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Forward optimization and real-time model adaptation with applications to portfolio management, indifference valuation and optimal liquidation

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The fundamental ingredients of stochastic optimization problems are the horizon, the payoff and the model. The horizon may be finite, $[0, T]$ or infinite $[0, \infty)$ (and even random) while the objective may be terminal, $U_T(\cdot)$ or "running", $U_{0,T}(\cdot)$. The model, denoted by $\mathcal{M}_{[0,T]}$, amounts in choosing a probability space and the dynamics of the controlled and uncontrolled state processes.

For convenience, we take $T < \infty$ and a terminal payoff $U_T(\cdot)$ depending exclusively on the terminal values of the state processes. We will think of a stochastic optimization problem as a triplet $\mathcal{P}\left(\mathcal{M}_{[0,T]}, [0, T], U_T(\cdot)\right)$ and we note that this triplet is chosen at initial time $t = 0$. The objective is then to find the optimal polices and the maximal expected utility (value function).

A widely used approach for finding the optimal policies is based on the celebrated Dynamic Programming Principle (DPP) which yields a "semi-group"-type optimality property for the value function. It also allows us to interpret the latter as the "intermediate utility" in arbitrary sub-horizons.

In Markovian models, the DPP allows us to work with the associated Hamilton-Jacobi-Bellman (HJB) equation, which is a fully non-linear and possibly degenerate partial differential equation. In most such models, under strong regularity assumptions, the first order conditions in the HJB equation are used to give the optimal policies in feedback form. Still, a general approach in establishing uniqueness, existence and regularity of the value function as well as a verification theorem for the candidate optimal polcies are missing, due to lack of compactness of the set of controls, degeneracies, state and control constraints, and others.

In general settings, the elegant duality approach is being used (in particular, in models with dynamics linear in controls) to study the dual instead of the primal problem. The dual problem has a much richer structure and may be easier to analyze.

Other approaches to study such problems rely on backward stochastic differential equations (especially problems with homogeneous payoffs), numerical approximations and others.

By far, the most challenging task in building accurate models to study real world applications is selecting the correct model. This is a tantamount task as it is well accepted that there is always model error and model decay.

The most popular approach to deal with model uncertainty is based on "robust optimization", where instead of choosing a specific model, one chooses a family $\mathcal{G}_{[0,T]}$ of possible models, together with penalty functionals that measure the plausibility of each member of this family (see, for example, [20], [29]). The HJB equation then characterizes a stochastic differential game.

Another approach is based on the so-called "adaptive control", where dynamic, "real-time" changes of the model are taken into account, by essentially "re-starting" the optimization problem for the remaining horizon(s) (see, among others, [37], [38]). This approach is quite popular in more applied aspects of optimization and in the engineering literature. It is, also, very popular in reinforcement learning where (near) optimal control policies are learned through adaptive interactions with the environment ([32], [58]).

These two approaches have distinct features. Robust optimization problems amount in choosing at initial time $t = 0$ a triplet $\mathcal{P}\left(\mathcal{G}_{[0,T]}, [0,T], U_T(\cdot)\right)$, instead of a single $\mathcal{P}\left(\mathcal{M}_{[0,T]}, [0,T], U_T(\cdot)\right)$. The problems are challenging due to their max-min features but are, nevertheless, amenable to the aforementioned stochastic optimization solutions approaches. Among others, they give rise to time-consistent policies across $[0,T]$. On the other hand, time consistency comes with a price, as no model revisions can be incorporated beyond $t = 0$. In other words, the choice $\mathcal{P}\left(\mathcal{G}_{[0,T]}, [0,T], U_T(\cdot)\right)$ is rather rigid and predominantly very conservative, as one tries to incorporate all the adverse scenaria in the modeling family.

Adaptive optimization is, by nature, time-inconsistent. Between revision times, time-consistency is naturally preserved since one deals with a single (locally in time valid) model. However, there is no global time-consistency, as sequential "real-time" model revisions occur, and the previously chosen model is abandoned. As a result, adaptive control is, from the one hand, not constrained to rigid, *a priori* model(s) commitment but, on the other, the associate solutions violate, by nature, the time-consistency property.

Both approaches are widely used, and have been extensively analyzed in a plethora of interesting theroetical and applied papers.

The goal of this thesis is to introduce a new, alternative approach to deal with model unceratinty and "real-time" model revisions and, in turn, develop a comparative study with existing approaches in the context of various applications in financial mathematics.

This new approach is based on the forward performance criteria which adapt in a time-consistent way to "real-time" model revisions. The novelty is that these revisions are genuinely "model-free" in that they occur in "real-time", without any modeling pre-commitment. For example, in the context of optimal liquidation (see Chapter 3 and Chapter 4), there is no *a priori* model for the evolution of the market impact parameter λ. It is rather assumed that this parameter switches at predictable times, to values only observable at the switching times. As such, the model revisions capture the evolving reality and allow for considerable flexibility.

This forward approach thus incorporates "real-time" model revisions and is, therefore, close to adaptive optimization. On the other hand, it produces, by construction, time-consistent policies and is, thus, close to the classical optimization with model(s) pre-commitment. In other words, it can be thought as a hybrid approach that accommodates dynamic model changes while preserving time-consistency.

We apply the forward approach with "real-time" model revisions in four distinct problems: portfolio management in discrete and continuous settings (binomial and lognormal, respectively), indifference valuation in lognormal models and optimal liquidation in the continuous time Almgren-Chriss model. We produce closed form solutions and characterize the optimal policies and optimal criteria. As the analysis shows, one needs to solve various sequential "inverse" optimal investment problems with random coefficients, corresponding to model revisions in real-time.

We develop a comparative study with the classical settings. A main novelty is the introduction of two performance metrics which measure the discrepancies between the actual performance, and the projected or the true optimal performances under the various criteria and behavior. We study these metrics for various scenaria, related to favorable and non-favorable market changes, and compare their performance. These metrics resemble the notion of "regret", which is now considered in a more dynamic and "real-time" manner. Among others, we show that the regret of the forward decision maker is always zero, independently of the upcoming model changes.

In what follows, we describe each application separately. For each application, we introduce the model, the forward and classical criteria, construct the corresponding solutions and policies, and compare them in detail.

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Regret is the biggest danger for financial health.

— Daniel Kahneman[1]

[1]Daniel Kahneman was awarded the 2002 Nobel Memorial Prize in Economics Sciences (shared with Vernon L. Smith). He is known for the work in prospect theory with Amos Tversky. The quote is from an interview of Dr. Kahneman; see https://finance.yahoo.com/news/invest-without-regrets-according-nobel-110010129.html

Chapter 1

Real-time model adaptation and investment behavior: the binomial case

1.1 Introduction

Classical stochastic optimization relies heavily on a pre-specified model (or a family of models) chosen at initial time, say $t = 0$. In turn, the solution is constructed with the full model (or a family of models) commitment for the entire optimization horizon $[0, T]$ or $[0, \infty)$. The main tool for this construction is the Dynamic Programming Principle (DPP), which yields the value function and the optimal policies via a backward-in-time recursive algorithm.

A direct consequence of this *backward construction* is that for each optimization period (discrete or infinitesimally small in continuous time) this *a priori* model commitment is fully embedded in both the value function and the optimal policies. Indeed, in discrete models, the solution is constructed from the last time step to the one before it and, then, recursively backwards till the initial time. In continuous time Markovian models, the Hamilton-Jacobi-Bellman (HJB) equation is a backward parabolic partial differential equation with a given terminal condition. Finally, in non-Markovian models, the value function process is constructed via backward stochastic differential equations, also reflecting the backward nature of the construction.

In practice, however, it is quite unrealistic to pre-commit to a model setting, especially if the horizon is long. Model revision is very often inevitable when the controller is interacting in real-time with the controlled system. Indeed, more accurate information about the underlying model typically becomes available while the controller exercises the control policies, and such information should be, thus, incorporated and exploited in upcoming times.

As a result, an important question arises, namely, how to incorporate this new information in a meaningful and tractable way. By far, the most popular approach is to work, from the beginning, with richer models aiming at incorporating most of the plausible future events and/or to involve criteria that minimize the effects of using the wrong model. Such methods involve robust criteria, models with many factors, hidden variables, linear or nonlinear filtering and others. Still, however, all these choices for the involved model setting are done at the very beginning. For example, one needs to pre-commit to a stochastic process on which filtering is being carried out.

Another equally imperative consideration when real-time model changes occur is whether, in the "information-adjusted" optimization problem, time-consistency must be preserved. In some settings, like for example, in mean-variance optimization, time-consistency is inherently absent due to the path dependent target. In the majority of the stochastic optimization problems, however, time-consistency follows directly from the DPP. As a matter of fact, time-consistency is the most important direct consequence of the DPP.

How does then one incorporate real-time model changes? Should time-consistency be a requirement for the new setting or not? Or shall one just "restart" the model, using the current value of the state processes, optimize in the remaining horizon and maintain the same terminal criterion that was chosen form the beginning? Clearly, one *cannot* in general achieve *both* real-time model revision and preservation of time-consistency under the original objective. What is then more important for real-time decision making? Shall one optimize the original criterion and violate the time-consistency, or preserve the time-consistency and revise the criterion?

Herein, the aim is not to defend one behavior in relation to the other. Rather, the aim is to initiate a comparative study among possible optimization behavior under real-time model revision with or without adaptation of the optimization criterion. This comparison is first cast in an investment problem with a single stock, represented by a simple yet rich enough binomial model. It is then extended to the Merton' s optimal investment problem setting in the next chapter. The "model knowledge" is revealed in real-time in the sense that, at the beginning of each decision making period, the investors only knows the one-period ahead conditional distribution of the stock return under the true physical measure. Such model assumption is in direct contrast with the typical classical formulation, where a full model (or a family of full models) needs to be specified at $t = 0$. We also note that we do not incorporate any other model component to describe how the stock return parameters would change, but solely require that these parameters are observable only at the time they change.

We consider four types of investors, and will refer to them as the "oblivious" investor, the "stubborn" investor, the "robust" investor and the "forward" investor.

i) *Oblivious investor*: he is ignorant of any future model changes. He perceives that the market environment over the investment horizon will not change and will remain as prescribed by the original (at $t = 0$) binomial model. He solves the optimization problem at $t = 0$ and applies the associated optimal policy for the entire horizon. Clearly, there will be a mismatch between the parameters entering in the calculation of the optimal feedback policy and the new, updated market parameters at later periods. Naturally, time-consistency is preserved but this is virtual since the involved market input is not accurate. Such type of investment behavior might correspond to the group of investors in the market who are less efficient in acquiring updated market information.

ii) *Stubborn investor*: she is able to observe the intermediate model change but chooses to optimize under the original criterion. Obviously, time-inconsistency occurs in this case, and the optimal investment performance before and after the model switching can be quite different. Such stubborn behavior is actually the most common way to incorporate model changes within the field of *adaptive control*, and represents the type of investors who revise their model very frequently in practice.

iii) *Robust investor*: she works under the classical robust control paradigm, where at $t = 0$ a family of possible models over the investment horizon is chosen. It is assumed that this family contains the true underlying (unknown) physical probability measure. However, like the oblivious investor, the robust investor, although already taking into account at $t = 0$ all possible scenarios of future model changes, does not revise this family of models once investment is initiated. The robust investor might give excessive weight to the worst case scenario even if the actual market model parameters turn out to be very distinct from the worst scenario.

iv) *Forward investor*: like the stubborn investor, she takes into account the model change at the intermediate time. However, contrary to the stubborn investor, she chooses to revise the terminal criterion for the remaining horizon in order to maintain the intertemporal consistency of the value function and the optimal policies. As a result, she is forced to revise the terminal criterion for the new shorter horizon according to a (predictable) forward performance process.

Comparisons among the four different types of investment behavior demonstrate their respective response to model changes. For instance, the stubborn behavior solely focuses on the optimization problem over the remaining horizon and can lead to increased or decreased ultimate performance, depending on the revealed new parameters. On the other hand, the forward behavior, regardless of the underlying (unknown) true model, preserves the same average performance when evaluated across times. This stability can be actually connected to *regret*, an important concept in real-time/on-line learning and optimization. Indeed, when assessed in retrospect, it is shown that the forward behavior has achieved *zero regret* with respect to the true model over the entire investment horizon. Due to the flexibility and the forward in time recursive construction of the optimization criteria, the forward approach allows dynamically revising the terminal criterion in real-time to minimize the regret. In contrast, zero regret is typically not attainable under the other three types of investment behavior.

1.2 Classical and forward views

Before getting into the comparative study on the four types of investment behavior, we would first present a very informal discussion of the classical and forward views for solving an optimal investment problem in a two-period binomial model. The classical backward reasoning underlies the first three types of investment behavior introduced above, whereas the forward optimization view gives rise to the forward behavior. The main purpose of this discussion is to highlight the *model commitment* issue that is ubiquitous in classical formulation through a motivation example, while at the same time, advocating the *flexibility* of the forward optimization approach in terms of model specification. Such model flexibility is one (and probably the most important one) property of the forward approach that distinguishes it from the classical approach.

We consider a market with a riskless asset whose interest rate is zero, and a risky asset whose return is modeled by the random variables R_1, R_2 at $t = T_1$ and $t = T_2$. We denote by $(\Omega, \mathcal{F})$ the measurable space that supports R_1, R_2, and $(\mathcal{F}_t)_{t=0,T_1,T_2}$ the filtration that represents the information available at the corresponding times. The investor would rebalance her portfolio at two times $t = 0$ and $t = T_1$, starting with initial wealth $x \in \mathbb{R}$ at $t = 0$. The investment strategies satisfy the usual self-financing condition, and lead to the wealth equations

$$X_{T_1} = X_0 + \pi_1(R_1 - 1), \quad \text{and} \quad X_{T_2} = X_{T_1} + \pi_2(R_2 - 1), \tag{1.1}$$

with $X_0 = x$.

We first consider the optimal investment problem for the backward investor with a terminal utility $U(x)$, set at $t = 0$ for the end of the horizon $t = T_2$. To better expose the idea and avoid unnecessary complexity, we assume that all possible values of the returns are known at time $t = 0$, and denote by R_1^u, R_1^d the two values of the random return R_1, and R_2^{uu}, R_2^{ud}, R_2^{du}, and R_2^{dd} the four possible values for R_2. At $t = 0$, the backward investor perceives the stock return dynamics under the measure $\widehat{\mathbb{P}}$, namely, $\widehat{\mathbb{P}}(R_1 = R_1^u) = p_1$,

$$\widehat{\mathbb{P}}(R_2 = R_2^{uu} | R_1 = R_1^u) = \hat{p}_2^{uu},$$

and

$$\widehat{\mathbb{P}}(R_2 = R_2^{du} | R_1 = R_1^d) = \hat{p}_2^{du}.$$

We also assume that the standard no-arbitrage condition satisfies, i.e., $0 < p_1, \hat{p}_2^{uu}, \hat{p}_2^{du} < 1$, $0 < R_1^d, R_2^{ud}, R_2^{dd} < 1 < R_1^u, R_2^{uu}, R_2^{du}$.

The optimal investment problem the backward investor intends to solve at $t = 0$ is

$$\widehat{V}_0(x) = \sup_{\pi_1,\pi_2} \mathbb{E}_{\widehat{\mathbb{P}}} \left[U(X_{T_2}) | X_0 = x \right],$$

where the expectation is taken under the ($t = 0$) perceived measure $\widehat{\mathbb{P}}$. To solve such problem based on the classical view, the backward investor first solves

$$\widehat{V}_{T_1}(x) = \operatorname*{esssup}_{\pi_2} \mathbb{E}_{\widehat{\mathbb{P}}} \left[U(X_{T_2}) \middle| X_{T_1} = x,\ \mathcal{F}_{T_1} \right] \in \mathcal{F}_{T_1}. \tag{1.2}$$

Expanding the conditional expectation at the right hand side of above equation yields

$$\begin{aligned}
&\mathbb{E}_{\widehat{\mathbb{P}}} \left[U(X_{T_2}) | X_{T_1} = x,\ \mathcal{F}_{T_1} \right] \\
&= \left(U(x + \pi_2(R_2^{uu} - 1))\widehat{p}_2^{uu} + U(x + \pi_2(R_2^{ud} - 1))(1 - \widehat{p}_2^{uu}) \right) \mathbb{1}_{\{R_1 = R_1^u\}} \\
&+ \left(U(x + \pi_2(R_2^{du} - 1))\widehat{p}_2^{du} + U(x + \pi_2(R_2^{dd} - 1))(1 - \widehat{p}_2^{du}) \right) \mathbb{1}_{\{R_1 = R_1^d\}}.
\end{aligned}$$

Hence, if $R_1 = R_1^u$ is realized at $t = T_1$, the investor with the current wealth $X_{T_1} = x$ would follow the policy (if it exists)

$$\widehat{\pi}^*_{2,R_1^u}(x) \in \operatorname*{argmax}_{\pi_2 \in \mathcal{A}_{R_1^u}(x)} \left(U(x + \pi_2(R_2^{uu} - 1))\widehat{p}_2^{uu} + U(x + \pi_2(R_2^{ud} - 1))(1 - \widehat{p}_2^{uu}) \right),$$

with $\mathcal{A}_{R_1^u}(x)$ being the admissible set for an investor with arbitrary wealth x at $t = T_1$, under the market condition that $R_1 = R_1^u$. Similar argument associated to the market condition $R_1 = R_1^d$ yields the optimizer $\widehat{\pi}^*_{2,R_1^d}(x)$, and we hence arrive at the optimal strategy at $t = T_1$ for an investor with wealth $X_{T_1} = x$

$$\widehat{\pi}^*_2(x) = \widehat{\pi}^*_{2,R_1^u}(x)\mathbb{1}_{\{R_1 = R_1^u\}} + \widehat{\pi}^*_{2,R_1^d}(x)\mathbb{1}_{\{R_1 = R_1^d\}} \in \mathcal{F}_{T_1}. \tag{1.3}$$

Plugging the optimal control $\widehat{\pi}_2^*(x)$ into the optimization problem (1.2), we obtain the value function at $t = T_1$ as

$$\widehat{V}_{T_1}(x) = \widehat{V}_{T_1,R_1^u}(x)\mathbb{1}_{\{R_1=R_1^u\}} + \widehat{V}_{T_1,R_1^d}(x)\mathbb{1}_{\{R_1=R_1^d\}} \in \mathcal{F}_{T_1},$$

where the deterministic function

$$\widehat{V}_{T_1,R_1^u}(x) := U\left(x + \widehat{\pi}_{2,R_1^u}^*(x)(R_2^{uu} - 1)\right)\widehat{p}_2^{uu} + U\left(x + \widehat{\pi}_{2,R_1^u}^*(x)(R_2^{ud} - 1)\right)(1-\widehat{p}_2^{uu}),$$

and

$$\widehat{V}_{T_1,R_1^d}(x) := U\left(x + \widehat{\pi}_{2,R_1^d}^*(x)(R_2^{du} - 1)\right)\widehat{p}_2^{du} + U\left(x + \widehat{\pi}_{2,R_1^d}^*(x)(R_2^{dd} - 1)\right)(1-\widehat{p}_2^{du}).$$

Note that both the optimal control $\widehat{\pi}_2^*(x)$ and the value function $\widehat{V}_{T_1}(x)$ depend on the time $t = 0$ specified model dynamics for the second period, i.e., $\widehat{p}_2^{uu}$, $\widehat{p}_2^{du}$, R_2^{uu}, R_2^{ud}, R_2^{du} and R_2^{dd}, in addition to the realization of the random return R_1 at $t = T_1$.

Now having found the value function $\widehat{V}_{T_1}(x)$, the backward induction procedure implies that

$$\begin{aligned}\widehat{V}_0(x) &= \sup_{\pi_1} \mathbb{E}_{\widehat{\mathbb{P}}}\left[\widehat{V}_{T_1}(X_{T_1})\middle| X_0 = x\right]\\ &= \sup_{\pi_1} \mathbb{E}_{\widehat{\mathbb{P}}}\left[\widehat{V}_{T_1}(x + \pi_1(R_1 - 1))\middle| X_0 = x\right]\\ &= \sup_{\pi_1}\left(\left(\widehat{V}_{T_1,R_1^u}(x + \pi_1(R_1^u - 1))\right)p_1 + \left(\widehat{V}_{T_1,R_1^d}(x + \pi_1(R_1^d - 1))\right)(1 - p_1)\right).\end{aligned} \tag{1.4}$$

Suppose there exists an maximizer $\widehat{\pi}_1^*(x)$ to equation (1.4) that is admissible for any initial wealth $X_0 = x$, then the value function $\widehat{V}_0(x)$ can be calculated by plugging $\widehat{\pi}_1^*(x)$ into (1.4). Note that $\widehat{V}_{T_1,R_1^u}(\cdot)$ and $\widehat{V}_{T_1,R_1^d}(\cdot)$ are present in the optimization problem (1.4), and they were computed earlier under the optimization problem (1.2). In turn, the optimal control $\widehat{\pi}_1^*(x)$ and the value function $\widehat{V}_0(x)$ would inevitably depend on the second period model dynamics under the $t = 0$ perceived measure $\widehat{\mathbb{P}}$, in addition to the first period model itself. We refer to this feature of the backward approach as model commitment.

We now turn to the forward approach in the same binomial setting, except that at time $t = 0$, we do not require any knowledge about the model characteristics of the second period stock return. Indeed, we follow the recent work in [3] on predictable forward performance processes. It is assumed that both the possible values and the probability distribution for $R_2 \in \mathcal{F}_{T_2}$ are only known at $t = T_1$. In other words, the values of return R_2^{uu}, R_2^{ud}, R_2^{du} and R_2^{dd} are $\mathcal{F}_{T_1}$-measurable random variables, and $p_2 = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{R_2=R_2^u\}}|\mathcal{F}_{T_1}]$, with $R_2^u := R_2^{uu}\mathbb{1}_{\{R_1=R_1^u\}} + R_2^{du}\mathbb{1}_{\{R_1=R_1^d\}}$, denotes the conditional probability of the second period stock return going up under the true underlying measure $\mathbb{P}$.

Different from the classical setting, however, in the forward setting, the forward investor only needs to know the model characteristics of the first period return in order to make investment decision at $t = 0$. Specifically, at time $t = 0$, she is aware of R_1^u and R_1^d, and p_1 under the true measure $\mathbb{P}$. In addition, she chooses an initial (deterministic) utility function $U_0(x)$, and looks for a deterministic utility $U_{T_1}(x)$ at $t = T_1$, such that the following intertemporal consistency property

$$U_0(x) = \sup_{\pi_1} \mathbb{E}_{\mathbb{P}} \left[U_{T_1}\left(X_{T_1}\right) \middle| X_0 = x \right], \tag{1.5}$$

for each initial wealth $X_0 = x$ holds. This is the inverse of the classical optimal investment problem discussed earlier, and the existence and uniqueness of its solution for completely monotonic cases have been extensively studied in [3]. If the inverse problem (1.5) indeed has a solution, one can readily see that the optimal allocation $\pi_1^*(\cdot)$ (with its resulting optimal wealth $X_{T_1}^*$) and the utility function $U_{T_1}(\cdot)$ depend exclusively on the first period model which is accurately known at $t = 0$, when problem (1.5) is solved. This is in direct contrast with the previous classical setting where we have shown that model commitment in $[0, T_2]$ is inevitable.

To solve the second period forward problem, we look for a random utility function $U_{T_2}(\cdot\,;\omega) \in \mathcal{F}_{T_1}$, with the analogous intertemporal consistency property to hold

$$U_{T_1}\left(x\right) = \operatorname*{ess\,sup}_{\pi_2} \mathbb{E}_{\mathbb{P}} \left[U_{T_2}(X_{T_2}) \middle| X_{T_1}^* = x, \mathcal{F}_{T_1} \right], \text{ a.s..} \tag{1.6}$$

Here, $X_{T_1}^* = x$ is the optimal wealth at $t = T_1$ obtained by following the optimal allocation $\pi_1^*(\cdot)$ over $[0, T_1]$. Notice that at $t = T_1$, the conditional probability p_2 under the true measure $\mathbb{P}$ is known, and the same holds for the possible values R_2^{uu}, R_2^{ud}, R_2^{du} and R_2^{dd} of the random return R_2. Condition (1.6) implies that given $\mathcal{F}_{T_1}$, on the set $\{R_1 = R_1^u\} \in \mathcal{F}_{T_1}$, we look for a (deterministic) utility function $U_{T_2,R_1^u}(\cdot)$, such that

$$U_{T_1}(x) = \sup_{\pi_2} \Big(U_{T_2,R_1^u}\Big(x + \pi_2(R_2^{uu} - 1)\Big)p_2 + U_{T_2,R_1^u}\Big(x + \pi_2(R_2^{ud} - 1)\Big)(1 - p_2)\Big), \tag{1.7}$$

with p_2, R_2^{uu} and R_2^{ud} now being known as constants, conditional on $\mathcal{F}_{T_1}$. This is an optimization problem similar to the first period problem (1.5), and the existence and uniqueness results from [3] readily apply. If such solution $U_{T_2,R_1^u}(\cdot)$ exists, then it would depend on the second period model characteristics as shown in the optimization problem (1.7), as well as the first period model through the presence of $U_{T_1}(\cdot)$. A similar argument can be made to obtain $U_{T_2,R_1^d}(\cdot)$ on the set $\{R_1 = R_1^d\} \in \mathcal{F}_{T_1}$ by solving an analogous inverse optimization problem as (1.7). The predictable forward utility is hence given by

$$U_{T_2}(x) = U_{T_2,R_1^u}(x)\mathbb{1}_{\{R_1=R_1^u\}} + U_{T_2,R_1^d}(x)\mathbb{1}_{\{R_1=R_1^d\}} \in \mathcal{F}_{T_1}.$$

It is now easy to see that both the forward utility $U_{T_2}(\cdot)$ and the optimal allocation $\pi_2^*(\cdot)$ (with its resulting optimal wealth $X_{T_2}^*$) depend on the model for the first and second periods under the true measure $\mathbb{P}$, whereas $U_{T_1}(\cdot)$ and and the optimal allocation $\pi_1^*(\cdot)$ (with its resulting optimal wealth $X_{T_1}^*$) only depend on the true model for the first period. The intuition behind such model flexibility is the compatibility between the sequential model revision and the utility process construction, both of which proceed forward in real-time.

1.3 Investment behavior in discrete time

In the previous section, we provided the (informal) discussion of the *model commitment* issue arising in the classical backward optimization paradigm, as well as how the forward approach can allow for *model flexibility*. This distinction between the two approaches has fundamental effects, one of which is on the *performance measurement* of the associated investment strategies. In this section, we introduce two metrics to quantitatively examine the performance for various investors following the backward approach (i.e., the oblivious investor, the stubborn investor and the robust investor), and for the investor adopting the forward approach. One of the observations is that, by adaptively revising the terminal criterion and seeking consistent investment behavior, the forward approach produces rather stable actual performance even under extreme unforeseen changes in the model. This stability is however not achievable for any of the investors within the backward approach paradigm.

1.3.1 Model setup and the two metrics

We work under the two-period binomial framework introduced in previous section. For completeness, we recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space that supports the random variables R_1, R_2, and that the filtration $(\mathcal{F}_t)_{t=0,T_1,T_2}$ represents the information available at the corresponding times. We denote by $\mathbb{E}$ the (conditional) expectation operator under the physical measure $\mathbb{P}$. The evolution of the stock return over the first period is modeled by $\mathbb{P}(R_1 = R_1^u) = p_1$, and by

$$\mathbb{P}\left(R_2 = R_2^{uu} \middle| R_1 = R_1^u\right) = p_2^{uu} \quad \text{and} \quad \mathbb{P}\left(R_2 = R_2^{ud} \middle| R_1 = R_1^u\right) = 1 - p_2^{uu}, \tag{1.8}$$

and

$$\mathbb{P}\left(R_2 = R_2^{du} \middle| R_1 = R_1^d\right) = p_2^{du} \quad \text{and} \quad \mathbb{P}\left(R_2 = R_2^{dd} \middle| R_1 = R_1^d\right) = 1 - p_2^{du}, \tag{1.9}$$

for the second period. Let $p_2^u := p_2^{uu}\mathbb{1}_{\{R_1 = R_1^u\}} + p_2^{du}\mathbb{1}_{\{R_1 = R_1^d\}} \in \mathcal{F}_{T_1}$ be the conditional probability of the second period stock return going up under the true model $\mathbb{P}$. The standard non-arbitrage assumption further implies that $0 < p_1, p_2^{uu}, p_2^{du} < 1$, and that $0 < R_1^d, R_2^{ud}, R_2^{dd} < 1 < R_1^u, R_2^{uu}, R_2^{du}$.

Notice that in reality, the underlying physical measure $\mathbb{P}$ is typically not fully known at $t = 0$. For a reasonable comparative study among different investors, we assume that they share the same amount of the initial knowledge about the underlying model. Specifically, all the investors are aware of the possible values for the stock returns R_1, R_2, and in addition, the true probability parameter p_1 at $t = 0$. The true parameters p_2^{uu} and p_2^{du}, however, only reveal to all the investors at $t = T_1$, conditional on $\mathcal{F}_{T_1}$. For instance, if $R_1 = R_1^u$ has occurred over the first period, then the investors would know the model (1.8) at $t = T_1$. It is worth noting that by our assumption, the true parameters p_2^{uu} and p_2^{du} are not known to any of the investors at time $t = 0$. Such assumption corresponds to the investment practice where the prediction power of any model typically decays as time moves into the future, and instead, more accurate model knowledge is actually updated in real-time.

The wealth equation under self-financing policies is given by (1.1) as in the previous section. For the three investors under the classical backward framework (i.e., the oblivious, the stubborn and the robust investors), we assume that they have exponential utility $U(x) = -e^{-\gamma x}$, $\gamma > 0$. The forward investor, on the other hand, takes a predictable forward criterion $U^F_{T_2}(x) \in \mathcal{F}_{T_1}$ that would be constructed in the sequel. To quantitatively compare the investment performance of the four types of investors in the true market (governed by the measure $\mathbb{P}$), we introduce two metrics as follows. For a generic type of investor, we define her actual performance under the true measure $\mathbb{P}$ as

$$V_0^{\text{Actual}}(x) = \mathbb{E}\left[U_{T_2}\left(X_{T_2}^{\widehat{\pi}_1^*,\widehat{\pi}_2^*}\right) \Big| X_0 = x\right], \tag{1.10}$$

where $U_{T_2}(\cdot)$ is the terminal utility, and $\widehat{\pi}_1^*, \widehat{\pi}_2^*$ are the investment policies derived under the associated type of investment behavior. The terminal wealth $X_{T_2}^{\widehat{\pi}_1^*,\widehat{\pi}_2^*}$ follows from the wealth equation (1.1), when the policies $\widehat{\pi}_1^*, \widehat{\pi}_2^*$ are applied at the two rebalancing times $t = 0$, $t = T_1$, respectively. Another quantity of interest is the $t = 0$ targeted average performance denoted by $V_0^{\text{Targeted}}(x)$ for a generic type of investor who starts with initial wealth $X_0 = x$. This value function measures the $t = 0$ perceived optimal performance under various investment behavior, and hence, it is in general not achievable. The discrepancy between the two quantities

$$m_{0,T_2}(x) := V_0^{\text{Actual}}(x) - V_0^{\text{Targeted}}(x)$$

is the first metric we introduce to measure the stability of investment performance under both the backward and forward framework. The second metric is the discrepancy

$$M_{0,T_2}(x) := V_0^{\text{Actual}}(x) - V_0^{\text{True}}(x),$$

where

$$V_0^{\text{True}}(x) := \sup_{\pi_1,\pi_2} \mathbb{E}\left[U_{T_2}(X_{T_2}^{\pi_1,\pi_2}) \middle| X_0 = x \right], \tag{1.11}$$

is the $t = 0$ true optimal performance, given the correct model $\mathbb{P}$ for the entire horizon $[0, T_2]$. We notice that $V_0^{\text{True}}(x)$ is also not achievable in reality, as it is computed in hindsight with the full knowledge of the underlying model. The metric $M_{0,T_2}(x)$ is directly motivated in spirit by the fundamental concept of *regret* from the online learning/optimization research field (see, e.g. [55]). Intuitively, it measures how much regret the investor undergoes for not having taken the genuine optimal policies under the true measure $\mathbb{P}$, which is not known at $t = 0$. It is expected that a stable investment process produce as minimal regret as possible, under various market conditions (i.e., corresponding to different $\mathbb{P}$).

Before we start discussing in detail the four types investment behavior, it is worth noting that many arguments would relate to the solution of a single-period binomial model investment problem under exponential type of utility/value functions. We hence consider such problem separately and analyze the existence and uniqueness of the optimal policy. Let R^u, R^d denote the two possible values of the stock return with $0 < R^d < 1 < R^u$, and $0 < p < 1$ denote the probability of the event $\{R = R^u\}$. Then, under an exponential type of terminal utility $U(x) = -e^{-\gamma x}$, $\gamma > 0$, the optimization problem is to solve

$$V_0(x) = \sup_{\pi} \left(-e^{-\gamma(x+\pi(R^u-1))} p - e^{-\gamma(x+\pi(R^d-1))}(1-p) \right), \tag{1.12}$$

for each initial wealth $x \in \mathbb{R}$. Direct computation yields the unique maximizer

$$\pi^*(x) = -\frac{1}{\gamma(R^u - R^d)} \ln \left(\frac{1-R^d}{R^u-1} \frac{1-p}{p} \right),$$

for all $x \in \mathbb{R}$, following from the first order condition and the second order condition for global concavity.

1.3.2 Oblivious investor (model non-adaptive/goal persistent)

We first consider the oblivious investor who is also known as the model non-adaptive/goal persistent investor. As discussed earlier, the backward approach requires a full model (or a family of full models, see section 2.4 for the robust investor) for both periods ahead. We refer to this model as the *perceived model* under the perceived measure $\widehat{\mathbb{P}}$, a measure that may not necessarily coincide with the genuine physical measure $\mathbb{P}$. The oblivious investor holds onto such perceived model $\widehat{\mathbb{P}}$ specified at $t=0$ for the entire horizon $[0, T_2]$ without revising the model at $t = T_1$ (i.e. non-adaptive), and pre-commits to the $t = 0$ specified terminal utility function $U(x) = -e^{-\gamma x}$ (i.e. goal persistent). The time $t = 0$ model he adopts under the perceived measure $\widehat{\mathbb{P}}$ is given by

$$\widehat{\mathbb{P}}(R_1 = R_1^u) = p_1, \quad \widehat{\mathbb{P}}(R_1 = R_1^d) = 1 - p_1,$$

for the first period, and

$$\widehat{\mathbb{P}}\left(R_2 = R_2^{uu} \middle| R_1 = R_1^u\right) = \hat{p}_2^{uu}, \quad \widehat{\mathbb{P}}\left(R_2 = R_2^{ud} \middle| R_1 = R_1^u\right) = 1 - \hat{p}_2^{uu},$$

and

$$\widehat{\mathbb{P}}\left(R_2 = R_2^{du} \middle| R_1 = R_1^d\right) = \hat{p}_2^{du}, \quad \widehat{\mathbb{P}}\left(R_2 = R_2^{dd} \middle| R_1 = R_1^d\right) = 1 - \hat{p}_2^{du},$$

for the second period. We also denote by $\hat{p}_2^u := \hat{p}_2^{uu}\mathbb{1}_{\{R_1=R_1^u\}} + \hat{p}_2^{du}\mathbb{1}_{\{R_1=R_1^d\}} \in \mathcal{F}_{T_1}$ the conditional probability for the second period return going up under $\widehat{\mathbb{P}}$. The parameters $0 < \hat{p}_2^{uu}, \hat{p}_2^{du} < 1$ are known at time $t = 0$ to the oblivious investor, who would solve the stochastic optimization problem at $t = 0$ under the perceived measure $\widehat{\mathbb{P}}$. Different from the stubborn and the forward investors, he would follow the $t = 0$ optimal policies $\widehat{\pi}_1^*$, $\widehat{\pi}_2^*$ all the way through the two periods, being oblivious (i.e. non-adaptive) to the accurate knowledge for the second period at $t = T_1$.

We now solve the classical optimization problem under the perceived measure $\widehat{\mathbb{P}}$ for the oblivious investor with the exponential utility $U(x) = -e^{-\gamma x}$. At $t = T_1$, given any wealth $X_{T_1} = x \in \mathbb{R}$, equation (1.2) yields the optimization problem

$$\sup_{\pi_2 \in \mathbb{R}} \left(-e^{-\gamma(x+\pi_2(R_2^{uu}-1))}\hat{p}_2^{uu} - e^{-\gamma(x+\pi_2(R_2^{ud}-1))}(1-\hat{p}_2^{uu})\right), \quad \text{if } R_1(\omega) = R_1^u,$$

and

$$\sup_{\pi_2 \in \mathbb{R}} \left(-e^{-\gamma(x+\pi_2(R_2^{du}-1))}\hat{p}_2^{du} - e^{-\gamma(x+\pi_2(R_2^{dd}-1))}(1-\hat{p}_2^{du})\right), \quad \text{if } R_1(\omega) = R_1^d.$$

According to (1.12), the unique optimal policy at $t = T_1$ is given by

$$\widehat{\pi}_2^* = -\frac{1}{\gamma(R_2^{uu} - R_2^{ud})} \ln\left(\frac{1 - R_2^{ud}}{R_2^{uu} - 1}\frac{1 - \hat{p}_2^{uu}}{\hat{p}_2^{uu}}\right)\mathbb{1}_{\{R_1=R_1^u\}}$$

$$-\frac{1}{\gamma(R_2^{du}-R_2^{dd})}\ln\left(\frac{1-R_2^{dd}}{R_2^{du}-1}\frac{1-\hat{p}_2^{du}}{\hat{p}_2^{du}}\right)\mathbb{1}_{\{R_1=R_1^d\}}$$

$$=-\frac{1}{\gamma(R_2^u-R_2^d)}\ln\left(\frac{q_2}{1-q_2}\frac{1-\hat{p}_2^u}{\hat{p}_2^u}\right)\in\mathcal{F}_{T_1}, \tag{1.13}$$

where $R_2^u := R_2^{uu}\mathbb{1}_{\{R_1=R_1^u\}}+R_2^{du}\mathbb{1}_{\{R_1=R_1^d\}} \in \mathcal{F}_{T_1}$, $R_2^d := R_2^{ud}\mathbb{1}_{\{R_1=R_1^u\}}+R_2^{dd}\mathbb{1}_{\{R_1=R_1^d\}} \in \mathcal{F}_{T_1}$ and the risk neutral probability

$$q_2 := q_2^u\mathbb{1}_{\{R_1=R_1^u\}} + q_2^d\mathbb{1}_{\{R_1=R_1^d\}}$$

$$=\frac{1-R_2^{ud}}{R_2^{uu}-R_2^{ud}}\mathbb{1}_{\{R_1=R_1^u\}}+\frac{1-R_2^{dd}}{R_2^{du}-R_2^{dd}}\mathbb{1}_{\{R_1=R_1^d\}}.$$

The optimal value function at $t=T_1$ under perceived measure $\widehat{\mathbb{P}}$ is hence given by

$$\widehat{V}_{T_1}(x) =$$

$$-e^{-\gamma x}\left(\hat{p}_2^{uu}\left(\frac{1-R_2^{ud}}{R_2^{uu}-1}\frac{1-\hat{p}_2^{uu}}{\hat{p}_2^{uu}}\right)^{\frac{R_2^{uu}-1}{R_2^{uu}-R_2^{ud}}}+(1-\hat{p}_2^{uu})\left(\frac{1-R_2^{ud}}{R_2^{uu}-1}\frac{1-\hat{p}_2^{uu}}{\hat{p}_2^{uu}}\right)^{\frac{R_2^{ud}-1}{R_2^{uu}-R_2^{ud}}}\right)\mathbb{1}_{\{R_1=R_1^u\}}$$

$$-e^{-\gamma x}\left(\hat{p}_2^{du}\left(\frac{1-R_2^{dd}}{R_2^{du}-1}\frac{1-\hat{p}_2^{du}}{\hat{p}_2^{du}}\right)^{\frac{R_2^{du}-1}{R_2^{du}-R_2^{dd}}}+(1-\hat{p}_2^{du})\left(\frac{1-R_2^{dd}}{R_2^{du}-1}\frac{1-\hat{p}_2^{du}}{\hat{p}_2^{du}}\right)^{\frac{R_2^{dd}-1}{R_2^{du}-R_2^{dd}}}\right)\mathbb{1}_{\{R_1=R_1^d\}}$$

$$=-e^{-\gamma x}\left(\frac{\hat{p}_2^u}{q_2}\right)^{q_2}\left(\frac{1-\hat{p}_2^u}{1-q_2}\right)^{1-q_2}\in\mathcal{F}_{T_1}. \tag{1.14}$$

By backward induction, we now need to solve the optimization problem (1.4) under the perceived measure $\widehat{\mathbb{P}}$, once we have computed $\widehat{V}_{T_1}(x)$ from (1.14). Indeed, according to (1.12), the unique optimal policy at $t=0$ is

$$\widehat{\pi}_1^* = -\frac{1}{\gamma(R_1^u-R_1^d)}$$

$$\times \ln \left(\frac{q_1}{p_1} \frac{1-p_1}{1-q_1} \left(\frac{\hat{p}_2^{du}}{q_2^d} \right)^{q_2^d} \left(\frac{q_2^u}{\hat{p}_2^{uu}} \right)^{q_2^u} \left(\frac{1-\hat{p}_2^{du}}{1-q_2^d} \right)^{1-q_2^d} \left(\frac{1-q_2^u}{1-\hat{p}_2^{uu}} \right)^{1-q_2^u} \right), \quad (1.15)$$

where $q_1 := \frac{1-R_1^d}{R_1^u - R_1^d}$ is the risk neutral probability for the first period.

The oblivious investor would follow $\widehat{\pi}_1^*$ over the first period and then $\widehat{\pi}_2^*$ for the second period, both in the underlying market whose genuine dynamics are described by the true measure $\mathbb{P}$. Since both policies $\widehat{\pi}_1^*$, $\widehat{\pi}_2^*$ are admissible in the genuine market, the oblivious investor, starting from $X_0 = x$, achieves terminal wealth

$$X_{T_2}^{\widehat{\pi}_1^*, \widehat{\pi}_2^*} = x + \widehat{\pi}_1^*(R_1 - 1) + \widehat{\pi}_2^*(R_2 - 1),$$

and hence, his actual $t = 0$ average performance under the physical measure $\mathbb{P}$ is given by $V_0^{\text{Actual}}(x)$, $x \in \mathbb{R}$, as in (1.10). On the other hand, under the $t = 0$ perceived measure $\widehat{\mathbb{P}}$, the targeted optimal performance of the oblivious investor is

$$V_0^{\text{Targeted}}(x) = \mathbb{E}_{\widehat{\mathbb{P}}} \left[U \left(X_{T_2}^{\widehat{\pi}_1^*, \widehat{\pi}_2^*} \right) \Big| X_0 = x \right].$$

The value function $V^{\text{True}}(x)$ is defined as in (1.11), given the full knowledge of the measure $\mathbb{P}$. We are now ready to provide the following quantitative comparison result for the oblivious investor under the two introduced metrics.

Proposition 1.3.1. *For any probability parameters p_2^{uu}, p_2^{du} under $\mathbb{P}$ (see (1.8), 1.9), the regret of the oblivious investor is always nonpositive, i.e.,*

$$M_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{True}(x) \leq 0.$$

The discrepancy

$$m_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{Targeted}(x) < 0,$$

if

- $p_2^{uu} > \hat{p}_2^{uu}$, *when* $\frac{q_2^u}{1-q_2^u}\frac{1-\hat{p}_2^{uu}}{\hat{p}_2^{uu}} > 1$;
- $p_2^{uu} < \hat{p}_2^{uu}$, *when* $\frac{q_2^u}{1-q_2^u}\frac{1-\hat{p}_2^{uu}}{\hat{p}_2^{uu}} < 1$;
- $p_2^{du} > \hat{p}_2^{du}$, *when* $\frac{q_2^d}{1-q_2^d}\frac{1-\hat{p}_2^{du}}{\hat{p}_2^{du}} > 1$;
- $p_2^{du} < \hat{p}_2^{du}$, *when* $\frac{q_2^d}{1-q_2^d}\frac{1-\hat{p}_2^{du}}{\hat{p}_2^{du}} < 1$.

Respectively, $m_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{Targeted}(x) > 0$, *if the above inequality in each regime is reversed.*

Proof. We first compute the optimal strategy π_1^*, π_2^* and the associated value function $V_0^{\text{True}}(x)$, given the true model $\mathbb{P}$ for both periods. The computation is essentially the same as for solving the problem under the perceived model $\widehat{\mathbb{P}}$, and we hence only present the corresponding optimal controls and value functions under $\mathbb{P}$. The unique optimal policy at $t = T_1$ is

$$\pi_2^* = -\frac{1}{\gamma(R_2^u - R_2^d)} \ln\left(\frac{q_2}{1-q_2}\frac{1-p_2^u}{p_2^u}\right) \in \mathcal{F}_{T_1}. \tag{1.16}$$

It follows that the value function at $t = T_1$ under the true model $\mathbb{P}$ is

$$V_{T_1}(x) = -e^{-\gamma x}\left(\frac{p_2^u}{q_2}\right)^{q_2}\left(\frac{1-p_2^u}{1-q_2}\right)^{1-q_2} \in \mathcal{F}_{T_1}. \tag{1.17}$$

In turn,

$$\pi_1^* = -\frac{1}{\gamma(R_1^u - R_1^d)}$$
$$\times \ln\left(\frac{q_1}{p_1}\frac{1-p_1}{1-q_1}\left(\frac{p_2^{du}}{q_2^d}\right)^{q_2^d}\left(\frac{q_2^u}{p_2^{uu}}\right)^{q_2^u}\left(\frac{1-p_2^{du}}{1-q_2^d}\right)^{1-q_2^d}\left(\frac{1-q_2^u}{1-p_2^{uu}}\right)^{1-q_2^u}\right), \quad (1.18)$$

Therefore, for each $x \in \mathbb{R}$,

$$V_0^{\mathrm{True}}(x) = \mathbb{E}\left[U(X_{T_2}^{\pi_1^*,\pi_2^*})\right] = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\left(\pi_1^*(R_1-1)+\pi_2^*(R_2-1)\right)}\right]$$
$$= -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\pi_1^*(R_1-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right]\right],$$

where the conditional expectation

$$\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right] = \left(e^{-\gamma\pi_2^*(R_2^{uu}-1)}p_2^{uu} + e^{-\gamma\pi_2^*(R_2^{ud}-1)}(1-p_2^{ud})\right)\mathbb{1}_{\{R_1=R_1^u\}}$$
$$+\left(e^{-\gamma\pi_2^*(R_2^{du}-1)}p_2^{du} + e^{-\gamma\pi_2^*(R_2^{dd}-1)}(1-p_2^{du})\right)\mathbb{1}_{\{R_1=R_1^d\}}$$

is indeed minimized by the unique minimizer π_2^* given in (1.16), according to (1.12). On the other hand, the $t = 0$ actual performance of the oblivious behavior under $\widehat{\pi}_1^*$, $\widehat{\pi}_2^*$ is

$$V_0^{\mathrm{Actual}}(x) = \mathbb{E}\left[U(X_{T_2}^{\widehat{\pi}_1^*,\widehat{\pi}_2^*})\right] = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\left(\widehat{\pi}_1^*(R_1-1)+\widehat{\pi}_2^*(R_2-1)\right)}\right]$$
$$= -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right]\right],$$

with the conditional expectation

$$\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right] \geq \mathbb{E}[e^{-\gamma\pi_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}].$$

As the consequence,

$$V_0^{\mathrm{Actual}}(x) = -e^{-\gamma x}\Big(p_1 e^{-\gamma\widehat{\pi}_1^*(R_1^u-1)}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right]\mathbb{1}_{\{R_1=R_1^u\}}$$

$$+(1-p_1)e^{-\gamma\widehat{\pi}_1^*(R_1^d-1)}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\mathbb{1}_{\{R_1=R_1^d\}}\Big)$$

$$\leq -e^{-\gamma x}\Big(p_1e^{-\gamma\widehat{\pi}_1^*(R_1^u-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\mathbb{1}_{\{R_1=R_1^u\}}$$

$$+(1-p_1)e^{-\gamma\widehat{\pi}_1^*(R_1^d-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\mathbb{1}_{\{R_1=R_1^d\}}\Big)$$

$$\leq -e^{-\gamma x}\Big(p_1e^{-\gamma\pi_1^*(R_1^u-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\mathbb{1}_{\{R_1=R_1^u\}}$$

$$+(1-p_1)e^{-\gamma\pi_1^*(R_1^d-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\mathbb{1}_{\{R_1=R_1^d\}}\Big)=V_0^{\text{True}}(x),$$

where the last inequality follows, since π_1^* from (1.18) is indeed the unique minimizer according to (1.12). Next, we compute the targeted optimal performance under $\widehat{\mathbb{P}}$

$$V_0^{\text{Targeted}}(x)=\mathbb{E}_{\widehat{\mathbb{P}}}\left[U(X_{T_2}^{\widehat{\pi}_1^*,\widehat{\pi}_2^*})\right]=-e^{-\gamma x}\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\left(\widehat{\pi}_1^*(R_1-1)+\widehat{\pi}_2^*(R_2-1)\right)}\right]$$

$$=-e^{-\gamma x}\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\right],$$

where the conditional expectation is

$$\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]=\left(e^{-\gamma\widehat{\pi}_2^*(R_2^{uu}-1)}\widehat{p}_2^{uu}+e^{-\gamma\widehat{\pi}_2^*(R_2^{ud}-1)}(1-\widehat{p}_2^{uu})\right)\mathbb{1}_{\{R_1=R_1^u\}}$$

$$+\left(e^{-\gamma\widehat{\pi}_2^*(R_2^{du}-1)}\widehat{p}_2^{du}+e^{-\gamma\widehat{\pi}_2^*(R_2^{dd}-1)}(1-\hat{p}_2^{du})\right)\mathbb{1}_{\{R_1=R_1^d\}}.$$

To compare $V_0^{\text{Targeted}}(x)$ with $V_0^{\text{Actual}}(x)$, we notice that over the first period, the same control $\widehat{\pi}_1^*$ and the same probability $0<p_1<1$ would be applied in the computation of the two value functions. It hence follows that if $\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]<\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]$, for all $\omega\in\Omega$, then $V_0^{\text{Actual}}(x)<V_0^{\text{Targeted}}(x)$, and vice versa. We next look at the case when $\frac{q_2^u}{1-q_2^u}\frac{1-\hat{p}_2^{uu}}{\hat{p}_2^{uu}}>1$.

Direct computation based on (1.13) yields $-\gamma\widehat{\pi}_2^*(R_2^{uu}-1) > -\gamma\widehat{\pi}_2^*(R_2^{ud}-1)$, and hence,

$$e^{-\gamma\widehat{\pi}_2^*(R_2^{uu}-1)}\hat{p}_2^{uu}+e^{-\gamma\widehat{\pi}_2^*(R_2^{ud}-1)}(1-\hat{p}_2^{uu}) < e^{-\gamma\widehat{\pi}_2^*(R_2^{uu}-1)}p_2^{uu}+e^{-\gamma\widehat{\pi}_2^*(R_2^{ud}-1)}(1-p_2^{uu}),$$

if $\hat{p}_2^{uu} < p_2^{uu} < 1$. Similarly, the case $\frac{q_2^d}{1-q_2^d}\frac{1-\hat{p}_2^{du}}{\hat{p}_2^{du}} > 1$ gives rise to $-\gamma\widehat{\pi}_2^*(R_2^{du}-1) > -\gamma\widehat{\pi}_2^*(R_2^{dd}-1)$, and consequently,

$$e^{-\gamma\widehat{\pi}_2^*(R_2^{du}-1)}\hat{p}_2^{du}+e^{-\gamma\widehat{\pi}_2^*(R_2^{dd}-1)}(1-\hat{p}_2^{du}) < e^{-\gamma\widehat{\pi}_2^*(R_2^{du}-1)}p_2^{du}+e^{-\gamma\widehat{\pi}_2^*(R_2^{dd}-1)}(1-p_2^{du}),$$

if $\hat{p}_2^{du} < p_2^{du} < 1$. The above results further lead to $\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right] < \mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]$, and hence $m_{0,T_2}(x) = V_0^{\text{Actual}}(x) - V_0^{\text{Targeted}}(x) < 0$. The other two cases in the proposition can be similarly derived. □

The conclusions in Proposition 1.3.1 are rather intuitive regarding the assessment of performance under both metrics. The regret metric $M_{0,T_2}(x)$ is nonpositive for all $x \in \mathbb{R}$, due to the fact that the oblivious investor made his decision fully based on the perceived measure $\widehat{\mathbb{P}}$ rather than the true measure $\mathbb{P}$, while his performance is evaluated in the true market under the measure $\mathbb{P}$.

The comparison between $V_0^{\text{Actual}}(x)$ and $V_0^{\text{Targeted}}(x)$ for the first metric $m_{0,T_2}(x)$, on the other hand, is more subtle. For fixed risk neutral probabilities, when the perceived probabilities $\hat{p}_2^{uu},\ \hat{p}_2^{du}$ are sufficiently higher than the true probabilities $p_2^{uu},\ p_2^{du}$, corresponding to the second and fourth cases in the proposition, the targeted performance is higher than the actual performance, as the oblivious investor overestimates the return of the market. However, when the perceived probabilities are significantly lower than the true probabilities, the targeted performance is also higher than the actual performance, corresponding to the first and third cases in the proposition. In these scenarios, one can notice that $\widehat{\pi}_2^*$ given by (1.13) is negative with a large absolute value, indicating substantial withdrawal from the investment in the stock. Failing to exploit the actually favorable investment opportunity over the second period in the market, the oblivious investor's actual performance is indeed worse than his $t = 0$ target.

1.3.3 Stubborn investor (model adaptive/goal persistent)

Like the oblivious investor, the stubborn investor also needs to specify the perceived measure for the entire horizon $[0, T_2]$ at $t = 0$, in order to compute the first period optimal strategy. For comparison purposes, we assume that the same perceived measure $\widehat{\mathbb{P}}$ is adopted by the stubborn investor at $t = 0$ and her terminal utility is also $U(x) = -e^{-\gamma x}$, $x \in \mathbb{R}$, for $\gamma > 0$. However, different from the oblivious investor, she is "model adaptive" in the sense that she reconsiders the optimization problem for the remaining horizon at $t = T_1$, as soon as she learns the accurate model for the second period at $t = T_1$. Therefore, conditional on $\mathcal{F}_{T_1}$, the stubborn investor solves the optimization problem

$$V_{T_1}(x) = \operatorname*{esssup}_{\pi_2} \mathbb{E}\left[U(X_{T_1} + \pi_2(R_2 - 1)) | X_{T_1} = x, \mathcal{F}_{T_1}\right],$$

under the true measure $\mathbb{P}$, and obtains the optimal control π_2^* given by (1.16). The terminal wealth is hence represented by $X_{T_2}^{\widehat{\pi}_1^*,\pi_2^*} = x + \widehat{\pi}_1^*(R_1 - 1) + \pi_2^*(R_2 - 1)$, with $\widehat{\pi}_1^*$ being derived at $t = 0$ under the perceived measure $\widehat{\mathbb{P}}$ as in (1.15). We have the following result on the performance of the stubborn investor under the two metrics.

Proposition 1.3.2. *For any probability parameters p_2^{uu}, p_2^{du} under $\mathbb{P}$ (see (1.8), 1.9), the regret of the stubborn investor is always nonpositive, i.e.,*

$$M_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{True}(x) \leq 0.$$

The discrepancy

$$m_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{Targeted}(x) < 0,$$

if $p_2^{uu} \in (\hat{p}_2^{uu}, q_2^u]$ or $p_2^{uu} \in [q_2^u, \hat{p}_2^{uu})$, and if $p_2^{du} \in (\hat{p}_2^{du}, q_2^d]$ or $p_2^{du} \in [q_2^d, \hat{p}_2^{du})$.

Proof. Working as in the proof of Proposition 1.3.1, we obtain that the $t = 0$ true optimal performance under measure $\mathbb{P}$ is

$$V_0^{\text{True}}(x) = \mathbb{E}\left[U(X_{T_2}^{\pi_1^*,\pi_2^*})\right] = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\pi_1^*(R_1-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\right],$$

whereas the $t = 0$ actual average performance of the stubborn investor is

$$V_0^{\text{Actual}}(x) = \mathbb{E}\left[U(X_{T_2}^{\widehat{\pi}_1^*,\pi_2^*})\right] = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\right].$$

According to (1.12), π_1^* from (1.18) is the unique minimizer of the quantity

$$\min_{\pi_1}\mathbb{E}\left[e^{-\gamma\pi_1(R_1-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\right],$$

given the second period optimizer π_2^* from (1.16). It hence leads to that $M_{0,T_2}(x) = V_0^{\text{Actual}}(x) - V_0^{\text{True}}(x) \leq 0$ under the genuine probability measure $\mathbb{P}$. To examine the metric $m_{0,T_2}(x)$, we notice that the $t = 0$ value function under the perceived measure $\widehat{\mathbb{P}}$ remains as

$$V_0^{\text{Targeted}}(x) = \mathbb{E}_{\widehat{\mathbb{P}}}\left[U_{T_2}(X_T^{\widehat{\pi}_1^*, \widehat{\pi}_2^*})\right] = -e^{-\gamma x}\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\right].$$

Recall that over the first period, the probability measures $\widehat{\mathbb{P}}$ and $\mathbb{P}$ share the same model parameter p_1, and the same policy $\widehat{\pi}_1^*$ is used. Hence, in order to compare $V_0^{\text{Targeted}}(x)$ and $V_0^{\text{Actual}}(x)$, if suffices to compare $\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]$ and $\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]$. Substituting the corresponding optimal policies $\widehat{\pi}_2^*$, π_2^* into the two conditional expectations, respectively, we obtain

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right] &= \left(\frac{\hat{p}_2^{uu}}{q_2^u}\right)^{q_2^u}\left(\frac{1-\hat{p}_2^{uu}}{1-q_2^u}\right)^{1-q_2^u}\mathbb{1}_{\{R_1=R_1^u\}}\\ &\quad+\left(\frac{\hat{p}_2^{du}}{q_2^d}\right)^{q_2^d}\left(\frac{1-\hat{p}_2^{du}}{1-q_2^d}\right)^{1-q_2^d}\mathbb{1}_{\{R_1=R_1^d\}},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right] &= \left(\frac{p_2^{uu}}{q_2^u}\right)^{q_2^u}\left(\frac{1-p_2^{uu}}{1-q_2^u}\right)^{1-q_2^u}\mathbb{1}_{\{R_1=R_1^u\}}\\ &\quad+\left(\frac{p_2^{du}}{q_2^d}\right)^{q_2^d}\left(\frac{1-p_2^{du}}{1-q_2^d}\right)^{1-q_2^d}\mathbb{1}_{\{R_1=R_1^d\}}.\end{aligned}$$

Therefore, we need to examine the monotonicity of the functions $f(p; q_2^u) := p^{q_2^u}(1-p)^{1-q_2^u}$ and $f(p; q_2^d) := p^{q_2^d}(1-p)^{1-q_2^d}$ for $0 < p < 1$, under fixed risk neutral probabilities $0 < p_2^u, p_2^d < 1$. Clearly, these functions have maximum values attained at q_2^u, q_2^d, respectively. We therefore easily conclude that if $p_2^{uu} \in (\hat{p}_2^{uu}, q_2^u]$ or $p_2^{uu} \in [q_2^u, \hat{p}_2^{uu})$, and if $p_2^{du} \in (\hat{p}_2^{du}, q_2^d]$ or $p_2^{du} \in [q_2^d, \hat{p}_2^{du})$, then $\mathbb{E}_{\widehat{\mathbb{P}}}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right] < \mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right]$, which leads to $m_{0,T_2}(x) < 0$. Analogously, $m_{0,T_2}(x) > 0$ if the second period stock return probabilities p_2^{uu}, p_2^{du} under the true model $\mathbb{P}$ stay outside the above regimes. □

Proposition 1.3.2 provides an intuitively correct comparison between the targeted performance and the actual performance. It shows that if the genuine market condition for the future period $[T_1, T_2]$, compared to the $t = 0$ perceived market condition for $[T_1, T_2]$, deviates substantially from the risk neutral case , then proper reaction by immediately taking into account such unanticipated deviation can lead to better overall performance than initially targeted. On the other hand, if the true market condition turns out to be close to the risk neutral case than initially expected, then even direct response to this correct knowledge at time $t = T_1$ can not enable the stubborn investor to achieve the targeted performance under the measure $\widehat{\mathbb{P}}$.

