1(X) \to 0. \end{aligned} \tag{2.4.3}$$

The last three terms in this sequence show that

$$b_1(U_1) + b_1(U_2) \leq b_1(Y) + b_1(X), \tag{2.4.4}$$

with equality if and only if the map induced by the inclusions

$$(i_1)_* \oplus (i_2)_* \colon H_1(Y) \to H_1(U_1) \oplus H_1(U_2)$$

is injective.

Since the Euler characteristic of an exact sequence is zero, (2.4.3) shows that

$$b_2(X) = 2b_1(X) + \sum_{i=1}^{2} (b_2(U_i) - b_1(U_i) - b_3(U_i)), \tag{2.4.5}$$

where we also used that $b_1(Y) = b_2(Y)$ and $b_1(X) = b_3(X)$.

By Novikov additivity, we have that $\sigma(X) = \sigma(U_1) + \sigma(U_2)$. So by summing the inequalities in (2.4.2) for $i = 1, 2$ and comparing with (2.4.5) we obtain

$$b_2(X) \geq 2(b_1(X) + b_2(Y) - b_1(U_1) - b_1(U_2)) + \sigma(X), \tag{2.4.6}$$

with equality if and only if we have equality in (2.4.2) for both $i = 1, 2$. Hence, X can be positive definite if and only if

$$b_1(U_1) + b_1(U_2) = b_2(Y) + b_1(X) \tag{2.4.7}$$

and we have equality in (2.4.2) for $i = 1, 2$. However we have already seen that equality occurs in (2.4.4) if and only if $(i_1)_* \oplus (i_2)_*$ is injective. □

This allows us to prove the main theorem.

Theorem 2.1.1. *Let $Y = S^2(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space over S^2 in standard form, that is, with $e \geq 0$, $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \dots, k\}$ and $\varepsilon(Y) \geq 0$. Suppose that Y bounds a smooth 4-manifold W such that $\sigma(W) = -b_2(W)$ and the inclusion induced map $H_1(Y; \mathbb{Q}) \to H_1(W; \mathbb{Q})$ is injective. Then there is a partition of $\{1, 2, \dots, k\}$ into at most e classes such that for each class C,*

$$\sum_{i \in C} \frac{q_i}{p_i} \leq 1.$$

Proof. Let X be the standard positive (semi-)definite plumbing 4-manifold with $\partial X = Y$, and let $Z = X \cup_Y -W$. It follows from Proposition 2.4.1 that Z is positive definite. Condition (b) of Proposition 2.4.1 holds, since

$\sigma(W) = -b_2(W)$ implies that $-W$ is positive definite. Thus, Z is a smooth positive definite 4-manifold, so by Donaldson's theorem Z has standard positive diagonal intersection form. The inclusion $X \subset Z$ induces a map $H_2(X) \to H_2(Z)$ which preserves the intersection pairing. Thus, there is a map of lattices $(H_2(X), Q_X) \to (\mathbb{Z}^m, \mathrm{Id})$ for some $m > 0$.

We construct a partition of $\{1, 2, \ldots, k\}$ into at most e classes as follows. Denote the orthonormal basis of coordinate vectors of $(\mathbb{Z}^m, \mathrm{Id})$ by $\{e_1, \ldots, e_m\}$. For $v \in (H_2(X), Q_X)$, we call $\{e_i : 1 \leq i \leq m, e_i \cdot v \neq 0\}$ the support of v. Without loss of generality, we may assume that the central vertex has support $\{e_1, e_2, \ldots, e_n\}$ where $n \leq e$. Let $v_1, v_2, \ldots, v_k$ be the vertices of the plumbing adjacent to the central vertex, so that v_i is a vertex belonging to the ith leg of the plumbing graph (with fraction $\frac{p_i}{q_i}$). For $i \in \{1, \ldots, n\}$, let $B_i = \{1 \leq j \leq k \mid v_j \cdot e_i \neq 0\}$ and define $B_0 = \emptyset$. Let $C_i = B_i \setminus \cup_{j<i} B_j$ for $i \in \{1, \ldots, n\}$. Then $C_1, \ldots, C_n$ are disjoint and $\cup_i C_i = \{1, \ldots, k\}$. Thus the non-empty classes $\{C_i : C_i \neq \emptyset\}$ form a partition of $\{1, 2, \ldots, k\}$ into at most e classes. By definition for each $i \in \{1, 2, \ldots, n\}$, the starting vertices of the linear chains indexed by C_i all have support containing the common unit vector e_i. Hence, by Theorem 2.3.2, we have that $\sum_{j \in C_i} \frac{q_j}{p_j} \leq 1$. $\square$

2.5 Neumann-Zagier's question

We prove Theorem 2.5.1 below which, when combined with Donaldson's theorem, immediately implies Theorem 2.1.2. Note that the following theorem also positively answers Neumann-Zagier's question stated in the introduction.

Theorem 2.5.1. *Let $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$, $k \geq 3$, be in standard form, that is, with $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \ldots, k\}$, $e > 0$ and with Y bounding a smooth positive definite plumbing X. Suppose that $|H_1(Y)| \in \{1, 2, 3, 5, 6, 7\}$ and the intersection lattice $(H_2(X), Q_X)$ embeds into a positive standard diagonal lattice. Then $e = 1$.*

Proof of Theorem 2.5.1. For sake of contradiction, assume that $e > 1$. We may apply Theorem 2.1.1, noting that the existence of W in the hypothesis of Theorem 2.1.1 is only required to ensure that there is a map of lattices of $(H_2(X), Q_X)$ into a positive standard diagonal lattice. Hence, there is a partition $\{C_1, \ldots, C_n\}$ of $\{1, \ldots, k\}$ into $n \leq e$ classes. Moreover, for each class C, $1 - \sum_{i \in C} \frac{q_i}{p_i} \geq 0$, and we call C *complementary* if equality occurs, and *non-complementary* otherwise.

We have

$$\begin{aligned} |H_1(Y)| &= p_1 \cdots p_k \cdot \varepsilon(Y) = p_1 \cdots p_k (e - \sum_{i=1}^{k} \frac{q_i}{p_i}) \\ &= p_1 \cdots p_k \left((e - n) + \sum_{i=1}^{n} (1 - \sum_{j \in C_i} \frac{q_j}{p_j}) \right) \\ &= p_1 \cdots p_k (e - n) + \sum_{i=1}^{n} a_i \prod_{\substack{1 \leq l \leq k \\ l \notin C_i}} p_l, \end{aligned} \tag{2.5.1}$$

where $a_i = (\prod_{j \in C_i} p_j) \cdot (1 - \sum_{l \in C_i} \frac{q_l}{p_l})$ is an integer for all $i \in \{1, \ldots, n\}$. Notice that all terms in (2.5.1) are non-negative integers. Since we are assuming that $|H_1(Y)| \in \{1, 2, 3, 5, 6, 7\}$, we must have $n = e$, otherwise $|H_1(Y)| \geq p_1 \cdots p_k \geq 2 \cdot 2 \cdot 2 = 8$ since $k \geq 3$.

We claim that $|C_i| \leq k-2$ for some $i \in \{1,\dots,e\}$ with C_i non-complementary. To see this, we argue as follows. There are $n = e \geq 2$ classes in the partition, and at least one non-complementary class since $|H_1(Y)| > 0$. If there are two non-complementary classes then at least one has size at most $k-2$ since $k \geq 3$. If there is only one non-complementary class, then there is a complementary class which necessarily has size at least 2, and hence the non-complementary class satisfies the claim.

Combining the above claim with (2.5.1), we see that $|H_1(Y)|$ is a sum of integers greater than 1, and at least one of these integers is not prime. For $|H_1(Y)| \in \{1,2,3,5,6,7\}$, this is only possible for $|H_1(Y)| = 7$ with decomposition $7 = 3 + 2 \cdot 2$, and for $|H_1(Y)| = 6$ with the two decompositions $|H_1(Y)| = 2 + 2 \cdot 2 = 2 \cdot 3$. We address these cases in turn. For $|H_1(Y)| = 7 = 3 + 2 \cdot 2$, comparing this decomposition with (2.5.1), we see that there must exist some non-complementary C_i with $|C_i| = 2$ and $p_j = 2$ for all $j \in C_i$. However, such a C_i must be complementary since $1 - \frac{1}{2} - \frac{1}{2} = 0$, a contradiction. A similar argument rules out the decomposition $|H_1(Y)| = 2 + 2 \cdot 2$. Finally, in the case $|H_1(Y)| = 2 \cdot 3$, the decomposition implies that there exists a complementary class $C_i = \{a, b\}$ with $p_a = 2$ and $p_b = 3$, which is impossible.

$\square$

We obtain the following corollary, answering a question of Lidman-Tweedy [LT18, Remark 4.3].

Corollary 2.1.3. *Let Y be a Seifert fibered integral homology sphere, and let $e \in \mathbb{Z}$ be the central weight in the standard definite plumbing graph for Y. If $|e| \neq 1$, then $d(Y) \neq 0$.*

Proof. We prove the contrapositive. Assume that $d(Y) = 0$. Note that reversing the orientation of Y simply changes the sign of the weight of the central vertex in the definite plumbing bounding Y. Thus, by reversing the orientation of Y if necessary we assume that Y bounds a smooth negative definite plumbing X^4. Let $\mathcal{C} = \{\xi \in H_2(X;\mathbb{Z}) \mid \xi \cdot v = v \cdot v \pmod 2 \text{ for all } v \in H_2(X;\mathbb{Z})\}$ be the set of characteristic vectors, and let $n = \mathrm{rk}(H_2(X))$. Elkies [Elk95] proved that $0 \leq n + \max_{\xi \in \mathcal{C}} \xi \cdot \xi$, with equality if and only if Q_X is diagonalizable over $\mathbb{Z}$. However, it follows from [OS03, Theorem 9.6] that $n + \max_{\xi \in \mathcal{C}} \xi \cdot \xi \leq 4d(Y) = 0$. Therefore Q_X is diagonalizable over $\mathbb{Z}$, in particular $(H_2(-X), Q_{-X})$ embeds into a positive standard diagonal lattice. Hence, Theorem 2.5.1 implies that $|e| = 1$. □

2.6 Seifert fibered spaces bounding rational homology $S^1 \times D^3$'s

In this section we prove Theorem 2.1.4, which in particular gives a classification of the Seifert fibered spaces which smoothly bound rational homology $S^1 \times D^3$'s. We note that the implication (2) implies (1) was proved by Donald [Don15, Proof of Theorem 1.3], and the equivalence of (1) and (2) was shown by Aceto [Ace15, Theorem 1.2].

Theorem 2.1.4. *Let Y be a Seifert fibered space over S^2 with $H_*(Y;\mathbb{Q}) \cong H_*(S^1 \times S^2;\mathbb{Q})$. The following are equivalent:*

1. *Y is of the form $S^2(k; \frac{p_1}{q_1}, \frac{p_1}{p_1-q_1}, \dots, \frac{p_k}{q_k}, \frac{p_k}{p_k-q_k})$, where $k \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \dots, k\}$.*

2. *$Y = \partial W$, where W is a smooth 4-manifold with $H_*(W;\mathbb{Q}) \cong H_*(S^1 \times D^3;\mathbb{Q})$.*

3. *Y is the boundary of smooth 4-manifolds W_+ and W_- such that $\sigma(W_\pm) = \pm b_2(W_\pm)$ and each of the inclusion-induced maps $H_1(Y;\mathbb{Q}) \to H_1(W_\pm;\mathbb{Q})$ is injective.*

Proof. First suppose that (1) holds, that is, $Y = S^2(k; \frac{p_1}{q_1}, \frac{p_1}{p_1-q_1}, \dots, \frac{p_k}{q_k}, \frac{p_k}{p_k-q_k})$, where $k \geq 0$ and $\frac{p_i}{q_i} \in \mathbb{Q}_{>1}$ for all $i \in \{1, \dots, k\}$. By Rolfsen twisting, Y can be put into the form $S^2(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}, -\frac{p_k}{q_k})$. Let $M = S^2(0; \frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_k}{q_k})$, let M° be the 3-manifold with torus boundary given by removing a tubular neighbourhood of a regular fiber of M and let $W = M \times [0,1]$. Then $\partial W = M^\circ \cup_\partial -M^\circ$ is the double of M°, which is precisely Y. Finally, notice that $H_*(W;\mathbb{Q}) = H_*(M^\circ;\mathbb{Q}) = H_*(S^1 \times D^3;\mathbb{Q})$, where the last equality follows from the fact that M is a rational homology S^3 and M° is obtained by removing a neighbourhood of a simple closed curve from M. This proves (2).

The implication (2) implies (3) holds by taking $W_\pm = W$ and noting that $H_1(Y;\mathbb{Q}) \to H_1(W;\mathbb{Q})$ is injective by the long exact sequence of the pair (Y, W).

Finally assume that (3) holds. Hence, Y is the boundary of smooth 4-manifolds W_+ and W_- satisfying $\sigma(W_\pm) = \pm b_2(W_\pm)$ and such that the inclusion induced maps $H_1(Y) \to H_1(W_\pm)$ are injective. Write Y as $S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ with $k \geq 3$ and $\frac{p_i}{q_i} \in \mathbb{Q}_{>1}$ for all $i \in \{1, \ldots, k\}$. Notice that $-Y = S^2(k - e; \frac{p_1}{p_1-q_1}, \ldots, \frac{p_k}{p_k-q_k})$, and Y is of the form given in (1) if and only if $-Y$ is of this form. Thus, by reversing the orientations of both Y and $W_\pm$ if necessary, we may assume that $e \geq \frac{k}{2}$.

By Theorem 2.1.1, there is a partition $\{C_1, \ldots, C_n\}$ of $\{1, \ldots, k\}$ into $n \leq e$ classes such that for each class C, $1 - \sum_{i \in C} \frac{q_i}{p_i} \geq 0$. Since Y is a rational homology $S^1 \times S^2$, we thus have

$$\begin{aligned} 0 = p_1 \cdots p_k \cdot \varepsilon(Y) &= p_1 \cdots p_k (e - \sum_{i=1}^{k} \frac{q_i}{p_i}) \\ &= p_1 \cdots p_k \left((e-n) + \sum_{i=1}^{n} (1 - \sum_{j \in C_i} \frac{q_j}{p_j}) \right), \end{aligned}$$

where all terms in the sum are non-negative. Hence, we must have $n = e$ and $1 - \sum_{i \in C} \frac{q_i}{p_i} = 0$, for all $i \in \{1, \ldots, n\}$. This implies that $|C_i| \geq 2$ for all $i \in \{1, \ldots, n\}$. Thus, there are at least $2n = 2e$ legs, so $e \leq \frac{k}{2}$. However, by assumption $e \geq \frac{k}{2}$ so $e = \frac{k}{2}$ and $|C_i| = 2$ for all $i \in \{1, \ldots, n\}$. Thus, $C_1, \ldots, C_n$ partition $\{1, \ldots, k\}$ into pairs of indices indexing pairs of fractions of the form $\frac{p}{q}, \frac{p}{p-q} \in \mathbb{Q}_{>1}$, and thus (1) holds. $\square$

Chapter 3

The classification of quasi-alternating Montesinos links[1]

3.1 Introduction

Quasi-alternating links were defined by Ozsváth-Szabó [OS05, Definition 3.1] as a natural generalisation of the class of alternating links.

Definition 3.1.1. *The set $\mathcal{Q}$ of quasi-alternating links is the smallest set of links satisfying the following:*

1. *The unknot U belongs to $\mathcal{Q}$.*
2. *If L is a link with a diagram containing a crossing c such that*
   (a) *both smoothings L_0 and L_1 of the link L at the crossing c, as in Figure 3.1, belong to $\mathcal{Q}$,*
   (b) *$det(L_0), det(L_1) \geq 1$, and*
   (c) *$det(L) = det(L_0) + det(L_1)$,*

   then L is in $\mathcal{Q}$. The crossing c is called a quasi-alternating crossing.

[1]This chapter is based on the paper: Ahmad Issa. *The classification of quasi-alternating Montesinos links.* Proc. Amer. Math. Soc., 146(9):4047–4057, 2018. The contents appear here by kind permission by the publisher.

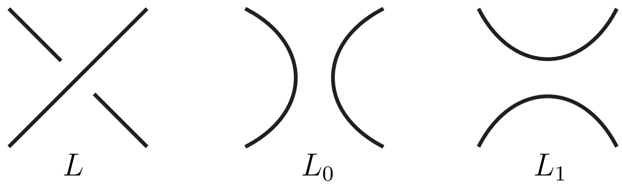

Figure 3.1: L and its two resolutions L_0 and L_1 in a neighbourhood of c.

Ozsváth-Szabó showed that the class of non-split alternating links is contained in $\mathcal{Q}$ [OS05, Lemma 3.2]. Moreover, quasi-alternating links share a number of properties with alternating links, we list a few of these. For a quasi-alternating link L:

(i) L is homologically thin for both Khovanov homology and knot Floer homology [MO08].

(ii) The double branched cover $\Sigma(L)$ of L is an L-space [OS05, Proposition 3.3].

(iii) The 3-manifold $\Sigma(L)$ bounds a smooth negative definite 4-manifold W with $H_1(W) = 0$ [OS05, Proof of Lemma 3.6].

For some further properties see [LO15], [QC15], [Ter15] and [ORS13, Remark after Proposition 5.2].

Due to their recursive definition, it is difficult in general to determine whether or not a link is quasi-alternating. For example, there still remain examples of 12-crossing knots with unknown quasi-alternating status [Jab14]. Champanerkar-Kofman [CK09] showed that the quasi-alternating property is preserved by replacing a quasi-alternating crossing with an alternating rational tangle extending the crossing. They used this to determine an infinite family of quasi-alternating pretzel links, which Greene later showed is the complete set of quasi-alternating pretzel links [Gre10].

Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerkar-Ording [CO15] independently generalised the sufficient conditions on pretzel links to obtain an infinite family of quasi-alternating Montesinos links. This family includes all examples of quasi-alternating Montesinos links found by Widmer [Wid09]. Furthermore, it was conjectured by Qazaqzeh-Chbili-Qublan that this family is the complete set of quasi-alternating Montesinos links. We mention that Watson [Wat11] gave an iterative surgical construction for constructing all quasi-alternating Montesinos links.

Some necessary conditions to be quasi-alternating in terms of the rational parameters of a Montesinos link were obtained in [QCQ15] and [CO15] based on the fact that a quasi-alternating link is homologically thin. Further conditions are described in [CO15] coming from the fact that the double branched cover of a quasi-alternating link is an L-space. Some additional restrictions were found in [QC15].

Our main result is the following theorem which states that the quasi-alternating Montesinos links are precisely those found by Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerkar-Ording [CO15]. See Figure 3.2 for our conventions on Montesinos links.

Theorem 3.1.2. *Let $L = M(e; t_1, \ldots, t_p)$ be a Montesinos link in standard form, that is, where $t_i = \frac{\alpha_i}{\beta_i} > 1$ and $\alpha_i, \beta_i > 0$ are coprime for all $i = 1, \ldots, p$. Then L is quasi-alternating if and only if*

(a) $e < 1$, *or*

(b) $e = 1$ *and* $\frac{\beta_i}{\alpha_i} + \frac{\beta_j}{\alpha_j} > 1$ *for some* i, j *with* $i \neq j$, *or*

(c) $e > p - 1$, *or*

(d) $e = p - 1$ *and* $\frac{\beta_i}{\alpha_i} + \frac{\beta_j}{\alpha_j} < 1$ *for some* i, j *with* $i \neq j$.

As a corollary of our proof we obtain the following characterisation of the Montesinos links L which are quasi-alternating in terms of their double branched covers $\Sigma(L)$:

Corollary 3.1.3. *A Montesinos link L is quasi-alternating if and only if*

(a) $\Sigma(L)$ *is an L-space, and*

(b) there exist a smooth negative definite 4*-manifold* W_1 *and a smooth positive definite* 4*-manifold* W_2 *with* $\partial W_i = \Sigma(L)$ *and* $H_1(W_i)$ *torsion-free for* $i = 1, 2$.

Note that in Corollary 3.1.3 and throughout this chapter, we assume all homology groups have $\mathbb{Z}$ coefficients.

In light of this corollary, Theorem 3.1.2 can also be seen as a classification of the L-space Seifert fibered spaces over S^2 which bound both positive and negative definite 4-manifolds with torsion-free first homology. To what extent Corollary 3.1.3 generalises to non-Montesinos links remains an interesting question.

This work also gives a classification of the Seifert fibered space formal L-spaces. The notion of a formal L-space was defined by Greene and Levine [GL16] as a 3-manifold analogue of quasi-alternating links. In fact, the double branched cover of a quasi-alternating link is an example of a formal L-space. In [LS17], Lidman and Sivek classified the quasi-alternating links of determinant at most 7. In fact, they show that the formal L-spaces M^3 with $|H_1(M)| \leq 7$ are precisely the double branched covers of quasi-alternating links with determinant at most 7. In this same direction, as a consequence of Corollary 3.1.3, we have the following.

Corollary 3.1.4. *A Seifert fibered space over S^2 is a formal L-space if and only if it is the double branched cover of a quasi-alternating link.*

Corollary 3.1.3 also seems significant given the recent independent characterisations of alternating knots by Greene [Gre17] and Howie [How17]. A non-split link is alternating if and only if it bounds negative definite and positive definite spanning surfaces (which are the checkerboard surfaces). The double branched cover of B^4 over such a surface is a definite 4-manifold of the appropriate sign. Generalising this, a quasi-alternating link has the property that it bounds a pair of surfaces in B^4 with double branched covers a positive definite and a negative definite simply connected 4-manifold (these surfaces cannot be embedded in S^3 in general). Corollary 3.1.3 shows that among Montesinos links with double branched covers which are L-spaces, this property characterises those which are quasi-alternating.

Our approach to proving Theorem 3.1.2 follows that of Greene [Gre10] on the determination of quasi-alternating pretzel links. One of Greene's main strategies is as follows. Suppose L is a quasi-alternating Montesinos link such that $\Sigma(L)$ is the oriented boundary of the standard negative definite plumbing X^4. Since the property of being quasi-alternating is closed under reflection, by property ((iii)) above, $-\Sigma(L) = \Sigma(\overline{L})$ bounds a negative definite 4-manifold W with $H_1(W) = 0$. By Donaldson's theorem [Don87], the smooth closed negative definite 4-manifold $X \cup W$ has diagonalisable intersection form. Hence, $H_2(X)/\mathrm{Tors} \hookrightarrow H_2(X \cup W)/\mathrm{Tors}$ is an embedding of the intersection lattice of X into the standard negative diagonal lattice. Moreover, using that $H_1(W)$ is torsion-free, it is shown that if A is an integer matrix representing the lattice embedding then A^T must be surjective over the integers.

When L is a pretzel link of a certain form, Greene analyses the possible embeddings of the intersection lattice of X into a negative diagonal lattice and shows that the aforementioned surjectivity condition cannot hold, and hence the link cannot be quasi-alternating. We analyse this surjectivity condition more generally and prove Theorem 3.1.5 below, which provides a strengthening of Theorem 2.1.1. In particular Theorem 3.1.5 when applied to $Y = \Sigma(\overline{L})$, combined with the condition that Y is an L-space, leads to the precise necessary conditions to complete the determination of quasi-alternating Montesinos links. Finally, we remark that our analysis of the surjectivity condition is rather different to, and much more general than the original argument given in the published article [Iss18] this chapter is based on. The original argument relied on some lattice embedding results by Lecuona-Lisca [LL11]. See Section 2.2 for our conventions on Seifert fibered spaces in Theorem 3.1.5 below.

Theorem 3.1.5. *Let* $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ *be a rational homology sphere Seifert fibered space over* S^2 *with* $e > 0$, $\frac{p_i}{q_i} > 1$ *for all* $i \in \{1, 2, \ldots, k\}$ *and* $\varepsilon(Y) := e - \sum_{i=1}^{k} \frac{q_i}{p_i} > 0$. *Suppose that* Y *bounds a negative definite smooth* 4*-manifold* W *with* $H_1(W)$ *torsion-free. Then there is a partition* P *of* $\{1, 2, \ldots, k\}$ *into at most* e *classes such that for each class* $C \in P$,

$$\sum_{i \in C} \frac{q_i}{p_i} < 1.$$

3.2 Preliminaries

We briefly recall some material on Montesinos links and plumbings. See [CO15] or [BZH14] for further detail on Montesinos links, and [NR78] for more on plumbings. The Montesinos link $M(e; t_1, \ldots, t_p)$, where $t_i = \frac{\alpha_i}{\beta_i} \in \mathbb{Q}$ with $\alpha_i > 1$ and β_i coprime integers, and e is an integer, is given by the diagram in Figure 3.2. In the figure, each box labelled t_i represents the corresponding rational tangle. The 0 rational tangle is shown in Figure 3.3. Introducing an additional positive (resp. negative) half-twist to the bottom of an a/b rational tangle produces a rational tangle represented by $a/b + 1$ (resp. $a/b - 1$), see Figure 3.3. Rotating (in either direction) a rational tangle represented by

$t \in \mathbb{Q} \cup \{1/0\}$ by $\pi/2$ produces the rational tangle represented by $-1/t$. The rational tangle represented by any $a/b \in \mathbb{Q} \cup \{1/0\}$ can be obtained from the 0 rational tangle by a sequence of these two operations. See [Cro04] for a more thorough treatment of rational links. Note however that an a/b rational tangle with our conventions corresponds to a b/a rational tangle in [Cro04].

We also note that with our conventions for a Montesinos link $M(e; t_1, \ldots, t_p)$, the integer e has opposite sign to that used by Champanerkar-Ording [CO15], and agrees with that of Qazaqzeh-Chbili-Qublan [QCQ15] and Greene [Gre10].

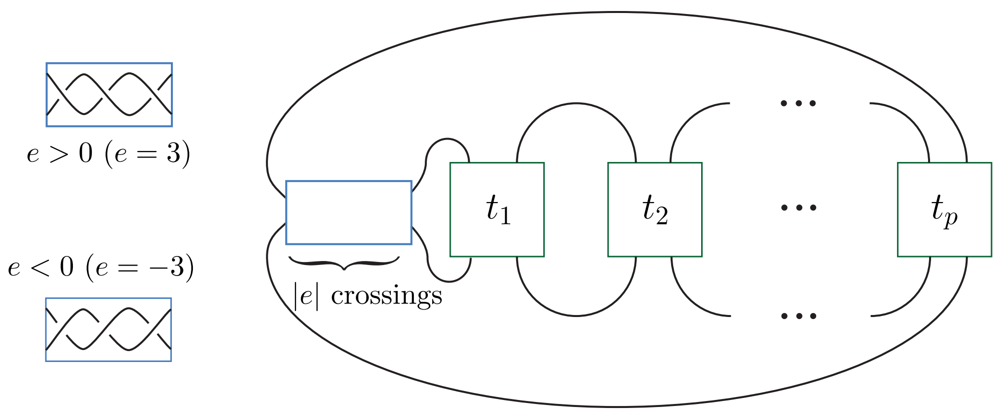

Figure 3.2: The Montesinos link $M(e; t_1, \ldots, t_p)$ where a box labelled t_i represents a rational tangle corresponding to t_i. The crossing type of the $|e|$ crossings depends on the sign of e, with the two possibilities shown on the left.

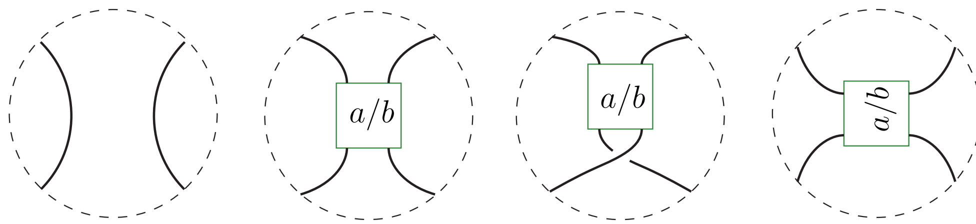

Figure 3.3: From left to right: the 0 rational tangle, an abstract representation of a a/b rational tangle, the $\frac{a}{b}+1$ rational tangle, and the $-b/a$ rational tangle.

The double branched cover of $M(e; t_1, \ldots, t_p)$ is the Seifert fibered space $Y := S^2(e; t_1, \ldots, t_p)$, cf. Section 2.2. By [Sav02, Section 1.2.3], we have that $\det(L) = |\mathrm{H}_1(Y)| = \left|\alpha_1 \ldots \alpha_p \left(e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i}\right)\right|$, where $|\mathrm{H}_1(Y)| = 0$ means that the homology group is infinite.

The Montesinos link $M(e; t_1, \ldots, t_p)$ is isotopic to $M(e+1; t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_p)$ where $t_i' = \frac{\alpha_i}{\beta_i + \alpha_i}$, and is also isotopic to $M(e-1; t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_p)$, where $t_i' = \frac{\alpha_i}{\beta_i - \alpha_i}$. Hence, a Montesinos link is isotopic to one in *standard form*, that is, of the form $M(e; t_1, \ldots, t_p)$ where $t_i > 1$ for all i.

Let $L = M(e; t_1, \ldots, t_p)$ where $t_i < -1$ for all i. Note that any Montesinos link can be put into this form. For each i, there is a unique continued fraction expansion

$$t_i = [a_1^i, \ldots, a_{h_i}^i] := a_1^i - \cfrac{1}{a_2^i - \cfrac{1}{\ddots \, a_{h_i-1}^i - \cfrac{1}{a_{h_i}^i}}},$$

where $h_i \geq 1$ and $a_j^i \leq -2$ for all $j \in \{1, \ldots, h_i\}$.

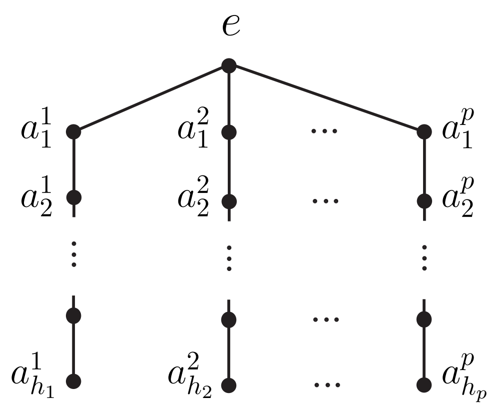

Figure 3.4: The (negative) weighted star-shaped plumbing graph Γ.

The double branched cover $\Sigma(L)$ of L is the oriented boundary of the 4-dimensional plumbing X_Γ of D^2-bundles over S^2 described by the weighted star-shaped graph Γ shown in Figure 3.4. We call Γ the standard negative star-shaped plumbing graph for L. The ith leg of Γ corresponding to t_i is the linear subgraph generated by the vertices labelled with weights $a_1^i, \ldots, a_{h_i}^i$. The degree p vertex labelled with weight e is called the central vertex. Denote the vertices of Γ by $v_1, v_2, \ldots, v_k$. The zero-sections of the D^2-bundles over S^2 corresponding to each of $v_1, \ldots, v_k$ in the plumbing together form a natural spherical basis for $H_2(X_\Gamma)$. With respect to this basis, the intersection form of X_Γ is given by the weighted adjacency matrix Q_Γ with entries Q_{ij}, $1 \leq i, j \leq k$ given by

$$Q_{ij} = \begin{cases} \mathrm{w}(v_i), & \text{if } i = j \\ 1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge ,} \\ 0, & \text{otherwise} \end{cases}$$

where $\mathrm{w}(v_i)$ is the weight of vertex v_i. We call $(\mathbb{Z}^k, Q_\Gamma)$ the intersection lattice of X_Γ (or of Γ).

3.3 Results

Equivalent sufficient conditions for a Montesinos link to be quasi-alternating were given in [CO15, Theorem 5.3] and [QCQ15, Theorem 3.5]. The goal of this section is to prove Theorem 3.1.2 which states that these sufficient conditions for a Montesinos link to be quasi-alternating are also necessary conditions.

Lemma 3.3.1. *Let $L = M(e; t_1, \ldots, t_p)$, $p \geq 3$, be a Montesinos link in standard form, i.e. where $t_i = \frac{\alpha_i}{\beta_i} > 1$ and $\alpha_i, \beta_i > 0$ are coprime for all i. Suppose that $e \leq p-2$ and $e - \sum_{i=1}^{p} \frac{1}{t_i} > 0$ (in particular $e \geq 1$). Then the double branched cover $\Sigma(L)$ is not an L-space, and therefore L is not quasi-alternating.*

Proof. The reflection of L is given by $\overline{L} = M(e'; t'_1, \ldots, t'_p) = M(-e; -t_1, \ldots, -t_p)$. The space $\Sigma(\overline{L})$ is the oriented boundary of a plumbing X_Γ corresponding to the standard star-shaped plumbing graph Γ for $\overline{L}$. Since $e' - \sum_{i=1}^{p} \frac{1}{t'_i} = -\left(e - \sum_{i=1}^{p} \frac{1}{t_i}\right) < 0$, by [NR78, Theorem 5.2], X_Γ has negative definite intersection form.

Since X_Γ is negative definite and Γ is almost-rational, by [Ném05, Theorem 6.3] we have that $\Sigma(\overline{L})$ is an L-space if and only if X_Γ is a rational surface singularity (more generally, see [Ném15]). Note that Γ is almost-rational since by sufficiently decreasing the weight of the central vertex we obtain a plumbing graph satisfying $-\mathrm{w}(v) \geq \deg(v)$ for all vertices v, where $\mathrm{w}(v)$ denotes the weight of v, and such a graph is rational (for details see [Ném05, Example 8.2(3)]).

Laufer's algorithm [Lau72, Section 4] can be used to determine whether the negative definite plumbing X_Γ is a rational surface singularity as follows. Let $v_1, \ldots, v_k$ be the vertices of Γ and for $i \in \{1, \ldots, k\}$, let $[\Sigma_{v_i}] \in H_2(X_\Gamma)$ be the spherical class naturally associated to v_i. The algorithm is as follows (see [Sti08, Section 3] for a similar formulation).

1. Let $K_0 = \sum_{i=1}^k [\Sigma_{v_i}] \in H_2(X_\Gamma)$.

2. In the ith step, consider the pairings $\langle PD[K_i], [\Sigma_{v_j}]\rangle$, for $j \in \{1, \ldots, k\}$. Note that these pairings may be evaluated using the adjacency matrix Q. If for some j the pairing is at least 2 then the algorithm stops and X_Γ is not a rational surface singularity. If for some j, the pairing is equal to 1, then set $K_{i+1} = K_i + [\Sigma_{v_j}]$ and go to the next step. Otherwise all pairings are non-positive, the algorithm stops and X_Γ is a rational surface singularity.

Applying Laufer's algorithm to X_Γ, we claim that the algorithm terminates at the 0^{th} step. To see this, note that for v the central vertex of Γ, $\langle PD[K_0], [\Sigma_v]\rangle = p - e$ (each vertex adjacent to v contributes 1, the central vertex contributes $-e$). By assumption $e \leq p-2$ so $\langle PD[K_0], [\Sigma_v]\rangle = p-e \geq 2$. Hence, the algorithm terminates, we conclude that X_Γ is not a rational surface singularity and hence $\Sigma(\overline{L})$ is not an L-space. Therefore $\Sigma(L)$ is not an L-space. $\square$

The following lemma will provide an obstruction to a Montesinos link being quasi-alternating.

Lemma 3.3.2 ([Gre10, Lemma 2.1])**.** *Suppose that X and W are a pair of 4-manifolds, $\partial X = -\partial W = Y$ is a rational homology sphere, and $H_1(W)$ is torsion-free. Express the map $H_2(X)/Tors \to H_2(X \cup W)/Tors$ with respect to a pair of bases by the matrix A. This map is an inclusion, and A^T is surjective.*

We now use Lemma 3.3.2 to prove the following strengthening of Theorem 2.1.1.

Theorem 3.1.5. *Let $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a rational homology sphere Seifert fibered space over S^2 with $e > 0$, $\frac{p_i}{q_i} > 1$ for all $i \in \{1, 2, \ldots, k\}$ and $\varepsilon(Y) := e - \sum_{i=1}^{k} \frac{q_i}{p_i} > 0$. Suppose that Y bounds a negative definite smooth 4-manifold W with $H_1(W)$ torsion-free. Then there is a partition P of $\{1, 2, \ldots, k\}$ into at most e classes such that for each class $C \in P$,*

$$\sum_{i \in C} \frac{q_i}{p_i} < 1.$$

Proof. Note that Theorem 2.1.1 implies the weaker conclusion with the inequality in Theorem 3.1.5 not strict. We will use that $H_1(W)$ torsion-free to rule out the equality case. For what follows it may be helpful for the reader to recall standard facts about Seifert fibered spaces from Section 2.2.

Let $X = X_\Gamma$ be the standard positive definite plumbing 4-manifold with $\partial X = Y$, where Γ is the (positive) weighted star-shaped graph. We recall the construction of the partition of $\{1, \ldots, k\}$ from the proof of Theorem 2.1.1. Inclusion $X \to X \cup_Y -W$ induces a map of lattices $\iota : (H_2(X), Q_X) \to (\mathbb{Z}^m, \mathrm{Id})$ for some $m > 0$, where we have used Donaldson's theorem. We may assume that the (image of the) central vertex is supported in the coordinate vectors $e_1, \ldots, e_n \in \mathbb{Z}^m$ for some $n \leq e$. Let $v_1, v_2, \ldots, v_k$ be the vertices of the plumbing adjacent to the central vertex, so that v_i is a vertex belonging to the ith leg of the plumbing graph (with fraction $\frac{p_i}{q_i}$). For $i \in \{1, \ldots, n\}$, let $B_i = \{1 \leq j \leq k \mid v_j \cdot e_i \neq 0\}$ and define $B_0 = \emptyset$. Let $C_i = B_i \setminus \cup_{j<i} B_j$ for $i \in \{1, \ldots, n\}$ be the "disjointification" of the B_i's. Then $C_1, \ldots, C_n$ are disjoint and $\cup_i C_i = \{1, \ldots, k\}$. Thus the non-empty classes $\{C_i : C_i \neq \emptyset\}$ form a partition of $\{1, 2, \ldots, k\}$ into at most e classes.

Suppose that for some class C, $\sum_{i \in C} \frac{q_i}{p_i} = 1$. We may write the map of lattices ι as an integer matrix. Denote the transpose of this matrix by A, so that the image of vertices of the plumbing are represented by rows of A. By Lemma 3.3.2, $H_1(W)$ torsion-free implies that A is surjective over the integers. Our goal will be to find a non-zero row vector $\overline{x} \in \mathbb{Z}^{|b_2(X)|}$ with coprime integer entries such that $\overline{x}A \equiv 0 \pmod{p}$ for some integer $p > 1$. Supposing the existence of $\overline{x}$ we argue as follows. Since $\overline{x}$ has coprime entries, there exists a column vector $y \in \mathbb{Z}^{|b_2(X)|}$ with $\overline{x}y = 1$. Since A is surjective, $Au = y$ for some $u \in \mathbb{Z}^m$. Hence, $0 \equiv (\overline{x}A)u = \overline{x}y = 1 \pmod{p}$, a contradiction.

We now construct such a vector $\overline{x}$. By construction, there exists some

$e_i \in \mathbb{Z}^m$ such that the starting vertex of each leg of Γ indexed by an element of C pairs non-trivially with e_i. We may assume that $e_i = e_1$. Let B be the matrix with first row $(1, 0, \ldots, 0) \in \mathbb{Z}^m$ and remaining rows given by the rows of A representing vertices of Γ belonging to legs indexed by C. Let r be the number of rows of B. By ordering the vertices of Γ, we may assume that the first r rows of A and B agree except for the first row. Denote the elements of C by $c_1, \ldots, c_{|C|}$. By reordering if necessary, we may assume that $Q_B := BB^T$ takes the form

$$\begin{pmatrix} 1 & w_1^T & \cdots & w_{|C|}^T \\ w_1 & A_1 & & 0 \\ \vdots & & \ddots & \\ w_{|C|} & 0 & & A_{|C|}, \end{pmatrix}$$

where each A_i is the standard intersection form matrix of the c_ith leg of Γ. Since $Q_B = BB^T$ and the first row of B equals $(1, 0, \ldots, 0)$, we get that the first column of Q_B and B agree. Thus, w_i encodes the pairings of e_1 with the vertices in the c_ith leg of Γ. Since $\sum_{i \in C} \frac{q_i}{p_i} = 1$, the equality case of Theorem 2.3.2 applied to the legs indexed by C, together with $w = e_1$, implies that each w_i takes one of two forms:

(1) $w_i = (\pm 1, 0, \ldots, 0)^T$ or,

(2) $w_i = \pm(1, -1, 0, \ldots, 0)^T$,

where form (2) occurs only if the starting vertex of the c_ith leg has weight 2.

Claim. $\det Q_B = 0$.

Proof of Claim. Suppose that w_i has form (1) for all $1 \leq i \leq |C|$. Then Q_B represents the intersection form of a star shaped plumbing graph with central weight 1 and legs given by fractions $\frac{p_{c_1}}{q_{c_1}}, \ldots, \frac{p_{c_{|C|}}}{q_{c_{|C|}}}$. The boundary of this star shaped plumbing is the space $S^2(1; \frac{p_{c_1}}{q_{c_1}}, \ldots, \frac{p_{c_{|C|}}}{q_{c_{|C|}}})$ which has generalised Euler invariant $\varepsilon := 1 - \sum_{i \in C} \frac{q_i}{p_i} = 0$. Since Q_B presents the first homology of this Seifert fibered space, this implies $\det Q_B = 0$. Now suppose that, say, w_1 has form (2) with $w_i = (-1, 1, 0, \ldots, 0)^T$. Consider the following identity obtained by adding the second row to the first row, and the second column to the first column:

$$\det \begin{pmatrix} 1 & -1 & 1 & \cdots & v \\ -1 & 2 & -1 & & 0 \\ 1 & -1 & a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & 0 & & -1 & a_l \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & \cdots & v \\ 1 & 2 & -1 & & 0 \\ 0 & -1 & a_2 & -1 & \\ \vdots & & -1 & \ddots & -1 \\ v & 0 & & -1 & a_l \end{pmatrix},$$

where $p_{c_1}/q_{c_1} = [2, a_2, \ldots, a_l]$, $a_j \geq 2$ for all j. We may use the above identity to modify Q_B, without changing the determinant, so that it is identical to the case where w_1 has form (1). Similarly, if w_1 has form (2) with $w_1 = (1, -1, 0, \ldots, 0)$ we can apply a similar identity (now subtracting the second row from the first row, and second column from the first column). By applying such an identity for each w_i of form (2), the claim then follows by the previous case with w_i of form (1) for all i. □

Motivated by the proof of the above claim, let P be obtained from the $r \times r$ identity matrix by adding ± 1 times row $i + 1$ to the first row for each w_i of form 2 with $w_i = \pm(-1, 1, 0, \ldots, 0)$, and let $Q = P Q_B P^T$. So Q can be thought of as the intersection form matrix of the star-shaped plumbing with central vertex 1 and legs given by fractions $\frac{p_{c_1}}{q_{c_1}}, \ldots, \frac{p_{c_{|C|}}}{q_{c_{|C|}}}$. Then $\det Q = 0$, so there exists a non-zero row vector $x = (x_1, \ldots, x_r) \in \mathbb{Z}^r$ such that $xQ = 0$. By dividing out by $\gcd(x_1, \ldots, x_r)$ we may assume that the entries of x are coprime.

Claim. *The first entry x_1 of x is divisible by an integer $p > 1$.*

First assume the claim. Since $xQ = 0$ we get that $xPQ_BP^Tx^T = 0$ and so $(xPB)(xPB)^T = 0$. Hence, $xPB = 0$. Let $x' = xP$ and write $x' = (x'_1, \ldots, x'_r)$. Since P is upper triangular with 1's on the diagonal, $x'_1 = x_1$ is also divisible by p. Let $\overline{x} = (x'_1, \ldots, x'_r, 0, \ldots, 0) \in \mathbb{Z}^{|\Gamma|}$. Then $\overline{x}A \equiv x'B = 0 \pmod p$, since A and B agree on the first r rows except for the first row, but $x'_1 = 0 \pmod p$. Thus, $\overline{x}$ is the required row-vector which finishes the proof.

Proof of Claim. For simplicity let $\frac{p}{q} = \frac{p_{c_1}}{q_{c_1}}$ and write $\frac{p}{q}$ as the standard (negative) continued fraction expansion $\frac{p}{q} = [a_1, \ldots, a_{\rho-1}]$ where $a_i \geq 2$ are integers for all i. We will show that p divides x_1. By appropriately ordering the vertices when we first defined B, we may assume that the first ρ rows of B are ordered so that the top-left $\rho \times \rho$ submatrix of $Q = PBB^TP^T$ is

$$\begin{pmatrix} 1 & \pm 1 & 0 & 0 & \cdots & 0 & 0 \\ \pm 1 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & a_2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{\rho-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_{\rho-1} \end{pmatrix}.$$

Let $Q' = \left(\begin{array}{ccc} \pm 1 & 0 & \cdots \\ \hline & Q_{p/q} & \end{array}\right)$ be the matrix obtained from the above matrix by removing the first column, where $Q_{p/q}$ is the intersection matrix of the linear chain representing p/q. We have

$$(x_1, \ldots, x_\rho) \cdot Q' = 0,$$

since $xQ = 0$ and the corresponding columns $2, \ldots, \rho$ of Q are supported in the first ρ rows. This implies that $(x_2, \ldots, x_\rho) \cdot Q_{p/q} = (\mp x_1, 0, \ldots, 0)$. Thus, we can change the last row of $Q_{p/q}$ to $(\mp x_1, 0, \ldots, 0)$, by first multiplying the last row of $Q_{p/q}$ by x_ρ, then for each $j \in \{1, \ldots, \rho-1\}$ adding x_j multiples of the jth row to the last row. The determinant of this new matrix is $x_{\rho-1} \cdot \det(Q_{p/q}) = x_{\rho-1} \cdot p$. However, by expanding the determinant along the final row we see that

$$\begin{vmatrix} a_1 & \pm 1 & 0 & 0 & \cdots & 0 & 0 \\ \pm 1 & a_2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & a_3 & -1 & \cdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & a_{\rho-2} & -1 \\ x_1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = \mp x_1.$$

Thus $x_1 = \mp x_{\rho-1} p$ is a multiple of p, proving the claim and completing the proof. □

□

Theorem 3.1.2. *Let $L = M(e; t_1, \ldots, t_p)$ be a Montesinos link in standard form, that is, where $t_i = \frac{\alpha_i}{\beta_i} > 1$ and $\alpha_i, \beta_i > 0$ are coprime for all $i = 1, \ldots, p$. Then L is quasi-alternating if and only if* *(a)* $e < 1$, *or*

(b) $e = 1$ *and* $\frac{\beta_i}{\alpha_i} + \frac{\beta_j}{\alpha_j} > 1$ *for some* i, j *with* $i \neq j$, *or*

(c) $e > p - 1$, *or*

(d) $e = p - 1$ *and* $\frac{\beta_i}{\alpha_i} + \frac{\beta_j}{\alpha_j} < 1$ *for some* i, j *with* $i \neq j$.

Proof. If one of the conditions (a)-(d) is satisfied then L is quasi-alternating by either of [CO15, Theorem 5.3] or [QCQ15, Theorem 3.5]. Now assume that L is quasi-alternating. If $p = 1$ then the conditions are automatically satisfied.

Recall that $\det(L) = \left|\alpha_1 \dots \alpha_p \left(e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i}\right)\right|$. Hence, for $p = 2$, conditions (a)-(d) are equivalent to $\det(L) \neq 0$ which is satisfied since L is quasi-alternating.

For the remainder of the argument we assume that $p \geq 3$. The reflection of L is given by

$$\overline{L} = M\left(-e, -\frac{\alpha_1}{\beta_1}, \dots, -\frac{\alpha_p}{\beta_p}\right) = M\left(p - e, \frac{\alpha_1}{\alpha_1 - \beta_1}, \dots, \frac{\alpha_p}{\alpha_p - \beta_p}\right),$$

where the latter is written in standard form. Notice that L satisfies (b) if and only if its reflection $\overline{L}$ satisfies (d), and similarly with conditions (a) and (c). Thus, conditions (a)-(d) are unchanged by a reflection. Moreover, we see that a reflection reverses the sign of $e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i}$ and thus by a reflection if necessary we may assume that $e - \sum_{i=1}^{p} \frac{\beta_i}{\alpha_i} > 0$ (we cannot have equality since $\det(L) \neq 0$). By Lemma 3.3.1, if $e \leq p - 2$ then $\Sigma(L)$ is not an L-space, so L is not quasi-alternating. Thus it remains to consider when $e = p - 1$.

Let $Y = \Sigma(L) = S^2(p-1; \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_p}{\beta_p})$. If L is quasi-alternating then Y bounds a negative definite manifold W with $H_1(W)$ torsion-free [OS05, Proof of Lemma 3.6]. Theorem 3.1.5 implies that there is a partition P of $\{1, \ldots, p\}$ into at most $p-1$ classes such for each class $C \in P$, $\sum_{i \in C} \frac{\beta_i}{\alpha_i} < 1$. By the pigeonhole principle there must be some class $C \in P$ of size at least two. Let $i, j \in C$ be distinct. Then we in particular have that $\frac{\beta_i}{\alpha_i} + \frac{\beta_j}{\alpha_j} < 1$. Hence, condition (d) is satisfied. □

Corollary 3.1.3. *A Montesinos link L is quasi-alternating if and only if*

(a) $\Sigma(L)$ *is an L-space, and*

(b) there exist a smooth negative definite 4-manifold W_1 and a smooth positive definite 4-manifold W_2 with $\partial W_i = \Sigma(L)$ and $H_1(W_i)$ torsion-free for $i = 1, 2$.

Proof. This is a corollary of the proof of Theorem 3.1.2. Suppose first that L is quasi-alternating. By [OS05, Proposition 3.3], $\Sigma(L)$ is an L-space. Furthermore, $\Sigma(L)$ must bound a negative definite 4-manifold W_1 with $H_1(W_1) = 0$ [OS05, Proof of Lemma 3.6]. Applying this to the reflection of L which is also quasi-alternating, we get that $\Sigma(L)$ also bounds a positive definite 4-manifold W_2 with $H_1(W_2) = 0$. For the converse, note that these two necessary conditions are the only conditions used to obstruct a Montesinos link from being quasi-alternating in the proof of Theorem 3.1.2. □

As a consequence, we obtain a classification of the Seifert fibered spaces which are formal L-spaces. Before stating it, we recall the definition of a formal L-space. We say that a triple (Y_1, Y_2, Y_3) of closed, oriented 3-manifolds form a *triad* if there is a 3-manifold M with torus boundary, and three oriented curves $\gamma_1, \gamma_2, \gamma_3 \subset \partial M$ at pairwise distance 1, such that Y_i is the result of Dehn filling M along γ_i, for $i = 1, 2, 3$.

Definition 3.3.3. *The set $\mathcal{F}$ of formal L-spaces is the smallest set of rational homology 3-spheres such that*

(1) $S^3 \in \mathcal{F}$, and

(2) if (Y, Y_0, Y_1) is a triad with $Y_0, Y_1 \in \mathcal{F}$ and

$$|H_1(Y)| = |H_1(Y_0)| + |H_1(Y_1)|,$$

then $Y \in \mathcal{F}$.

Corollary 3.1.4. *A Seifert fibered space over S^2 is a formal L-space if and only if it is the double branched cover of a quasi-alternating link.*

Proof. Let L be a quasi-alternating Montesinos link. Then the double branched cover of L is a Seifert fibered space over S^2. Ozsváth and Szabó show that the double branched cover of a quasi-alternating link is an L-space [OS05, Proposition 3.3]. Their proof in fact shows that the double branched cover of a quasi-alternating link is a formal L-space. Hence $\Sigma(L)$ is a formal L-space Seifert fibered space over S^2.

Now let M be a formal L-space Seifert fibered space over S^2. Then M is the double branched cover of a Montesinos link L. Ozsváth and Szabó's in [OS05, Proof of Lemma 3.6] show that the double branched cover of a quasi-alternating link bounds both a positive definite, and a negative definite 4-manifold with vanishing first homology. However, their proof in fact shows this for all formal L-spaces. Hence $M = \Sigma(L)$ is a formal L-space bounding positive and negative definite 4-manifolds with vanishing first homology. Thus, Corollary 3.1.3 implies that L is quasi-alternating. $\square$

Chapter 4

Embedding Seifert fibered spaces in 4-space[1]

4.1 Introduction

It is known that every closed 3-manifold smoothly embeds in S^5 [Roh65, Wal65, Hir61]. However, the question of which closed 3-manifolds embed in S^4 is far more subtle. Not every 3-manifold embeds in S^4 and, in fact, the existence of embeddings often depends on whether one is working in the smooth or topological category. The question of which closed orientable 3-manifolds embed in S^4 appears as Problem 3.20 on Kirby's list. Over the years many different techniques and obstructions have been developed to address the question. For example, Hantzsche [Han38] proved that if Y embeds in S^4 then the torsion part of $H_1(Y)$ must split as a direct double, that is, $\operatorname{tor} H_1(Y) \cong G \oplus G$ for some abelian group G. There have also been applications of topological obstructions based on linking forms [Hil09], Casson-Gordon signatures [GL83] and the G-index theorem [CH98], as well as smooth obstructions based on Rokhlin's theorem, the Neumann-Siebenman invariant, Furuta's 10/8 theorem, Donaldson's theorem and the Ozsváth-Szabó d-invariants, see e.g. [BB12]

[1] This chapter is primarily based on the preprint *Smoothly embedding Seifert fibered spaces in* S^4, https://arxiv.org/abs/1810.04770, 2018 which is joint work with Duncan McCoy. Lemma 4.9.2, Proposition 4.9.3 and Proposition 4.9.7 are not part of the aforementioned preprint, and consists entirely of work of my own.

and [Don15]. For a nice introduction to the subject of embedding 3-manifolds in S^4 see [BB12].

In this chapter we study the question of which Seifert fibered spaces over an orientable base surface smoothly embed in S^4. We use $Y = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ to denote the Seifert fibered space over orientable base surface F which is obtained by surgery as in Figure 4.2. After possibly changing orientation, Y may be assumed to be in *standard form*, where $\frac{p_i}{q_i} > 1$ for all i and with non-negative generalized Euler invariant $\varepsilon(Y) := e - \sum_{i=1}^{k} \frac{q_i}{p_i} \geq 0$.[2]

By using an obstruction based on Donaldson's theorem [Don87], we show that if Y embeds smoothly in S^4 then $e \leq \frac{k+1}{2}$ and classify precisely which embed when $e = \frac{k+1}{2}$.