1.3.4 Robust investor (model robust/goal persistent)

In the context of optimization under model ambiguity, different formulations and solutions have been proposed within the paradigm of backward approach, among which the readers can find the seminal works including the (backward) robust approach in [29] and the multiple priors formulation in [20]. A more unified discussion in the dynamic setting is provided in [40]. The robust control approach takes into account a family of possible models and typically leads to conservative investment behavior for an ambiguity averse investor. This attitude is reflected in the underlying mathematical formulation that solves a maxmin problem; an optimal control law is sought to mainly respond the worst scenario that the decision maker can ever possibly encounter from the (presumed malevolent) nature. As a consequence, the robust approach provides an insurance on the investor's performance against the most unfavorable market condition by not exploring other investment opportunities that may also possibly occur. In particular, if the true market condition turns out not to be so adverse, or even favorable for investment, then the conservative behavior implied by the robust approach may, in retrospect, induce large losses. The regret concept we introduced earlier turns out to be an appropriate metric to gauge the loss of the investor being too conservative when facing model uncertainty. Indeed, under the current two-period binomial model setting, we show that the well documented non-participation effect could occur for the second period if the ambiguity set of the investor is large enough. Such conservative behavior leads to zero allocation in the risky asset. Although the robust approach protects the investor from potentially harmful market scenarios in this way, it can in turn cause more regret in hindsight if the genuine market opportunity turns out to be actually favorable for more active investment behavior.

As before, we focus on the representative two-period binomial model with uncertainty on the second period stock return probabilities, with only the first period probability revealed to the robust investor at $t = 0$. Different from how the other investors respond to model uncertainty, the robust investor would impose *a family of probabilities* for the second period stock return going up, conditional on the first period return going up or down. We denote the assocaited ambiguity sets as $\hat{p}_2^{uu} \in [\varepsilon^u, 1-\varepsilon^u]$ and $\hat{p}_2^{du} \in [\varepsilon^d, 1-\varepsilon^d]$, respectively. We emphasize that not only the second period stock return probabilities can be conditioned on the first period return movement, but also the ambiguity sets may be dependent on the outcome of the first period stock return. The latter captures adaptive learning incorporated into the robust framework (see, e.g. [7], or [21]). As time evolves, the ambiguity sets may shrink ([7]) or expand depending on the nature of the signal and the underlying quantity to learn ([21]), while if learning is not applied, the ambiguity sets would remain constant bandwidth across different periods. Although adaptive learning can be included in the robust control framework, we note that this seemingly online attribute of such model-based learning does not change the backward reasoning nature of the robust approach. Still, the issue of model commitment prevails even with such kind of learning, as decision made for today is still contingent on the $t = 0$ prescribed reaction rule to the future stock returns through the model-based learning. In fact, one can soon recognize that the learning rule is actually part of the model state dynamics, and that it has been taken into account by the backward induction method in discrete time to generate time-consistent optimal strategy.

To avoid the unnecessary complexity due to the model-based learning that does not genuinely change the backward reasoning nature, we formulate the robust control problem in its original form without the component of learning. The $t = 0$ problem to solve is the maxmin problem under the exponential utility $U(x) = -e^{-\gamma x}$,

$$V_0^{\text{Targeted}}(x) = \sup_{\pi_1,\pi_2} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}} \left[U_{T_2}(X_{T_2}^{\pi_1,\pi_2}) | X_0 = x \right], \tag{1.19}$$

where the family of possible measures $\mathcal{Q}$ is defined as

$$\mathcal{Q} := \Big\{ \mathbb{Q} : \mathbb{Q}(R_2 = R_2^{uu} | R_1 = R_1^u) = \hat{p}_2^{uu},\ \mathbb{Q}(R_2 = R_2^{du} | R_1 = R_1^d) = \hat{p}_2^{du},$$

$$\text{with } \hat{p}_2^{uu} \in [\varepsilon^u, 1-\varepsilon^u],\ \hat{p}_2^{du} \in [\varepsilon^d, 1-\varepsilon^d],\ \text{and } \mathbb{Q}(A) = \mathbb{P}(A),\ \forall A \in \mathcal{F}_{T_1} \Big\}.$$

In the above formulation, we assume $0 < \varepsilon^u, \varepsilon^d < \frac{1}{2}$ to be constants known at $t = 0$. Note that such assumption complies with the model commitment discussed earlier, which requires the pre-specification of one or a family of possible models for the future stock returns at $t = 0$.

To solve problem (1.19), we first solve

$$V_{T_1}^{\text{Targeted}}(X_{T_1}) = \operatorname*{esssup}_{\pi_2} \operatorname*{essinf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}} \left[-e^{-\gamma(X_{T_1}+\pi_2(R_2-1))} \Big| X_{T_1},\ \mathcal{F}_{T_1} \right] \tag{1.20}$$

$$= \max_{\pi_2} \min_{\substack{\varepsilon^u \le \hat{p}_2^{uu} \le 1-\varepsilon^u \\ \varepsilon^d \le \hat{p}_2^{du} \le 1-\varepsilon^d}} \Bigg(\Big(-\exp\left(-\gamma\left(X_{T_1} + \pi_2(R_2^{uu}-1)\right)\right) \hat{p}_2^{uu}$$

$$-\exp\Big(-\gamma\Big(X_{T_1} + \pi_2(R_2^{ud}-1)\Big)\Big)(1-\hat{p}_2^{uu})\Big) \mathbb{1}_{\{R_1=R_1^u\}}$$

$$+\Big(-\exp\Big(-\gamma\Big(X_{T_1} + \pi_2(R_2^{du}-1)\Big)\Big) \hat{p}_2^{du}$$

$$-\exp\Big(-\gamma\Big(X_{T_1} + \pi_2(R_2^{dd}-1)\Big)\Big)(1-\hat{p}_2^{du})\Big) \mathbb{1}_{\{R_1=R_1^d\}} \Bigg).$$

The above maxmin problem can be separately considered on the sets $\{R_1 = R_1^u\}$ and $\{R_1 = R_1^d\}$; for instance, the problem conditional on $\{R_1 = R_1^u\}$ is to solve

$$\max_{\pi_2} \min_{\varepsilon^u \le \hat{p}_2^{uu} \le 1-\varepsilon^u} f(\pi_2, \hat{p}_2^{uu}) := \max_{\pi_2} \min_{\varepsilon^u \le \hat{p}_2^{uu} \le 1-\varepsilon^u} \Big(-\exp\left(-\gamma \left(X_{T_1} + \pi_2(R_2^{uu} - 1)\right)\right) \hat{p}_2^{uu} - \exp\Big(-\gamma \Big(X_{T_1} + \pi_2(R_2^{ud} - 1)\Big)\Big) (1 - \hat{p}_2^{uu})\Big),$$

with the function $f(\pi_2, \hat{p}_2^{uu})$ clearly being a convex-concave function, and conditions for applying the Minmax theorem can be easily verified (see e.g. [41]). Hence, we turn to solve

$$\min_{\varepsilon^u \le \hat{p}_2^{uu} \le 1-\varepsilon^u} \max_{\pi_2} \Big(-\exp\Big(-\gamma \left(X_{T_1} + \pi_2(R_2^{uu} - 1)\right) \Big) \hat{p}_2^{uu} - \exp\Big(-\gamma \Big(X_{T_1} + \pi_2(R_2^{ud} - 1)\Big)\Big) (1 - \hat{p}_2^{uu})\Big). \tag{1.21}$$

For any fixed $\hat{p}_2^{uu} \in [\varepsilon^u, 1 - \varepsilon^u]$, according to (1.12), the unique maximizer $\widehat{\pi}_2^*$ on the set $\{R_1 = R_1^u\}$ is given by

$$\widehat{\pi}_2^* \mathbb{1}_{\{R_1 = R_1^u\}} = -\frac{1}{\gamma(R_2^{uu} - R_2^{ud})} \ln\left(\frac{1 - R_2^{ud}}{R_2^{uu} - 1} \frac{1 - \hat{p}_2^{uu}}{\hat{p}_2^{uu}} \right). \tag{1.22}$$

Similar argument on the set $\{R_1 = R_1^d\}$ yields

$$\widehat{\pi}_2^* \mathbb{1}_{\{R_1 = R_1^d\}} = -\frac{1}{\gamma(R_2^{du} - R_2^{dd})} \ln\left(\frac{1 - R_2^{dd}}{R_2^{du} - 1} \frac{1 - \hat{p}_2^{du}}{\hat{p}_2^{du}} \right), \tag{1.23}$$

for any fixed $\hat{p}_2^{du} \in [\varepsilon^d, 1 - \varepsilon^d]$. It remains to solve the outer minimization problem in (1.21) after we substitute (1.22) into the objective function to be minimized. Direct computation leads to the minimization problem

$$\min_{\varepsilon^u \le \hat{p}_2^{uu} \le 1-\varepsilon^u} -e^{-\gamma x} \left(\left(\frac{q_2^u}{1 - q_2^u} \frac{1 - \hat{p}_2^{uu}}{\hat{p}_2^{uu}} \right)^{1 - q_2^u} \hat{p}_2^{uu} + \left(\frac{q_2^u}{1 - q_2^u} \frac{1 - \hat{p}_2^{uu}}{\hat{p}_2^{uu}} \right)^{-q_2^u} (1 - \hat{p}_2^{uu}) \right)$$

with $X_{T_1} = x \in \mathbb{R}$, whose minimizer is

$$\hat{p}_2^{uu*} = \begin{cases} 1-\varepsilon^u, & \text{if } q_2^u \geq 1-\varepsilon^u; \\ \varepsilon^u, & \text{if } q_2^u \leq \varepsilon^u; \\ q_2^u, & \text{if } \varepsilon^u < q_2^u < 1-\varepsilon^u. \end{cases}$$

Accordingly, from (1.22), we can conclude that, conditional on $\{R_1 = R_1^u\}$,

- if $0 < q_2^u < \varepsilon^u$, then $\hat{p}_2^{uu*} = \varepsilon^u$ and

  $$\widehat{\pi}_2^* \mathbb{1}_{\{R_1 = R_1^u\}} = -\frac{1}{\gamma(R_2^{uu} - R_2^{ud})} \ln\left(\frac{1-R_2^{ud}}{R_2^{uu}-1}\frac{1-\varepsilon^u}{\varepsilon^u}\right) > 0,$$

  i.e., the robust investor would long the risky asset during the second period;

- if $0 < 1-\varepsilon^u < q_2^u$, then $\hat{p}_2^{uu*} = 1-\varepsilon^u$ and

  $$\widehat{\pi}_2^* \mathbb{1}_{\{R_1 = R_1^u\}} = -\frac{1}{\gamma(R_2^{uu} - R_2^{ud})} \ln\left(\frac{1-R_2^{ud}}{R_2^{uu}-1}\frac{\varepsilon^u}{1-\varepsilon^u}\right) < 0,$$

  i.e., the robust investor would short the risky asset during the second period;

- if $\varepsilon^u \leq q_2^u \leq 1-\varepsilon^u$, then $\hat{p}_2^{uu*} = q_2^u$ and

  $$\widehat{\pi}_2^* \mathbb{1}_{\{R_1 = R_1^u\}} = 0,$$

  i.e., the robust investor would hold zero position in the risky asset during the second period.

Similar conclusions hold on the set $\{R_1 = R_1^d\}$. The above results derived under the robust control framework have rather intuitive interpretations. Indeed, the first two scenarios correspond to a not too large ambiguity set (bounded from above or below by the risk neutral probability), and hence the investor would be relatively confident about what action to take. When all the perceived probabilities for the stock return going up over the second period exceed the risk neutral probability, the investor should long the risky asset. On the other hand, if the investor perceives the probability of future return going up to be definitely lower than the risk neutral probability, she should short the risky asset. In the third scenario where the ambiguity set is too large (containing the risk neutral probability), the ambiguity averse investor has not accumulated sufficient information to make investment decisions and, hence, the non-participation effect occurs. Such conservative behavior under robust control framework has been well documented in both theoretical and empirical studies (see e.g. [17], [13]).

When the ambiguity set is large enough to induce the non-participation behavior over the second period, the value function $V_{T_1}^{\text{Targeted}}(\cdot)$ coincides with the terminal exponential utility, since all wealth would be put into the riskless asset with zero interest rate for the second period. Accordingly,

$$V_0^{\text{Targeted}}(x) = \max_{\pi_1} \mathbb{E}_{\mathbb{P}} \left[V_{T_1}^{\text{Targeted}}(X_{T_1}) \middle| X_0 = x \right]$$

$$= \max_{\pi_1} \mathbb{E}_{\mathbb{P}} \left[-e^{-\gamma(x+\pi_1(R_1-1))} \middle| X_0 = x \right],$$

where we have applied the fact $\mathbb{Q}(A) = \mathbb{P}(A)$, for any $\mathbb{Q} \in \mathcal{Q}$ and any $A \in \mathcal{F}_{T_1}$.

Direct computation then gives the first period optimal policy

$$\widehat{\pi}_1^* = -\frac{1}{\gamma(R_1^u - R_1^d)} \ln\left(\frac{1-R_1^d}{R_1^u-1}\frac{1-p_1}{p_1}\right), \tag{1.24}$$

as well as the $t=0$ targeted value function

$$\begin{aligned} V_0^{\text{Targeted}}(x) &= -e^{-\gamma x}\left(\left(\frac{q_1}{1-q_1}\frac{1-p_1}{p_1}\right)^{1-q_1} p_1 + \left(\frac{q_1}{1-q_1}\frac{1-p_1}{p_1}\right)^{-q_1}(1-p_1)\right) \\ &= -e^{-\gamma x}\left(\frac{q_1}{1-q_1}\frac{1-p_1}{p_1}\right)^{-q_1}\frac{1-p_1}{1-q_1}. \end{aligned} \tag{1.25}$$

The non-participation effect guarantees that the robust investor can still do relatively well even under the worst scenario, but typically at the cost of giving up the opportunity to exploit possibly beneficial market conditions. In the next proposition, we quantitatively measure the loss in terms of regret after showing that the regret for the robust investor is also nonpositive. Our result demonstrates that the conservative behavior under the robust control approach can produce more regret if the reality turns out to be further from the worst scenario.

Proposition 1.3.3. *For any probability parameters p_2^{uu}, p_2^{du} under $\mathbb{P}$ (see (1.8), (1.9)), that satisfy $p_2^{uu} \in [\varepsilon^u, 1-\varepsilon^u]$, $p_2^{du} \in [\varepsilon^d, 1-\varepsilon^d]$, the regret of the robust investor is nonpositive, i.e.,*

$$M_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{True}(x) \le 0.$$

Furthermore, if $q_2^u \in [\varepsilon^u, 1-\varepsilon^u]$ and $q_2^d \in [\varepsilon^d, 1-\varepsilon^d]$, then there exists a function $C : [\varepsilon^u, 1-\varepsilon^u] \times [\varepsilon^d, 1-\varepsilon^d] \mapsto \mathbb{R}_+$, such that the discrepancy

$$M_{0,T_2}(x) = m_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{True}(x)$$

$$= V_0^{Targeted}(x) - V_0^{True}(x) = (1 - C(p_2^{uu}, p_2^{du}))V_0^{Actual}(x),$$

for all initial wealth $x \in \mathbb{R}$, with

$$\min_{\substack{\varepsilon^u \le p_2^{uu} \le 1-\varepsilon^u \\ \varepsilon^d \le p_2^{du} \le 1-\varepsilon^d}} C(p_2^{uu}, p_2^{du}) \to 0, \qquad \textit{as} \quad \varepsilon^u,\ \varepsilon^d \to 0. \tag{1.26}$$

Proof. The regret of the robust investor being nonpositive follows easily from the fact that the optimal robust controls $\widehat{\pi}_1^*$, $\widehat{\pi}_2^*$ are in general only admissible controls, rather than the genuine optimal controls that yield the value function $V_0^{\text{True}}(x)$. Indeed, let π_1^* and π_2^* be the optimal controls of the classical stochastic optimization problem in hindsight under the exponential utility, with the full knowledge of the true measure $\mathbb{P}$ given. Then clearly,

$$V_0^{\text{True}}(x) = \mathbb{E}\left[-e^{-\gamma\left(x+\pi_1^*(R_1-1)+\pi_2^*(R_2-1)\right)}\right]$$

$$\geq \mathbb{E}\left[-e^{-\gamma\left(x+\widehat{\pi}_1^*(R_1-1)+\widehat{\pi}_2^*(R_2-1)\right)}\right] = V_0^{\text{Actual}}(x),$$

for all $x \in \mathbb{R}$, since both strategies (π_1^*, π_2^*) and $(\widehat{\pi}_1^*, \widehat{\pi}_2^*)$ are evaluated under the same true measure $\mathbb{P}$, with the former being the optimizer under $\mathbb{P}$. To quantify the loss in regret, we notice that under the additional assumption $\varepsilon^u \le q_2^u \le 1 - \varepsilon^u$ and $\varepsilon^d \le q_2^d \le 1 - \varepsilon^d$, the optimal robust controls $\widehat{\pi}_1^*$ is given by (1.24) while $\widehat{\pi}_2^* = 0$. It hence yields that

$$V_0^{\text{Actual}}(x) = \mathbb{E}\left[-e^{-\gamma\left(x+\widehat{\pi}_1^*(R_1-1)+0(R_2-1)\right)}\right] = V_0^{\text{Targeted}}(x),$$

for any $x \in \mathbb{R}$. On the other hand,

$$V_0^{\text{True}}(x) = \mathbb{E}\left[-e^{-\gamma\left(x+\pi_1^*(R_1-1)+\pi_2^*(R_2-1)\right)}\right]$$

$$= \mathbb{E}\left[-e^{-\gamma\left(x+\pi_1^*(R_1-1)\right)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\right].$$

According to (1.12), we have

$$\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right] = \min_{\pi_2}\mathbb{E}\left[e^{-\gamma\pi_2(R_2-1)}\middle|\mathcal{F}_{T_1}\right]$$

$$= \left(\left(\frac{q_2^u}{1-q_2^u}\frac{1-p_2^{uu}}{p_2^{uu}}\right)^{1-q_2^u} p_2^{uu} + \left(\frac{q_2^u}{1-q_2^u}\frac{1-p_2^{uu}}{p_2^{uu}}\right)^{-q_2^u}(1-p_2^{uu})\right)\mathbb{1}_{\{R_1=R_1^u\}}$$

$$+ \left(\left(\frac{q_2^d}{1-q_2^d}\frac{1-p_2^{du}}{p_2^{du}}\right)^{1-q_2^d} p_2^{du} + \left(\frac{q_2^d}{1-q_2^d}\frac{1-p_2^{du}}{p_2^{du}}\right)^{-q_2^d}(1-p_2^{du})\right)\mathbb{1}_{\{R_1=R_1^d\}}.$$

Let

$$A^u := \left(\left(\frac{q_2^u}{1-q_2^u}\frac{1-p_2^{uu}}{p_2^{uu}}\right)^{1-q_2^u} p_2^{uu} + \left(\frac{q_2^u}{1-q_2^u}\frac{1-p_2^{uu}}{p_2^{uu}}\right)^{-q_2^u}(1-p_2^{uu})\right), \quad (1.27)$$

and

$$A^d := \left(\left(\frac{q_2^d}{1-q_2^d}\frac{1-p_2^{du}}{p_2^{du}}\right)^{1-q_2^d} p_2^{du} + \left(\frac{q_2^d}{1-q_2^d}\frac{1-p_2^{du}}{p_2^{du}}\right)^{-q_2^d}(1-p_2^{du})\right). \quad (1.28)$$

It then follows that

$$V_0^{\text{True}}(x) = \max_{\pi_1}\left[-e^{-\gamma\left(x+\pi_1(R_1^u-1)\right)}A^u p_1 - e^{-\gamma\left(x+\pi_1(R_1^d-1)\right)}A^d(1-p_1)\right].$$

Again, by (1.12), we obtain

$$\pi_1^* = -\frac{1}{\gamma(R_1^u-R_1^d)}\ln\left(\frac{q_1}{1-q_1}\frac{1-p_1}{p_1}\frac{A^d}{A^u}\right), \quad (1.29)$$

and

$$V_0^{\text{True}}(x) = -e^{-\gamma x}\left(\frac{q_1}{1-q_1}\frac{1-p_1}{p_1}\right)^{-q_1}\frac{1-p_1}{1-q_1}\left(\frac{A^d}{A^u}\right)^{-q_1}A^d. \quad (1.30)$$

Now recalling $V_0^{\text{Targeted}}(x)$ as in (1.25), we then have

$$M_{0,T_2}(x) = V_0^{\text{Actual}}(x) - V_0^{\text{True}}(x)$$

$$= V_0^{\text{Targeted}}(x) - V_0^{\text{True}}(x) = (1 - C(p_2^{uu}, p_2^{du}))V_0^{\text{Actual}}(x), \tag{1.31}$$

with the function $C(p_2^{uu}, p_2^{du})$ given by

$$C(p_2^{uu}, p_2^{du}) = C(p_2^{uu}, p_2^{du}; q_2^u, q_2^d) = \left(\frac{A^d}{A^u}\right)^{-q_1} A^d.$$

Note that $V_0^{\text{Actual}}(x)$ does not depend on the second period true probability parameters p_2^{uu}, p_2^{du}, nor the ambiguity sets parameters ε^u, ε^d. It is therefore sufficient to analyze the single quantity $C(p_2^{uu}, p_2^{du})$ to determine when the robust investor would experience the most regret. Direct computation shows

$$C(p_2^{uu}, p_2^{du}) =$$

$$\left(1 - p_2^{uu}\right)^{q_1(1-q_2^u)} \left(p_2^{uu}\right)^{q_1 q_2^u} \left(1 - p_2^{du}\right)^{(1-q_1)(1-q_2^d)} \left(p_2^{du}\right)^{(1-q_1)q_2^d} C_1(q_2^u, q_2^d), \tag{1.32}$$

where $C_1(q_2^u, q_2^d)$ is some known constant that depends only on the fixed risk neutral probabilities q_2^u, q_2^d. Minimization of $C(p_2^{uu}, p_2^{du})$ over the ambiguity sets yields

$$\underset{\substack{\varepsilon^u \le p_2^{uu} \le 1-\varepsilon^u \\ \varepsilon^d \le p_2^{du} \le 1-\varepsilon^d}}{\arg\min}\, C(p_2^{uu}, p_2^{du}) = \begin{cases} 1-\varepsilon^u, \quad 1-\varepsilon^d, & \text{if} \quad 0 < q_2^u < \frac{1}{2}, \quad 0 < q_2^d < \frac{1}{2}, \\ \varepsilon^u, \quad \varepsilon^d, & \text{if} \quad \frac{1}{2} < q_2^u < 1, \quad \frac{1}{2} < q_2^d < 1, \\ 1-\varepsilon^u, \quad \varepsilon^d, & \text{if} \quad 0 < q_2^u < \frac{1}{2}, \quad \frac{1}{2} < q_2^d < 1, \\ \varepsilon^u, \quad 1-\varepsilon^d, & \text{if} \quad \frac{1}{2} < q_2^u < 1, \quad 0 < q_2^d < \frac{1}{2}. \end{cases}$$

In all of these cases, it is easy to verify that as ε^u, $\varepsilon^d \to 0$,

$$\min_{\substack{\varepsilon^u \le p_2^{uu} \le 1-\varepsilon^u \\ \varepsilon^d \le p_2^{du} \le 1-\varepsilon^d}} C(p_2^{uu}, p_2^{du}) \to 0,$$

and hence, according to (1.31), the regret of the robust investor approaches its largest negative value $V_0^{\text{Actual}}(x)$, as $\varepsilon^u,\ \varepsilon^d \to 0$. $\square$

In the proof of Proposition 1.3.3, we can actually see that the largest negative regret occurs if reality is most distinct from the worst scenario that the robust control approach originally intended to tackle. Notice that the distinctiveness between reality and the worst scenario is characterized by the Euclidean distance between their associated probability parameters, or equivalently the distance between the vectors (p_2^{uu}, p_2^{du}) and (q_2^u, q_2^d), in the current finite-dimensional parametric binomial model setting. Indeed, the minimizer of the quantity $C(p_2^{uu}, p_2^{du})$ will be always attained at one of the boundaries of the ambiguity set that is furthest from the risk neutral probability. It hence implies that if reality turns out to be the most favorable scenario (i.e., furthest from the risk neutral probabilities), then the robust investor would feel most regretful for her non-participation in the stock market during the second period, an intuitive result that complies with most investors' investment psychology. It might be interesting to study in more general settings when the robust control approach can induce the most regret, under some appropriate distance metric (e.g., Wasserstein distance) on the infinite-dimensional space of measures.

When non-participation behavior occurs, the robust investor should typically experience more regret than the stubborn investor in the same backward paradigm, as a consequence of the excessive weight put on the worst scenario by the robust control approach. Another interesting observation from

Proposition 1.3.3 is that more regret would be induced for a non-participating robust investor who is less confident in the second period stock return profile (corresponding to larger ambiguity sets, as $\varepsilon^u, \ \varepsilon^d \to 0$). Hence, it might be recommended that the investors take some actions rather than completely withdraw from investing in the risky asset when there is too much ambiguity, if the investment goal is to have less regret in hindsight.

1.3.5 Forward investor (model adaptive/goal consistent)

The forward investor, unlike any of the backward investors, does *not* pre-commit at $t = 0$ to a perceived measure (or a family of perceived measures) for the entire horizon $[0, T_2]$. Rather, at $t = 0$, he starts with some admissible initial performance, and solves for the optimal policy $\widehat{\pi}_1^*$ based only on the probablity model over the first period under the genuine measure $\mathbb{P}$. At the begining of the second period, however, a consistent terminal criterion, denoted by $U_{T_2}^F(\cdot)$, together with the corresponding optimal polciy $\widehat{\pi}_2^*$ would be determined based on the genuine sub-model (1.8) or (1.9) under $\mathbb{P}$, which is fully knwon to the forward investor at $t = T_1$. For reasonable comparative analysis, we choose the initial performance to be the targeted optimal value function at $t = 0$ under the same perceived measure $\widehat{\mathbb{P}}$ of the oblivious and stubborn investors, i.e., $U_0^F(x) = V_0^{\text{Targeted}}(x)$, but we emphasize that in general, the forward approach can be applied to a much larger class of initial performances (see [3] for more details). Such flexibility for initial datum accounts for both subjective optimistic and pessimistic views (corresponding to different perceived measures $\widehat{\mathbb{P}}$) at initial time $t = 0$. The strength of the forward approach, as depicted in the sequel, is the ability to deliver the performance that is on the average consistent with the initial view, even if reality turns out to be rather different from the subjective belief in the beginning.

Time-consistency of performance process in dynamic setting has been studied in the context of model ambiguity. For example, in the work of [40], the nature is allowed to choose any probability model from a family but under certain cost. Their result shows that the (backward) dynamic variational performance process is time-consistent if and only if the cost functions satisfy the so called no-gain condition and Bayes Rule is applied for model update. In other words, a decision maker in their context is dynamically consistent if and only if she has a way to impose (hypothetical) costs on the nature's choice of probability measures such that (she thinks that) the nature is also dynamically consistent. Our formulation under the forward approach differs in several aspects. First of all, consistency of model choice of the nature is essential to the time-consistency of the decision making process in the classical backward framework, as demonstrated in the Dynamic Programming Principle, for instance. In the current two-period binomial model setting, model consistency amounts to claiming that if the nature has chosen the true measure $\mathbb{P}$ over $[0, T_2]$, then the model for the second period is the conditional probability $\mathbb{P}|_{\mathcal{F}_{T_1}}$. However, the decision maker/investor in our setting knows for sure that she would experience time-inconsistency at $t = T_1$, since her subjective belief $\widehat{\mathbb{P}}$ (or a family of subjective beliefs of her) would in general differ from the true conditional probability measure over the second period. Such inconsistency is unavoidable and is due to the discrepancy between the investor's view and the nature's true consistent behavior. The forward investor, on the other hand, does not assume any subjective or objective cost on the nature's choice, but adaptively revises her goal and beliefs forward in real-time, knowing that such model inconsistency is due to the other backward investors' initial limited knowledge. This also gives rise to the second difference between the forward approach and the existing backward methods; namely, the forward investor only needs to react in real-time to the single model that is actually chosen by the nature, whereas the existing robust control framework takes into account *a priori* a family of possible measures that can be selected by the nature and neglects the subsequent interactions between the decision maker and the nature. In this sense, we can view the classical backward approach as being *proactive* to model ambiguity while the forward approach has a clear *reactive* perspective in real-time. Lastly, Bayes formula typically serves as the fundamental update rule in the classical backward framework, which leads to the time-consistent decision making process by augmenting the state space with a belief state. Being one of the model-based learning rules, it however does not genuinely resolve the model commitment issue inherent in the backward framework (see the discussion in section 2.3). In contrast, the learning mechanism that is compatible with the forward approach can be rather general; in particular, unlike the Bayesian update, it is not necessary to prescribe at $t = 0$ how to learn the second period model under the forward framework. Such flexibility makes it possible for us to resort to other sophisticated model-free learning mechanisms for which analytic update rules may not be available (e.g. deep learning). Also, as pointed out in [19], the classical backward framework, with or without model-based learning, cannot successfully incorporate both time-consistent planning and surprising events, a fact in accordance with the backward model commitment issue bought up earlier. The forward approach, relieved from the pre-commitment to the future model or learning rule, can capture the unforeseen "surprises" into the revised decision making criterion in real-time and, still, generate consistent performance process.

The forward performance process theory is rather general. Here, we choose to work with a specific family known as the predictable forward performance processes introduced in [3]. Starting from $U_0^F(x) = V_0^{\text{Targeted}}(x)$, the forward investor seeks a criterion $U_{T_1}^F(\cdot) \in \mathcal{F}_0$ that is consistent with $U_0^F(x)$ in the sense

$$V_0^{\text{Targeted}}(x) = \sup_{\pi_1} \mathbb{E}\left[U_{T_1}^F\left(X_{T_1}^{\pi_1}\right) \Big| X_0 = x\right], \tag{1.33}$$

with $X_{T_1}^{\pi_1} = x + \pi_1(R_1 - 1)$. Notice that in the above formulation (1.33), only the knowledge of first period stock return distribution $\mathbb{P}(R_1 = R_1^u) = p_1$ is needed, and recall that it is known to all investors at $t = 0$, including the forward investor. This gives the fundamental difference in decision making between the forward investor and the other three investors who adopts the backward approach. To solve the forward problem (1.33), since the $t = 0$ targeted performance has the form $V_0^{\text{Targeted}}(x) = -e^{-\gamma x}A$, for some known constant $A > 0$, we look for $U_{T_1}^F(\cdot)$ in the similar form $U_{T_1}^F(x) = -e^{-\gamma x}B$, for some $B > 0$ to be determined. Equation (1.33) gives that

$$V_0^{\text{Targeted}}(x) = \max_{\pi_1} \left(p_1 U_{T_1}^F (x + \pi_1(R_1^u - 1)) + (1 - p_1) U_{T_1}^F (x + \pi_1(R_1^d - 1)) \right), \tag{1.34}$$

whose maximizer is unique and is given by

$$\widehat{\pi}_1^* = -\frac{1}{\gamma(R_1^u - R_1^d)} \ln \left(\frac{q_1}{1 - q_1} \frac{1 - p_1}{p_1} \right). \tag{1.35}$$

Upon substituting the maximizer (1.35) into the equation (1.33), we obtain

$$B = \frac{A \frac{p_1}{1-p_1} \frac{1-q_1}{q_1}}{p_1 e^{1-q_1} + (1 - p_1) e^{-q_1}} \in \mathcal{F}_0,$$

and hence the consistent criterion $U_{T_1}^F(\cdot)$. Then, at $t = T_1$, the genuine sub-model (1.8) or (1.9) for the second period stock return is fully available (conditional on the first period stock return), and the goal is to find a $U_{T_2}^F(\cdot) \in \mathcal{F}_{T_1}$ such that

$$U_{T_1}^F(x) = \operatorname*{esssup}_{\pi_2} \mathbb{E} \left[U_{T_2}^F(x + \pi_2(R_2 - 1)) \middle| X_{T_1} = x,\ \mathcal{F}_{T_1} \right], \text{ a.s..} \tag{1.36}$$

Assuming such terminal criterion has the form $U_{T_2}^F(x) = -e^{-\gamma x} C$, for some random variable $C \in \mathcal{F}_{T_1}$ that is almost surely positive under the true measure $\mathbb{P}$, we have

$$\begin{aligned} -e^{-\gamma x} B = \max_{\pi_2} \Bigg(& \left(-e^{-\gamma(x + \pi_2(R_2^{uu} - 1))} p_2^{uu} - e^{-\gamma(x + \pi_2(R_2^{ud} - 1))} (1 - p_2^{uu}) \right) C \mathbb{1}_{\{R_1 = R_1^u\}} \\ & + \left(-e^{-\gamma(x + \pi_2(R_2^{du} - 1))} p_2^{du} - e^{-\gamma(x + \pi_2(R_2^{dd} - 1))} (1 - p_2^{du}) \right) C \mathbb{1}_{\{R_1 = R_1^d\}} \Bigg). \end{aligned} \tag{1.37}$$

As before, we determine the unique optimal policy for the second period as

$$\widehat{\pi}_2^* = -\frac{1}{\gamma(R_2^u - R_2^d)} \ln \left(\frac{q_2}{1 - q_2} \frac{1 - p_2^u}{p_2^u} \right), \tag{1.38}$$

and the random quantity $C \in \mathcal{F}_{T_1}$ is determined after we substitute the optimal policy $\widehat{\pi}_2^*$ back to (1.36),

$$C = \frac{B^{\frac{p_2^u}{1-p_2^u}\frac{1-q_2}{q_2}}}{e^{1-q_2}p_2^u + e^{-q_2}(1-p_2^u)} \in \mathcal{F}_{T_1}.$$

We recall that here p_2^u and q_2 are the genuine physical probability and the risk neutral probability for the second period stock return defined as before by

$$p_2^u = p_2^{uu}\mathbb{1}_{\{R_1=R_1^u\}} + p_2^{du}\mathbb{1}_{\{R_1=R_1^d\}},$$

$$q_2 = q_2^u\mathbb{1}_{\{R_1=R_1^u\}} + q_2^d\mathbb{1}_{\{R_1=R_1^d\}},$$

respectively.

From above real-time construction of the forward criteria $U_0^F(\cdot)$, $U_{T_1}^F(\cdot)$, $U_{T_2}^F(\cdot)$, we can see how the gradually acquired knowledge of the true underlying model $\mathbb{P}$ enters into the forward optimization problem. Indeed, as shown next, the appropriate real-time modification of the optimization criteria results in both the two metrics $M_{0,T_2}(x)$ and $m_{0,T_2}(x)$ being identically zero, regardless of the true measure $\mathbb{P}$.

Proposition 1.3.4. *For any probability parameters p_2^{uu}, p_2^{du} under $\mathbb{P}$ (see (1.8), (1.9)), the regret of the forward investor is identically zero, i.e.,*

$$M_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{True}(x) = 0.$$

Moreover, the discrepancy

$$m_{0,T_2}(x) = V_0^{Actual}(x) - V_0^{Targeted}(x) = 0.$$

Proof. We start by computing the $t = 0$ actual average performance under the true measure $\mathbb{P}$ for the forward investor. We have

$$V_0^{\text{Actual}}(x) = \mathbb{E}\left[U_{T_2}^F(X_{T_2}^{\widehat{\pi}_1^*,\widehat{\pi}_2^*})\right] = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\left(\widehat{\pi}_1^*(R_1-1)+\widehat{\pi}_2^*(R_2-1)\right)}C\right]$$
$$= -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}C\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]\right].$$

Recall that the random variable $C \in \mathcal{F}_{T_1}$ is determined such that the equation (1.37) is satisfied by the unique optimizer (1.38). We hence have $B = C\mathbb{E}\left[e^{-\gamma\tilde{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right]$ from (1.37). Using that $B \in \mathcal{F}_0$, we get

$$V_0^{\text{Actual}}(x) = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}\right]B = -e^{-\gamma x}A = V_0^{\text{Targeted}}(x),$$

where the second equality follows because $\widehat{\pi}_1^*$ is the unique optimizer to the equation (1.34). This concludes that $m_{0,T_2}(x) = 0$, for any p_2^{uu}, p_2^{du} under $\mathbb{P}$. We now consider the $t = 0$ optimal performance under the true model, in hindsight, for the forward investor with terminal utility $U_{T_2}^F(x)$. Notice that we are essentially solving the classical backward problem under $U_{T_2}^F(x)$ with full knowledge of the true measure $\mathbb{P}$. It follows that

$$V_{T_1}(x) = \operatorname*{esssup}_{\pi_2}\mathbb{E}\left[U_{T_2}^F(x+\pi_2(R_2-1))\middle|X_{T_1}=x,\ \mathcal{F}_{T_1}\right]$$
$$= -e^{-\gamma x}C\operatorname*{essinf}_{\pi_2}\mathbb{E}\left[e^{-\gamma\pi_2(R_2-1)}\middle|\mathcal{F}_{T_1}\right] = -e^{-\gamma x}B,$$

where we used that $C \in \mathcal{F}_{T_1}$ satisfies (cf. (1.37))

$$C\operatorname*{essinf}_{\pi_2}\mathbb{E}\left[e^{-\gamma\pi_2(R_2-1)}\middle|\mathcal{F}_{T_1}\right] = C\,\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\middle|\mathcal{F}_{T_1}\right] = B.$$

By backward induction,

$$V_0^{\text{True}}(x) = \sup_{\pi_1}\mathbb{E}\left[V_{T_1}(x+\pi_1(R_1-1))\middle|X_0=x\right]$$

$$= -e^{-\gamma x} B \inf_{\pi_1} \mathbb{E}\left[e^{-\gamma \pi_1 (R_1 - 1)}\right] = -e^{-\gamma x} A,$$

where we used that $B \in \mathcal{F}_0$ satisfies (cf. (1.34))

$$B \inf_{\pi_1} \mathbb{E}\left[e^{-\gamma \pi_1 (R_1 - 1)}\right] = B\, \mathbb{E}\left[e^{-\gamma \widehat{\pi}_1^* (R_1 - 1)}\right] = A.$$

We hence obtain that $M_{0,T_2}(x) = 0$, for any $p_2^{uu},\ p_2^{du}$ under $\mathbb{P}$. □

1.4 Comparison of regret for various types of investors

In previous sections, we examined four types of possible investment behavior under model uncertainty, from the perspective of the metric $M_{0,T_2}(x)$ that characterizes the regret in hindsight, and the metric $m_{0,T_2}(x)$ that characterizes the discrepancy from the targeted performance. It was shown that all three investors adopting the backward approach, the oblivious, stubborn and robust investors, endure negative regret in general, while the forward investor always achieves zero regret regardless of the true underlying measure $\mathbb{P}$. It is thus interesting to quantify and compare the negative regret for the three backward investors. Intuitively, each of the three investment types has its relative strength facing different realities and, hence, the regret of one type of investor can dominate or stay underneath the regret of the others. For instance, we expect the stubborn investor to experience less regret in most scenarios of reality, since he has taken into account the new model when it is available at $t = T_1$. On the other hand, the robust investor should be able to benefit from the conservative non-participation strategy if reality coincides with the worst scenario. In this section, we provide detailed analysis on the magnitude of regret for the three types of investors belonging to the classical backward paradigm.

We first recall that the value function $V_0^{\text{True}}(x)$ for the three backward investors in the regret metric $M_{0,T_2}(x)$ does not change. It is the true optimal performance in hindsight, computed under the terminal utility $U(x) = -e^{-\gamma x}$ and the true measure $\mathbb{P}$,

$$V_0^{\text{True}}(x) = \sup_{\pi_1,\pi_2} \mathbb{E}\left[U(X_{T_2}^{\pi_1,\pi_2})\Big| X_0 = x\right],$$

for any initial wealth $x \in \mathbb{R}$. From (1.30), we recall that

$$V_0^{\text{True}}(x) = -e^{-\gamma x}\left(\frac{q_1}{1-q_1}\frac{1-p_1}{p_1}\right)^{-q_1}\frac{1-p_1}{1-q_1}\left(\frac{A^d}{A^u}\right)^{-q_1}A^d,$$

with A^u, A^d given in (1.27) and (1.28), respectively. The model parameters p_1, q_1, q_2^u and q_2^d in A^u, A^d are fixed and known to all backward investors at $t = 0$, while p_2^{uu}, p_2^{du} are generic probability parameters for the second period stock return under the true measure $\mathbb{P}$. The value function $V_0^{\text{True}}(x)$ hence would solely depend on p_2^{uu} and p_2^{du} that correspond to different realities. Following from (1.32), we obtain that for fixed initial wealth $x \in \mathbb{R}$,

$$V_0^{\text{True}}(x) = O\left(\left(1-p_2^{uu}\right)^{q_1(1-q_2^u)}\left(p_2^{uu}\right)^{q_1 q_2^u}\left(1-p_2^{du}\right)^{(1-q_1)(1-q_2^d)}\left(p_2^{du}\right)^{(1-q_1)q_2^d}\right). \tag{1.39}$$

We next quantify the term $V_0^{\text{Actual}}(x)$ in the regret metric for the oblivious, stubborn and robust investors. According to Proposition 1.3.3, it is easy to see that for the robust investor who favors non-participation in the second period (corresponding to the case with large ambiguity sets), the regret is given by

$$M_{0,T_2}(x) = V_0^{\text{Actual}}(x) - V_0^{\text{True}}(x)$$

$$= -e^{-\gamma x}\left(\frac{q_1}{1-q_1}\frac{1-p_1}{p_1}\right)^{-q_1}\frac{1-p_1}{1-q_1} - V_0^{\text{True}}(x) = O(1),$$

where $O(1)$ denotes a constant, as p_2^{uu}, p_2^{du} goes to 0 or 1. For the oblivious investor, we have

$$V_0^{\text{Actual}}(x) = \mathbb{E}\left[U(X_{T_2}^{\widehat{\pi}_1^*,\widehat{\pi}_2^*})\Big|X_0 = x\right],$$

where the expectation is taken under the true measure $\mathbb{P}$. Applying now the (perceived) optimal policies $\widehat{\pi}_1^*$, $\widehat{\pi}_2^*$, given by (1.15) and (1.13), under the true measure $\mathbb{P}$ yields

$$V_0^{\text{Actual}}(x) = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right]\right]$$

$$= -e^{-\gamma x}\left(\frac{p_1}{q_1}\right)^{q_1}\frac{(1-p_1)^{1-q_1}}{(1-q_1)^{-q_1}}\left(\frac{\hat{p}_2^{uu}}{q_2^u}\right)^{q_1 q_2^u}\left(\frac{\hat{p}_2^{du}}{q_2^d}\right)^{(1-q_1)q_2^d}\left(\frac{1-\hat{p}_2^{du}}{1-q_2^d}\right)^{q_1 q_2^d - q_1 - q_2^d}$$

$$\times\left(\frac{1-q_2^u}{1-\hat{p}_2^{uu}}\right)^{q_1 q_2^u - q_1}\left(\left(\frac{q_1}{1-q_1}\frac{1-\hat{p}_2^{du}}{1-q_2^d}\frac{q_2^u-\hat{p}_2^{uu}}{\hat{p}_2^{uu}(1-\hat{p}_2^{uu})}\right)p_2^{uu}\right.$$

$$\left.+\left(\frac{q_2^d-\hat{p}_2^{du}}{(1-q_2^d)\hat{p}_2^{du}}\right)p_2^{du} + \left(\frac{q_1}{1-q_1}\frac{1-\hat{p}_2^{du}}{1-q_2^d}\frac{1-q_2^u}{1-\hat{p}_2^{uu}}+1\right)\right).$$

Noticing that the last term in the above expression is a strictly positive quantity independent of p_2^{uu}, p_2^{du}, we obtain $V_0^{\text{Actual}}(x) = O(1)$, which yields that the regret of the oblivious investor is also of order $O(1)$ as the robust investor.

We now turn to the stubborn behavior whose actual average performance is given by

$$V_0^{\text{Actual}}(x) = \mathbb{E}\left[U(X_{T_2}^{\widehat{\pi}_1^*,\pi_2^*})\Big|X_0 = x\right],$$

where, again, the expectation is taken under the true measure $\mathbb{P}$ and $\widehat{\pi}_1^*$ is obtained under the perceived measure $\widehat{\mathbb{P}}$, while π_2^* is the optimal control (1.16) revised under the true sub-models (1.8), (1.9) at $t = T_1$. Further computation then gives

$$V_0^{\text{Actual}}(x) = -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma\widehat{\pi}_1^*(R_1-1)}\mathbb{E}\left[e^{-\gamma\pi_2^*(R_2-1)}\Big|\mathcal{F}_{T_1}\right]\right]$$

$$= -e^{-\gamma x}(1-p_1)\left(\frac{p_1}{q_1}\right)^{q_1}\frac{(q_2^d)^{(q_1-1)q_2^d}}{(\hat{p}_2^{du})^{q_1 q_2^d}}\left(\frac{\hat{p}_2^{uu}}{q_2^u}\right)^{q_1 q_2^u}\frac{(1-q_2^d)^{(q_1-1)(1-q_2^d)}}{(1-\hat{p}^{du})^{q_1(1-q_2^d)}}$$

$$\times\left(\frac{1-\hat{p}_2^{uu}}{1-q_2^u}\right)^{q_1(1-q_2^u)}\left(\frac{q_1}{1-q_1}\frac{(p_2^{uu})^{q_2^u}(1-p_2^{uu})^{1-q_2^u}}{(\hat{p}_2^{uu})^{q_2^u}(1-\hat{p}_2^{uu})^{1-q_2^u}}+(p_2^{du})^{q_2^d}(1-p_2^{du})^{1-q_2^d}\right).$$

Denote $Z_1 := (p_2^{uu})^{q_2^u}(1-p_2^{uu})^{1-q_2^u}$ and $Z_2 := \left(p_2^{du}\right)^{q_2^d}\left(1-p_2^{du}\right)^{1-q_2^d}$. Then, for any fixed initial wealth $x \in \mathbb{R}$, the regret for stubborn investor is of order

$$V_0^{\text{Actual}}(x) - V_0^{\text{True}}(x) = O\left((p_2^{uu})^{q_2^u}(1-p_2^{uu})^{1-q_2^u}+\left(p_2^{du}\right)^{q_2^d}\left(1-p_2^{du}\right)^{1-q_2^d}\right)$$

$$-O\left(\left(1-p_2^{uu}\right)^{q_1(1-q_2^u)}\left(p_2^{uu}\right)^{q_1 q_2^u}\left(1-p_2^{du}\right)^{(1-q_1)(1-q_2^d)}\left(p_2^{du}\right)^{(1-q_1)q_2^d}\right)$$

$$= O\left(Z_1+Z_2\right) - O\left(Z_1^{q_1}Z_2^{1-q_1}\right) = O\left(Z_1+Z_2\right),$$

where we have used the fact that the (lower) limit of the quantity

$$\frac{Z_1+Z_2}{Z_1^{q_1}Z_2^{1-q_1}} = \left(\frac{Z_1}{Z_2}\right)^{1-q_1}+\left(\frac{Z_2}{Z_1}\right)^{q_1}$$

approaches infinity if $Z_1 \neq O(Z_2)$, and approaches a positive constant if $Z_1 = O(Z_2)$, as Z_1, $Z_2 \to 0$, both leading to $O\left(Z_1+Z_2\right) - O\left(Z_1^{q_1}Z_2^{1-q_1}\right) = O\left(Z_1+Z_2\right)$. Table 1.1 summarizes the regret comparison results for different investment behavior types.

Table 1.1: Regret Comparison of Different Investment Behavior

Investment Behavior	Sign/Magnitude of Regret	
Oblivious	Nonpositive	$O(1) - O\left(Z_1^{q_1} Z_2^{1-q_1}\right) = O(1)$
Stubborn	Nonpositive	$O\left(Z_1 + Z_2\right) - O\left(Z_1^{q_1} Z_2^{1-q_1}\right) = O\left(Z_1 + Z_2\right)$
Robust	Nonpositive	$O(1) - O\left(Z_1^{q_1} Z_2^{1-q_1}\right) = O(1)$
Forward	Zero	Zero

$Z_1 := \left(p_2^{uu}\right)^{q_2^u} \left(1 - p_2^{uu}\right)^{1-q_2^u}$ and $Z_2 := \left(p_2^{du}\right)^{q_2^d} \left(1 - p_2^{du}\right)^{1-q_2^d}$

We now can see that the stubborn behavior generally induces less regret compared to the oblivious and robust behavior, especially when the reality of the future turns out to be extreme. This is when Z_1 and Z_2 are close to zero, corresponding to extremely good future market conditions (i.e., p_2^{uu}, p_2^{du} close to 1), or extremely adverse conditions (i.e., p_2^{uu}, p_2^{du} close to 0), or the combination of the two (e.g., p_2^{uu} close to 1, but p_2^{du} close to 0). The stubborn investor would unsurprisingly benefit from these extreme scenarios, thanks to prompt reaction to the new model knowledge at $t = T_1$. Hence, she experiences less regret in retrospect when facing extreme cases. On the other hand, depending on the interlinked connections between p_2^{uu}, p_2^{du} and the rest of model parameters, the stubborn behavior may cause less or more regret than the oblivious and robust behavior in moderate scenarios. In particular, if the genuine probabilities p_2^{uu}, p_2^{du} for the second period stock return are close to the risk neutral probabilities, then the worst scenario indeed happens, and less regret would be induced if the investor follows the robust behavior (see Figure

1.1). This is another interesting observation, which suggests that, in moderate scenarios, incorporating the true knowledge “half way through” the decision making process becomes less important than being initially conservative and cautious when facing model ambiguity.

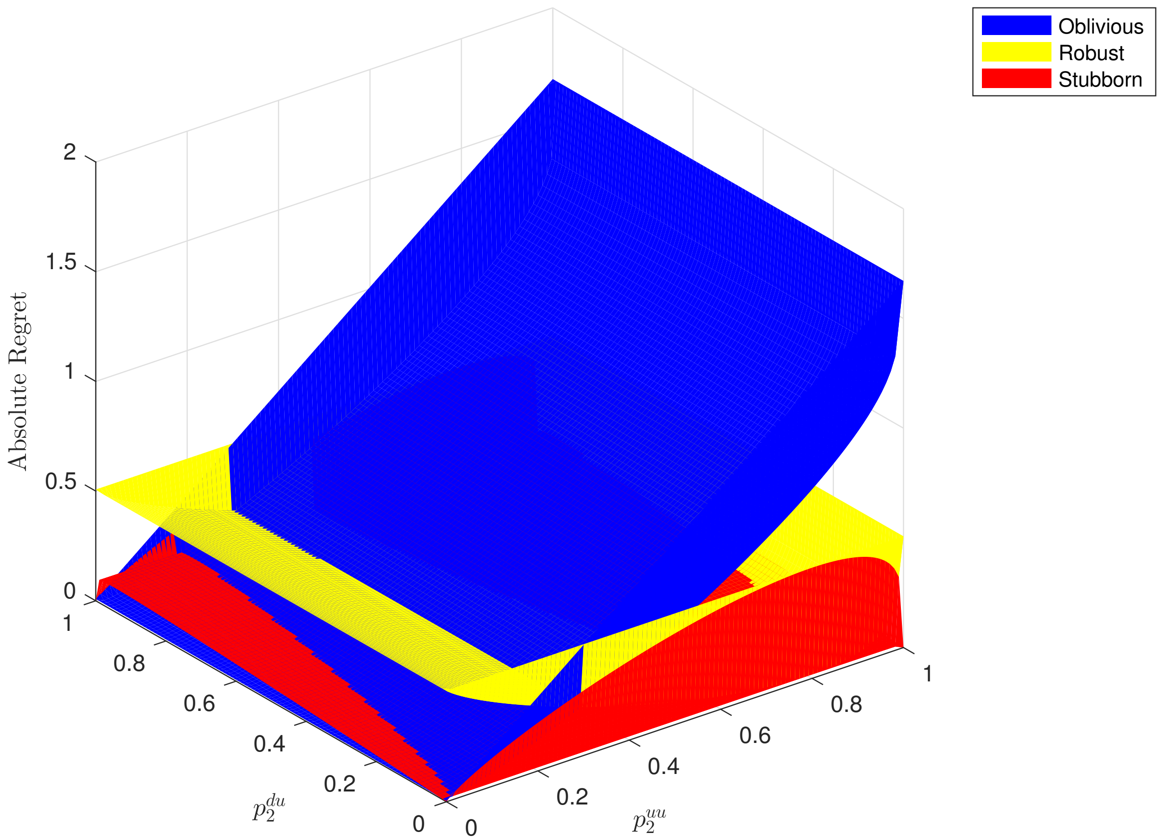

Figure 1.1: Regret comparison of various types of investment behavior (initial wealth $x = 0$). Model parameters: $p_1 = \frac{1}{2}$, $q_1 = \frac{1}{500}$, $q_2^u = \frac{4}{5}$, $q_2^d = \frac{4}{5}$, $\hat{p}_2^{uu} = \frac{1}{100}$, and $\hat{p}_2^{du} = \frac{1}{100}$. The absolute regret of oblivious and robust investors generally dominate that of the stubborn investor. Nevertheless, the stubborn behavior can induce more regret than the robust behavior, especially when the true probabilities for the second period stock return p_2^{uu}, p_2^{du} are close to the risk neutral probabilities q_2^u, q_2^d.

1.5 A unified discussion

Under the setting of model ambiguity and real-time learning, we discussed various types of investors in previous sections. The stubborn investor and the forward investor take the advantage of the progressively revealed knowledge of the underlying model in real-time, and produce their respective optimal strategies accordingly over each period. The other two types of investors, the oblivious and the robust investor, neglect the further acquisition of new (unexpected) knowledge about the true model after $t = 0$. In this sense, we may not view the associated strategies as solutions to the real-time optimal investment problem, although they can still be considered as alternative solutions when an investor faces model ambiguity.

In this section, we will provide a unified analysis of the real-time optimal investment problems under general criteria $U_{T_1}(\cdot) \in \mathcal{F}_{T_1}$ and $U_{T_2}(\cdot) \in \mathcal{F}_{T_2}$. The measurability condition for the criteria imposed here is flexible enough to incorporate most investment behavior including the stubborn and forward behavior. Indeed, $U_{T_1}(\cdot) = V_{T_1}(\,\cdot\,;\widehat{\mathbb{P}})$ and $U_{T_2}(\cdot) = U(\cdot)$ in the stubborn case, where $U(\cdot)$ is a classical (deterministic) terminal utility function specified at $t = 0$, and $V_{T_1}(\,\cdot\,;\widehat{\mathbb{P}})$ is the associated value function under $U(\cdot)$ and the ($t = 0$) perceived measure $\widehat{\mathbb{P}}$. In the forward behavior setting, $U_{T_1}(\cdot) = U^F_{T_1}(\cdot)$ and $U_{T_2}(\cdot) = U^F_{T_2}(\cdot)$, with $U^F_{T_1}(\cdot)$ and $U^F_{T_2}(\cdot)$ being the predictable forward performance at $t = T_1$ and $t = T_2$, respectively. Herein, the two forward criteria are restricted to be predictable, but in general, forward performance process (in discrete time or continuous time) needs not to be necessarily predictable *a* *priori*. Our unified framework includes the general forward processes as well as other (state-dependent or independent) performance criteria derived based on possible approaches under model uncertainty.

We next introduce the decision making process under the generic criteria $U_{T_1}(\cdot) \in \mathcal{F}_{T_1}$ and $U_{T_2}(\cdot) \in \mathcal{F}_{T_2}$ set for the first period and the second in the current binomial model setting. To emphasize the real-time feature of the decision making process, we assume as before that the genuine underlying measure $\mathbb{P}$ is only known to the investor/decision maker progressively. Precisely, only one-period ahead probability under $\mathbb{P}$ would be revealed at each decision making time, conditional on the information up to that time (see (1.8), (1.9)). Under this assumption, the first period policy selected is any admissible π_1^* that optimizes (assuming it exists)

$$\sup_{\pi_1} \mathbb{E}\left[U_{T_1}\left(X_{T_1}^{\pi_1}\right)\Big|X_0 = x\right] = \sup_{\pi_1} \mathbb{E}\left[U_{T_1}\left(x + \pi_1(R_1 - 1)\right)\Big|X_0 = x\right], \quad (1.40)$$

where the probability over the first period under $\mathbb{P}$ is sufficient and used for the computation of the above expectations. The second period decision making process yields, similarly, the optimal admissible policy π_2^* that maximizes (assuming it exists)

$$\begin{aligned} &\operatorname*{esssup}_{\pi_2} \mathbb{E}\left[U_{T_2}\left(X_{T_2}^{\pi_1^*,\pi_2}\right)\Big|X_{T_1}^{\pi_1^*},\ \mathcal{F}_{T_1}\right] \\ &= \operatorname*{esssup}_{\pi_2} \mathbb{E}\left[U_{T_2}\left(X_{T_1}^{\pi_1^*} + \pi_2(R_2 - 1)\right)\Big|X_{T_1}^{\pi_1^*},\ \mathcal{F}_{T_1}\right]. \end{aligned} \quad (1.41)$$

Here, $X_{T_1}^{\pi_1^*} \in \mathcal{F}_{T_1}$ is the wealth at $t = T_1$ obtained by following the optimal control π_1^* in (1.40), and the one-period ahead conditional probability under the true measure $\mathbb{P}$, given $\mathcal{F}_{T_1}$, is applied for the computation of the above conditional expectations.