Theorem 4.1.1. *Let $Y = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $\varepsilon(Y) > 0$ and $\frac{p_i}{q_i} > 1$ for all i. If Y embeds smoothly in S^4, then $e \leq \frac{k+1}{2}$. Moreover, if $e = \frac{k+1}{2}$ then Y smoothly embeds in S^4 if and only if Y takes the form*

$$Y = F\left(e; \frac{a}{a-1}, \left\{a, \frac{a}{a-1}\right\}^{\times(e-1)}\right) = F\left(\frac{k+1}{2}; \frac{a}{a-1}, a, \frac{a}{a-1}, a, \dots, \frac{a}{a-1}\right)$$

where $e \geq 1$ and $a \geq 2$ is an integer.

This upper bound is one example of the difference between smooth and topological embeddings. The optimal upper bound for topological embeddings is $e \leq k - 1$ (see Proposition 4.4.6).

[2] With these conventions the Poincaré homology sphere oriented to bound the positive definite E_8 plumbing is $S^2(2; 2, \frac{3}{2}, \frac{5}{4})$.

Classifying which Seifert fibered spaces embed smoothly in S^4 becomes increasingly difficult as e decreases relative to k. For $e = \frac{k}{2}$, we are able to obtain a partial classification.

Theorem 4.1.2. *Let $Y = F(\frac{k}{2}; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $\frac{p_i}{q_i} > 1$ for all i, k even and $\varepsilon(Y) > 0$. If Y smoothly embeds in S^4 then there exist positive integers p, q, r, s with $\frac{p}{q}, \frac{r}{s} > 1$, $(p, q) = (r, s) = 1$ and $\frac{s}{r} + \frac{q}{p} = 1 - \frac{1}{pr}$ such that Y takes the form*

1.
$$Y = F\left(\frac{k}{2}; \frac{p}{q}, \frac{r}{s}, \left\{\frac{p}{p-q}, \frac{p}{q}\right\}^{\geq 0}, \left\{\frac{r}{r-s}, \frac{r}{s}\right\}^{\geq 0}\right), \text{ or}$$

2.
$$Y = F\left(\frac{k}{2}; \frac{p}{q}, \frac{r}{s}, \left\{pr, \frac{pr}{pr-1}\right\}^{\geq 1}, \left\{\frac{p}{q}, \frac{p}{p-q}\right\}^{\geq 0}, \left\{\frac{r}{s}, \frac{r}{r-s}\right\}^{\geq 0}\right).$$

Moreover, in case (1) Y embeds smoothly in S^4. Here the notation $\{\frac{a}{b}, \frac{a}{a-b}\}^{\geq m}$ means that there are at least m pairs of fractions of this form.

Both Theorem 4.1.1 and Theorem 4.1.2 are derived from a more general result, Theorem 4.1.4 below, stating that a Seifert fibered space which smoothly embeds in S^4 must satisfy a strong condition which we call *partitionable.*

Definition 4.1.3. *Let $Y = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $\varepsilon(Y) > 0$ and $\frac{p_i}{q_i} > 1$ for all i. We say that Y is* partitionable *if* tor $H_1(Y) \cong G \oplus G$ *for some finite abelian group G, and there* *exist partitions P_1 and P_2 of $\{1,\ldots,k\}$, each into precisely e classes, such that the following hold. For each partition $P \in \{P_1, P_2\}$:*

(a) There exists a unique class $C \in P$ such that $\displaystyle\sum_{j\in C}\frac{q_j}{p_j} = 1 - \frac{1}{\operatorname{lcm}(p_1,\ldots,p_k)}$.

(b) For each other class $C' \in P$, $\displaystyle\sum_{j\in C'}\frac{q_j}{p_j} = 1$.

(c) No non-empty union of a proper subset of classes in P_1 is equal to a union of classes in P_2.

The classes satisfying condition (b) are said to be complementary.

With this definition in place, the general obstruction we derive from Donaldson's theorem can be stated as the following.

Theorem 4.1.4. *Let $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ with F an orientable surface, $\frac{p_i}{q_i} > 1$ for all i, and $\varepsilon(Y) > 0$. If Y smoothly embeds in S^4 then Y is partitionable.*

In this chapter we focus on Seifert fibered spaces over an orientable base surface and with non-zero generalized Euler invariant, i.e. $\varepsilon \neq 0$. We point out that there are already relatively strong results known when $\varepsilon = 0$ or the base surface is non-orientable. For orientable base surface and $\varepsilon = 0$, Donald showed that if Y smoothly embeds in S^4 then it can be written in the form $Y = F(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_k}{q_k})$ [Don15, Theorem 1.3], see also [Hil09]. Donald also obtained similar results when the base surface is non-orientable [Don15, Theorem 1.2] and further results in the non-orientable case can be found in [CH98].

In the course of applying Theorem 4.1.4 it becomes natural to define an operation on Seifert fibered spaces, which we call *expansion*.

Definition 4.1.5. *Let* $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ *be a Seifert fibered space with* $k \geq 1$. *The Seifert fibered space* Y' *is obtained from* Y *by* expansion *if it takes the form*

$$Y' = F\left(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_j}{q_j}, \frac{p_j}{q_j}\right) = F\left(e+1; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, \frac{p_j}{p_j - q_j}, \frac{p_j}{q_j}\right),$$

for some j *in the range* $1 \leq j \leq k$.

With this definition, notice that the spaces in Theorem 4.1.1 and Theorem 4.1.2 are precisely those obtained by a sequence of expansions from spaces of the form $F(1; \frac{a}{a-1})$, $F(1; \frac{p}{q}, \frac{r}{s})$, or $F(2; \frac{p}{q}, \frac{r}{s}, pr, \frac{pr}{pr-1})$. In fact, we prove these results by showing that whenever e is large relative to k, any space which is partitionable is obtained by expansion from some other space which is also partitionable.

In the opposite direction, the notion of expansion also proves to be useful for constructing embeddings into S^4.

Lemma 4.1.6. *If* Y' *is obtained by expansion from* Y, *then* Y' *smoothly embeds in* $Y \times [0,1]$. *In particular, if* Y *embeds smoothly in* S^4, *then so does* Y'.

This easily shows that the Seifert fibered spaces in Theorem 4.1.1 and Theorem 4.1.2(1) smoothly embed in S^4. Since Seifert fibered spaces of the form $S^2(1; \frac{a}{a-1})$ and $S^2(1; \frac{p}{q}, \frac{r}{s})$, where $\frac{s}{r} + \frac{q}{p} = 1 - \frac{1}{pr}$ and $a > 1$ is an integer, are homeomorphic to S^3, they embed in S^4. When the base surface is S^2 the spaces we wish to embed are precisely those obtained by expansion from these descriptions of S^3, so their embeddings can be constructed via Lemma 4.1.6. The higher genus base surface case follows from this case by a result of Crisp-Hillman [CH98, Lemma 3.2], see Proposition 4.7.2.

The family of Seifert fibered spaces in Theorem 4.1.2(2) which we are unable to completely resolve arises when Y is partitionable with a partition containing a complementary class of size three. When the base surface is $F = S^2$, we have further tools at our disposal, namely the Neumann-Siebenmann invariant $\overline{\mu}$. An analysis of this invariant gives further restrictions.

Proposition 4.1.7. *In Theorem 4.1.2 with $F = S^2$, if the space Y smoothly embeds in S^4 then in family (2) p and r must both odd.*

We conjecture that for Y to smoothly embed, not only must Y be partitionable as in Theorem 4.1.4, but that each complementary class in the partitions must have size two. This would rule out the spaces in Theorem 4.1.2(2) from embedding. It would also imply that if $e > 1$ and Y smoothly embeds in S^4, then Y is necessarily an expansion of a partitionable space (see Lemma 4.6.1(i)). This suggests the following conjecture.

Conjecture 4.1.8. *A Seifert fibered space Y over S^2 with $\varepsilon(Y) > 0$ smoothly embeds in S^4 if and only if it is obtained by a (possibly empty) sequence of* *expansions from some Y' of the form $Y' = S^2(1; \frac{p_1}{q_1}, \dots, \frac{p_l}{q_l})$ with $\frac{p_i}{q_i} > 1$ for all i which also smoothly embeds in S^4.*

Notice that the "if" direction of this conjecture is provided by Lemma 4.1.6. Since expansion preserves the generalized Euler invariant, the space Y' in this conjecture necessarily satisfies $\varepsilon(Y') = \varepsilon(Y) > 0$.

As well as the behaviour in the case $e \geq \frac{k}{2}$ discussed above, we have further evidence for the "only if" direction. We find that expansions naturally arise from the partitionable condition. For example, when $e \geq \frac{2k+3}{5}$ a partitionable space is obtained by expansion from some smaller partitionable space (see Lemma 4.6.1). We also consider the case of Y with all exceptional fibers of even multiplicity. For such spaces the $\overline{\mu}$ invariant is particularly effective and shows that if Y smoothly embeds in S^4, then in the induced partitions there can only be one complementary class of size larger than two and this class has size three (see Proposition 4.8.8). It may be possible that further analysis can rule out the existence a complementary class of size three.

If true, Conjecture 4.1.8 would reduce the problem of which Y (over base surface S^2) smoothly embed in S^4 to the case when $e = 1$, which we now briefly discuss. When $k = 3$ and Y is an integer homology sphere several infinite families of examples, as well as some sporadic examples, are known to bound Mazur manifolds and thus to embed in S^4 [AK79, CH81, FS81, Fic84]. Donald showed that the rational homology sphere $S^2(1; 4, 4, \frac{12}{5})$ smoothly embeds in S^4 [Don15, Example 2.14]. In Section 4.9, we give two further examples of Seifert fibered spaces which embed, namely $S^2(1; \frac{7}{2}, \frac{7}{2}, \frac{7}{2})$ and $S^2(1; 3, \frac{15}{4}, 3)$. Despite the many examples known to embed, a conjectural picture of precisely which Seifert fibered spaces embed in the $k = 3$ case remains unclear. It is an interesting open question whether there exist any examples which embed with $k \geq 4$ and $e = 1$. There appears to be some evidence towards a negative answer to this question, particularly when Y is an integer homology sphere. Note that an integral homology sphere which embeds in S^4 necessarily bounds an acyclic manifold. The question of which Seifert fibered integral homology spheres can bound integral homology balls has arisen in relation to the Montgomery-Yang problem on pseudo-free circle actions on S^5 [FS87]. In this setting it is conjectured that a Seifert fibered homology sphere bounding an acyclic manifold can have at most three exceptional fibers. More recently, consideration of algebraic geometry led Kollár to make a similar conjecture [Kol08, Conjecture 20]. These considerations along with the upper bound from Theorem 4.1.1 lead us to a further conjecture, which in particular, implies a negative answer to the aforementioned question.

Conjecture 4.1.9. *If $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ smoothly embeds in S^4, where $\frac{p_i}{q_i} > 1$ for all i and $\varepsilon(Y) > 0$, then $e \in \{\frac{k+1}{2}, \frac{k}{2}, \frac{k-1}{2}\}$.*

We prove this conjecture in the special case where every exceptional fiber has even multiplicity. More generally, using the Neumann-Siebenmann invariant we prove a lower bound on e, which complements the upper bound given in Theorem 4.1.1.

Theorem 4.1.10. *Let $Y = S^2(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space with $\varepsilon(Y) > 0$ and $\frac{p_i}{q_i} > 1$ for all i. If Y smoothly embeds in S^4 then $\dim H^1(Y; \mathbb{Z}_2) \leq 2e$.*

If p_i is even for precisely $N \geq 1$ different values of $i \in \{1, \dots, k\}$, then $\dim H^1(Y; \mathbb{Z}_2) = N - 1$ (see Lemma 4.4.5). So when p_i is even for all i, Theorem 4.1.10 provides the lower bound $e \geq \frac{k-1}{2}$ as stipulated by Conjecture 4.1.9.

It is worth noting that the obstructions considered in this chapter use only the fact that S^4 is an integer homology sphere. So all of our results could be be restated in terms of Seifert fibered spaces embedding in integer homology 4-spheres. It is an interesting open question whether there is a 3-manifold which does not embed in S^4, but does embed in some other integer homology sphere.

In another direction, we give a construction for doubly slice links analogous to the "expansion" construction in Lemma 4.1.6 (see Proposition 4.9.3), which in particular shows that the Seifert fibered spaces over S^2 in Theorem 4.1.1 and Theorem 4.1.2(1) are double branched covers of doubly slice Montesinos links. A link in S^3 is (smoothly) doubly slice if it arises as the cross-section of an unknotted smoothly embedded 2-sphere in S^4. It is an easy consequence of this definition that the double branched cover of a doubly slice link smoothly embeds in S^4.

Note, however, that not every embedding of Seifert fibered spaces can arise in this manner. The integer homology sphere $S^2(1; 3, \frac{5}{2}, \frac{34}{9})$ bounds a Mazur manifold and therefore smoothly embeds in S^4 [Fic84]. However, it is the double branched cover of precisely one Montesinos knot, and this knot is not doubly slice (in fact, it is not even slice as it fails the Fox-Milnor condition).

As a consequence of Theorem 4.1.1 and these constructions of doubly slice links, we obtain a classification of the smoothly doubly slice odd pretzel knots up to mutation. An odd pretzel knot is one of the form $P(c_1, \ldots, c_n)$, where the c_i are odd integers, see Figure 4.17.

Theorem 4.1.11. *If K is an odd pretzel knot, then the following are equivalent:*

(i) *$\Sigma(K)$ embeds smoothly in S^4,*

(ii) *K is a mutant of a smoothly doubly slice odd pretzel knot,*

(iii) *and K is a mutant of $P(a, -a, \ldots, a)$ for some odd a with $|a| \geq 3$.*

In the special case where the odd pretzel knot has 3 or 4 strands, Theorem 4.1.11 follows from earlier work of Donald [Don15, Theorem 1.5]. We note that a very recent preprint of Clayton [McD19] shows that all mutants in Theorem 4.1.11(iii) are doubly slice. Together with Theorem 4.1.11, this gives a complete classification of the doubly slice odd pretzels.

We also note one further easy application of our results to doubly slice Montesinos links. Although we were unable to find it stated in the literature, it seems possible that the following result was already known in the alternating case.

Proposition 4.1.12. *A quasi-alternating Montesinos link is never topologically doubly slice.*

Structure

The first three sections of this chapter are primarily background material. Section 4.2 discusses background material on Seifert fibered spaces and the plumbings they bound. Section 4.3 recounts some homological consequences of embedding 3-manifolds into S^4. Section 4.4 is devoted to calculating various homological properties of Seifert fibered spaces. The analysis of the obstruction based on Donaldson's theorem is given in Section 4.5, where we prove Theorem 4.1.4. In Section 4.6, we study partitionable spaces and show that under various circumstances partitionable spaces can be obtained by expansion from smaller partitionable spaces. This allows us to prove the obstruction part of Theorem 4.1.1 and Theorem 4.1.2. The proofs of Theorem 4.1.1 and Theorem 4.1.2 are completed in Section 4.7 by providing embeddings of the necessary spaces. The proof of Lemma 4.1.6 is contained in this section, as well as some observations about the $\varepsilon = 0$ case. In Section 4.8 our attention turns to the $\overline{\mu}$ invariant, allowing us to prove Theorem 4.1.10, as well as give various restrictions in the presence of exceptional fibers of even multiplicity. Finally, Section 4.9 contains the results relating to doubly slice links.

Conventions and notation

Throughout this chapter F will always denote an orientable surface. We will sometimes use $\mathbb{Z}_n$ to denote the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Unless explicitly stated otherwise all homology and cohomology groups are assumed to have integer coefficients.

4.2 Seifert fibered spaces and plumbings

In this section we briefly recall some standard facts on Seifert fibered spaces and definite manifolds which they bound, as well as establish notation and conventions. See [NR78] for a more in depth treatment on Seifert fibered spaces and plumbings.

Given a rational number $r > 1$, there is a unique (negative) continued fraction expansion

$$r = [a_1, \dots, a_n]^- := a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots \, a_{n-1} - \cfrac{1}{a_n}}},$$

where $n \geq 1$ and $a_i \geq 2$ are integers for all $i \in \{1, \dots, n\}$. We associate to r the weighted linear graph (or linear chain) given in Figure 4.1. We call the vertex with weight labelled by a_i the ith vertex of the linear chain associated to r, so that the vertex labelled with weight a_1 is the first, or starting vertex of the linear chain.

We denote by $Y_g = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ the Seifert fibered space over the

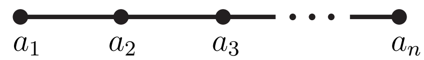

Figure 4.1: Weighted linear chain representing $r = [a_1, \ldots, a_n]^-$.

closed orientable genus g surface F given in Figure 4.2, where $e \in \mathbb{Z}$, and $\frac{p_i}{q_i} \in \mathbb{Q}$ is non-zero for all $i \in \{1, \ldots, k\}$. When $g = 0$, this is the usual surgery presentation for a Seifert fibered space over S^2. In general, each of the g pairs of 0-framed components increases the genus of the base space by one, see [CH98, Appendix].

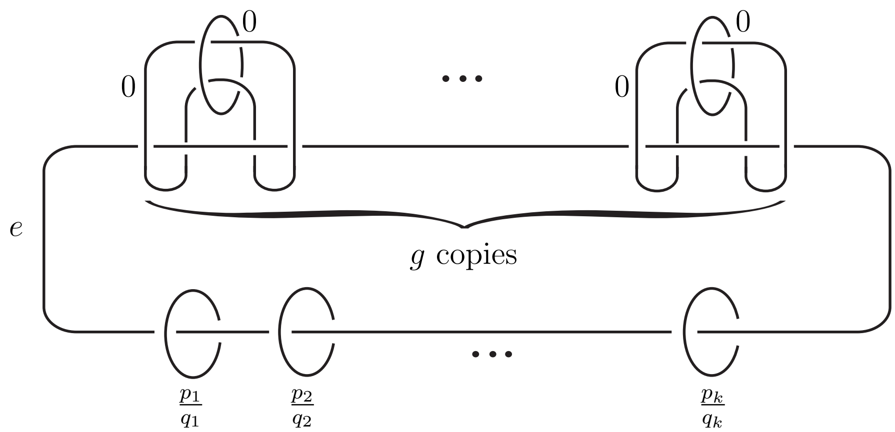

Figure 4.2: Surgery presentation of the Seifert fibered space $F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$, where F is an orientable genus g surface.

The generalised Euler invariant of Y_g is given by $\varepsilon(Y) = e - \sum_{i=1}^{k} \frac{q_i}{p_i}$. Every Seifert fibered space Y_g is (possibly orientation reversing) homeomorphic to one in standard form, i.e. such that $\varepsilon(Y_g) \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \ldots, k\}$. When in standard form, we call e the normalized central weight of Y_g.

We henceforth assume that Y_g is in standard form. Then Y_g bounds a positive semi-definite 4-manifold which we now describe. We first describe the case Y_0 where the base surface is S^2. If $\varepsilon(Y_0) \neq 0$ then Y_0 is a rational homology sphere with $|H_1(Y_0)| = |(p_1 \cdots p_k)\varepsilon(Y_0)|$, and if $\varepsilon(Y_0) = 0$ then Y_0 is a rational homology $S^1 \times S^2$.

For each $i \in \{1, \ldots, k\}$, we have the unique continued fraction expansion $\frac{p_i}{q_i} = [a_1^i, \ldots, a_{h_i}^i]^-$ where $h_i \geq 1$ and $a_j^i \geq 2$ are integers for all $j \in \{1, \ldots, h_i\}$. We associate to $Y_0 = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ the weighted star-shaped graph in Figure 4.3. The ith leg (sometimes also called the ith arm) of the star-shaped graph is the weighted linear subgraph for p_i/q_i generated by the vertices labelled with weights $a_1^i, \ldots, a_{h_i}^i$. The degree k vertex labelled with weight e is called the central vertex.

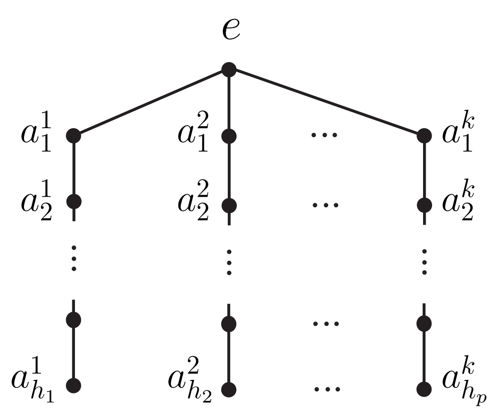

Figure 4.3: The weighted star-shaped plumbing graph Γ.

Let Γ be either the weighted star-shaped graph for Y_0, or a disjoint union of weighted linear graphs. There is an oriented smooth 4-manifold X_Γ given by plumbing D^2-bundles over S^2 according to the weighted graph Γ. We denote by $|\Gamma|$ the number of vertices in Γ. Let $m = |\Gamma|$ and denote the vertices of Γ by $v_1, v_2, \ldots, v_m$. The zero-sections of the D^2-bundles over S^2 corresponding to each of $v_1, \ldots, v_m$ in the plumbing together form a natural spherical basis for $H_2(X_\Gamma)$. With respect to this basis, which we call the vertex basis, the intersection form of X_Γ is given by the weighted adjacency matrix Q_Γ with entries Q_{ij}, $1 \leq i, j \leq m$ given by

$$Q_{ij} = \begin{cases} \mathrm{w}(v_i), & \text{if } i = j \\ -1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge ,} \\ 0, & \text{otherwise} \end{cases}$$

where $\mathrm{w}(v_i)$ is the weight of vertex v_i. Denoting by Q_X the intersection form of X, we call $(H_2(X), Q_X) \cong (\mathbb{Z}^m, Q_\Gamma)$ the intersection lattice of X_Γ (or of Γ). We denote the intersection pairing of two elements $x, y \in \mathbb{Z}^m$ by $x \cdot y = x^T\, Q_\Gamma\, y$ and the norm $x \cdot x$ by $\|x\|^2$. Now assume that Γ is the star-shaped plumbing for Y_0. If $\varepsilon(Y) > 0$ then X_Γ is a positive definite 4-manifold and Γ is the standard positive definite plumbing graph for Y_0. If $\varepsilon(Y_0) = 0$, then X_Γ is a positive semi-definite manifold.

Generalising the case above for Y_g over an orientable genus g surface, we have that Y_g is the boundary of the 4-manifold $X_{\Gamma,g}$ in Figure 4.4. Since the 2-handles do not homologically link any 1-handles, the intersection form of $X_{\Gamma,g}$ is independent of g. In particular, $(H_2(X_{\Gamma,g}), Q_{X_{\Gamma,g}}) \cong (\mathbb{Z}^m, Q_\Gamma)$ where Γ is the weighted star-shaped graph in Figure 4.3.

Let $\iota : (\mathbb{Z}^m, Q_\Gamma) \to (\mathbb{Z}^r, \mathrm{Id})$, $r > 0$, be a map of lattices, i.e. a $\mathbb{Z}$-linear map preserving pairings, where $(\mathbb{Z}^r, \mathrm{Id})$ is the standard positive diagonal

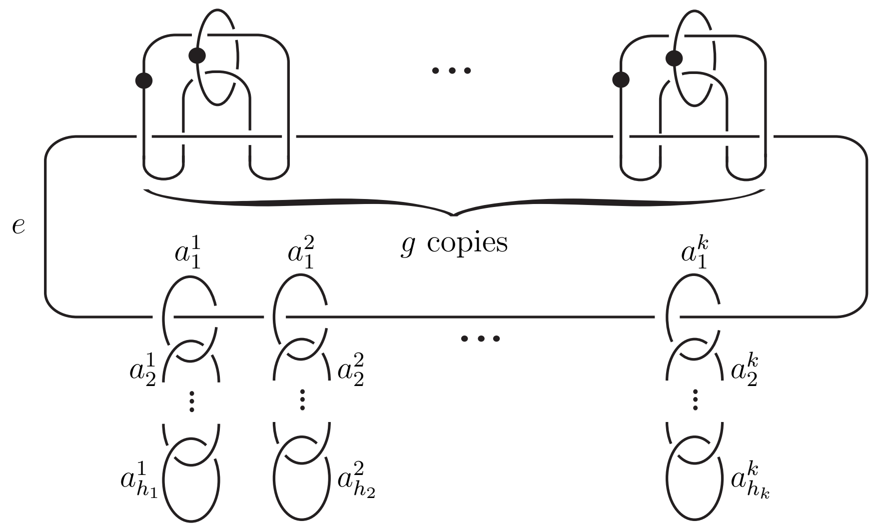

Figure 4.4: Kirby diagram for the positive semi-definite 4-manifold $X_{\Gamma,g}$ with boundary the Seifert fibered space $Y_g = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ over the orientable genus g surface F. Recall that $\frac{p_i}{q_i} = [a_1^i, \ldots, a_{h_i}^i]^-$ for all $i \in \{1, \ldots, k\}$. The intersection form of $X_{\Gamma,g}$ is isomorphic to $(\mathbb{Z}^m, Q_\Gamma)$ where Γ is the weighted star-shaped graph in Figure 4.3.

lattice. We denote the orthonormal coordinate vectors of $(\mathbb{Z}^r, \mathrm{Id})$ by $e_1, \ldots, e_r$. We call ι a lattice embedding if it is injective. We adopt the following standard abuse of notation. First, for each $i \in \{1, \ldots, m\}$, we identify the vertex v_i with the corresponding ith basis element of $(\mathbb{Z}^m, Q_\Gamma)$. Moreover, we shall identify an element $v \in (\mathbb{Z}^m, Q_\Gamma)$ with its image $\iota(v) \in (\mathbb{Z}^r, \mathrm{Id})$.

4.3 Homological consequences of embedding in S^4

We recall some well-known consequences of a 3-manifold embedding into S^4.

Proposition 4.3.1. *Let Y be a closed orientable 3-manifold topologically lo-*

cally flatly embedded in S^4*. Then* S^4 *can be decomposed as* $S^4 = U_1 \cup_Y -U_2$*, where* $\partial U_1 = \partial U_2 = -Y$*, such that*

1. *the restriction map* $H^2(U_1;\mathbb{Z}) \oplus H^2(U_2;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ *is an isomorphism,*

2. $H^3(U_1;\mathbb{Z}) \cong H^3(U_2;\mathbb{Z}) \cong 0$*,*

3. $\operatorname{tor} H^2(U_1;\mathbb{Z}) \cong \operatorname{tor} H^2(U_2;\mathbb{Z})$*, and*

4. $\sigma(U_i) = b_2(U_i) + b_1(U_i) - b_3(U_i) - b_2(Y) = 0$*.*

Proof. Since S^4 has trivial first homology, any embedded connected 3-manifold must separate S^4 into two submanifolds which we call U_1 and U_2. Consider the Mayer-Vietoris sequence for $S^4 = U_1 \cup_Y -U_2$. This contains within it the exact sequence,

$$0 \to H^2(U_1;\mathbb{Z}) \oplus H^2(U_2;\mathbb{Z}) \to H^2(Y;\mathbb{Z}) \to 0,$$

which proves the restriction map in (1) is an isomorphism. It also contains the exact sequence,

$$0 \to H^3(U_1;\mathbb{Z}) \oplus H^3(U_2;\mathbb{Z}) \to H^3(Y;\mathbb{Z}) \to H^4(S^4;\mathbb{Z}) \to 0.$$

Since the map $H^3(Y) \to H^4(S^4)$ is surjective from $\mathbb{Z}$ to $\mathbb{Z}$ it is an isomorphism, implying (2).