The criteria $U_{T_1}(\cdot) \in \mathcal{F}_{T_1}$ and $U_{T_2}(\cdot) \in \mathcal{F}_{T_2}$ that produce the optimal policies π_1^* and π_2^*, respectively, in general have no intertemporal connections *a priori*, as the investor/decision maker can have arbitrary (short-term) objectives based on her own preference and personal view of the future market at each time. However, as we will show in the next theorem, in order to achieve zero regret, the criterion $U_{T_1}(\cdot)$ has to satisfy certain consistency condition with its successor $U_{T_2}(\cdot)$, a condition we refer to as *forward consistency.* Before stating the precise result, we first introduce the following stochastic optimization problem in hindsight at $t = T_2$. In retrospect at $t = T_2$, the underlying measure $\mathbb{P}$ is assumed to be fully known to the investor, and the problem is to solve

$$\sup_{\pi_1,\pi_2} \mathbb{E}\left[U_{T_2}(X_{T_2}^{\pi_1,\pi_2})\right] = \sup_{\pi_1,\pi_2} \mathbb{E}\left[U_{T_2}\left(x + \pi_1(R_1 - 1) + \pi_2(R_2 - 1)\right)\right], \quad (1.42)$$

where the expectation is computed under the true measure $\mathbb{P}$ on $[0, T_2]$. We note that in general $U_{T_2}(\cdot) \in \mathcal{F}_{T_2}$. Assuming that problem (1.42) can be solved via backward induction, we can then define the value function (in hindsight) at $t = T_1$ as

$$V_{T_1}(x) = \operatorname*{esssup}_{\pi_2} \mathbb{E}\left[U_{T_2}\left(X_{T_1} + \pi_2(R_2 - 1)\right) \Big| X_{T_1} = x,\ \mathcal{F}_{T_1}\right], \quad \text{a.s.}, \quad (1.43)$$

and, in general, $V_{T_1}(x) \in \mathcal{F}_{T_1}$, for all admissible x. The value function at $t = 0$ (in hindsight) is

$$V_0(x) = \sup_{\pi_1} \mathbb{E}\left[V_{T_1}\left(x + \pi_1(R_1 - 1)\right) \Big| X_0 = x\right], \quad (1.44)$$

and backward induction gives that $V_0(x)$ is the optimal value of the problem (1.42).

Theorem 1.5.1 (Forward consistency)**.** *The criteria pair* (U_{T_1}, U_{T_2}) *together with the corresponding optimal controls* π_1^*, π_2^* *obtained in real-time by (1.40) and (1.41), respectively, generate zero (total) regret if and only if* π_1^* *is also the maximizer of the first period problem in hindsight, i.e.,*

$$\pi_1^*(x) \in \underset{\pi_1}{argmax} \mathbb{E}\left[V_{T_1}\left(x + \pi_1(R_1 - 1)\right)\Big| X_0 = x\right], \tag{1.45}$$

for any admissible initial wealth x*. In particular, the (not necessarily predictable) forward performance criteria pair* $\left(U_{T_1}^F, U_{T_2}^F\right)$ *yields zero regret.*

Proof. ($\Longrightarrow$) Assume that the pair (U_{T_1}, U_{T_2}) and the optimal controls π_1^*, π_2^* generate zero regret; that is, by definition of the regret, the $t = 0$ genuine value function in hindsight coincides with the actual average performance,

$$V_0(x) = \mathbb{E}\left[U_{T_2}\left(x + \pi_1^*(R_1 - 1) + \pi_2^*(R_2 - 1)\right)\right].$$

We note that π_1^* is not necessarily the optimizer to the hindsight problem (1.44) *a priori*. For example, within the forward behavior setting we considered in section 2.5, π_1^* is solely determined by the first period model under $\mathbb{P}$, together with the admissible initial criterion $U_0^F(\cdot)$. However, in general, the optimizer of the hindsight problem (1.44) depends not only on the first period model, but also on $V_{T_1}(\cdot)$ which in turn depends on the second period conditional model. It follows from the assumption of zero regret that

$$V_0(x) = \mathbb{E}\left[U_{T_2}\left(x + \pi_1^*(R_1 - 1) + \pi_2^*(R_2 - 1)\right)\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[U_{T_2}\left(X_{T_1}^{\pi_1^*} + \pi_2^*(R_2 - 1)\right)\Big| X_{T_1}^{\pi_1^*},\ \mathcal{F}_{T_1}\right]\right]$$

$$= \mathbb{E}\left[V_{T_1}\left(X_{T_1}^{\pi_1^*}\right)\right] = \mathbb{E}\left[V_{T_1}\left(x + \pi_1^*(R_1 - 1)\right)\right] \leq V_0(x),$$

where we used the fact that π_2^* is the optimizer for both the real-time problem (1.41) and the hindsight problem (1.43)[1], while π_1^* from the real-time problem (1.40) is in general only one admissible policy for the first period hindsight problem. We hence conclude that π_1^* is the maximizer to the hindsight problem

$$\sup_{\pi_1} \mathbb{E}\left[V_{T_1}\left(x + \pi_1(R_1 - 1)\right)\Big| X_0 = x\right].$$

($\Longleftarrow$) Suppose the criteria pair (U_{T_1}, U_{T_2}) is determined such that the first period optimal control π_1^* which solves the real-time problem (1.40) also solves the hindsight problem (1.45). It is then straightforward to see that the actual average performance by following such π_1^* from (1.40) and π_2^* from (1.41) satisfies

$$\mathbb{E}\left[U_{T_2}\left(x + \pi_1^*(R_1 - 1) + \pi_2^*(R_2 - 1)\right)\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[U_{T_2}\left(X_{T_1}^{\pi_1^*} + \pi_2^*(R_2 - 1)\right)\Big| X_{T_1}^{\pi_1^*},\ \mathcal{F}_{T_1}\right]\right]$$

$$= \mathbb{E}\left[V_{T_1}\left(X_{T_1}^{\pi_1^*}\right)\right] = V_0(x),$$

where, again, we used that π_2^* is the optimizer for both the real-time problem (1.41) and the hindsight problem (1.43), as well as the assumption that π_1^* also

[1]When solving the last period problem in real-time or in hindsight, the available knowledge of the probability model for the decision maker is identical and correct. In other words, the regret of the *last* period is always zero by problem formulation.

maximizes the hindsight problem (1.45). This completes the proof of showing zero regret under the forward consistency condition (1.45).

It remains to show that the general (not necessarily predictable) forward performance criteria pair $\left(U^F_{T_1}, U^F_{T_2}\right)$ achieves the forward consistency, and hence generates zero regret. Indeed, by the definition of forward performance processes, $U^F_{T_1}$ and $U^F_{T_2}$ are directly connected through the equation

$$U^F_{T_1}\left(X_{T_1}\right) = \operatorname*{esssup}_{\pi_2} \mathbb{E}\left[U^F_{T_2}\left(X_{T_1} + \pi_2(R_2 - 1)\right) \middle| \mathcal{F}_{T_1}\right], \text{ a.s.}. \tag{1.46}$$

It therefore yields, by the uniqueness of the (essential) supremum, $U^F_{T_1}\left(X_{T_1}\right) = V_{T_1}\left(X_{T_1}\right)$ a.s., with $X_{T_1} = x + \pi_1(R_1 - 1)$ for any admissible control π_1 over the first period. A direct consequence is that any π_1^* that solves the real-time problem (1.40) under $U^F_{T_1}$ also solves the hindsight problem (1.45), i.e., the forward consistency condition (1.45) holds. We hence conclude that the generic forward criteria pair $\left(U^F_{T_1}, U^F_{T}\right)$ generates zero total regret. □

Theorem 1.5.1 basically states that the decision maker can achieve zero regret if and only if the past decision made at $t = 0$ for the first period remains valid when viewed in hindsight under the full model knowledge, an intuitive result that is almost self-explanatory. Obviously, to have such forward consistency, certain connection between the intermediate criterion U_{T_1} and its successor U_{T_2} has to be established. An interesting fact, however, is that such connection does not to need to be as strong as the classical definition for the froward performance process as given in (1.46). The reason is, although forward consistency requires that π_1^* optimizes both the real-time problem

(1.40) and the hindsight problem (1.44), it does not necessarily require the same optimal value for the two problems, in order to have zero regret. In other words, it may be true that for some admissible x,

$$\sup_{\pi_1} \mathbb{E}\left[U_{T_1}\left(x+\pi_1(R_1-1)\right)\Big|X_0=x\right] \neq \sup_{\pi_1} \mathbb{E}\left[V_{T_1}\left(x+\pi_1(R_1-1)\right)\Big|X_0=x\right],$$

but zero regret still holds for all admissible x. On the other hand, since the definition of the forward performance process (1.46) is certainly stronger than the forward consistency (1.45), the forward criteria pair $\left(U^F_{T_1}, U^F_{T_2}\right)$ generates zero regret, regardless of the underlying measure $\mathbb{P}$, with the special case of predictable forward performance process already discussed separately in Proposition 1.3.4.

It is easy to see that the unified framework in Theorem 1.5.1 can account for both the stubborn behavior and the forward behavior described in previous sections. It nails down the fundamental reason, i.e., the violation of forward consistency, that causes the classical (backward) adaptive control approach to generally yield non-zero regret (see Proposition 1.3.2 for more details). It also addresses why the other decision making behavior, the forward behavior, can eliminate any regret induced by model knowledge that is revealed in real-time. This theorem can also incorporate classical stochastic optimization problems when model knowledge is fully available at $t=0$, i.e., the scenario when decision is made under the *known unknowns* instead of the *unknown unknowns*. There, the regret is clearly always zero, and the forward consistency is certainly satisfied as soon as one recognizes that the criteria pair

$(U_{T_1}(\cdot), U_{T_2}(\cdot))$ coincides with $(V_{T_1}(\cdot\,;\mathbb{P}), U(\cdot))$, where $V_{T_1}(\cdot\,;\mathbb{P})$ is the classical value function under the known true measure $\mathbb{P}$ and the terminal utility $U(\cdot)$.

Chapter 2

Real-time model adaptation and investment behavior: the Merton case

2.1 Introduction

As disucssed in the previous chapter, model commitment is ubiquitous in the classical stochastic optimization framework, and therefore, excludes the flexibility of allowing dynamic model revision, a concept by nature incompatible with any preassigned commitment at $t = 0$. However, from practical point of view, model revision is inevitable as one may obtain updated information that leads to more accurate estimates of the underlying model. Such new knowledge accumulated in real-time should be exploited to solve the control problem at hand. In other scenarios, the environment itself may be non-stationary and has changed after some time, making it necessary for the decision maker to closely track the environment in order to make better decisions. No consensus, however, has been reached regarding the best approach to handle the dynamic (unanticipated) model changes in control theory and practice. Adaptive control methodology is probably the main tool in theory and practice to address this problem (see, e.g., [38], [12]). It basically at each time re-solves the control problem for remaining horizon, given the updated estimates of the parameters in the model dynamics. It is clearly time-inconsistent and could induce fluctuations in the performance of the decision maker as time enfolds. The same idea is employed for more practical applications of control theory; for instance, in reinforcement learning (RL) field, the learning agent would typically re-adapt to the changed environment through continuing interaction, while fixing the original $t = 0$ optimization objective. This behavior in turn leads to volatile performance, as a steep decrease of learning performance usually occurs in such non-stationary learning contxt (see, e.g., [16]).

In this work, we aim to compare two control approaches arising under the circumstance of real-time model revision in a "Merton type" investment setting. The first one is rooted in the adaptive control paradigm, which we call the "stubborn" method in view of the fixed terminal criterion regardless of any changes to the market environment. The advantage of this method is clearly the preservation of the original goal, which motivates the name "stubborn", as well as the adaptation to progressively realized market conditions in real-time. However, it violates the time-consistency that is not only fundamental to the sound definition of classical optimality, but also crucial to have a non-volatile overall performance. The second approach we consider is the forward performance approach. This methodology, by relaxing the stringent commitment to a fixed terminal criterion (and/or a fixed terminal horizon), introduces greater flexibility to control problems under real-time model revision. Intertemporal consistency is guaranteed directly due to the construction of the forward performance process. Moreover, when evaluated in retrospect, the performance of the forward investor would maintain unchanged on average even after the unanticipated change of the market environment, yielding less variability for the actual performance. Whether sticking to a fixed terminal criterion leads to different degree of model choice flexibility (specifically for the long-term), and distinguishes the two fundamentally different optimization approaches. Nevertheless, the stubborn and forward methods are comparable in some aspects, and it is one of the goals of this work to conduct a comprehensive comparison analysis. As the forward performance process theory is rather general, we choose to work with a specific family, namely the zero volatility forward performance process. In addition to the comparisons, we also seek to reconcile the two approaches in the last section. Precisely, we would construct a forward "bridge" process to preserve both the original $t = 0$ objective and intertemporal consistency in a real-time model revision setting. As we shall see, this in general could only be possible if we go beyond the zero volatility forward processes, and introduce a none-zero volatility to the performance process.

2.2 Classical approach

In this section we consider the classical adaptive control approach, which is "stubborn" to a fixed $t = 0$ optimization objective under real-time model revision. For simplicity, we focus on the Merton's optimal investment problem in a single stock market over a fixed horizon $[0, T]$, assuming that interest rate is zero. The stock price under the genuine but unknown physical measure $\mathbb{P}$ is modeled by

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t),$$

with $S_0 > 0$. The process W_t, $0 \leq t \leq T$, is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the filtration $\mathcal{F}_t$, $0 \leq t \leq T$, satisfying the usual conditions. The coefficients μ_t, σ_t, $0 \leq t \leq T$, are $\mathcal{F}_t$-adapted processes, and assumed for simplicity to be

$$\mu_t = \mu \mathbb{1}_{\{0 \leq t \leq \tau_1\}} + \mu_1 \mathbb{1}_{\{0 < \tau_1 \leq T\}},$$

and

$$\sigma_t = \sigma \mathbb{1}_{\{0 \leq t \leq \tau_1\}} + \sigma_1 \mathbb{1}_{\{0 < \tau_1 \leq T\}},$$

with $\mu,\ \sigma \in \mathcal{F}_0$ and $\mu_1,\ \sigma_1 \in \mathcal{F}_{\tau_1}$. It is assumed that $\sigma,\ \sigma_1 > 0$ almost surely under the true measure $\mathbb{P}$. We further define the Sharpe ratio as $\lambda = \frac{\mu}{\sigma}$ for $[0, \tau_1]$ and $\lambda_1 = \frac{\mu_1}{\sigma_1}$ for $(\tau_1, T]$. The model parameters for the stock price hence only change at $t = \tau_1 \in \mathcal{F}_0$, with $0 < \tau_1 < T$. It is worth noting that, different form most other works, we do not specify another (hyper-) model at $t = 0$ to describe how those parameters may actually change in the future. In other words, the investor is unaware of the full model under the true measure $\mathbb{P}$ at $t = 0$, and new parameters can only be observed at the model revision time $t = \tau_1$.

We next formulate the knowledge of the stubborn investor under her subjective belief at $t = 0$. Knowing that the market parameters would certainly shift at $t = \tau_1$ (e.g., due to scheduled announcement in the market or self-planned market condition reassessment, etc), but unclear about the genuine switching dynamics in advance, the investor at $t = 0$ is assumed for simplicity to perceive the model parameters for the whole horizon as piecewise constants $\mathcal{M}_{[0,\tau_1]} = \{\mu, \sigma, \lambda\}$ and $\widehat{\mathcal{M}}_{(\tau_1,T]} = \{\widehat{\mu}, \widehat{\sigma}, \widehat{\lambda}\}$, with $\widehat{\sigma} > 0$ and $\widehat{\lambda} = \frac{\widehat{\mu}}{\widehat{\sigma}}$. Precisely, the stock price under her perceived measure $\widehat{\mathbb{P}}$ is given by

$$d\widehat{S}_t = \widehat{S}_t(\widehat{\mu}_t dt + \widehat{\sigma}_t d\widehat{W}_t),$$

with $\widehat{S}_0 = S_0$, where $\widehat{W}_t$, $0 \leq t \leq T$, is a standard Brownian motion on a filtered probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$, with the filtration $\widehat{\mathcal{F}}_t$, $0 \leq t \leq T$, satisfying the usual conditions and $\widehat{\mathcal{F}}_0 = \mathcal{F}_0$. The coefficients under her perceived model are hence assumed to be correct only over the first sub-horizon $[0, \tau_1]$, and differ from the truth in the remaining horizon $(\tau_1, T]$. This assumption complies with most prediction mechanism in investment practice whose prediction power typically decays as time moves into the far future.

It is important to note that the very reason for the investor to have a model for the whole horizon at $t = 0$ is clearly due to the backward model commitment of the classical approach discussed in previous chapter. This feature of the classical approach inevitably and undesirably forces the investor at $t = 0$ to commit to a probably vague model for probably remote future time period $(\tau_1, T]$. We stress, however, that such $t = 0$ perceived model under the measure $\widehat{\mathbb{P}}$ is introduced only for the purpose of computing the optimal policy over the first sub-horizon $[0, \tau_1]$. Once arriving at the intermediate time $t = \tau_1$,

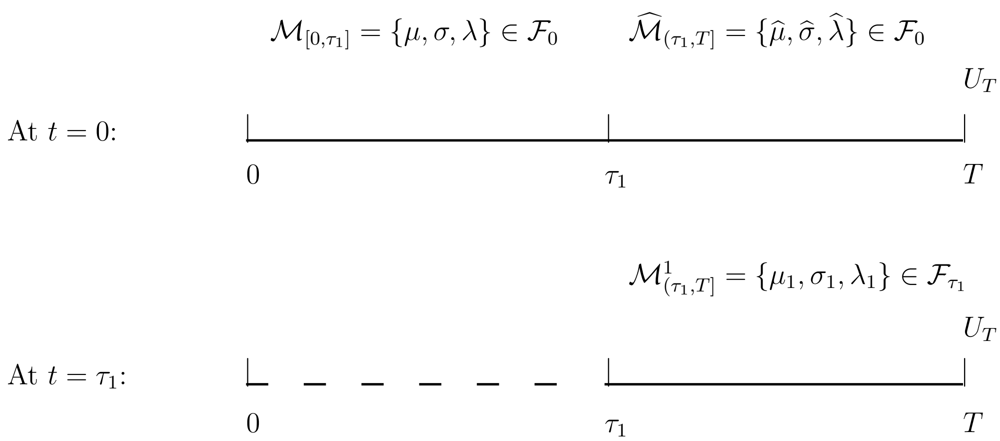

Figure 2.1: Classical approach with model revision.

the investor has a chance to review the model for the remaining horizon $(\tau_1, T]$ and obtains the revised (true) model parameters $\mathcal{M}^1_{(\tau_1,T]} = \{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$. She is then able to take corresponding actions under the revised model, but the terminal objective is not allowed to change, i.e., the terminal utility function is fixed to be a $\mathcal{F}_0$-measurable function $U_T : \mathbb{R}_+ \to \mathbb{R}$, a strictly increasing and strictly concave function satisfying Inada's conditions $\lim_{x\downarrow 0} U_T'(x) = \infty$ and $\lim_{x\uparrow\infty} U_T'(x) = 0$. The inverse marginal of the terminal utility is defined as usual $I : \mathbb{R}_+ \to \mathbb{R}_+$, with $I(x) = (U_T')^{(-1)}(x)$. The problem setting is summarized in Figure 2.2.

The investor at $t = 0$ would then solve a classical optimal control problem with the backward induction argument, under the perceived model. In particular, for $\tau_1 < t \leq T$, the wealth process under the standard self-financing condition has the dynamics given by

$$d\widehat{X}_s = \widehat{\mu}\pi_s ds + \widehat{\sigma}\pi_s d\widehat{W}_s,$$

for $t \le s \le T$, with $\widehat{X}_t = x$. The set of admissible strategies under the perceived model is defined as

$$\widehat{\mathcal{A}}_{[0,T]} = \Big\{ \pi \,:\, \text{self-financing with } \pi_t \in \widehat{\mathcal{F}}_t \text{ and } \mathbb{E}_{\widehat{\mathbb{P}}}\Big[\int_0^T \pi_t^2 dt \Big] < \infty \Big\}.$$

To solve the $t = 0$ problem following the backward induction, let the value function over the sub-horizon $(\tau_1, T]$ be defined as

$$\widehat{V}(x, t; \widehat{\lambda}) = \sup_{\pi} \mathbb{E}_{\widehat{\mathbb{P}}} \left[U_T(\widehat{X}_T) | \widehat{X}_t = x \right],$$

where the expectation is taken under the $(\tau_1, T]$ marginal probability measure of $\widehat{\mathbb{P}}$ associated to the perceived model $\widehat{\mathcal{M}}_{(\tau_1,T]}$. It is then well known that the function $\widehat{V}(x, t; \widehat{\lambda})$ is the strictly increasing and strictly concave solution (in the spatial variable) to the HJB equation (see, e.g., [35])

$$\widehat{V}_t - \frac{\widehat{\lambda}^2}{2} \frac{\widehat{V}_x^2}{\widehat{V}_{xx}} = 0, \tag{2.1}$$

with terminal condition $\widehat{V}(x, T) = U_T(x)$. The optimal portfolio process is given by

$$\widehat{\pi}^*(\widehat{X}_t^*, t) = -\frac{\widehat{\lambda}}{\widehat{\sigma}} \frac{\widehat{V}_x(\widehat{X}_t^*, t)}{\widehat{V}_{xx}(\widehat{X}_t^*, t)},$$

for $\tau_1 < t \le T$. It is also convenient to define the local absolute risk tolerance function $\widehat{r}(x, t) = -\frac{\widehat{V}_x(x,t)}{\widehat{V}_{xx}(x,t)}$ to write the optimal portfolio process as

$$\widehat{\pi}^*(\widehat{X}_t^*, t) = \frac{\widehat{\lambda}}{\widehat{\sigma}} \widehat{r}(\widehat{X}_t^*, t).$$

We now consider the following transformation that is essential for the analytical representation of the above quantities

$$\widehat{V}_x\left(\widehat{H}(x,t),t\right) = \exp\Big(-x - \frac{1}{2}\widehat{\lambda}^2(T-t)\Big). \tag{2.2}$$

It is then well known (see [33]) that the function $\widehat{H}(x,t)$ is the solution to the classical backward heat equation

$$\widehat{H}_t + \frac{\widehat{\lambda}^2}{2}\widehat{H}_{xx} = 0, \tag{2.3}$$

for $\tau_1 < t \leq T$, with terminal condition $\widehat{H}(x,T) = I(e^{-x})$. The local absolute risk tolerance function can then be rewritten as $\widehat{r}(x,t) = \widehat{H}_x(\widehat{H}^{(-1)}(x,t),t)$, and the optimal portfolio process as well as the optimal wealth process are

$$\widehat{\pi}_t^* = \frac{\widehat{\lambda}}{\widehat{\sigma}}\widehat{H}_x\Big(\widehat{H}^{(-1)}(x,\tau_1) + \widehat{\lambda}^2(t-\tau_1) + \widehat{\lambda}(\widehat{W}_t - \widehat{W}_{\tau_1}),t\Big), \tag{2.4}$$

and

$$\widehat{X}_t^* = \widehat{H}\Big(\widehat{H}^{(-1)}(x,\tau_1) + \widehat{\lambda}^2(t-\tau_1) + \widehat{\lambda}(\widehat{W}_t - \widehat{W}_{\tau_1}),t\Big), \quad \widehat{X}_{\tau_1}^* = x, \tag{2.5}$$

for $\tau_1 < t \leq T$, respectively.

The backward induction reasoning implies, knowing that the value function $\widehat{V}\left(X_{\tau_1},\tau_1;\widehat{\lambda}\right)$ is the best achievable performance over $(\tau_1,T]$ under the perceived measure $\widehat{\mathbb{P}}$ starting from any admissible wealth level X_{τ_1} at $t=\tau_1$, the investor would take it as the "short-term" objective and solve the optimization problem for $0 \leq t \leq \tau_1$ as following

$$\widehat{V}(x,t;\lambda,\widehat{\lambda}) = \sup_{\pi} \mathbb{E}_{\widehat{\mathbb{P}}}\left[\widehat{V}\left(\widehat{X}_{\tau_1},\tau_1;\widehat{\lambda}\right)|\widehat{X}_t = x\right], \tag{2.6}$$

where the expectation is taken under the $[0, \tau_1]$ marginal of $\widehat{\mathbb{P}}$ associated to the accurate model $\mathcal{M}_{[0,\tau_1]}$. To solve this first sub-horizon problem, we can derive a similar HJB equation as (2.1), i.e., for $0 \leq t \leq \tau_1$,

$$\widehat{V}_t - \frac{\lambda^2}{2} \frac{\widehat{V}_x^2}{\widehat{V}_{xx}} = 0, \tag{2.7}$$

with terminal condition being $\widehat{V}(x, \tau_1) = \widehat{V}\left(x, \tau_1; \widehat{\lambda}\right)$, for all admissible x. If a strictly increasing and strictly concave classical solution (in the state variable) can be found, we can derive the optimal portfolio as

$$\widehat{\pi}^*(\widehat{X}_t^*, t) = -\frac{\lambda}{\sigma} \frac{\widehat{V}_x(\widehat{X}_t^*, t)}{\widehat{V}_{xx}(\widehat{X}_t^*, t)}, \tag{2.8}$$

where $\widehat{X}_t^*$, $0 \leq t \leq \tau_1$, is the corresponding optimal wealth process over the first sub-horizon.

The above existing results correspond to the $t = 0$ classical Merton's problem without any *unanticipated* model parameter changes. However, the true model indeed changes at $t = \tau_1$ as we have formulated under the physical measure $\mathbb{P}$, and such change cannot be captured by the investor's $t = 0$ subjective belief under $\widehat{\mathbb{P}}$. The investor would hence only follow the optimal feedback policy (2.8) derived under the $t = 0$ perceived measure $\widehat{\mathbb{P}}$ up to time $t = \tau_1$ in the true market governed by the physical measure $\mathbb{P}$. Then at the interface $t = \tau_1$ of the two investment sub-horizons, the investor would recognize that the true realized model for the second sub-horizon is $\mathcal{M}^1_{(\tau_1,T]} = \{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$. It is reasonable for her to reconsider the decision by taking into account this (unanticipated) new knowledge about the market, i.e., under the adaptive control framework, she would solve the (conditional) optimal control problem over $(\tau_1, T]$

$$V(x,t;\lambda_1) = \operatorname*{esssup}_{\pi} \mathbb{E}\Big[U_T(X_T)\Big|\mathcal{F}_{\tau_1},\ X_t = x\Big] \in \mathcal{F}_{\tau_1},\ \text{a.s.}, \tag{2.9}$$

where the conditional expectation is taken under the true measure $\mathbb{P}$. The terminal utility is not changed, since in this section we focus on the "stubborn" behavior. Conditional on $\mathcal{F}_{\tau_1}$, the problem (2.9) is still a Merton's problem for a shorter horizon. Standard argument therefore yields that the random value function $V(x,t;\lambda_1)$ is the strictly increasing and strictly concave solution (in the spatial variable) to the HJB equation with random coefficient

$$V_t - \frac{\lambda_1^2}{2}\frac{V_x^2}{V_{xx}} = 0, \quad \text{a.s.}, \tag{2.10}$$

for $\tau_1 < t \le T$ with terminal condition $V(x,T;\lambda_1) = U_T(x)$, a.s. under $\mathbb{P}$. The solution to equation (2.10) as well as the associated optimal portfolio process and optimal wealth process are analogous to their previous counterparts (2.1), (2.4) and (2.5). Indeed, such problem is known as the adaptive control problem, for which two phases are typically involved, the optimization phase and adaptation phase (see, e.g., [12]). In the optimization phase, the control problem with the unknown model parameters is solved and the associated optimal strategy is obtained. Then one would complete the adaptation phase by substituting the estimated model parameters into the optimal strategy at each model reassessment time. We now illustrate the details in the following example under a terminal power utility.

2.2.1 Classical approach: the power utility case

Consider the power utility $U_T(x) = \frac{1}{\gamma}x^\gamma$, with $0 < \gamma < 1$. Then the strictly increasing and strictly concave solution to the HJB equation (2.1) associated to the second sub-horizon model $\widehat{\mathcal{M}}_{(\tau_1,T]}$ is

$$\widehat{V}(x,t;\widehat{\lambda}) = \frac{x^\gamma}{\gamma}\exp\Big(\frac{\widehat{\lambda}^2\gamma}{2(\gamma-1)}(t-T)\Big),$$

for all $(x,t) \in [0,\infty) \times (\tau_1, T]$. By backward induction, the value function corresponding to the first sub-horizon model $\mathcal{M}_{[0,\tau_1]}$ is the strictly increasing and strictly concave solution to HJB equation (2.7)

$$\widehat{V}(x,t;\lambda,\widehat{\lambda}) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda^2(t-\tau_1)+\widehat{\lambda}^2(\tau_1-T)\right)\right),$$

for all $(x,t) \in [0,\infty)\times[0,\tau_1]$. We stress that since the investor would eventually realize at $t = \tau_1$ that the model for the second sub-horizon is $\mathcal{M}^1_{(\tau_1,T]}$ rather than $\widehat{\mathcal{M}}_{(\tau_1,T]}$, the time $t = 0$ perceived optimal portfolio process would only be followed up to $t = \tau_1$ in the true market under the physical measure $\mathbb{P}$. This process is given by $\widehat{\pi}^*_t = \frac{\lambda}{\sigma(1-\gamma)}X^*_t$, for $0 \le t \le \tau_1$, according to (2.8), and the resulting optimal wealth process under the true measure $\mathbb{P}$ is the unique strong solution to the stochastic differential equation (SDE)

$$dX^*_t = \frac{\lambda^2}{1-\gamma}X^*_t dt + \frac{\lambda}{1-\gamma}X^*_t dW_t, \tag{2.11}$$

with $X^*_0 = x > 0$, and $0 \le t \le \tau_1$. The solution is given by

$$X^*_t = x\exp\Big(\frac{(1-2\gamma)\lambda^2}{2(1-\gamma)^2}t + \frac{\lambda}{1-\gamma}W_t\Big),$$

for $0 \le t \le \tau_1$. We also compute the value function process along this genuine wealth process for later discussion

$$\widehat{V}(X_t^*, t; \lambda, \widehat{\lambda}) = \frac{x^\gamma}{\gamma} \exp\left(\frac{\gamma}{2(\gamma-1)}\left(\frac{\gamma\lambda^2}{1-\gamma}t + (\widehat{\lambda}^2 - \lambda^2)\tau_1 - \widehat{\lambda}^2 T\right) + \frac{\lambda\gamma}{1-\gamma}W_t\right), \tag{2.12}$$

for $0 \le t \le \tau_1$. At time $t = \tau_1$, knowing that the true realized model for the second sub-horizon is $\mathcal{M}^1_{(\tau_1, T]} = \{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$, the investor solves the (conditional) HJB equation (2.10) and obtains the solution

$$V(x, t; \lambda_1) = \frac{x^\gamma}{\gamma} \exp\left(\frac{\lambda_1^2 \gamma}{2(\gamma-1)}(t - T)\right) \in \mathcal{F}_{\tau_1},$$

for $\tau_1 < t \le T$. The optimal portfolio process is given by $\pi_t^* = \frac{\lambda_1}{\sigma_1(1-\gamma)} X_t^*$ for $\tau_1 < t \le T$. The SDE for the optimal wealth process under the true measure $\mathbb{P}$ now becomes

$$dX_t^* = \frac{\lambda_1^2}{1-\gamma} X_t^* dt + \frac{\lambda_1}{1-\gamma} X_t^* dW_t, \tag{2.13}$$

with $X_{\tau_1}^* = x\exp\left(\frac{(1-2\gamma)\lambda^2}{2(1-\gamma)^2}\tau_1 + \frac{\lambda}{1-\gamma}W_{\tau_1}\right)$. Conditional on $\mathcal{F}_{\tau_1}$, this is an SDE with (conditionally) independent initial condition, and the solution yields

$$X_t^* = x\exp\left(\frac{1-2\gamma}{2(1-\gamma)^2}\left((\lambda^2 - \lambda_1^2)\tau_1 + \lambda_1^2 t\right) + \frac{\lambda - \lambda_1}{1-\gamma}W_{\tau_1} + \frac{\lambda_1}{1-\gamma}W_t\right), \tag{2.14}$$

for $\tau_1 < t \le T$. The value function process over $(\tau_1, T]$ at the optimum hence can be computed as

$$V(X_t^*, t; \lambda_1) = \frac{x^\gamma}{\gamma} \exp\Bigg(\frac{\gamma}{2(1-\gamma)^2}\left((1-2\gamma)(\lambda^2 - \lambda_1^2)\tau_1 - (\gamma-1)\lambda_1^2 T - \gamma\lambda_1^2 t\right) + \frac{\gamma(\lambda - \lambda_1)}{1-\gamma}W_{\tau_1} + \frac{\gamma\lambda_1}{1-\gamma}W_t\Bigg). \tag{2.15}$$

Finally, we note that the above solutions corresponding to the $t = 0$ and $t = \tau_1$ optimization problems can also be recovered through the transformation (2.2). For example, at $t = 0$, it yields the function

$$\widehat{H}(x,t) = \exp\Big(\frac{x}{1-\gamma} + \frac{\widehat{\lambda}^2}{2(1-\gamma)^2}(T-t)\Big),$$

for $\tau_1 < t \le T$. Then, the $t = 0$ perceived optimal portfolio and wealth processes follow from the analytic representations (2.4), (2.5).

2.3 Forward approach

At the intermediate model revision time $t = \tau_1$, the classical adaptive control method basically lets the stubborn investor forget what she has achieved during $[0, \tau_1]$, and restart solving a stochastic optimization problem for the remaining horizon, given the market information $\mathcal{M}^1_{(\tau_1,T]} = \{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$ and the achieved optimal wealth $X^*_{\tau_1}$. It is clearly time-inconsistent over the whole horizon $[0, T]$, and it simply puts together two optimization problems without establishing any intertemporal connection. The forward approach, on the other hand, is based on the forward performance process theory which is built to maintain time-consistency and to achieve less volatile optimal performance along the time. At $t = 0$, the forward investor has the same correct view as the stubborn investor about the market for the first sub-horizon, i.e., $\mathcal{M}_{[0,\tau_1]} = \{\mu, \sigma, \lambda\} \in \mathcal{F}_0$, and also at $t = \tau_1$, she has the same correct view for the second sub-horizon $\mathcal{M}^1_{(\tau_1,T]} = \{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$. The main difference is that the forward investor is allowed to choose a revised terminal utility at

$t = \tau_1$, based on her correct view of the market for the remaining sub-horizon, so that certain intertemporal consistency can be preserved across the first and second sub-horizons. We will see that although both investors have access to the same true information, the extra flexibility to the forward investor for not being "stubborn" will give rise to consistent investment behavior and more stable performance processes. Another flexibility of the forward approach is that it allows to start with a family of admissible initial utility, corresponding to different initial views about the market (e.g., optimistic or pessimistic), but in this work we choose the initial utility to be $\widehat{V}(x, 0; \lambda, \widehat{\lambda})$, the initial value function of the $t = 0$ problem of the stubborn investor based on her $t = 0$ belief $\widehat{\mathbb{P}}$, in order to have a comparable analysis between the two types of investment behavior. By choosing $\widehat{V}(x, 0; \lambda, \widehat{\lambda})$, the forward investor intends to achieve the same level of performance as the stubborn investor at $t = 0$, but different from the stubborn investor, she can maintain the same level of performance even at later times through the preservation of optimality and time-consistency under model revision. In terms of model specification, the forward investor enjoys a third flexibility that allows her to only commit to a model for the current sub-horizon in real-time, i.e., a model only for $[0, \tau_1]$ at $t = 0$ and a model for $(\tau_1, T]$ once at $t = \tau_1$. There is hence no model commitment issue present in the forward framework, and the impact of a misspecified model for remote future is minimal[1]. A summary of the model revision under

[1]The inaccurate model $\widehat{\mathcal{M}}_{(\tau_1, T]} = \{\widehat{\mu}, \widehat{\sigma}, \widehat{\lambda}\}$ still affects the forward solution through the initial condition $\widehat{V}(x, 0; \lambda, \widehat{\lambda})$. However, such long-term model is not necessary to specify in

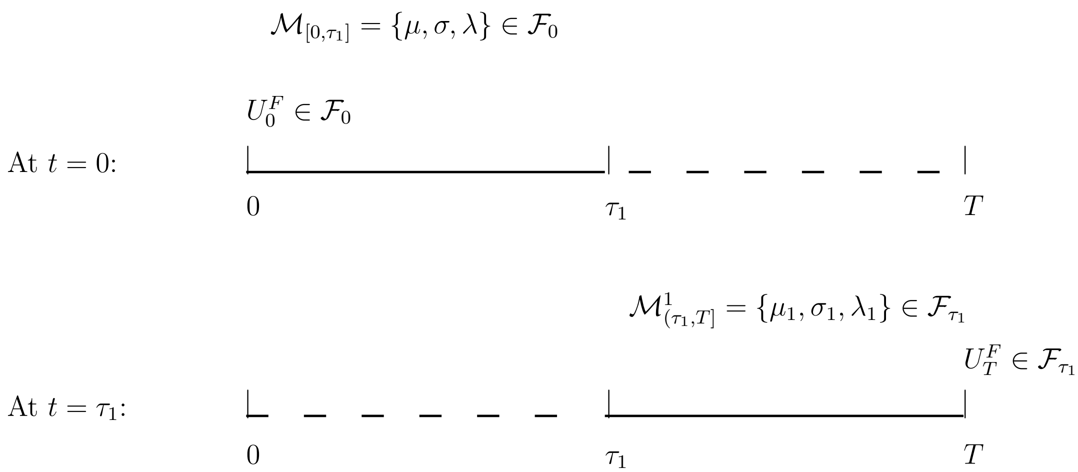

Figure 2.2: Forward approach with model revision.

the forward approach is given in Figure 2.3.

At $t = 0$, the forward investor starts with $U_0^F(x) = \widehat{V}(x, 0; \lambda, \widehat{\lambda})$, and the true model for the first sub-horizon $\mathcal{M}_{[0,\tau_1]} = \{\mu, \sigma, \lambda\} \in \mathcal{F}_0$. The goal is to construct a forward performance process $U_t^F(x)$, $0 \leq t \leq \tau_1$, such that $U_t^F(X_t^*)$, $0 \leq t \leq \tau_1$, is a martingale along the forward optimal wealth process X^*, and $U_t^F(X_t)$, $0 \leq t \leq \tau_1$, is a supermartingale along any admissible wealth process. In this section, we would concentrate on a specific family of forward processes, namely the zero volatility forward performance process, which may be seen as the closet analogue of the classical counterpart. In the next section, the non-zero volatility forward processes will be discussed for the

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general, as we remind the reader that a generic forward process does not need to start from $\widehat{V}(x, 0; \lambda, \widehat{\lambda})$. This choice is only for comparable analysis in the current work.

reconciliation of the two approaches. Similar to the classical scenario, for the first sub-horizon problem, the zero volatility forward process satisfies the same HJB equation (2.1), but with an initial condition $U_0^F(x) = \widehat{V}(x, 0; \lambda, \widehat{\lambda})$, which in turn makes it an ill-posed problem. Existing result about zero volatility forward processes (see [45]) shows that if the function $u(x,t)$ is a strictly increasing and strictly concave solution (in the spatial variable) to the fully nonlinear partial differential equation (PDE)

$$u_t - \frac{1}{2}\frac{u_x^2}{u_{xx}} = 0, \tag{2.16}$$

with $u(x,0) = \widehat{V}(x, 0; \lambda, \widehat{\lambda})$, then the process $U_t^F(x) = u(x, \lambda^2 t)$, $0 \leq t \leq \tau_1$, is a forward performance process over the first sub-horizon $[0, \tau_1]$. By virtue of the transformation

$$u_x(h(x,t), t) = e^{-x+\frac{t}{2}}, \tag{2.17}$$

it is known that the function $h : \mathbb{R} \times [0, \infty) \to \mathbb{R}_+$ is the strictly increasing in x solution to the ill-posed heat equation

$$h_t + \frac{1}{2}h_{xx} = 0, \tag{2.18}$$

with initial condition defined through $h(x,0) = I_{\widehat{V}}(e^{-x})$, where $I_{\widehat{V}} : \mathbb{R}_+ \to \mathbb{R}_+$, with $I_{\widehat{V}}(x) = (\widehat{V}_x)^{(-1)}(x, 0; \lambda, \widehat{\lambda})$ being the inverse marginal of the initial value function $\widehat{V}(x, 0; \lambda, \widehat{\lambda})$. The optimal portfolio and wealth processes under the forward performance process can then be represented by

$$\pi_t^* = \frac{\lambda}{\sigma} h_x\Big(h^{(-1)}(X_t^*, \lambda^2 t), \lambda^2 t\Big), \tag{2.19}$$

and

$$X_t^* = h\Big(h^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, \lambda^2 t\Big), \quad X_0^* = x, \tag{2.20}$$

respectively, for $0 \leq t \leq \tau_1$. Notice the difference between (2.4), (2.5) and (2.19), (2.20) are twofold, i.e., the $\widehat{H}(x,t)$ function for the classical problem is constructed with a normalization time T implicitly embedded, and no time rescaling is involved in the classical optimal portfolio and wealth processes.

The forward problem over the first sub-horizon $[0, \tau_1]$ would be completely solved once the function $h(x,t)$ is found. At $t = \tau_1$, the forward investor has access to the true model for the remaining horizon $\mathcal{M}^1_{(\tau_1,T]} = \{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$. Given this new information and her optimal wealth $X^*_{\tau_1}$ obtained by following the forward optimal strategy over sub-horizon $[0, \tau_1]$, she would seek a terminal criterion $U_T^F(x)$ at $t = \tau_1$ in order to achieve intertemporal consistency. Recall that by committing to the initial condition $U_0^F(x) = \widehat{V}(x, 0; \lambda, \widehat{\lambda})$, the forward investor at $t = 0$ has indirectly committed to the terminal utility $U_T(x)$ and the initial subjective belief $\widehat{\mathbb{P}}$ of the stubborn investor. From this perspective, the forward investor can be seen as revising her original investment objective, $U_T(x)$ that she implicitly shares with the stubborn investor at $t = 0$, after receiving real-time updated information from the market at $t = \tau_1$. Such revised objective would be a terminal utility $U_T^F(x)$ that is determined through

$$U^F_{\tau_1}\left(X^*_{\tau_1}\right) = \operatorname*{esssup}_{\pi} \mathbb{E}\Big[U_T^F(X_T)\Big|\mathcal{F}_{\tau_1}\Big], \text{ a.s.}, \tag{2.21}$$

for any $X^*_{\tau_1} \in \mathcal{F}_{\tau_1}$, yielding from the forward optimal strategy π_t^*, $0 \leq t \leq \tau_1$, given by (2.19) and all admissible $X_0^* = x$. Conditional on $\mathcal{F}_{\tau_1}$, the question boils down to looking for a forward performance process $U_t^F(x)$, $\tau_1 < t \leq T$, with initial condition $U_{\tau_1}^F(x)$, such that $U_t^F(X_t^*)$, $\tau_1 < t \leq T$, is a martingale along the forward optimal wealth process and that $U_t^F(X_t)$, $\tau_1 < t \leq T$, is a supermartingale along any admissible wealth process. It can be similarly shown as for the first sub-horizon problem that

$$U_t^F(x) = u\left(x, (\lambda^2 - \lambda_1^2)\tau_1 + \lambda_1^2 t\right),$$

$\tau_1 < t \leq T$, is a forward performance process where the function $u(x,t)$ is the strictly increasing and strictly concave solution to PDE (2.16)[2]. We conclude this section by revisiting the power utility scenario under the forward approach.

2.3.1 Forward approach: the power utility case

Recall that the value function for the stubborn investor over the first sub-horizon $[0, \tau_1]$, given the full perceived model $\mathcal{M}_{[0,\tau_1]}$, $\widehat{\mathcal{M}}_{(\tau_1,T]}$ and the terminal utility $U_T(x) = \frac{x^\gamma}{\gamma}$, for $0 < \gamma < 1$, is

$$\widehat{V}(x,t;\lambda,\widehat{\lambda}) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda^2(t-\tau_1) + \widehat{\lambda}^2(\tau_1 - T)\right)\right),$$

[2]Unlike the existing results as in [45], where a forward performance process can be constructed by $U_t^F(x) = u(x, \int_0^t \lambda_s^2 ds)$ for the whole horizon $[0,T]$ with time-varying λ_s, $0 \leq s \leq T$, we instead construct it locally forward in real-time, since at $t = 0$, we only have the knowledge of the true model up to $t = \tau_1$. Nevertheless, due to the forward recursive nature of the forward performance process, the two constructions coincide as expected. This is not valid though for classical stochastic optimization problems because of model commitment.

for all $(x,t) \in [0,\infty) \times [0,\tau_1]$. Hence, the forward investor takes

$$U_0^F(x) = \widehat{V}(x,0;\lambda,\widehat{\lambda}) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left((\widehat{\lambda}^2-\lambda^2)\tau_1 - \widehat{\lambda}^2 T\right)\right)$$

as the initial condition. The function $u(x,t)$ therefore solves the PDE (2.16) with initial condition $u(x,0) = U_0^F(x)$. A strictly increasing and strictly concave solution (in spatial variable) is given by

$$u(x,t) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left(t + (\widehat{\lambda}^2-\lambda^2)\tau_1 - \widehat{\lambda}^2 T\right)\right).$$

The zero volatility (or the time-monotone) forward performance process is

$$U_t^F(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda^2 t + (\widehat{\lambda}^2-\lambda^2)\tau_1 - \widehat{\lambda}^2 T\right)\right), \tag{2.22}$$

for $0 \le t \le \tau_1$, and

$$U_t^F(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda_1^2 t + (\widehat{\lambda}^2-\lambda_1^2)\tau_1 - \widehat{\lambda}^2 T\right)\right), \tag{2.23}$$

for $\tau_1 < t \le T$. We also notice that the function $h(x,t)$ defined by (2.17) is

$$h(x,t) = \exp\left(-\frac{x}{\gamma-1} - \frac{1}{2(\gamma-1)^2}\left(t + \gamma(\widehat{\lambda}^2-\lambda^2)\tau_1 - \gamma\widehat{\lambda}^2 T\right)\right),$$

which is clearly a strictly positive solution to the ill-posed heat equation (2.18). The forward optimal portfolio process and optimal wealth process over the first sub-horizon $[0,\tau_1]$ in turn are

$$\pi_t^* = -\frac{\lambda}{\sigma(\gamma-1)}X_t^*,$$

$$X_t^* = x\exp\left(-\frac{(2\gamma-1)\lambda^2}{2(\gamma-1)^2}t - \frac{\lambda}{\gamma-1}W_t\right), \quad X_0^* = x,$$

respectively. The forward performance along the optimal wealth process over $[0,\tau_1]$ can be computed as

$$U_t^F(X_t^*) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(1-\gamma)}\left(\frac{\gamma\lambda^2}{\gamma-1}t + (\widehat{\lambda}^2-\lambda^2)\tau_1 - \widehat{\lambda}^2 T\right) + \frac{\lambda\gamma}{1-\gamma}W_t\right),$$

and hence it coincides pointwise in (t,ω) with its counterpart (2.12) derived under the classical approach. We can see that based on the zero volatility forward performance process that starts from $\widehat{V}(x,0;\lambda,\widehat{\lambda})$, the forward investor achieves exactly the same optimal portfolio, optimal wealth and optimal performance processes as the stubborn investor over the first sub-horizon $[0,\tau_1]$. For the second sub-horizon $(\tau_1,T]$, the construction of the optimal portfolio and wealth process is still through the function $h(x,t)$. Indeed, as shown in [45], the optimal wealth process under the zero volatility forward performance process $U_t^F(x)$, for $\tau_1 < t \le T$, is given by

$$X_t^* = h\left(h^{(-1)}(x,0) + \int_0^t \lambda_s^2 ds + \int_0^t \lambda_s dW_s, \int_0^t \lambda_s^2 ds\right)$$

$$= x\exp\left(\frac{(1-2\gamma)}{2(\gamma-1)^2}\left(\lambda_1^2 t + (\lambda^2-\lambda_1^2)\tau_1\right) - \frac{\lambda-\lambda_1}{\gamma-1}W_{\tau_1} - \frac{\lambda_1}{\gamma-1}W_t\right), \quad (2.24)$$

where $\lambda_s := \lambda\mathbb{1}_{\{0\le s\le\tau_1\}} + \lambda_1\mathbb{1}_{\{\tau_1<s\le T\}}$. The corresponding forward performance along the optimal wealth is therefore

$$U_t^F(X_t^*) = \frac{x^\gamma}{\gamma}\exp\Big(\frac{\gamma}{2(\gamma-1)^2}\Big(-\gamma\lambda_1^2 t + \left(\gamma\lambda_1^2 + (1-2\gamma)\lambda^2 + (\gamma-1)\widehat{\lambda}^2\right)\tau_1$$
$$+(1-\gamma)\widehat{\lambda}T\Big) - \frac{\gamma(\lambda-\lambda_1)}{\gamma-1}W_{\tau_1} - \frac{\gamma\lambda_1}{\gamma-1}W_t\Big),$$

for $\tau_1 < t \le T$. Several observations then follow after we obtain the above explicit expressions. First, due to the choice of the predictable model $\mathcal{M}^1_{(\tau_1,T]} =$

$\{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$ over $(\tau_1, T]$, the terminal utility $U_T^F(x)$ implied form the zero volatility forward performance process (2.23) is actually $\mathcal{F}_{\tau_1}$-measurable. Hence, the forward investor is indeed aware of the consistent revised objective at $t = \tau_1$ based on the updated information about the market; in other words, the zero volatility forward performance leads to a predictable forward utility $U_T^F(x) \in \mathcal{F}_{\tau_1}$. However, it is clear that $U_T^F(x)$ is different from $U_T(x)$, showing that the zero volatility forward family cannot reconcile with the stubborn approach. Second, for sub-horizon $(\tau_1, T]$, the forward performance along the optimum $U_t^F(X_t^*)$ does not coincide with the $t = 0$ (perceived) optimal value along the optimum $\widehat{V}(X_t^*, t; \widehat{\lambda})$, $\tau_1 < t \leq T$, neither would it agree with the $t = \tau_1$ (genuine) optimal value along the optimum $V(X_t^*, t; \lambda_1)$, $\tau_1 < t \leq T$, given by (2.15). Nevertheless, conditional on $\mathcal{F}_{\tau_1}$, due to the fact $\lambda_1 \in \mathcal{F}_{\tau_1}$, it is straightforward to show that

$$\mathbb{E}\left[U_t^F(X_t^*)\middle|\mathcal{F}_{\tau_1}\right] = \widehat{V}\left(X_{\tau_1}^*, \tau_1; \lambda, \widehat{\lambda}\right), \text{ a.s.,}$$

for $\tau_1 < t \leq T$. On the other hand,

$$\mathbb{E}\left[U_t^F(X_t^*)\right] = \mathbb{E}\left[\mathbb{E}\left[U_t^F(X_t^*)\middle|\mathcal{F}_{\tau_1}\right]\right] = \mathbb{E}\left[U_{\tau_1}^F\left(X_{\tau_1}^*, \tau_1; \lambda, \widehat{\lambda}\right)\right] = \widehat{V}(x, 0; \lambda, \widehat{\lambda}), \tag{2.25}$$

for $\tau_1 < t \leq T$, following from the construction of the forward performance process. We now can conclude that whether assessed at $t = 0$ or at $t = \tau_1$, the forward investor in the second sub-horizon $(\tau_1, T]$ performs equally well on average under the genuine model $\mathbb{P}$ as the stubborn investor under the $t = 0$ perceived model $\widehat{\mathbb{P}}$. This is one of the stability properties we observed for the forward approach, which is mainly due to the selection of the $t = 0$ classical value function as the initial condition as well as the consistent construction of the performance process afterwards. Such stability is obviously not achievable in general by the time-inconsistent stubborn approach, as one can compute its time $t = \tau_1$ average performance, which typically holds that with positive probability under $\mathbb{P}$,

$$\mathbb{E}\left[V(X_t^*, t; \lambda_1)\middle|\mathcal{F}_{\tau_1}\right] \neq \widehat{V}\left(X_{\tau_1}^*, \tau_1; \lambda, \widehat{\lambda}\right),$$

for $\tau_1 < t \leq T$, where $V(X_t^*, t; \lambda_1)$ is the genuine performance along optimum of the stubborn approach under $\mathbb{P}$ over the second sub-horizon (i.e., (2.15)). Similar inequality

$$\mathbb{E}\left[V(X_t^*, t; \lambda_1)\right] \neq \widehat{V}\left(x, 0; \lambda, \widehat{\lambda}\right),$$

for $\tau_1 < t \leq T$, holds as well for the $t = 0$ average performance comparison. Indeed, to compute explicitly the above (conditional) expectations, we need to know the exact probability correlation between $\lambda_1 \in \mathcal{F}_{\tau_1}$ and the Brownian motion W_t, $0 \leq t \leq T$, under the genuine physical measure $\mathbb{P}$. In the special case where the model parameter is independent of the underlying Brownian motion, we can compute the $t = 0$ actual performance of the stubborn investor under power utility as

$$\begin{aligned}\mathbb{E}\left[\left(\frac{X_T^*}{\gamma}\right)^\gamma\right] &= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{X_T^*}{\gamma}\right)^\gamma \middle| \lambda_1\right]\right] \\ &= \mathbb{E}\left[\frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(1-\gamma)}\left(\lambda^2\tau_1 + \lambda_1^2(T-\tau_1)\right)\right)\right], \qquad (2.26)\end{aligned}$$

where X_T^* is the genuinely achieved terminal wealth (2.14) by the stubborn investor. Indeed, the equality (2.26) follows from the fact that

$$\mathbb{E}\left[\left(\frac{X_T^*}{\gamma}\right)^\gamma \Big| \lambda_1\right]$$

$$= \mathbb{E}\left[\frac{x^\gamma}{\gamma}\exp\left(\frac{(1-2\gamma)\gamma}{2(1-\gamma)^2}\left((\lambda^2-\lambda_1^2)\tau_1+\lambda_1^2 T\right)+\frac{(\lambda-\lambda_1)\gamma}{1-\gamma}W_{\tau_1}+\frac{\lambda_1\gamma}{1-\gamma}W_T\right)\Big|\lambda_1\right]$$

$$= \frac{x^\gamma}{\gamma}\exp\left(\frac{(1-2\gamma)\gamma}{2(1-\gamma)^2}\left((\lambda^2-\lambda_1^2)\tau_1+\lambda_1^2 T\right)\right)$$

$$\times\mathbb{E}\left[\exp\left(\frac{\lambda\gamma}{1-\gamma}W_{\tau_1}\right)\mathbb{E}\left[\exp\left(\frac{\lambda_1\gamma}{1-\gamma}(W_T-W_{\tau_1})\right)\Big|\mathcal{F}_{\tau_1},\ \lambda_1\right]\Big|\lambda_1\right]$$

$$= \frac{x^\gamma}{\gamma}\exp\left(\frac{(1-2\gamma)\gamma}{2(1-\gamma)^2}\left((\lambda^2-\lambda_1^2)\tau_1+\lambda_1^2 T\right)\right)\exp\left(\frac{\gamma^2}{2(1-\gamma)^2}\left(\lambda^2\tau_1+\lambda_1^2(T-\tau_1)\right)\right)$$

$$= \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(1-\gamma)}\left(\lambda^2\tau_1+\lambda_1^2(T-\tau_1)\right)\right), \tag{2.27}$$

by the independence of $\lambda_1 \in \mathcal{F}_{\tau_1}$ and the Brownian motion $W_t,\ 0 \le t \le T$. Now if we recall that the targeted time $t = 0$ performance under the perceived model $\widehat{\mathbb{P}}$ is given by

$$\widehat{V}(x,0;\lambda,\widehat{\lambda}) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(1-\gamma)}\left(\lambda^2\tau_1+\widehat{\lambda}^2(T-\tau_1)\right)\right), \tag{2.28}$$

then a comparison between (2.26) and (2.28) yields the intuitive conclusion: the stubborn investor should perform better than originally perceived (at $t = 0$), if $\lambda_1 \in \mathcal{F}_{\tau_1}$ has a high probability outweighing its counterpart $\widehat{\lambda}$ over the same sub-horizon $(\tau_1, T]$, corresponding to a higher Sharpe ratio, or if the model correction from the inaccurate $\widehat{\lambda}$ to the true parameter λ_1 happens earlier, corresponding to a smaller τ_1. We also stress that the difference between (2.27) and (2.28) is actually the first metric $m_{[0,T]}(x)$ we introduced in previous chapter to gauge the discrepancy between the actual and targeted investment performance, given the realization of the parameter λ_1 in the current Merton's case. The above observation clearly extends to the comparison between (2.27) and (2.28), when each $\lambda_1(\omega)$, for $\omega \in \Omega$, is considered rather than on the average.

2.4 Regret of investment behavior

We have introduced two different types of investment behavior based on the classical and the forward approaches, in face of the same model change at $t = \tau_1$. By following their respective optimal strategies, the two appraoches typically generate different terminal wealth, denoted by $X_T^{S,*}$ and $X_T^{F,*}$ at $t = T$, in the true underlying market. It is hence reasonable to review the performance according to a suitable baseline in retrospect at $t = T$. Motivated by the important concept *regret* in online learning/optimization literature (see, e.g., [55]), we introduce the similar performance regret as in the previous chapter to examine the two types of investment behavior under real-time (unanticipated) model changes.

Definition 2.4.1 (Performance Regret)**.** Suppose that $\mathcal{M}^1_{(\tau_1,T]}$ is the set of realized model parameters over $(\tau_1, T]$, and let $U_T^A(\cdot)$, $X_T^{A,*}$ be the terminal utility and the corresponding terminal wealth, respectively, associated to the investment behavior of type A in the true market. The performance regret (PR) of behavior type A is defined as the discrepancy between the expected utility of the genuine terminal wealth $X_T^{A,*}$ and that of the optimal terminal wealth in hindsight[3], given the knowledge of $\mathcal{M}^1_{(\tau_1,T]}$, i.e.,

$$M_{[0,T]}(x) = \mathbb{E}\left[U_T^A\left(X_T^{A,*}\right)\Big|\mathcal{M}^1_{(\tau_1,T]}\right] - \operatorname*{esssup}_{\pi}\mathbb{E}\left[U_T^A\left(X_T^{\pi}\right)\Big|\mathcal{M}^1_{(\tau_1,T]}\right], \text{a.s.,} \tag{2.29}$$

with $X_0^{A,*} = X_0^{\pi} = x$, for every x that is admissible.

In the above definition, the two conditional expectations in (2.29) are taken with respect to the true underlying physical measure $\mathbb{P}$, assumed to be completely known at $t = T$ in retrospect. The performance regret $M_{[0,T]}(x)$ is in general a random variable, as it obviously depends on the realized model parameters $\mathcal{M}^1_{(\tau_1,T]}$ whose distribution are governed by $\mathbb{P}$ (e.g., the stochastic factors model). Nonetheless, we would next show that for the forward behavior, the performance regret $M_{[0,T]}(x)$ is zero ω-almost surely, for each admissible x. This demonstrates the path-wise robustness in terms of *zero regret* for the forward behavior, a property that is typically not attainable for other types of investment behavior within the classical (backward) stochastic optimization paradigm.

Recall that the forward approach yields the consistent terminal utility given by

$$U_T^F(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}(\lambda_1^2 - \widehat{\lambda}^2)(T-\tau_1)\right), \tag{2.30}$$

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[3]Alternatively, this can be interpreted as the discrepancy between the $t = 0$ performance of two type A investors (i.e., the mortal and the genie), with one being an expert (i.e., the genie) who has the accurate knowledge about what parameters would be realized over $(\tau_1, T]$ at $t = 0$.

according to (2.23). Similar to (2.25), it can be further shown that for any realized $\lambda_1 \in \mathcal{F}_{\tau_1}$,

$$\mathbb{E}\left[U_T^F\left(X_T^{F,*}\right)\Big|\,\lambda_1\right] = \mathbb{E}\left[\mathbb{E}\left[U_T^F\left(X_T^{F,*}\right)\Big|\mathcal{F}_{\tau_1}\right]\Big|\,\lambda_1\right] = \mathbb{E}\left[U_{\tau_1}^F\left(X_{\tau_1}^{F,*}\right)\Big|\,\lambda_1\right]$$
$$= \mathbb{E}\left[U_{\tau_1}^F\left(X_{\tau_1}^{F,*}\right)\right] = \widehat{V}(x,0;\lambda,\widehat{\lambda}), \tag{2.31}$$

where we have resorted to the fact that the first sub-horizon forward process along optimum $U_t^F(X_t^{F,*})$, $0 \le t \le \tau_1$, is constructed independently of $\lambda_1 \in \mathcal{F}_{\tau_1}$. To compute the performance regret for the forward behavior, we only need to solve the classical backward stochastic optimization problem under $U_T^F(\cdot)$ in hindsight at $t = T$, knowing that the true realized parameters are $\mathcal{M}^1_{(\tau_1,T]} = \{\mu_1, \sigma_1, \lambda_1\}$. To make the hindsight problem tractable, we assume, given any realized model parameters $\mathcal{M}_{(\tau_1,T]}$, the underlying log-normal dynamics remain valid over $[0,T]$. This could include the case, for example, when the model parameters are driven by a Markov chain that is independent of the Brownian motion W_t, $0 \le t \le T$, under the true physical measure $\mathbb{P}$.