As U_1, U_2 are subsets of S^4, Alexander duality shows that $H_1(U_1;\mathbb{Z}) \cong H^2(U_2;\mathbb{Z})$. However by the universal coefficient theorems $H_1(U_1;\mathbb{Z})$ and $H^2(U_1;\mathbb{Z})$

have isomorphic torsion subgroups. We have $\sigma(U_i) = 0$ since both U_i and $-U_i$ can be glued to form a positive-definite manifold, but the required value for $\sigma(U_i)$ given by Proposition 2.4.1 is invariant under change of orientations. $\square$

The following corollary, first due to Hantzsche [Han38], follows immediately from (1) and (3) of Proposition 4.3.1.

Corollary 4.3.2. *If a 3-manifold Y embeds topologically locally flatly in S^4, then $\operatorname{tor} H_1(Y;\mathbb{Z})$ splits as a direct double, that is, $\operatorname{tor} H_1(Y;\mathbb{Z}) \cong G \oplus G$ for some finite abelian group G.*

In Section 4.8, the following well-known variant of Proposition 4.3.1 will also be useful.

Lemma 4.3.3. *Let Y be a rational homology sphere smoothly embedded in S^4 which decomposes S^4 as $S^4 = U_1 \cup_Y -U_2$ with U_1 and U_2 as in Proposition 4.3.1. Then*

1. *$|\mathrm{Spin}(Y)| = d^2$ for some integer $d \geq 1$, and*
2. *for $i = 1, 2$, the manifold U_i is a spin rational homology ball with $|\mathrm{Spin}(U_i)| = d$ and the restriction map $\mathrm{Spin}(U_i) \to \mathrm{Spin}(Y)$ is injective.*

Proof. First notice that U_1 and U_2 are spin since they are submanifolds of S^4. As Y is a rational homology sphere, it follows immediately from the relevant Mayer-Vietoris sequence that U_1 and U_2 are rational homology balls.

By Proposition 4.3.1(3) we see that $H^2(U_1;\mathbb{Z}) \cong H^2(U_2;\mathbb{Z})$ and hence that $H_1(U_1;\mathbb{Z}) \cong H_1(U_2;\mathbb{Z})$. Applying the universal coefficient theorem shows that $H^1(U_1;\mathbb{Z}_2) \cong H^1(U_2;\mathbb{Z}_2)$. The Mayer-Vietoris sequence with $\mathbb{Z}_2$ coefficients shows that the restriction maps yield an isomorphism

$$H^1(U_1;\mathbb{Z}_2) \oplus H^1(U_2;\mathbb{Z}_2) \to H^1(Y;\mathbb{Z}_2).$$

Since spin structures on a spin manifold M form a torsor over the group $H^1(M;\mathbb{Z}_2)$, this shows that $|\mathrm{Spin}(Y)| = d^2$ where $d = |\mathrm{Spin}(U_1)| = |\mathrm{Spin}(U_2)|$. The restriction map $\mathrm{Spin}(U_i) \to \mathrm{Spin}(Y)$ is injective since the restriction map $H^1(U_i;\mathbb{Z}_2) \to H^1(Y;\mathbb{Z}_2)$ is injective. □

4.4 Homology of Seifert fibered spaces

In this section we prove several useful statements about the homology of Seifert fibered spaces over orientable base surfaces.

Lemma 4.4.1. *The Seifert fibered space $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$, where F is an orientable genus g surface, has homology*

$$H_1(Y;\mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \bigoplus_{i=1}^{k} \frac{\mathbb{Z}}{D_i\mathbb{Z}},$$

where for $i \in \{1, \ldots, k\}$, $D_i = d_{i+1}/d_i$ where

$$d_j = \begin{cases} 1 & \text{if } j = 1, 2 \\ \gcd\{p_{\sigma(1)}p_{\sigma(2)} \cdots p_{\sigma(j-2)} \mid \sigma \in S_k\} & \text{if } 3 \le j \le k \\ (p_1 \cdots p_k)\varepsilon(Y) & \text{if } j = k+1. \end{cases}$$

Proof. From the surgery description of Y in Figure 4.2, we see that $H_1(Y)$ has a presentation matrix given by a diagonal block matrix with g blocks of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and a block of the form

$$A := \begin{pmatrix} e & 1 & \dots & 1 \\ q_1 & p_1 & & 0 \\ \vdots & & \ddots & \\ q_k & 0 & & p_k \end{pmatrix}.$$

This shows that $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \operatorname{coker} A$. For each $i \in \{1, \dots, k+1\}$, let d_i be the ith determinantal divisor of A, that is, the greatest common divisor of all $i \times i$ minors of A. Then it is a standard algebraic fact that $\operatorname{coker} A \cong \bigoplus_{i=1}^{k} \frac{\mathbb{Z}}{D_i \mathbb{Z}}$, where $D_i = d_{i+1}/d_i$ for all $1 \leq i \leq k$. We will compute $d_1, \dots, d_k$ for our particular A. Since A contains an entry equal to one, we have $d_1 = 1$. Since A has a 2×2 minor with determinant one, we have $d_2 = 1$.

Let $i \in \{3, 4, \dots, k\}$. The $i \times i$ submatrices of A

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ p_1 & 0 & & 0 & 0 \\ 0 & p_2 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & & p_{i-1} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e & 1 & \dots & 1 & 1 \\ q_1 & p_1 & 0 & 0 & 0 \\ q_2 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & p_{i-2} & 0 \\ q_{i-1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

show that (up to signs) $p_1 p_2 \cdots p_{i-1}$ and $p_1 p_2 \cdots p_{i-2} q_{i-1}$ appear as $i \times i$ minors of A, and so d_i divides their greatest common divisor, which is $p_1 p_2 \cdots p_{i-2}$. Similarly, one can get that d_i divides $p_{\sigma(1)} \cdots p_{\sigma(i-2)}$ for any permutation $\sigma \in S_k$. However, notice that in any $i \times i$ submatrix A' of A, a non-zero product of i entries of A', one from each column and row, must necessarily be a multiple of a product of $i-2$ of $p_1, \dots, p_k$. Hence, $\det(A')$ is a multiple of

$\gcd\{p_{\sigma(1)}p_{\sigma(2)}\dots p_{\sigma(j-2)} \mid \sigma \in S_k\}$. Thus, $d_i = \gcd\{p_{\sigma(1)}p_{\sigma(2)}\dots p_{\sigma(j-2)} \mid \sigma \in S_k\}$.

The final statement in the lemma follows by observing that $d_{k+1} = p_1 \cdots p_k \varepsilon(Y)$, and so $D_k = d_{k+1}/d_k$ is non-zero for $\varepsilon(Y) \neq 0$. $\square$

For a positive prime p we use $V_p(\alpha)$ to denote the p-adic valuation of α.[3] Recall that any finitely generated abelian group can be decomposed as a direct sum

$$G \cong \mathbb{Z}^m \oplus \bigoplus_{p \,\text{prime}} G_p,$$

where G_p is the p-primary part of G. For a cyclic group $\mathbb{Z}/n\mathbb{Z}$ the p-primary part is cyclic of order $p^{V_p(n)}$. It will be useful to consider such a decomposition for the homology of Seifert fibered spaces.

Lemma 4.4.2. *Let $Y = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ with F an orientable surface and $\varepsilon(Y) \neq 0$. For a prime p, let $v_1 \leq \cdots \leq v_k$ denote the p-adic valuations $V_p(p_1), \dots, V_p(p_k)$ ordered so as to be increasing. Then the p-primary part of $H_1(Y;\mathbb{Z})$ is isomorphic to*

$$\frac{\mathbb{Z}}{p^v\mathbb{Z}} \oplus \bigoplus_{i=1}^{k-2} \frac{\mathbb{Z}}{p^{v_i}\mathbb{Z}},$$

where $v = v_k + v_{k-1} + V_p(\varepsilon(Y))$. Moreover we have $v \geq v_{k-1}$ and if $v_k > v_{k-1}$, then $v = v_{k-1}$.

[3] That is $V_p(\alpha) = n$ if α can be written in the form $\alpha = p^n \frac{a}{b}$ with $a, b \in \mathbb{Z}$ both coprime to p.

Proof. We can write

$$H_1(Y;\mathbb{Z}) = \mathbb{Z}^{2g} \oplus \bigoplus_{i=1}^{k} \frac{\mathbb{Z}}{D_i\mathbb{Z}},$$

with the $D_i = d_{i+1}/d_i$ as defined in Lemma 4.4.1. By definition the d_j are such that

$$V_p(d_j) = \begin{cases} 0 & \text{if } j = 1 \text{ or } 2, \\ v_1 + \cdots + v_{j-2} & \text{if } 3 \leq j \leq k, \\ v_1 + \cdots + v_k + V_p(\varepsilon(Y)) & \text{if } j = k+1. \end{cases}$$

Therefore we have that

$$V_p(D_j) = \begin{cases} 0 & \text{if } j = 1, \\ v_{j-1} & \text{if } 1 < j < k, \\ v_{k-1} + v_k + V_p(\varepsilon(Y)) & \text{if } j = k. \end{cases}$$

The statements about the p-primary part is immediate from these p-adic valuation computations. Notice that $\varepsilon(Y) \neq 0$ can be expressed as a fraction with denominator $\operatorname{lcm}(p_1, \ldots, p_k)$. Since $V_p(\operatorname{lcm}(p_1, \ldots, p_k)) = v_k$, this shows that $V_p(\varepsilon(Y)) \geq -v_k$, which shows that $v = V_p(D_k) \geq v_{k-1}$. Finally suppose that $v_k > v_{k-1}$. In this case when we write each summand of $\varepsilon(Y) = e - \sum \frac{q_i}{p_i}$ as a fraction over the common denominator $\operatorname{lcm}(p_1, \ldots, p_k)$, the numerators will all be divisible by p with the exception of the numerator of corresponding to the unique summand $\frac{q_j}{p_j}$ where $V_p(p_j) = v_k$, which will not be divisible by p. Thus when we write $\varepsilon(Y)$ as a fraction over $\operatorname{lcm}(p_1, \ldots, p_k)$, the numerator will not be divisible by p and hence $V_p(\varepsilon(Y)) = -V_p(\operatorname{lcm}(p_1, \ldots, p_k)) = -v_k$. So $v = v_{k-1}$ as required, in this case. □

We use this to determine the effect of expansion on homology. Although we only deal with the $\varepsilon(Y) \neq 0$ case, it is not hard to see that a similar result holds when $\varepsilon(Y) = 0$.

Lemma 4.4.3. *Let $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $\varepsilon(Y) \neq 0$. If $Y' = F(e+1; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, \frac{p_k}{p_k - q_k}, \frac{p_k}{q_k})$ is obtained from Y by expansion, then*

$$H_1(Y'; \mathbb{Z}) \cong H_1(Y; \mathbb{Z}) \oplus \frac{\mathbb{Z}}{p_k\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_k\mathbb{Z}}.$$

In particular $\operatorname{tor} H_1(Y;\mathbb{Z})$ *is a direct double if and only if* $\operatorname{tor} H_1(Y';\mathbb{Z})$ *is a direct double.*

Proof. Since expansion preserves the generalized Euler invariant, we have $\varepsilon(Y) = \varepsilon(Y')$. For a fixed prime p, let $v_1 \leq \cdots \leq v_k$ denote the p-adic valuations of $p_1, \ldots, p_k$ ordered to be increasing. By Lemma 4.4.2 the p-primary part of $H_1(Y; \mathbb{Z})$ is

$$\frac{\mathbb{Z}}{p^{v_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{v_{k-2}}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{v}\mathbb{Z}},$$

where $v = v_k + v_{k-1} + V_p(\varepsilon(Y))$. Now let $w_1 \leq \cdots \leq w_{k+2}$ be the p-adic valuations of $p_1, \ldots, p_k, p_k, p_k$ in increasing order. Notice that this sequence is obtained from the v_i by inserting two extra copies of $V_p(p_k)$ at the appropriate point. First suppose that $V_p(p_k) = v_j$ for some $j \leq k-1$. Calculating the p-primary part of $H_1(Y'; \mathbb{Z})$ using Lemma 4.4.2 we obtain

$$\frac{\mathbb{Z}}{p^{v_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{v_{k-2}}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{v}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{v_j}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{v_j}\mathbb{Z}},$$

since $w_{k+2} = v_k$, $w_{k+1} = v_{k-1}$ and $\varepsilon(Y) = \varepsilon(Y')$. Thus consider the case that $v_k = V_p(p_k) > v_{k-1}$. In this case, we showed in Lemma 4.4.2 that $v = v_{k-1}$

and $V_p(\varepsilon(Y)) = -v_k$. Thus calculating the p-primary part of $H_1(Y';\mathbb{Z})$ yields

$$\frac{\mathbb{Z}}{p^{v_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{v_{k-1}}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{v_k}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{v_k}\mathbb{Z}},$$

since $v = v_{k-1} = w_{k-1}$, $w_k = v_k$ and $w_{k+2} + w_{k+1} + V_p(\varepsilon(Y')) = v_k$. In either case, the p-primary part of $H_1(Y';\mathbb{Z})$ is obtained from the p-primary part of $H_1(Y;\mathbb{Z})$ by adding a $\frac{\mathbb{Z}}{p^{V_p(p_k)}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{V_p(p_k)}\mathbb{Z}}$ summand. Since this is true for all primes we see that

$$H_1(Y';\mathbb{Z}) \cong H_1(Y;\mathbb{Z}) \oplus \frac{\mathbb{Z}}{p_k\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_k\mathbb{Z}},$$

as required. □

The following is a key ingredient in the proof of Theorem 4.1.4.

Lemma 4.4.4. *Let $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $\varepsilon(Y) > 0$ and $p_i/q_i > 1$ for all i. Suppose that* $\operatorname{tor} H_1(Y) \cong G \oplus G$ *for some finite abelian group G. If $\mathcal{P} = \{C_1, \ldots, C_n\}$ is a partition of $\{1, \ldots, k\}$ into $n \leq e$ classes such that*

$$\sum_{j \in C_i} \frac{q_j}{p_j} \leq 1 \tag{4.4.1}$$

for all $i \in \{1, \ldots, n\}$, then $n = e$ and there is precisely one value $i \in \{1, \ldots, n\}$ for which the inequality in (4.4.1) is strict and this satisfies

$$1 - \sum_{j \in C_i} \frac{q_i}{p_i} = \frac{1}{\operatorname{lcm}(p_1, \ldots, p_k)}.$$

Moreover, if k is even then $\gcd(p_1, \ldots, p_k) = 1$.

Proof. Since $\operatorname{tor} H_1(Y;\mathbb{Z})$ is a direct double, then for each prime p the p-primary part of $\operatorname{tor} H_1(Y;\mathbb{Z})$ must also be a direct double. Let $v_1 \leq \cdots \leq v_k$ be the p-adic valuations of the p_i in increasing order. By Lemma 4.4.2 the relevant p-primary part is isomorphic to

$$\frac{\mathbb{Z}}{p^{v_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p^{v_{k-2}}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^{v}\mathbb{Z}}, \tag{4.4.2}$$

where $v = v_k + v_{k-1} + V_p(\varepsilon(Y)) \geq v_{k-1}$. Since the v_i are increasing, this can be a direct double only if $v = v_{k-1} = v_{k-2}$. This implies that

$$V_p(\varepsilon(Y)) = -v_k = -V_p(\operatorname{lcm}(p_1, \dots, p_k)).$$

Notice also that we must have an even number of non-trivial summands in (4.4.2). Thus if k is even, we necessarily have $v_1 = V_p(\gcd(p_1, \dots, p_k)) = 0$. Since our choice of prime was arbitrary, it follows that

$$\varepsilon(Y) = \frac{1}{\operatorname{lcm}(p_1, \dots, p_k)}$$

and, if k is even, that

$$\gcd(p_1, \dots, p_k) = 1.$$

Now suppose that we have a partition $\mathcal{P}$ as in the statement of the lemma. We may split $\varepsilon(Y)$ up as follows:

$$\varepsilon(Y) = e - n + \sum_{k=1}^{n} \left(1 - \sum_{i \in C_k} \frac{q_i}{p_i}\right) = \frac{1}{\operatorname{lcm}(p_1, \dots, p_k)},$$

where $1 - \sum_{i \in C_k} \frac{q_i}{p_i} \geq 0$ for all k. Thus we see immediately that $e = n$. However notice that if $1 - \sum_{i \in C_k} \frac{q_i}{p_i} > 0$, then

$$1 - \sum_{i \in C_k} \frac{q_i}{p_i} \geq \frac{1}{\operatorname{lcm}(p_1, \dots, p_k)}.$$

Consequently we must have $1-\sum_{i\in C_k}\frac{q_i}{p_i}=0$ for all but one k for which we have the required equality. $\square$

The following will be useful in Section 4.8.

Lemma 4.4.5. *Let $Y=S^2(e;\frac{p_1}{q_1},\dots,\frac{p_k}{q_k})$ be a Seifert fibered space with $\varepsilon(Y)\neq 0$ and N exceptional fibers of even multiplicity. If $N\geq 1$, then*

$$\dim H^1(Y;\mathbb{Z}_2)=N-1.$$

If $N=0$, then

$$\dim H^1(Y;\mathbb{Z}_2)\leq 1.$$

Proof. Since $\mathbb{Z}_2$ is a field $\dim H_1(Y;\mathbb{Z}_2)=\dim H^1(Y;\mathbb{Z}_2)$. Thus we will compute $\dim H_1(Y;\mathbb{Z}_2)$. By the universal coefficient theorem, $\dim H_1(Y;\mathbb{Z}_2)$ is equal to the number of summands in the 2-primary part of $H_1(Y;\mathbb{Z})$. Let $0\leq v_1\leq\cdots\leq v_k$ be the 2-adic valuations of the p_i ordered to be increasing. By Lemma 4.4.2 this 2-primary part can be written as

$$\frac{\mathbb{Z}}{2^{v_1}\mathbb{Z}}\oplus\cdots\oplus\frac{\mathbb{Z}}{2^{v_{k-2}}\mathbb{Z}}\oplus\frac{\mathbb{Z}}{2^{v}\mathbb{Z}},$$

where $v\geq v_{k-1}$. By assumption precisely N of $v_1,\dots,v_k$ are non-zero. So if $N\geq 2$, then $N-1$ values of $v_1,\dots,v_{k-1}$ are non-zero, giving the desired number of summands. If $N\leq 1$, then only v can be non-zero, so $\dim H^1(Y;\mathbb{Z}_2)\leq 1$ in this case. However, if $N=1$, then $v_k>v_{k-1}=0$, so Lemma 4.4.2 also shows that $v=v_{k-1}=0$ in this case, as required. $\square$

We end the section with one easy topological application of Proposition 4.4.4, which is the topological analogue of the upper bound in Theorem 4.1.1.

Proposition 4.4.6. *Let $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ with $\frac{p_i}{q_i} > 1$ for all i and $\varepsilon(Y) > 0$. If Y embeds topologically in S^4, then $e \leq k-1$.*

Proof. If Y embeds into S^4, then Proposition 4.3.1 shows that $\operatorname{tor} H^2(Y) \cong \operatorname{tor} H_1(Y)$ is a direct double. This implies that $e \leq k-1$. For if $e \geq k$, the partition $\{\{1\}, \{2\}, \ldots, \{k\}\}$ would violate the conditions of Lemma 4.4.4 since there would be $k > 1$ classes for which the inequality (4.4.1) of Lemma 4.4.4 is strict. $\square$

Remark 4.4.7. *The bound in Proposition 4.4.6 is sharp. It follows from the work of Freedman that every integer homology sphere embeds topologically locally flatly in S^4 [Fre82b]. For a given $k \geq 3$, there exist Seifert fibered integer homology spheres for any value of e in the range $1 \leq e \leq k-1$.*

4.5 Obstruction to smoothly embedding Seifert fibered spaces

In this section we analyse an obstruction to smoothly embedding a Seifert fibered space Y over an orientable base surface in S^4 coming from Donaldson's theorem, culminating in a proof of Theorem 4.1.4.

For the duration of this section we will use the following notation. Let

$$Y = F\left(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}\right)$$

be a Seifert fibered space over an orientable base surface of genus g with $\varepsilon(Y) > 0$ and $\frac{p_i}{q_i} > 1$ for all i. As in Figure 4.4, there is a positive-definite X with boundary Y and intersection form $(\mathbb{Z}^n, Q_\Gamma)$, where Γ is a weighted star-shaped graph as in Figure 4.3 and $n = |\Gamma|$ is the number of vertices in Γ.

Before embarking on the proof, we summarise the idea behind the obstruction based on Donaldson's theorem as follows. A smooth embedding of Y into S^4 splits S^4 into two 4-manifolds U_1 and U_2 with boundary $-Y$. The smooth manifold $W_i = X \cup U_i$ is positive definite, so Donaldson's theorem implies that it has standard diagonal intersection form. The inclusion map $X \hookrightarrow W_i$ induces maps of intersection lattices $\iota_i : (H_2(X), Q_X) \to (H_2(W_i), \mathrm{Id})$, which we can write as the transpose of an integer matrix A_i. Following Greene-Jabuka [GJ11], Donald proved that the image of the restriction map $H^2(W_i) \to H^2(Y)$ is isomorphic to $\frac{\operatorname{im} A_i}{\operatorname{im} Q_X}$ [Don15, Theorem 3.6]. Combining this with the fact that the restriction-induced map $H^2(U_1) \oplus H^2(U_2) \to H^2(Y)$ is an isomorphism, he showed that $\frac{\operatorname{im} A_1}{\operatorname{im} Q_X} \oplus \frac{\operatorname{im} A_2}{\operatorname{im} Q_X} = \operatorname{coker} Q_X$. This condition implies that the augmented matrix $(A_1|A_2)$ is surjective over the integers, see Theorem 4.5.1.

Using the fact that $H_1(Y)$ must split as a direct double, we are able to prove some structural results about the form any lattice embedding $(H_2(X), Q_X) \to (\mathbb{Z}^{b_2(X)}, \mathrm{Id})$ must take. An important ingredient in this proof is the lattice inequality given in Theorem 2.3.2. It is this result which makes an analysis of the obstruction based on Donaldson's theorem feasible.

The following theorem is the key obstruction to smoothly embedding

Seifert fibered spaces in S^4 derived from Donaldson's theorem. It is a slight variation of [Don15, Corollary 3.9].

Theorem 4.5.1. *If Y embeds smoothly into S^4, then there exist lattice embeddings*

$$\iota_i : (\mathbb{Z}^n, Q_\Gamma) \to (\mathbb{Z}^n, Id)$$

for $i = 1, 2$, such that the augmented matrix $(A_1|A_2)$ is surjective, where A_i is the transpose of the integer matrix representing ι_i for $i = 1, 2$.

Proof. Unless explicitly stated otherwise, all homology and cohomology groups in this proof are taken with coefficients in $\mathbb{Z}$. If Y embeds smoothly into S^4, then Proposition 4.3.1 shows that it splits into two 4-manifolds U_1 and U_2, with $\partial U_1 \cong \partial U_2 \cong -Y$. Let W_i be the closed manifold $W_i = X \cup_Y U_i$. We claim that Proposition 2.4.1 implies this is positive definite with $b_2(W_i) = b_2(X)$. To see this, note that in Proposition 2.4.1 injectivity condition (a) is satisfied since the map $H_1(Y;\mathbb{Q}) \to H_1(X;\mathbb{Q})$ is injective, and the signature condition (b) follows from $b_1(X) + b_2(X) - b_3(X) - b_2(Y) = 2g + n - 0 - 2g = n = \sigma(X)$, together with Proposition 4.3.1(4). Thus, Donaldson's diagonalization theorem implies that the intersection form of W_i is diagonalizable. Hence, the inclusion $H_2(X) \to H_2(W_i)$ induces an embedding of lattices $\iota_i : (\mathbb{Z}^{b_2}, Q_\Gamma) \to (\mathbb{Z}^{b_2}, I)$, for $i \in \{1, 2\}$.

Now fix $i \in \{1, 2\}$. By considering the long exact sequences of pairs and the inclusion $(X, Y) \hookrightarrow (W_i, U_i)$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^2(W_i,U_i) \arrow[r, "\alpha"] \arrow[d, "\cong i_1"] & H^2(W_i) \arrow[r, "\beta"] \arrow[d, "i_2"] & H^2(U_i) \arrow[r] \arrow[d, "i_3"] & H^3(W_i,U_i) \arrow[d, "\cong i_4"] \\
H^2(X,Y) \arrow[r, "\gamma"] & H^2(X) \arrow[r, "\delta"] & H^2(Y) \arrow[r, "\epsilon"] & H^3(X,Y).
\end{tikzcd}$$

It follows by excision that the maps i_1 and i_4 are isomorphisms.

Recall that we have a basis for $H_2(X)$ for which the intersection form of X is given by the matrix Q_Γ. By the universal coefficient theorems, $\operatorname{tor} H^2(X) \cong \operatorname{tor} H_1(X) = 0$, so we may choose the dual basis for $H^2(X) \cong \operatorname{Hom}(H_2(X), \mathbb{Z})$. We choose the Poincaré dual basis for $H^2(X,Y)$. With respect to these bases the map γ is represented by Q_Γ. By Donaldson's theorem, we can choose a basis for $H^2(W_i)/\operatorname{tor} \cong \operatorname{Hom}(H_2(W_i), \mathbb{Z})$ for which $Q_W = \operatorname{Id}$. The map $H^2(W_i)/\operatorname{tor} \to H^2(X)$ is dual to the inclusion induced map $H_2(X) \to H_2(W)/\operatorname{tor}$, and is therefore given by A_i with respect to these choices of bases.

Now notice that $H^3(X,Y) \cong H_1(X)$ is torsion free. Thus $\operatorname{tor} H^2(Y) \subseteq \ker \epsilon$. However since $H^3(X) = 0$, the map ϵ is surjective. Since $H^2(Y)$ and $H^3(X,Y)$ have the same rank we see that $\operatorname{im} \delta = \operatorname{tor} H^2(Y) = \ker \epsilon$. This allows us to identify $\operatorname{tor} H^2(Y)$ with $\operatorname{coker} \gamma$ via δ. Since i_1 is an isomorphism, we see that $\operatorname{im} \gamma \subset \operatorname{im} i_2$. In turn this shows that δ induces an injective map $\frac{\operatorname{im} i_2}{\operatorname{im} \gamma} \to \operatorname{tor} H^2(Y)$. We have that $\operatorname{im}(i_3 \circ \beta) = \operatorname{im}(\delta \circ i_2) \subset \operatorname{tor} H^2(Y)$, which we may identify with $\frac{\operatorname{im} i_2}{\operatorname{im} \gamma}$ via δ. Since $H^2(X)$ is torsion free, i_2 maps finite order elements of $H^2(W)$ to 0. Thus, in coordinates with respect to the bases given earlier $\frac{\operatorname{im} i_2}{\operatorname{im} \gamma}$ is given by $\frac{\operatorname{im} A_i}{\operatorname{im} Q_\Gamma}$.

We claim that $\mathrm{im}(i_3 \circ \beta) = \mathrm{im}(i_3)$. It suffices to check these finite groups have the same order. By Proposition 4.3.1(1), the order of $\mathrm{im}(i_3)$ is the square root of $|\operatorname{tor} H_1(Y)| = |\operatorname{coker} Q_\Gamma|$, where in this last equality uses the fact that Q_Γ presents $\operatorname{tor} H_1(Y)$. Using the fact that $Q_\Gamma = A_i A_i^T$ we see that this is also the order of $\mathrm{im}(i_3 \circ \beta) \cong \frac{\mathrm{im}\, A_i}{\mathrm{im}\, Q_\Gamma}$, proving the claim. Thus, we can identify $\operatorname{tor} H^2(Y)$ with $\mathbb{Z}^{b_2}/\,\mathrm{im}\, Q_\Gamma$, and under this identification the image of $\operatorname{tor} H^2(U_i) \to \operatorname{tor} H^2(Y)$ is $\mathrm{im}\, A_i/\,\mathrm{im}\, Q_\Gamma$.

By Proposition 4.3.1(3) the map $\operatorname{tor} H^2(U_1) \oplus \operatorname{tor} H^2(U_1) \to \operatorname{tor} H^2(Y)$ is an isomorphism. Thus

$$\frac{\mathbb{Z}^{b_2}}{\mathrm{im}\, Q_X} \cong \frac{\mathrm{im}\, A_1}{\mathrm{im}\, Q_X} \oplus \frac{\mathrm{im}\, A_2}{\mathrm{im}\, Q_X}, \tag{4.5.1}$$

where the direct sum is an internal direct sum as subspaces of $\operatorname{coker} Q_X$.

It suffices to show that (4.5.1) implies $\mathrm{im}(A_1 \mid A_2) = \mathbb{Z}^{b_2(X)}$. Let $x \in \mathbb{Z}^{b_2(X)}$ and let $q : \mathbb{Z}^{b_2(X)} \to \operatorname{coker} Q_X$ be the quotient map. By Equation (4.5.1), $q(a_1)+q(a_2) = q(x)$ for some $a_1 \in \mathrm{im}(A_1)$ and $a_2 \in \mathrm{im}(A_2)$. Thus, $a_1 + a_2 = x + k$ for some $k \in \mathrm{im}(Q_X)$. Since $Q_X = A_1 A_1^T$, we have $\mathrm{im}\, Q_X \subset \mathrm{im}(A_1)$. Therefore $(a_1 - k) + a_2 = x$ shows that $x \in \mathrm{im}(A_1 \mid A_2)$, as required. $\square$

With Y, X and Γ as defined at the beginning of this section, we have the following lemma which in particular shows that from an embedding of lattices we can define a partition.

Lemma 4.5.2. *Let $\iota : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^N, Id)$, where $N > 0$ be a lattice embedding. Let $\{e_1, \dots, e_N\}$ be an orthonormal basis for $(\mathbb{Z}^N, Id)$. If* $\operatorname{tor} H_1(Y; \mathbb{Z}) =$

$G \oplus G$ for some abelian group G, then upto an automorphism of $\mathbb{Z}^N$ we may assume the following. The image of the central vertex is $e_1 + \cdots + e_e$. For each $i \in \{1, \dots, e\}$ let C_i be the subset of $\{1, \dots, k\}$ consisting of j such that the first vertex of the linear chain p_j/q_j pairs non-trivially with e_i. Then

1. *$\{C_1, \dots, C_e\}$ is a partition of $\{1, \dots, k\}$ such that*

$$\sum_{j \in C_i} \frac{q_i}{p_i} = 1$$

   for $i = 1, \dots, e-1$ and

$$\sum_{j \in C_e} \frac{q_i}{p_i} = 1 - \frac{1}{\operatorname{lcm}(p_1, \dots, p_k)}$$

2. *and for $i \in \{1, \dots, e\}$ the vertices with which e_i pairs non-trivially are precisely the leading vertices of the arms in C_i and the central vertex.*

Proof. Let $p_i/q_i = [a_1^i, \dots, a_{l_i}^i]^-$, where $a_j^i \geq 2$. Let v_j^i denote the image of the jth vertex in the linear chain corresponding to p_i/q_i. So $\|v_j^i\|^2 = a_j^i$. Let ν be the image of the central vertex. By applying an automorphism of $\mathbb{Z}^N$ we may assume that ν takes the form $\nu = \alpha_1 e_1 + \cdots + \alpha_n e_n$ with $\alpha_i > 0$ and $n \leq e$. Let $C_1, \dots, C_n$ be the sets defined by

$$C_i = \{j \in \{1, \dots, k\} \mid e_i \cdot v_1^j \neq 0\}$$

as in the statement of the lemma. Since $\nu \cdot v_1^j = -1$, each j in the range $1 \leq j \leq k$ is contained in at least one C_i. *A priori* the C_i may not be a partition, since they may not be pairwise disjoint and some C_i's may be empty.

However by discarding repetitions, we can obtain C_i' such that $C_i' \subseteq C_i$ and the non-empty C_i' form a genuine partition of $\{1, \ldots, k\}$.

By Theorem 2.3.2 we can conclude that for each i we have

$$\sum_{j \in C_i'} \frac{q_j}{p_j} \leq \sum_{j \in C_i} \frac{q_j}{p_j} \leq 1.$$

Thus by Lemma 4.4.4 the partition consisting of the C_i' has precisely e non-empty classes. It follows that ν must take the form $\nu = e_1 + \cdots + e_e$ as required. Furthermore Lemma 4.4.4 also implies that after permuting the e_i if necessary, we can assume that

$$\sum_{j \in C_i'} \frac{q_j}{p_j} = \sum_{j \in C_i} \frac{q_j}{p_j} = 1 \tag{4.5.2}$$

for $i = 1, \ldots, e-1$ and

$$1 - \frac{1}{\operatorname{lcm}(p_1, \ldots, p_k)} = \sum_{j \in C_e'} \frac{q_j}{p_j} \leq \sum_{j \in C_e} \frac{q_j}{p_j} \leq 1. \tag{4.5.3}$$

This shows that $C_i = C_i'$ for $i = 1, \ldots, e-1$. To show that the C_i form a partition, it remains to verify that $C_e = C_e'$. We will use the following claim to complete the proof.

Claim. *Let v_s^j be a vertex such that $j \notin C_l'$ but $v_s^j \cdot e_l \neq 0$ for some l in the range $1 \leq l \leq e$. Then $s = 1$, $l = e$ and $v_s^j \cdot e_e = \pm 1$.*

Proof. Since $j \notin C_l'$ the vector v_s^j is orthogonal to all vertices in the linear chains corresponding to elements of C_l'. As we can consider a single vertex as a linear chain in its own right, Theorem 2.3.2 applies to show that

$$\frac{1}{\|v_s^j\|^2} + \sum_{i \in C_l'} \frac{q_i}{p_i} \leq 1.$$

By (4.5.2) and (4.5.3) we see that this is only possible if $l = e$ and $\|v_s^j\|^2 \geq \mathrm{lcm}(p_1, \dots, p_k)$. However since $\|v_s^j\|^2 = a_s^j$ appears in the continued fraction expansion for p_j/q_j, we see that $\|v_s^j\|^2 \leq p_j$ with equality only if $p_j/q_j = \|v_s^j\|^2$ is an integer, in which case $s = 1$. As $\mathrm{lcm}(p_1, \dots, p_k) \geq p_j$, this implies that $s = 1$ and $\|v_s^j\|^2 = \mathrm{lcm}(p_1, \dots, p_k)$. However, by (4.5.3) we have that $\frac{1}{\|v_s^j\|^2} + \sum_{i \in C_l'} \frac{q_i}{p_i} = 1$. Thus we can apply the equality case of Theorem 2.3.2 to conclude that $v_s^j \cdot e_e = \pm 1$. $\square$

We will now check that $C_e' = C_e$. If not, then there would be a vertex v_1^j for some $j \notin C_e'$ such that $v_1^j \cdot e_e \neq 0$. By the claim, such a vertex satisfies $v_1^j \cdot e_e = \pm 1$. However we have $j \in C_l$ for some unique $1 \leq l < e$. By the equality case of Theorem 2.3.2, this implies that $v_1^j \cdot e_l = \pm 1$. Thus $\nu \cdot v_1^j = v_1^j \cdot e_l + v_1^j \cdot e_e$ must be even, contradicting $v_1^j \cdot \nu = -1$. Thus we can conclude that $C_e' = C_e$ completing the proof that $C_1, \dots, C_e$ are a partition.

Finally, we check that the non-leading vertices cannot pair non-trivially with e_l for any $l \in \{1, \dots, e\}$. Since the non-leading vertices have trivial pairing with the central vertex ν, it suffices to check that a non-leading vertex can pair non-trivially with e_l for at most one $l \in \{1, \dots, e\}$. However, this follows easily from the above claim, which shows that for $s > 1$ a vertex v_s^j can pair non-trivially with e_l only if $j \in C_l' = C_l$. This completes the proof. $\square$

For the following lemma let Y, X and Γ as defined at the beginning of this section, and recall that a class $C \subset \{1, \dots, k\}$ is called complementary if $\sum_{i \in C} \frac{q_i}{p_i} = 1$.

Lemma 4.5.3. *Suppose that* $\operatorname{tor} H_1(Y) \cong G \oplus G$ *for some abelian group* G. *For* $i = 1, 2$, *let* $\iota_i : (\mathbb{Z}^n, Q_\Gamma) \to (\mathbb{Z}^n, Id)$, *where* $n = |\Gamma|$, *be a map of lattices, and represent* ι_i *as an integer matrix by the transpose of* A_i. *Let* $A = (A_1|A_2)$, *and suppose that the column space of* A *is all of* $\mathbb{Z}^n$. *For* $i \in \{1, 2\}$, *let* P_i *be the partition of* $\{1, \dots, k\}$ *induced by* ι_i *as in Lemma 4.5.2. Then, no non-empty union of complementary classes of* P_1 *is a union of complementary classes of* P_2.

Proof. We are assuming that both ι_1 and ι_2 satisfy the conclusions of Lemma 4.5.2. For $i \in \{1, 2\}$, let $C^i_1, \dots, C^i_{\ell_i}$ be a non-empty collection of complementary classes in P_i. Suppose for sake of contradiction that $\cup_{i=1}^{\ell_1} C_i^1 = \cup_{i=1}^{\ell_2} C_i^2$ and denote their common union by $H \subset \{1, \dots, k\}$. Since $\sum_{i \in C} \frac{q_i}{p_i} = 1$ for a complementary class C, we have $\ell_1 = \sum_{i \in H} \frac{q_i}{p_i} = \ell_2$ and we denote their common value by ℓ. Our goal will be to find a non-zero row vector $\overline{x}$ with coprime integer entries and an integer $p > 1$ such that $\overline{x} A_i \equiv 0 \bmod p$ for both $i = 1$ and $i = 2$. Given such a vector we will use that $\overline{x} A \equiv 0 \bmod p$ to show that A is not surjective over $\mathbb{Z}$.

Let R be the weighted star-shaped graph with central weight ℓ and legs given by the legs of Γ indexed by elements of H. For $i = 1, 2$, there is a map of lattices $q_i : (\mathbb{Z}^{|R|}, Q_R) \to (\mathbb{Z}^n, \mathrm{Id})$ which is the restriction of ι_i on the non-central vertices of R and maps the central vertex of R to $e_1 + \cdots + e_\ell$. That q_i is a map of lattices follows from the structure imposed by Lemma 4.5.2. The classes $C^i_1, \dots, C^i_\ell$ are precisely the ones whose leading vertices pair non-trivially with exactly one of the unit vectors $e_1, \dots, e_\ell$ and this non-trivial pairing is necessarily -1 in all cases. Furthermore no non-leading vertex pairs non-trivially with any of $e_1, \dots, e_\ell$.

Recall that the image of the vertices of Γ under ι_i are given by the rows of A_i. By ordering the vertices, we may assume that the first row of A_i corresponds to the central vertex ν, and the next $|R|-1$ rows correspond to the non-central vertices of Γ that appear in R. Let B_i be the transpose of the integer matrix representing q_i. With the above choice of vertex ordering, B_i is obtained by taking the first $|R|$ rows of A_i, and replacing the first row by the vector $(\underbrace{1,1,\dots,1}_{\ell \text{ ones}},0,\dots 0)$.

For both $i=1,2$, we have $B_iB_i^T = Q$, where $Q = Q_R$ is the matrix representing the intersection lattice $(\mathbb{Z}^{|R|}, Q_R)$ with respect to the vertex basis. Since the classes $C_1^i, \dots, C_\ell^i$ are complementary, the boundary of the plumbing with weighted graph R is a Seifert fibered space Y' with $\varepsilon(Y') = 0$. Thus $\det Q = 0$, implying that there exists a non-zero row vector $x = (x_1, \dots, x_{|R|}) \in \mathbb{Z}^{|R|}$ such that $xQ = 0$. Hence, $(xB_i)(xB_i)^T = xQx^T = 0$, implying $xB_i = 0 \in \mathbb{Z}^n$. Thus we have obtained x such that $xB_1 = xB_2 = 0 \in \mathbb{Z}^n$. By dividing out by any common factors we may assume that $\gcd(x_1, \dots, x_{|R|}) = 1$.

Claim. *The entry x_1 is divisible by an integer $p > 1$.*

With this claim, the proof concludes as follows. Consider the vector $\bar{x} = (x_1, \dots, x_{|R|}, 0, \dots, 0) \in \mathbb{Z}^n$. Since B_i is obtained from A_i by taking the first $|R|$ rows and modifying the first row, we see that every entry of $\bar{x}A_i$ is a multiple of x_1. This shows that $\bar{x}A = \bar{x} \cdot (A_1 \mid A_2) \equiv 0 \pmod{p}$, where p is the integer from the claim.

Since $\gcd(x_1, \ldots, x_{|R|}) = 1$, we can write 1 as an integer combination of the x_i. This implies there is a column vector $v \in \mathbb{Z}^n$ such that $\bar{x}v = 1$. If A were surjective, then there would be a vector u such that $v = Au$. This would show $0 \equiv \bar{x}Au = \bar{x}v = 1 \pmod{p}$, which is a contradiction. We complete the proof by proving the claim.

Proof of Claim. Consider a leg in R with corresponding fraction p/q. We will show that p divides x_1. Suppose that the continued fraction expansion of p/q is $\frac{p}{q} = [a_1, \ldots, a_{\rho-1}]^-$, where $a_j \geq 2$ are integers for all j. By ordering the vertices we may assume that the first ρ rows of B_i correspond to the central vertex followed by the vertices of our chosen leg in R. Thus, the top-left $\rho \times \rho$ submatrix of $Q = B_i B_i^T$ is

$$\begin{pmatrix} \ell & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & a_2 & -1 & \cdots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{\rho-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_{\rho-1} \end{pmatrix}.$$

Let $Q' = \left(\begin{array}{c} -1 \quad 0 \quad \cdots \\ \hline Q_{p/q} \end{array}\right)$ be the matrix obtained from the above matrix by removing the first column, where $Q_{p/q}$ is the intersection matrix of the linear chain representing p/q. We have

$$(x_1, \ldots, x_\rho) \cdot Q' = 0,$$

since $xQ = 0$ and the corresponding columns $2, \ldots, \rho$ of Q are supported in the first ρ rows. This implies that $(x_2, \ldots, x_\rho) \cdot Q_{p/q} = (x_1, 0, \ldots, 0)$. Thus, we can change the last row of $Q_{p/q}$ to $(x_1, 0, \ldots, 0)$, by first multiplying the last row of $Q_{p/q}$ by x_ρ, then for each $j \in \{1, \ldots, \rho-1\}$ adding x_j multiples of the jth row to the last row. The determinant of this new matrix is $x_{\rho-1} \cdot \det(Q_{p/q}) = x_{\rho-1} \cdot p$. However, by expanding the determinant along the final row we see that

$$\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & a_2 & -1 & \cdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & a_{\rho-2} & -1 \\ x_1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = x_1.$$

Thus $x_1 = x_{\rho-1}p$ is a multiple of p, proving the claim. □

□

This allows us to prove our main obstruction to embedding Seifert fibered spaces in S^4.

Theorem 4.1.4. *Let $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ with F an orientable surface, $\frac{p_i}{q_i} > 1$ for all i, and $\varepsilon(Y) > 0$. If Y smoothly embeds in S^4 then Y is partitionable.*

Proof. Suppose that Y smoothly embeds in S^4. By Corollary 4.3.2, tor $H_1(Y)$ splits as a direct double. Theorem 4.5.1 implies that there are lattice embeddings $\iota_i : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^{|\Gamma|}, \mathrm{Id})$, where $i \in \{1, 2\}$ and Γ is the weighted star-shaped graph describing the intersection lattice of the standard positive-definite 4-manifold bounding Y. Moreover, $(A_1|A_2)$ is surjective, where A_i

is the transpose of the integer matrix representing ι_i for $i \in \{1,2\}$. As in Lemma 4.5.2, for $i \in \{1,2\}$, there is a partition P_i induced by ι_i satisfying properties (a) and (b) of Definition 4.1.3. Lemma 4.5.3 shows that no non-empty union of any subset of complementary classes of P_1 is a union of any subset of complementary classes of P_2.

For $i = 1, 2$, let $\mathcal{C}_i$ be a non-empty proper subset of P_i. For sake of contradiction suppose that $\cup_{C \in \mathcal{C}_1} C = \cup_{C \in \mathcal{C}_2} C$, and let $H \subset \{1, \ldots, k\}$ be their common union. Properties (a) and (b) imply that for $i \in \{1,2\}$, $\mathcal{C}_i$ contains a non-complementary class if and only if $\sum_{j \in H} \frac{q_i}{p_i}$ is not an integer. Thus, $\mathcal{C}_1$ and $\mathcal{C}_2$ either both contain a non-complementary class or both do not. Thus either P_1 and P_2, or $P_1 \backslash \mathcal{C}_1$ and $P_2 \backslash \mathcal{C}_2$ contain only complementary classes. This shows that property (c) of Definition 4.1.3 holds. □

4.6 Applications of Theorem 4.1.4

Now we consider which spaces can pass the obstruction given by Theorem 4.1.4 when $e \geq \frac{k}{2}$. We will prove the obstruction halves of Theorem 4.1.1 and Theorem 4.1.2, leaving the construction of the embeddings into S^4 to Section 4.7.

Theorem 4.1.1. *Let $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $\varepsilon(Y) > 0$ and $\frac{p_i}{q_i} > 1$ for all i. If Y embeds smoothly in S^4, then $e \leq \frac{k+1}{2}$. Moreover, if $e = \frac{k+1}{2}$ then Y smoothly embeds in S^4 if* *and only if Y takes the form*

$$Y = F\left(e; \frac{a}{a-1}, \left\{a, \frac{a}{a-1}\right\}^{\times(e-1)}\right) = F\left(\frac{k+1}{2}; \frac{a}{a-1}, a, \frac{a}{a-1}, a, \dots, \frac{a}{a-1}\right)$$

where $e \geq 1$ and $a \geq 2$ is an integer.

Proof. Let P_1 and P_2 of $\{1, \dots, k\}$ be the partitions from Theorem 4.1.4, each into e classes. For each partition, there are e classes and at most one class of size one, since a size one class must be non-complementary. Thus, $k \geq 1+2(e-1)$, and so $e \leq \frac{k+1}{2}$. Now assume that $e = \frac{k+1}{2}$, in particular k is odd. For each partition all but one class has size 2, and the remaining class has size 1. Using that no non-empty proper subset of classes in P_1 is a union of classes in P_2, we without loss of generality assume that $P_1 = \{\{1\}, \{2,3\}, \{4,5\}, \dots, \{k-1,k\}\}$ and $P_2 = \{\{1,2\}, \{3,4\}, \dots, \{k-2,k-1\}, \{k\}\}$. By Lemma 4.4.4, $1 - \frac{q_1}{p_1} = \frac{1}{\operatorname{lcm}(p_1,\dots,p_k)}$, and thus $\frac{p_1}{q_1} = \frac{a}{a-1}$ where $a = \operatorname{lcm}(p_1, \dots, p_k)$. For a complementary classes $\{i,j\}$ we have $\frac{q_i}{p_i} + \frac{q_j}{p_j} = 1$. Applying this to the complementary classes in P_1 and P_2 allows us to write the remaining fractions in terms of a, which shows that M is of the required form.

Finally, the fact that the Seifert fibered spaces of this form smoothly embed in S^4 follows from Proposition 4.7.3 proved in Section 4.7. □

We now analyse the $e = \frac{k}{2}$ case. The reader may find it helpful to recall definitions of *expansion* (Definition 4.1.5) and *partitionable* (Definition 4.1.3) stated in the introduction. We first prove the following lemma.

Lemma 4.6.1. *Let $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $k \geq 3$, $\frac{p_i}{q_i} > 1$ for all i, and $\varepsilon(Y) > 0$. Suppose that Y is partitionable with partitions P_1 and P_2 such that either*

(i) $m_1 + m_2 \geq e$ where m_i is the number of complementary pairs in P_i for $i \in \{1, 2\}$, or

(ii) both P_1 and P_2 contain a class of size one, or

(iii) $e \geq \frac{2k+3}{5}$.

Then Y is an expansion of a partitionable Seifert fibered space Y'.

Proof. By Lemma 4.4.3 the property that $\operatorname{tor} H_1$ is a direct double is not changed by expansions. Thus, in order to show that Y' is partitionable it suffices to come up with partitions satisfying the three remaining conditions in Definition 4.1.3.

Suppose first that (i) holds. We claim that there are complementary pairs $\{a, b\} \in P_1$ and $\{b, c\} \in P_2$ with a, b, c distinct. Suppose otherwise, then $\sum_{i=1}^{k} \frac{q_i}{p_i} \geq m_1 + m_2 \geq e$ since each complementary pair contributes one and there are $m_1 + m_2$ disjoint complementary pairs, but this contradicts Definition 4.1.3 which implies that $\sum_{i=1}^{k} \frac{q_i}{p_i} < e$.

By permuting the fractions $\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}$, we may assume that $b = k$, $a = k-1$, $c = k-2$. Since $\{k-1, k\}$ and $\{k-2, k\}$ are complementary, we have that $\frac{p_{k-2}}{q_{k-2}} = \frac{p_{k-1}}{q_{k-1}} = \frac{p_k}{p_k - q_k}$. Thus Y is an expansion of $Y' = F(e-1; \frac{p_1}{q_1}, \ldots, \frac{p_{k-2}}{q_{k-2}})$. Let

$P_1' = P_1 \setminus \{\{k-1,k\}\}$ and let P_2' be obtained from $P_2 \setminus \{\{k-2,k\}\}$ by replacing $k-1$ with $k-2$ in the class C containing $k-1$, and call this new class C'. We claim that P_1' and P_2' satisfy the conditions in Definition 4.1.3, showing that Y' is partitionable. These conditions follow from the corresponding conditions for P_1 and P_2. Conditions (a) and (b) follow immediately. To see condition (c) let $S_1 \subsetneq P_1'$ and $S_2 \subset P_2'$ be non-empty with the union of classes in S_1 equal to the union of classes in S_2. We denote their common union by $H \subset \{1, \ldots, k-2\}$. If $k-2 \notin H$ then this would contradict condition (c) for P_1, P_2 since $S_1 \subset P_1$ and $S_2 \subset P_2$. Similarly, if $k-2 \in H$ then $S_1 \cup \{\{k-1,k\}\}$ and $(S_2 \cup \{C\}) \setminus \{C'\}$ would contradict condition (c) for P_1, P_2. This proves the conclusion if (i) holds.

Now suppose that (ii) holds. If $k = 3$ then P_1 and P_2 each contain a complementary class of size two and (i) holds. Thus we can assume that $k \geq 4$ and by permuting the fractions we may assume that $\{k\} \in P_1$ and $\{k-2\} \in P_2$. In particular these are the non-complementary classes so $\frac{p_k}{q_k} = \frac{p_{k-2}}{q_{k-2}} = m/(m-1)$, where $m = \operatorname{lcm}(p_1, \ldots, p_k)$. Let $C \in P_2$ be the complementary class containing k, and let $i \in C$ with $i \neq k$. Since C is complementary $\frac{m-1}{m} + \frac{q_i}{p_i} \leq 1$ with equality only if C has size two. Rearranging this gives $\frac{q_i}{p_i} \leq \frac{1}{m}$. However, $\frac{q_i}{p_i} \geq \frac{1}{m}$ since $m = \operatorname{lcm}(p_1, \ldots, p_k) \geq p_i$. Thus we must have equality and so $|C| = 2$. Similarly the complementary class in P_1 containing $k-2$ has size two. Since $k > 3$, this implies that we can assume that P_1 and P_2 take the form

$$P_1 = \{\ldots, \{\ldots, k-3\}, \{k-2\}, \{k-1, k\}\},$$

$$P_2 = \{\ldots, \{\ldots, k-1\}, \{k-3, k-2\}, \{k\}\}.$$

Then Y is an expansion of $Y' = F(e-1; \frac{p_1}{q_1}, \ldots, \frac{p_{k-2}}{q_{k-2}})$ with partitions $P_1' = P_1 \setminus \{\{k-1, k\}\}$ and P_2' obtained from $P_2 \setminus \{\{k\}\}$ by replacing the class C containing $k-1$ by $C' := C \setminus \{\{k-1\}\}$. We check the conditions of Definition 4.1.3. First (a) and (b) are immediate, noting that $C' \in P_2'$ is the non-complementary class. To verify (c), let $S_1 \subsetneq P_1'$ and $S_2 \subset P_2'$ be non-empty with the union of classes in S_1 equal to the union of classes in S_2. If S_2 does not contain C' then $S_1 \subset P_1$, $S_2 \subset P_2$ contradicting (c) for P_1, P_2. If S_2 contains C' then $S_1 \cup \{\{k-1, k\}\}$ and $(S_2 \cup \{\{k\}, \{k-3, k-2\}, C\}) \setminus \{C'\}$ would contradict (c) for P_1, P_2. This completes the proof if (ii) holds.

Now suppose that (iii) holds, so $e \geq \frac{2k+3}{5}$. If (ii) holds then we are done, so we may assume that the non-complementary class of P_2 has size at least two. We now show that (i) holds. Let m_i be the number of complementary pairs in P_i for $i \in \{1, 2\}$. Thus there are $e - m_i - 1$ complementary classes in P_i of size at least 3, for $i \in \{1, 2\}$. Hence,

$$k \geq 1 + 2m_1 + 3(e - m_1 - 1), \text{ and}$$
$$k \geq 2 + 2m_2 + 3(e - m_2 - 1).$$

Adding these inequalities give $2k \geq 6e - (m_1 + m_2) - 3$. Rearranging gives

$$m_1 + m_2 \geq 6e - 2k - 3 \geq e + (5e - 2k) - 3 \geq e,$$

since $e \geq \frac{2k+3}{5}$. This completes the proof. □

Now we are ready to analyze the $e = \frac{k}{2}$ case.

Theorem 4.1.2. *Let $Y = F(\frac{k}{2}; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a Seifert fibered space over orientable base surface F with $\frac{p_i}{q_i} > 1$ for all i, k even and $\varepsilon(Y) > 0$. If Y smoothly embeds in S^4 then there exist positive integers p, q, r, s with $\frac{p}{q}, \frac{r}{s} > 1$, $(p,q) = (r,s) = 1$ and $\frac{s}{r} + \frac{q}{p} = 1 - \frac{1}{pr}$ such that Y takes the form*

1.
$$Y = F\left(\frac{k}{2}; \frac{p}{q}, \frac{r}{s}, \left\{\frac{p}{p-q}, \frac{p}{q}\right\}^{\geq 0}, \left\{\frac{r}{r-s}, \frac{r}{s}\right\}^{\geq 0}\right), \text{ or}$$

2.
$$Y = F\left(\frac{k}{2}; \frac{p}{q}, \frac{r}{s}, \left\{pr, \frac{pr}{pr-1}\right\}^{\geq 1}, \left\{\frac{p}{q}, \frac{p}{p-q}\right\}^{\geq 0}, \left\{\frac{r}{s}, \frac{r}{r-s}\right\}^{\geq 0}\right).$$

Moreover, in case (1) Y embeds smoothly in S^4. Here the notation $\{\frac{a}{b}, \frac{a}{a-b}\}^{\geq m}$ means that there are at least m pairs of fractions of this form.