At terminal time $t = T$, the solution to the classical Merton's problem under the utility function $U_T^F(\cdot)$ in hindsight can be obtained through DPP over the two sub-horizons $(\tau_1, T]$ and then $[0, \tau_1]$ with the corresponding parameters applied. Indeed, over the period $(\tau_1, T]$, the value function $\widetilde{V}(x,t)$ is the unique strictly increasing and strictly concave (in the spatial variable) classical solution to the HJB equation

$$\widetilde{V}_t - \frac{\lambda_1^2}{2}\frac{\widetilde{V}_x^2}{\widetilde{V}_{xx}} = 0, \text{ a.s.},$$

with the terminal condition $\widetilde{V}(x,T) = U_T^F(x)$ given by (2.30). The value function is easily obtained as

$$\widetilde{V}(x,t) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda_1^2(t-\tau_1) - \widehat{\lambda}^2(T-\tau_1)\right)\right),$$

for $\tau_1 \le t \le T$. By DPP, the value function $\widetilde{V}(x,t)$ over $[0,\tau_1]$ satisfies

$$\widetilde{V}(x,t) = \sup_\pi \mathbb{E}\left[\widetilde{V}\left(X_{\tau_1},\tau_1\right)\Big|X_t = x\right],$$

with λ being applied over this first sub-horizon. Here,

$$\widetilde{V}\left(x,\tau_1\right) = \frac{x^\gamma}{\gamma}\exp\left(-\frac{\gamma\widehat{\lambda}^2}{2(\gamma-1)}(T-\tau_1)\right), \tag{2.32}$$

according to the solution for the second sub-horizon problem. We hence have the following HJB equation over $[0,\tau_1]$

$$\widetilde{V}_t - \frac{\lambda^2}{2}\frac{\widetilde{V}_x^2}{\widetilde{V}_{xx}} = 0,$$

with terminal condition given by (2.32). Again, this is the classical HJB equation for the Merton's problem over $[0,\tau_1]$ and it completely coincides with the HJB equation (2.7) in terms of both the equation and the terminal condition at $t=\tau_1$. Uniqueness result on its classical solution hence leads to that

$$\widetilde{V}(x,0) = \widehat{V}(x,0;\lambda,\widehat{\lambda}).$$

Recalling (2.31), we conclude that the performance regret of the forward behavior achieves zero regret given any realized parameter set $\mathcal{M}_{(\tau_1,T]}$, i.e., $M_{[0,T]}(x) = 0$, a.s. under $\mathbb{P}$, for any admissible x. This remarkable robustness in terms of path-wise zero regret is mainly due to the forward performance process that completely incorporates any unexpected model changes along real-time, a stability not shared by the stubborn behavior in general.

The stubborn investor, on the other hand, sticks to the fixed terminal criterion $U_T^S(x) = \frac{x^\gamma}{\gamma}$, for $0 < \gamma < 1$, when she makes decisions in real-time. At terminal time $t = T$, the Merton's problem in hindsight is also solved under $U_T^S(\cdot)$. The procedure to obtain the the value functions is almost the same as that for the forward behavior demonstrated earlier, with the only difference arising in the terminal criterion. Indeed, the value function over $(\tau_1, T]$, given $M^1_{(\tau_1,T]} = \{\mu_1, \sigma_1, \lambda_1\} \in \mathcal{F}_{\tau_1}$, is

$$\widetilde{V}(x,t) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\lambda_1^2\gamma}{2(\gamma-1)}(t-T)\right),$$

whereas the value function over $[0, \tau_1]$ is

$$\widetilde{V}(x,t) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda^2 t + (\lambda_1^2 - \lambda^2)\tau_1 - \lambda_1^2 T\right)\right).$$

We hence obtain

$$\widetilde{V}(x,0) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(\gamma-1)}\left((\lambda_1^2 - \lambda^2)\tau_1 - \lambda_1^2 T\right)\right), \tag{2.33}$$

a quantity depending on both λ and λ_1 due to the backward model commitment nature of classical approach as expected. On the other hand, the first term $\mathbb{E}[U_T^S(X_T^{S,*})|\mathcal{M}^1_{(\tau_1,T]}]$ in definition (2.29) is computed under the true physical measure $\mathbb{P}$, given the knowledge of realized parameters $\mathcal{M}_{(\tau_1,T]}$. Unless we have more specific knowledge about the correlation between λ_1 and the underlying Brownian motion under the true physical measure $\mathbb{P}$, we cannot have explicit result for such quantity. Nonetheless, it is easy to see that

$M_{[0,T]}(x)$ as defined in (2.29) is indeed a random variable with $M_{[0,T]}(x) \leq 0$ a.s. under $\mathbb{P}$, for all $x \geq 0$, since the policy that yields $X_T^{S,*}$ based on the stubborn behavior is only one admissible policy, and it does not necessarily coincide with the optimal policy in general, except for special situations. One of such situations is when λ_1 is independent of the Brownian motion under the genuine measure $\mathbb{P}$, for which we can actually conclude, based on the explicit computation (2.27) and (2.33), that $M_{[0,T]}(x) = 0$, a.s., for all $x \geq 0$. This can be seen as a degenerated case, since the optimal strategy induced by the stubborn behavior over $[0, T]$ is the same as that of the optimization problem with full knowledge in hindsight. Such degeneracy arises due to the optimality of *myopic* strategy for Merton's problem under power utility, as well as the current formulation of model knowledge that is revealed *locally* in real-time. In general, however, by the definition for the performance regret (2.29), it is expected that $M_{[0,T]}(x) \leq 0$, a.s. under $\mathbb{P}$, for each admissible x, under the stubborn behavior, whereas for the forward behavior, as we have shown, $M_{[0,T]}(x) = 0$, a.s..

2.5 The forward bridge problem

In the previous two sections, we have seen the respective advantages and disadvantages of the two approaches in the model revision setting. Specifically, the stubborn approach maintains a fixed objective $U_T(x) \in \mathcal{F}_0$, regardless of any unanticipated model changes in the future. This commitment may be sometimes desirable when an investor targets at a certain investment goal. However, this stubbornness gives rise to time-inconsistent investment behavior with ill-defined optimality across different time periods, and volatile investment performance as shown in previous sections. The ill-posedness of the stochastic optimization problem depicts as the following. The $t = 0$ optimal control rule for the second sub-horizon $(\tau_1, T]$ is no longer optimal when the investor reconsiders the optimization problem at $t = \tau_1$, and also, at $t = \tau_1$, the $t = 0$ optimal control rule for the first sub-horizon $[0, \tau_1]$, reassessed under the updated model knowledge at $t = \tau_1$, turns out to be actually suboptimal. Such future and past inconsistency indicate the failure of classical optimization approach, in that a decision made today for the remote future would inevitably be revised when the future comes, and an investor would inevitably regret both her decisions for the past and those for the future at *each* time instant. The forward approach, on the other hand, leads to a well-defined optimization problem as time enfolds, i.e., the decision made in the past is still optimal as the investor gains more new information. It also relieves the investor from making decisions for the remote future, as she is no longer committed to an optimization objective at the future time T, and therefore, the (probably vague and inaccurate) specification of any inflexible model for the far future becomes unnecessary. The forward approach, however, achieves these desirable flexibilities at the cost of abandoning a fixed target $U_T(x) \in \mathcal{F}_0$ that is specified at $t = 0$. Dynamically changing one's objective in a consistent way may be reasonable in real world where model knowledge at $t = 0$ is typically insufficient for decision making for long-term, but as mentioned earlier, stubbornness is required in some scenarios. It is the purpose of this section to reconcile the two approaches, and therefore, to have the advantages of both the two optimization approaches. Specifically, we will construct a consistent forward performance process that ultimately recovers the original objective $U_T(x) \in \mathcal{F}_0$ when time reaches $t = T$. As we have seen in the previous section, the zero volatility forward process in general cannot achieve such reconciliation (except for special cases, e.g., see Remark 2.5.2), and hence, it is necessary to consider general non-zero volatility forward processes. We will provide results for constructing such forward performance processes in the power, exponential and logarithmic utility scenarios under suitable conditions on the market parameters.

2.5.1 Power utility case

As demonstrated in section 3, the forward investor starts at $t = 0$ from the value function $\widehat{V}(x, 0; \lambda, \widehat{\lambda})$ and fully recovers the performance and optimal portfolio and wealth processes of the stubborn investor up to $t = \tau_1$. In particular, the forward criterion at $t = \tau_1$ is $U^F_{\tau_1}(x) = \widehat{V}\left(x, \tau_1; \lambda, \widehat{\lambda}\right)$ for all $x \geq 0$. Similar as before, at $t = \tau_1$, the goal is to determine a forward performance process $U^F_t(x)$, for $\tau_1 < t \leq T$, that satisfies the martingale (supermartingale, respectively) property along the optimal wealth process (along any admissible wealth process, respectively), as well as the two "bridge" conditions $U^F_{\tau_1}(x) = \widehat{V}\left(x, \tau_1; \lambda, \widehat{\lambda}\right)$ and $U^F_T(x) = U_T(x)$. We refer to this problem as the *forward bridge problem.* In this section, we restrict the filtration $\mathcal{F}_t$, $0 \leq t \leq T$, to be the filtration generated by the Brownian motion W_t, $0 \leq t \leq T$, satisfying the usual conditions. The following proposition gives a sufficient condition to construct such a forward performance process in the power utility scenario.

Proposition 2.5.1. *Let μ_t, σ_t and λ_t, $\tau_1 \leq t \leq T$, be the (conditional) model parameter processes for the second sub-horizon $[\tau_1, T]$, and denote by $\widetilde{W}_t = W_t - W_{\tau_1}$ the standard Brownian motion for $\tau_1 \leq t \leq T$, conditional on $\mathcal{F}_{\tau_1}$. Suppose there exists a progressively measurable process a_t^f, $\tau_1 \leq t \leq T$, such that the (conditional) stochastic differential equation (SDE)*

$$df_t = \left(\frac{\gamma}{2(\gamma-1)}(\lambda_t + a_t^f)^2 - \frac{(a_t^f)^2}{2} \right) dt + a_t^f d\widetilde{W}_t, \quad \tau_1 < t < T, \tag{2.34}$$

with $f_{\tau_1} = \frac{\gamma\widehat{\lambda}^2}{2(1-\gamma)}(T-\tau_1)$ and $f_T = 0$ is well defined and has a strong solution $f_t \in \mathcal{F}_t$, for $\tau_1 \leq t \leq T$. Then the forward bridge problem for a terminal power utility $U_T^F(x) = \frac{x^\gamma}{\gamma}$ has a solution

$$U_t^F(x) = \frac{x^\gamma}{\gamma} e^{f_t},$$

for $\tau_1 \leq t \leq T$.

Proof. In the Itô's diffusion market considered herein, it is reasonable to conjecture that the forward performance process with non-zero volatility satisfies

$$dU_t^F(x) = b(x,t)dt + a(x,t)d\widetilde{W}_t,$$

for $\tau_1 < t < T$, conditional on $\mathcal{F}_{\tau_1}$. Recall also the wealth dynamics

$$dX_t = \mu_t \pi_t dt + \sigma_t \pi_t d\widetilde{W}_t,$$

with $X_{\tau_1} = x \geq 0$ over the same sub-horizon, conditional on $\mathcal{F}_{\tau_1}$. Then, under suitable conditions, standard argument (see, e.g., [46]) suggests that $U_t^F(x)$ is the solution to the fully nonlinear stochastic partial differential equation (SPDE)

$$dU_t^F(x) = \frac{\left(\mu_t \frac{\partial U_t^F(x)}{\partial x} + \sigma_t \frac{\partial a(x,t)}{\partial x}\right)^2}{2\sigma_t^2 \frac{\partial^2 U_t(x)}{\partial x^2}} dt + a(x,t) d\widetilde{W}_t, \tag{2.35}$$

with initial condition $U_{\tau_1}^F(x) = \widehat{V}\left(x, \tau_1; \lambda, \widehat{\lambda}\right) = \frac{x^\gamma}{\gamma} \exp\left(\frac{\gamma \widehat{\lambda}^2}{2(1-\gamma)}(T - \tau_1)\right)$ and terminal condition $U_T^F(x) = \frac{x^\gamma}{\gamma}$. The power utility types of boundary conditions suggest the scaling property in the spatial variable, leading to a candidate forward performance process $U_t^F(x) = \frac{x^\gamma}{\gamma} e^{f_t}$ for some $\mathcal{F}_t$-adapted process f_t, $\tau_1 \leq t \leq T$. Notice that such forward performance process is indeed strictly increasing and strictly concave in x at each time $\tau_1 \leq t \leq T$, satisfying Inada's conditions. Now we further assume that the process f_t has the Itô's decomposition

$$df_t = b_t^f dt + a_t^f d\widetilde{W}_t,$$

for some admissible processes a_t^f and b_t^f such that the above diffusion process is well defined. Then direct computation yields that

$$\begin{aligned} dU_t^F(x) &= \frac{x^\gamma}{\gamma}\left(e^{f_t} df_t + \frac{1}{2} e^{f_t} \left(a_t^f\right)^2 dt\right) \\ &= \frac{x^\gamma}{\gamma} e^{f_t} \left(\left(b_t^f + \frac{1}{2}(a_t^f)^2\right) dt + a_t^f d\widetilde{W}_t\right) &(2.36) \\ &= \frac{\left(x^{\gamma-1} e^{f_t} \mu_t + a_x(x,t)\sigma_t\right)^2}{2(\gamma-1)x^{\gamma-2} e^{f_t} \sigma_t^2} dt + a(x,t) d\widetilde{W}_t. &(2.37) \end{aligned}$$

Now comparing the volatility parts of expressions (2.36) and (2.37), we obtain

$$a(x,t) = \frac{x^\gamma}{\gamma} e^{f_t} a_t^f,$$

and comparison of the drift parts yields

$$\frac{x^\gamma}{\gamma} e^{f_t} \left(b_t^f + \frac{1}{2}(a_t^f)^2 \right) = \frac{x^\gamma}{2(\gamma-1)} e^{f_t} \left(\lambda_t + a_t^f \right)^2 ,$$

which leads to $b_t^f = \frac{\gamma}{2(\gamma-1)} \left(\lambda_t + a_t^f \right)^2 - \frac{(a_t^f)^2}{2}$. The construction of the solution to the forward bridge problem therefore boils down to looking for a_t^f, $\tau_1 \le t \le T$, an admissible volatility process for the process f_t, such that the SDE (2.34) has an $\mathcal{F}_t$-adapted well defined strong solution, with the two boundary conditions $f_{\tau_1} = \frac{\gamma \widehat{\lambda}^2}{2(1-\gamma)}(T-\tau_1)$ and $f_T = 0$ being satisfied. This completes the proof for the power utility scenario. □

Proposition 2.5.1 claims that, once a solution to the SDE (2.34) is found, a forward performance process that recovers $U_T(x)$ at $t = T$ exists, and therefore, the stubborn approach and the forward approach reconcile. We next provide sufficient conditions on the parameter processes to prove the existence and uniqueness of the strong solution to the SDE (2.34), and therefore give a full characterization of the solution to the forward bridge problem in the power utility scenario.

Proposition 2.5.2. *Let the (conditional) SDE for the forward bridge problem be given by (2.34), and the process λ_t, $\tau_1 \le t \le T$, be uniformly bounded in (t, ω) and satisfy*

$$\exp\left(\frac{\gamma \widehat{\lambda}^2}{2(1-\gamma)^2}(T-\tau_1) \right) =$$

$$\mathbb{E}_{\mathbb{P}} \left[\exp\left(\frac{\gamma}{2(1-\gamma)} \int_{\tau_1}^{T} \lambda_s^2 ds + \frac{\gamma}{1-\gamma} \int_{\tau_1}^{T} \lambda_s d\widetilde{W}_s \right) \Big| \mathcal{F}_{\tau_1} \right]. \qquad (2.38)$$

Then, the SDE (2.34) has a unique uniformly bounded strong solution given by

$$f_t = (1-\gamma)\ln \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\gamma}{2(1-\gamma)}\int_t^T \lambda_s^2 ds + \frac{\gamma}{1-\gamma}\int_t^T \lambda_s d\widetilde{W}_s\right)\middle|\mathcal{F}_t\right], \quad (2.39)$$

for $\tau_1 \leq t \leq T$*, with* $\int_{\tau_1}^t a_s^f d\widetilde{W}_s$*,* $\tau_1 \leq t \leq T$*, being a BMO (Bounded Mean Oscillation) martingale under the measure* $\mathbb{P}$*.*

Proof. The SDE (2.34) can be seen as a quadratic backward stochastic differential equaiton (BSDE) with an extra initial condition. Well established result on the existence and uniqueness of the solution (f, a^f) to quadratic BSDEs applies here (see, e.g., Chapter 10 of [59]). Indeed, by the assumption that the process λ_t, $\tau_1 \leq t \leq T$, is uniformly bounded, conditional on $\mathcal{F}_{\tau_1}$, there exists a unique solution (f, a^f) to the BSDE

$$df_t = \left(\frac{\gamma}{2(\gamma-1)}(\lambda_t + a_t^f)^2 - \frac{(a_t^f)^2}{2}\right)dt + a_t^f d\widetilde{W}_t, \quad \tau_1 < t < T, \quad f_T = 0,$$

such that f is uniformly bounded and $\left(\int_{\tau_1}^{\cdot} a_t^f d\widetilde{W}_t\right)$ is a BMO martingale under $\mathbb{P}$. Our next step is to identify the process f with the explicit representation in (2.39). To this end, we first, by a change of measure, reduce the (conditional) SDE (2.34) to

$$\begin{aligned} df_t &= \left(\frac{\gamma}{2(\gamma-1)}\lambda_t^2 + \frac{\gamma}{\gamma-1}\lambda_t a_t^f + \frac{1}{2(\gamma-1)}\left(a_t^f\right)^2\right)dt + a_t^f d\widetilde{W}_t \\ &= \left(\frac{\gamma}{2(\gamma-1)}\lambda_t^2 + \frac{1}{2(\gamma-1)}\left(a_t^f\right)^2\right)dt + a_t^f d\widetilde{W}_t^{\mathbb{Q}}, \end{aligned}$$

where $d\widetilde{W}_t^{\mathbb{Q}} = d\widetilde{W}_t + \frac{\gamma}{\gamma-1}\lambda_t dt$. Under the assumption that λ_t, $\tau_1 < t \leq T$, is uniformly bounded, the Novikov's condition applies, and the process $\widetilde{W}_t^{\mathbb{Q}} =$

$\widetilde{W}_t + \int_{\tau_1}^{t} \frac{\gamma}{\gamma-1}\lambda_s ds$, $\tau_1 \le t \le T$, is a Brownian motion under the measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\int_{\tau_1}^{T} \frac{\gamma}{\gamma-1}\lambda_s d\widetilde{W}_s - \frac{1}{2}\int_{\tau_1}^{T}\left(\frac{\gamma}{\gamma-1}\right)^2 \lambda_s^2 ds\right), \qquad (2.40)$$

with $\widetilde{W}_{\tau_1}^{\mathbb{Q}} = 0$. Next, we define the process $\widetilde{f}_t = f_t - \int_{\tau_1}^{t} \frac{\gamma}{2(\gamma-1)}\lambda_s^2 ds$. Then, the new process $\widetilde{f}$ is also uniformly bounded and satisfies the quadratic BSDE

$$d\widetilde{f}_t = \frac{1}{2(\gamma-1)}\left(a_t^f\right)^2 dt + a_t^f d\widetilde{W}_t^{\mathbb{Q}}, \qquad (2.41)$$

with terminal condition $\widetilde{f}_T = -\int_{\tau_1}^{T} \frac{\gamma}{2(\gamma-1)}\lambda_s^2 ds$. Notice that $\widetilde{f}_T$ is bounded, and hence by the property of quadratic BSDE with bounded terminal condition (see, e.g., Lemma 10.2 of [59]), we can claim that $\left(\int_{\tau_1}^{\cdot} a_t^f d\widetilde{W}_t^{\mathbb{Q}}\right)$ is a BMO martingale under the measure $\mathbb{Q}$, due to that $\widetilde{f}$ is uniformly bounded and it solves the BSDE (2.41) by construction. Applying Itô's lemma then yields

$$\begin{aligned} d\left(e^{\frac{1}{1-\gamma}\widetilde{f}_t}\right) &= \frac{1}{1-\gamma}e^{\frac{1}{1-\gamma}\widetilde{f}_t}\left(\frac{1}{2(r-1)}\left(a_t^f\right)^2 dt + a_t^f d\widetilde{W}_t^{\mathbb{Q}}\right) + \frac{1}{2(1-\gamma)^2}e^{\frac{1}{1-\gamma}\widetilde{f}_t}\left(a_t^f\right)^2 dt \\ &= \frac{1}{1-\gamma}e^{\frac{1}{1-\gamma}\widetilde{f}_t} a_t^f d\widetilde{W}_t^{\mathbb{Q}}. \end{aligned}$$

Since $\widetilde{f}$ is uniformly bounded and $\left(\int_{\tau_1}^{\cdot} a_t^f d\widetilde{W}_t^{\mathbb{Q}}\right)$ is a BMO martingale under $\mathbb{Q}$ (hence also square integrable), we claim that the process $e^{\frac{1}{1-\gamma}\widetilde{f}_t}$, $\tau_1 \le t \le T$, is a genuine (square integrable) martingale under $\mathbb{Q}$. It hence follows that

$$\begin{aligned} e^{\frac{1}{1-\gamma}\widetilde{f}_t} &= \mathbb{E}_{\mathbb{Q}}\left[e^{\int_{\tau_1}^{T} \frac{\gamma}{2(1-\gamma)^2}\lambda_s^2 ds}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[e^{\int_{\tau_1}^{T} \frac{\gamma}{2(1-\gamma)^2}\lambda_s^2 ds} e^{-\int_t^T \frac{\gamma}{\gamma-1}\lambda_s d\widetilde{W}_s - \frac{1}{2}\int_t^T\left(\frac{\gamma}{\gamma-1}\right)^2\lambda_s^2 ds}\middle|\mathcal{F}_t\right] \end{aligned}$$

$$= \mathbb{E}_{\mathbb{P}} \left[\exp \left(\frac{\gamma}{2(1-\gamma)} \int_t^T \lambda_s^2 ds + \frac{\gamma}{1-\gamma} \int_t^T \lambda_s d\widetilde{W}_s + \frac{\gamma}{2(1-\gamma)^2} \int_{\tau_1}^t \lambda_s^2 ds \right) \Big| \mathcal{F}_t \right],$$

which gives rise to

$$\widetilde{f}_t = (1-\gamma) \ln \mathbb{E}_{\mathbb{P}} \Big[\exp \Big(\frac{\gamma}{2(1-\gamma)} \int_t^T \lambda_s^2 ds + \frac{\gamma}{1-\gamma} \int_t^T \lambda_s d\widetilde{W}_s + \frac{\gamma}{2(1-\gamma)^2} \int_{\tau_1}^t \lambda_s^2 ds \Big) \Big| \mathcal{F}_t \Big],$$

for $\tau_1 \leq t \leq T$. Under assumption (2.38) on the parameter process λ_t, $\tau_1 \leq t \leq T$, we can easily verify that the initial condition $\widetilde{f}_{\tau_1} = \frac{\gamma \widehat{\lambda}^2}{2(1-\gamma)}(T-\tau_1)$ is automatically satisfied by the process $\widetilde{f}$. Finally, the unique uniformly bounded strong solution can be derived as

$$f_t = (1-\gamma) \ln \mathbb{E}_{\mathbb{P}} \left[\exp \left(\frac{\gamma}{2(1-\gamma)} \int_t^T \lambda_s^2 ds + \frac{\gamma}{1-\gamma} \int_t^T \lambda_s d\widetilde{W}_s \right) \Big| \mathcal{F}_t \right].$$

□

Remark 2.5.1. A further look at the condition (2.38) for the parameter process λ yields that, under the measure $\mathbb{Q}$ defined in (2.40),

$$\mathbb{E}_{\mathbb{P}} \left[\exp \left(\frac{\gamma}{2(1-\gamma)} \int_{\tau_1}^T \lambda_s^2 ds + \frac{\gamma}{1-\gamma} \int_{\tau_1}^T \lambda_s d\widetilde{W}_s \right) \Big| \mathcal{F}_{\tau_1} \right]$$

$$= \mathbb{E}_{\mathbb{Q}} \left[\exp \left(\frac{\gamma}{2(1-\gamma)^2} \int_{\tau_1}^T \lambda_s^2 ds \right) \Big| \mathcal{F}_{\tau_1} \right] = \exp \left(\frac{\gamma \widehat{\lambda}^2}{2(1-\gamma)^2} (T-\tau_1) \right).$$

It hence leads to that

$$\mathbb{E}_{\mathbb{Q}} \left[\exp \left(\frac{\gamma}{2(1-\gamma)^2} \int_{\tau_1}^T \left(\lambda_s^2 - \widehat{\lambda}^2 \right) ds \right) \Big| \mathcal{F}_{\tau_1} \right] = 1. \tag{2.42}$$

The equivalent condition (2.42) implies that, under the measure $\mathbb{Q}$, the discrepancy process $\Delta_s := \lambda_s^2 - \widehat{\lambda}^2$, $\tau_1 \leq s \leq T$, conditional on $\mathcal{F}_{\tau_1}$, should not deviate from zero too much on average under exponential weighting. See a more detailed discussion for the deterministic parameter case in Remark 2.5.2.

We now summarize the results for the forward bridge problem in the power utility setting, and also provide the verification theorem to show that the process $U_t^F(x)$, $\tau_1 \leq t \leq T$, we have constructed is indeed a forward performance process.

Theorem 2.5.3. *Suppose that the process λ_t, $\tau_1 \leq t \leq T$, is uniformly bounded and satisfies the condition (2.38). Let (f, a^f) be the unique solution to the (conditional) SDE (2.34) with f being uniformly bounded. Then, the process $U_t^F(x) = \frac{x^\gamma}{\gamma} e^{f_t}$, $\tau_1 \leq t \leq T$, is a forward performance process that achieves power utility $U_T^F(x) = \frac{x^\gamma}{\gamma}$ at terminal time $t = T$, where f_t, $\tau_1 \leq t \leq T$, is given by (2.39). The forward performance process in addition satisfies the following Itô's decomposition*

$$dU_t^F(x) = \frac{x^\gamma e^{f_t}\left(\lambda_t + a_t^f\right)^2}{2(\gamma-1)}dt + \frac{x^\gamma e^{f_t} a_t^f}{\gamma} d\widetilde{W}_t, \quad \tau_1 < t < T, \tag{2.43}$$

with initial and terminal conditions being $U_{\tau_1}^F(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma\widehat{\lambda}^2}{2(1-\gamma)}(T-\tau_1)\right)$ and $U_T^F(x) = \frac{x^\gamma}{\gamma}$, respectively. The optimal investment strategy is

$$\pi_t^* = -\frac{\lambda_t + a_t^f}{(\gamma-1)\sigma_t} X_t^*, \quad \tau_1 \leq t \leq T, \tag{2.44}$$

with the optimal wealth process being

$$X_t^* = X_{\tau_1}^* \exp\left(\int_{\tau_1}^t \frac{\left((1-2\gamma)\lambda_s - a_s^f\right)\left(\lambda_s + a_s^f\right)}{2(1-\gamma)^2} ds + \int_{\tau_1}^t \frac{\lambda_s + a_s^f}{1-\gamma} d\widetilde{W}_s\right), \tag{2.45}$$

for $\tau_1 \le t \le T$. *Here,* $X^*_{\tau_1} = x\exp\left(\frac{(1-2\gamma)\widehat{\lambda}^2}{2(1-\gamma)^2}\tau_1 + \frac{\widehat{\lambda}}{1-\gamma}W_{\tau_1}\right)$ *is the optimal wealth at* $t = \tau_1$.

Proof. It is easy to see that $U^F_t(x) = \frac{x^\gamma}{\gamma}e^{f_t}$ is $\mathcal{F}_t$-adapted, and also, for each fixed $t \in [\tau_1, T]$, the mapping $x \mapsto U^F_t(x)$ is strictly increasing and strictly concave, almost surely, with Inada's conditions being satisfied. One can also directly verify that the initial and terminal conditions for the bridge problem are satisfied by the process $U^F_t(x)$ at time $t = \tau_1$ and $t = T$, respectively, based on the expression (2.39) for f_t and the condition (2.38) on λ_t. It hence remains to prove that for any wealth process X_t generated by admissible policy π_t, $\mathbb{E}\left[U^F_s(X_s)|\mathcal{F}_t\right] \le U^F_t(X_t)$, for $\tau_1 \le t \le s \le T$, and for the wealth process X^*_t given by (2.45), $\mathbb{E}\left[U^F_s(X^*_s)|\mathcal{F}_t\right] = U^F_t(X^*_t)$, for $\tau_1 \le t \le s \le T$. To this end, we first notice that the process $U^F_t(x) = \frac{x^\gamma}{\gamma}e^{f_t}$ has the Itô's decomposition

$$dU^F_t(x) = \frac{x^\gamma}{\gamma}e^{f_t}\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda_t + a^f_t\right)^2\right)dt + \frac{x^\gamma}{\gamma}e^{f_t}a^f_t d\widetilde{W}_t,$$

since the process f_t is the unique uniformly bounded solution to the SDE (2.34) by Proposition 2.5.2. Moreover, the process $U^F_t(x)$ is smooth enough so that the Itô-Ventzel's formula can be applied to yield, for any wealth process X_t generated by admissible policy $\pi_t \in \mathcal{A}$,

$$dU^F_t(X_t) = \frac{X^\gamma_t}{\gamma}e^{f_t}\left(\frac{\gamma}{2(\gamma-1)}\left(\lambda_t + a^f_t\right)^2\right)dt + \frac{X^\gamma_t}{\gamma}e^{f_t}a^f_t d\widetilde{W}_t$$

$$+X^{\gamma-1}_t e^{f_t}dX_t + \frac{1}{2}(\gamma-1)X^{\gamma-2}_t e^{f_t}\langle dX_t\rangle + \left\langle \frac{\partial}{\partial x}\left(\frac{x^\gamma}{\gamma}e^{f_t}a^f_t\right)d\widetilde{W}_t, dX_t\right\rangle\Big|_{x=X_t}$$

$$= \left(-\frac{1-\gamma}{2}\sigma^2_t\pi^2_t + \left(\mu_t + \sigma_t a^f_t\right)X_t\pi_t + \frac{X^2_t}{2(\gamma-1)}\left(\lambda_t + a^f_t\right)^2\right)X^{\gamma-2}_t e^{f_t}dt$$

$$+\frac{X_t^{\gamma-1}e^{f_t}}{\gamma}\left(X_t a_t^f + \gamma\sigma_t\pi_t\right)d\widetilde{W}_t$$

$$= -\frac{1-\gamma}{2}\left(\sigma_t\pi_t + \frac{\lambda_t + a_t^f}{\gamma-1}X_t\right)^2 X_t^{\gamma-2}e^{f_t}dt + \frac{X_t^{\gamma-1}e^{f_t}}{\gamma}\left(X_t a_t^f + \gamma\sigma_t\pi_t\right)d\widetilde{W}_t.$$

Note that $-\frac{1-\gamma}{2} < 0$ and hence the drift of the process $U_t^F(X_t)$, $\tau_1 \leq t \leq T$, would achieve maximum value zero at $\pi_t^* = -\frac{\lambda_t + a_t^f}{(\gamma-1)\sigma_t}X_t^*$, where X_t^* is the wealth process generated under such policy π_t^*, for $\tau_1 \leq t \leq T$. Define

$$\tau^n := \inf\left\{s \geq t : \int_t^s \Big|\frac{X_u^{\gamma-1}e^{f_u}}{\gamma}\left(X_u a_u^f + \gamma\sigma_u\pi_u\right)\Big|^2 du \geq n\right\} \wedge T.$$

Then, it holds that for any admissible $\pi_t \in \mathcal{A}$,

$$\mathbb{E}\left[U_{s\wedge\tau^n}^F(X_{s\wedge\tau^n})\Big|\mathcal{F}_t\right] \leq U_t^F(X_t), \quad \tau_1 \leq t \leq s \leq T. \tag{2.46}$$

Notice that $\mathbb{E}\left[\sup_{\tau_1\leq t\leq T}|X_t|^\gamma\right] \leq \mathbb{E}\left[\sup_{\tau_1\leq t\leq T}|X_t|^2\right] + (T-\tau_1)$, and the fact that for any admissible $\pi_t \in \mathcal{A}$, the following estimate holds

$$\mathbb{E}\left[\sup_{\tau_1\leq t\leq T}|X_t|^2\right] \leq \mathbb{E}\left[\sup_{\tau_1\leq t\leq T}\left|X_{\tau_1}^* + \int_{\tau_1}^t \mu_t\pi_t dt + \int_{\tau_1}^t \sigma_t\pi_t d\widetilde{W}_t\right|^2\right]$$

$$\leq 3\left(\mathbb{E}|X_{\tau_1}^*|^2 + (T-\tau_1)\mathbb{E}\left[\int_{\tau_1}^T |\mu_t\pi_t|^2 dt\right] + \mathbb{E}\left[\sup_{\tau_1\leq t\leq T}\left|\int_{\tau_1}^t \sigma_t\pi_t d\widetilde{W}_t\right|^2\right]\right)$$

$$\leq 3\left(\mathbb{E}|X_{\tau_1}^*|^2 + (T-\tau_1)\mathbb{E}\left[\int_{\tau_1}^T |\mu_t\pi_t|^2 dt\right] + 4\mathbb{E}\left[\int_{\tau_1}^T |\sigma_t\pi_t|^2 dt\right]\right),$$

where we have applied Doob's maximal inequality. Since

$$\pi_t \in \mathcal{A} := \Big\{\pi : \pi_t \text{ is self-financing and } \mathcal{F}_t - \text{progressively measurable}$$

$$\text{with } \mathbb{E}\left[\int_{\tau_1}^T |\sigma_t\pi_t|^2 dt\right] < \infty \text{ and } X_t \geq 0,\ \tau_1 \leq t \leq T\Big\},$$

we have shown that $\mathbb{E}\left[\sup_{\tau_1 \le t \le T} |X_t|^2\right] < \infty$, under the assumption that the process λ_t, $\tau_1 \le t \le T$, is uniformly bounded. It hence follows that $\mathbb{E}\left[\sup_{\tau_1 \le t \le T} |X_t|^\gamma\right] < \infty$ as well. Let $n \to \infty$ in (2.46), and by dominated convergence theorem and recalling that the process f_t, $\tau_1 \le t \le T$, is uniformly bounded, we obtain $\mathbb{E}\left[U_s^F(X_s)\middle|\mathcal{F}_t\right] \le U_t^F(X_t)$, $\tau_1 \le t \le s \le T$, for any admissible $\pi_t \in \mathcal{A}$, and equality holds when X_t is replaced by X_t^* in (2.45) that is generated by the policy π_t^* in (2.44). The admissibility of the policy π_t^* given in (2.44) can be easily verified, following from the assumption that the parameter process λ_t is uniformly bounded, and the conclusion from Proposition 2.5.2 that $\int_{\tau_1}^t a_s^f d\widetilde{W}_s$ is a BMO martingale (hence square integrable) under $\mathbb{P}$. □

Remark 2.5.2. In the case when the process λ_t, $\tau_1 \le t \le T$, is deterministic, conditional on $\mathcal{F}_{\tau_1}$, then condition (2.38) reduces to

$$\frac{1}{T-\tau_1}\int_{\tau_1}^T \lambda_t^2 dt = \widehat{\lambda}^2, \quad \text{a.s.,} \tag{2.47}$$

indicating that the average of the process λ_t over the second sub-horizon should not be very different from the perceived parameter $\widehat{\lambda}$ for the same sub-horizon. Within this setting, the forward performance process that achieves power utility at terminal time $t = T$ is given by

$$U_t^F(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{\gamma}{2(1-\gamma)}\int_t^T \lambda_s^2 ds\right), \quad \tau \le t \le T,$$

whose Itô's decomposition is

$$dU_t^F(x) = \frac{x^\gamma e^{f_t}\lambda_t^2}{2(\gamma-1)}dt, \quad \tau \le t \le T$$

by (2.43). Notice that this is a time-decreasing zero volatility forward performance process, and it includes the special case when there is no unanticipated model switch at $t = \tau_1$, i.e., the case when $\lambda_t(\omega) = \widehat{\lambda}$, $\tau_1 \leq t \leq T$, for almost all $\omega \in \Omega$.

2.5.2 Exponential utility case

The forward bridge problem and the solution for the exponential utility $U(x) = -e^{-\gamma x}$, $\gamma > 0$, basically state in the same way as for the power utility case, except for a different constant appearing in the drift of the SDE (2.34). Indeed, following the similar argument as in the power utility setting, one can derive that at the model switching time $t = \tau_1$, an exponential utility investor has the forward criterion $U^F_{\tau_1}(x) = \widehat{V}\left(x, \tau_1; \lambda, \widehat{\lambda}\right) = -\exp\left(-\gamma x - \frac{\widehat{\lambda}^2}{2}(T-\tau_1)\right)$. The goal is then to construct the forward performance process $U^F_t(x)$, for $\tau_1 < t \leq T$, conditional on $\mathcal{F}_{\tau_1}$, while recovering the terminal exponential utility $U^F_T(x) = -e^{-\gamma x}$ as time reaches $t = T$. Argument similar to Proposition 2.5.1 states as following.

Proposition 2.5.4. *Let μ_t, σ_t and λ_t, $\tau_1 \leq t \leq T$, be the (conditional) model parameter processes for the second sub-horizon $[\tau_1, T]$, and denote by $\widetilde{W}_t = W_t - W_{\tau_1}$ the standard Brownian motion for $\tau_1 \leq t \leq T$, conditional on $\mathcal{F}_{\tau_1}$. Suppose there exists a progressively measurable process a^f_t, $\tau_1 \leq t \leq T$, such that the (conditional) stochastic differential equation*

$$df_t = \frac{1}{2}\left(\lambda_t^2 + 2\lambda_t a^f_t\right) dt + a^f_t d\widetilde{W}_t, \quad \tau_1 < t < T, \tag{2.48}$$

with $f_{\tau_1} = -\frac{\widehat{\lambda}^2}{2}(T-\tau_1)$ *and* $f_T = 0$ *is well defined and has a strong solution* $f_t \in \mathcal{F}_t$*, for* $\tau_1 \leq t \leq T$*. Then the forward bridge problem for a terminal exponential utility* $U_T^F(x) = -e^{-\gamma x}$ *has a solution*

$$U_t^F(x) = -e^{-\gamma x}e^{f_t},$$

for $\tau_1 \leq t \leq T$*.*

Proof. The proof for the exponential utility case is mostly identical to that of Proposition 2.5.1. Recall that in the current case, the SPDE satisfied by the forward performance process is given by (2.35), for $\tau_1 < t < T$, with initial and terminal conditions being $U_{\tau_1}^F(x) = \widehat{V}\left(x, \tau_1; \lambda, \widehat{\lambda}\right) = -\exp\left(-\gamma x - \frac{\widehat{\lambda}^2}{2}(T-\tau_1)\right)$ and $U_T^F(x) = -e^{-\gamma x}$, respectively. The exponential scaling in the boundary conditions suggests a candidate forward performance process $U_t^F(x) = -e^{-\gamma x}e^{f_t}$ for some $\mathcal{F}_t$-adapted process f_t, $\tau_1 \leq t \leq T$. Notice that such forward performance process is indeed strictly increasing and strictly concave in x and satisfies Inada' s conditions at each time $\tau_1 \leq t \leq T$. Now we further assume that the process f_t has the Itô's decomposition

$$df_t = b_t^f dt + a_t^f d\widetilde{W}_t,$$

for some admissible processes a_t^f and b_t^f such that the above diffusion process is well defined. Then direct computation yields that

$$\begin{aligned} dU_t^F(x) &= -e^{-\gamma x}\left(e^{f_t}df_t + \frac{1}{2}e^{f_t}\left(a_t^f\right)^2 dt\right) \\ &= -e^{-\gamma x}e^{f_t}\left(\left(b_t^f + \frac{1}{2}(a_t^f)^2\right)dt + a_t^f d\widetilde{W}_t\right) &\quad (2.49) \\ &= \frac{\left(\gamma e^{-\gamma x}e^{f_t}\mu_t + a_x(x,t)\sigma_t\right)^2}{-2\gamma^2 e^{-\gamma x}e^{f_t}\sigma_t^2}dt + a(x,t)d\widetilde{W}_t. &\quad (2.50) \end{aligned}$$

Now comparing the volatility parts of expression (2.49) and (2.50), we obtain

$$a(x,t) = -e^{-\gamma x} e^{f_t} a_t^f,$$

and comparison of the drift parts yields

$$-e^{-\gamma x} e^{f_t} \left(b_t^f + \frac{1}{2}(a_t^f)^2 \right) = \frac{-e^{-\gamma x} e^{f_t}}{2} \left(\lambda_t + a_t^f \right)^2,$$

which gives rise to $b_t^f = \frac{1}{2}\left(\lambda_t^2 + 2\lambda_t a_t^f\right)$. The construction of the forward bridge problem solution therefore boils down to looking for a_t^f, $\tau_1 \leq t \leq T$, an admissible volatility process for the process f_t, such that the SDE (2.48) has an $\mathcal{F}_t$-adapted well defined strong solution, with the two boundary conditions $f_{\tau_1} = -\frac{\widehat{\lambda}^2}{2}(T - \tau_1)$ and $f_T = 0$ being satisfied. This completes the proof for the exponential utility scenario. □

The existence and uniqueness of solution to the SDE (2.48) can be similarly handled as in Proposition 2.5.2. It is actually slightly easier in the current exponential utility case, since the involved BSDE has the affine generator instead of a quadratic generator. We hence resort to the well established results on such BSDEs (see, e.g., Chapter 9 of [59]) and obtain the following proposition.

Proposition 2.5.5. *Let the (conditional) SDE for the forward bridge problem be given by (2.48), and the process λ_t, $\tau_1 \leq t \leq T$, be uniformly bounded in (t, ω) and satisfy*

$$\mathbb{E}_{\mathbb{P}}\left[\int_{\tau_1}^T \frac{\lambda_s^2}{2} ds \exp\left(-\int_{\tau_1}^T \frac{\lambda_s^2}{2} ds - \int_{\tau_1}^T \lambda_s d\widetilde{W}_s \right) \middle| \mathcal{F}_{\tau_1} \right] = \frac{\widehat{\lambda}^2}{2}(T - \tau_1). \qquad (2.51)$$

Then, the SDE (2.48) has a unique solution that satisfies $\mathbb{E}_{\mathbb{P}}\left[\sup_{\tau_1 \le t \le T} |f_t|^2 | \mathcal{F}_{\tau_1}\right] < \infty$ *and it is given by*

$$f_t = \mathbb{E}_{\mathbb{P}}\left[-\int_{\tau_1}^{T} \frac{\lambda_s^2}{2} ds \exp\left(-\int_t^T \lambda_s d\widetilde{W}_s - \int_t^T \frac{\lambda_s^2}{2} ds\right) \Big| \mathcal{F}_t\right], \tag{2.52}$$

for $\tau_1 \le t \le T$, *with* $\int_{\tau_1}^t a_t^f d\widetilde{W}_t$, $\tau_1 \le t \le T$, *being a square integrable martingale under the measure* $\mathbb{P}$.

Proof. As in the proof of Proposition 2.5.2, we first examine the BSDE

$$df_t = \frac{1}{2}\left(\lambda_t^2 + 2\lambda_t a_t^f\right) dt + a_t^f d\widetilde{W}_t, \quad f_T = 0,$$

which has an affine generator. Under the assumption that λ_t, $\tau_1 \le t \le T$, is uniformly bounded, well established existence and uniqueness result leads to that there is a unique solution (f, a^f), with $\mathbb{E}_{\mathbb{P}}\left[\sup_{\tau_1 \le t \le T} |f_t|^2 | \mathcal{F}_{\tau_1}\right] < \infty$ and $\int_{\tau_1}^t a_t^f d\widetilde{W}_t$, $\tau_1 \le t \le T$, being a square integrable martingale under $\mathbb{P}$. Our next step is to identify this solution f_t with the expression in (2.52). We first, by a change of measure, turn the BSDE into

$$df_t = \frac{\lambda_t^2}{2} dt + a_t^f d\widetilde{W}_t^{\mathbb{Q}}, \quad f_T = 0,$$

with $\widetilde{W}_t^{\mathbb{Q}} := \widetilde{W}_t + \int_{\tau_1}^t \lambda_s ds$ being a standard Brownian motion under $\mathbb{Q}$, with $\widetilde{W}_{\tau_1}^{\mathbb{Q}} = 0$. Here, the equivalent measure $\mathbb{Q}$ is defined by, on $\mathcal{F}_T$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\left(-\int_{\tau_1}^T \lambda_s d\widetilde{W}_s - \frac{1}{2}\int_{\tau_1}^T \lambda_s^2 ds\right).$$

Next, let $\widetilde{f}_t := f_t - \int_{\tau_1}^t \frac{\lambda_s^2}{2} ds$, then $\widetilde{f}_t$, $\tau_1 \le t \le T$, solves the BSDE

$$d\widetilde{f}_t = a_t^f d\widetilde{W}_t^{\mathbb{Q}}, \quad \widetilde{f}_T = -\int_{\tau_1}^T \frac{\lambda_s^2}{2} ds.$$

By existence and uniqueness of solution to the above BSDE, we claim that $(\widetilde{f}, a^f)$ is the unique solution with $\int_{\tau_1}^{\cdot} a_t^f d\widetilde{W}_t^{\mathbb{Q}}$ being a square integrable martingale under measure $\mathbb{Q}$. Thus, $\widetilde{f}$ is a genuine martingale, and it has representation

$$\widetilde{f}_t = \mathbb{E}_{\mathbb{Q}}\left[-\int_{\tau_1}^{T} \frac{\lambda_s^2}{2} ds \Big| \mathcal{F}_t \right]$$
$$= \mathbb{E}_{\mathbb{P}}\left[-\int_{\tau_1}^{T} \frac{\lambda_s^2}{2} ds \exp\left(-\int_t^T \lambda_s d\widetilde{W}_s - \int_t^T \frac{\lambda_s^2}{2} ds \right) \Big| \mathcal{F}_t \right].$$

Direct verification gives that under assumption (2.51), the initial condition $f_{\tau_1} = -\frac{\widehat{\lambda}^2}{2}(T - \tau_1)$ is satisfied. □

Remark 2.5.3. Similar to Remark 2.5.1, we could rewrite the condition (2.51) under the measure $\mathbb{Q}$, and hence obtain

$$\mathbb{E}_{\mathbb{P}}\left[\int_{\tau_1}^{T} \frac{\lambda_s^2}{2} ds \exp\left(-\int_{\tau_1}^T \frac{\lambda_s^2}{2} ds - \int_{\tau_1}^T \lambda_s d\widetilde{W}_s \right) \Big| \mathcal{F}_{\tau_1} \right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{T} \frac{\lambda_s^2}{2} ds \Big| \mathcal{F}_{\tau_1} \right] = \frac{\widehat{\lambda}^2}{2}(T - \tau_1),$$

which leads to

$$\mathbb{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{T} \frac{\lambda_s^2 - \widehat{\lambda}^2}{2} ds \Big| \mathcal{F}_{\tau_1} \right] = 0, \quad \text{a.s..} \tag{2.53}$$

The interpretation of the condition (2.53) is similar to that of (2.42), i.e., the discrepancy process $\Delta_s = \lambda_s^2 - \widehat{\lambda}^2$, $\tau_1 \leq s \leq T$, on average should be zero. Note that, however, it is different from the power utility case (2.42), as there is no exponential weighting due to the risk aversion parameter on such average.

We next summarize the results for the exponential utility forward bridge problem in the following theorem, whose proof is similar to that of Theorem 2.5.3 and hence is omitted.

Theorem 2.5.6. *Suppose that the process λ_t, $\tau_1 \le t \le T$, is uniformly bounded and satisfies the condition (2.51). Let (f, a^f) be the unique solution to the (conditional) SDE (2.48) with f being given by (2.52). Then, the process $U_t^F(x) = -e^{-\gamma x} e^{f_t}$, $\tau_1 \le t \le T$, is a forward performance process that achieves exponential utility $U_T^F(x) = -e^{-\gamma x}$ at terminal time $t = T$. The forward performance process in addition satisfies the following Itô's decomposition*

$$dU_t^F(x) = -e^{-\gamma x} e^{f_t} \frac{(\lambda_t + a_t^f)^2}{2} dt - e^{-\gamma x} e^{f_t} a_t^f d\widetilde{W}_t, \quad \tau_1 < t < T, \tag{2.54}$$

with initial and terminal conditions being $U_{\tau_1}^F(x) = -\exp\left(-\gamma x - \frac{\widehat{\lambda}^2}{2}(T - \tau_1)\right)$ and $U_T^F(x) = -e^{-\gamma x}$, respectively. The optimal investment strategy is

$$\pi_t^* = \frac{\lambda_t + a_t^f}{\gamma \sigma_t}, \quad \tau_1 \le t \le T, \tag{2.55}$$

with the optimal wealth process being

$$X_t^* = X_{\tau_1}^* + \int_{\tau_1}^t \frac{\lambda_s^2 + \lambda_s a_s^f}{\gamma} ds + \int_{\tau_1}^t \frac{\lambda_s + a_s^f}{\gamma} d\widetilde{W}_s, \tag{2.56}$$

for $\tau_1 \le t \le T$. Here, $X_{\tau_1}^*$ is the optimal wealth at $t = \tau_1$.

2.5.3 Logarithmic utility case

In the next proposition, we provide the result for the logarithmic utility scenario, where a condition for the parameter process λ_t, $\tau_1 \le t \le T$, based on the Martingale Representation Theorem is also needed.

Proposition 2.5.7. *Let μ_t, σ_t and λ_t, $\tau_1 \le t \le T$, be the (conditional) model parameter processes for the second second sub-horizon $[\tau_1, T]$, and denote by*

$\widetilde{W}_t = W_t - W_{\tau_1}$ *the standard Brownian motion for* $\tau_1 \leq t \leq T$*, conditional on* $\mathcal{F}_{\tau_1}$*. If* $\int_{\tau_1}^{T} \frac{\lambda_t^2}{2} dt$ *is square integrable, and that*

$$\mathbb{E}\left[\int_{\tau_1}^{T} \frac{\lambda_t^2}{2} dt \middle| \mathcal{F}_{\tau_1}\right] = \frac{\widehat{\lambda}^2}{2}(T - \tau_1), \ \ a.s.. \tag{2.57}$$

Then the forward bridge problem for a terminal logarithmic utility $U_T^F(x) = \ln x$*,* $x > 0$*, has a solution*

$$U_t^F(x) = \ln x + \frac{\widehat{\lambda}^2}{2}(T - \tau_1) - \int_{\tau_1}^{t} \frac{\lambda_s^2}{2} ds + \int_{\tau_1}^{t} a_s^f d\widetilde{W}_s, \tag{2.58}$$

for $\tau_1 \leq t \leq T$*, with* $\int_{\tau_1}^{t} a_s^f d\widetilde{W}_s$*,* $\tau_1 \leq t \leq T$*, being a square integrable martingale under the measure* $\mathbb{P}$*.*

Proof. The proof is mostly similar to the one for Proposition 2.5.1. We therefore only highlight the main differences. First, similar to the power and exponential utility scenarios, over the first sub-horizon $[0, \tau_1]$, the zero volatility forward performance process fully recovers the value function process given by the stubborn approach under the $t = 0$ perceived model $\widehat{\mathbb{P}}$. In particular, $U_{\tau_1}^F(x) = \widehat{V}\left(x, \tau_1; \lambda, \widehat{\lambda}\right) = \ln x + \frac{\widehat{\lambda}^2}{2}(T - \tau_1)$, following from the similar computations for the power utility scenario. The same SPDE (2.35) is satisfied by $U_t^F(x)$ for $\tau_1 < t < T$, with the initial and terminal conditions being $U_{\tau_1}^F(x) = \ln x + \frac{\widehat{\lambda}^2}{2}(T - \tau_1)$ and $U_T^F(x) = \ln x$, respectively. The logarithmic scaling in the boundary conditions suggests a candidate forward performance process given by $U_t^F(x) = \ln x + f_t$ for some $\mathcal{F}_t$-adapted process f_t, $\tau_1 \leq t \leq T$. Notice that such forward performance process is indeed strictly increasing and strictly concave in x and satisfies Inada's conditions at each time $\tau_1 \leq t \leq T$.

Let the Itô's decomposition for f_t be $df_t = b_t^f dt + a_t^f d\widetilde{W}_t$, for some admissible processes a_t^f and b_t^f such that this diffusion process is well defined. Direct computation then leads to

$$dU_t^F(x) \;=\; df_t = b_t^f dt + a_t^f d\widetilde{W}_t \tag{2.59}$$

$$\;=\; \frac{\left(\frac{\mu_t}{x} + \sigma_t a_x(x,t)\right)^2}{-\frac{2\sigma_t^2}{x^2}} dt + a(x,t) d\widetilde{W}_t. \tag{2.60}$$

Comparing the volatility parts of expression (2.59) and (2.60), we have $a_t^f = a(x,t)$, and comparison of drift parts gives rise to $b_t^f = -\frac{\lambda_t^2}{2}$. Therefore, the construction of the forward performance process $U_t^F(x)$ for the bridge problem boils down to looking for an admissible volatility process a_t^f, such that the SDE

$$df_t = -\frac{\lambda_t^2}{2} dt + a_t^f d\widetilde{W}_t, \tag{2.61}$$

with $f_{\tau_1} = \frac{\widehat{\lambda}^2}{2}(T - \tau_1)$ and $f_T = 0$ has a well defined $\mathcal{F}_t$-adapted strong solution for $\tau_1 \le t \le T$. Clearly, the solution to (2.61) is

$$f_t = \frac{\widehat{\lambda}^2}{2}(T - \tau_1) - \int_{\tau_1}^{t} \frac{\lambda_s^2}{2} ds + \int_{\tau_1}^{t} a_s^f d\widetilde{W}_s.$$

The terminal condition $f_T = 0$ implies

$$\int_{\tau_1}^{T} a_t^f d\widetilde{W}_t = \int_{\tau_1}^{T} \frac{\lambda_t^2}{2} dt - \frac{\widehat{\lambda}^2}{2}(T - \tau_1). \tag{2.62}$$

Under the conditions on the process λ_t, $\tau_1 \le t \le T$, given in the assumption, a unique admissible volatility process a_t^f, $\tau_1 \le t \le T$, exists, by virtue of the Martingale Representation Theorem. □

Example 2.5.8. *We give an example to explicitly construct the solution to equation (2.62), and hence obtain the forward bridge solution (2.58). Given a specific market parameter process λ_t, $\tau_1 \leq t \leq T$, we will find the volatility process a_t^f, $\tau_1 \leq t \leq T$, such that equation (2.62) is satisfied. Recall that conditional on $\mathcal{F}_{\tau_1}$, the process $\widetilde{W}_t$, $\tau_1 \leq t \leq T$, is a standard Brownian motion. Let $Y_t = \widetilde{W}_t^2 - t$, then $dY_t = 2\widetilde{W}_t d\widetilde{W}_t$. Let also $X_t = \frac{t^3}{3}$. Integrating by parts, we obtain*

$$\begin{aligned}
\frac{T^3}{3}(\widetilde{W}_T^2 - T) &= X_T Y_T = X_{\tau_1} Y_{\tau_1} + \int_{\tau_1}^T X_t dY_t + \int_{\tau_1}^T Y_t dX_t \\
&= -\frac{{\tau_1}^4}{3} + \int_{\tau_1}^T \frac{2}{3} t^3 \widetilde{W}_t d\widetilde{W}_t + \int_{\tau_1}^T t^2(\widetilde{W}_t^2 - t)dt \\
&= -\left(\frac{T^4}{4} + \frac{{\tau_1}^4}{12}\right) + \int_{\tau_1}^T \frac{2}{3} t^3 \widetilde{W}_t d\widetilde{W}_t + \int_{\tau_1}^T t^2 \widetilde{W}_t^2 dt.
\end{aligned}$$

We also note that

$$\frac{T^3}{3}(\widetilde{W}_T^2 - T) = \frac{T^3}{3}\left(\int_{\tau_1}^T 2\widetilde{W}_t d\widetilde{W}_t - \tau_1\right).$$

It hence follows that

$$\int_{\tau_1}^T \frac{2}{3}(T^3 - t^3)\widetilde{W}_t d\widetilde{W}_t = \int_{\tau_1}^T t^2 \widetilde{W}_t^2 dt - \frac{3T^4 + \tau_1^4 - 4T^3\tau_1}{12}. \qquad (2.63)$$

Now, if $\lambda_t = \sqrt{2C} t \widetilde{W}_t$, for $\tau_1 \leq t \leq T$, where $C := \frac{6\widehat{\lambda}^2(T-\tau_1)}{3T^4+\tau_1^4-4T^3\tau_1}$, then equation (2.63) yields

$$\int_{\tau_1}^T a_t^f d\widetilde{W}_t = \int_{\tau_1}^T \frac{\lambda_t^2}{2} dt - \frac{\widehat{\lambda}^2}{2}(T - \tau_1),$$

with

$$a_t^f = \frac{2}{3} C (T^3 - t^3)\widetilde{W}_t \in \mathcal{F}_t,$$

for $\tau_1 \leq t \leq T$. We can also easily verify that the constant $C \geq 0$ and the condition (2.57) is satisfied, since $\mathbb{E}\left[\widetilde{W}_t^2 | \mathcal{F}_{\tau_1}\right] = t - \tau_1$, for all $\tau_1 \leq t \leq T$.

Remark 2.5.4. In section 3 where the model parameter process $\lambda_t \equiv \lambda_1 \in \mathcal{F}_{\tau_1}$, a.s. under $\mathbb{P}$, for all $\tau_1 < t \leq T$, we can actually see from condition (2.57) that $\lambda_1 = \widehat{\lambda}$, a.s., must hold in order to construct the forward bridge process. This basically corresponds to a market without any intermediate unanticipated model changes, as the model specified at $t = 0$ will remain valid for the whole horizon $[0, T]$. It is then obvious that one can have a forward bridge solution starting with $\widehat{V}(x, 0; \lambda, \widehat{\lambda})$ by just following the classical value function process for $0 \leq t \leq T$. This result also implies that it is typically not possible to construct a forward bridge solution under the predictable model assumption, except for some degenerated scenario.