Proof. We will prove that if Y embeds then it takes the desired form. We leave the proof that the family in (1) smoothly embeds to the next section, see Proposition 4.7.3.

Suppose that $Y = F(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ with $k = 2e$ is partitionable. Since the property that $k = 2e$ is preserved under expansions, we can assume that Y is obtained by a (possibly empty) sequence of expansions from a partitionable space that is minimal in the sense that it is not obtained by expansion from any other partitionable space. Assume that Y is such a minimal space. By

Lemma 4.6.1(iii) minimality implies that $e \leq \frac{2k+2}{5} = \frac{4e+2}{5}$. This shows that $e \leq 2$.

If $e = 1$, then $Y = F(1; \frac{p}{q}, \frac{r}{s})$ for some $\frac{p}{q}, \frac{r}{s}$ such that $\frac{q}{p} + \frac{s}{r} = 1 - \frac{1}{\operatorname{lcm}(p,r)}$. However Lemma 4.4.4 implies that p and r are coprime so $\operatorname{lcm}(p,r) = pr$.

If $e = 2$, then $Y = F(2; \frac{p_1}{q_1}, \dots, \frac{p_4}{q_4})$. We consider the possible partitions, $P_1 = \{C_1, C_2\}$ and $P_2 = \{D_1, D_2\}$ of such a Y. We assume that C_1 and D_1 are the complementary classes and C_2 and D_2 are the non-complementary classes. By Lemma 4.6.1 the minimality of Y shows that we cannot have $|C_2| = |D_2| = 1$ or $|C_1| = |C_2| = 2$. Thus we can assume that $|C_1| = 3$, $|C_2| = 1$, $|D_1| = 2$ and $|D_2| = 2$. Suppose that $C_2 = \{1\}$. This implies that $\frac{q_1}{p_1} = 1 - \frac{1}{\operatorname{lcm}(p_1,\dots,p_4)}$. We may assume that $\{1,2\}$ is a class in P_2. Since $\frac{p_2}{q_2} \leq \operatorname{lcm}(p_1, \dots, p_4)$, we have that $\frac{q_1}{p_1} + \frac{q_2}{p_2} \geq 1$. Thus $D_1 = \{1,2\}$ is the complementary class and $\frac{p_2}{q_2} = \operatorname{lcm}(p_1, \dots, p_4)$. By Lemma 4.4.4, we have $\gcd(p_1, \dots, p_4) = 1$. Since $p_1 = p_2 = \operatorname{lcm}(p_1, \dots, p_4)$, it follows that p_3 and p_4 must be coprime. Since the complementary class C_1 is $C_1 = \{2,3,4\}$, it follows that $\frac{q_3}{p_3} + \frac{q_4}{p_4} + \frac{1}{\operatorname{lcm}(p_1,\dots,p_4)} = 1$. This implies that $\operatorname{lcm}(p_1, \dots, p_4) = p_3 p_4$. Thus by taking $\frac{p_3}{q_3} = \frac{p}{q}$ and $\frac{p_4}{q_4} = \frac{r}{s}$ we see that Y takes the form $Y = F(2; \frac{p}{q}, \frac{r}{s}, \frac{pr}{pr-1}, pr)$, where $\frac{q}{p} + \frac{s}{r} + \frac{1}{pr} = 1$.

Thus if Y is partitionable and $e = \frac{k}{2}$, then Y is obtained by a sequence of expansions from either $F(1; \frac{p}{q}, \frac{r}{s})$ or $F(2; \frac{p}{q}, \frac{r}{s}, \frac{pr}{pr-1}, pr)$, where $\frac{q}{p} + \frac{s}{r} + \frac{1}{pr} = 1$. By Theorem 4.1.4, this shows that if Y smoothly embeds in S^4, then it is of the form required by the theorem. $\square$

Remark 4.6.2. *We remark that the family* (2) *in Theorem 4.1.2 arises only when one of the partitions has a complementary class indexing fractions of the form* $\frac{p}{q}, \frac{r}{s}, pr$. *The above proof shows this when* $e = 2$, *and it follows inductively for larger* e *from the way the partitions for* Y' *are obtained from* P_1 *and* P_2 *in the proof of Lemma 4.6.1.*

4.7 Constructing embeddings of Seifert fibered spaces

In this section we construct embeddings of the families of Seifert fibered spaces in Theorem 4.1.1 and Theorem 4.1.2(1). We also recall what is known in the $\varepsilon(Y) = 0$ case and make some observations which give some new embeddings.

Lemma 4.1.6. *If* Y' *is obtained by expansion from* Y, *then* Y' *smoothly embeds in* $Y \times [0, 1]$. *In particular, if* Y *embeds smoothly in* S^4, *then so does* Y'.

Proof. Let $Y = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ and $Y' = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}, -\frac{p_k}{q_k}, \frac{p_k}{q_k})$ a space obtained by expansion from Y. We will explicitly find a subset of $Y \times [0, 1]$ which is homeomorphic to Y'. Let $N_1 \subset Y$ be a Seifert fibered neighbourhood of the exceptional fiber corresponding to p_k/q_k, that is, a set homeomorphic to $S^1 \times D^2$ whose boundary is a union of regular fibres. Consider the set $M = N_1 \times [\frac{1}{4}, \frac{3}{4}]$. The boundary ∂M is homeomorphic to $S^1 \times S^2$ and it naturally inherits a Seifert fibred structure of the form $\partial M = S^2(0; -\frac{p_k}{q_k}, \frac{p_k}{q_k})$. On $N_1 \times \{\frac{1}{4}\}$ and $N_1 \times \{\frac{3}{4}\}$ this structure is a translate of the one on N_1, giving the two exceptional fibres, and is the obvious product structure on $\partial N_1 \times [\frac{1}{4}, \frac{3}{4}]$. Now let $N_2 \subseteq N_1$ be a Seifert fibred neighbourhood of a regular fiber. We take X to be the subset

$$X = (Y \setminus \operatorname{int} N_2) \times \{0\} \cup \partial N_2 \times [0, \frac{1}{4}] \cup (\partial M \setminus \operatorname{int}(N_2 \times \{\frac{1}{4}\})).$$

As a manifold, X is obtained by taking Y and M, deleting open fibred neighbourhoods of regular fibers in both and gluing the two resulting manifolds along their boundaries so that the boundary fibers match up. From this description X is clearly homeomorphic to Y'. Thus by smoothing the corners of X we can obtain a smooth embedding of Y' into $Y \times [0, 1]$. $\square$

Remark 4.7.1. *Although all our applications are for Seifert fibered spaces over orientable surfaces, both the definition of expansion and Lemma 4.1.6 work perfectly well over non-orientable surfaces.*

The following proposition is due to Crisp-Hillman [CH98, Lemma 3.2].

Proposition 4.7.2. *Let $Y_g = F_g(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ where F_g is an orientable genus $g \geq 0$ surface. If Y_g smoothly embeds in S^4, then Y_{g+1} smoothly embeds in S^4.*

Proof. We follow the approach due to Donald [Don13, Lemma 2.23]. We prove that Y_{g+1} smoothly embeds in $Y_g \times [0, 1]$ via Kirby calculus. Start with a surgery presentation for Y_g as in Figure 4.2. Take a relative handle decomposition of $Y_g \times [0, 1]$ by attaching handles around the meridian of the curve representing the central vertex (the e framed curve) as shown in Figure 4.5.

To see the embedding of Y_{g+1} in this manifold observe that the dotted circle and one of the 0-framed unknots form a Whitehead double, so their boundary along with the surgery presentation for Y_g provide the embedding into $Y_g \times [0,1]$. To see that the Kirby diagram is $Y_g \times [0,1]$, observe that 0-framed handle in the Whitehead double can be unlinked from the dotted curve by sliding over the meridional 0-framed unknot. This curve can then be cancelled with the 3-handle, leaving the 1-handle and 2-handle which form a cancelling pair.

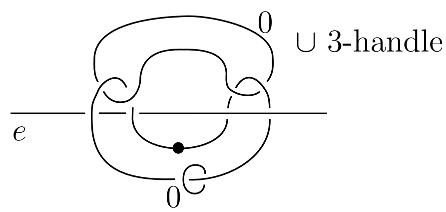

Figure 4.5: Increasing the genus

□

Together these allow us to find the embeddings required for Theorem 4.1.1 and Theorem 4.1.2.

Proposition 4.7.3. *Let Y be a Seifert fibered space over orientable base surface F, with $k > 2$ exceptional fibers, in either of the following two families:*

(a) $F\left(\frac{k+1}{2}; \frac{a}{a-1}, a, \ldots, \frac{a}{a-1}\right) = F(0; -a, a, \ldots, -a)$, *where $a > 1$ is an integer, or*

(b) $F\left(\frac{k}{2}; \frac{p}{q}, \frac{p}{p-q}, \cdots, \frac{p}{q}, \frac{r}{s}, \frac{r}{r-s}, \cdots, \frac{r}{s}\right) = F\left(0; \frac{p}{q}, -\frac{p}{q}, \ldots, \frac{p}{q}, \frac{r}{s}, -\frac{r}{s}, \ldots, \frac{r}{s}\right)$ *where* $\frac{p}{q}, \frac{r}{s} > 1$ *and* $\frac{q}{p} + \frac{s}{r} = 1 - \frac{1}{pr}$.

Then Y smoothly embeds in S^4.

Proof. Observe that S^3 admits Seifert fibered structures of the form $S^2(1; \frac{a}{a-1})$ and $S^2(1; \frac{p}{q}, \frac{r}{s})$, where $\frac{q}{p} + \frac{s}{r} = 1 - \frac{1}{pr}$. Since S^3 smoothly embeds in S^4 and each of the families is obtained from one of these structures on S^3 by a sequence of expansions and possibly increasing the genus of the base surface, Lemma 4.1.6 and Proposition 4.7.2 allow us to build the necessary embeddings. □

Remark 4.7.4. *Some of the Seifert fibered spaces in Proposition 4.7.3 were already known to embed in S^4. Crisp-Hillman [CH98, Section 3a] showed that the manifolds in (a) embed in S^4. Donald [Don15] showed that for $k = 3, 4$, the manifolds in family (a) and a subfamily of those in (b) embed in S^4 as the double branched cover of doubly slice links.*

We now recall what is known about and make some brief observations on smoothly embedding Seifert fibered spaces Y over an orientable base surface with $\varepsilon(Y) = 0$.

Donald [Don15, Theorem 1.3] used Donaldson's theorem to prove that in order for Y to smoothly embed the Seifert invariants must occur in complementary pairs. More precisely, he shows the following.

Theorem 4.7.5. *Let Y be a Seifert fibered space over a closed orientable surface F with $\varepsilon(Y) = 0$. If Y smoothly embeds in S^4 then Y is of the form*

$$F\left(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_k}{q_k}\right) = F\left(k; \frac{p_1}{q_1}, \frac{p_1}{p_1 - q_1}, \ldots, \frac{p_k}{q_k}, \frac{p_k}{p_k - q_k}\right),$$

where $k \geq 0$ and $\frac{p_i}{q_i} > 1$ for all $i \in \{1, \ldots, k\}$.

We remark that a proof of Theorem 4.7.5 also follows from Theorem 2.1.4. It is still not known precisely which Seifert fibered spaces Y of the form given in Theorem 4.7.5 smoothly embed in S^4. Crisp-Hillman [CH98, Remark following Lemma 3.1] showed that if p_i is odd for all $i \in \{1, \dots, k\}$ then Y smoothly embeds. Donald [Don15] showed that $S^2(0; a, -a, b, -b)$, where $a, b \in \mathbb{Z}$ are non-zero, embeds if a is even and b is odd. If a and b are both even and $a \neq b$, then he used Furuta's 10/8 theorem to show that the Seifert fibered space does not embed. It turns out that embedding Seifert fibered spaces with $\varepsilon = 0$ is closely related to embedding Seifert fibered spaces over D^2. We will make use of the following easy observation.

Lemma 4.7.6. *Let* $Y = F(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$, *then for any subset* $\{i_1, \dots, i_l\} \subseteq \{1, \dots, k\}$, Y *contains a submanifold homeomorphic to* $D^2(\frac{p_{i_1}}{q_{i_1}}, \dots, \frac{p_{i_l}}{q_{i_l}})$.

Proof. Consider the projection of Y onto its base orbifold $\widehat{F}$. Choose a disk in $\widehat{F}$ containing the cone points corresponding to the exceptional fibers given by the fractions $\frac{p_{i_1}}{q_{i_1}}, \dots, \frac{p_{i_l}}{q_{i_l}}$ in its interior. The pre-image of this disk in Y is the desired submanifold. $\square$

This allows us to characterize when a Seifert fibered space with $\varepsilon = 0$ embeds in S^4 in terms of the existence of an embedding for a Seifert fibered space over D^2. This characterization shows that existence of an embedding is independent of the genus of the base surface. This is in contrast to the situation for spaces with $\varepsilon \neq 0$, where it is unknown how important the genus of the base surface is to the existence of an embedding into S^4.

Proposition 4.7.7. *The Seifert fibered space* $Y = F(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_k}{q_k})$ *over orientable base surface* F *embeds smoothly in* S^4 *if and only if the Seifert fibered space* $\widetilde{Y} = D^2(\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ *smoothly embeds in* S^4.

Proof. By Lemma 4.7.6, Y contains $\widetilde{Y}$ as a submanifold, so an embedding of Y gives an embedding of $\widetilde{Y}$. This proves the "only if" direction. In the opposite direction notice that the manifold $Y' = \widetilde{Y} \cup_\partial -\widetilde{Y}$ we obtain by doubling $\widetilde{Y}$ along its boundary is homeomorphic to $S^2(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_k}{q_k})$. If $\widetilde{Y}$ embeds in S^4 then it has a tubular neighbourhood $\widetilde{Y} \times [0,1] \subseteq S^4$. The boundary of this tubular neighbour is homeomorphic to $Y' \cong S^2(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_k}{q_k})$. By applying Proposition 4.7.2 to raise the genus of the base surface if necessary, this shows that Y embeds smoothly in S^4. $\square$

We also extend the result of Crisp-Hillman described above.

Proposition 4.7.8. *Let* $Y = S^2(0; \frac{p_1}{q_1}, -\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}, -\frac{p_k}{q_k})$ *where* p_i *is even for at most one* i*. Then* Y *smoothly embeds in* S^4.

Proof. If precisely one of the p_i is even, then let $Y' = S^2(0; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$. If all the p_i are odd, then define Y' by

$$Y' = \begin{cases} S^2(0; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}) & \text{if } q_1 + \cdots + q_k \equiv 1 \bmod 2 \\ S^2(1; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}) & \text{if } q_1 + \cdots + q_k \equiv 0 \bmod 2 \end{cases}$$

These are chosen to ensure that $|H_1(Y')|$ is odd. Therefore Y' is the double branched cover of a Montesinos knot K, and Zeeman's twist-spinning theorem [Zee65] implies that $Y' \setminus \{pt\}$ smoothly embeds in S^4 as a fiber of the complement of the 2-twist spin of K. However Lemma 4.7.6 shows that $Y' \setminus \{pt\}$

contains a submanifold homeomorphic to $D^2(\frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$. Therefore Y embeds in S^4 by Proposition 4.7.7. $\square$

Further variations on these ideas are also possible.

Example 4.7.9. *There is a smooth embedding of $S^2(0; 4, -4, \frac{12}{5}, -\frac{12}{5})$ into S^4. In [Don15, Example 2.14], Donald showed that $S^2(1; 4, 4, \frac{12}{5})$ embeds smoothly in S^4. This contains a $D^2(4, \frac{12}{5})$ submanifold, giving an embedding of $S^2(0; 4, -4, \frac{12}{5}, -\frac{12}{5})$.*

4.8 The Neumann-Siebenmann invariant

In this section, we apply the $\overline{\mu}$ invariant to the question of when a Seifert fibered space can embed smoothly into S^4. The main result of this section is Proposition 4.8.8, which allows us to add further conditions to partitions arising from Theorem 4.1.4 when there is an exceptional fiber of even multiplicity. This allows us to prove Theorem 4.1.10 and Proposition 4.1.7. Throughout this section let $Y = S^2(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space with $\varepsilon(Y) > 0$ and $\frac{p_i}{q_i} > 1$ for all i. Let Γ be the canonical plumbing graph corresponding to Y with vertex set V and X the positive definite manifold obtained by plumbing according to Γ.

We say that a subset $C \subseteq V$ is *characteristic* if $x = \sum_{v \in C} v$ is characteristic when considered as a vector in the intersection lattice $(\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$. Recall that a vector x in an integer lattice is characteristic if

$$x \cdot z \equiv z \cdot z \bmod 2$$

for all z in the lattice. It is well known that there is a bijective correspondence between characteristic subsets of Γ and $\mathrm{Spin}(Y)$ [GS99, Proposition 5.7.11][4].

The following definition of the $\overline{\mu}$ invariant is due to Neumann [Neu80]. Siebenmann also gave an equivalent definition in [Sie80].

Definition 4.8.1. *Given a spin structure $\mathfrak{s}$ on Y, the Neumann-Siebenmann invariant $\overline{\mu}(Y, \mathfrak{s})$ is defined as*

$$\overline{\mu}(Y, \mathfrak{s}) = |\Gamma| - \|w\|^2,$$

where $w = \sum_{v \in C} v$ and C is the characteristic subset corresponding to $\mathfrak{s}$ and $|\Gamma| = |V|$ is the number of vertices in Γ.

Remark 4.8.2. *Some comments on this definition are in order:*

1. *We have chosen to define $\overline{\mu}$ in terms of the positive definite plumbing. There is a more general definition that allows $\overline{\mu}$ to be calculated from any plumbing cobounding Y.*

2. *It is not hard to see that any characteristic subset of $C \subset V$ must consist of isolated vertices,[5] that is, no pair of adjacent vertices are both in C. So we can equivalently define*

$$\overline{\mu}(Y, \mathfrak{s}) = |\Gamma| - \sum_{v \in C} \|v\|^2.$$

[4]This correspondence is much more general than we are using here: it applies whenever we have a 3-manifold with a given surgery presentation. It is usually described in terms of characteristic sublinks of a surgery diagram.

[5]The characteristic condition implies that any vertex in a characteristic set must have an even number of neighbours in the set. Since Γ is a tree this forces the subset to be isolated.

It is known that for Seifert fibered spaces over S^2, $\overline{\mu}$ is a spin rational homology cobordism invariant [Ue05] and that $\overline{\mu}(Y,\mathfrak{s}) = 0$ whenever $(Y,\mathfrak{s})$ is the boundary of a spin rational homology ball.

In order to apply $\overline{\mu}$ effectively we need to understand which characteristic subsets correspond to spin structures which extend over a given cobounding spin rational homology ball. We can do this by studying lattice embeddings.

Proposition 4.8.3. *Suppose that Y bounds a smooth spin rational homology 4-ball W with $H^3(W;\mathbb{Z}) = 0$. The inclusion map $X \hookrightarrow X \cup -W$ induces a map on second homology, which we identify with $\iota : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^{|\Gamma|}, Id)$. Let $e_1, \dots, e_{|\Gamma|}$ be an orthonormal basis for $(\mathbb{Z}^{|\Gamma|}, Id)$. Let $\mathfrak{s}$ be a spin structure on Y with corresponding characteristic subset $C \subset V$. Then $\mathfrak{s}$ extends over W if and only if $\sum_{v\in C} \iota(v)$ is characteristic in $\mathbb{Z}^{|\Gamma|}$, that is*

$$\sum_{v\in C} \iota(v) \cdot e_i \equiv 1 \bmod 2$$

for all basis elements e_i.

Proof. Let $Z = X \cup -W$. Since $H^3(W;\mathbb{Z}) = 0$ and $H_1(X;\mathbb{Z}) = 0$, the Mayer-Vietoris sequence and Poincaré-Lefschetz duality imply that $H_1(Z;\mathbb{Z}) = 0$, and thus $H_2(Z;\mathbb{Z})$ is torsion free. Hence, $H_2(Z;\mathbb{Z}) \cong \mathbb{Z}^{|\Gamma|}$. Since Z is positive definite, Donaldson's theorem implies that $(H_2(Z;\mathbb{Z}), Q_Z) \cong (\mathbb{Z}^{|\Gamma|}, \mathrm{Id})$.

Let $F \subset X$ be a closed connected oriented surface, such that $[F] \in H_2(X;\mathbb{Z})$ represents $\sum_{v\in C} v \in (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \cong H_2(X;\mathbb{Z})$. Then F is the obstruction to extending $\mathfrak{s}$ over X, that is, $\mathfrak{s}$ extends to a spin structure $\mathfrak{s}_X$ on $X \backslash F$ which does not extend across F.

Suppose that $\mathfrak{s}$ extends to a spin structure $\mathfrak{s}_W$ on W. Then gluing the spin structures $\mathfrak{s}_W$ and $\mathfrak{s}_X$ along Y gives a spin structure $\mathfrak{s}_Z$ on $Z \backslash F$ which does not extend across F. Thus, the mod 2 reduction of $[F] \in H_2(Z;\mathbb{Z})$ is Poincaré dual to the second Stiefel-Whitney class $w_2(Z) \in H^2(Z;\mathbb{Z}_2)$. However, the Wu formula states that $PD(w_2(Z)) \in H_2(Z;\mathbb{Z}_2)$ is the unique element satisfying $PD(w_2(Z)) \cdot x = x \cdot x$ for all $x \in H_2(Z;\mathbb{Z}_2)$. Thus, we see that $PD(w_2(Z))$ is the mod 2 reduction of a characteristic element of $H_2(Z;\mathbb{Z})$. This implies that $\sum_{v\in C} \iota(v) \cdot e_i \equiv 1 \bmod 2$, as required.

Conversely, suppose that $\sum_{v\in C} \iota(v) \cdot e_i \equiv 1 \bmod 2$ for all e_i. This shows that $\sum_{v\in C} \iota(v)$ reduced mod 2 is Poincaré dual to $w_2(Z)$. Then $Z \backslash F$ admits a spin structure $\mathfrak{s}_Z$. The bijection between characteristic sublinks and spin structures on Y then shows that $\mathfrak{s}_Z$ restricts to $\mathfrak{s}$ on Y. Restricting $\mathfrak{s}_Z$ to $W \subset Z$ then shows that $\mathfrak{s}$ extends to a spin structure on W. $\square$

This allows us to obtain further restrictions on the image of the characteristic subsets corresponding to spin structures that extend over a homology ball.

Proposition 4.8.4. *Suppose that Y bounds a spin rational homology ball W with $H^3(W;\mathbb{Z}) = 0$. Let $\iota : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^{|\Gamma|}, Id)$ be the lattice embedding induced by the inclusion $X \hookrightarrow X \cup -W$. For any choice of orthonormal basis $\{e_i\}$, the following are true:*

1. *Let C be a characteristic subset corresponding to a spin structure which extends over W. Then for all $v \in C$, we have $|\iota(v) \cdot e_i| \leq 1$ for all e_i and*

for each e_i there is precisely one $v \in C$ with $|\iota(v) \cdot e_i| = 1$.

2. *For any $m \in \{1, \ldots, |\Gamma|\}$, there are at most two distinct vertices with the property that the image of each vertex under ι pairs non-trivially with e_m and each vertex belongs to a characteristic subset corresponding to a spin structure that extends over W.*

Proof. We will abuse notation by identifying each vertex of Γ with its image under ι. If the spin structure corresponding to C extends over W, then the corresponding $\overline{\mu}$ invariant vanishes. This implies that

$$\sum_{v \in C} v = \sum_{v \in C} \sum_{i=1}^{|\Gamma|} (v \cdot e_i)^2 = |\Gamma|.$$

By Proposition 4.8.3, we have $\sum_{v \in C} e_i \cdot v$ is odd for all i. Thus there is at least one vertex in C satisfying $v \cdot e_i \neq 0$. However by the above equation we see that there is at most one such v and it satisfies $|v \cdot e_i| = 1$. This verifies (1).

Now suppose that we have characteristic subsets C_1, C_2 and C_3 corresponding to spin structures that extend over W. Suppose that v_1, v_2 and v_3 are distinct vertices satisfying $v_i \cdot e_m \neq 0$ and $v_i \in C_i$ for $i \in \{1, 2, 3\}$. It follows from (1) that $v_i \in C_j$ if and only if $i = j$. Now define C_4 to be the set of vertices such that v is in C_4 if and only it is contained in precisely one or three of C_1, C_2 or C_3. We have that v_1, v_2 and v_3 are all in C_4. It is easy to verify that not only is C_4 a characteristic subset, but that for any unit basis vector e_i, we have

$$\sum_{v \in C_4} v \cdot e_i \equiv \sum_{v \in C_1} v \cdot e_i + \sum_{v \in C_2} v \cdot e_i + \sum_{v \in C_3} v \cdot e_i \equiv 1 \bmod 2.$$

So by Proposition 4.8.3 we see that C_4 also corresponds to a spin structure that extends over W. Thus by (1) we see that at most one of $v_1 \cdot e_m, v_2 \cdot e_m$ and $v_3 \cdot e_m$ can be non-zero, a contradiction. This proves (2). $\square$

We now need to understand the characteristic subsets of Γ. When p_i is even for at least one i, these are determined by choosing characteristic subsets on the linear chains corresponding to the fibers of Y. Thus we need to understand the characteristic subsets on linear chains first.

Lemma 4.8.5. *Let Δ be the linear chain corresponding to $p/q = [a_1, \ldots, a_l]^-$, where $a_j \geq 2$ for all j.*

1. *If p is odd, then Δ has a unique characteristic subset.*
2. *If p is even, then Δ has two characteristic subsets, where one contains the first vertex and the other does not.*

Proof. The characteristic subsets on Δ are in bijection with spin structures on the lens space $L(p, q)$. Thus there is precisely one if p is odd and precisely two if p is even. Now suppose that p is even and we will justify the statement concerning the leading vertex. Consider the matrix

$$M = \begin{pmatrix} a_1 & -1 & \\ -1 & \ddots & -1 \\ & -1 & a_l \end{pmatrix} \bmod 2.$$

We can think of a characteristic subset of Δ as a vector $w \in \mathbb{Z}_2^l$ such that

$$Mw \equiv \begin{pmatrix} a_1 \\ \vdots \\ a_l \end{pmatrix} \bmod 2.$$

Thus if w and w' are the vectors in $\mathbb{Z}_2^l$ corresponding to the two distinct characteristic subsets, then the vector $w - w'$ is a non-zero element of $\ker M$ mod two. However, if $v = \begin{pmatrix} v_1 \\ \vdots \\ v_l \end{pmatrix}$ is a non-zero element of the kernel of M mod two, then v_1 is non-zero. Otherwise, suppose that $v_1 = \cdots = v_{k-1} = 0$ and $v_k \neq 0$ for some $k \leq l$, this would imply that the $(k-1)$-st row of Mv is non-zero. Thus precisely one of the two characteristic subsets contains the first vertex. $\square$

Remark 4.8.6. *Although we will not need this fact, one can show that if p is odd, then the unique characteristic subset on Δ contains the leading vertex if and only if q is odd.*

This allows us to construct the characteristic subsets on Γ when at least one p_i is even.