Remark 2.5.5. Beyond the predictable model assumption, we can consider more general model parameter process λ_t for the second sub-horizon $[\tau_1, T]$. For instance, consider the stochastic volatility model for $\tau_1 \leq t \leq T$

$$\frac{dS_t}{S_t} = \mu(Y_t)dt + \sigma(Y_t)d\widetilde{W}_t^1,$$

$$dY_t = b(Y_t)dt + d(Y_t)\left(\rho d\widetilde{W}_t^1 + \sqrt{1-\rho^2}d\widetilde{W}_t^2\right),$$

with $|\rho| < 1$. Here, the process Y_t is the stochastic factor that drives the stock price process S_t over the sub-horizon $[\tau_1, T]$, and $\widetilde{W}_t = (\widetilde{W}_t^1, \widetilde{W}_t^2)$, conditional on $\mathcal{F}_{\tau_1}$, is the two-dimensional Brownian motion with its natural filtration $\mathcal{F}_t$ satisfying the usual conditions. The deterministic functions $\mu(\cdot), \sigma(\cdot), b(\cdot), d(\cdot)$

are such that the two SDEs have unique strong solutions. Then in this incomplete Itô's diffusion market, it is reasonable to consider the forward performance process with the decomposition

$$dU_t^F(x) = b(x,t)dt + a_1(x,t)d\widetilde{W}_t^1 + a_2(x,t)d\widetilde{W}_t^2 = b(x,t)dt + a(x,t)\cdot d\widetilde{W}_t,$$

where $a(x,t) = (a_1(x,t), a_2(x,t))$. The argument then follows exactly as that given in the proof of Proposition 2.5.7; namely, the question boils down to looking for an admissible volatility process $a_t^f = \left(a_t^{f,1}, a_t^{f,2}\right)$ such that the SDE

$$df_t = -\frac{\lambda^2(Y_t)}{2}dt + a_t^f \cdot d\widetilde{W}_t,$$

with $f_{\tau_1} = \frac{\widehat{\lambda}^2}{2}(T-\tau_1)$ and $f_T = 0$ has a well defined $\mathcal{F}_t$-adapted solution for $\tau_1 \le t \le T$. Then similar as before, under the conditions that $\int_{\tau_1}^T \frac{\lambda^2(Y_t)}{2}dt$ is square integrable under $\mathbb{P}$ and that

$$\mathbb{E}\left[\int_{\tau_1}^T \frac{\lambda^2(Y_t)}{2}dt \middle| \mathcal{F}_{\tau_1}\right] = \frac{\widehat{\lambda}^2}{2}(T-\tau_1), \text{ a.s.},$$

the Martingale Representation Theorem guarantees the existence and uniqueness of such volatility process a_t^f, for $\tau_1 < t \le T$, and therefore also the existence of a solution to the forward bridge problem with the terminal logarithmic utility.

Chapter 3

Forward optimal liquidation with market parameter shift: the quadratic case

3.1 Introduction

Trade execution has been taken as an important component of the investment process ([23], [18]), since a poorly executed large order can consume profits from investment in an illiquid market. It is well known that institutional traders typically face a dilemma of trading speed. A trading that completes quickly may yield lower revenue due to the insufficient liquidity provided by the market. However, the trader is relieved from uncertainty of future asset price movement ([2], [18]). On the other hand, a slow trading may bear more uncertainty as the execution horizon extends, but benefits from low trading cost. A possible approach to address the best trade-off between fast and slow trading resorts to the expected utility optimization paradigm, where the trade-off between risk and return is characterized by a single utility function. Various criteria have already been considered in the optimal execution literature, see [6], [48] for the risk-neutral criterion; [2], [39] for the mean-variance criterion;

[53] for exponential utility and [52] for more general utility functions.

The optimal liquidation problem, as any other classical expected utility optimization problems, basically requires two inputs: a model specified by the investor and a criterion set for the end of a fixed trading horizon. Most existing works assume that these two elements are given *a priori* and, therefore, they can determine the optimal execution strategy (adapted or deterministic) at time $t = 0$. In principle, the agent should follow this strategy until the end of the trading horizon, but in practice, an unexpected market event or new trading opportunity may occur at any intermediate time, and this should lead the agent to revise the underlying model specification, trading volume as well as trading horizon specification. In other words, intermediate reoptimization due to unexpected model changes is a more realistic and necessary issue to address. The first contribution of this work is to propose a consistent and reasonable extension of the classical single-optimization problem by following the forward performance processes theory. More precisely, through incorporating the unanticipated market information, we determine the updated trading horizon and the updated performance criterion in real-time; together the two yield a revised optimal trading strategy that is consistent with previous strategies.

When it comes to model specification, typically two price impact components are considered in the optimal execution literature. The permanent impact is independent of current trading rate and can encompass asymmetric information or the total order flow from other agents, while the temporary impact measures instantaneous premium of liquidity and has been interpreted as a transaction cost ([8], [11], [49]). It is well known that the intraday pattern of trading volume, liquidity and volatility is time-varying, and the variability over different days within a week also exists ([15], [14]). [1] further pointed out that smaller capitalization stocks are generally more difficult to trade as their liquidity and volatility profiles, unlike those of the large capitalization stocks, are generally hard to model in advance. To address the evolution of market parameters, [26], [36], [29] proposed deterministic functions of time to characterize the change of these parameters, while [2] and [8] imposed a probability distribution over possible updated values of the parameters at a single future time. Still within the Markovian framework, [25] and [1] considered market parameters driven by various stochastic processes and essentially worked with stochastic factor models. More general non-Markovian model can be found in [4], where the optimal liquidation problem was solved by analyzing a backward stochastic differential equation with a singular terminal condition. See also [28] for the inclusion of an uncontrolled factor process in the dynamics of liquidity and volatility, as well as the associated stochastic Hamilton-Jacobi-Bellman equation with a singular terminal condition.

Our work contributes in this direction. Different from the works mentioned above where the deterministic or stochastic market profiles are prespecified and committed to through the entire trading horizon, however, our framework based on the forward approach can accommodate the sequentially updated model knowledge that is unanticipated. Empirical findings in [24] suggest the low predictability of market impact models (typically $< 5\%$ R^2), which in turn points to the necessity of updating the parameters in real-time to compensate for the low accuracy[1]. However, the classical expected utility optimization problem (including all the works mentioned above) are committed to the pre-specified model at $t = 0$ for the entire horizon, and cannot accommodate such model revision procedure with intertemporal consistent trading behavior. Nonetheless, we consider a trading behavior in the classical framework that naively re-optimizes based on the revised model and violates time-consistency. Under suitable metrics we introduced, it can be shown that compared to the naive behavior, the performance of execution under the forward approach is more stable, and remains higher especially in unanticipated adverse market scenarios (e.g., the Flash Crash). We also present a convergence result of the forward performance process when the model revision is done continuously in the limit.

3.2 Classical full-liquidation problem

For completeness, we first review the classical optimal liquidation problem studied in [53]. For easy exposition, we focus on the liquidation of a single stock. Assume an arbitrary but *prechosen* finite liquidation time, say $T < \infty$. The stock price process solves

$$P_t = P_0 + \sigma_0 W_t + \gamma_0 (X_t - X_0) + \lambda_0 \dot{X}_t, \tag{3.1}$$

[1]See [15] for a robust regression model that holds locally for short time interval and changes along with the well-known intraday seasonality effects.

$t \in [0,T]$, where W_t is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with $\{\mathcal{F}_t\}$ being the natural filtration satisfying the usual conditions. Without loss of generality, it is assumed that $\sigma_0 = 1$. The parameters λ_0, γ_0 model the *temporary* and *permanent* price impacts and are positive constants chosen at $t = 0$. The inventory process X_t models the amount of stock shares held at time t. It is taken to be absolutely continuous and solves, for $t \in [0,T]$,

$$X_t = x - \int_0^t \xi_u du, \tag{3.2}$$

with initial inventory $X_0 = x > 0$. The control process ξ_t represents the rate of liquidation.

The revenue process R_t is, in turn, given by

$$R_t = \int_0^t \xi_u P_u du = P_0 x - \frac{\gamma_0}{2} x^2 + \int_0^t X_u dW_u - \lambda_0 \int_0^t \xi_u^2 du. \tag{3.3}$$

The control set $\mathcal{A}_{[0,T]}$ is defined as the set of $\mathcal{F}_t$-progressively measurable processes ξ_t such that $x = \int_0^T \xi_s ds$, $\xi_t \geq 0$, $t \in [0,T]$ and $\int_0^T \xi_s^2 ds < \infty$, a.s., and the associated process X_t is bounded uniformly in (t, ω), with upper and lower bounds possibly depending on ξ_t (see [53], [51]).

The manager is risk averse and seeks, from the one hand, to maximize the expected utility of terminal revenue and, from the other, to fully liquidate by T. The authors in [53] considered the stochastic optimization problem

$$V(x, r, 0; T) := \sup_{\mathcal{A}_{[0,T]}} \mathbb{E}\left(v\left(X_T, R_T\right)\right), \tag{3.4}$$

with $r = P_0 x - \frac{\gamma_0}{2} x^2$ and singular terminal datum

$$v(x,r) = \begin{cases} -e^{-r}, & \text{if } x = 0, \\ -\infty, & \text{if } x > 0. \end{cases} \tag{3.5}$$

We will be using the self-evident notation $\mathcal{L}(\lambda_0; 0, T)$ to denote the above liquidation problem.

The related Hamilton-Jacobi-Bellman (HJB) equation is

$$V_t + \frac{1}{2} x^2 V_{rr} + \sup_{\xi} \left(-\lambda_0 \xi^2 V_r - \xi V_x \right) = 0, \tag{3.6}$$

$(x,r,t) \in \mathbb{R}^+ \times \mathbb{R} \times [0,T]$, with $V(x,r,T;T) = v(x,r)$. It turns out that the value function is given by

$$V(x,r,t;T) = -\exp\left(-r + \sqrt{\frac{\lambda_0}{2}} x^2 \coth \frac{T-t}{\sqrt{2\lambda_0}} \right), \tag{3.7}$$

and the optimal feedback liquidation control function is given by

$$\xi^*(x,r,t) = \frac{1}{\sqrt{2\lambda_0}} x \coth \frac{T-t}{\sqrt{2\lambda_0}}.$$

Therefore, at initial time,

$$V(x,r,0;T) = -\exp\left(-r + \sqrt{\frac{\lambda_0}{2}} \coth \frac{T}{\sqrt{2\lambda_0}} x^2 \right),$$

and, for $t \in [0,T]$, the optimal liquidation and inventory processes are given explicitly by

$$X_t^* = x \frac{\sinh \frac{T-t}{\sqrt{2\lambda_0}}}{\sinh \frac{T}{\sqrt{2\lambda_0}}} \qquad \text{and} \qquad \xi_t^* = \frac{1}{\sqrt{2\lambda_0}} x \frac{\cosh \frac{T-t}{\sqrt{2\lambda_0}}}{\sinh \frac{T}{\sqrt{2\lambda_0}}}. \tag{3.8}$$

From (3.8), it is easy to see that the optimal process ξ_t^* indeed leads to full liquidation, as $X_T^* = 0$. It is worth noticing that this full liquidation is *implicitly "forced"* through the singular form of the terminal datum (3.5). Note also that both X_t^* and ξ_t^* are deterministic, and they depend on the $t = 0$ pre-specified temporary price impact parameter λ_0.

A variation of the above problem has been solved in an infinite horizon setting ($T = \infty$), using a similar stochastic optimization approach in [52]. Therein, when the terminal utility is the same exponential utility, the optimal inventory process is $X_t^* = xe^{-\frac{t}{\sqrt{2\lambda_0}}}$, where, again, λ_0 is the $t = 0$ pre-specified price impact parameter for the entire horizon $[0, \infty)$. We denote such infinite horizon liquidation problem by $\mathcal{L}(\lambda_0; 0, \infty)$.

3.2.1 Inverse liquidation problem

In this section, we introduce a new problem which will serve as the building block in the method we propose herein. As in the classical case, the manager starts at $t = 0$ with a given liquidation model, as in (3.2) and (3.3), for some arbitrary but fixed price impact parameters. To facilitate the discussion later on, we only focus on the temporary price impact parameter, and denote it by λ. Then, we have the model dynamics

$$dX_t^\zeta = -\zeta_t dt \quad \text{and} \quad dR_t^\zeta = -\lambda \zeta_t^2 dt + X_t^\zeta dW_t, \tag{3.9}$$

with $X_0 = x > 0$ and $R_0 = r \in \mathbb{R}$. Here, W_t is a standard Brownian motion defined for all $t \geq 0$ on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with $\{\mathcal{F}_t\}$ being the natural filtration satisfying the usual conditions. We work with the set of admissible policies $\mathcal{A}$ that consists of all $\mathcal{F}_t$-progressively measurable processes ζ, with $\zeta_t \geq 0$, for all $t \in \left[0, T^{\zeta}\right)$, $T^{\zeta} = \inf\left\{t > 0 : \; x = \int_0^t \zeta_s ds\right\}$, $\int_0^{T^{\zeta}} \zeta_s^2 ds < L\left(\zeta\right)$, and $\mathbb{E}\int_0^{T^{\zeta}} (X_s^{\zeta})^2 ds < \infty$. Here, $L\left(\zeta\right) > 0$ is a constant that only depends on ζ.

We now introduce the new liquidation problem.

Problem $\mathcal{P}\left(\lambda, k; 0\right)$: *Let* $\lambda > 0$, and *assume that the inventory and revenue processes satisfy (3.9). Let* $k > 0$ *and introduce the function* $u : \mathbb{R}^{+} \times \mathbb{R} \longrightarrow \mathbb{R}^{-}$,

$$u\left(x, r, 0\right) := -e^{-r + kx^2}. \tag{3.10}$$

Find the longest deterministic time $T\left(\lambda, k\right) \geq 0$ *and a deterministic function* $U\left(x, r, t\right) : \mathbb{R} \times \mathbb{R}^{+} \times \left[0, T\left(\lambda, k\right)\right) \rightarrow \mathbb{R}^{-}$, *of the separable form*

$$U\left(x, r, t\right) = -e^{-r + h(x,t)}, \tag{3.11}$$

for $h \in C^{1,1}\left(\mathbb{R}^{+} \times \left[0, T\left(\lambda, k\right)\right)\right)$, *with the following properties:*

i) $U\left(x, r, 0\right) = u(x, r, 0)$,

ii) for any $\zeta \in \mathcal{A}$, *the process* $U\left(X_t^{\zeta}, R_t^{\zeta}, t\right)$ *is a supermartingale, for* $t \in \left[0, T\left(\lambda, k\right) \wedge T^{\zeta}\right)$,

iii) there exists $\zeta^{*} \in \mathcal{A}$ *such that the process* $U\left(X_t^{\zeta^*}, R_t^{\zeta^*}, t\right)$ *is a martingale, for* $t \in \left[0, T\left(\lambda, k\right) \wedge T^{\zeta}\right)$.

In other words, the manager trades in a similar market environment as in the classical case but she does not pre-determine a time at which full liquidation must occur. Rather, she chooses an initial datum of form (3.10) and seeks the longest time such that the conditions (i)-(iii) are satisfied.

While this problem might for now look artificial, we will show in the sequel that such problem becomes the building block for constructing the forward performance process under real-time model revision. Moreover, the aforementioned classical settings can be also recast as $P(\lambda, k; 0)$ problems with suitable initial conditions.

We proceed with the solution of the above problem. Let the parameter

$$m := k\sqrt{\frac{2}{\lambda}}, \tag{3.12}$$

and the auxiliary functions $F, G : \mathbb{R}^+ \to \mathbb{R}$,

$$F(t; m, \lambda) := \cosh\frac{t}{\sqrt{2\lambda}} - m\sinh\frac{t}{\sqrt{2\lambda}}, \qquad G(t; m, \lambda) := \cosh\frac{t}{\sqrt{2\lambda}} - \frac{1}{m}\sinh\frac{t}{\sqrt{2\lambda}}. \tag{3.13}$$

Clearly, $F(t; 1, \lambda) = G(t; 1, \lambda)$ and, more generally, $F(t; m, \lambda) = G\left(t; \frac{1}{m}, \lambda\right)$, $m > 0$. Also, $F(0; m, \lambda) = G(0; m, \lambda) = 1$.

Furthermore, direct calculations yield that

$$G\left(\sqrt{\frac{\lambda}{2}}\ln\frac{1+m}{1-m}; m, \lambda\right) = 0 \qquad \text{and} \qquad F\left(\sqrt{\frac{\lambda}{2}}\ln\frac{1+m}{1-m}; m, \lambda\right) > 0, \tag{3.14}$$

if $m \in (0, 1)$, and $F\left(\sqrt{\frac{\lambda}{2}}\ln\frac{m+1}{m-1}; m, \lambda\right) = 0$, if $m > 1$.

We start with a result about the candidate function(s) $h(x, t)$ that will appear in (3.11).

Lemma 3.2.1. *Let $\lambda > 0, k > 0$ and $m = k\sqrt{\frac{2}{\lambda}}$ (cf. (3.12)). Define*

$$T(\lambda, k) := \begin{cases} \sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}, & \text{if } m < 1, \\ \infty, & \text{if } m = 1, \\ \sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{m-1}, & \text{if } m > 1. \end{cases} \tag{3.15}$$

Then, for $t \in [0, T(\lambda, k))$, the Hamilton-Jacobi equation

$$h_t - \frac{1}{4\lambda} h_x^2 + \frac{1}{2} x^2 = 0, \tag{3.16}$$

with $h(x, 0) = kx^2$, $x > 0$ and $h(0, t) = 0$, has a unique non-decreasing in x solution, given by

$$h(x, t) = kx^2 \frac{G(t; m, \lambda)}{F(t; m, \lambda)}, \tag{3.17}$$

with F, G as in (3.13). Furthermore, for $x \geq 0$,

$$\lim_{t \uparrow T(\lambda, k)} h(x, t) = \begin{cases} 0, & \text{if } m < 1, \\ kx^2, & \text{if } m = 1, \\ \infty \mathbf{1}_{\{x > 0\}} + 0 \mathbf{1}_{\{x = 0\}}, & \text{if } m > 1. \end{cases} \tag{3.18}$$

Proof. We solve equation (3.16) using the method of characteristics. These curves, denoted by $X(t), P(t)$, satisfy

$$\frac{dX(t)}{dt} = -\frac{1}{2\lambda} P(t), \quad \frac{dP(t)}{dt} = -X(t), \tag{3.19}$$

and

$$\frac{dh(X(t), t)}{dt} = -\frac{1}{4\lambda} P^2(t) - \frac{1}{2} X^2(t), \tag{3.20}$$

with $P(t) = h_x(X(t), t)$, $X(0) = x$. Therefore, for $t \geq 0$,

$$X(t) = C_1 e^{\frac{t}{\sqrt{2\lambda}}} + C_2 e^{-\frac{t}{\sqrt{2\lambda}}} \quad \text{and} \quad P(t) = \sqrt{2\lambda} \left(-C_1 e^{\frac{t}{\sqrt{2\lambda}}} + C_2 e^{-\frac{t}{\sqrt{2\lambda}}} \right). \tag{3.21}$$

The initial condition of (3.16) yields $P(0) = h_x(x,0) = 2kx$. Thus, we must have

$$C_1 = \frac{x}{2}\left(1 - k\sqrt{\frac{2}{\lambda}}\right) \qquad \text{and} \qquad C_2 = \frac{x}{2}\left(1 + k\sqrt{\frac{2}{\lambda}}\right) \tag{3.22}$$

and, in turn, for $t \geq 0$,

$$X\left(t\right) = x\left(\cosh\frac{t}{\sqrt{2\lambda}} - m\sinh\frac{t}{\sqrt{2\lambda}}\right) = xF\left(t; m, \lambda\right). \tag{3.23}$$

Therefore, for $t \geq 0$,

$$\begin{aligned} h(X\left(t\right), t) &= h(x,0) - \int_0^t \left(\frac{1}{4\lambda}P^2(s) + \frac{1}{2}X^2(s)\right) ds \\ &= kx^2 + \sqrt{\frac{\lambda}{2}}\left(C_2^2 e^{-\sqrt{\frac{2}{\lambda}}t} - C_1^2 e^{\sqrt{\frac{2}{\lambda}}t}\right) + \sqrt{\frac{\lambda}{2}}\left(C_1^2 - C_2^2\right) \\ &= x^2\left(k\cosh\left(\sqrt{\frac{2}{\lambda}}t\right) - \sqrt{\frac{\lambda}{2}}\left(\frac{1}{2} + \frac{k^2}{\lambda}\right)\sinh\left(\sqrt{\frac{2}{\lambda}}t\right)\right) \end{aligned}$$

We then seek the *maximal* time $T\left(\lambda, k\right)$ such that a well defined solution $h\left(x, t\right)$ exists, for each $x \geq 0$ and $t \in \left[0, T\left(\lambda, k\right)\right)$, that is also nondecreasing in x and satisfies $h\left(0, t\right) = 0$. For this, we first need to invert the characteristic curve (3.23), insuring that for each $X\left(t\right) > 0$, with $t \in \left[0, T\left(\lambda, k\right)\right)$, there exists a unique $x > 0$ that satisfies (3.23).

We look at the following cases:

If $m = 1$, then $F\left(t; 1, \lambda\right) = e^{-\frac{t}{\sqrt{2\lambda}}}$ while, if $m < 1$, then $F(t; m, \lambda) > e^{-\frac{t}{\sqrt{2\lambda}}}$. Thus, for $m \leq 1$, $F\left(t; m, \lambda\right) > 0$, $t \geq 0$ and therefore, (3.23) can be inverted for all times $t > 0$.

If, on the other hand, $m > 1$, curve (3.23) can be inverted only up to the first zero of $F(t; m, \lambda)$, which occurs at the (finite) time, say T_1, given by

$$T_1 = \sqrt{2\lambda} \operatorname{arc\,coth} m = \sqrt{\frac{\lambda}{2}} \ln \frac{m+1}{m-1}.$$

Therefore, if (with a slight abuse of notation) we define time $T_1(\lambda, k)$ as

$$T_1(\lambda, k) = \infty, \text{ if } m \leq 1 \quad \text{and} \quad T_1(\lambda, k) = \sqrt{\frac{\lambda}{2}} \ln \frac{m+1}{m-1}, \text{ if } m > 1, \tag{3.24}$$

we deduce that a well-defined solution is given, for $t \in [0, T_1(\lambda, k))$, by

$$h(x,t) = x^2 \frac{k \cosh \sqrt{\frac{2}{\lambda}} t - \sqrt{\frac{\lambda}{2}} \left(\frac{1}{2} + \frac{k^2}{\lambda} \right) \sinh \sqrt{\frac{2}{\lambda}} t}{\left(\cosh \frac{t}{\sqrt{2\lambda}} - k\sqrt{\frac{2}{\lambda}} \sinh \frac{t}{\sqrt{2\lambda}} \right)^2}$$

$$= kx^2 \frac{\left(\cosh \frac{t}{\sqrt{2\lambda}} - \frac{1}{m} \sinh \frac{t}{\sqrt{2\lambda}} \right)}{\left(\cosh \frac{t}{\sqrt{2\lambda}} - m \sinh \frac{t}{\sqrt{2\lambda}} \right)} = kx^2 \frac{G(t; m, \lambda)}{F(t; m, \lambda)}.$$

Note, however, that the above function might not be spatially increasing.

It remains to insure the spatial monotonicity of $h(x,t)$. To this end, let

$$T_2(\lambda, k) := \sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}, \text{ if } m < 1 \quad \text{and} \quad T_2(\lambda, k) := \infty, \text{ if } m \geq 1. \tag{3.25}$$

Then both $F(t; m, \lambda), G(t; m, \lambda) > 0$, for $t \in [0, T_2(\lambda, k))$, and combining (3.24) and (3.25), we easily conclude.

To show uniqueness, we assume that there are two solutions that are non-decreasing in x, $h \in C^{1,1}(\mathbb{R}^+ \times [0, T))$ and $\tilde{h} \in C^{1,1}\left(\mathbb{R}^+ \times \left[0, \widetilde{T}\right)\right)$, with

$\widetilde{T} > T$, satisfying $h(x,0) = \tilde{h}(x,0) = kx^2$, $x > 0$, $h(0,t) = \tilde{h}(0,t)$, $t \in [0,T)$. Then, $H := h - \tilde{h}$ satisfies, for $(x,t) \in \mathbb{R}^+ \times [0,T)$,

$$H_t - \frac{1}{4}\left(h_x^2 - \tilde{h}_x^2\right) = H_t - \frac{1}{4}H_x\left(h_x + \tilde{h}_x\right) = 0,$$

with $H(x,0) = 0$ and $H(0,t) = 0$. For the characteristics we have $\frac{dX(t)}{dt} = -\frac{h_x(X(t),t)+\tilde{h}_x(X(t),t)}{4\lambda_0}$, with $h_x(X(t),t) + \tilde{h}_x(X(t),t) \geq 0$. It hence implies that for any $X(t) = x \geq 0$, $t \in [0)$, the initial value $X(0) = x_0 \geq x \geq 0$. We conclude, using $H(X(t),t) = H(x_0,0) = 0$, with $x_0 \geq 0$, that $H \equiv 0$ is the unique solution up to time T. It then follows $T = \widetilde{T}$, and $h(x,t) = \tilde{h}(x,t)$, for $(x,t) \in \mathbb{R}^+ \times [0,T)$.

It remains to show (3.18). The case $m = 1$ is trivial. If $m < 1$, then

$$\lim_{t \uparrow T(\lambda,k)} h(x,t) = \lim_{t \uparrow \sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}} kx^2 \frac{G(t;m,\lambda)}{F(t;m,\lambda)},$$

and using that $G\left(\sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}; m, \lambda\right) = 0$ and $F\left(\sqrt{\frac{\lambda}{2}} \ln \frac{1+m}{1-m}; m, \lambda\right) > 0$, we conclude. The case $m > 1$, follows similarly. □

The next result states that in the class of separable functions (3.11), the inverse liquidation problem $\mathcal{P}(\lambda, k; 0)$ has, for each pair $(\lambda, k) \in \mathbb{R}^+ \times \mathbb{R}^+$, a *unique* solution, which is also explicitly constructed.

Theorem 3.2.2. *Let $(\lambda, k) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $m := k\sqrt{\frac{2}{\lambda}}$. Let also F and G as in (3.13), $T(\lambda, k)$ as in (3.15) and h as in (3.17). Then, the following assertions hold:*

i) The problem $\mathcal{P}(\lambda, k; 0)$ *has a solution, given by the pair* $(T(\lambda,k), U(x,r,t))$ *with* $U(x,r,t) : \mathbb{R}^+ \times \mathbb{R} \times [0, T(\lambda,k)) \longrightarrow \mathbb{R}^-$ *given by*

$$U(x,r,t) := -e^{-r+h(x,t)}. \tag{3.26}$$

This solution is unique in the class of separable functions (3.11).

Furthermore, for each $(x,r) \in \mathbb{R}^+ \times \mathbb{R}$,

$$\lim_{t \uparrow T(\lambda,k)} U(x,r,t) = \begin{cases} -e^{-r}, & \text{if } m < 1, \\ -e^{-r+kx^2}, & \text{if } m = 1, \\ -\infty \mathbf{1}_{\{x>0\}} - e^{-r}\mathbf{1}_{\{x=0\}}, & \text{if } m > 1. \end{cases} \tag{3.27}$$

ii) The optimal policy ζ^* *and optimal inventory* X^**, are given, respectively, by*

$$\zeta_t^* = \frac{1}{2\lambda} h_x(X_t^*, t) = x \frac{k}{\lambda} \frac{G(t; m, \lambda)}{F(t; m, \lambda)}, \tag{3.28}$$

and

$$X_t^* = xF(t; m, \lambda). \tag{3.29}$$

iii) For each $x > 0$,

$$\lim_{t \uparrow T(\lambda,k)} X_t^* = \begin{cases} x\sqrt{(1-m)(1+m)}, & \text{if } m < 1, \\ 0, & \text{if } m \geq 1. \end{cases} \tag{3.30}$$

Therefore, if $m \geq 1$*, the optimal policy* ζ^* *is also a full liquidation policy at the solvability time* $T(\lambda, k)$.

Proof. It follows trivially that $U(x,r,0) = u(x,r,0)$.

To show property (ii) and (iii), we work as follows.

Let $\zeta \in \mathcal{A}$. Then for $t \in [0, T(\lambda, k))$, the functions $h(x, s)$ and $U(x, r, s)$ are well defined for $s \in [0, t]$. On the other hand, Ito's formula yields

$$U(X^{\zeta}_{t\wedge T^{\zeta}}, R^{\zeta}_{t\wedge T^{\zeta}}, t \wedge T^{\zeta}) = u(r, x, 0)+$$

$$\int_0^{t\wedge T^{\zeta}} \left(U_t \left(X^{\zeta}_s, R^{\zeta}_s, s \right) - \zeta_s U_x \left(X^{\zeta}_s, R^{\zeta}_s, s \right) - \lambda U_r \left(X^{\zeta}_s, R^{\zeta}_s, s \right) \zeta_s^2 + \frac{1}{2} U_{rr} \left(X^{\zeta}_s, R^{\zeta}_s, s \right) \left(X^{\zeta}_s \right)^2 \right) ds$$

$$+ \int_0^{t\wedge T^{\zeta}} U_r(X^{\zeta}_s, R^{\zeta}_s, s) X^{\zeta}_s dW_s$$

$$= \int_0^{t\wedge T^{\zeta}} \left(\zeta_s - \frac{1}{2\lambda} h_x \left(X^{\zeta}_s, s \right) \right)^2 U \left(X^{\zeta}_s, R^{\zeta}_s, s \right) ds + \int_0^{t\wedge T^{\zeta}} U_r(X^{\zeta}_s, R^{\zeta}_s, s) X^{\zeta}_s dW_s$$

$$= - \int_0^{t\wedge T^{\zeta}} \left(\zeta_s - \frac{1}{2\lambda} h_x \left(X^{\zeta}_s, s \right) \right)^2 e^{-R^{\zeta}_s + h\left(X^{\zeta}_s, s\right)} ds + \int_0^{t\wedge T^{\zeta}} U_r(X^{\zeta}_s, R^{\zeta}_s, s) X^{\zeta}_s dW_s, \tag{3.31}$$

where we used (3.26). Next, we show that the process

$$\int_0^{t\wedge T^{\zeta}} U_r(X^{\zeta}_s, R^{\zeta}_s, s) X^{\zeta}_s dW_s = \int_0^{t\wedge T^{\zeta}} e^{-R^{\zeta}_s + h(X^{\zeta}_s, s)} X^{\zeta}_s dW_s$$

is a genuine martingale, for $t \in [0, T(\lambda, k))$. To this end, we have

$$\mathbb{E} \int_0^t \left(e^{-R^{\zeta}_s + h(X^{\zeta}_s, s)} X^{\zeta}_s \right)^2 ds \leq x^2 \mathbb{E} \int_0^t e^{-2R^{\zeta}_s + 2kx^2 \frac{G(s;m,\lambda)}{F(s;m,\lambda)}} ds.$$

Furthermore, if $m \geq 1$, (3.13) yields $\frac{G(s;m,\lambda)}{F(s;m,\lambda)} \leq \frac{G(t;m,\lambda)}{F(t;m,\lambda)}$, while if $m < 1$, $\frac{G(s;m,\lambda)}{F(s;m,\lambda)} \leq \frac{G(0;m,\lambda)}{F(0;m,\lambda)} = 1$.

Therefore, it suffices to show that $\mathbb{E} \int_0^{t\wedge T^{\zeta}} e^{-2R^{\zeta}_s} ds < \infty$. By admissibility of $\zeta \in \mathcal{A}$, there exist constants $L(\zeta)$, $K(\zeta) > 0$, such that $\int_0^{T^{\zeta}} \zeta_s^2 ds < L(\zeta)$, a.s., and $\mathbb{E} \int_0^{T^{\zeta}} (X^{\zeta}_s)^2 ds < K(\zeta)$. Hence, using the dynamics (3.9) for R^{ζ}, we obtain

$$\mathbb{E} \int_0^{t\wedge T^{\zeta}} e^{-2R^{\zeta}_s} ds = \mathbb{E} \int_0^{t\wedge T^{\zeta}} \exp \left(-2r - 2 \int_0^s X^{\zeta}_u dW_u + 2\lambda \int_0^s \zeta_u^2 du \right) ds$$

$$\leq e^{-2r+2\lambda L(\zeta)}\mathbb{E}\int_0^{t\wedge T^\zeta} e^{-2\int_0^s X_u^\zeta dW_u} ds = e^{-2r+2\lambda L(\zeta)}\int_0^t \mathbb{E}\Big[e^{-2\int_0^{s\wedge T^\zeta} X_u^\zeta dW_u}\Big] ds$$

$$\leq e^{-2r+2\lambda L(\zeta)}\int_0^t e^{2K(\zeta)} ds = e^{-2r+2\lambda L(\zeta)+2K(\zeta)} t < e^{-2r+2\lambda L(\zeta)+2K(\zeta)} T(\lambda,k) \leq \infty,$$

where we have used the fact that $\int_0^{s\wedge T^\zeta} X_u^\zeta dW_u$, for $s \in [0,t]$, is a square integrable martingale with quadratic variation at most $K(\zeta)$.

Next, consider for $t \in [0, T(\lambda,k))$ the feedback policy $\zeta_t^* = \frac{1}{2\lambda} h_x(X_t^*, t) > 0$. Then, (3.13) and (3.17) give

$$dX_t^* = -\frac{k}{\lambda} X_t^* \frac{\cosh\frac{t}{\sqrt{2\lambda}} - \frac{1}{m}\sinh\frac{t}{\sqrt{2\lambda}}}{\cosh\frac{t}{\sqrt{2\lambda}} - m\sinh\frac{t}{\sqrt{2\lambda}}}, \qquad X_0^* = x.$$

We claim that the solution (3.29) follows. In turn, ζ_t^* is given by the deterministic function in (3.28). Note that for $t \in [0, T(\lambda,k))$, all involved quantities are well defined. We then easily deduce that this policy is admissible. Its optimality then follows from (3.31).

We now look at $\lim_{t\uparrow T(\lambda,k)} X_t^*$. If the parameters (λ,k) are such that $m < 1$, then (3.29), (3.13) and (3.15) give

$$\lim_{t\uparrow T(\lambda,k)} X_t^* = x\left(\cosh\left(\tanh^{-1} m\right) - m\sinh\left(\tanh^{(-1)} m\right)\right)$$

$$= x\left(1-m^2\right)\cosh\left(\tanh^{(-1)} m\right) = x\left(1-m^2\right)\cosh\left(\ln\sqrt{\frac{1+m}{1-m}}\right)$$

$$= \frac{1}{2}x\left(1-m^2\right)\left(\sqrt{\frac{1+m}{1-m}} + \sqrt{\frac{1-m}{1+m}}\right) = x\sqrt{(1-m)(1+m)} > 0.$$

If $m = 1$, then $T(\lambda,k) = \infty$ and (3.29) gives $\lim_{t\uparrow\infty} X_t^* = \lim_{t\uparrow\infty} xe^{-\frac{k}{\lambda}t} = 0$, and thus $T^{\zeta^*} = T(\lambda,k) = \infty$.

Finally, if $m > 1$, direct calculations in (3.29) yield that $\lim_{t \uparrow T(\lambda,k)} X_t^* = 0$, and that $T^{\zeta^*} = T(\lambda, k) < \infty$. Therefore, the optimal policy is also a perfect liquidation policy at time $T(\lambda, k)$. $\square$

Corollary 3.2.3. *For $x > 0$ and $t \in [0, T(\lambda, k))$, we have*

$$h_t(x,t) \gtrless 0,\ m \gtrless 1 \quad \textit{and} \quad h_t(x,t) = 0,\ m = 1. \tag{3.32}$$

Moreover, the optimal liquidation policy ζ^* satisfies

$$\frac{d}{dt}\zeta_t^* = \frac{1}{2}\frac{k(m^2-1)}{F^2(t; m, \lambda)}. \tag{3.33}$$

Therefore, if $m > 1$ $(m < 1)$, then ζ^* is strictly increasing (resp. decreasing) in time.

As shown in Theorem 3.2.2, the inverse liquidation problem $\mathcal{P}(\lambda_0, k; 0)$ gives rise to different liquidation strategies and different horizons, for various market conditions characterized by λ. The classical liquidation problems, on the other hand, lacks in such flexibility, once a fixed terminal singular condition and the horizon were pre-specified. Beyond this, we will also compare the solutions of the two problems under suitable metrics, and show that the trading behavior given by the problem $\mathcal{P}(\lambda_0, k; 0)$ is indeed superior.

3.2.2 Reconciling the classical and the inverse liquidation problems

We conclude this section by showing that the classical full-liquidation problems $\mathcal{L}(\lambda_0; 0, T)$ or $\mathcal{L}(\lambda_0; 0, \infty)$ are special instances of the inverse liquidation problem $\mathcal{P}(\lambda_0, k; 0)$.

As we saw earlier, every classical liquidation problem is parametrized by the market parameter λ_0 and the targeted full-liquidation horizon T (finite or not). We now show that for each such problem, $\mathcal{L}(\lambda_0; 0, T)$ or $\mathcal{L}(\lambda_0; 0, \infty)$, there exists an inverse liquidation problem $\mathcal{P}(\lambda_0, k; 0)$ that has the same optimal policies and same full-liquidation times.

3.2.2.1 Problem $\mathcal{L}(\lambda_0; 0, T)$ - finite full-liquidation horizon

With λ_0 and $T < \infty$ given, introduce the constant

$$k_0 := \sqrt{\frac{\lambda_0}{2}} \coth \frac{T}{\sqrt{2\lambda_0}}, \tag{3.34}$$

and consider the inverse liquidation problem $\mathcal{P}(\lambda_0, k_0; 0)$. Then, (3.12) gives $m_0 = \coth \frac{T}{\sqrt{2\lambda_0}} > 1$. In turn, (3.15) yields

$$T(\lambda_0, k_0) = \sqrt{2\lambda_0} \ln \sqrt{\frac{1 + m_0}{1 - m_0}} = T.$$

Therefore, the full liquidation time T of the classical problem $\mathcal{L}(\lambda_0; 0, T)$ coincides with the solvability time $T(\lambda_0, k_0)$ of the inverse problem $\mathcal{P}(\lambda_0, k_0; 0)$. Furthermore, for $t \in [0, T)$, (3.28) and (3.34) give,

$$\zeta_t^* = x \frac{k_0}{\lambda_0} \frac{\cosh \frac{t}{\sqrt{2\lambda_0}} - \sinh \frac{t}{\sqrt{2\lambda_0}}}{\cosh \frac{t}{\sqrt{2\lambda_0}} - \coth \frac{T}{\sqrt{2\lambda_0}} \sinh \frac{t}{\sqrt{2\lambda_0}}}$$

$$= x \frac{k_0}{\lambda_0} \frac{\coth \frac{T}{\sqrt{2\lambda_0}} \cosh \frac{t}{\sqrt{2\lambda_0}} - \sinh \frac{t}{\sqrt{2\lambda_0}}}{\coth \frac{T}{\sqrt{2\lambda_0}} \left(\cosh \frac{t}{\sqrt{2\lambda_0}} - \coth \frac{T}{\sqrt{2\lambda_0}} \sinh \frac{t}{\sqrt{2\lambda_0}}\right)} = x \frac{1}{\sqrt{2\lambda_0}} \frac{\cosh \frac{T-t}{\sqrt{2\lambda_0}}}{\sinh \frac{T}{\sqrt{2\lambda_0}}} = \xi_t^*.$$

Obviously $X_t^{*,\mathcal{P}} = X_t^{*,\mathcal{L}}$, with full liquidation at T.

Notice that full liquidation at $T(\lambda_0, k_0)$ for $\mathcal{P}(\lambda_0, k_0; 0)$ is expected since $m_0 > 1$, which further implies that λ_0 is relatively small and favorable for liquidation of the asset.

Tedious but direct calculations also show that, for $t \in [0, T)$, $U(x, r, t) = V(x, r, t; T)$ and $\lim_{t \uparrow T(\lambda_0, k)} U(x, r, t) = \lim_{t \uparrow T} V(x, r, t)$.

Remark 3.2.1. The choice of the constant k_0 in (3.34) is not the only one that yields $T(\lambda_0, k_0) = T$. Indeed, for $k_0' := \sqrt{\frac{\lambda_0}{2}} \tanh \frac{T}{\sqrt{2\lambda_0}}$, we also have $T(\lambda_0, k_0) = T$. In this case, however, $m_0' = \tanh \frac{T}{\sqrt{2\lambda_0}} < 1$, and as we have seen in (3.30), the optimal policy for $\mathcal{P}(\lambda_0, k_0'; 0)$ does not lead to full liquidation. Therefore, the problems $\mathcal{L}(\lambda_0; 0, T)$ and $\mathcal{P}(\lambda_0, k_0'; 0)$ do not have the same solution.

3.2.2.2 Problem $\mathcal{L}(\lambda_0; 0, \infty)$ - infinite full-liquidation horizon

With λ_0 given, let $k_0 := \sqrt{\frac{\lambda_0}{2}}$ and consider the inverse liquidation problem $\mathcal{P}(\lambda_0, k_0; 0)$. Then (3.12) gives $m_0 = 1$ and, thus, $T(\lambda_0, k_0) = T = \infty$. In turn, for $t \geq 0$,

$$\zeta_t^* = x \frac{1}{\sqrt{2\lambda_0}} = x \frac{k_0}{\lambda_0} = \xi_t^*.$$

Furthermore, $h(x, t) = \sqrt{\frac{\lambda_0}{2}} x^2$ and, thus, $U(x, r, t) = -\exp\left(-r + \sqrt{\frac{\lambda_0}{2}} x^2\right)$. We easily deduce that the problems $\mathcal{L}(\lambda_0; 0, \infty)$ and $\mathcal{P}\left(\lambda_0, \sqrt{\frac{\lambda_0}{2}}; 0\right)$ have the same solution.

Note that, contrary to the previous case of finite full-liquidation horizon, there is a *unique* choice of the constant k that gives $T(\lambda_0, k) = T = \infty$.

3.3 "Real-time" single parameter shift

We now consider the following extension of the classical liquidation setting. At $t = 0$, we allow for the market impact parameter to change at a given (deterministic) time, say $\tau_1 < T$. While, however, we know a priori that a change in this parameter will occur at τ_1, we *do not know a priori* its new value, say λ_1, nor its probability distribution at $t = 0$. In other words, $\tau_1 \in \mathcal{F}_0$ and $\lambda_1 \in \mathcal{F}_1$.

We also consider two trading agents, whom we, respectively, call *"naive"* and *"forward"*. They both have access to the information that the parameter λ_0 will change at τ_1, and will take (an unknown at $t = 0$) value λ_1.

The two agents exhibit different behavior with regards to this knowledge. We describe this behavior below and analyze the differences and similarities. Essentially, the naive and the forward agents will solve the (conditional) variants of the problem $\mathcal{L}(\lambda; 0, T)$ and problem $\mathcal{P}(\lambda, k; 0)$, respectively. It is worth noting that the basic form of the problem $\mathcal{P}(\lambda, k; 0)$, with the analysis presented in Theorem 3.2.2, is interesting on its own right.

3.3.1 The naive agent

At $t = 0$, the agent pre-determines a full-liquation time T and assumes terminal utility (3.5). In order to solve the related optimization problem (3.4), he needs to pre-specify at $t = 0$ a model for the *entire* horizon $[0, T]$, for *both* periods $[0, \tau_1)$ and $[\tau_1, T]$. Since he is aware that the market parameter λ_0 will be *revised* at the deterministic time τ_1, he chooses dynamics,

$$dX_t^{\lambda_0} = -\xi_t dt \quad \text{and} \quad dR_t^{\lambda_0} = -\lambda_0 \xi_t^2 dt + X_t^{\lambda_0} dW_t, \quad \text{for } t \in [0, \tau_1) \tag{3.35}$$

with $X_0 = x, R_0 = r$, and

$$dX_t^{\hat{\lambda}} = -\xi_t dt \quad \text{and} \quad dR_t^{\hat{\lambda}} = -\hat{\lambda} \xi_t^2 dt + X_t^{\hat{\lambda}} dW_t, \quad \text{for } t \in [\tau_1, T], \tag{3.36}$$

with $X_{\tau_1}^{\hat{\lambda}} = X_{\tau_1}^{\lambda_0}, R_{\tau_1}^{\hat{\lambda}} = R_{\tau_1}^{\lambda}$.

The value $\hat{\lambda}$ can be interpreted as *his best, at* $t = 0$, *estimate* for the *future* new value of the market impact parameter, to be realized at τ_1 and to remain accurate in $[\tau_1, T]$. In general, of course, $\hat{\lambda}$ might *not be* the correct revised value, $\lambda_1(\omega)$, since the latter will be realized only at τ_1.

The agent starts trading at $t = 0$ till the predictable revision time τ_1. Then, once the true value $\lambda_1(\omega)$ is revealed, he starts a *new* liquidation problem in $[\tau_1, T]$, *still committed to fully liquidate at the originally chosen (i.e. at* $t = 0$*) time* T.

For the remaining trading period $(\tau_1, T]$, he now uses the accurately revised model dynamics

$$dX_t^{\lambda_1} = -\xi_t dt \quad \text{and} \quad dR_t^{\lambda_1} = -\lambda_1 \xi_t^2 dt + X_t^{\lambda_1} dW_t, \quad \text{for } t \in (\tau_1, T], \tag{3.37}$$

with $X_{\tau_1}^{\lambda_1} = X_{\tau_1}^* > 0$ and $R_{\tau_1}^{\lambda_1} = R_{\tau_1}^*$, where $X_{\tau_1}^*, R_{\tau_1}^*$ are the optimal inventory and revenue realized at τ_1. Naturally, the values $X_{\tau_1}^*$ and $R_{\tau_1}^*$ have inherited the model mispecification error, $\hat{\lambda}$ instead of λ_1 in $[\tau_1, T]$, as the explicit expressions below show.

To construct the solution in $[0, \tau_1)$, the agent needs to first solve problem $\mathcal{L}\left(\hat{\lambda}; \tau_1, T\right)$, as required by backward induction. Of course, a posteriori, this will be a "virtual" problem, since the agent will never encounter it, for he will switch to the correct model (3.37) at the model revision time τ_1. However, the solution in $[0, \tau_1)$ does depend on $\mathcal{L}\left(\hat{\lambda}; \tau_1, T\right)$ due to the backward construction, and, thus, on the initial choice of the model input $\hat{\lambda}$, which at τ_1 will turn out to be inaccurate.

To apply the backward induction, we first solve $\mathcal{L}\left(\hat{\lambda}; \tau_1, T\right)$. In analogy to (3.7), the value function and optimal policy are given, for $(x, r) \in \mathbb{R}^+ \times \mathbb{R}$ and $t \in [\tau_1, T]$, by

$$\hat{V}(x, r, t; T) = -e^{-r+\sqrt{\frac{\hat{\lambda}}{2}}x^2 \coth \frac{T-t}{\sqrt{2\hat{\lambda}}}} \qquad \text{and} \qquad \hat{\xi}_t^* = X_{\tau_1}^* \frac{\cosh \frac{T-t}{\sqrt{2\hat{\lambda}}}}{\sinh \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}}}. \tag{3.38}$$

In $[0, \tau_1]$, we solve an analogous optimal liquidation problem - but without requiring full liquidation at time τ_1 - with dynamics as in (3.35) and terminal utility

$$V(x, r, \tau_1; \tau_1) = \hat{V}(x, r, \tau_1; T) = -e^{-r+x^2\sqrt{\frac{\hat{\lambda}}{2}} \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}}}.$$

The associated HJB equation is the same as (3.6) with the above terminal condition (instead of (3.5)). We again look for separable solutions of the form $V(x, r, t; \tau_1) = -e^{-r+h(x,t)}$, with $h(x, t) = x^2 g(t)$, for some function g. Then, for $t \in [0, \tau_1)$, h will satisfy (3.16) with terminal condition

$h(x,\tau_1) = \sqrt{\frac{\hat{\lambda}}{2}} x^2 \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}}$ and, thus, g must solve

$$g'(t) = \frac{1}{\lambda_0} g^2(t) - \frac{1}{2} \quad \text{with} \quad g(\tau_1) = \sqrt{\frac{\hat{\lambda}}{2}} \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}}.$$

Let $c := \sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}} = \sqrt{\frac{2}{\lambda_0}} g(\tau_1)$. We have the following cases:

i) Let $c > 1$, or equivalently, $g(\tau_1) > \sqrt{\frac{\lambda_0}{2}}$. Then, setting $C^1 := \tau_1 + \sqrt{2\lambda_0} \coth^{(-1)} \left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}} \right)$, we have

$$g(t) = \sqrt{\frac{\lambda_0}{2}} \coth \frac{C^1 - t}{\sqrt{2\lambda_0}} = \sqrt{\frac{\lambda_0}{2}} \coth \left(\frac{\tau_1 - t}{\sqrt{2\lambda_0}} + \coth^{(-1)} \left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}} \right) \right)$$

$$= \sqrt{\frac{\lambda_0}{2}} \coth \left(\frac{\tau_1 - t}{\sqrt{2\lambda_0}} + \coth^{(-1)} \left(\sqrt{\frac{2}{\lambda_0}} g(\tau_1) \right) \right).$$

Then,

$$V(x,r,t;\tau_1) = -e^{-r + x^2 \sqrt{\frac{\lambda_0}{2}} \coth \frac{C^1 - t}{\sqrt{2\lambda_0}}},$$

and the feedback control function is given by $\xi^*(x,t) = \frac{x}{\sqrt{2\lambda_0}} \coth \frac{C^1 - t}{\sqrt{2\lambda_0}}$.

The optimal inventory and revenue processes are given by

$$X_t^* = x \frac{\sinh \frac{C^1 - t}{\sqrt{2\lambda_0}}}{\sinh \frac{C^1}{\sqrt{2\lambda_0}}},$$

$$R_t^* = r - \int_0^t \lambda_0 (\xi_s^*)^2 \, ds + \int_0^t X_s^* dW_s$$

$$= r - \frac{x^2}{4 \left(\sinh \frac{C^1}{\sqrt{2\lambda_0}} \right)^2} \left(t + \sqrt{\frac{\lambda_0}{2}} \left(\sinh \left(\sqrt{\frac{2}{\lambda_0}} C^1 \right) - \sinh \left(\sqrt{\frac{2}{\lambda_0}} (C^1 - t) \right) \right) \right)$$

$$+ \frac{x}{\sinh \frac{C^1}{\sqrt{2\lambda_0}}} \int_0^t \sinh \frac{C^1 - s}{\sqrt{2\lambda_0}} dW_s,$$

for $t \in [0, \tau_1)$.

ii) Let $c < 1$, or, equivalently, $g(\tau_1) < \sqrt{\frac{\lambda_0}{2}}$. Then, setting $C_1 := \tau_1 + \sqrt{2\lambda_0} \tanh^{(-1)} \left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}} \right)$, we have

$$g(t) = \sqrt{\frac{\lambda_0}{2}} \tanh\left(\frac{C_1 - t}{\sqrt{2\lambda_0}}\right) = \sqrt{\frac{\lambda_0}{2}} \tanh\left(\frac{\tau_1 - t}{\sqrt{2\lambda_0}} + \tanh^{(-1)}\left(\sqrt{\frac{\hat{\lambda}}{\lambda_0}} \coth \frac{T - \tau_1}{\sqrt{2\hat{\lambda}}}\right)\right)$$

$$= \sqrt{\frac{\lambda_0}{2}} \tanh\left(\frac{\tau_1 - t}{\sqrt{2\lambda_0}} + \tanh^{(-1)}\left(\sqrt{\frac{2}{\lambda_0}} g(\tau_1)\right)\right).$$

Then,

$$V(x, r, t; \tau_1) = -e^{-r + x^2 \sqrt{\frac{\lambda_0}{2}} \tanh \frac{C_1 - t}{\sqrt{2\lambda_0}}}$$

and the feedback control $\xi^*(x, t) = \frac{x}{\sqrt{2\lambda_0}} \tanh \frac{C_1 - t}{\sqrt{2\lambda_0}}$. Then,

$$X_t^* = x \frac{\cosh \frac{C_1 - t}{\sqrt{2\lambda_0}}}{\sinh \frac{C_1}{\sqrt{2\lambda_0}}},$$

$$R_t^* = r - \int_0^t \lambda_0 (\xi_s^*)^2 ds + \int_0^t X_s^* dW_s$$

$$= r + \frac{x^2}{4\left(\sinh \frac{C_1}{\sqrt{2\lambda_0}}\right)^2} \left(t - \sqrt{\frac{\lambda_0}{2}} \left(\sinh\left(\sqrt{\frac{2}{\lambda_0}} C_1\right) - \sinh\left(\sqrt{\frac{2}{\lambda_0}} (C_1 - t)\right)\right)\right)$$

$$+ \frac{x}{\sinh \frac{C_1}{\sqrt{2\lambda_0}}} \int_0^t \cosh \frac{C_1 - s}{\sqrt{2\lambda_0}} dW_s,$$

for $t \in [0, \tau_1)$.

iii) Let $c = 1$, or equivalently, $g(\tau_1) = \sqrt{\frac{\lambda_0}{2}}$.

Then, $V(x, r, t; \tau_1) = -e^{-r + x^2 \sqrt{\frac{\lambda_0}{2}}}$ and $\xi^*(x, t) = \frac{x}{\sqrt{2\lambda_0}}$. It follows that $X_t^* = x e^{-\frac{t}{\sqrt{2\lambda_0}}}$, and

$$R_t^* = r - \int_0^t \lambda_0 (\xi_s^*)^2 ds + \int_0^t X_s^* dW_s$$

$$= r + x^2\sqrt{\frac{\lambda_0^3}{2}}\left(e^{-\sqrt{\frac{2}{\lambda_0}}t} - 1\right) + x\int_0^t e^{-\frac{s}{\sqrt{2\lambda_0}}}dW_s,$$

for $t \in [0, \tau_1)$.

At time τ_1, the *true* value $\lambda_1 \in \mathcal{F}_{\tau_1}$ is revealed. If $\lambda_1 = \hat{\lambda}$, then the solution in $[\tau_1, T]$ is given by $\hat{V}$ and the the optimal policy in (3.38).

If, on the other hand, $\lambda_1 \neq \hat{\lambda}$, the agent adjusts his model dynamics to (3.37) and starts a *new* liquidation problem $\mathcal{L}(\lambda_1; \tau_1, T)$ with initial inventory and revenue given by $X^*_{\tau_1}, R^*_{\tau_1}$ obtained above for each case.

For this new problem, $\mathcal{L}(\lambda_1; \tau_1, T)$, we have, for $t \in [\tau_1, T]$,

$$V^1(x, r, t; \tau_1, T) = -\exp\left(-r + \sqrt{\frac{\lambda_1}{2}}x^2 \coth\frac{T-t}{\sqrt{2\lambda_1}}\right), \tag{3.39}$$

with $V^1(x, r, T; \tau_1, T) = v(x, r)$, v as in (3.5), and

$$\xi_t^{1,*} = \frac{1}{\sqrt{2\lambda_1}}X_t^* \coth\frac{T-t}{\sqrt{2\lambda_1}} \qquad \text{and} \qquad X_t^{1,*} = X^*_{\tau_1}\frac{\sinh\frac{T-t}{\sqrt{2\lambda_1}}}{\sinh\frac{T-\tau_1}{\sqrt{2\lambda_1}}}.$$

As expected, $X_T^{1,*} = 0$.

Notice that even though the agent considers an entirely new liquidation model in $[\tau_1, T]$, the initial wrong assessment $\hat{\lambda}$ - instead of the true, in hindsight, λ_1 - still enters in the solution of $\mathcal{L}(\lambda_1; \tau_1, T)$ through the initial condition $X^*_{\tau_1}$, as it depends on $g(\cdot)$, which itself depends on $\lambda_0, \hat{\lambda}$ through $g(\tau_1)$ above.

In summary, if $\hat{\lambda} \neq \lambda_1$, the *realized strategy* of the *naive* agent, denoted by ξ^a, is given by

$$\xi_t^a = \xi_t^* \mathbf{1}_{\{t<\tau_1\}} + \xi_t^{1,*}\mathbf{1}_{\{\tau_1 \le t \le T\}}.$$

It is *discontinuous* at τ_1, with the discontinuity $\Delta^*_{\tau_1}\left(\lambda_0, \hat{\lambda}, \lambda_1\right) := \lim_{t\downarrow\tau_1} \xi^{1,*}_t - \lim_{t\uparrow\tau_1} \xi^*_t$ given by

$$\Delta^*_{\tau_1}\left(\lambda_0, \hat{\lambda}, \lambda_1\right) = X^*_{\tau_1}\left(\frac{1}{\sqrt{2\lambda_1}}\coth\frac{T-t}{\sqrt{2\lambda_1}} - \frac{1}{\sqrt{2\lambda}}\coth\frac{T-t}{\sqrt{2\lambda_0}}\right),$$

with $X^*_{\tau_1}$ being given as in each case considered above. Naturally, if $\hat{\lambda} = \lambda_1$, $\Delta^*_{\tau_1}(\lambda_0, \lambda_1, \lambda_1) = 0$. Also, it is easy to check that

$$\Delta^*_{\tau_1}\left(\lambda_0, \hat{\lambda}, \lambda_1\right) \gtreqless 0, \quad \text{if } \lambda_0 \gtreqless \lambda_1,$$

which indicates the intuitive behavior of accelerating (decelerating) liquidation if the unanticipated market condition becomes favorable (unfavorable, respectively).

The *realized inventory* is given by

$$X^a_t = \begin{cases} x\frac{\sinh\frac{C^1-t}{\sqrt{2\lambda_0}}}{\sinh\frac{C^1}{\sqrt{2\lambda_0}}}, & \text{if } c>1,\ 0\le t<\tau_1, \\ x\frac{\cosh\frac{C_1-t}{\sqrt{2\lambda_0}}}{\sinh\frac{C_1}{\sqrt{2\lambda_0}}}, & \text{if } c<1,\ 0\le t<\tau_1, \\ xe^{-\frac{t}{\sqrt{2\lambda_0}}}, & \text{if } c=1,\ 0\le t<\tau_1, \\ X^a_{\tau_1}\frac{\sinh\frac{T-t}{\sqrt{2\lambda_1}}}{\sinh\frac{T-\tau_1}{\sqrt{2\lambda_1}}}, & \text{if } \tau_1\le t\le T. \end{cases}$$

with $X^a_T = 0$. It is continuous in $[0, T]$.

The value function process associated to the above strategy ξ^a_t is given by

$$V^a(x, r, t) = V(x, r, t; \tau_1, T)\,\mathbf{1}_{\{0\le t<\tau_1\}} + V^1(x, r, t; \tau_1, T)\,\mathbf{1}_{\{\tau_1\le t\le T\}}$$

with V computed for each case above, and V^1 as in (3.39).

Naturally, for $\hat{\lambda} \neq \lambda_1$, $V^a(x, r, t)$ is *discontinuous* at τ_1, which results from the fact that the agent *totally discards* the previously realized performance as soon as the model dynamics change at time τ_1. Indeed,

$$\lim_{t \uparrow \tau_1} V^a(x, r, t; \tau_1, T) = \lim_{t \uparrow \tau_1} V(x, r, t; \tau_1, T)$$
$$= \lim_{t \downarrow \tau_1} \hat{V}(x, r, t; T) = -\exp\left(-r + \sqrt{\frac{\hat{\lambda}}{2}} x^2 \coth \frac{T - \tau_1}{\sqrt{2\hat{\lambda}}}\right),$$

while

$$V^1(x, r, \tau_1; \tau_1, T) = -\exp\left(-r + \sqrt{\frac{\lambda_1}{2}} x^2 \coth \frac{T - \tau_1}{\sqrt{2\lambda_1}}\right).$$

3.3.2 The forward agent

The forward agent starts at $t = 0$ with initial inventory x. She also assesses the level of the market impact parameter, λ_0, and, like the pre-committed management, she is aware that λ_0 will change at τ_1 without knowing at $(t = 0)$ its upcoming new level.