Lemma 4.8.7. *Suppose that p_i is even for at least one i. Then no characteristic subset of Γ contains the central vertex and any characteristic subset on Γ is uniquely determined by the set of the vertices adjacent to the central vertex it contains. In fact, it suffices to determine which of the leading vertices on arms corresponding to even p_i it contains.*

Proof. We prove this by constructing all characteristic subsets. Suppose that $N \geq 1$ of the p_i are even. By Lemma 4.4.5, Y admits $|H^1(Y;\mathbb{Z}_2)| = 2^{N-1}$ spin structures. We may construct a characteristic subset C as follows. For each arm of Γ corresponding to p_i/q_i with p_i odd include the vertices corresponding to the unique characteristic subset on that linear chain. Suppose that α of these chains include the leading vertex. Now choose a subset S of the arms corresponding to even p_i such that $|S| \equiv \alpha + e \bmod 2$. For each arm in S choose the characteristic subset containing its leading vertex. For all other arms choose the characteristic subset on the linear chain not containing the leading vertex. This defines a characteristic subset since it is characteristic on the arms by construction and does not contain the central vertex. Moreover, it is chosen so that it contains $|S| + \alpha \equiv e \bmod 2$ vertices adjacent to the central vertex. Notice however that of the set of N arms corresponding to even p_i, there are 2^{N-1} even subsets and 2^{N-1} odd subsets. Thus we can construct all the characteristic subsets this way irrespective of the parity of α. □

We can now add further conditions to the partitions in Theorem 4.1.4. The following proposition, although sufficient for our applications, is certainly not the most general statement that can be proven. For example, using Remark 4.8.6, one could also add further conditions relating to the parity of the q_i.

Proposition 4.8.8. *Let $Y = S^2(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ be a Seifert fibered space with $\varepsilon(Y) > 0$, $\frac{p_i}{q_i} > 1$ for all i and p_j even for at least one j. Suppose that Y smoothly embeds in S^4 and let P be one of the partitions of $\{1, \dots, k\}$ given by Theorem 4.1.4. Then the following further conditions apply to P.*

1. *There is precisely one class containing an odd number of i for which p_i is even and there are one or three such i.*

2. In all other classes there are zero or two values of i such that p_i is even.

Moreover suppose that $C = \{1, \dots, l\}$ is a complementary class such that p_1 and p_2 are even and p_i is odd for all $3 \leq i \leq l$, then

$$\lceil p_1/q_1 \rceil \leq 1 + \sum_{i=2}^{l}(p_i - 1). \tag{4.8.1}$$

Proof. Recall that these partitions are constructed by taking the splitting $S^4 = U_1 \cup_Y -U_2$ and $\iota_1 : (\mathbb{Z}^{|\Gamma|}, Q_\Gamma) \to (\mathbb{Z}^{|\Gamma|}, \mathrm{Id})$ be the lattice embedding induced by the inclusion $X \hookrightarrow X \cup -U_i$ for $i = 1$ or 2. Without loss of generality, we will work with $\iota = \iota_1$. We will abuse notation and identify each vertex of Γ with its image under ι. As shown in Lemma 4.5.2 we may assume that the central vertex is given by $\nu = e_1 + \cdots + e_e$ and for $i = 1, \dots, e$ the class C_i is taken to be the subset of $\{1, \dots, k\}$ such that the first vertex of the linear chain corresponding to p_i/q_i pairs non-trivially with e_i.

Suppose that Y has $N \geq 1$ exceptional fibers of even order, so that $\dim H^1(Y; \mathbb{Z}_2) = N - 1$ by Lemma 4.4.5. Let n_i be the number of fibers of even order in each class of the partition. Let C be a characteristic set corresponding to a spin structure which extends over the ball U_1. By Proposition 4.3.1(2) we have $H^3(U_1; \mathbb{Z}) = 0$, so Proposition 4.8.4(1) applies, implying that for each class there is precisely one arm from each class whose leading vertex is in C. Moreover Proposition 4.8.4(2) shows that for each class in the partition there are at most two choices for the arm whose leading vertex can appear in any such C. However since characteristic subsets all coincide on arms corresponding to odd p_i, we see that two choices for the leading vertex from arms in a class C_i can only be realized if $n_i \geq 2$. Thus, if there are m values of n_i such that $n_i \geq 2$, then at most 2^m spin structures extend over U_1. However by Lemma 4.3.3, we know that $2^{(N-1)/2}$ spin structures extend over U_1. This shows that

$$2m \geq N - 1 = n_1 + \cdots + n_e - 1.$$

This shows that with exactly one exception $n_i \in \{0, 2\}$ and for this exception we must have $n_i \in \{1, 3\}$, which completes the count of even p_i in each class.

Now we establish (4.8.1). Suppose that we have the class $C_1 = \{1, \ldots, l\}$ is complementary with p_1 and p_2 even and all other p_i is this class odd, that is $n_1 = 2$. The argument in the previous paragraph shows that the leading vertices of both the arms corresponding to p_1/q_1 and p_2/q_2 must appear in characteristic subsets corresponding to spin structures that extend over U_1. In particular if v is the leading vertex of the arm corresponding to p_1/q_1, then v satisfies $|v \cdot e_i| \leq 1$ for all i by Proposition 4.8.4(2) and $\|v\|^2 = \lceil p_1/q_1 \rceil$ by definition. So to bound $\|v\|^2$ above it suffices to bound above the number of basis elements e_i for which $|v \cdot e_i| \neq 0$. To do this notice that if $|v \cdot e_i| \neq 0$, then $w \cdot e_i \neq 0$ for some other vertex w appearing in one of the other chains in the class C_1. Otherwise we could consider the vector $v' = v - (v \cdot e_i)e_i$ to obtain an embedding of linear chains with corresponding fractions $\lceil p_1/q_1 \rceil - 1, p_2/q_2, \ldots, p_l/q_l$. Since $\lceil p_1/q_1 \rceil - 1 < p_1/q_1$, this would contradict Theorem 2.3.2. However, by inducting on the length of the continued fraction, one can see that an embedding of the linear chain corresponding to r/s can use at most r distinct orthonormal basis vectors. Thus we see that

$$\lceil p_1/q_1 \rceil = \|v\|^2 \leq 1 + \sum_{i=2}^{l}(p_i - 1),$$

where $p_i - 1$ terms come from observing that by definition all the linear chains in C_1 have at least one common basis element with which they pair non-trivially. This is the required upper bound. □

We now have the tools to establish our lower bound on e.

Theorem 4.1.10. *Let $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ be a Seifert fibered space with $\varepsilon(Y) > 0$ and $\frac{p_i}{q_i} > 1$ for all i. If Y smoothly embeds in S^4 then $\dim H^1(Y; \mathbb{Z}_2) \leq 2e$.*

Proof. First note that if Y has no exceptional fibers of even order and Y embeds in S^4, then $H^1(Y; \mathbb{Z}_2) = 0$. So we may suppose that Y has at least one exceptional fiber of even order. Proposition 4.8.8 shows that there can be at most $2e + 1 = 3 + 2(e - 1)$ such fibers. Thus by Lemma 4.4.5 we have $H^1(Y; \mathbb{Z}_2) \leq 2e$ in this case too. □

Remark 4.8.9. *Donald showed that $S^2(1; 4, 4, \frac{12}{5})$ smoothly embeds in S^4 [Don15, Example 2.14]. This Seifert fibered space and its expansions show that the bound in Theorem 4.1.10 is sharp.*

We conclude with the following lemma which justifies Proposition 4.1.7. To see this, note that the Seifert fibered spaces in Theorem 4.1.2(1) only arise when applying Theorem 4.1.4 when there is a partition containing a complementary class of the form $\{\frac{p}{q}, \frac{r}{s}, rp\}$ (cf. Remark 4.6.2). The following lemma shows that rp must be odd.

Lemma 4.8.10. *If $Y = S^2(e; \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k})$ embeds smoothly into S^4, then neither of the partitions given by Theorem 4.1.4 can contain a complementary class of the form $\{\frac{p}{q}, \frac{r}{s}, rp\}$ with rp even.*

Proof. Suppose that we had such a class. Since the class is complementary, we have $\frac{s}{r} + \frac{q}{p} + \frac{1}{rp} = 1$. This implies that p and r are coprime so precisely one of r or p is even. Thus (4.8.1) from Proposition 4.8.8 applies to show that $rp \leq r + p - 1$. This is easily seen to be impossible as $r, p > 1$. □

4.9 Doubly slice Montesinos links

In this section we turn our attention to doubly slice links. We prove that the Seifert fibered spaces over S^2 in Theorem 4.1.1 and Theorem 4.1.2(1) are double branched covers of Montesinos links. We also prove Theorem 4.1.11 which provides a classification of the smoothly doubly slice odd pretzel knots up to mutation. We then prove Proposition 4.1.12 showing that no non-trivial quasi-alternating Montesinos link is doubly slice. Finally, we show that the Seifert fibered spaces $S^2(1; \frac{7}{2}, \frac{7}{2}, \frac{7}{2})$, $S^2(1; 4, \frac{12}{5}, 4)$ and $S^2(1; 3, \frac{15}{4}, 3)$ are double branched covers of doubly slice Montesinos links.

Proposition 4.9.1. *Let Y be a Seifert fibered space over S^2, with $k > 2$ exceptional fibers, in either of the following two families:*

(a) $S^2\left(\frac{k+1}{2}; \frac{a}{a-1}, a, \ldots, \frac{a}{a-1}\right) = S^2(0; -a, a, \ldots, -a)$, *where* $a > 1$ *is an integer, or*

(b) $S^2\left(\frac{k}{2}; \frac{p}{q}, \frac{p}{p-q}, \cdots, \frac{p}{q}, \frac{r}{s}, \frac{r}{r-s}, \cdots, \frac{r}{s}\right) = S^2\left(0; \frac{p}{q}, -\frac{p}{q}, \ldots, \frac{p}{q}, \frac{r}{s}, -\frac{r}{s}, \ldots, \frac{r}{s}\right)$ *where* $\frac{p}{q}, \frac{r}{s} > 1$ *and* $\frac{s}{r} + \frac{q}{p} = 1 - \frac{1}{pr}$.

Then Y *is the double branched cover of a smoothly doubly slice Montesinos link.*

As discussed in Remark 4.7.4, some special cases of Proposition 4.9.1 were previously known by work of Donald [Don15]. We give two proofs of Proposition 4.9.1. The first proof gives a method for constructing new doubly slice links out of old ones. The second proof uses a doubly slice criterion due to Donald [Don15].

The first proof

At the heart of our first proof of Proposition 4.9.1 is the following lemma which allows us to modify a doubly slice link to construct new doubly slice links.

Lemma 4.9.2. *Let* $L \subset S^3$ *be a link with planar diagram* D_L *and suppose that the diagram* D_L *contains a disk* D *intersecting the link in a* 2*-tangle* T*. Let* L' *be the link obtained by modifying* D_L *inside of* D *as shown in Figure 4.6. If* L *is smoothly doubly slice then* L' *is also smoothly doubly slice.*

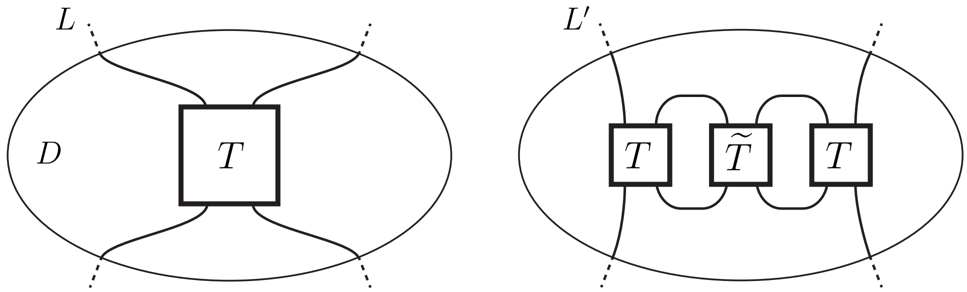

Figure 4.6: The 2-tangle $\widetilde{T}$ is the mirror image of T. More precisely, $\widetilde{T}$ is obtained from T by first doing a π-rotation (in $\mathbb{R}^3$) about the vertical axis which cuts through the center of T and lies on the projection plane, and then changing every crossing of the resulting tangle.

Proof. Since L is doubly slice, there is an embedded 2-sphere $F \subset S^4$ with $F \cap S^3 = L$. Hence, we can find a neighbourhood of (S^3, L) in (S^4, F) homeomorphic to $(S^3 \times [0,1], L \times [0,1])$. We will show that we can isotope $S^3 \times \{0\}$ inside this neighbourhood to intersect $L \times [0,1]$ in precisely L', thus showing that L' is doubly slice.

It may be helpful to refer to Figure 4.8 while reading the following construction. We may assume that L sits inside $S^2 \times [0, \epsilon] \subset S^3$, with planar projection onto S^2 giving the diagram D_L. Let $B_L = D \times [0, \epsilon] \subset S^2 \times [0, \epsilon] \subset S^3$. Then $B_L \times [\frac{1}{2}, \frac{3}{4}] \subset S^3 \times [0, 1]$ is a 4-ball.

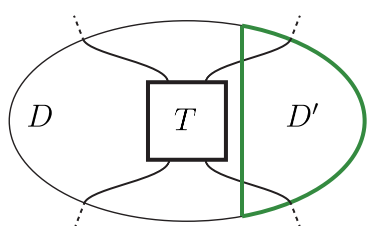

Figure 4.7: The disk $D' \subset D$ with boundary shown in green.

Let $D' \subset D$ be the disk as shown in Figure 4.7, and let $B'_L = D' \times [0, \epsilon] \subset$

$S^2 \times [0, \epsilon] \subset S^3$. Consider the 4-manifold

$$X = (S^3 \times [0, \frac{1}{4}]) \cup (B'_L \times [\frac{1}{4}, \frac{1}{2}]) \cup (B_L \times [\frac{1}{2}, \frac{1}{4}])$$

sitting inside $S^3 \times [0,1]$. It has two boundary components, namely $S^3 \times \{0\}$ and another boundary component Y homeomorphic to S^3. The intersection of Y with $L \times [0,1]$ is precisely L', see Figure 4.8. Finally, note that X is homeomorphic to $S^3 \times [0,1]$ as it is obtained from $S^3 \times [0, \frac{1}{4}]$ by gluing on a 4-ball along a 3-ball in its boundary. Hence Y and $S^3 \times \{0\}$ are ambient isotopic in S^4. This argument also works in the smooth category by appropriately rounding corners.

□

Proposition 4.9.3. *If the Montesinos link $L := M(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k})$ is doubly slice then $L_i := M(e; \frac{p_1}{q_1}, \dots, \frac{p_{i-1}}{q_{i-1}}, \frac{p_i}{q_i}, -\frac{p_i}{q_i}, \frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}}, \dots, \frac{p_k}{q_k})$ is also doubly slice, for $1 \leq i \leq k$.*

Proof. This follows immediately by applying Lemma 4.9.2 to the standard Montesinos diagram of L (see Figure 3.2) and taking D to be the disk containing precisely the rational tangle $\frac{p_i}{q_i}$. □

Proof of Proposition 4.9.1. It suffices to prove that the Montesinos links

1. $M(0; -a, a, \dots, -a)$, where $a > 1$ is an integer and,
2. $M\left(0; \frac{p}{q}, -\frac{p}{q}, \dots, \frac{p}{q}, \frac{r}{s}, -\frac{r}{s}, \dots, \frac{r}{s}\right)$ where $\frac{p}{q}, \frac{r}{s} > 1$ and $\frac{s}{r} + \frac{q}{p} = 1 - \frac{1}{pr}$,

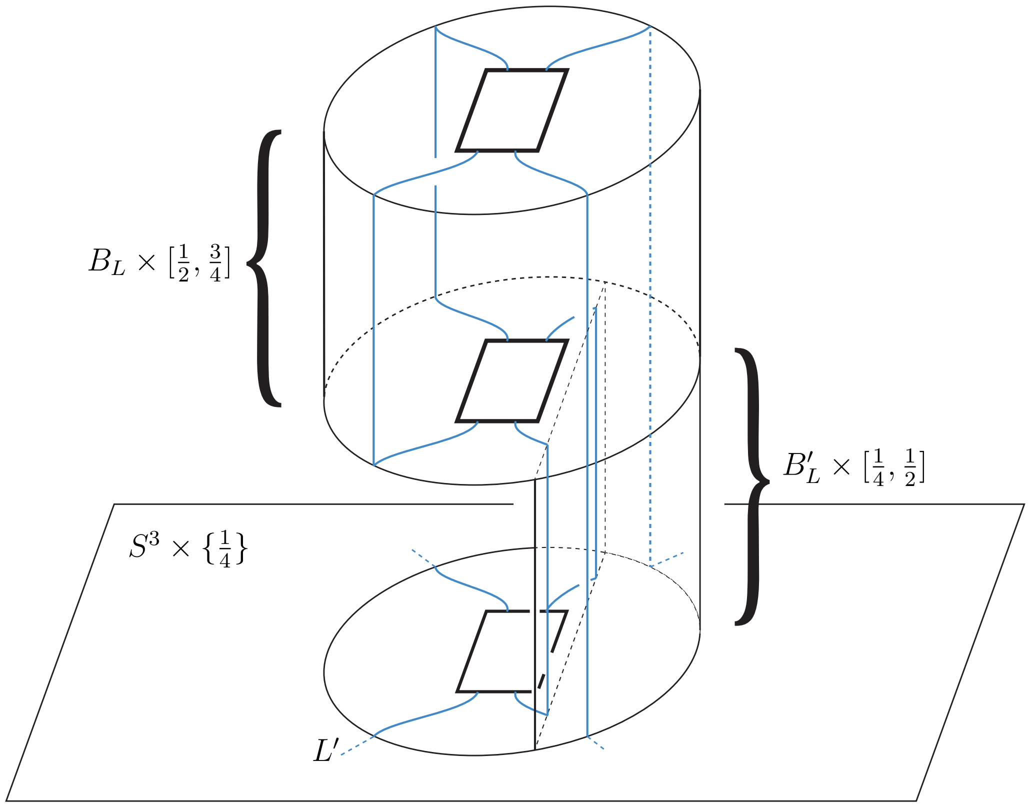

Figure 4.8: Schematic diagram of $L' = Y \cap F$. The link L' is shown in blue.

are doubly slice. We prove this by induction on k (the number of rational parameters). If $k = 1$ or $k = 2$ then every Montesinos link in families (1) and (2) is the unknot (they are 2-bridge links with determinant 1) and hence is doubly slice. The induction step then follows by applying Proposition 4.9.3.

□

The second proof

We now give a second proof of Proposition 4.9.1. We will use the following doubly slice criterion of his [Don15, Corollary 2.5] to prove Proposition 4.9.1.

Theorem 4.9.4. *Suppose L is a link in S^3 and there are two sets of band moves $\{A_i\}_{1\leq i\leq k}$ and $\{B_j\}_{1\leq j\leq l}$ such that performing the moves:*

- $\{A_i\}_{1\leq i\leq k} \cup \{B_j\}_{1\leq j\leq l}$ *gives the unknot,*
- $\{A_i\}_{1\leq i\leq k} \cup \{B_j\}_{1\leq j\leq l-n}$ *gives an $(n+1)$-component unlink for all $n \in \{1, 2, \ldots, l\}$,*
- $\{A_i\}_{1\leq i\leq k-n} \cup \{B_j\}_{1\leq j\leq l}$ *gives an $(n+1)$-components unlink for all $n \in \{1, 2, \ldots, k\}$.*

Then L is smoothly doubly slice.

The collection of band moves that we will use can be quite complicated when viewed in (S^3, L). Instead, these band moves can be more naturally viewed as corresponding to certain 2-handle attachments in the double branched cover of (S^3, L). The following theorem of Montesinos will allow us to make this correspondence.

Theorem 4.9.5 (Theorem 3 of [Mon78])**.** *Consider a handle representation $W^4 = H^0 \cup nH^2$ of a 4-manifold with boundary given by attaching n 2-handles to the 4-ball. If the n 2-handles are attached along a strongly invertible link in S^3, then W is a 2-fold cyclic covering space of D^4 branched over a 2-manifold.*

Montesinos [Mon78] describes how to obtain the branched surface in D^4 from the attaching link and involution. We now describe this construction in the case of interest to us. This is also described in [Lec12], where Lecuona used similar ideas to show certain Montesinos knots are ribbon.

Suppose that the 2-handles in Theorem 4.9.5 are attached along a framed link $L \subset S^3$, where the strong involution is a rotation by π about an axis in S^3. Suppose furthermore that each component of L is an unknot which is given by a trivial arc above and below the rotation axis, see left of Figure 4.9. The branch surface in Theorem 4.9.5 has a simple description as follows. Replace each arc below the rotation axis with a twisted band following the arc, with twisting such that the signed number of crossings in the band is equal to the framing of the link component containing the arc, see Figure 4.9. These bands are attached to a rectangular disc with an edge lying on the axis of rotation. The bands and rectangular disc form a surface in S^3. Pushing this surface into D^4 gives the branch surface in Theorem 4.9.5.

Observe that if $L = L' \cup \{K\}$ as framed links then the branched surface S for L is obtained from the branched surface S' for L' by a band attachment. In particular, the link ∂S is obtained from $\partial S'$ by a band or ribbon move. If L is the integer surgery presentation of a Seifert fibered space Y over S^2 coming from the plumbing graph, then the boundary of the branch surface S is a Montesinos link.

Example 4.9.6. *Consider the Seifert fibered space $Y = S^2(0; 3, -3, 2)$ with surgery presentation and strong involution as in Figure 4.9. Interpreting the surgery presentation as a Kirby diagram for the plumbing 4-manifold X, we see that X is the double branched cover of (D^4, S), where S is the surface in the*

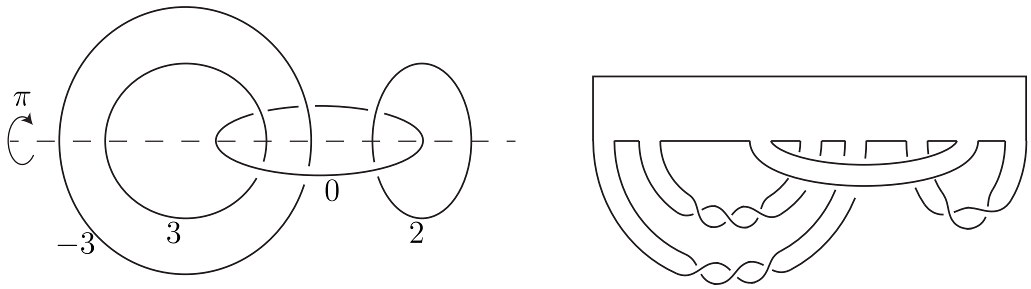

Figure 4.9: Left: Kirby diagram of a 4-manifold with boundary $S^2(0; 3, -3, 2)$. Right: corresponding branch surface with boundary a Montesinos knot.

right of Figure [4.9](#) pushed into the 4*-ball. The knot* $\partial S \subset S^3$ *is the Montesinos knot with double branched cover* Y*.*

Attaching an additional 2*-handle to* X *which respects the strong involution, as shown in bold in the left of Figure [4.10](#), gives a* 4*-manifold* X' *which is the double branched cover of the surface* S' *in the right of Figure [4.10](#). We see that* S' *is obtained from* S *by attaching a* 2*-dimensional* 1*-handle. Hence, the link* $\partial S'$ *is obtained from* $\partial S'$ *by a band, or ribbon move. One can check that* $\partial X' = S^2 \times S^1$*. Since the* 2*-component unlink is the only link in* S^3 *with double branched cover* $S^2 \times S^1$ *[KT80], we get that* $\partial S'$ *is the* 2*-component unlink (one can also see this directly) and the Montesinos knot* ∂S *is ribbon.*

We are now ready to prove Proposition 4.9.1.

Proof of Proposition 4.9.1((a)). Let $Y = S^2(0; -a, a, \ldots, -a)$ with k fibers, where $k \geq 1$ is odd and $a > 2$ is an integer. If $k = 1$ then Y is S^3 which is the double branched cover of the unknot which is trivially doubly slice. Assume that $k > 1$. Then Y is the boundary of the 4-manifold X given by attaching

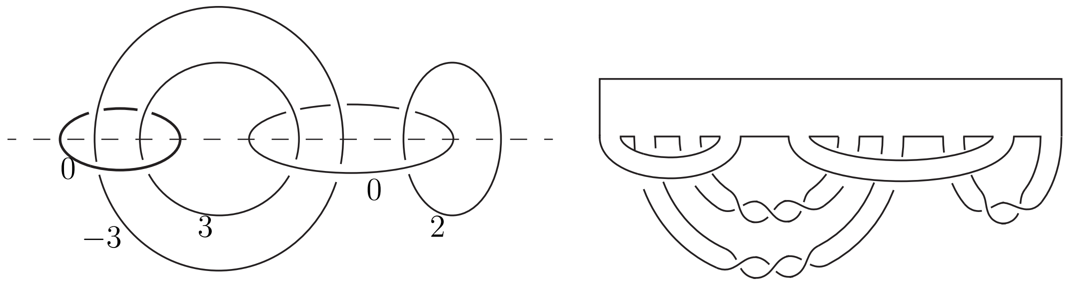

Figure 4.10: Left: The Kirby diagram with the extra 2-handle. Right: The corresponding band in the link.

2-handles to the 4-ball, as shown in Figure 4.11 for $k = 5$ (ignoring for now the 2-handles with labels A_1, A_2, B_1 and B_2). The 2-handles are attached along a strongly invertible link in Figure 4.11, where the involution is given by a π rotation about the dotted axis. Thus, Theorem 4.9.5 implies that X is the double branched cover of D^4 over a properly embedded surface S where $L = \partial S \subset S^3$ is the Montesinos link with double branched cover $\Sigma(L) = Y$.

In Figure 4.11, there are $2m := k-1$ ($k = 5$ shown) additional 0-framed 4-dimensional 2-handles, shown in bold, which are attached equivariantly with respect to the strong involution. By the discussion above Example 4.9.6, there are $2m := k - 1$ disjoint bands $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ defining band moves on L such that doing any subset S of these band moves changes $L \mapsto L'$ in such a way that $\Sigma(L') = \partial X_S$, where X_S is the 4-manifold given by attaching the correspondingly labeled subset of 0-framed 2-handles to X, as in Figure 4.11, or by an isotopy, as in Figure 4.12.