However, at $t = 0$, she neither pre-specifies a value for the market parameter in $[\tau_1, T]$ nor a full-liquidation time. Rather, she only specifies an initial criterion $U(x, r, 0) = -e^{-r + kx^2}$ of form (3.11), for some constant $k > 0$, and solves the inverse liquidation problem $\mathcal{P}(\lambda_0, k; 0)$.

One interpretation of the choice $U(x, r, 0)$ follows from the result of Theorem 3.2.2. Indeed, according to (3.28), the optimal trading rate under the performance criterion $U(x,r,0)$ at $t=0$ is $\zeta_0^* = \frac{k}{\lambda_0}x$. Therefor, for fixed $\lambda_0 > 0$ and total inventory $x > 0$ to liquidate at $t = 0$, the initial performance criterion has a one to one correspondence to the initial trading rate. In other words, by specifying a criterion $U(x,r,0)$ to the forward trading agent, the client proposes an initial trading profile through the trading rate that she seeks to consistently preserve in the future.

The fact that k is, for now, arbitrary is only for mere generality and for showing how we can construct the solution for any given initial condition. In the sequel, when we *compare* the performance of the two managers, we will choose k accordingly, for meaningful comparisons.

Let $T(\lambda_0, k)$ be the solvability time of problem $\mathcal{P}(\lambda_0, k; 0)$ and let $m := k\sqrt{\frac{2}{\lambda_0}}$.

Case 1: Model parameter is revised *before* the solvability time: $\tau_1 < T(\lambda_0, k)$

If $m < 1$, then no full liquidation occurs in $[0, T(\lambda_0, k))$ and, thus, neither in $[0, \tau_1]$. Then, equations (3.29), (3.9) and (3.17) yield, with F, G as in (3.13), that, for $t \in [0, \tau_1]$,

$$X_t^* = xF(t; m, \lambda_0) > 0 \qquad \text{and} \qquad R_t^* = r - \lambda_0 \int_0^t \left(\zeta_s^*\right)^2 ds + \int_0^t X_s^* dW_s, \quad (3.40)$$

and

$$U(x,r,t) = -\exp\left(-r + k\frac{G(t; m, \lambda_0)}{F(t; m, \lambda_0)}x^2\right).$$

Therefore, $U(x, r, \tau_1)$ can be written as

$$U(x, r, \tau_1) = -e^{-r+k_1 x^2} \qquad \text{with} \quad k_1 := k \frac{G(\tau_1; m, \lambda_0)}{F(\tau_1; m, \lambda_0)}.$$

At τ_1, the agent learns the *true* value $\lambda_1 \in \mathcal{F}_{\tau_1}$ and considers a *new* inverse liquidation problem, in complete analogy to $\mathcal{P}(\lambda_0, k; 0)$. Specifically, she solves problem $\mathcal{P}(\lambda_1, k_1; \tau_1)$ with *initial datum* $U(x, r, \tau_1)$, setting $x = X^*_{\tau_1}$ and $r = R^*_{\tau_1}$, where (cf. (3.40)),

$$X^*_{\tau_1} = xF(\tau_1; m, \lambda_0) > 0 \qquad \text{and} \qquad R^*_{\tau_1} = r - \lambda_0 \int_0^{\tau_1} \left(\zeta^*_s\right)^2 ds + \int_0^{\tau_1} X^*_s dW_s.$$

Observe that this new inverse liquidation problem $\mathcal{P}(\lambda_1, k_1; \tau_1)$, introduced at time τ_1, captures the "real-time" change λ_1 at τ_1, but also incorporates the "past", since its initial condition $U\left(X^*_{\tau_1}, R^*_{\tau_1}, \tau_1\right)$ depends, through its form and each of its arguments, on the model input (k, λ_0, τ_1), which was *chosen at initial time* $t = 0$.

We can now solve $\mathcal{P}(\lambda_1, k_1; \tau_1)$ using arguments similar to the ones in the proof of Theorem 3.2.2.

To this end, let

$$m_1 := k_1 \sqrt{\frac{2}{\lambda_1}} = k_0 \frac{G(\tau_1; m_0, \lambda_0)}{F(\tau_1; m_0, \lambda_0)} \sqrt{\frac{2}{\lambda_1}} = k_0 \frac{\cosh \frac{\tau_1}{\sqrt{2\lambda_0}} - \frac{1}{m_0} \sinh \frac{\tau_1}{\sqrt{2\lambda_0}}}{\cosh \frac{\tau_1}{\sqrt{2\lambda_0}} - m_0 \sinh \frac{\tau_1}{\sqrt{2\lambda_0}}} \sqrt{\frac{2}{\lambda_1}}.$$

If $m_1 < 1$, the solvability time $T(\lambda_1, k_1; \tau_1)$ of the new problem is given (cf. (3.15)) by $T(\lambda_1, k_1; \tau_1) = \sqrt{2\lambda_1} \ln \sqrt{\frac{1+m_1}{1-m_1}}$.

For convenience, we set

$$T_1 := \tau_1 + T(\lambda_1, k_1; \tau_1) = \tau_1 + \sqrt{2\lambda_1} \ln \sqrt{\frac{1+m_1}{1-m_1}}. \tag{3.41}$$

Then, for $t \in [\tau_1, T_1)$, the solution U^1 is given by

$$U^1(x, r, t) = -\exp\left(-r + k_1 x^2 \frac{G(t-\tau_1; m_1, \lambda_1)}{F(t-\tau_1; m_1, \lambda_1)}\right) \in \mathcal{F}_{\tau_1},$$

which, by construction, satisfies at revision time τ_1 the pasting condition

$$U(x, r, \tau_1) = U^1(x, r, \tau_1).$$

Furthermore, the optimal inventory $X_t^{1,*}$ and liquidation strategy $\zeta_t^{1,*}$ are given, for $t \in [\tau_1, T_1)$, by

$$X_t^{1,*} = xF(\tau_1; m, \lambda_0) F(t-\tau_1; m_1, \lambda_1) \in \mathcal{F}_{\tau_1}$$

and

$$\zeta_t^{1,*} = X_{\tau_1}^* \frac{k_1}{\lambda_1} \frac{G(t-\tau_1; m_1, \lambda_1)}{F(t-\tau_1; m_1, \lambda_1)} = xF(\tau_1; m, \lambda_0) \frac{k_1}{\lambda_1} \frac{G(t-\tau_1; m_1, \lambda_1)}{F(t-\tau_1; m_1, \lambda_1)}.$$

Combining the above and (3.30), we deduce that there is *non-zero* optimal inventory left at T_1, given by

$$X_{T_1}^{1,*} = xF(\tau_1; m, \lambda_0) \sqrt{(1-m_1)(1+m_1)} > 0.$$

If $m_1 = 1$, then $T(\lambda_1, k_1; \tau_1) = \infty$ and, for $t \in [\tau_1, \infty)$,

$$X_t^{1,*} = X_{\tau_1}^* e^{-\frac{k_1}{\lambda_1}(t-\tau_1)} = xF(\tau_1; m, \lambda_0) F(t-\tau_1; 1, \lambda_1) \in \mathcal{F}_{\tau_1}$$

and

$$\zeta_t^{1,*} = X_{\tau_1}^* \frac{k_1}{\lambda_1} = xF\left(\tau_1; m, \lambda_0\right) F\left(t - \tau_1; 1, \lambda_1\right).$$

Finally, if $m_1 > 1$, then $T\left(\lambda_1, k_1; \tau_1\right) = \sqrt{2\lambda_1} \ln \sqrt{\frac{1+m}{m-1}}$. Setting

$$T_1 := \tau_1 + T\left(\lambda_1, k_1; \tau_1\right) = \tau_1 + \sqrt{2\lambda_1} \ln \sqrt{\frac{1+m_1}{m_1 - 1}},$$

then, the results in Theorem 3.2.2 yield that the optimal inventory and liquidation policies are given, for $t \in [\tau_1, T_1)$ by

$$X_t^{1,*} = \; xF\left(\tau_1; m, \lambda_0\right) F\left(t - \tau_1; m_1, \lambda_1\right) \in \mathcal{F}_{\tau_1}$$

and

$$\zeta_t^{1,*} = X_{\tau_1}^* \frac{k_1}{\lambda_1} \frac{G\left(t - \tau_1; m_1, \lambda_1\right)}{F(t - \tau_1; m_1, \lambda_1)} = xF\left(\tau_1; m, \lambda_0\right) \frac{k_1}{\lambda_1} \frac{G\left(t - \tau_1; m_1, \lambda_1\right)}{F(t - \tau_1; m_1, \lambda_1)}.$$

We deduce that *full-liquidation* occurs at T_1, i.e., $X_{T_1}^{1,*} = 0$.

Case 2: Model parameter is revised *after* or *at* the solvability time: $T\left(\lambda_0, k\right) \leq \tau_1 < \infty$

This case is viable only if $m \neq 1$. The parameter revision per se is irrelevant, for the inverse liquidation problem $\mathcal{P}\left(\lambda_0, k; 0\right)$ is not well defined beyond time $T\left(\lambda_0, k\right)$. Trading stops at $T\left(\lambda_0, k\right)$, and the final inventory is given by (3.30).

Therefore, if $m < 1$, there is non-zero inventory left, given by $X_{T(\lambda_0,k)}^* = x\sqrt{\left(1-m\right)\left(1+m\right)}$.

On the other hand, if $m > 1$, problem $\mathcal{P}\left(\lambda_0, k; 0\right)$ is not well defined for times beyond $T\left(\lambda_0, k\right)$. However, full-liquidation does occur at time $T\left(\lambda_0, k\right)$, $X_{T(\lambda_0,k)}^* = 0$.

3.4 Comparative analysis for the *naive* and *forward* liquidation strategies

Both agents start at $t = 0$, having the same initial inventory, same assessement and knowledge about the upcoming change of the market impact parameter.

The pre-committed agent chooses to liquidate at T, and thus he is obliged to choose at $t = 0$ a specific model for the entire $[0, T]$. As mentioned earlier, he chooses (3.35) and (3.36), with $\hat{\lambda}$ reflecting his best guess at $t = 0$ for the value of the parameter in the future period $(\tau_1, T]$.

The forward agent exhibits different behavior, choosing not to commit at $t = 0$ to any parameter selection beyond the (a priori known) revision time τ_1. Furthermore, she does not impose any full-liquidation horizon but, rather, chooses an initial criterion $U(x, r, 0)$.

To draw meaningful comparisons for the two agents, we assume that their *initial conditions coincide*, i.e. for $(x, r) \in \mathbb{R}^+ \times \mathbb{R}$,

$$U(x, r, 0) = V(x, r, 0; T),$$

with $V(x, r, 0; T)$ as in previous section for the cases (i)-(iii), corresponding to different regimes of $\hat{\lambda}$. In other words, the forward agent chooses as her initial datum to be the value function of the naive one. Note that this initial choice for the forward agent does induce indirect dependence on both $\hat{\lambda}$ and T, since $V(x, r, 0; T)$ is the solution of the backward optimization problem over

$[0, T]$ for model (3.35), (3.36). This is unavoidable, if we want to compare the behaviors and policies of the two agents.

3.4.1 Comparative performance metrics

To draw comparisons between the naive and the forward agents, we introduce two *performance metrics*, denoted by $\mathcal{R}_0^1(x,r)$ and $\mathcal{R}_0^2(x,r)$.

- Metric $\mathcal{R}_0^1(x,r)$ is defined as

$$\mathcal{R}_0^1(x,r) := \mathbb{E}\Big[U_\tau(X_\tau^{\zeta^a}, R_\tau^{\zeta^a})\Big] - V\left(x,r,0\right),$$

  and measures the discrepancy between the actual average performance and the perceived optimal performance at $t=0$.

- Metric $\mathcal{R}_0^2(x,r)$ is defined as

$$\mathcal{R}_0^2(x,r) := \mathbb{E}\Big[U_\tau(X_\tau^{\zeta^a}, R_\tau^{\zeta^a})\Big] - \sup_{\zeta} \mathbb{E}\left[U_\tau(X_\tau^{\zeta}, R_\tau^{\zeta})\right],$$

  and measures the discrepancy between the actual average performance and the true optimal performance in hindsight, with the full knowledge of the underlying model. The metric $\mathcal{R}_0^2(x,r)$ can be interpreted as the "regret" of the trading agent for not having taken the genuine optimal policy under the true measure $\mathbb{P}$ that is not fully known at $t=0$.

Here, the expectation is taken with respect to $\mathbb{P}$, under which $\lambda_1 \in \mathcal{F}_{\tau_1}$ is correctly modeled. The criterion $U_\tau(x,r)$ and the evaluation horizon τ vary for different problems. For instance, for the naive agent, $\tau = T$ is the pre-specified liquidation horizon, and $U_\tau(x,r) = -e^{-r}$, since $X_\tau^\zeta = 0$ always holds for any admissible ζ, under the singular terminal condition $v(x,r)$ in (3.5).

On the other hand, for a forward agent with model revision time $\tau_1 < T(\lambda_0, k)$, the evaluation horizon $\tau = T_1 \in \mathcal{F}_{\tau_1}$ is the *revised* liquidation horizon, whereas the criterion $U_\tau(x,r) = U^1(x,r,T_1) \in \mathcal{F}_{\tau_1}$ is the corresponding forward performance criterion at T_1.

We first examine $\mathcal{R}_0^1(x,r)$ for the naive agent. Recall that the perceived value function $V(x,r,0)$ is calculated for cases (i)-(iii) under the $t=0$ perceived model (3.35) and (3.36).

To compute the actual average performance of the naive agent, we notice that under the true measure $\mathbb{P}$,

$$\begin{aligned}\mathbb{E}\left[-e^{-R_T^{\zeta^a}}\right] &= \mathbb{E}\left[\mathbb{E}\left[-e^{-R_T^{\zeta^a}}\middle|\mathcal{F}_{\tau_1}\right]\right] = \mathbb{E}\left[V^1\left(X_{\tau_1}^{\zeta^a}, R_{\tau_1}^{\zeta^a}, \tau_1\right)\right] \\ &= \mathbb{E}\left[-\exp\left(-R_{\tau_1}^{\zeta^a} + \sqrt{\frac{\lambda_1}{2}}\left(X_{\tau_1}^{\zeta^a}\right)^2 \coth\left(\frac{T-\tau_1}{\sqrt{2\lambda_1}}\right)\right)\right], \qquad (3.42)\end{aligned}$$

where $R_T^{\zeta^a}$ is the terminal revenue generated by the strategy ζ^a over $[0,T]$, and $R_{\tau_1}^{\zeta^a}, X_{\tau_1}^{\zeta^a}$ are the revenue and remaining inventory to liquidate at τ_1.

The exact value of the expectation in (3.42) is in general unknown, unless we further specify how the Brownian motion over $[0,\tau_1)$ (as it appears in $R_{\tau_1}^{\zeta^a}$) is *correlated* with λ_1 under the true measure $\mathbb{P}$.

Nevertheless, we can make qualitative comparisons between this true average performance and the perceived performance. For instance, if $\lambda_1 = \hat{\lambda}$

a.s., under $\mathbb{P}$, then, the true performance coincides with the perceived one, as expected.

Another intuitive observation is that, since the function

$$f(\lambda_1) = \sqrt{\frac{\lambda_1}{2}} \left(X_{\tau_1}^{\zeta^a}\right)^2 \coth\left(\frac{T-\tau_1}{\sqrt{2\lambda_1}}\right)$$

is increasing in λ_1 (notice that $X_{\tau_1}^{\zeta^a}$ is a deterministic constant that does *not* depend on λ_1), we can conclude that if $\lambda_1 \geq \hat{\lambda}$ a.s., the actual performance would be dominated by the perceived performance on average, and vice versa.

This is an intuitive fact, as in a market with unfavorable liquidation condition (e.g., price impact would increase with high probability), even if the agent makes direct response to the realized market condition $\lambda_1 \in \mathcal{F}_{\tau_1}$, he may still undergo tremendous loss compared to what he has perceived at $t = 0$. Such instability of performance is due to the "stubbornness to a fixed criterion" that cannot incorporate the unexpected market changes along real-time in a consistent manner.

3.4.2 Regret and the forward approach

As it was shown in the previous section, the metric $\mathcal{R}_0^1(x, r)$ for the naive agent can be positive or negative, depending on whether the true future market condition is sufficiently better or worse than initially perceived in a reasonable way.

Next, we consider the forward liquidation behavior under the two introduced metrics and eventually demonstrate its stability property under both of the two metrics.

For a forward agent who revises the model parameter before the solvability time, i.e., $\tau_1 < T(\lambda_0, k)$, we have derived the actual strategy she would follow, ζ_t^* for $t \in [0, \tau_1)$ and $\zeta_t^{1,*}$ for $t \in [\tau_1, T_1)$. Denote such strategy by $\zeta^{a,*}$, with

$$\zeta_t^{a,*} := \zeta_t^* \mathbf{1}_{\{t<\tau_1\}} + \zeta_t^{1,*} \mathbf{1}_{\{\tau_1 \le t < T_1\}}.$$

We first consider the metric $\mathcal{R}_0^1(x, r)$ for the forward liquidation behavior. Recall that we have chosen $U(x, r, 0) = V(x, r, 0)$,

On the other hand, the actual average performance of the forward agent is

$$\mathbb{E}\left[U\left(X_{T_1}^{\zeta^{a,*}}, R_{T_1}^{\zeta^{a,*}}, T_1\right)\right] = \mathbb{E}\left[\mathbb{E}\left[U\left(X_{T_1}^{\zeta^{a,*}}, R_{T_1}^{\zeta^{a,*}}, T_1\right) \,\middle|\, \mathcal{F}_{\tau_1}\right]\right]$$
$$= \mathbb{E}\left[U\left(X_{\tau_1}^{\zeta^{a,*}}, R_{\tau_1}^{\zeta^{a,*}}, \tau_1\right) \,\middle|\, X_0^{\zeta^{a,*}} = x,\ R^{\zeta^{a,*}} = r\right] = U(x, r, 0).$$

It hence follows that $\mathcal{R}_0^1(x, r) = 0$ for the forward agent.

To examine the regret metric, we now focus on the hypothetical value function $V^{\text{True}}(x, r, t)$, which can be computed based on the true full model under $\mathbb{P}$.

We solve this virtual problem with backward induction, as it is applicable in the current scenario; i.e., we first solve the problem

$$V^{\text{True}}(x, r, \tau_1; \lambda_1) := \operatorname*{esssup}_{\zeta} \mathbb{E}\left[U\left(X_{T_1}^{\zeta}, R_{T_1}^{\zeta}, T_1\right) \,\middle|\, X_{\tau_1}^{\zeta} = x,\ R_{\tau_1}^{\zeta} = r,\ \mathcal{F}_{\tau_1}\right]. \tag{3.43}$$

Here, the forward utility $U(x,r,T_1)$ is constructed forward in time, such that

$$U(x,r,\tau_1) = \operatorname*{esssup}_{\zeta} \mathbb{E}\left[U\left(X^{\zeta}_{T_1}, R^{\zeta}_{T_1}, T_1\right) \Big|\ X^{\zeta}_{\tau_1} = x,\ R^{\zeta}_{\tau_1} = r,\ \mathcal{F}_{\tau_1}\right],\ \text{a.s..} \tag{3.44}$$

According to Theorem 3.2.2, conditional on $\mathcal{F}_{\tau_1}$, the solution $U(x,r,T_1)$ to problem (3.44) exists, and the essential supremum is attained by $\zeta^{1,*}$. It follows, by the uniqueness of essential supremum, $V^{\text{true}}(x,r,\tau_1) = U(x,r,\tau_1)$ a.s, under $\mathbb{P}$.

By backward induction, solving the problem over $[0,\tau_1)$ using such intermediate value function $V^{\text{true}}(x,r,\tau_1)$ gives the true optimal value at $t=0$, namely,

$$\begin{aligned} V^{\text{True}}(x,r,0) &= \sup_{\zeta} \mathbb{E}\left[V^{\text{True}}\left(X^{\zeta}_{\tau_1}, R^{\zeta}_{\tau_1}, \tau_1\right)\Big|\ X^{\zeta}_0 = x,\ R^{\zeta}_0 = r\right] \\ &= \sup_{\zeta} \mathbb{E}\left[U\left(X^{\zeta}_{\tau_1}, R^{\zeta}_{\tau_1}, \tau_1\right)\Big|\ X^{\zeta}_0 = x,\ R^{\zeta}_0 = r\right] = U(x,r,0), \end{aligned}$$

with the last equality, again, follows from Theorem 3.2.2. Hence, we have shown *zero regret* for the forward liquidation behavior.

The study of the regret metric $\mathcal{R}^2_0(x,r)$ for the naive agent, similar to that of $\mathcal{R}^1_0(x,r)$, requires more specific knowledge of the interaction between the Brownian motion and λ_1, under the true measure $\mathbb{P}$. Nonetheless, it is clear that although the naive agent reacts promptly at τ_1 to the true model when it is revealed, the overall policy ζ^a implemented is only one admissible policy. Therefore, in general, such policy cannot outperform the optimal policy computed under the true measure $\mathbb{P}$, which leads to $\mathcal{R}_0^2(x,r) \leq 0$ for the naive agent, in contrast with the zero regret stability under the forward approach.

3.4.3 Comparison under adverse market conditions

The forward approach has demonstrated sound stability under the two metrics $\mathcal{R}_0^1(x,r)$ and $\mathcal{R}_0^2(x,r)$, which is generally lacking under the classical optimization framework. In this section, we will further show that the forward agent outperforms the naive agent in terms of the liquidation revenue, under unanticipated adverse market condition that mostly concerns the trading agents (e.g., the 2010 Flash Crash). Such unfavorable market condition corresponds to that a large price impact $\lambda_1 \in \mathcal{F}_{\tau_1}$ is realized at $t = \tau_1$. The naive agent would nonetheless have a full liquidation at T, regardless of the adverse market condition. The forward agent, on the other hand, has the ability to endogenously determine the revised liquidation horizon and the volume to trade, which allows her to obtain higher expected liquidation revenue with comparable variance, as we will show next.

Recall that we have taken $U(x,r,0) = V(x,r,0)$ for the forward agent. Then, for case 1 in section 3.2 (i.e., $\tau_1 < T(\lambda_0,k)$), direct computation yields that the forward and the naive agents have the same optimal policy , i.e., $\zeta_t^* = \xi_t^*$, for $t \in [0,\tau_1)$. Moreover, both of them execute their policies over $[0,\tau_1)$ in the underlying market with the common parameter λ_0. It follows that the two agents have the same (non-zero) inventory and revenue at $t = \tau_1$, denoted by $X_{\tau_1}^*$ and $R_{\tau_1}^*$, respectively. We also denote their terminal revenue by $R_T^{n,*}$ and $R_{T_1}^{f,*}$, with T_1 given as in (3.41) corresponding to $m_1 < 1$ (i.e., λ_1 large). Notice that

$$R_T^{n,*} = R_{\tau_1}^* - \lambda_1 \int_{\tau_1}^T \left(\xi_t^{1,*}\right)^2 dt + \int_{\tau_1}^T X_t^{n,*} dW_t,$$

and

$$R_T^{f,*} = R_{\tau_1}^* - \lambda_1 \int_{\tau_1}^{T_1} \left(\zeta_t^{1,*}\right)^2 dt + \int_{\tau_1}^{T_1} X_t^{f,*} dW_t,$$

with $X_t^{n,*}$, $X_t^{f,*}$ being the inventory processes for the naive and forward agent, after the model revision time τ_1. As the revenue at τ_1 is the same for both agents, we aim to compare the conditional mean revenue $\mathbb{E}[R_T^{n,*}|\mathcal{F}_{\tau_1}]$ and $\mathbb{E}[R_{T_1}^{f,*}|\mathcal{F}_{\tau_1}]$, as well as the conditional variance $\mathrm{Var}[R_T^{n,*}|\mathcal{F}_{\tau_1}]$ and $\mathrm{Var}[R_{T_1}^{f,*}|\mathcal{F}_{\tau_1}]$. For the former, due to $\lambda_1, T_1 \in \mathcal{F}_{\tau_1}$, we obtain

$$\mathbb{E}\Big[R_{T_1}^{f,*}|\mathcal{F}_{\tau_1}\Big] - \mathbb{E}\Big[R_T^{n,*}|\mathcal{F}_{\tau_1}\Big] = \mathbb{E}\Big[\lambda_1 \int_{\tau_1}^T \left(\xi_t^{1,*}\right)^2 dt \Big| \mathcal{F}_{\tau_1}\Big] - \mathbb{E}\Big[\lambda_1 \int_{\tau_1}^{T_1} \left(\zeta_t^{1,*}\right)^2 dt \Big| \mathcal{F}_{\tau_1}\Big]$$

$$= \frac{1}{2}\left(X_{\tau_1}^*\right)^2 \left(\int_{\tau_1}^T \frac{\cosh^2 \frac{T-t}{\sqrt{2\lambda_1}}}{\sinh^2 \frac{T-\tau_1}{\sqrt{2\lambda_1}}} dt - \int_{\tau_1}^{T_1} \frac{\sinh^2 \frac{T_1-t}{\sqrt{2\lambda_1}}}{\cosh^2 \frac{T_1-\tau_1}{\sqrt{2\lambda_1}}} dt \right)$$

$$= \frac{\sqrt{2\lambda_1}}{4}\left(X_{\tau_1}^*\right)^2 \left(\coth \frac{T-\tau_1}{\sqrt{2\lambda_1}} - \tanh \frac{T_1-\tau_1}{\sqrt{2\lambda_1}} + \frac{\frac{T-\tau_1}{\sqrt{2\lambda_1}}}{\sinh^2 \frac{T-\tau_1}{\sqrt{2\lambda_1}}} + \frac{\frac{T_1-\tau_1}{\sqrt{2\lambda_1}}}{\cosh^2 \frac{T_1-\tau_1}{\sqrt{2\lambda_1}}} \right).$$

Using that

$$T_1 = \tau_1 + \sqrt{2\lambda_1} \tanh^{(-1)} \left(\sqrt{\frac{\hat{\lambda}}{\lambda_1}} \coth \frac{T-\tau_1}{\sqrt{2\hat{\lambda}}} \right), \tag{3.45}$$

from (3.41), for the case $m_1 < 1$, we conclude with

$$\mathbb{E}\Big[R_{T_1}^{f,*}|\mathcal{F}_{\tau_1}\Big] - \mathbb{E}\Big[R_T^{n,*}|\mathcal{F}_{\tau_1}\Big] \to \infty, \quad \text{as } \lambda_1 \to \infty.$$

Therefore, the (conditional) mean revenue of the forward agent is higher than that of the naive agent, if the unanticipated price impact is large. To examine the (conditional) variance, we first notice that, due to $\lambda_1, T_1 \in \mathcal{F}_{\tau_1}$,

$$\mathrm{Var}[R^{f,*}_{T_1}|\mathcal{F}_{\tau_1}] = \mathbb{E}\left[\int_{\tau_1}^{T_1} \left(X_t^{f,*}\right)^2 dt \middle| \mathcal{F}_{\tau_1}\right] = \left(X^*_{\tau_1}\right)^2 \int_{\tau_1}^{T_1} \frac{\cosh^2 \frac{T_1-t}{\sqrt{2\lambda_1}}}{\cosh^2 \frac{T_1-\tau_1}{\sqrt{2\lambda_1}}} dt$$

$$= \frac{1}{2}\left(X^*_{\tau_1}\right)^2 \left(\sqrt{2\lambda_1}\tanh\frac{T_1-\tau_1}{\sqrt{2\lambda_1}} + \frac{T_1-\tau_1}{\cosh^2\frac{T_1-\tau_1}{\sqrt{2\lambda_1}}}\right).$$

Again, by (3.45), we obtain the limit

$$\mathrm{Var}[R^{f,*}_{T_1}|\mathcal{F}_{\tau_1}] \to \left(X^*_{\tau_1}\right)^2 \sqrt{2\hat{\lambda}}\coth\frac{T-\tau_1}{\sqrt{2\hat{\lambda}}}, \quad \text{as } \lambda_1 \to \infty.$$

On the other hand, for the naive agent, we have

$$\mathrm{Var}[R^{n,*}_{T}|\mathcal{F}_{\tau_1}] = \mathbb{E}\left[\int_{\tau_1}^{T} (X_t^{n,*})^2 dt \middle| \mathcal{F}_{\tau_1}\right] = \left(X^*_{\tau_1}\right)^2 \int_{\tau_1}^{T} \frac{\sinh^2 \frac{T-t}{\sqrt{2\lambda_1}}}{\sinh^2 \frac{T-\tau_1}{\sqrt{2\lambda_1}}} dt$$

$$\longrightarrow \left(X^*_{\tau_1}\right)^2 \frac{T-\tau_1}{3}, \quad \text{as } \lambda_1 \to \infty,$$

by dominated convergence theorem. The above results suggest that the forward agent outperforms the naive agent in unanticipated catastrophic market conditions (i.e., $\lambda_1 \to \infty$), by achieving higher expected liquidation revenue with comparable variance. The variance of her revenue as well as that of the naive agent both approach to some pre-determined constants, as the new market price impact becomes significantly large.

One intuitive explanation for such superiority of the forward approach is based on the metric $\mathcal{R}^1_0(x,r) = 0$, as shown before. Regardless of the the future market condition, the forward agent can always deliver a pre-chosen performance $V(x, r, 0)$, by consistently revising the liquidation criterion, the liquidation horizon and the volume to trade. However, the naive agent typically experiences $\mathcal{R}_0^1(x, r) < 0$ when facing unanticipated adverse market conditions, due to the stringent commitment to the terminal criterion (3.5) set at $t = 0$.

Another possible explanation relates to the discontinuity of the realized strategy $\zeta^{a,*}$ at τ_1 for the forward agent. Indeed, similar to the discontinuity $\Delta^*_{\tau_1}(\lambda_0, \hat{\lambda}, \lambda_1)$ defined for the naive agent, we define

$$\Delta^{f,*}_{\tau_1}(\lambda_0, \hat{\lambda}, \lambda_1) = \lim_{t \downarrow \tau_1} \zeta_t^{1,*} - \lim_{t \uparrow \tau_1} \zeta_t^*,$$

for the forward agent. It follows from direct computation that

$$\Delta^{f,*}_{\tau_1}(\lambda_0, \hat{\lambda}, \lambda_1) \gtreqless \Delta^*_{\tau_1}\left(\lambda_0, \hat{\lambda}, \lambda_1\right) \gtreqless 0, \quad \text{if } \lambda_0 \gtreqless \lambda_1.$$

This result implies that, compared to the naive agent, the forward agent can take more advantage of the new market conditions, by increasing (decreasing) the trading rate with a larger magnitude if the market turns out to be favorable (adverse, respectively) for the liquidation activity.

3.5 Sequential "real-time" model updating and forward liquidation

We now present the construction of the forward performance process in general multi-period setting. It is a direct extension of the previous two-period setting where the market parameter shifts once at τ_1.

- At $t = 0$, the trading agent starts with an initial criterion $U(x, r, 0)$, and assesses the market impact parameter λ_0 for $[0, \tau_1)$, with $\tau_1 \in \mathcal{F}_0$. The time τ_1 is the first time that the market impact parameter will be reassessed, and it is known at $t = 0$. The time period $[0, \tau_1)$ is subjective, as it reflects how long the agent would remain confident in the $t = 0$ estimated market parameter λ_0.

The initial criterion is taken to be of the form

$$U(x, r, 0) = -\exp\left(-r + k_0 x^2\right),$$

$(x, r) \in \mathbb{R}^+ \times \mathbb{R}$, for some $k_0 > 0$.

The choice of such initial criterion is flexible enough to cover several interesting scenarios. For instance, it may be taken to coincide with the initial condition $V(x, r, 0)$, indicating that the client would like to achieve a pre-specified performance. The forward approach allows the agent to deliver such performance to the client, due to the first metric $\mathcal{R}_0^1(x, r) = 0$ as we have shown. It is also possible for the criterion $U(x, r, 0)$ to have implicit dependence on some pre-chosen liquidation time T, through the parameter $k_0 > 0$. This follows from the reconciliation of the forward and the classical liquidation problems discussed earlier. A third interesting choice for $U(x, r, 0)$, as we have observed, is to take into account the initial trading profile the client preferred. Indeed, given $\lambda_0 > 0$ and $x > 0$ at $t = 0$, k_0 (hence $U(x, r, 0)$) is uniquely determined by the initial (preferred) trading rate ζ_0^* of the client (cf. (3.28)). The forward agent then takes it as an input and outputs a consistent trading pattern for later times, by solving sequentially the forward liquidation problems in real-time.

Starting at $t = 0$, the agent solves the first inverse liquidation problem $\mathcal{P}(\lambda_0, k_0; 0)$. According to Theorem 3.2.2, if its solvability time $T(\lambda_0, k_0) \leq \tau_1$, then trading stops at $T(\lambda_0, k_0)$. Let $m_0 = k_0\sqrt{\frac{2}{\lambda_0}}$, then, if $m_0 < 1$, there is non-zero inventory left, $X^*_{T(\lambda_0,k_0)} > 0$, while if $m_0 \geq 1$, full liquidation occurs optimally with $X^*_{T(\lambda_0,k_0)} = 0$, and the liquidation program stops.

The more interesting case is when model revision happens *before* the liquidation problem stops. That is $T(\lambda_0, k_0) > \tau_1$, then clearly $X^*_{\tau_1} > 0$. and there will be non-zero inventory left at the first model revision time τ_1 at which the market parameter $\lambda_1 \in \mathcal{F}_{\tau_1}$ is revealed. We will continue with this case.

- At $t = \tau_1 \in \mathcal{F}_0$, the agent considers the inverse liquidation problem $\mathcal{P}(\lambda_1, k_1; 0)$, with $\lambda_1 \in \mathcal{F}_{\tau_1}$ being the actual, realized value of the market impact parameter and constant $k_1 = k_0 \frac{G(\tau_1; m_0, \lambda_0)}{F(\tau_1; m_0, \lambda_0)} \in \mathcal{F}_0$.

From Theorem 3.2.2, we have that conditional on $\mathcal{F}_{\tau_1}$, the solution is given, for $t \in [\tau_1, \tau_1 + T(\lambda_1, k_1))$, by

$$U^1(x, r, t; \omega) = -\exp\left(-r + h^1(x, t; \omega)\right) \in \mathcal{F}_{\tau_1},$$

where h^1 solves, for $t \in [\tau_1, \tau_1 + T(\lambda_1, k_1))$ the HJ equation

$$h_t - \frac{1}{4\lambda_1} h_x^2 + \frac{1}{2} x^2 = 0,$$

with $h^1(x,\tau_1) = k_1 x^2$. Denote $m_1 := k_1\sqrt{\frac{2}{\lambda_1}}$, then, according to Lemma 3.2.1,

$$h^1(x,t) = k_1 x^2 \frac{G(t-\tau_1; m_1, \lambda_1)}{F(t-\tau_1; m_1, \lambda_1)},$$

for $t \in [\tau_1, \tau_1 + T(\lambda_1, k_1))$. The solvability horizon $T(\lambda_1, k_1) \in \mathcal{F}_{\tau_1}$ is given by (3.15) in Lemma 3.2.1.

At $t = \tau_1$, the agent also needs to choose the next model revision time $\tau_2 \in \mathcal{F}_{\tau_1}$. If $m_1 \geq 1$, and $\tau_2 \geq \tau_1 + T(\lambda_1, k_1)$, then full liquidation occurs with $X^*_{\tau_1+T(\lambda_1,k_1)} = 0$, and the liquidation program stops. Notice that $m_1 \geq 1$ implies that λ_1 is relatively small, while τ_2 being large indicates that the agent is confident that the current market condition with small price impact would last. It is hence intuitively reasonable to complete the liquidation program in such long-standing favorable market conditions.

On the other hand, if $m_1 < 1$, we assume that the agent chooses $\tau_2 < \tau_1 + T(\lambda_1, k_1)$ and, therefore, the forward liquidation program continues, with the remaining inventory

$$X^*_{\tau_2} = X^*_{\tau_1} F(\tau_2 - \tau_1; m_1, \lambda_1) > 0.$$

This assumption is reasonable, since $m_1 < 1$ corresponds to a relatively large λ_1 that indicates adverse market condition for liquidation. The agent typically would not commit to such λ_1 for a long time, but rather, revise it before the solvability horizon.

The forward liquidation program continues for $n \geq 3$ as depicted above, whenever at each model revision time $\tau_n \in \mathcal{F}_{\tau_{n-1}}$, there exists non-zero inventory $X^*_{\tau_n} > 0$ left.

3.6 Continuous time forward liquidation with market parameters update

In this section, we consider the continuous time forward performance process for the liquidation problem. Le W_t, $t \geq 0$ be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\mathcal{F}_t$ satisfying the usual conditions.We consider the forward performance process, as well as the inventory and revenue processes, given by, for $t \geq 0$,

$$dU(x,r,t) = b(x,r,t)dt + a(x,r,t)dW_t,$$

and

$$dX_t^\zeta = -\zeta_t dt, \quad dR_t^\zeta = \sigma_t X_t^\zeta dW_t - \lambda_t \zeta_t^2 dt, \tag{3.46}$$

with $U(x,r,0) = u(x,r,0)$, $X_0^\zeta = x \in \mathbb{R}^+$, and $R_0^\zeta = r \in \mathbb{R}$. The processes $\lambda_t > 0$ and $\sigma_t > 0$ are $\mathcal{F}_t$-progressively measurable price impact process and volatility process, respectively. We also assume that $a(x,r,t)$ is $\mathcal{F}_t$-progressively measurable and continuously differentiable in the variable r.

Assuming that $U(x,r,t)$ is smooth enough so that the Itô-Ventzell formula can be applied to $U(X_t^\zeta, R_t^\zeta, t)$, for each admissible policy ζ, we then obtain

$$dU(X_t^\zeta, R_t^\zeta, t) = b(X_t^\zeta, R_t^\zeta, t)dt + a(X_t^\zeta, R_t^\zeta, t)dW_t - U_x(X_t^\zeta, R_t^\zeta, t)\zeta_t dt$$

$$+U_r(X_t^\zeta, R_t^\zeta, t)\sigma_t X_t dW_t - U_r(X_t^\zeta, R_t^\zeta, t)\lambda_t \zeta_t^2 dt$$

$$+\frac{1}{2}U_{rr}(X_t^\zeta, R_t^\zeta, t)\sigma_t^2 X_t^2 dt + a_r(X_t^\zeta, R_t^\zeta, t)\sigma_t X_t dt$$

$$= \Big(b - U_x\zeta_t - U_r\lambda_t\zeta_t^2 + \frac{1}{2}U_{rr}\sigma_t^2 X_t^2 + a_r\sigma_t X_t\Big)dt + \Big(U_r\sigma_t X_t + a\Big)dW_t,$$

where we have suppressed the arguments in the last equality.

For the process $U(x, r, t)$ to be a forward performance process, we need to further assume that the mapping $r \mapsto U(x, r, t)$ is strictly concave and increasing, for fixed $(x, t) \in \mathbb{R}^+ \times [0, \tau)$, almost surely, with $t = \tau$ being the solvability horizon to be determined. By the first and the second order condition (since $U_r > 0$), we calculate the optimal trading rate as

$$\zeta_t^* = -\frac{U_x(X_t^*, R_t^*, t)}{2\lambda_t U_r(X_t^*, R_t^*, t)}, \tag{3.47}$$

and it should be nonnegative in a liquidation program. Since $U_r > 0$, we hence obtain the constraint that $U_x \leq 0$. Note that such constraint already exists in the construction of the forward performance process in discrete time, through the condition $h_x(x, t) \geq 0$, for all $(x, t) \in \mathbb{R}^+ \times [0, T(\lambda, k))$.

Unlike the backward scenario, with the forward formulation, we are allowed to choose the volatility process $a(x, r, t)$ which determines the drift process $b(x, r, t)$ and, in turn, the dynamics of the performance process $U(x, r, t)$. Indeed, based on the (local) martingale property of $U(X_t^*, R_t^*, t)$ and the (local) supermartingale property of $U(X_t^\zeta, R_t^\zeta, t)$, we can deduce that the drift satisfies

$$b(x, r, t) = -\frac{U_x(x, r, t)^2}{4\lambda_t U_r(x, r, t)} - \frac{1}{2}U_{rr}(x, r, t)\sigma_t^2 x^2 - a_r(x, r, t)\sigma_t x^2.$$

Therefore, the forward performance process $U(x, r, t)$ satisfies the stochastic partial differential equation (SPDE)

$$dU(x,r,t) = \Big(-\frac{U_x(x,r,t)^2}{4\lambda_t U_r(x,r,t)} - \Big(\frac{\sigma_t}{2}U_{rr}(x,r,t) + a_r(x,r,t)\Big)\sigma_t x^2\Big)dt + a(x,r,t)dW_t, \tag{3.48}$$

with the terminal datum $U(x,r,0) = u(x,r,0)$. The first natural step is to consider the zero volatility case, i.e., $a(x,r,t) \equiv 0$. In this case, the SPDE(3.48) reduces to

$$dU(x,r,t) = -\Big(\frac{U_x(x,r,t)^2}{4\lambda_t U_r(x,r,t)} + \frac{1}{2}U_{rr}(x,r,t)\sigma_t^2 x^2\Big)dt. \tag{3.49}$$

It is easy to see that the zero volatility case does not necessarily yield a time monotone forward performance process. Another observation is that in the constant parameters scenario, i.e., $\lambda_t \equiv \lambda > 0$ and $\sigma_t \equiv \sigma > 0$, the solution to equation (3.48) is the deterministic function satisfying the HJB equation

$$U_t(x,r,t) + \frac{U_x(x,r,t)^2}{4\lambda U_r(x,r,t)} + \frac{1}{2}U_{rr}(x,r,t)\sigma^2 x^2 = 0, \tag{3.50}$$

with initial datum $U(x,r,0) = u(x,r,0)$, whose solvability has been studied in Theorem 3.2.2. Next, we present another scenario where the SPDE (3.48) has a unique well defined solution, under suitable conditions on the involved parameter processes.

3.6.1 The coordinated variation parameters case

[1] studied the coordinated variation case, namely when $\sigma_t^2\lambda_t = constant$, a case typically considered normal for periods where largest fraction of the trading happens. Without loss of generality, we assume that $\sigma_t^2\lambda_t = 1$. Also,

to make reasonable connection to the classical liquidation problem, we choose the initial datum for the forward performance process to be the $t = 0$ value function of the classical liquidation problem with $\lambda = \sigma = 1$. It is easy to see that the following result holds for any $u(x, r, 0) = -e^{-r+kx^2}$, $k > 0$, as usual.

Proposition 3.6.1. *Suppose that the coefficients in equation (3.49) satisfy the coordinated variation condition*

$$\sigma_t^2 \lambda_t = 1, \forall\ t > 0\ a.s., \tag{3.51}$$

and the initial datum is given by

$$u(x, r, 0) = -\exp\left(-r + \frac{x^2}{\sqrt{2}} \coth\left(\frac{T}{\sqrt{2}}\right)\right), \tag{3.52}$$

for some constant $T > 0$. Then, for $0 < t < \tau := \inf\{s > 0 | \int_0^s \frac{1}{\lambda_u} du = T\}$,

$$U(x, r, t) = -\exp\left(-r + \frac{x^2}{\sqrt{2}} \coth\left(\frac{T - \int_0^t \frac{1}{\lambda_s} ds}{\sqrt{2}}\right)\right) \tag{3.53}$$

is the unique solution to the equation (3.48) with the separable form, and the optimal admissible inventory process is

$$X_t^* = x \exp\left(-\int_0^t \frac{1}{\sqrt{2}\lambda_s} \coth\left(\frac{T - \int_0^s \frac{1}{\lambda_u} du}{\sqrt{2}}\right) ds\right). \tag{3.54}$$

Proof. We consider rescaling of time, i.e., let $U(x, r, t) = u(x, r, \int_0^t \sigma_s^2 ds)$, for a smooth deterministic function $u(x, r, t)$ that satisfies (3.52). The equation (3.49) and the coordinated variation condition (3.51) direct yield that

$$u_t + \frac{u_x^2}{4u_r} + \frac{1}{2} u_{rr} x^2 = 0,$$

with initial condition (3.52). Within the separable family $u(x,r,t) = -e^{-r+h(x,t)}$, we obtain the unique solution according to Lemma 3.2.1

$$u(x,r,t) = -e^{-r+\frac{x^2}{\sqrt{2}}\coth(\frac{T-t}{\sqrt{2}})},$$

with $T > 0$ being the solvability horizon. The solution (3.53) then follows easily. The optimal trading rate can be derived from (3.47), i.e,

$$\zeta_t^* = -\frac{U_x(X_t^*, R_t^*, t)}{2\lambda_t U_r(X_t^*, R_t^*, t)} = \frac{X_t^*}{\sqrt{2}\lambda_t}\coth\left(\frac{T - \int_0^t \frac{1}{\lambda_s}ds}{\sqrt{2}}\right),$$

which is admissible as it is clearly nonnegative. This leads to the optimal inventory process given in (3.54). □

It is easy to see from (3.54) that the optimal trading strategy of the forward agent does not necessarily lead to a full liquidation before or at the desired calender time T, which is set by the client based on the information (i.e., $\lambda = \sigma = 1$) at $t = 0$, and the solution of the classical problem (cf. (3.52)). This is actually reasonable in a market with stochastic market coefficients. Indeed, if

$$\int_0^t \frac{1}{\lambda_s}ds < T - \varepsilon,\ \forall t > 0,\ a.s., \tag{3.55}$$

for some small $\varepsilon > 0$. Then, $\tau = \infty,\ a.s.$, and we obtain a forward performance process defined for all time. It is easy to show that there exists a positive constant C, such that $X_t^* > C > 0, \forall t > 0$, a.s., based on (3.54). The condition (3.55) may hold in a market with large price impact where the consistent (optimal) strategy aiming to complete liquidation in finite time is no longer available. On the other hand, if $0 < C_1 < \lambda_t < C_2$ uniformly in

(t, ω)), then $\tau < \infty$, and we can show $X^*_\tau = 0$. This implies, with moderate market impact, it is always possible to complete liquidation in finite time and maintain intertemporal consistency. However, τ may no longer coincide with the pre-determined time T.

3.6.2 Convergence to the continuous time zero volatility forward process

The multi-period forward performance process constructed in section 5 gives the criteria $U(x, r, \tau_n)$, $n \geq 1$, provided that each model revision time $\tau_n \in \mathcal{F}_{\tau_{n-1}}$ is strictly before the solvability horizon $T(\lambda_{n-1}, k_{n-1}) \in \mathcal{F}_{\tau_{n-1}}$. Notice that the initial condition and the solution to the HJ equation (3.16) are both quadratic in the spatial variable within the solvability horizon and, hence, such desirable construction of the performance criteria for all $n \geq 1$ becomes feasible.

The continuing construction of the forward criteria allows us to study the limiting process, as the update of the price impact parameters λ_n at each τ_n is done more and more frequently. Indeed, we will show that under suitable conditions on the parameter processes in the continuous time problem (3.46), the discrete time forward criteria sequence $U(x, r, \tau_n)$ converges to the zero volatility forward performance process that solves the equation (3.49). Recall that $U(x, r, \tau_n) \in \mathcal{F}_{\tau_{n-1}}$, for $n \geq 1$. Our convergence result hence shows the close connection between the discrete time *predictable* forward performance process and the continuous time *zero volatility* forward performance process, as the model revision period vanishes.

To establish such connection, we assume $\sigma_t = 1$, $t \geq 0$, as in the multi-period setting. It follows that in analogy to Proposition 3.6.1, the equation (3.49) with the initial datum $U(x, r, 0) = -e^{-r+k_0 x^2}$, $k_0 > 0$, has an admissible solution given by $U(x, r, t) = -e^{-r+k(t;\omega)x^2}$, if the nonegative function $k(t; \omega)$ solves the (random) Riccati equation almost surely

$$\frac{dk(t)}{dt} = \frac{k^2(t)}{\lambda_t} - \frac{1}{2}, \tag{3.56}$$

with $k(0) = k_0$. We make the following assumption for λ_t in equation (3.56).

Assumption 1. λ_t, $t \geq 0$, *is continuous and satisfies* $\inf_{t \geq 0} \lambda_t > 0$, *a.s..*

We next introduce the sequence of strictly increasing model revision times τ_n^N, $n \geq 0$, that satisfies

$$\lim_{N \to \infty} \sup_{n \geq 0} |\tau_{n+1}^N - \tau_n^N| = 0,$$

and $\tau_0^N \equiv 0$, for all $N \geq 1$. Indeed, for each $N \geq 1$ and each $n \geq 0$, let $\tau_{n+1}^N \in \mathcal{F}_{\tau_n^N}$ be given by

$$\tau_{n+1}^N = \tau_n^N + \frac{T(\lambda_n^N, k_n^N) \wedge 1}{N+1},$$

with $T(\lambda_n^N, k_n^N)$ being the solvability horizon for the $(n+1)$-th period. Here, $\lambda_n^N = \lambda_{\tau_n^N} \in \mathcal{F}_{\tau_n^N}$ is the value of the price impact process λ_t at time τ_n^N, while k_n^N is constructed recursively forward in real-time through $m_n^N = k_n^N \sqrt{\frac{2}{\lambda_n^N}}$ and

$$k_{n+1}^N = k_n^N \frac{G(\tau_{n+1}^N - \tau_n^N; m_n^N, \lambda_n^N)}{F(\tau_{n+1}^N - \tau_n^N; m_n^N, \lambda_n^N)},$$

for $n \geq 0$, with $k_0^N \equiv k_0$, for all $N \geq 1$. Note that because each τ_n^N is strictly before the corresponding solvability horizon, it follows that $k_n^N > 0$ is well defined, for all $N \geq 1$, $n \geq 0$. Finally, for each $N \geq 1$, denote by k^N the mapping form $[0, \lim_{n\to\infty} \tau_n^N)$ into $\mathbb{R}^+$, obtained as the the linear interpolation of the function $\tau_n^N \mapsto k_n^N$, $n \geq 0$. Then, we have the following convergence result.

Theorem 3.6.2. *Let Assumption 1 hold. Then, there exist $T^* > 0$ and a continuous function $k : [0, T^*) \mapsto \mathbb{R}^+$, such that, for any $t \in (0, T^*)$,*

$$\lim_{N\to\infty} \sup_{s\in[0,t]} |k^N(s) - k(s)| = 0, \ \ a.s..$$

Moreover, k is uniquely determined by the Riccati equation (3.56) for $t \in [0, T^*)$, and

$$T^* = \sup\{t > 0 : \text{there exists a bounded nonnegative solution}$$
$$\text{to equation (3.56) for } s \in [0, t]\}. \tag{3.57}$$

Proof. We conduct the proof for each fixed $\omega \in \Omega$ that does not belong to the null set. Let $C_1 > k_0$ be a constant, and for each $N \geq 1$, we construct the sequence $\{\hat{k}_n^N\}_{n\geq 0}$ as follows

$$\hat{k}_n^N = k_{n\wedge\tau^N}^N, \quad \tau^N = \inf\{l \geq 0 : k_l^N > C_1\},$$

with the convention $\inf \emptyset = \infty$. Then, it easily follows that $0 \leq \hat{k}_n^N \leq C_1$, for all $N \geq 1$ and $n \geq 0$. Notice also that $\hat{k}_n^N = \hat{k}_{n-1}^N$ for $n > \tau^N$. Hence, we only consider $1 \leq n \leq \tau^N$, and obtain

$$\left|\hat{k}_n^N - \hat{k}_{n-1}^N\right| = \left|\hat{k}_{n-1}^N \frac{G(\tau_n^N - \tau_{n-1}^N; m_{n-1}^N, \lambda_{n-1}^N)}{F(\tau_n^N - \tau_{n-1}^N; m_{n-1}^N, \lambda_{n-1}^N)} - \hat{k}_{n-1}^N\right|$$

$$= \hat{k}_{n-1}^N \left| \frac{\cosh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}} - \frac{1}{m_{n-1}^N} \sinh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}}}{\cosh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}} - m_{n-1}^N \sinh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}}} - 1 \right|$$

$$= \hat{k}_{n-1}^N \left| m_{n-1}^N - \frac{1}{m_{n-1}^N} \right| \frac{\sinh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}}}{\cosh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}} - \frac{\sqrt{2}\hat{k}_{n-1}^N}{\sqrt{\lambda_{n-1}^N}} \sinh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}}}$$

$$\leq \left(\frac{\sqrt{2}(\hat{k}_{n-1}^N)^2}{\sqrt{\lambda_{n-1}^N}} + \sqrt{\frac{\lambda_{n-1}^N}{2}} \right) \frac{\sinh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}}}{\cosh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}} - \frac{\sqrt{2}\hat{k}_{n-1}^N}{\sqrt{\lambda_{n-1}^N}} \sinh \frac{\tau_n^N - \tau_{n-1}^N}{\sqrt{2\lambda_{n-1}^N}}}.$$

Hence, we can find some constant $C > 0$ that only depends on C_1 and $\inf_{t \geq 0} \lambda_t$, such that $|\hat{k}_n^N - \hat{k}_{n-1}^N| \leq C(\tau_n^N - \tau_{n-1}^N)$, for all $n \geq 0$, as $N \to \infty$. Denote $\delta_1 := \liminf_{N \to \infty} \tau^N$. Then, since we have shown that the linear interpolation functions $\{k^N\}_{N \geq 1}$ are uniformly Lipschitz, it follows that $\delta_1 > 0$. We assume for now that $\delta_1 < \infty$. By Arzelà-Ascoli Theorem, we conclude that $\{k^N\}_{N \geq 1}$ is compact in $C([0, \delta_1])$, and $k^N(\delta_1) \to C_1$, as $N \to \infty$.

Now consider any convergent subsequence of $\{k^N\}_{N \geq 1}$, and denote its limit function by $k(t)$, for $t \in [0, \delta_1]$. For any fixed $t \in (0, \delta_1)$ and $N \geq 1$ that is sufficiently large, denote $j(N) = \max\{n \geq 0 : \tau_n^N < t\}$. Then, we divide by $\tau_{j(N)+1}^N - \tau_{j(N)}^N$ on both sides of the recursive equation that connects $\hat{k}_{j(N+1)}^N$

and $\hat{k}^N_{j(N)}$

$$\hat{k}^N_{j(N+1)} - \hat{k}^N_{j(N)} = \hat{k}^N_{j(N)} \left(\frac{\cosh \frac{\tau^N_{j(N)+1} - \tau^N_{j(N)}}{\sqrt{2\lambda^N_{j(N)}}} - \frac{1}{m^N_{j(N)}} \sinh \frac{\tau^N_{j(N)+1} - \tau^N_{j(N)}}{\sqrt{2\lambda^N_{j(N)}}}}{\cosh \frac{\tau^N_{j(N)+1} - \tau^N_{j(N)}}{\sqrt{2\lambda^N_{j(N)}}} - m^N_{j(N)} \sinh \frac{\tau^N_{j(N)+1} - \tau^N_{j(N)}}{\sqrt{2\lambda^N_{j(N)}}}} - 1 \right),$$

and let $N \to \infty$, to obtain that $k(t)$ satisfies the Riccati equation (3.56) at t. Therefore, we conclude that $k(t)$ solves the equation (3.56) for $t \in [0, \delta_1]$. Notice that the solution to equation (3.56) is unique in the family of bounded nonnegative functions, as follows from the standard contraction argument. Therefore, we conclude that k^N converges to k in $C([0, \delta_1])$, the unique bounded nonnegative solution to equation (3.56), as $N \to \infty$.

Choosing an increasing sequence $\{C_m\}_{m \geq 1}$, with $\lim_{m \to \infty} C_m = \infty$, and repeating the above constructions, we obtain an increasing sequence $\{\delta_m\}_{m \geq 1}$, such that k^N converges to k in $C([0, \delta_m])$, as $N \to \infty$, and k satisfies equation (3.56) for $t \in [0, \delta_m]$. Let $T^* := \lim_{m \to \infty} \delta_m$, we conclude that k satisfies equation (3.56), for $t \in [0, T^*)$. Assume that there exists a bounded nonnegative solution to equation (3.56) for $t \in [0, T']$, with $T' > T^*$. Then, it follows that $T^* < \infty$ and hence, $\delta_m < \infty$, for all $m \geq 0$, and such solution has to coincide with k on every $[0, \delta_m]$, due to the uniqueness of a bounded nonnegative solution to equation (3.56). However, when $\delta_m < \infty$, we have $k^N(\delta_m) \to k(\delta_m) = C_m$, which converges to infinity as $m \to \infty$. This leads to a contradiction and thus, T^* satisfies (3.57). It is clear that T^* is also uniquely determined by (3.57). $\square$

Chapter 4

Forward optimal liquidation with market parameter shift: the general case

4.1 Introduction

In this chapter, we provide the companion work to the forward optimal liquidation problem discussed in the previous chapter. The contributions of the current work are twofold. In terms of the solution to the forward liquidation problem, we present more general results that fully accommodate the quadratic case in the previous chapter for various formulations, namely, the single inverse problem formulation, the multi-period forward optimal liquidation formulation and finally, the continuous time forward performance process formulation. These generalizations reveal that the initial performance criterion in the previous work, which includes the $t = 0$ value function of the classical optimal liquidation problem, is only one specific choice from a much larger family of admissible initial conditions presented in this work for the the forward processes.

The second contribution of this work is to present new insights on classical optimal liquidation problem that has been studied extensively in recent years. The literature on the liquidation problem has shown various interests, including both the single agent optimal liquidation in rather general market settings ([1],[4], [11], [26] and [28]) and the mean filed game formulation ([9], [31]). In the classical optimal liquidation setting, it becomes almost conventional to impose a singular terminal condition to guarantee full liquidation by a fixed time T, which yields an optimal strategy that unwinds all possible shares by T. However, in reality, the total amount of shares of any stock is finite in the market, and hence an agent should only be concerned about full liquidation of initial inventory with a finite upper bound. Under the condition that the initial inventory is bounded from above, the forward optimal liquidation formulation gives rise to a classical optimal liquidation problem with a *regular* terminal condition to guarantee full liquidation by any fixed time T. Moreover, the $t = 0$ value function is higher under the regular terminal condition than that under the singular terminal condition. This is a reasonable consequence, since choosing to fully liquidate any amount of initial inventory, even it is virtual, is a stringent requirement on the agent's optimal strategy and, hence, decreases the optimal value. Another interesting fact under the forward formulation is that the liquidation horizon is endogenously determined by the initial normalized trading rate and the market price impact parameter. We obtained the intuitive result that liquidation can be complete earlier if the initial trading is relatively fast, and if the market is relatively liquid. It is in contrast with the typical classical setting (see e.g., [52], [53]) where liquidation horizon coincides with the pre-determined T for any initial inventory and any market condition.

Our analysis is based on the study of the existence and uniqueness of the classical solution to a Hamilton-Jacobi equation with a state dependent Hamitonian that is concave in the gradient, and with a not necessarily convex initial condition. In particular, we examine in detail under what conditions the method of characteristics can be applied to give a smooth solution. Obviously, a global solution (up to any finite time T) does not exist in general, and one contribution of the current paper is to give a class of admissible initial datum under which the HJ equation has a unique classical solution up to an explicitly determined time horizon. This time horizon for solvability depends on both the shape of the initial datum and the parameters in the HJ equation. Moreover, in the special case of quadratic initial datum, including the finite and infinite horizon classical optimal liquidation problems as in [53] and [52], we obtain the tight bound on the solvability horizon, and fully recover the existing results. Working with the suitable class of initial conditions, we can also show that the classical solution to the HJ equation has the same properties as its initial condition at any time within the solvability horizon. This self-similarity enables us to provide a continuing construction of the multi-period forward performance process recursively forward in real-time, similar to the quadratic case discussed in the previous chapter.

The organization of this chapter is as follows. In section 4.2, we restate the results of the classical finite horizon liquidation problem for completeness and also set up the model dynamics for later discussion. Section 4.3.1 provides the main result for the single inverse liquidation problem and the its connection to the classical liquidation problem with non-singular terminal condition. In section 4.3.2, we incorporate real-time model updating and recursively construct the intertemporally consistent forward performance process in discrete time. Finally, section 4.3.3 discusses the convergence of the discrete time forward performance process to the continuous time zero volatility forward performance process, in the limit case as the model revision period shrinks to zero.

4.2 Classical approach

The optimal liquidation problem in continuous time has been analyzed for finite and infinite horizon by Schied et al. [53] and [52], respectively. In this section, we briefly recall the results in [53], and will address more on the connections between the classical (backward) scenario and the forward scenario in the sequel. For simplicity, we consider liquidation of only one single asset within a finite horizon $T > 0$. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F}_t$, $0 \leq t \leq T$, that satisfies the usual conditions, the price follows the dynamics

$$P_t = P_0 + \sigma W_t + \gamma(X_t - X_0) + \lambda \dot{X}_t,$$

where W_t, $0 \leq t \leq T$, is the standard Brownian motion, while γ and λ are the permanent and temporary price impact parameters, respectively. Under the assumption that the inventory process X_t is absolutely continuous, i.e., $X_t = x - \int_0^t \xi_s ds$ for some admissible trading rate ξ, and that the liquidation completes at T, the terminal revenue $R_T(\xi)$ can be calculated as

$$R_T(\xi) = \int_0^T \xi_t P_t dt = P_0 x - \frac{\gamma}{2}x^2 + \sigma \int_0^T X_t dW_t - \lambda \int_0^T \xi_t^2 dt.$$

Therefore, the two processes involved in the stochastic control problem are

$$X_t^\xi := x - \int_0^t \xi_s ds, \tag{4.1}$$

$$R_t^\xi := r + \sigma \int_0^t X_s^\xi dW_s - \lambda \int_0^t \xi_s^2 ds, \tag{4.2}$$

where $0 \leq t \leq T$, $x > 0$ is the initial inventory and $r \in \mathbb{R}$ is the initial revenue. Now consider the exponential utility $U(r) = -e^{-r}$, and we want to maximize the expected utility of terminal revenue, i.e., define

$$V(x, r, 0; T) = \sup_\xi \mathbb{E}\Big[- e^{-R_T^\xi} \Big| X_0^\xi = x, R_0^\xi = r \Big].$$

Applying Dynamic Programming Principle and Itô's lemma to the process $V(X_t^\xi, R_t^\xi, t; T)$, we derive that the deterministic function $V(x, r, t)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$V_t + \frac{1}{2}\sigma^2 x^2 V_{rr} + \sup_\xi(-\lambda \xi^2 V_r - \xi V_x) = 0, \tag{4.3}$$

with the terminal conditions

$$\begin{cases} V(0, r, T) = -e^{-r}, \\ V(x, r, T) = -\infty, \quad \text{for } x > 0. \end{cases} \tag{4.4}$$

The first condition is due to definition of value function V, whereas the second condition follows from the fact that the liquidation should be completed before

T; otherwise an infinite penalty will be imposed. The HJB equation (4.3) gives the value function (setting $\sigma = 1$)

$$V(x, r, t; T) = -\exp\Big(-r + \sqrt{\frac{\lambda}{2}} x^2 \coth \frac{T-t}{\sqrt{2\lambda}}\Big). \tag{4.5}$$

In addition, the optimal trading rate in the classical setting is given by

$$\xi_t^* = -\frac{V_x(X_t^*, R_t^*, t)}{2\lambda V_r(X_t^*, R_t^*, t)} = \frac{1}{\sqrt{2\lambda}} X_t^* \coth \frac{T-t}{\sqrt{2\lambda}}.$$

This quantity is positive for $0 \le t \le T$, a desired property of the trading rate in the liquidation problem. Solving (4.1), we get the optimal inventory process

$$X_t^* = \frac{x \sinh\left(\frac{T-t}{\sqrt{2\lambda}}\right)}{\sinh\left(\frac{T}{\sqrt{2\lambda}}\right)}. \tag{4.6}$$

It satisfies the condition $X_T = 0$.

We notice that the above classical backward problem is a single evaluation problem, in the sense that at $t = 0$ the agent is given the trading horizon T, the terminal utility function and the dynamics of the market parameter processes (deterministic or stochastic), all of which are fixed over the entire horizon $[0, T]$. The optimal trading rule and the intermediate value functions are then completely determined *a priori* at $t = 0$, as a consequence of the backward reasoning of the classical approach. Such framework would fail in the case where the agent is in a volatile market with unanticipated time-varying market parameters, or his estimates of the market parameters are not correct at $t = 0$ even if their true values stay unchanged.

To overcome the model commitment issue under the single evaluation backward formulation, we adopt the forward performance process approach in the following sections. The forward approach allows to incorporate the real-time updates of the unanticipated market information into the trading criterion, and gives rise to interteomparrly consistent trading behavior.

4.3 Forward approach

4.3.1 Single inverse problem

We present the idea, formulation and results regarding the optimal liquidation problem under the forward performance process. In the first part, we focus on the first period of the liquidation activity. In contrast to the classical backward formulation, we aim to find a consistent terminal utility for a given initial perfomance, and hence the problem considered herein can be seen as the inverse of the classical problem. This problem also serves as the foundation of the multi-period problem. Indeed, the general forward optimal liquidation problem under discrete time model revision boils down to addressing how to solve each single inverse liquidation problem, and how to concatenate them to obtain a multi-period forward performance process. The next theorem states the main result of the first inverse problem. For simplicity, we take $\sigma \equiv 1$ henceforth.

Theorem 4.3.1. *Assume that* $U(x, r, 0) = -e^{-r+g(x)}$*, with the function* $g \in C^2(\mathbb{R}^+)$ *satisfying* $g'(0), g''(0) \in \mathbb{R}$*, and that there exist positive constants* $a \geq$

$b > 0$ [1] *such that*

$$\sup_{x>0} g''(x) \leq a \quad \textit{and} \quad \inf_{x>0} \frac{g'(x)}{x} \geq b. \tag{4.7}$$

Then, the initial condition $U(x,r,0)$ *is admissible in the sense that an optimal pure liquidation policy exists under the forward performance process, for* $0 \leq t < T^g(\lambda)$*, where*

$$T^g(\lambda) := \sqrt{2\lambda} \min \Big(\tanh^{(-1)} \big(\frac{b}{\sqrt{2\lambda}} \wedge 1 \big), \coth^{(-1)} \big(\frac{a}{\sqrt{2\lambda}} \vee 1 \big) \Big), \tag{4.8}$$

with the convention that $\tanh^{(-1)}(1) = \coth^{(-1)}(1) = \infty$.

Proof. The time $T^g(\lambda)$ is the solvability horizon for the inverse liquidation problem under the initial condition $U(x,r,0) = -e^{-r+g(x)}$ and the price impact parameter that would appear in the HJB equation. To characterize $T^g(\lambda)$ more specifically, we note that in the current formulation, the (deterministic) forward performance process satisfies

$$U(x,r,t) = \sup_{\xi} \mathbb{E}\Big[U(X_s^\xi, R_s^\xi, s) | X_t^\xi = x, R_t^\xi = r\Big],$$

with $U(0,x,r) = -e^{-r+g(x)}$, for $0 \leq t \leq s < T^g(\lambda)$. Since the initial condition $U(x,r,0)$ is of exponential form, it is reasonable to expect a similar function form for the forward process $U(x,r,t) = -e^{-r+h(x,t)}$, with some function h to be determined. Based on the definition for the forward performance process, $U(x,r,t)$ satisfies the HJB equation (4.3) with the initial condition $U(0,x,r) =$

[1] $a \geq b$ must hold; otherwise the condition $g'(0) \in \mathbb{R}$ would be violated

$-e^{-r+g(x)}$. Hence, the function $h(x,t)$ is the solution to the following Hamilton-Jacobi equation

$$\begin{cases} h_t - \frac{1}{4\lambda}h_x^2 + \frac{1}{2}x^2 = 0, \quad x > 0,\ 0 < t < T^g(\lambda), \\ h(x,0) = g(x), \quad x > 0. \end{cases} \tag{4.9}$$

First order condition yields the optimal strategy $\xi_t^* = \frac{h_x(X_t^*,t)}{2\lambda}$, for $0 \leq t < T^g(\lambda)$, and it is expected to be nonnegative in a liquidation program. Therefore, we aim to look for the solvability horizon $T^g(\lambda)$ up to which the Hamilton-Jacobi equation (4.9) has a unique $C^{1,1}(\mathbb{R}^+ \times [0, T^g(\lambda)))$ solution with nonnegative spatial derivative for all $0 \leq t < T^g(\lambda)$.

We apply the method of characteristics (see, e.g., section 3.2.5 in [22]) to system (4.9), and obtain the following characteristic ODEs

$$\begin{cases} \frac{dX(s)}{ds} = -\frac{1}{2\lambda}P(s); \\ \frac{dP(s)}{ds} = -X(s); \\ \frac{dh(X(s),s)}{ds} = -\frac{1}{4\lambda}P^2(s) - \frac{1}{2}X^2(s), \end{cases} \tag{4.10}$$

where $P(s) = h_x(X(s), s)$, which yields that

$$\begin{cases} X(s) = C_1 e^{\frac{s}{\sqrt{2\lambda}}} + C_2 e^{-\frac{s}{\sqrt{2\lambda}}}, \\ P(s) = -\sqrt{2\lambda}C_1 e^{\frac{s}{\sqrt{2\lambda}}} + \sqrt{2\lambda}C_2 e^{-\frac{s}{\sqrt{2\lambda}}}. \end{cases} \tag{4.11}$$

Now, since $P(0) = h_x(X(0), 0) = g'(X(0)) = g'(x_0)$ following from the initial condition of (4.9), let $s = 0$ in (4.11), and we can get the two constants C_1 and C_2

$$\begin{cases} C_1 = \frac{x_0 - \frac{g'(x_0)}{\sqrt{2\lambda}}}{2}, \\ C_2 = \frac{x_0 + \frac{g'(x_0)}{\sqrt{2\lambda}}}{2}. \end{cases} \tag{4.12}$$

Then for any $(x,t) \in \mathbb{R}^+ \times (0, T^g(\lambda))$, integration along the characteristic curve gives

$$
\begin{aligned}
h(x,t) &= h(x_0, 0) + \int_0^t \Big(-\frac{1}{4\lambda} P^2(s) - \frac{1}{2} X^2(s) \Big) ds \\
&= g(x_0) + \frac{\sqrt{2\lambda}}{2} \Big(C_2^2 e^{-\frac{2t}{\sqrt{2\lambda}}} - C_1^2 e^{\frac{2t}{\sqrt{2\lambda}}} \Big) + \frac{\sqrt{2\lambda}}{2} \Big(C_1^2 - C_2^2 \Big) \\
= g(x_0) - &\frac{\sqrt{2\lambda}}{2} \Big(\frac{x_0^2}{2} + \frac{g'(x_0)^2}{4\lambda} \Big) \sinh\Big(\frac{2t}{\sqrt{2\lambda}} \Big) + \frac{x_0 g'(x_0)}{2} \Big(\cosh\Big(\frac{2t}{\sqrt{2\lambda}} \Big) - 1 \Big).
\end{aligned}
\tag{4.13}
$$

We notice that in expression (4.13), to have a solution for any $(x,t) \in \mathbb{R}^+ \times (0, T^g(\lambda))$, it is necessary to represent the initial state $x_0 \in \mathbb{R}^+$ by a unique function of $(x,t) \in \mathbb{R}^+ \times (0, T^g(\lambda))$. Under the assumption (4.7), we can prove the existence and uniqueness of such a function. Indeed, following from (4.11) and (4.12), the characteristic curve is

$$
X(s) = \frac{x_0 - \frac{g'(x_0)}{\sqrt{2\lambda}}}{2} e^{\frac{s}{\sqrt{2\lambda}}} + \frac{x_0 + \frac{g'(x_0)}{\sqrt{2\lambda}}}{2} e^{-\frac{s}{\sqrt{2\lambda}}},
$$

and to have $X(t) = x$ for a given pair of (x,t), it is clear that

$$
x_0 \cosh\Big(\frac{t}{\sqrt{2\lambda}} \Big) - \frac{g'(x_0)}{\sqrt{2\lambda}} \sinh\Big(\frac{t}{\sqrt{2\lambda}} \Big) = x. \tag{4.14}
$$

Now it remains to show that for any given $(x,t) \in \mathbb{R}^+ \times (0, T^g(\lambda))$, $x_0 \in \mathbb{R}^+$ is uniquely determined through the equation

$$
G(x_0, t) = x,
$$

where the function

$$
G(x_0, t) := x_0 \cosh\Big(\frac{t}{\sqrt{2\lambda}} \Big) - \frac{g'(x_0)}{\sqrt{2\lambda}} \sinh\Big(\frac{t}{\sqrt{2\lambda}} \Big),
$$

for any $x_0 \in \mathbb{R}^+$, $0 < t < T^g(\lambda)$.

Let $0 < t < T^g(\lambda)$ be fixed. We consider the single variable function $G(\cdot, t) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined above. From $\sup_{x>0} g''(x) \leq a$ in condition (4.7) and $T^g(\lambda)$ defined in (4.8), we know for $x_0 \in \mathbb{R}^+$, $0 < t < T^g(\lambda)$,

$$G_{x_0}(x_0, t) = \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{g''(x_0)}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right) > 0. \tag{4.15}$$

Condition (4.7) $\inf_{x>0} \frac{g'(x)}{x} \geq b > 0$ further leads to $0 \leq g'(0) < \infty$, yielding that $G(0, t) \leq 0$. It remains to show that

$$G(x_0, t) \to \infty, \quad \text{as} \quad x_0 \to \infty \tag{4.16}$$

Proposition (B.1.1) in Appendix implies that for each $\varepsilon > 0$, there exist $d_\varepsilon > 0$, such that $g'(x) \leq d_\varepsilon(x + \varepsilon)$, for all $x > 0$. Hence,

$$G(x_0, t) \geq x_0\Big(\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{d_\varepsilon}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right)\Big) - \frac{\varepsilon d_\varepsilon}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right). \tag{4.17}$$

Therefore, a sufficient condition for (4.16) to be valid is

$$\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) \geq \frac{d_\varepsilon}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right) \tag{4.18}$$

for $0 < t < T^g(\lambda)$. Recall from Proposition (B.1.1) $d_\varepsilon = \sqrt{2aK}$ with $K = \max\{\frac{a}{2}, \frac{g'(0)}{2\varepsilon}, -\frac{c_0}{\varepsilon^2}\}$. Then for large enough $\varepsilon > 0$, we obtain $K = \frac{a}{2}$ and $d_\varepsilon = a$. The sufficient condition therefore reduces to

$$a \leq \sqrt{2\lambda} \coth\left(\frac{t}{\sqrt{2\lambda}}\right)$$

for $0 < t < T^g(\lambda)$, which is implied by the definition (4.8) of $T^g(\lambda)$. Therefore, given any $(x, t) \in \mathbb{R}^+ \times (0, T^g(\lambda))$, we can find a unique $x_0 \in \mathbb{R}^+$, the starting point of the characteristic curve passing through (x,t), and integrate along the characteristic curve to obtain the function $h(x,t)$ through (4.13).

After showing that the Hamilton-Jacobi equation (4.9) has a unique classical solution up to time $T^g(\lambda)$, the last constraint we need to consider is to have nonnegative spatial derivative $h_x(x,t)$ for all $(x,t)\in\mathbb{R}^+\times(0,T^g(\lambda))$. Let $f(x,t):=h_x(x,t)$, then the function $f(x,t)$ is the solution to the quasilinear equation

$$\begin{cases} f_t-\frac{1}{2\lambda}ff_x+x=0, & x>0,\ 0<t<T^g(\lambda),\\ f(x,0)=g'(x), & x\geq 0. \end{cases} \tag{4.19}$$

Equation (4.19) can be solved by the method of characteristics as (4.9). Direct calculation yields the same characteristic curve as in (4.14), and the solution

$$f(x(x_0,t),t)=g'(x_0)\cosh\left(\frac{t}{\sqrt{2\lambda}}\right)-\sqrt{2\lambda}x_0\sinh\left(\frac{t}{\sqrt{2\lambda}}\right). \tag{4.20}$$

Then, a sufficient condition to have nonnegative value for $f(x(x_0,t),t)$ given any $(x,t)\in\mathbb{R}^+\times(0,T^g(\lambda))$ is therefore

$$g'(x_0)\cosh\left(\frac{t}{\sqrt{2\lambda}}\right)-\sqrt{2\lambda}x_0\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)\geq 0. \tag{4.21}$$

Definition of $T^g(\lambda)$ in (4.8) as well as the conditions (4.7) satisfied by the function $g(\cdot)$ imply that (4.21) is valid for all $(x_0,t)\in\mathbb{R}^+\times(0,T^g(\lambda))$.

We conclude the proof by showing the uniqueness of the classical solution to the Hamilton-Jacobi equation (4.9). To this end, we assume that there are two solutions that are non-decreasing in x, $h\in C^{1,1}(\mathbb{R}^+\times[0,T))$ and $\tilde{h}\in C^{1,1}(\mathbb{R}^+\times[0,\widetilde{T}))$, with $\widetilde{T}>T$, satisfying $h(x,0)=\tilde{h}(x,0)=g(x)$,

$x > 0$. Then, $H := h - \tilde{h}$ satisfies, for $(x,t) \in \mathbb{R}^+ \times [0,T)$,

$$H_t - \frac{1}{4}\left(h_x^2 - \tilde{h}_x^2\right) = H_t - \frac{1}{4}H_x\left(h_x + \tilde{h}_x\right) = 0,$$

with $H(x,0) = 0$. For the characteristics we have $\frac{dX(t)}{dt} = -\frac{h_x(X(t),t)+\tilde{h}_x(X(t),t)}{4\lambda_0}$, with $h_x(X(t),t) + \tilde{h}_x(X(t),t) \geq 0$. It hence implies that for any $X(t) = x \geq 0$, $t \in [0,T)$, the initial value $X(0) = x_0 \geq x \geq 0$. We conclude, using $H(X(t),t) = H(x_0,0) = 0$, with $x_0 \geq 0$, that $H \equiv 0$ is the unique solution up to time T. It then follows $T = \widetilde{T}$, and $h(x,t) = \tilde{h}(x,t)$, for $(x,t) \in \mathbb{R}^+ \times [0,T)$. □

Remark 4.3.1. The condition (4.7) can be interpreted as the following. Since $g'(x) = h_x(x,0)$ following from the equation (4.9), and $h_x(x,0)$ is the trading rate at time $t = 0$ with initial inventory $x > 0$, the condition

$$\inf_{x>0} \frac{g'(x)}{x} \geq b > 0$$

then basically requires that the normalized initial trading rate (or the percentage with respect to initial inventory) should be uniformly bounded away from zero. This is reasonable from practical point of view, as $b > 0$ can be taken as $\delta/\Delta t$, where δ is the minimal percentage of shares that are allowed to trade in the market, and Δt is the time discretization of the continuous time model. If the initial trading rate is zero, then the starting time of trading is actually postponed to some later time at which the above condition is satisfied.

Theorem 4.3.1 gives the condition for the ill-posed inverse liquidation problem to be solvable over the time interval $[0, T^g(\lambda))$, based on the analysis of the existence and uniqueness of solution to the HJ equation (4.9). In the next proposition, we provide a verification argument to the inverse liquidation problem and therefore complete the discussion for single-period forward liquidation problem.

Theorem 4.3.2. *Assume that $U(x,r,0) = -e^{-r+g(x)}$, with g satisfying the assumption in Theorem 4.3.1. Then, the process $U(x,r,t) = -e^{-r+h(x,t)}$ is a forward performance process, for $0 \le t < T^g(\lambda)$, where the function $h(x,t)$ is the unique classical solution to the Hamilton-Jacobi equation with nonnegative spatial derivative*

$$\begin{cases} h_t - \frac{1}{4\lambda} h_x^2 + \frac{1}{2}x^2 = 0, \quad x > 0, \ 0 < t < T^g(\lambda), \\ h(x,0) = g(x), \quad x > 0, \end{cases} \tag{4.22}$$

and $T^g(\lambda)$ is given by (4.8). Moreover, the optimal liquidation strategy under this forward performance process is

$$\xi_t^* = \frac{h_x(X_t^*, t)}{2\lambda}, \tag{4.23}$$

and the corresponding optimal inventory process is

$$X_t^* = X_0 - \int_0^t \xi_s^* ds = X_0 \cosh \frac{t}{\sqrt{2\lambda}} - \frac{g'(X_0)}{\sqrt{2\lambda}} \sinh \frac{t}{\sqrt{2\lambda}}, \tag{4.24}$$

with initial inventory $X_0 > 0$. In particular, full liquidation can be achieved under the forward optimal trading strategy (4.24) if and only if

$$T^* := \sqrt{2\lambda} \coth^{(-1)} \left(\frac{g'(X_0)}{\sqrt{2\lambda} X_0} \vee 1 \right) \le T^g(\lambda). \tag{4.25}$$

Proof. Let $\mathcal{A}$ be the admissible set that consists of all $\mathcal{F}_t$-progressively measurable processes ξ, such that $\xi_t \geq 0$, for all $t \in \left[0, T^\xi\right)$, $\int_0^{T^\xi} \xi_s^2 ds < L(\xi)$ a.s., and $\mathbb{E} \int_0^{T^\xi} (X_s^\xi)^2 ds < \infty$, with $T^\xi = \inf\left\{t > 0 : X_0 = \int_0^t \xi_s ds\right\}$, and $L(\xi) > 0$ being a constant that only depends on ξ.

The verification argument aims to show, for any $0 < T < T^g(\lambda) \leq \infty$ and any admissible $\xi \in \mathcal{A}$, the process $U(X^\xi_{t\wedge T^\xi}, R^\xi_{t\wedge T^\xi}, t \wedge T^\xi)$, $t \in [0, T]$ is a supermartingale, while for the specific ξ^* and T^* given in (4.23) and (4.25), respectively, the process $U(X^*_{t\wedge T^*}, R^*_{t\wedge T^*}, t \wedge T^*)$, $t \in [0, T]$ is a martingale. The upper bound of the horizon $T^\xi \wedge T^g(\lambda)$ or $T^* \wedge T^g(\lambda)$ for the verification argument is needed, since the function $h(x, t)$ in the forward performance process $U(x, r, t)$ is only well defined for $(x, t) \in \mathbb{R}^+ \times [0, T^g(\lambda))$. We start by applying Itô's lemma, and obtain, for every $\xi \in \mathcal{A}$, and $0 \leq t \leq T < T^g(\lambda)$, that

$$
\begin{aligned}
U(X^\xi_{t\wedge T^\xi}, R^\xi_{t\wedge T^\xi}, t \wedge T^\xi) &= U(x, r, 0) + \int_0^{t\wedge T^\xi} U_s(X^\xi_s, R^\xi_s, s) ds \\
&- \int_0^{t\wedge T^\xi} U_x(X^\xi_s, R^\xi_s, s)\xi_s ds - \lambda \int_0^{t\wedge T^\xi} U_r(X^\xi_s, R^\xi_s, s)\xi_s^2 ds \\
&+ \frac{1}{2}\int_0^{t\wedge T^\xi} U_{rr}(X^\xi_s, R^\xi_s, s)(X^\xi_s)^2 ds + \int_0^{t\wedge T^\xi} U_r(X^\xi_s, R^\xi_s, s) X^\xi_s dW_s \\
&= U(x, r, 0) + \int_0^{t\wedge T^\xi} \left(U_s - U_x\xi_s - \lambda U_r \xi_s^2 + \frac{1}{2} U_{rr}(X^\xi_s)^2\right) ds + \int_0^{t\wedge T^\xi} U_r X^\xi_s dW_s
\end{aligned}
$$

where we have suppressed the arguments of U in the last equality. It follows from the last equality and the fact $h(x, t)$ solves the HJ equation (4.22) that for $\xi^*_t = \frac{h_x(X^*_t, t)}{2\lambda}$, the drift vanishes. For any other $\xi \in \mathcal{A}$, the drift remains nonpositive, giving the supermartingale property away from the optimum once we have shown the stochastic integral is a true martingale. We next show the stochastic integral

$$\int_0^{t\wedge T^\xi} U_r(X_s^\xi, R_s^\xi, s) X_s^\xi dW_s = \int_0^{t\wedge T^\xi} e^{-R_s^\xi + h(X_s^\xi, s)} X_s^\xi dW_s,$$

is a true martingale, for $0 \le t \le T < T^g(\lambda)$. It suffices to show the square integrability

$$\mathbb{E} \int_0^{T\wedge T^\xi} e^{-2R_s^\xi + 2h(X_s^\xi, s)} (X_s^\xi)^2 ds < \infty,$$

for each admissible $\xi \in \mathcal{A}$. Notice that for any $0 \le s \le T^\xi$, the inventory process $|X_s^\xi| \le X_0$ uniformly in (s, ω), and

$$\left| h(X_{s\wedge T^\xi}^\xi, s \wedge T^\xi) \right| \le \max_{(x,s)\in[0,X_0]\times[0,T]} |h(x,s)| < \infty,$$

as $T < T^g(\lambda)$. It hence remains to show $\mathbb{E} \int_0^{T\wedge T^\xi} e^{-2R_s^\xi} ds < \infty$. By admissibility of $\xi \in \mathcal{A}$, there exist constants L_ξ, $K_\xi > 0$, such that $\int_0^{T^g(\lambda)} \xi_{s\wedge T^\xi}^2 ds < L_\xi$, a.s., and $\mathbb{E} \int_0^{T^g(\lambda)} (X_{s\wedge T^\xi}^\xi)^2 ds < K_\xi$. Therefore, we obtain

$$\mathbb{E} \int_0^{T\wedge T^\xi} e^{-2R_s^\xi} ds = \mathbb{E} \int_0^{T\wedge T^\xi} \exp\left(-2r - 2\int_0^s X_u^\xi dW_u + 2\lambda \int_0^s \xi_u^2 du \right) ds$$

$$\le e^{-2r+2\lambda L_\xi} \mathbb{E} \int_0^{T\wedge T^\xi} e^{-2\int_0^s X_u^\xi dW_u} ds = e^{-2r+2\lambda L_\xi} \int_0^T \mathbb{E}\left[e^{-2\int_0^s X_{u\wedge T^\xi}^\xi dW_u} \right] ds$$

$$\le e^{-2r+2\lambda L_\xi} \int_0^T e^{2K_\xi} ds = e^{-2r+2\lambda L_\xi + 2K_\xi} T < \infty,$$

where we have used the fact that the process $\int_0^s X_{u\wedge T^\xi}^\xi dW_u$, $0 \le s \le T < T^g(\lambda)$ is a square integrable martingale with quadratic variation at most K_ξ.

This completes the proof of showing the genuine martingality of the stochastic integral for any $\xi \in \mathcal{A}$. We next complete the verification argument by deriving the candidate optimal strategy ξ^* and showing its admissibility. Indeed, recall from the characteristic ODEs (4.10) that

$$X(t) - X(0) = -\frac{1}{2\lambda}\int_0^t P(s)ds = -\int_0^t \frac{h_x(X(s), s)}{2\lambda}ds. \tag{4.26}$$

The proof of Theorem 4.3.1 guarantees that for any $0 \leq t < T^g(\lambda)$ and any $X(t) = x \geq 0$, there exists a unique initial value $X(0) = x_0 \geq 0$, such that $X(t)$ in (4.26) can be alternatively obtained through the characteristic curve $x = G(x_0, t)$ as following

$$X(t) = X(0)\cosh\frac{t}{\sqrt{2\lambda}} - \frac{g'(X(0))}{\sqrt{2\lambda}}\sinh\frac{t}{\sqrt{2\lambda}}. \tag{4.27}$$

On the other hand, the above verification argument has shown that the (candidate) optimal inventory process X_t^* starting from initial inventory $X_0^* = X_0$ should satisfy, for $0 \leq t \leq T \leq T^* \wedge T^g(\lambda)$, that

$$X_t^* - X_0 = -\int_0^t \xi_s^* ds = -\int_0^t \frac{h_x(X_s^*, s)}{2\lambda}ds. \tag{4.28}$$

A comparison of (4.26) and (4.28) therefore yield that the optimal inventory X_t^* can be represented by (4.27) with a uniquely determined initial inventory $X_0^* = X(0) \geq 0$. Conversely, given any initial inventory $X_0 > 0$, the process defined by (4.27) with $X(0) = X_0$ is the unique optimal inventory process under the forward performance process $U(x, r, t) = -e^{-r+h(x,t)}$. Hence, we obtain the conclusion (4.24) and in particular (4.25) after we show the admissibility of ξ^* for generic initial function $g(x)$ satisfying (4.7). When $T^g(\lambda) = T^* = \infty$ for a specific function $g(x)$ and a specific initial inventory $X_0 > 0$, we can directly check by (4.24) that $\xi^* = \frac{X_0}{\sqrt{2\lambda}}e^{-\frac{t}{\sqrt{2\lambda}}}$ and, hence, admissibility easily follows.

For any other choices of $g(x)$ and inventory $X_0 > 0$, we have $T^* \wedge T^g(\lambda) < \infty$. If $T^* < T^g(\lambda) \leq \infty$ or $T^* = T^g(\lambda) < \infty$, then according to (4.24), both X^* and ξ^* are uniformly bounded over the finite interval $[0, T^*]$, and hence $\int_0^{T^*} \xi_t^{*2} dt < \infty$ and $\mathbb{E} \int_0^{T^*} \left(X_t^*\right)^2 dt < \infty$. On the other hand, if $T^* > T^g(\lambda)$, it follows from (4.24) that $T^* = \infty$, and that X^* and ξ^* are only defined over the finite interval $[0, T^g(\lambda))$ and remain uniformly bounded. The conditions $\int_0^{T^*} \xi_t^{*2} dt = \int_0^{T^g(\lambda)} \xi_t^{*2} dt < \infty$ and $\mathbb{E} \int_0^{T^*} \left(X_t^*\right)^2 dt = \mathbb{E} \int_0^{T^g(\lambda)} \left(X_t^*\right)^2 dt < \infty$ are also satisfied. Finally, in all above cases, $\xi_t^* \geq 0$ for $0 \leq t < T^*$ is guaranteed by (4.24) and the construction of $T^g(\lambda)$ as in Theorem 4.3.1. □

Remark 4.3.2. A simple scenario is when $g'(0) = 0$, which leads to $T^* \geq T^g(\lambda)$. Indeed, if $g'(0) = 0$, we have $g'(x) \leq ax$, for all $x > 0$ and all $a \geq \sup_{x>0} g''(x)$. Comparing the expressions (4.8) and (4.25), we obtain

$$T^* = \sqrt{2\lambda} \coth^{(-1)} \left(\frac{g'(X_0)}{\sqrt{2\lambda} X_0} \vee 1 \right) \geq \sqrt{2\lambda} \coth^{(-1)} \left(\frac{a}{\sqrt{2\lambda}} \vee 1 \right) \geq T^g(\lambda).$$

The scenario $g'(0) = 0$ includes both the classical finite and infinite horizon liquidation problems under exponential utility ([53], [52]), as well as other possible choices for the function $g(x)$, for example, $g(x) = ax^2 + \frac{1}{x^2+c}$, with $c > 0$, or $g(x) = ax^2 - x^2 e^{-cx}$, with $c > 0$ and properly chosen constant a, among others. In such scenario, there would be non-zero inventory at any time strictly before the solvability horizon of the inverse liquidation problem.

In classical finite and infinite horizon liquidation problems, the full liquidation time is independent of the initial inventory X_0. This is also obvious from (4.25), as for quadratic function $g(x) = ax^2$, we have $\frac{g'(X_0)}{X_0} = 2a$ being a constant. However, intuitively, the quantity to liquidate should have an effect on the liquidation time. In the forward framework, this is true, since the full liquidation time T^* depends on the normalized initial trading rate $\frac{g'(X_0)}{X_0}$ (see Remark 4.3.1) and its magnitude relative to the price impact parameter λ. It is easy to see that the higher the normalized initial trading rate, the sooner the liquidation would be completed. This is in compliance with the widely observed "front-loaded" characteristic of most trading strategies. From the perspective of market liquidity conditions, (4.25) indicates that the higher the price impact parameter λ, i.e., less liquidity available in the market, the longer the liquidation horizon would be, and full liquidation would only be possible if $T^* \leq T^g(\lambda)$. In the forward framework, these qualitative properties agree well with practical intuition.

As shown by (4.25), it is possible to have $T^* < T^g(\lambda)$ or $T^* \geq T^g(\lambda)$ by properly choosing $g(\cdot)$, X_0 and λ. An interesting consequence of the scenario $T^* < T^g(\lambda)$ is that it is not necessary to impose a singular terminal condition to guarantee a full liquidation as in most existing works (e.g., [53], [4], [28], etc). Indeed, we may take $U(x, r, T^*) = -e^{-r+h(x,T^*)}$ as the terminal utility function where $h(x,t)$ is the solution to the Hamilton-Jacobi equation (4.22) with an appropriate initial condition $g(\cdot)$. This is a classical expected utility maximization problem with the same optimal trading strategy (4.24) that fully unwinds a range of initial inventory at T^*. The terminal utility is non-singular since Theorem 4.3.1 guarantees well-posedness of $h(x,t)$ up to $T^g(\lambda) > T^*$. In fact, the possibility of full liquidation under non-singular terminal condition basically results from $g'(0) > 0$ (see Remark 4.3.2 for otherwise). In the next proposition, we examine the maximum initial inventory that could be fully unwound by the associated time horizon $T^g(\lambda)$, given an admissible $g(\cdot)$ function and a fixed market impact λ.

Another interesting observation is that specifying a function $g(\cdot)$ is equivalent to specifying an initial trading rate (see Remark 4.3.1), which is practically meaningful as a client may only know her preferred trading profile at $t = 0$ when she comes to the trading agent, rather than being fully aware of her future utility function. The agent can then come up with the consistent trading behavior following this initial profile by solving the inverse liquidation problem, and as a side result, we can also infer the non-singular terminal criterion that is consistent with the client's initial preference.

Proposition 4.3.3. *Assume that the function g satisfies the assumption in Theorem 4.3.1, and in addition, $g'(0) > 0$. Then, there exist $0 < \overline{X} \leq \infty$ and an increasing function $Z : [0, T^g(\lambda)) \to [0, \overline{X})$, such that, for any $0 < T < T^g(\lambda)$, and any initial inventory $X_0 \in [0, Z(T)]$, the full liquidation can be achieved under the classical non-singular terminal utility $U(x, r, T) = -e^{-r+h(x,T)}$.*

Proof. For a given admissible $g(x)$ with $g'(0) > 0$, we define the function

$$f(X_0) := X_0 \cosh \frac{T}{\sqrt{2\lambda}} - \frac{g'(X_0)}{\sqrt{2\lambda}} \sinh \frac{T}{\sqrt{2\lambda}},$$

for any $X_0 \geq 0$, $0 \leq T < T^g(\lambda)$. Then clearly, $f(0) \leq 0$ and $f(X_0)$ is strictly increasing in X_0 due to the construction of $T^g(\lambda)$ in Theorem 4.3.1

and that $T^g(\lambda) > T$. Next, we show $f(X_0) \to \infty$, as $X_0 \to \infty$. Indeed, since (4.8) implies $a \le \sqrt{2\lambda}\coth\frac{T^g(\lambda)}{\sqrt{2\lambda}}$, we can find a constant $d > 0$ such that $a < d < \sqrt{2\lambda}\coth\frac{T}{\sqrt{2\lambda}}$ for $0 \le T < T^g(\lambda)$. We also know from Proposition (A.1) that $g'(x) \le d_\varepsilon(x+\varepsilon)$, for all $x > 0$, $\varepsilon > 0$, and for large enough $\varepsilon_0 > 0$, we obtain $g'(x) \le a(x+\varepsilon_0)$. Now it is easy to check for $x > \frac{a\varepsilon_0}{d-a}$, we have $g'(x) \le a(x+\varepsilon_0) < dx$. Hence, for X_0 sufficiently large, we have

$$f(X_0) \ge X_0\left(\cosh\frac{T}{\sqrt{2\lambda}} - \frac{d}{\sqrt{2\lambda}}\sinh\frac{T}{\sqrt{2\lambda}}\right) \to \infty$$

as $X_0 \to \infty$, for $0 \le T < T^g(\lambda)$. We therefore conclude that there exists a unique function $Z(T;g) \in [0,\infty)$, such that $f(Z(T;g)) = 0$ for any $0 \le T < T^g(\lambda)$, and any admissible $g(x)$ with $g'(0) > 0$. Hence, $Z(\cdot;g):\ [0, T^g(\lambda)) \to [0,\infty)$ is well defined, and $Z(0;g) = 0$. Also, due to the implicit function theorem and the fact $f'(X_0) > 0$ for all $0 \le T < T^g(\lambda)$, we know $\frac{dZ(T;g)}{dT}$ exists. Differentiation of the equation $f(Z(T;g)) = 0$ with respect to T gives rise to

$$\frac{dZ(T;g)}{dT}\left(\cosh\frac{T}{\sqrt{2\lambda}} - \frac{g''(Z(T;g))}{\sqrt{2\lambda}}\sinh\frac{T}{\sqrt{2\lambda}}\right)$$

$$= \frac{1}{\sqrt{2\lambda}}\left(\frac{g'(Z(T;g))}{\sqrt{2\lambda}}\cosh\frac{T}{\sqrt{2\lambda}} - Z(T;g)\sinh\frac{T}{\sqrt{2\lambda}}\right).$$

Direct check of the terms in the two parentheses shows that they are strictly positive for $0 \le T < T^g(\lambda)$, giving that $\frac{dZ(T;g)}{dT} > 0$, for any $0 \le T < T^g(\lambda)$, and any admissible $g(\cdot)$ with $g'(0) > 0$. Therefore, $\lim_{T\uparrow T^g(\lambda)} Z(T;g)$ exists. Finally, we notice by (4.24) that for fixed admissible $g(\cdot)$ with $g'(0) > 0$ and fixed λ, if the initial inventory $\widehat{X_0} > 0$ can be fully unwound by some time horizon $T < T^g(\lambda)$ under the optimal strategy (4.24), then for any initial inventory $0 < X_0 \le \widehat{X}_0$, its corresponding full liquidation time stays within $[0, T]$. $\square$

Obviously, in Proposition 4.3.3, $\overline{X} = \lim_{T \uparrow T^g(\lambda)} Z(T; g)$ is the maximum initial inventory that could be liquidated under the pre-chosen function $g(\cdot)$ and the market condition $\lambda > 0$. This maximum initial inventory can be infinite in some scenarios, indicating that any initial inventory can be fully liquidated by $0 < T < T^g(\lambda)$, under the non-singular terminal utility $U(x, r, T) = -e^{-r+h(x,T)}$. For instance, taking $g(x) = x^2 + x$ and $0 < \lambda < 2$, then

$$Z(T; g) = \frac{1}{\sqrt{2\lambda} \coth \frac{T}{\sqrt{2\lambda}} - 2},$$

and clearly, $\lim_{T \uparrow T^g(\lambda)} Z(T; g) = \infty$. Nevertheless, our finding does not contradict with the classical results under the singular terminal condition. Indeed, in our framework, the full liquidation time T^* increases as the initial inventory X_0 increases, whereas in the classical setting, a fixed common liquidation horizon $\widehat{T}$ is imposed for all initial inventory $X_0 > 0$. If there exists such fixed horizon $\widehat{T} > 0$ in our framework, such that $T^* \le \widehat{T} < T^g(\lambda)$ for all $X_0 > 0$, then this amounts to imposing a finite penalty to achieve full liquidation for any inventory by a fixed time. If this could happen, then a comparison of (4.8) and (4.25) yields $\inf_{x>0} \frac{g'(x)}{x} > \sup_{x>0} g''(x)$. Taking two constants C_1, C_2, such that $\inf_{x>0} \frac{g'(x)}{x} > C_1 > C_2 > \sup_{x>0} g''(x)$, it is then easy to see

$$C_1 x - g'(0) < g'(x) - g'(0) < C_2 x,$$

which leads to $(C_1 - C_2)x < g'(0)$, for all $x > 0$. This violates the assumption that $g'(0)$ is finite in Theorem 4.3.1 and Proposition 4.3.2.

Our results bring new insights to the classical optimal liquidation problem, in that, instead of specifying a singular terminal criterion, we can specify an initial trading profile, and infer from that the consistent trading horizon and the (possibly non-singular) consistent terminal criterion. The flexibility to choose an initial trading profile is rooted in the flexibility of forward performance process in terms of the initial condition, which is not possible in the classical framework due to the backward construction.

Another interesting question related to the classical problem under the forward formulation is as follows. Given any fixed time horizon $T > 0$, and any initial inventory $X_0 \in [0, \overline{X}]$, with $\overline{X}$ being the finite total number of shares in the market, determine whether it is possible to choose a non-singular terminal criterion at T, such that on one hand, full liquidation is guaranteed by T for any $X_0 \in [0, \overline{X}]$, and on the other, this criterion yields higher $t = 0$ value compared to that under the classical singular terminal criterion. We provide a positive answer following Proposition 4.3.3. First recall the fact that if an initial inventory $\widehat{X}_0$ can be fully liquidated by some time $0 < T < T^g(\lambda)$ following the forward optimal strategy (4.24), then any other initial inventory $0 < X_0 \leq \widehat{X}_0$ can also be fully liquidated by T following the strategy (4.24). This observation implies that we can look for an admissible function $g(\cdot)$, such that the given pair $(\overline{X}, T)$ satisfies both

$$\overline{X}\cosh\frac{T}{\sqrt{2\lambda}} - \frac{g'\left(\overline{X}\right)}{\sqrt{2\lambda}}\sinh\frac{T}{\sqrt{2\lambda}} = 0 \tag{4.29}$$

and $0 < T < T^g(\lambda)$, simultaneously. With such admissible function $g(\cdot)$, according to Theorem (4.3.1) and Proposition (4.3.2), we have a unique non-singular terminal criterion $U(x, r, T) = -e^{-r+h(x,T)}$ for all $x \geq 0$ at the fixed time T. Moreover, the corresponding optimal inventory process (4.24) achieves full liquidation for any $X_0 \in [0, \overline{X}]$ by T.

The solution $g(\cdot)$ to (4.29) is clearly not unique, and we only focus on the quadratic case $g(x) = ax^2 + bx + c$, with $a, b > 0$ and $c \in \mathbb{R}$. The condition $b = g'(0) > 0$ is necessary, as discussed in Remark 4.3.2. Notice this case is fundamentally different from the quadratic case considered in the previous chapter and the existing works (e.g., [53], [52]), due to the condition $g'(0) > 0$. With the function $g(\cdot)$, the condition (4.29) reduces to $2a + \frac{b}{\overline{X}} = \sqrt{2\lambda}\coth\left(\frac{T}{\sqrt{2\lambda}}\right)$. This together with the condition $0 < T < T^g(\lambda)$ gives rise to one family of solutions among others, provided that

$$\sqrt{2\lambda} < 2a < \sqrt{2\lambda}\coth\frac{T}{\sqrt{2\lambda}},$$

$$2a + \frac{b}{\overline{X}} = \sqrt{2\lambda}\coth\frac{T}{\sqrt{2\lambda}},$$

and $b > 0$ are satisfied simultaneously. It is easy to check the above system of equations are compatible and solutions exist. We next compare the $t = 0$ performance under the the non-singular terminal criterion associated to the solution $g(\cdot)$ and the classical singular terminal criterion (4.4). First, for any initial inventory $X_0 \in [0, \overline{X}]$, the full liquidation time under the optimal trading trajectory (4.24) is a deterministic time $T^*(X_0) \leq T < T^g(\lambda)$ due to (4.25) and the condition (4.29). The martingale property along the optimum under the forward performance process therefore implies the $t = 0$ optimal value is

$$-e^{-r+g(x)} = \mathbb{E}\left[-e^{-R^*_{T^*(X_0)}+h(0,T^*(X_0))}\middle| R_0 = r, X_0 = x\right]$$

for all $X_0 = x \in [0, \overline{X}]$, where we have applied the fact $X^*_{T^*(X_0)} = 0$. On the other hand, the classical problem takes the optimal strategy (4.6) which unwinds all initial inventory $X_0 \in [0, \overline{X}]$ exactly at the fixed time T. Hence, its optimal $t = 0$ value is

$$-e^{-r+\tilde{g}(x)} = \mathbb{E}\left[-e^{-R^*_T}\middle| R_0 = r, X_0 = x\right],$$

where the function $\tilde{g}(x) = \frac{\sqrt{2\lambda}}{2}x^2 \coth \frac{T}{\sqrt{2\lambda}}$, according to the classical value function (4.5). Direct computation then shows that for any solution (a, b) that satisfies the system of equations, if

$$c \leq \frac{b^2}{4a - \sqrt{2\lambda}\coth\left(\frac{T}{\sqrt{2\lambda}}\right)} < 0,$$

then $g(x) \leq \tilde{g}(x)$, which yields that given the same initial revenue and inventory, the $t = 0$ optimal value under the non-singular terminal criterion exceeds the optimal value under the classical singular terminal criterion, while both criteria lead to full liquidation of all initial inventory $X_0 \in [0, \overline{X}]$ by a common fixed time T.

4.3.2 Multi-period problem

Following [3], once we have the result for the single inverse liquidation problem, the multi-period forward performance process can be constructed recursively forward in time. In the multi-period setting, we also incorporate model revision as in the previous chapter. The success of a continuing construction of the forward performance process in the previous chapter is based on the nice property that the Hamilton-Jacobi equation (4.9) with quadratic initial condition $g(\cdot)$ has a quadratic solution $h(\cdot, t)$, for any $0 \leq t < T^g(\lambda)$. This self-similarity makes it possible to concatenate each single inverse liquidation problem after a conditioning argument. In this section, we show that for general initial datum $g(\cdot)$ that is not necessarily quadratic, the same self-similarity property holds as well. More precisely, the unique classical solution to the Hamilton-Jacobi equation (4.9) stays in the same class as its initial datum, which leads to a feasible concatenation of the solution to the single inverse liquidation problem.

Proposition 4.3.4. *Assume that the function g satisfies the conditions in Theorem 4.3.1. Then, the unique classical solution h of the Hamilton-Jacobi equation (4.9) with nonnegative spatial derivative satisfies, for every $0 \leq t < T^g(\lambda)$, that*

$$\inf_{x>0} \frac{h_x(x,t)}{x} \geq \frac{b\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)}{\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{b}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)} > 0, \tag{4.30}$$

and also that

$$\sup_{x>0} h_{xx}(x,t) \leq \frac{a\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)}{\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{a}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)} > 0. \tag{4.31}$$

Proof. In Theorem 4.3.1, we have shown that for each $0 \le t < T^g(\lambda)$, the mapping $G(\cdot,t) : \mathbb{R}^+ \to \mathbb{R}$ is well defined and strictly increasing, where

$$G(x_0,t) = x_0 \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{g'(x_0)}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right),$$

for all $x_0 \in \mathbb{R}^+$. Denote the spatial inverse of $G(\cdot,t)$ by $x_0 = x_0(x,t)$, for each $x > 0$, and $0 \le t < T^g(\lambda)$, then the mapping $x_0(\cdot,t) : \mathbb{R}^+ \to \mathbb{R}^+$ is well defined and continuously differentiable with $\frac{\partial x_0}{\partial x} > 0$, due to the construction of $T^g(\lambda)$ in Theorem 4.3.1. Differentiating both sides of the characteristic curve equation $G(x_0,t) = x$ with respect to x, we obtain

$$\frac{\partial x_0}{\partial x} = \frac{1}{\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{g''(x_0)}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right)} \le \frac{1}{\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{a}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right)}. \tag{4.32}$$

Now, recall that $h_x(x,t) = f(x,t)$ in the proof of Theorem 4.3.1, with

$$f(x(x_0,t),t) = g'(x_0) \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda} x_0 \sinh\left(\frac{t}{\sqrt{2\lambda}}\right), \tag{4.33}$$

and the same characteristic curve equation as for $h(x,t)$

$$x(x_0,t) = x_0 \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{g'(x_0)}{\sqrt{2\lambda}} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right),$$

for all $x_0 \in \mathbb{R}^+$, and $0 \le t < T^g(\lambda)$. Therefore, for all $x > 0$, and $0 \le t < T^g(\lambda)$, we have $x_0 > 0$ and

$$\begin{aligned} \frac{h_x(x,t)}{x} &= \frac{g'(x_0) \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda} x_0 \sinh\left(\frac{t}{\sqrt{2\lambda}}\right)}{x} \\ &= \frac{x_0}{x}\left(\frac{g'(x_0)}{x_0} \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda} \sinh\left(\frac{t}{\sqrt{2\lambda}}\right)\right) \end{aligned}$$

$$\geq \frac{b\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)}{\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{b}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)} > 0, \tag{4.34}$$

following from $g'(x_0) \geq bx_0$, for all $x_0 \in \mathbb{R}^+$, and that

$$\begin{aligned} 0 < \frac{x}{x_0} &= \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{g'(x_0)}{x_0\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) \\ &\leq \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{b}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) \end{aligned}$$

from the characteristic curve equation. The numerator

$$b\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) > 0$$

is due to the construction of $T^g(\lambda)$ (cf. (4.8)). The denominator

$$\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{b}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) > 0$$

follows from the fact $a \geq b > 0$ and, hence,

$$\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{b}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) \geq \cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{a}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) > 0,$$

again due to the construction of $T^g(\lambda)$. The proof for (4.30) is therefore complete. We next prove (4.31). Indeed, differentiating both sides of equation (4.33) with respect to x, we obtain

$$\begin{aligned} h_{xx}(x,t) &= \frac{\partial x_0}{\partial x}\left(g''(x_0)\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)\right) \\ &\leq \frac{a\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)}{\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{a}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right)} > 0, \end{aligned}$$

following from $g''(x_0) \leq a$, for all $x_0 \in \mathbb{R}^+$, and the inequality (4.32). The denominator

$$\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \frac{a}{\sqrt{2\lambda}}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) > 0$$

is due to the construction of $T^g(\lambda)$, while the numerator

$$a\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) > 0$$

follows from the fact $a \geq b > 0$ and, hence,

$$a\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) \geq b\cosh\left(\frac{t}{\sqrt{2\lambda}}\right) - \sqrt{2\lambda}\sinh\left(\frac{t}{\sqrt{2\lambda}}\right) > 0,$$

again due to the construction of $T^g(\lambda)$. □

4.3.2.1 General result

In this section, we provide the result for constructing the general forward performance process in a model switching scenario. The model revision is the same as in the previous chapter, but we allow more general initial performance datum for the forward process. The argument of Theorem 4.3.5 is based on the desirable self-similarity property of the solution to the Hamilton-Jacobi equation discussed in Proposition 4.3.4.

Theorem 4.3.5. *Assume that $U(x, r, 0) = -e^{-r+g(x)}$, with g satisfying the assumption in Theorem 4.3.1. Then, for any predictable time $\tau_n \in \mathcal{F}_{\tau_{n-1}}$, $n \geq 1$, such that $\tau_0 = 0$, and $\tau_{n-1} < \tau_n < \tau_{n-1} + T^g(\lambda_1, \cdots, \lambda_n)$, with*

$$T^g(\lambda_1, \cdots, \lambda_n) := \sqrt{2\lambda_n}\min\left(\tanh^{(-1)}\left(\frac{b_{n-1}(\lambda_1, \cdots, \lambda_{n-1})}{\sqrt{2\lambda_n}} \wedge 1\right),\right.$$

$$\coth^{(-1)}\left(\frac{a_{n-1}(\lambda_1,\cdots,\lambda_{n-1})}{\sqrt{2\lambda_n}}\vee 1\right)\Bigg)\in\mathcal{F}_{\tau_{n-1}},\tag{4.35}$$

the process

$$U(x,r,\tau_n)=-e^{-r+h^{(n)}(x,\tau_n)}\in\mathcal{F}_{\tau_{n-1}},\tag{4.36}$$

is the unique predictable forward performance process in the separable form, where the random function $h^{(n)}$ is the unique classical solution with nonnegative spatial derivative to the Hamilton-Jacobi equation with random coefficient $\lambda_n\in\mathcal{F}_{\tau_{n-1}}$, $n\geq 1$,

$$h_t^{(n)}-\frac{1}{4\lambda_n}h_x^{(n)^2}+\frac{1}{2}x^2=0,\quad x>0,\ \tau_{n-1}<t<\tau_{n-1}+T^g(\lambda_1,\cdots,\lambda_n),\tag{4.37}$$

with initial condition $h^{(n)}(x,\tau_{n-1})=h^{(n-1)}(x,\tau_{n-1})$, for $n\geq 1$, and $h^{(0)}(x,0)=g(x)$. For $n\geq 2$, the positive random variables a_{n-1}, b_{n-1} in (4.35) are $\mathcal{F}_{\tau_{n-2}}$-measurable, and satisfy

$$a_{n-1}(\lambda_1,\cdots,\lambda_{n-1})\geq\sup_{x>0}h_{xx}^{(n-1)}(x,\tau_{n-1}),$$

and

$$b_{n-1}(\lambda_1,\cdots,\lambda_{n-1})\leq\inf_{x>0}\frac{h_x^{(n-1)}(x,\tau_{n-1})}{x},$$

while $a_0=a$, and $b_0=b$, with a,b as in (4.7).

Proof. We prove by induction. Clearly, for $n=1$, we have $U(x,r,0)=-e^{-r+g(x)}$, and Theorem 4.3.1 together with the verification argument of Theorem 4.3.2 guarantee the existence and uniqueness of a forward performance process in the separable form up to the deterministic time $T^g(\lambda_1)$ given by (4.8). Assume now the deterministic time $0<\tau_1<T^g(\lambda_1)$ is chosen at τ_0, then

obviously $U(x, r, \tau_1) = -e^{-r+h^{(1)}(x,\tau_1)}$, with $h^{(1)}(x, t)$ being the unique classical solution that has nonnegative spatial derivative to the Hamilton-Jacobi equation (4.37) with deterministic coefficient $\lambda_1 \in \mathcal{F}_0$. The forward optimal inventory process X^* is given by (4.24), and we assume $X^*_{\tau_1} > 0$ to make the subsequent continuation still interesting (otherwise, the forward optimal liquidation stops at the deterministic time $T^*(X_0) \leq T^g(\lambda_1)$ by (4.25)).

Now assume the conclusion of the proposition is true for $k = 1, 2, \ldots, n$, with $n \geq 1$, i.e., assume we have determined $\tau_n \in \mathcal{F}_{\tau_{n-1}}$ and obtained the forward performance criterion

$$U(x, r, \tau_n) = -e^{-r+h^{(n)}(x,\tau_n)}$$

at τ_n, and $X^*_{\tau_n} > 0$. Then at τ_n, according to the definition of the predictable forward performance process in [3], the goal is to seek a predictable time $\tau_{n+1} \in \mathcal{F}_{\tau_n}$ and a predictable utility function $U(x, r, \tau_{n+1}) \in \mathcal{F}_{\tau_n}$, for $(x, r) \in \mathbb{R}^+ \times \mathbb{R}$, such that

$$U\Big(X^*_{\tau_n}, R^*_{\tau_n}, \tau_n\Big) = \operatorname*{esssup}_{\xi} \mathbb{E}\Big[U\Big(X^{\xi}_{\tau_{n+1}}, R^{\xi}_{\tau_{n+1}}, \tau_{n+1}\Big)\Big|\mathcal{F}_{\tau_n}\Big], \text{ a.s..} \tag{4.38}$$

where $R^*_{\tau_n} \in \mathbb{R}$, $X^*_{\tau_n} > 0$ are the optimal revenue and optimal inventory at time τ_n, respectively, due to the previous forward optimal trading strategies. Suggested by the scaling property of the criterion $U(x, r, \tau_n)$, we look for $U(x, r, \tau_{n+1})$ with a similar separable form and rewrite the above definition as

$$\begin{aligned} &-e^{-R^*_{\tau_n}+h^{(n)}(X^*_{\tau_n},\tau_n)} \\ &= \operatorname*{esssup}_{\xi} \mathbb{E}\Big[-e^{-R^{\xi}_{\tau_{n+1}}+h^{(n+1)}\left(X^{\xi}_{\tau_{n+1}},\tau_{n+1};\omega\right)}\Big|\mathcal{F}_{\tau_n}\Big], \text{ a.s.} \end{aligned}$$

with some $\mathcal{F}_{\tau_n}$-measurable function $h^{(n+1)}(x,t;\omega)$. By martingality and Itô's lemma, conditional on $\mathcal{F}_{\tau_n}$, the random function $h^{(n+1)}(x,t;\omega)$ solves the following Hamilton-Jacobi equation almost surely

$$\begin{cases} h_t^{(n+1)} - \frac{1}{4\lambda_{n+1}}(h_x^{(n+1)})^2 + \frac{1}{2}x^2 = 0, \quad x > 0,\ \tau_n < t < \tau_{n+1}, \\ h^{(n+1)}(x,\tau_n) = h^{(n)}(x,\tau_n), \quad x > 0. \end{cases} \tag{4.39}$$

By Proposition 4.3.4, the initial condition $h^{(n)}(\cdot,\tau_n)$ satisfies the desired property to be an admissible initial condition and, hence, a repeated application of Theorem 4.3.1 is possible, conditional on $\mathcal{F}_{\tau_n}$. That is, for almost every $\omega \in \Omega$, a well defined unique solution $h^{(n+1)}(x,t;\omega)$ exists for $(x,t) \in \mathbb{R}^+ \times [\tau_n, \tau_{n+1}]$, provided $\tau_n < \tau_{n+1} < \tau_n + T^g(\lambda_1,\ldots,\lambda_{n+1})$. The solvability horizon $T^g(\lambda_1,\ldots,\lambda_{n+1})$ is determined by the bounds on the first and second order derivatives of the admissible initial condition $h^{(n)}(\cdot,\tau_n)$, following Theorem 4.3.1. Moreover, conditional on $\mathcal{F}_{\tau_n}$, the verification argument in Theorem 4.3.2 shows the optimality condition (4.38) for any sub-horizon $[\tau_n,\tau_{n+1}]$, such that $\tau_{n+1} \in \mathcal{F}_{\tau_n}$ and $\tau_n < \tau_{n+1} < \tau_n + T^g(\lambda_1,\cdots,\lambda_{n+1})$. □

4.3.3 Continuous time problem

Theorem 4.3.5 presents the result for general multi-period forward optimal liquidation problem along with discrete time model revision. The update of the criterion and the update of the model parameter λ both take place at the same frequency in discrete time. A natural question to ask is what the limit would be as the updating frequency goes to infinity. It is easier to see that, as model revision is conducted more and more often, we could observe a path of the realized market parameter process $(\lambda_t)_{t\geq 0}$, instead of a sequence of realized values of the random variables $\{\lambda_n\}_{n\geq 1}$. The limit of the forward performance process (4.36) in discrete time, however, is less clear, since we need first to show the limit indeed exists before we can identify it with any known process.