We now show that the two sets of bands $\{A_i\}_{1\leq i\leq m}$ and $\{B_i\}_{1\leq j\leq m}$ satisfy the doubly slice hypotheses of Theorem 4.9.4, thereby showing that

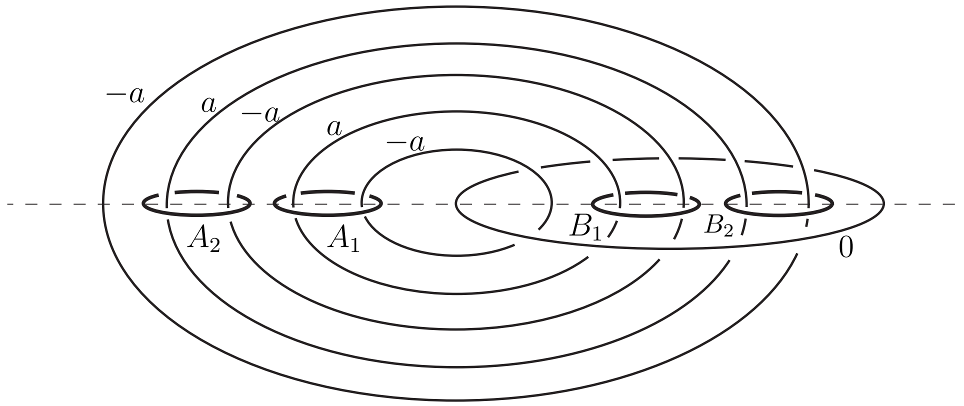

Figure 4.11: Ignoring the curves in bold, $Y = S^2(0; -a, a, \ldots, -a)$ is the doubly branched cover of the link $L \subset S^3$ given by quotienting out by the involution given by rotating about the dotted axis. The case where Y has 5 exceptional fibers is shown.

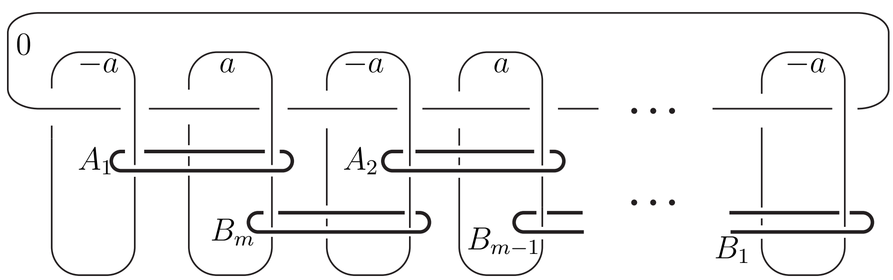

Figure 4.12: Ignoring 2-handles in bold, this is a Kirby diagram of 4-manifold with boundary $S^2(0; -a, a, -a, \ldots, -a)$ containing $k = 2m + 1$ fibers.

L is doubly slice. First, let $S_n = \{A_i\}_{1\leq i\leq m} \cup \{B_j\}_{1\leq j\leq m-n}$, where $n \in \{0, 1, 2, \ldots, m\}$. We can realise X_{S_n} as a union of linear plumbings, by handlesliding the central 0-framed 2-handle over each of the handles labeled $A_1, \ldots, A_m$ as shown in Figure 4.13.

We claim that ∂X_{S_n} is a connected sum of n copies of $S^1\times S^2$. Assuming the claim, by [KT80], the $(n+1)$-component unlink is the unique link in S^3 with double branched cover $\#_n(S^1 \times S^2)$. This implies that performing band moves S_n results in the $(n+1)$-component unlink, for all $n \in \{1, 2, \ldots, m\}$. To show that $\partial X_{S_n} = \#_n(S^1 \times S^2)$, note that ∂X_{S_n} consists of $n+1$ disjoint linear chains of unknots, where n of these chains have length 3 with components having framings (in linear order) $-a, 0, a$ giving an $S^1 \times S^2$ summand. Similarly, the remaining chain has framings $0, -a, 0, a, \ldots, -a, 0, a$ which represents S^3. Thus, $\partial X_{S_n} = \#_n(S^1 \times S^2)$.

By symmetry we may interchange the roles of the $\{A_i\}$ and $\{B_i\}$ bands in the argument given above, which shows that the remaining hypothesis of Theorem 4.9.4 is satisfied, where band moves are performed on $S'_n = \{A_i\}_{1 \leq i \leq m-n} \cup \{B_j\}_{1 \leq j \leq m}$, for $n \in \{1, 2, \ldots, m\}$.

$\square$

Proof of Proposition 4.9.1((b)). Let $Y = S^2(0; \frac{p}{q}, -\frac{p}{q}, \ldots, \frac{p}{q}, \frac{r}{s}, -\frac{r}{s}, \ldots, \frac{r}{s})$ with k fibers, where k is even and $\frac{s}{r} + \frac{q}{p} = 1 - \frac{1}{pr}$. When $k = 2$, we have that Y is a lens space with trivial first homology, so $Y = S^3$ and Y is the doubly branched cover of the unknot, which is doubly slice. Assume that $k > 2$ and let ℓ be the number of fibers of the form $\pm\frac{p}{q}$ and $b = n - \ell$ be the number of fibers of the form $\pm\frac{r}{s}$. Observe that ℓ and b are both odd. Let $[a_1, a_2, \ldots, a_g]^-$ (resp. $[b_1, b_2, \ldots, b_h]^-$) be the continued fraction expansion for $\frac{p}{q}$ (resp. $\frac{r}{s}$).

We follow the same strategy as in the proof of Proposition 4.9.1(a) above to show that Y is the double branched cover of a doubly slice link. The Seifert fibered space Y is the boundary of a star-shaped plumbing 4-

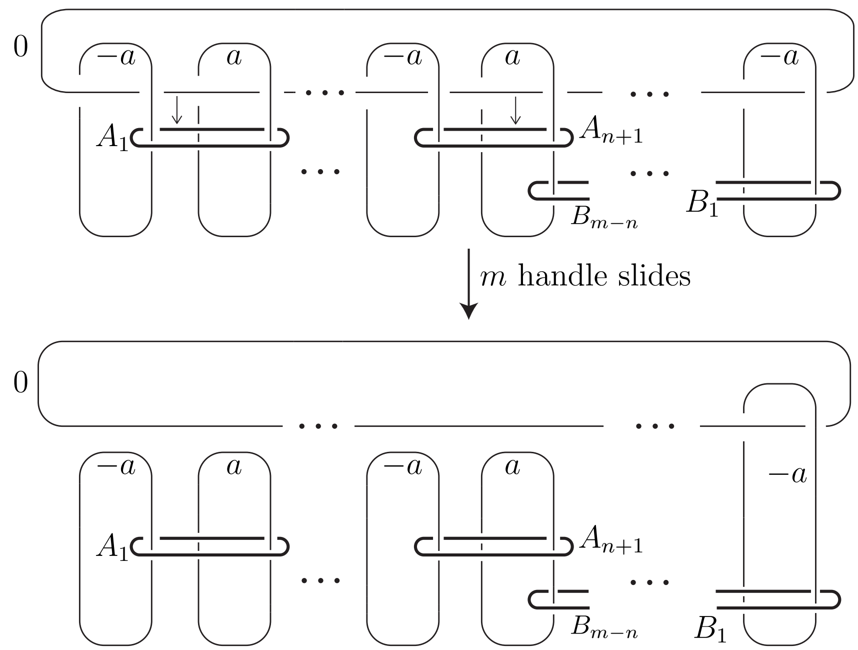

Figure 4.13: The 4-manifold X_{S_n}. Handleslide the 0-framed central 2-handle over each of the handles labelled $A_1, \ldots, A_m$.

manifold X as shown in Figure 4.15 (ignoring the 2-handles in bold). By Theorem 4.9.5, X is the double branched cover of (D^4, S) where S is a surface. Then $Y = \partial X$ is the double branched cover of S^3 branched over the Montesinos link $L = \partial S$. There are bands $A_1, \ldots, A_m, B_1, \ldots, B_m$ which may be attached to L, where $m = \frac{k}{2} - 1$, such that performing a subset S of these band moves changes $L \mapsto L'$ such that $\Sigma(L') = \partial X_S$, where X_S is the 4-manifold obtained by attaching the 0-framed 2-handles with labels in S to X in Figure 4.15. Figure 4.14, obtained by an isotopy of the link in Figure 4.15, shows that the 2-handles may be attached equivariantly with respect to the involution.

We check the hypotheses of Theorem 4.9.4. First let $S_n = \{A_i\}_{1 \leq i \leq m} \cup$

$\{B_j\}_{1\leq j\leq m-n}$, where $n \in \{0, 1, 2, \ldots, m\}$. We can realise X_{S_n} as a plumbing of a union of linear chains, by handle sliding the central 0-framed handle over each of the handles labeled $A_1, \ldots, A_m$ in Figure 4.15. This union of linear chains consists of:

1. n linear chains of one of two forms, either with framings $-a_1, -a_2, \ldots, -a_g, 0, a_g, \ldots, a_1$ or with framings $b_1, b_2, \ldots, b_h, 0, -b_h, \ldots, b_1$, and
2. a linear chain with framings

$$a_g, \ldots, a_1, 0, -a_1, \ldots, -a_g, \ldots, 0, a_g, \ldots, a_1, 0, b_1, \ldots, b_h, 0, \\ \ldots, -b_h, \ldots, -b_1, 0, b_1 \ldots, b_h.$$

Each linear chain in (1) contributes an $S^1 \times S^2$ summand to ∂X_{S_n}, and the linear chain in (2) contributes an S^3 summand to ∂X_{S_n}. In order to see this, we repeatedly use the fact that a subchain with framings $r, 0, -r$ where $r \in \mathbb{Z}$, can be replaced by a single 0 framed component. This fact follows by handlesliding the r framed component over the $-r$ framed component, then cancelling the $-r$ framed component and its 0 framed meridian. Repeatedly applying this fact, in case (2), we will be left with a linear chain $a_g, \ldots, a_1, 0, b_1, \ldots, b_h$ representing the Seifert fibered space $S^2(0; \frac{p}{q}, \frac{r}{s})$ which is homeomorphic S^3 since the condition $\frac{s}{r} + \frac{q}{p} = 1 - \frac{1}{pr}$ implies that it is a lens space with trivial first homology. This verifies that $\partial X_{S_n} = \#_n(S^1 \times S^2)$.

Now let $S'_n = \{A_i\}_{1\leq i\leq m-n} \cup \{B_j\}_{1\leq j\leq m}$, for $n \in \{1, 2, \ldots, m\}$. Each 2-handle attached to X corresponding to a band of the form B_j links two

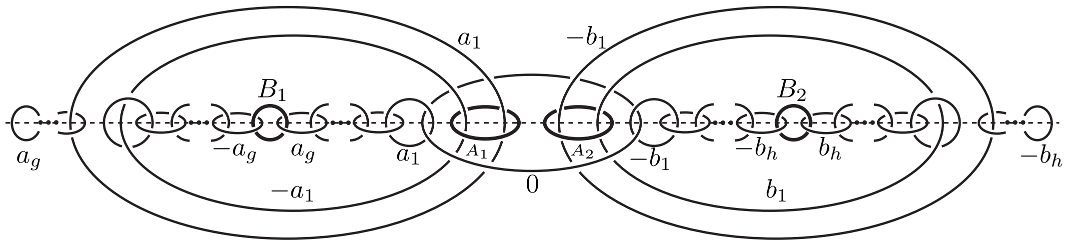

Figure 4.14: Kirby diagram for $X_{S'_n}$. Ignoring the components in bold gives a Kirby diagram for X with boundary Y. For simplicity only the case with $k = 6$ and $\ell = 3$ is shown. The strong involution is rotation by π about the dotted axis.

unknotted components with framings $-a_g$ and a_g. We use the same fact as above, that is, handlesliding the a_g framed component over the $-a_g$ framed component leads to the B_j labelled 2-handle linking the a_g framed component as a meridian, and hence we can cancel these two components without changing $\partial X_{S'_n}$. We see a 0-framed unknot linking components with framings $-a_{g-1}$ and $-a_{g-1}$ and we can again handleslide the a_{g-1} component over the $-a_{g-1}$ and remove the $-a_{g-1}$ framed components and its 0-framed meridian. Repeating this procedure leads to the surgery presentation for $\partial X_{S'_n}$ shown in Figure 4.16.

Next, we handleslide the 0-framed central curve in Figure 4.16 over the m 0-framed components as indicated by the arrows (note that the handleslides here are thought of merely as a move on surgery presentations for $\partial X_{S'_n}$). This gives a presentation for $\partial X_{S'_n}$ from which, by an analogous computation to the previous case, one can check that $\partial X_{S'_n} = \#_n(S^1 \times S^2)$.

□

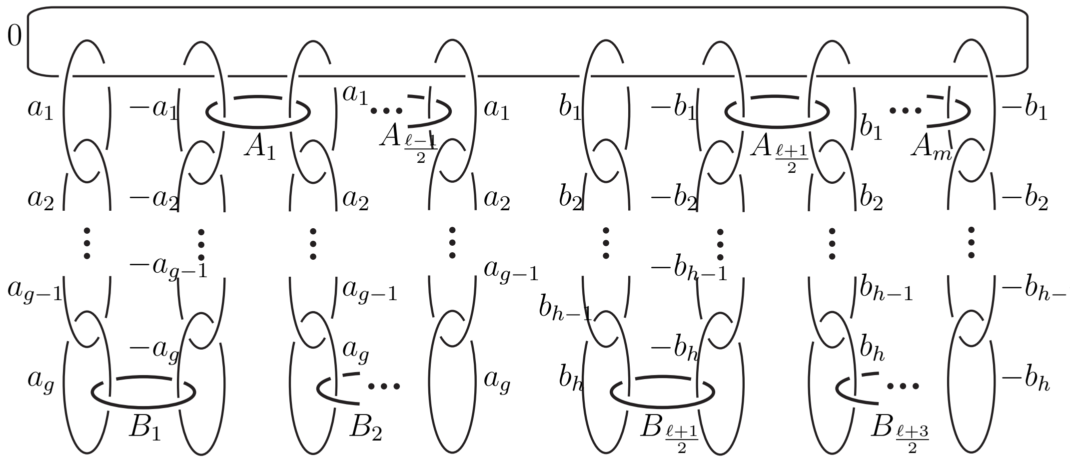

Figure 4.15: Kirby diagram for $X_{S'n}$. Ignoring the components in bold gives a Kirby diagram for X with boundary Y.

Doubly slice odd pretzels up to mutation

Our constructions of doubly slice Montesinos links along with the obstructions from earlier in the chapter allows us to prove the following theorem which classifies the smoothly doubly slice odd pretzel knots up to mutation. For 3 or 4-strand odd pretzel knots this was proved by Donald [Don15, Theorem 1.5].

Theorem 4.1.11. *If K is an odd pretzel knot, then the following are equivalent:*

(i) $\Sigma(K)$ embeds smoothly in S^4,

(ii) K is a mutant of a smoothly doubly slice odd pretzel knot,

(iii) and K is a mutant of $P(a, -a, \ldots, a)$ for some odd a with $|a| \geq 3$.

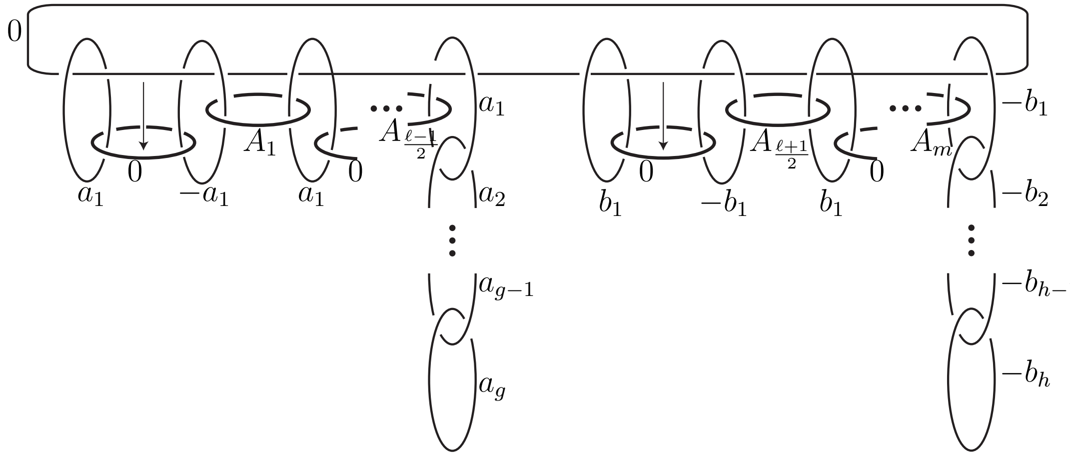

Figure 4.16: Surgery presentation for $X_{S'_m}$. In the next step, we handleslide the central 0-framed component over each of the m 0-framed components indicated by the arrows. The general case $X_{S'_n}$, $1 \leq n \leq m$, is analogous.

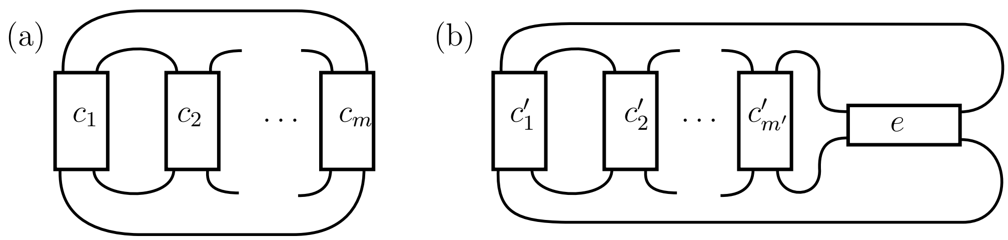

Figure 4.17: Two diagrams for pretzel knots, where the labelled boxes are used to denote twist regions with the corresponding number of crossings. In the right hand side, we may assume $|c'_i| > 1$ for all i.

Proof. The implication (iii)⇒(ii) follows from the proof of Proposition 4.9.1. In order to see this, following Example 4.9.6, one can check that the doubly slice pretzel knot corresponding to quotienting out Figure 4.11 by the strong involution indicated is precisely $P(-a, a, \dots, -a)$. The implication (ii)⇒(i) is well-known. The content of this proof is in the implication (i)⇒(iii), which we prove now.

Consider a pretzel knot $K = P(c_1, \ldots, c_k)$ as depicted in Figure 4.17(a), where the c_i are all odd. Notice that if $|c_i| = 1$, for some i, then the corresponding twist region is just a single crossing. By performing flypes and Reidemeister II moves if necessary we can assume that these crossings are in a single twist region as in Figure 4.17(b). That is, we can assume K takes the form

$$K = P(c'_1, \ldots, c'_{m'}, \underbrace{\varepsilon, \ldots, \varepsilon}_{|e|}),$$

where $|c'_i| > 1$ for all i and $e = \varepsilon|e|$ is an integer. For such a K double branched cover $\Sigma(K)$ takes the form

$$\Sigma(K) = S^2(e; a_1, \ldots, a_n, -b_1, \ldots, -b_m).$$

Assume, by reflecting K if necessary, that $\varepsilon(\Sigma(K)) > 0$. So writing $\Sigma(K)$ in standard form we obtain,

$$\Sigma(K) = S^2(m + e; a_1, \ldots, a_n, \frac{b_1}{b_1 - 1}, \ldots, \frac{b_m}{b_m - 1}).$$

Now assume that $\Sigma(K)$ embeds smoothly in S^4. First consider a partition as given by Theorem 4.1.4. Note that since $\frac{b_i-1}{b_i} > \frac{1}{2}$ for all i, each class in the partition contains at most one of the fibers corresponding to $\frac{b_i}{b_i-1}$. This shows that there are at least m such classes, implying that $e \geq 0$.

Now consider the condition that $\overline{\mu}(\Sigma(K)) = 0$. Consider the standard positive definite plumbing for $\Sigma(K)$. Since $\frac{b_i}{b_i-1}$ has continued fraction

$$\frac{b_i}{b_i - 1} = [\underbrace{2, \ldots, 2}_{b_i-1}]^-,$$

each of the arms corresponding to $\frac{b_i}{b_i-1}$ has $b_i - 1$ vertices. Thus the plumbing has $1 - m + n + \sum_{i=1}^{m} b_i$ vertices. Now it is easily checked that the (unique) characteristic subset on this plumbing is obtained by taking the central vertex along with $\frac{b_i-1}{2}$ vertices of norm two from each of the arms corresponding to $\frac{b_i}{b_i-1}$. Thus the sum of norms in the characteristic subset is $e + m + \sum_{i=1}^{m}(b_i - 1) = e + \sum_{i=1}^{m} b_i$. Thus we have

$$\overline{\mu}(\Sigma(K)) = n - m + 1 - e = 0.$$

Thus $e = n - m + 1 \geq 0$. However notice that $\Sigma(K)$ has $n + m$ exceptional fibers. Thus by Theorem 4.1.1 we have $m + e \leq \frac{n+m+1}{2}$. Altogether this shows

$$0 \leq e \leq \frac{n - m + 1}{2} = \frac{e}{2},$$

which implies that $e = 0$. Thus $\Sigma(K)$ has $n + m = 2m - 1$ exceptional fibers. Thus Theorem 4.1.1 implies that $b_i = a_j > 1$ for all i and j. Thus K is of the desired form. $\square$

Doubly slice quasi-alternating Montesinos links

We now prove the following result on doubly slice quasi-alternating Montesinos links.

Proposition 4.1.12. *A quasi-alternating Montesinos link is never topologically doubly slice.*

Proof. Let K be a quasi-alternating Montesinos link. The double branched covers of quasi-alternating Montesinos links have been classified in Chapter 3.

After possibly reflecting K, we can assume that

$$\Sigma(K) = S^2(e; \frac{p_1}{q_1}, \dots, \frac{p_k}{q_k}),$$

where $\varepsilon(\Sigma(K)) > 0$ and $\frac{p_i}{q_i} > 1$ and either

1. $e \geq k$ or

2. $e = k - 1$ and $\frac{q_{k-1}}{p_{k-1}} + \frac{q_k}{p_k} < 1$

holds. However notice that in the first case we have a partition

$$\mathcal{P} = \{\{1\}, \dots, \{k\}\}$$

violating Lemma 4.4.4, and in the second case we have a partition

$$\mathcal{P} = \{\{1\}, \dots, \{k-2\}, \{k-1, k\}\}$$

violating Lemma 4.4.4. Thus in neither case can $H_1(\Sigma(K))$ split as a direct double. This shows that $\Sigma(K)$ cannot embed topologically locally flatly in S^4 and hence that K is not topologically doubly slice. $\square$

A few sporadic examples

Finally, we show that the Seifert fibered spaces $S^2(1; \frac{7}{2}, \frac{7}{2}, \frac{7}{2})$, $S^2(1; 4, \frac{12}{5}, 4)$ and $S^2(1; 3, \frac{15}{4}, 3)$ are double branched covers of doubly slice Montesinos links. We note that $S^2(1; 4, \frac{12}{5}, 4)$ is known to embed by work of Donald [Don15, Example 2.14], however his proof is via a Kirby calculus argument. We refer the reader to Figure 3.2 for our conventions for Montesinos links in the following proposition.

Proposition 4.9.7. *The Montesinos links* $M(1; \frac{7}{2}, \frac{7}{2}, \frac{7}{2})$, $M(1; 4, \frac{12}{5}, 4)$ *and* $M(1; 3, \frac{15}{4}, 3)$ *are doubly slice.*

Proof. For each link, we demonstrate a pair of bands, an A-band and a B-band such that performing either of the two band moves results in the 2-component unlink, and performing both band moves simultaneously results in the unknot. Theorem 4.9.4 then implies that the link is doubly slice. We now demonstrate the bands, it is straightforward to check that they have the required properties, see Figures 4.18, 4.19 and 4.20.

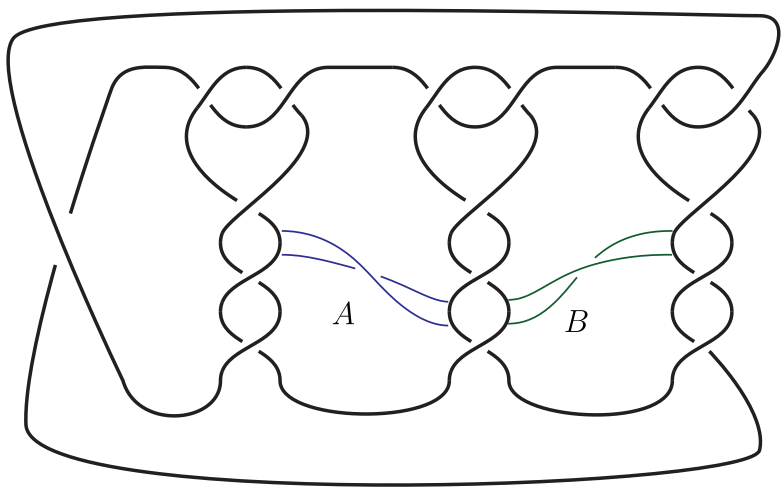

Figure 4.18: The Montesinos knot $M(1; \frac{7}{2}, \frac{7}{2}, \frac{7}{2})$.

□

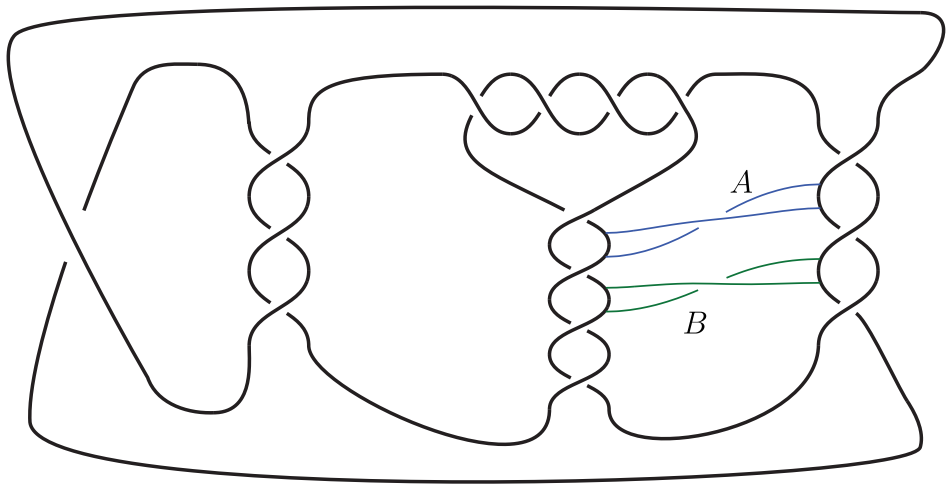

Figure 4.19: The Montesinos knot $M(1; 3, \frac{15}{4}, 3)$.

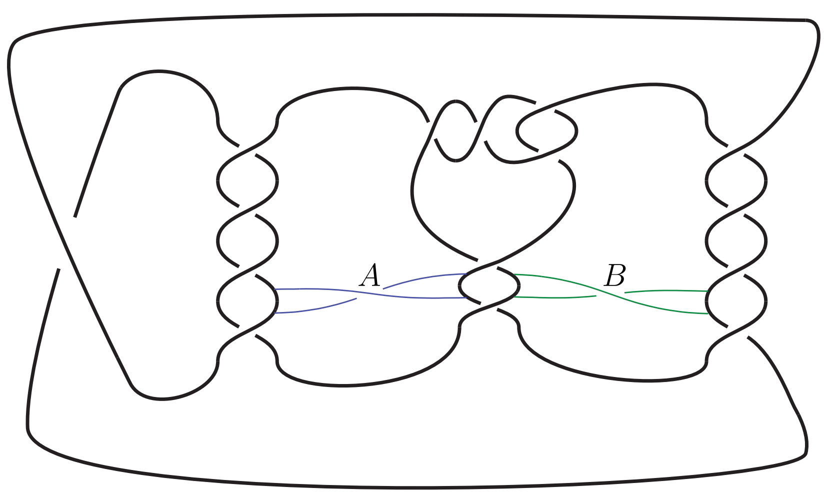

Figure 4.20: The Montesinos link $M(1; 4, \frac{12}{5}, 4)$.

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