In the previous chapter, we have studied this problem in detail under quadratic initial condition $g(x) = kx^2$, $k > 0$, and identified the limit as the continuous time zero volatility forward performance process in the optimal liquidation context. The continuous time forward theory has been developed since the initiation of the study on forward performance processes, with the zero volatility case extensively analyzed in the work [45], among others. This family of forward performance processes is more convenient to tackle, compared to the general non-zero volatility forward processes, although it still has the challenging ill-posedness issue for the the associated Hamilton-Jacobi-Bellman equations. In the previous chapter, we have shown that in addition to its tractability, the zero volatility forward process is the the limit of a sequence of well defined forward performance process in discrete time. Such result brings new insight into the zero volatility forward performance process family, beyond its sound mathematical properties.

In this section, we present a more general convergence argument under initial condition $g(\cdot)$ that is not necessarily quadratic. The success of tje similar argument in the previous chapter is partly because we have the explicit solution for the discrete time forward performance process under quadratic initial condition. For general admissible $g(\cdot)$, such explicit representation is no longer available. Nevertheless, based on the semi-explicit expression (4.13) and the characteristic curve equation (4.14), we can obtain the limit that still coincides with the zero volatility forward performance process under general admissible $g(\cdot)$. For complete discussion, we recall from the previous chapter that the zero volatility forward process in the optimal liquidation context satisfies the equation

$$dU(x,r,t) = -\Big(\frac{U_x(x,r,t)^2}{4\lambda_t U_r(x,r,t)} + \frac{1}{2}U_{rr}(x,r,t)\sigma_t^2 x^2\Big)dt, \tag{4.40}$$

with initial condition $U(x,r,0) = -e^{-r+g(x)}$, for admissible $g(\cdot)$. Here, the market parameter processes λ_t and σ_t are assumed to be general progressively measurable stochastic processes. In some very special cases, including constant λ and σ and the coordinated variation scenario considered in the previous chapter, the equation (4.40) has an explicit solution under quadratic initial condition $g(\cdot)$, and an existence and uniqueness result under other $g(\cdot)$ that satisfies the assumption in Theorem 4.3.1. For more general parameter processes, the existence and uniqueness of solution to (4.40) is not clear. Hence, we only present the heuristic argument and consider the solutions of the separable form $U(x,r,t) = -e^{-r+h(x,t)}$. Direct computation yields the Hamilton-Jacobi equation with random coefficient (taking $\sigma_t = 1$, $t \geq 0$ for simplicity)

$$h_t(x,t) - \frac{1}{4\lambda_t}h_x^2(x,t) + \frac{1}{2}x^2 = 0, \text{ a.s.}$$

with initial condition $h(x,0) = g(x)$.

We now turn to the discrete time forward performance process defined as follows. For each $\omega \in \Omega$, and each admissible initial condition $g(\cdot)$, define the model revision times $\{\tau_n^N\}_{n\geq 1}$ and the the functions $\{h^{(n,N)}\}_{n\geq 1}$ for every integer $N \geq 1$ as

- $\tau_n^N = \tau_{n-1}^N + \frac{T^g(\lambda_1,\cdots,\lambda_n)\wedge 1}{N+1}$, for all $n \geq 1$, with $T^g(\lambda_1, \cdots, \lambda_n)$ given by (4.35); set also $\tau_0^N = 0$ for all N;
- $h^{(n,N)}(x,t;\omega)$ is the unique classical solution with nonnegative spatial derivative to the Hamilton-Jacobi equation

$$h_t^{(n,N)} - \frac{1}{4\lambda_n(\omega)}(h_x^{(n,N)})^2 + \frac{1}{2}x^2 = 0, \quad x > 0,\ \tau_{n-1}^N < t < \tau_n^N, \qquad (4.41)$$

  with initial condition $h^{(n,N)}(x, \tau_{n-1}^N) = h^{(n-1,N)}(x, \tau_{n-1}^N)$, for $n \geq 1$, and $h^{(0,N)}(x, 0) = g(x)$ for all N.

Notice that the above recursive construction is well defined for all integer $n \geq 1$ and $N \geq 1$, due to the self-similarity property of the solution to the Hamilton-Jacobi equation under admissible $g(\cdot)$ (cf. Proposition 4.3.4). The random variable λ_n, $n \geq 1$, for the n-th period is the price impact parameter given by $\lambda_n = \lambda_{\tau_{n-1}^N} \in \mathcal{F}_{\tau_{n-1}^N}$ and, hence, the realization $\lambda_n(\omega)$ is known at the beginning of each interval $[\tau_{n-1}^N, \tau_n^N]$. Finally, for each $N \geq 1$, we denote by h^N the continuous mapping from $\mathbb{R}^+ \times [0, T^N)$ to $\mathbb{R}$, obtained from the concatenation of the functions $\{h^{(n,N)}\}_{n\geq 0}$ across each τ_n, $n \geq 1$. Here, $T^N := \lim_{n\to\infty} \tau_n^N$ is clearly well defined for every $N \geq 1$. We then have the following convergence result.

Theorem 4.3.6. *Assume that the function g satisfies the assumption in Theorem 4.3.1, and λ_t, $t \geq 0$, is continuous with $\inf_{t\geq 0} \lambda_t > 0$, a.s. Then, for any subsequence of $\{h^N\}_{N\geq 1}$, there exist a convergent subsequence $\{\tilde{h}^N\}_{N\geq 1}$, a $T^* > 0$, and a continuous function $\tilde{h} : \mathbb{R}^+ \times [0, T^*) \mapsto \mathbb{R}$, such that for any compact subset $D \subset \mathbb{R}^+$ and any $0 < T < T^*$,*

$$\lim_{N\to\infty} \max_{(x,t)\in D\times[0,T]} \left|\tilde{h}^N(x,t) - \tilde{h}(x,t)\right| = 0, \ \ a.s.. \tag{4.42}$$

Furthermore, if for any convergent subsequence $\{\tilde{h}^N\}_{N\geq 1}$, it holds that $\tilde{h}^N_x \to \tilde{h}_x$ and $\tilde{h}^N_t \to \tilde{h}_t$ uniformly on $D \times [0, T]$, as $N \to \infty$. Then, convergence in (4.42) also holds for the original sequence $\{h^N\}_{N\geq 1}$, and $\tilde{h}$ and T^* are determined by

$$\tilde{h}_t(x,t) - \frac{1}{4\lambda_t}\tilde{h}^2_x(x,t) + \frac{1}{2}x^2 = 0, \ \ a.s. \tag{4.43}$$

with initial condition $\tilde{h}(x, 0) = g(x)$, and

$$T^* = \sup\{t > 0 : \text{ The Hamilton-jacobi equation (4.43) has a unique}$$

$$\text{classical solution with nonnegative spatial derivative for } s \in [0,t]\}.$$

Proof. We provide the proof for each fixed $\omega \in \Omega$ that does not belong to the null set. First, let $C_1 > |g(x)|$ and $C_1 > |g'(x)|$ for all $x \in D \subset \mathbb{R}^+$. Denote

$$\tau^N(\omega) := \inf_{n\geq 1}\{\left|h^{(n,N)}(x,\tau^N_n)\right| > C_1 \text{ or } \left|h^{(n,N)}_x(x,\tau^N_n)\right| > C_1 \text{ for some } x \in D\},$$

with the convention $\inf \emptyset = \infty$, and the truncated sequence

$$\hat{h}^{(n,N)}(x,\tau^N_n) := h^{(n,N)}(x,\tau^N_{n\wedge\tau^N})$$

for all $N \geq 1$. Now for $1 \leq n \leq \tau^N$, we have that for any $x, y \in D$,

$$\hat{h}^{(n,N)}(x, \tau_n^N) - \hat{h}^{(n-1,N)}(y, \tau_{n-1}^N) = \hat{h}^{(n,N)}(x, \tau_n^N) - \hat{h}^{(n,N)}(y, \tau_{n-1}^N)$$

$$= \hat{h}_t^{(n,N)} \cdot (\tau_n^N - \tau_{n-1}^N) + \hat{h}_x^{(n,N)} \cdot (x - y),$$

with the first equality following from the multi-period concatenation (cf. Theorem 4.3.5), and the second one due to the Mean Value Theorem. Since the function $h^{(n,N)}(x,t)$ satisfies the Hamilton-Jacobi equation (4.41), we have that for $1 \leq n \leq \tau^N$, the temporal derivative $\hat{h}_t^{(n,N)}$ is also uniformly bounded, due to the uniform boundedness of the spatial derivative $\hat{h}_x^{(n,N)}$ and the compactness of D. Hence, the family of continuous functions $\{\hat{h}^N\}_{N\geq 1}$ is uniformly bounded and equicontinuous. Denote $T^1(\omega) := \liminf_{N\to\infty} \tau^N(\omega)$ and, since $\{\hat{h}^N\}_{N\geq 1}$ is uniformly Lipschitz in (x,t), it is direct to see $T^1 > 0$. Finally, by the Arcelà-Ascoli Theorem, we can conclude, up to a subsequnce, $\hat{h}^N$ converges uniformly on $D \times [0,T]$ for any $0 < T < T^1$, as $N \to \infty$.

Now consider a converging subsequence over some compact domain $D \times [0,T]$ and denote its limit as $\tilde{h}$. For any $t \in (0,T]$, denote $j(N) = \max\{n \geq 1 : \tau_{n-1}^N < t\}$. Then clearly, as $N \to \infty$, $\lambda_{j(N)} \to \lambda_t$. Next, by (4.13), we have over the interval $[\tau_{j(N)-1}^N, \tau_{j(N)}^N]$ that,

$$\begin{aligned} \hat{h}^{(j(N),N)}(x, \tau_{j(N)}^N) &= \hat{h}^{(j(N),N)}(x_0, \tau_{j(N)-1}^N) \\ &- \frac{\sqrt{2\lambda_{j(N)}}}{2}\left(\frac{x_0^2}{2} + \frac{\hat{h}_x^{(j(N),N)^2}(x_0, \tau_{j(N)-1}^N)}{4\lambda_{j(N)}}\right)\sinh\left(\frac{2(\tau_{j(N)}^N - \tau_{j(N)-1}^N)}{\sqrt{2\lambda_{j(N)}}}\right) \\ &+ \frac{x_0 \hat{h}_x^{(j(N),N)}(x_0, \tau_{j(N)-1}^N)}{2}\left(\cosh\left(\frac{2(\tau_{j(N)}^N - \tau_{j(N)-1}^N)}{\sqrt{2\lambda_{j(N)}}}\right) - 1\right), \end{aligned} \tag{4.44}$$

with x_0 and x being connected through the characteristic equation (4.14), i.e.,

$$x_0 \cosh\left(\frac{(\tau^N_{j(N)} - \tau^N_{j(N)-1})}{\sqrt{2\lambda_{j(N)}}}\right) - \frac{\hat{h}^{(j(N),N)}_x(x_0, \tau^N_{j(N)-1})}{\sqrt{2\lambda_{j(N)}}} \sinh\left(\frac{(\tau^N_{j(N)} - \tau^N_{j(N)-1})}{\sqrt{2\lambda_{j(N)}}}\right) = x. \tag{4.45}$$

Notice that

$$\begin{aligned}
&\hat{h}^{(j(N),N)}(x, \tau^N_{j(N)}) - \hat{h}^{(j(N),N)}(x_0, \tau^N_{j(N)-1}) \\
&= \left[\hat{h}^{(j(N),N)}(x, \tau^N_{j(N)}) - \hat{h}^{(j(N),N)}(x_0, \tau^N_{j(N)})\right] \\
&+ \left[\hat{h}^{(j(N),N)}(x_0, \tau^N_{j(N)}) - \hat{h}^{(j(N),N)}(x_0, \tau^N_{j(N)-1})\right].
\end{aligned}$$

Dividing both sides of (4.44) by $\tau^N_{j(N)} - \tau^N_{j(N)-1}$ and letting $N \to \infty$, we obtain that at $t \in (0, T]$, the limit function $\tilde{h}$ satisfies, due to the uniform convergence of $\{\hat{h}^N\}_{N \geq 1}$, and the assumption $\tilde{h}^N_x \to \tilde{h}_x$ and $\tilde{h}^N_t \to \tilde{h}_t$ uniformly,

$$\lim_{N\to\infty} \left(\hat{h}^{(j(N),N)}_x \frac{x - x_0}{\tau^N_{j(N)} - \tau^N_{j(N)-1}}\right) + \tilde{h}_t(x,t) + \left(\frac{x^2}{2} + \frac{\tilde{h}^2_x(x,t)}{4\lambda_t}\right) = 0,$$

where we have used that $x_0 \to x$, as $\tau^N_{j(N)} - \tau^N_{j(N)-1} \to 0$. Moreover, we also have

$$\lim_{N\to\infty} \frac{x - x_0}{\tau^N_{j(N)} - \tau^N_{j(N)-1}} = -\frac{\tilde{h}_x(x,t)}{2\lambda_t},$$

after an application of the characteristic curve equation (4.45), and the fact that $x_0 \to x$, as $\tau^N_{j(N)} - \tau^N_{j(N)-1} \to 0$. Combining the above results, we conclude that, for any converging subsequence $\{\hat{h}\}_{N\geq 1}$, the limit is a continuously differentiable function $\tilde{h}$ that satisfies (4.43) with nonnegative spatial derivative. Moreover, if the Hamilton-Jacobi equation (4.43) has a unique solution, then the original family of functions $\{\hat{h}\}_{N\geq 1}$ (not just subsequence) converge uniformly on any compacts to the solution of (4.43).

More precisely, consider the above construction for an unbounded increasing sequence $C_m > C_1$ and define analogously the increasing times $\{T^m\}_{m\geq 1}$ and the limit $T^* = \lim_m T^m$. Then the above argument still holds for each pair (C_m, T^m), and the limit function $\tilde{h}$ has the property that $\tilde{h}(x, T^m) = C_m$ or $\tilde{h}_x(x, T^m) = C_m$, if T^m is finite. If equation (4.43) has a unique classical solution with nonnegative spatial derivative up to some time $\widehat{T} > T^*$, then it has to coincide with the limit function $\tilde{h}$ over every $[0, T^m]$, leading to $\tilde{h}(x, T^m)$ or $\tilde{h}_x(x, T^m)$ exceeding C^m. This is a contradiction since $C^m \to \infty$ as m increases, while a classical solution obviously has uniformly bounded function values and spatial derivatives on the compact domain $D \times [0, T^*]$. It is also obvious that $\widehat{T} < T^*$ cannot happen, due to the assumption that a unique classical solution with nonnegative spatial derivative exists up to $\widehat{T}$, and the fact that the limit function $\tilde{h}$ is such a classical solution to (4.43) up to T^*. Hence, we conclude $T^* = \widehat{T}$. □

Chapter 5

Relative forward indifference valuation of real-time incoming projects

5.1 Introduction

Options pricing/projects evaluation, as one of the core areas of mathematical finance, is well understood in complete market. When the market is incomplete, there is no unique arbitrage free price as it is no longer possible to fully eliminate the risk through replication. One approach to price options in incomplete market, including real options, is to resort to expected utility maximization methodology, which is commonly known as the utility indifference valuation approach (see, e.g., [10], [50], [30]). The investor would accept a price today such that she is indifferent to proceed optimally under the current investment opportunity with and without a liability at the terminal time $t = T$. This price is known as indifferent price of the option.

The classical backward indifference valuation methodology can apply to the evaluation of a single real option, a portfolio of options, or a single option relative to an existing portfolio of options, the latter of which is known as relative indifference valuation (see, e.g., [5], [56] and [57]). Nonetheless, as other utility-based optimal control problems, the backward reasoning is subject to substantial commitment at $t = 0$, which restricts the class of projects that can be priced. One of such restrictions is that a model that describes the underlying market for employing (partial) hedging strategies needs to be specified to a full extent at $t = 0$, and fixed thereafter. In contrast, information in reality unfolds along real-time, as the underlying market may experience unanticipated favorable or unfavorable conditions for hedging purpose after $t = 0$. Moreover, for real options pricing or projects evaluation, another commitment inherent to the classical approach arises, which we refer to as *projects commitment*. Since the classical optimization/valuation approach solves the problem backwards in time, the investor has to know at $t = 0$ the complete profile of all the incoming projects with their characteristics (e.g., initiation, expiry and payoff functional, etc), and no new projects can be included once the valuation and hedging procedures start at $t = 0$ in order to maintain time-consistency and exclude pricing discrepancy. Again, this may not be realistic; instead, project investors in practice decide on the risk exposure of a new project typically based on the performance of existing ones rather than to make an inflexible overall evaluation ahead of time. For instance, a drug company may decide on the risk of developing another new drug only at its initiation, based on the progress of the concurrent R&D of the drugs already under development, or on the market conditions at that future time. All of such information would be generally hard to know or model at the initial time

$t = 0$.

In the current work, we study the indifference valuation of real options in real-time within the forward performance process framework. The real-time feature is in direct contrast with both the model commitment and the projects commitment due to the backward reasoning under the classical stochastic optimization methodology. The forward approach, on the other hand, can address the unanticipated real-time changes in both the market investment opportunity and the projects profile. In particular, we first consider the evaluation of a single real option, but with unanticipated model change before the expiry of the option. Under classical framework, such real-time model change could lead to pricing discrepancy under the fixed terminal evaluation criterion, as well as time-inconsistency for the underlying stochastic control problems. We develop the forward indifference valuation scheme to overcome the model commitment issue of the classical approach, and demonstrate that both pricing discrepancy and time-inconsistency would not occur if the evaluation criterion is adaptive enough to capture the unanticipated model switch along real-time.

We then examine the relative indifference valuation problem of two real options by adopting the forward approach. To demonstrate the flexibility of the forward approach and the absence of the projects commitment, we work under less restrictive assumption; that is, at $t = 0$, we don't assume any knowledge of the full characteristics of the second option, except its initiation time. The expiry and the payoff functional of the second option are only observable at the initiation time of this option. In other words, different from what is typically assumed under the classical backward approach, the investor only prices the first option at $t = 0$, without knowing the full profile of the second option. Such relaxed assumption accounts for real world problems where multiple phases of a long-term project (e.g., R&D project, drug development, oil exploration, etc.) can be regarded as separate real options, while the risk exposures of the options in the remote future are typically difficult to be accurately modeled/predicted at $t = 0$.

Due to projects commitment discussed earlier, time-inconsistency would arise if the terminal valuation criterion is not revised after the arrival of the second option. We hence adopt the forward performance approach to seek consistent terminal criterion under which the original valuation of the first option would remain valid even after the arrival of the second option, i.e., to exclude intertemporal pricing discrepancy due to unanticipated incoming new options. The revised criterion at $t = T$ would typically depend on the characteristics of the second option. In this work, we consider two families of forward performance processes, the predictable family and the adaptive family, and also compute their respective relative indifference prices of the first option given the risk exposure of the second. It is interesting to notice that although the two types of forward criteria have different measurability, they give rise to the same relative indifference price that is consistent with the initially settled price for the first option. Such robustness of relative indifference valuation together with the greater flexibility to incorporate unanticipated model/projects profile changes along real-time make the forward performance process approach more appealing in real world applications.

We then turn to the valuation of the second option once it is introduced with its full profile available at the initiation time. In the classical framework where the full profile of both options is available at $t = 0$, the relative indifference valuation of the second option given the first is essentially the same as that of the first option given the second. However, in the current asynchronous information arrival setting, the relative indifference valuation of the new option requires both model extension and criterion extension, beyond the expiry of the first option. We extend the (relative) valuation criterion following the forward performance process theory and discuss the additivity property of the resulting relative indifference prices, the residual optimal wealth processes and the residual risk processes.

5.2 Single real option with model revision

In this section, we consider the indifference valuation of a single project/real option with model revision. As we have mentioned, since the real-time model revision is not anticipated at $t = 0$, following the classical backward indifference valuation methodology would result in time-inconsistency and pricing discrepancy (see also the discussion in [44]). We therefore consider indifference valuation under the forward performance approach, aiming at achieving intertemporal consistency along with real-time model revision. Here, we consider the two-period model revision extension of the dynamic market environment proposed in [43]. Precisely, the investment universe consists of a riskless asset and two risky assets. We assume for simplicity that the riskless asset is given by a zero interest Bond $B_t = 1$, for all $t \geq 0$. The first risky asset is a stock that can be traded, whose price follows the log-normal diffusion prior to the model revision time $t = \tau_1$

$$dS_s = \mu_1 S_s ds + \sigma_1 S_s dW_s^1,$$

with $S_t = s > 0$, and $0 \leq t \leq s \leq \tau_1$. The second asset is a nontraded asset whose value is modeled by the diffusion process

$$dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s,$$

with $Y_t = y \in \mathbb{R}$, and $0 \leq t \leq s \leq T$. The two Brownian motions W^1, W are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the filtration $\mathcal{F}_t$, $0 \leq t \leq T$, generated by (W^1, W) and satisfying the usual conditions. We suppose the correlation between W^1, W is $\rho \in (-1, 1)$, and that the deterministic functions $b(\cdot, \cdot)$, $a(\cdot, \cdot)$ are such that the stochastic differential equation for Y has a unique strong solution.

The model revision time $0 < \tau_1 < T$ is a known deterministic time at which the investor would re-estimate the model parameters μ_1 and $\sigma_1 > 0$, probably due to scheduled market information release or self-planned model reassessment procedure. During $(\tau_1, T]$, the investor will change her view on the market condition and hold onto $\mu_2, \sigma_2 \in \mathcal{F}_{\tau_1}$, with $\sigma_2 > 0$, a.s., under $\mathbb{P}$. We denote the respective Sharpe ratios as $\lambda_1 := \frac{\mu_1}{\sigma_1} \in \mathcal{F}_0$ and $\lambda_2 := \frac{\mu_2}{\sigma_2} \in \mathcal{F}_{\tau_1}$.

A European type real option is initiated at $t = 0$ and expires at $t = T$, with payoff being $G(Y_T)$ for a bounded function $G(\cdot)$.

As in the classical framework, the indifference valuation mechanism involves two competing investors, with one being only investing optimally in the stock market, and the other being holding the real option and proceeding optimally in the market. We refer to the first investor as the plain investor and the second the writer of the option (in the case she pays $G(Y_T)$ at expiry). The plain investor maintains the following forward performance process by, on one hand, optimally investing in the stock market, and on the other, taking into account the unanticipated model change in real-time,

$$U(x,t) = -e^{-\gamma x + \frac{1}{2}\int_0^t \lambda_s^2 ds} \tag{5.1}$$

with $\lambda_s = \lambda_1$ for $0 \le s \le \tau_1$ and $\lambda_s = \lambda_2$ for $\tau_1 < s \le T$.

The writer, also starting with initial utility $U_0(x) = -e^{-\gamma x}$, maintains optimality in the stock market but with an extra liability $G(Y_T)$ at the terminal time[1]. During the first period $[0, \tau_1]$, she has the same correct view about the market as the plain investor, i.e., the Sharpe ratio is λ_1. To proceed optimally from $U_0(x) = -e^{-\gamma x}$, she aims to find a consistent indirect utility $V^{W,\lambda_1}(x, \tau_1)$

[1]Here, both the plain investor and the writer start with an initial utility $U_0(x) = -e^{-\gamma x}$, $\gamma > 0$, instead of a terminal utility. However, we mention that both of them can choose the original terminal utility $U_T(x) = -e^{-\gamma x}$ and start with $U_0(x) = V(x, 0; \lambda_1)$, the value function at $t = 0$ with the best estimated model parameter λ_1 up to the initiation time of the option. The rest of the argument would still follow.

such that

$$- e^{-\gamma(x-h_0(y))} = \sup \mathbb{E}\Big[V^{W,\lambda_1}(X_{\tau_1}, \tau_1)|X_0 = x, Y_0 = y\Big], \tag{5.2}$$

where the expectation is taken under the $[0, \tau_1]$ marginal of the true underlying measure $\mathbb{P}$, and we note that such marginal is known to the writer at $t = 0$. The process X denotes the wealth process over the first period $[0, \tau_1]$, and is given by

$$dX_s = \mu\pi_s ds + \sigma\pi_s dW_s^1,$$

with $X_t = x \in \mathbb{R}$, and $0 \leq t \leq s \leq \tau_1$. The quantity $h_0(y)$ is the indifference price of the real option at $t = 0$. In general, the plain investor and the writer can agree on an initial price $h_0(y) \geq 0$ and then both proceed optimally forward in time starting from a common initial utility $U_0(x) = -e^{-\gamma x}$. A more reasonable choice for $h_0(y)$ is the classical indifference price at $t = 0$ when both investors view the market over the whole horizon $[0, T]$ with $\lambda_1 \in \mathcal{F}_0$ and take the common terminal utility $U_T(x) = -e^{-\gamma x}$. Then after $t = \tau_1$, both investors take into account the new realized market condition $\lambda_2 \in \mathcal{F}_{\tau_1}$ and seek to determine their respective revised terminal utility that is consistent with their individual optimal investment behavior during $[0, \tau_1]$. We follow this reasoning and choose

$$h_0(y) = \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_T)}|Y_0 = y\Big], \tag{5.3}$$

whit $\widehat{\mathbb{Q}}$ being the minimal relative entropy martingale measure with respect to the hypothetical measure $\widehat{\mathbb{P}}$ that models the market with λ_1 being applied for the whole horizon (see Theorem 2 in [43]). The remaining step is to find $V^{W,\lambda_1}(x,\tau_1)$ such that (5.2) is true. Notice that the procedure involves solving an inverse problem, as in equation (5.2), the initial value function is given while the goal is to find the (indirect) utility function $V^{W,\lambda_1}(x,\tau_1)$. The induced Hamilton-Jacobi-Bellman (HJB) equation is therefore ill-posed and existence and uniqueness of solution is typically lacking. To find one solution $V^{W,\lambda_1}(x,\tau_1)$ in (5.2), we consider the distortion transformation as in [43]

$$V^{W,\lambda_1}(x,t) = -e^{-\gamma x} v(Y_t,t)^{\frac{1}{1-\rho^2}},$$

for all $0 \le t \le \tau_1$. Direct computation then yields that the deterministic function $v(y,t)$ solves the ill-posed linear parabolic partial differential equation (PDE)

$$v_t + \frac{1}{2}a^2(y,t)v_{yy} + [b(y,t) - \rho\lambda_1 a(y,t)]v_y = \frac{1}{2}(1-\rho^2)\lambda_1^2 v, \tag{5.4}$$

for $0 \le t \le \tau_1$, with initial condition $v(y,0) = e^{r(1-\rho^2)h_0(y)}$. Although the ill-posedness in general leads to extra difficulty in terms of obtaining existence and uniqueness results, we can actually determine one positive classical solution to (5.4), under the proper choice (5.3). Indeed, consider the hypothetical problem

$$\widehat{V}(x,y,t) := \sup \mathbb{E}_{\widehat{\mathbb{P}}}\Big[U^{W,\lambda_1}(X_T - G(Y_T),T)|X_t = x, Y_t = y\Big],$$

for $0 \le t \le T$, where the conditional expectation is taken under the hypothetical measure $\widehat{\mathbb{P}}$ over $[0,T]$, and $U^{W,\lambda_1}(x,T) = -e^{-\gamma x + \frac{T}{2}\lambda_1^2}$. Then by the same distortion transformation

$$\widehat{V}(x,y,t) = -e^{-\gamma x}\widehat{v}(y,t)^{\frac{1}{1-\rho^2}},$$

for $0 \leq t \leq T$, one can deduce a well-posed problem for function $\widehat{v}(y,t)$, i.e., $\widehat{v}(y,t)$ solves the linear PDE (5.4) for $0 \leq t \leq T$, with terminal condition $\widehat{v}(y,T) = e^{(1-\rho^2)(\gamma G(y)+\frac{1}{2}\lambda_1^2 T)}$. Classical rigorous result regarding this well-posed problem the applies (see [60]) and a unique positive solution is obtained by Feynman-Kac representation

$$\widehat{v}(y,t) = \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)\big(G(Y_T)+\frac{1}{2}\lambda_1^2 t\big)}|Y_t = y\Big]. \tag{5.5}$$

It is then easy to see that $\widehat{v}(y,0) = e^{r(1-\rho^2)h_0(y)} = v(y,0)$, for all $y \in \mathbb{R}$. We therefore can take $V^{W,\lambda_1}(x,\tau_1) = -e^{-\gamma x}\widehat{v}(Y_{\tau_1},\tau_1)$. Also, it follows that the indifference price over the first period $[0,\tau_1]$ under $V^{W,\lambda_1}(x,\tau_1)$ is

$$h_t(y) = \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\widehat{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_T)}|Y_t = y\Big]. \tag{5.6}$$

As we can see, the indifference price under the forward performance process approach before the market model revision coincides with its classical counterpart, as a result of the particular choice of $h_0(y)$ in (5.3).

At the reassessment time $\tau_1 \in \mathcal{F}_0$, both investors change their views on the market and realize the risk-premium has changed to $\lambda_2 \in \mathcal{F}_{\tau_1}$. Up to $t = \tau_1$, the plain investor has preserved her performance up to $U(x,\tau_1) = -e^{-\gamma x+\frac{\lambda_1^2}{2}\tau_1}$, while the writer, holding the real option and proceeding optimally, has achieved $V^{W,\lambda_1}(x,\tau_1)$. The goal for both investors is to choose their respective terminal utilities, taking into account that $\lambda_2 \in \mathcal{F}_{\tau_1}$, to be consistent with their individual optimality they have preserved so far. As we already know, the consistent terminal utility for the plain investor is given by (5.1) $U(x,T) = -e^{-\gamma x + \frac{\lambda_1^2}{2}\tau_1 + \frac{\lambda_2^2}{2}(T-\tau_1)}$. The remaining step is to determine $U^{W,\lambda_2}(x,T)$, such that

$$V^{W,\lambda_1}(x,\tau_1) = \operatorname{esssup} \mathbb{E}\Big[U^{W,\lambda_2}(X_T - G(Y_T), T)\Big|\, \mathcal{F}_{\tau_1}, X_{\tau_1} = x\Big], \text{ a.s..} \tag{5.7}$$

Notice that the conditional expectation is taken under the $(\tau_1, T]$ marginal of the underlying physical measure $\mathbb{P}$, conditional on $\mathcal{F}_{\tau_1}$, which is known to the writer when she solves the problem (5.7) at $t = \tau_1$. To find one solution $U^{W,\lambda_2}(x,T)$ to the inverse problem (5.7), similar as before, we define the value function for the remaining time period $(\tau_1, T]$ as

$$V(x,y,t;\omega) = \operatorname{esssup} \mathbb{E}\Big[U^{W,\lambda_2}(X_T - G(Y_T), T)\Big|\, \mathcal{F}_{\tau_1},\ X_t = x, Y_t = y\Big], \text{a.s..} \tag{5.8}$$

By the same distortion transformation $V(x,y,t;\omega) = -e^{-\gamma x}\tilde{v}(y,t;\omega)^{\frac{1}{1-\rho^2}}$, we obtain that $\tilde{v}(y,t;\omega)$ solves almost surely the ill-posed linear parabolic PDE

$$\tilde{v} + \frac{1}{2}a^2(y,t)\tilde{v}_{yy} + (b(y,t) - \rho\lambda_2 a(y,t))\,\tilde{v}_y = \frac{1}{2}(1-\rho^2)\lambda_2^2\tilde{v} \tag{5.9}$$

with initial condition $\tilde{v}(y,\tau_1) = \mathbb{E}_{\widehat{\mathbb{Q}}}[e^{r(1-\rho^2)G(Y_T) + \frac{1}{2}(1-\rho^2)\lambda_1^2\tau_1}|Y_{\tau_1} = y]$. The initial condition follows from that at $t = \tau_1$, $V^{W,\lambda_1}(x,\tau_1) = V(x, Y_{\tau_1}, \tau_1)$, a.s., according to requirement for consistency (5.7). Again, it is not clear whether a positive solution exists for the ill-posed equation (5.9). However, following the same argument as before, we study a well-posed problem which produces the initial condition $\tilde{v}(y,\tau_1)$. To be specific, suppose $U^{W,\lambda_2}(x,T) = -e^{-\gamma x + F(\tau_1,T;\omega)}$ with $F(\tau_1,T;\omega) \in \mathcal{F}_{\tau_1}$. Then $\tilde{v}(y,t)$ solves the random linear PDE (5.9) over

$(\tau_1, T]$, with a terminal condition $\tilde{v}(y,T) = e^{(1-\rho^2)(\gamma G(y)+F(\tau_1,T))} \in \mathcal{F}_{\tau_1}$. Conditional on $\mathcal{F}_{\tau_1}$, by the Feynman-Kac representation, we derive for $\tau_1 < t \leq T$,

$$\tilde{v}(y,t) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{(1-\rho^2)(\gamma G(Y_T)+F(\tau_1,T))-\frac{1}{2}(1-\rho^2)\lambda_2^2(T-t)}|Y_t = y\Big],$$

where conditional on $\mathcal{F}_{\tau_1}$, the measure $\widetilde{\mathbb{Q}}$ on $\mathcal{F}_T$ is defined by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = e^{-\lambda_2(W_T^1 - W_{\tau_1}^1)-\frac{1}{2}\lambda_2^2(T-\tau_1)} \tag{5.10}$$

and $\widetilde{W}_s := W_s - W_{\tau_1} + \lambda_2\rho(s-\tau_1)$, $\tau_1 \leq s \leq T$, is a standard Brownian motion under $\widetilde{\mathbb{Q}}$. Finally, the consistency condition (5.7) leads to $V^{W,\lambda_1}(x,\tau_1) = V(x, Y_{\tau_1}, \tau_1)$, which implies $\tilde{v}(y,\tau_1) = v(y,\tau_1)$. Under the predictable assumption that $F(\tau_1,T) \in \mathcal{F}_{\tau_1}$, we derive that

$$F(\tau_1,T) = \frac{1}{2}\Big(\lambda_1^2\tau_1 + \lambda_2^2(T-\tau_1)\Big) + \frac{1}{1-\rho^2}\ln\frac{\mathbb{E}_{\widehat{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_T)}|Y_{\tau_1}]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_T)}|Y_{\tau_1}]} \in \mathcal{F}_{\tau_1}.$$

Hence, the consistent terminal utility for the writer would be

$$\begin{aligned} U^{W,\lambda_2}(x,T) = -\exp\Big(&-\gamma x + \frac{1}{2}(\lambda_1^2\tau_1 + \lambda_2^2(T-\tau_1)) \\ &+ \frac{1}{1-\rho^2}\ln\frac{\mathbb{E}_{\widehat{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_T)}|Y_{\tau_1}]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_T)}|Y_{\tau_1}]}\Big) \in \mathcal{F}_{\tau_1}. \end{aligned} \tag{5.11}$$

The indifference price of the project during $(\tau_1, T]$ therefore follows from the equilibrium between two investors

$$U(x - h_t(y;\omega), t;\omega) = V(x,y,t;\omega), \quad a.s.,$$

from which we derive

$$h_t(y;\omega) = \frac{1}{\gamma(1-\rho^2)}\left(\ln\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_T)}|Y_t = y\Big] + \ln\frac{\mathbb{E}_{\widehat{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_T)}|Y_{\tau_1}\Big]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_T)}|Y_{\tau_1}\Big]}\right), \tag{5.12}$$

for $\tau_1 < t \leq T$. Compared with $U(x,T)$, the terminal utility of the plain investor, it is clear that $U^{W,\lambda_2}(x,T)$ has an additional correction term which is $\mathcal{F}_{\tau_1}$-measurable. This is different form [44] where the forward terminal utility $U(x,T)$ is used for both the plain investor and the writer. Therein, under the forward criterion $U(x,T)$, the indifference valuation problem is solved backwards. In particular, to solve for the first period $[0,\tau_1]$, one needs to know/commit to the dynamics of market parameters during the second period $(\tau_1, T]$ (as shown in the HJB of Proposition 12 in [44]). However, in the scenario of real-time model revision as we modeled here, the investor would not have been able to know the necessary information of $\lambda_2 \in \mathcal{F}_{\tau_1}$ in order to settle the indifference price as well as the partial hedging strategy during the first period. The forward performance process approach when applied in real-time therefore can allow for more flexibility in terms of model pre-specification, by solving the valuation problem period by period forward in time.

Remark 5.2.1. A simple scenario is when $\lambda_2 = \lambda_1$, a.s., i.e., no model revision is necessary. Then the forward indifference valuation problem should collapse to the classical one proposed in [43]. Indeed, one can easily check that both plain investor and the writer's utilities reduce to $U(x,T) = -e^{-\gamma x + \frac{\lambda_1^2}{2}T}$ that corresponds to $U_0(x) = -e^{-\gamma x}$ (they will reduce to $U(x,T) = -e^{-\gamma x}$ if the initial utility $U_0(x) = V(x,0;\lambda_1)$). Moreover, the indifference price (5.12) for the second period also reduces to its classical counterpart in [43].

5.3 Forward evaluation of a flow of real options

The classical backward indifference valuation methodology is subject to both model commitment and projects commitment, due to the backward reasoning used to solve the associated underlying stochastic optimization problems. In previous section, we have demonstrated how the forward performance process approach can overcome the model commitment issue, in a real-time model revision scenario where a single real option is evaluated. We now continue to show that the forward approach can also handle the projects commitment issue, and develop the relative forward indifference valuation scheme for a flow of real options that arrive with asynchronous information of their risk profiles.

5.3.1 Relative forward indifference valuation of the first option

To expose the main idea, we work with two projects in a market with *a priori* known probabilistic dynamics (i.e., no model revision is needed, for simplicity) over a fixed horizon $[0, T]$. The difference between our scenario and classical multiple projects (relative) indifference valuation is that, we do not assume full knowledge of the second project at $t = 0$, deterministically or probabilistically; in particular, the expiry and payoff functional of the second project only reveal at the its initiation time $0 < \tau_1 < T$ which we assume to be known at $t = 0$. This model setup can describe more general scenarios in practice; for instance, a drug company may know when to start developing the second drug, but it is not clear today how long the process would take and how the risk of the development would be for the second drug, as these characteristics may mostly depend on other factors, including the R&D outcome of the first drug, and they are more possible to be known when the second drug development officially starts.

Formally, we extend the one project valuation model in [43] in the following way. Let the horizon $[0, T]$ be given, and assume the first project is initiated at $t = 0$ with expiry at $t = T$. It pays $H(Y_T)$ at expiry with $H(\cdot) \in \mathcal{F}_0$. A second project arrives at $\tau_1 \in \mathcal{F}_0$ and expires at $\tau_2 \in \mathcal{F}_{\tau_1}$ with $0 < \tau_1 < \tau_2 \leq T$. It pays $G(Y_{\tau_2})$ with payoff functional $G(\cdot) \in \mathcal{F}_{\tau_1}$. The investors involved (i.e., both the plain investor and the writer) are assumed to take exponential utility $U_T(x) = -e^{-\gamma x}$ at time $t = 0$, but they are allowed to update this terminal criterion (and intermediate criteria) based on the arrival of the second project. Hence, this $\mathcal{F}_0$-measurable terminal utility $U_T(x) = -e^{-\gamma x}$ would only be used for the valuation of the first project before $t = \tau_1$; after this time, both investors would have enough knowledge to revise their respective performance criterion in a consistent way.

We now start with pricing the first project. During the period $[0, \tau_1)$, both the writer and the plain investor have no clue about the profile of the second project, and they can only price the first project under the common terminal utility $U_T(x) = -e^{\gamma x}$. Therefore, essentially, a classical indifference valuation is done for this period. We define the following classical value function processes for the plain investor and the writer over $[0, \tau_1]$,

$$V^0(x, t) = \operatorname{esssup} \mathbb{E}\left[- e^{-\gamma X_T} \middle| \mathcal{F}_t,\ X_t = x \right], \tag{5.13}$$

$$V^{P_1}(x,t) = \operatorname{esssup} \mathbb{E}\left[-e^{-\gamma(X_T - H(Y_T))} \Big| \mathcal{F}_t,\ X_t = x \right], \tag{5.14}$$

respectively. The classical results give that (see [43])

$$V^0(x,t) = -e^{-\gamma x - \frac{1}{2}\lambda^2 (T-t)},$$

and

$$V^{P_1}(x,t) = u^{P_1}(x, Y_t, t),$$

with

$$u^{P_1}(x,y,t) := -e^{-\gamma x}\left(\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T) - \frac{1}{2}(1-\rho^2)\lambda^2(T-t)} \Big| Y_t = y \right] \right)^{\frac{1}{1-\rho^2}},$$

for $0 \le t \le \tau_1$, where $\mathbb{Q}$ is the minimal relative entropy martingale measure with respect to $\mathbb{P}$. At $t = \tau_1$, the writer has preserved her individual optimality under the risk exposure of the first project up to $V^{P_1}(x, \tau_1)$, and the plain investor has achieved $V^0(x, \tau_1)$. Also, both investors realize the arrival of second project with its profile, i.e, τ_2, $G(\cdot) \in \mathcal{F}_{\tau_1}$. Assuming that the investment in the second project is for sure to happen, both investors would evaluate the first project during the life-span of the second one following a relative indifference valuation reasoning. More precisely, during $[\tau_1, \tau_2]$, the goal for the writer is to find a valuation criterion $U^W(x, \tau_2)$ such that consistency along the optimality of investment in first project is preserved, under the extra liability to pay $G(Y_{\tau_2})$ at $t = \tau_2$. In particular, at $t = \tau_1$, she solves

$$V^{P_1}(x,\tau_1) = \operatorname{esssup} \mathbb{E}\Big[U^W(X_{\tau_2} - G(Y_{\tau_2}), \tau_2) \Big| \mathcal{F}_{\tau_1},\ X_{\tau_1} = x \Big], \text{ a.s..} \tag{5.15}$$

Similarly, the plain investor would also undertake the liability $G(Y_{\tau_2})$ of the second project but without the liability $H(Y_T)$ of the first one. The goal is to determine the valuation criterion $U^0(x,\tau_2)$ to maintain intertemporal consistency along optimality and exclude pricing discrepancy. At $t=\tau_1$, the plain investor solves

$$V^0(x,\tau_1) = \operatorname{esssup} \mathbb{E}\left[U^0(X_{\tau_2} - G(Y_{\tau_2}),\tau_2)\Big|\mathcal{F}_{\tau_1},\ X_{\tau_1}=x\right],\ \text{a.s..} \tag{5.16}$$

Once the two utility functions U^W and U^0 are determined, we can define the value function processes similarly as in (5.14) and (5.13) for $\tau_1 \le t \le \tau_2$, conditional on $\mathcal{F}_{\tau_1}$,

$$V^{P_1,P_2}(x,t;\omega) := \operatorname{esssup} \mathbb{E}\Big[U^W(X_{\tau_2} - G(Y_{\tau_2}),\tau_2)\Big|\mathcal{F}_{\tau_1},\ X_t = x\Big], \tag{5.17}$$

and

$$V^{P_2}(x,t;\omega) := \operatorname{esssup} \mathbb{E}\Big[U^0(X_{\tau_2} - G(Y_{\tau_2}),\tau_2)\Big|\mathcal{F}_{\tau_1},\ X_t = x\Big]. \tag{5.18}$$

The forward indifference price of the first project relative to the second project during period $[\tau_1,\tau_2]$ would be naturally defined as the "break-even" process $H_t^{P_1|P_2}$, $\tau_1 \le t \le \tau_2$, that satisfies

$$V^{P_2}(X_t - H_t^{P_1|P_2}, t) = V^{P_1,P_2}(X_t,t), \quad \text{a.s..}$$

The problem now boils down to looking for the respective forward criterion U^W and U^0 for the writer and the plain investor, such that the consistency conditions (5.15) and (5.16) hold. Notice that as in the continuous time framework for the forward performance processes, the consistent $(t=\tau_2)$ forward criterion in general is not unique. In the following sections, we consider two types of forward performance criteria, namely the predictable criteria (i.e., $U^W(x,\tau_2),\ U^0(x,\tau_2) \in \mathcal{F}_{\tau_1}$) and the adaptive criteria (i.e., $U^W(x,\tau_2),\ U^0(x,\tau_2)\in\mathcal{F}_{\tau_2}$). As we will see, different class of forward criteria lead to different relative indifference prices, but time-inconsistency and pricing discrepancy are excluded in both cases.

5.3.1.1 Predictable forward criteria and relative indifference valuation

We first consider the predictable forward family and the associated relative indifference price of the first project during $[\tau_1,\tau_2]$. Suppose the writer has the consistent forward criteria of the form $U^W(x,\tau_2)=-e^{-\gamma x+F_{\tau_2}}$ with $F_{\tau_2}\in\mathcal{F}_{\tau_1}$. We apply the distortion transformation

$$V^{P_1,P_2}(x,t;\omega)=-e^{-\gamma x}v(Y_t,t;\omega)^{\frac{1}{1-\rho^2}},$$

then equation (5.17) can rewrite as

$$-e^{-\gamma x}v(y,t;\omega)^{\frac{1}{1-\rho^2}}=\operatorname{esssup}\mathbb{E}\Big[U^W(X_T-G(Y_{\tau_2}),\tau_2)\Big|\mathcal{F}_{\tau_1},\ X_t=x,\ Y_t=y\Big],\ \text{a.s..}$$

The function $v(y,t;\omega)$ solves almost surely the random linear parabolic equation

$$v_t+\frac{1}{2}a^2(y,t)v_{yy}+\Big(b(y,t)-\rho\lambda a(y,t)\Big)v_y=\frac{1}{2}(1-\rho^2)\lambda^2 v \qquad (5.19)$$

for $\tau_1<t<\tau_2$ with terminal condition $v(y,\tau_2;\omega)=e^{\gamma(1-\rho^2)G(y)+(1-\rho^2)F_{\tau_2}}\in\mathcal{F}_{\tau_1}$. Conditional on $\mathcal{F}_{\tau_1}$, the solution to (5.19) has the Feynman-Kac representation for $\tau_1\le t\le\tau_2$,

$$v(y,t;\omega)=\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F_{\tau_2}-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\Big|\mathcal{F}_{\tau_1},\ Y_t=y\Big],\ \text{a.s..} \qquad (5.20)$$

Conditional on $\mathcal{F}_{\tau_1}$, the measure $\widetilde{\mathbb{Q}}$ is defined on $\mathcal{F}_{\tau_2}$ as

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\bigg|_{\tau_2} = e^{-\lambda(W^1_{\tau_2}-W^1_{\tau_1})-\frac{1}{2}\lambda^2(\tau_2-\tau_1)},$$

and $\widetilde{W}_s = W_s - W_{\tau_1} + \rho\lambda(s-\tau_1)$ is a standard Brownian motion for $\tau_1 \leq s \leq \tau_2$. In particular,

$$v(y,\tau_1) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F_{\tau_2}-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-\tau_1)}\Big|\,\mathcal{F}_{\tau_1}, Y_{\tau_1}=y\Big],$$

while the consistency condition (5.15) suggests on the other hand that

$$v(y,\tau_1) = \mathbb{E}_{\mathbb{Q}}\Big[e^{\gamma(1-\rho^2)H(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-\tau_1)}\Big|Y_{\tau_1}=y\Big]. \tag{5.21}$$

Under the fact $F_{\tau_2} \in \mathcal{F}_{\tau_1}$, we derive from (5.21) that

$$F_{\tau_2} = -\frac{1}{2}\lambda^2(T-\tau_2) + \frac{1}{1-\rho^2}\ln\frac{\mathbb{E}_{\mathbb{Q}}[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}|Y_{\tau_1}]} \in \mathcal{F}_{\tau_1}. \tag{5.22}$$

Hence, the consistent predictable forward criterion for the writer at the expiry of the second project would be

$$U^W(x,\tau_2) = -\exp\left(-\gamma x - \frac{1}{2}\lambda^2(T-\tau_2) + \frac{1}{1-\rho^2}\ln\frac{E_{\mathbb{Q}}[e^{\gamma(1-\rho^2)H(Y_T)}|Y_{\tau_1}]}{E_{\widetilde{\mathbb{Q}}}[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}|Y_{\tau_1}]}\right) \in \mathcal{F}_{\tau_1}. \tag{5.23}$$

The next step is to determine the predictable forward criterion $U^0(x,\tau_2)$ for the plain investor who only pays $G(Y_{\tau_2})$ at $t=\tau_2$. The procedure follows closely to the derivation of $U^W(x,\tau_2)$. To be specific, suppose $U^0(x,\tau_2) = -e^{-\gamma x+\widetilde{F}_{\tau_2}}$, with $\widetilde{F}_{\tau_2} \in \mathcal{F}_{\tau_1}$, and consider as usual the distortion transformation $V^{P_2}(x,t;\omega) = -e^{-\gamma x}\widetilde{v}(y,t;\omega)^{\frac{1}{1-\rho^2}}$. Then from equation (5.18), we can conclude that the function $\widetilde{v}(y,t;\omega)$ solves almost surely the linear parabolic PDE

$$\widetilde{v}_t + \frac{1}{2}a^2(y,t)\widetilde{v}_{yy} + \Big(b(y,t)-\rho\lambda a(y,t)\Big)\widetilde{v}_y = \frac{1}{2}(1-\rho^2)\lambda^2\widetilde{v} \tag{5.24}$$

with terminal condition $\widetilde{v}(y,\tau_2;\omega) = e^{\gamma(1-\rho^2)G(y)+(1-\rho^2)\widetilde{F}_{\tau_2}} \in \mathcal{F}_{\tau_1}$. Conditional on $\mathcal{F}_{\tau_1}$, the Feynman-Kac representation of the solution to (5.24) is

$$\widetilde{v}(y,t;\omega) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)\widetilde{F}_{\tau_2}-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\Big|\mathcal{F}_{\tau_1},\ Y_t = y\Big],\ \text{a.s.}, \quad (5.25)$$

for $\tau_1 \le t \le \tau_2$. Finally, consistency condition (5.16) for the plain investor implies on the other hand that

$$\tilde{v}(y,\tau_1) = e^{-\frac{1}{2}(1-\rho^2)\lambda^2(T-\tau_1)},$$

and under the assumption $\widetilde{F}_{\tau_2} \in \mathcal{F}_{\tau_1}$, we can derive that

$$\widetilde{F}_{\tau_2} = -\frac{1}{2}\lambda^2(T-\tau_2) - \frac{1}{1-\rho^2}\ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}\Big|Y_{\tau_1}\Big] \in \mathcal{F}_{\tau_1}. \quad (5.26)$$

The consistent predictable forward criterion for the plain investor therefore is

$$U^0(x,\tau_2) = -\exp\left(-\gamma x - \frac{1}{2}\lambda^2(T-\tau_2) - \frac{1}{1-\rho^2}\ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)G(Y_{\tau_2})}|Y_{\tau_1}\right]\right) \in \mathcal{F}_{\tau_1}. \quad (5.27)$$

Next we are ready to derive the relative indifference price $H_t^{P_1|P_2}$ for the first project given the second project over $[\tau_1,\tau_2]$. From the distortion transformation and the (relative) indifference price definition

$$V^{P_2}(X_t - H_t^{P_1|P_2}, t) = V^{P_1,P_2}(X_t,t), \quad \text{a.s.},$$

we have

$$-e^{-\gamma(x-h(y,t;\omega))}\widetilde{v}(y,t;\omega)^{\frac{1}{1-\rho^2}} = -e^{-\gamma x}v(y,t;\omega)^{\frac{1}{1-\rho^2}},\ \text{a.s.}, \quad (5.28)$$

where we have assumed $H_t^{P_1|P_2} = h(Y_t, t; \omega)$, due to the exponential utility in terms of wealth. Then it easily follows from (5.28) that for $\tau_1 \leq t \leq \tau_2$,

$$h(y,t;\omega) = \frac{1}{\gamma(1-\rho^2)} \ln \frac{v(y,t;\omega)}{\widetilde{v}(y,t;\omega)} = \frac{1}{\gamma(1-\rho^2)} \ln e^{(1-\rho^2)(F_{\tau_2} - \widetilde{F}_{\tau_2})}$$
$$= \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\mathbb{Q}}\Big[e^{\gamma(1-\rho^2)H(Y_T)} | Y_{\tau_1}\Big], \text{ a.s.}. \tag{5.29}$$

We notice that after $t = \tau_1$, i.e., the arrival/initiation time of the second project, the consistent price for the first project over $[\tau_1, \tau_2]$ remains constant (conditional on $\mathcal{F}_{\tau_1}$) under the predictable assumption of the utility functions for the two investors. The price of the first project would stay on the level exactly before the arrival of the second project. Such constant extension of the valuation problem over $[0, \tau_1]$ is probably the simplest way to maintain pricing consistency before and after the appearance of a new project. As we will see in the next section, even under a different class of forward performance processes that are not predictable, the same conditionally constant indifference price can be derived to excludes time-inconsistency and pricing discrepancy.

5.3.1.2 Adaptive forward criteria and relative indifference valuation

In this section, we work with the consistent forward criteria that are adaptive, i.e., $U^W(x, \tau_2)$, $U^0(x, \tau_2) \in \mathcal{F}_{\tau_2}$. The main argument will follow closely as in the previous section, except that we consider factor form forward criteria $U^W(x, \tau_2) = -e^{-\gamma x + F(Y_{\tau_2}, \tau_2)}$ for the writer, and $U^0(x, \tau_2) = -e^{-\gamma x + \widetilde{F}(Y_{\tau_2}, \tau_2)}$ for the plain investor, where $F(y, \tau_2; \omega)$ and $\widetilde{F}(y, \tau_2; \omega)$ are both

$\mathcal{F}_{\tau_1}$-measurable. Under this assumption and applying the distortion transformation, we derive form (5.17) that $v(y,t;\omega)$ solves almost surely the linear parabolic PDE (5.19) with terminal condition $v(y,\tau_2;\omega) = e^{\gamma(1-\rho^2)G(y)+(1-\rho^2)F(y,\tau_2)} \in \mathcal{F}_{\tau_1}$. Conditional on $\mathcal{F}_{\tau_1}$, the Feynman-Kac representation of the solution is

$$v(y,t;\omega) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\Big|\mathcal{F}_{\tau_1},\ Y_t = y\Big],\ \text{a.s.,} \tag{5.30}$$

for $\tau_1 \leq t \leq \tau_2$, where the measure $\widetilde{\mathbb{Q}}$ is defined as in the previous section. Consistency condition (5.15) for the writer then writes as

$$\begin{aligned} &\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-\tau_1)}\Big|\,\mathcal{F}_{\tau_1}, Y_{\tau_1} = y\Big] \\ &\quad = \mathbb{E}_{\mathbb{Q}}\Big[e^{\gamma(1-\rho^2)H(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-\tau_1)}\Big|Y_{\tau_1} = y\Big],\ \text{a.s..} \end{aligned} \tag{5.31}$$

One can directly verify that

$$F(Y_{\tau_2},\tau_2) = -\left(\gamma G(Y_{\tau_2}) + \frac{1}{2}\lambda^2(T-\tau_2)\right) + \frac{1}{1-\rho^2}\ln \mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T)}\Big|Y_{\tau_1}\right] \tag{5.32}$$

would satisfy the consistency condition (5.31). Hence, the writer's adaptive forward criterion at the expiry of the second project is

$$U^W(x,\tau_2) = -e^{-\gamma(x+G(Y_{\tau_2}))-\frac{1}{2}\lambda^2(T-\tau_2)}\left(\mathbb{E}_{\mathbb{Q}}\Big[e^{\gamma(1-\rho^2)H(Y_T)}\Big|Y_{\tau_1}\Big]\right)^{\frac{1}{1-\rho^2}} \in \mathcal{F}_{\tau_2}. \tag{5.33}$$

We next derive the plain investor's adaptive forward criterion $U^0(x,\tau_2) \in \mathcal{F}_{\tau_2}$. Following the same argument as above, we propose the factor form $U^0(x,\tau_2) = -e^{-\gamma x+\widetilde{F}(Y_{\tau_2},\tau_2)}$ with $\widetilde{F}(y,\tau_2) \in \mathcal{F}_{\tau_1}$. After the application of the distortion transformation and the (conditional) Feynman-Kac representation, the consistency requirement (5.16) for the plain investor gives that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)\widetilde{F}(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-\tau_1)}\Big|\mathcal{F}_{\tau_1}, Y_{\tau_1}=y\Big] = e^{-\frac{1}{2}(1-\rho^2)\lambda^2(T-\tau_1)}. \tag{5.34}$$

One can then verify that

$$\widetilde{F}(Y_{\tau_2},\tau_2) = -\gamma G(Y_{\tau_2}) - \frac{1}{2}\lambda^2(T-\tau_2) \in \mathcal{F}_{\tau_2} \tag{5.35}$$

satisfies the consistency equation (5.34). The plain investor's consistent forward utility is therefore

$$U^0(x,\tau_2) = -e^{-\gamma(x+G(Y_{\tau_2}))-\frac{1}{2}\lambda^2(T-\tau_2)} \in \mathcal{F}_{\tau_2}. \tag{5.36}$$

The relative indifference price of the first project given the second project over $[\tau_1,\tau_2]$ again follows from (5.28)

$$-e^{-\gamma(x-h(y,t;\omega))}\widetilde{v}(y,t;\omega)^{\frac{1}{1-\rho^2}} = -e^{-\gamma x}v(y,t;\omega)^{\frac{1}{1-\rho^2}}, \text{ a.s.},$$

where

$$\widetilde{v}(y,t;\omega) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)\widetilde{F}(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\Big|\mathcal{F}_{\tau_1}, Y_t=y\Big],$$

and

$$v(y,t;\omega) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\Big[e^{\gamma(1-\rho^2)G(Y_{\tau_2})+(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda^2(\tau_2-t)}\Big|\mathcal{F}_{\tau_1},\ Y_t=y\Big],$$

respectively. It follows that

$$h(y,t;\omega) = \frac{1}{\gamma(1-\rho^2)}\ln\frac{v(y,t;\omega)}{\widetilde{v}(y,t;\omega)}$$

$$= \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\mathbb{Q}}\Big[e^{\gamma(1-\rho^2)H(Y_T)}\Big|Y_{\tau_1}\Big], \text{ a.s.,} \tag{5.37}$$

for $\tau_1 \le t \le \tau_2$, which yields the same (conditionally) constant relative indifference price of the first project over $[\tau_1, \tau_2]$ as in the previous section

$$H_t^{P_1|P_2} = h(Y_t, t) = \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\mathbb{Q}}\Big[e^{\gamma(1-\rho^2)H(Y_T)}\Big|Y_{\tau_1}\Big].$$

5.3.2 Relative forward indifference valuation of the second real option

In this section, we discuss the relative indifference valuation of the second real option/project given the first under the forward approach. As before, it is assumed that the second project has an initiation time $0 < \tau_1 < T$ with expiry $\tau_2 \in \mathcal{F}_{\tau_1}$ and payoff $G(Y_{\tau_2})$ and $G(\cdot) \in \mathcal{F}_{\tau_1}$. The case we are mainly interested in is when $\tau_2 > T$ a.s., where $t = T$ is the expiry of the first project whose payoff is $H(Y_T)$; the other case when the second project expires before the first project is easier to handle. We continue to work with the log-normal model in [43] and denote the Sharpe ratio over $[0, T]$ by $\lambda \in \mathcal{F}_0$. At $t = \tau_1$, the second project is introduced and the investor is aware of its expiry and payoff structure. It is then necessary for her to extend the current log-normal model at $t = \tau_1$ to cover the life-span of the new project for the purpose of (relative) indifference valuation. We assume that conditional on $\mathcal{F}_{\tau_1}$, the extended model over $[T, \tau_2]$ still follows the log-normal dynamics with the Sharpe ratio $\lambda_1 \in \mathcal{F}_{\tau_1}$. For simplicity, we also assume that the model for the nontraded asset Y would remain the same after extension to $[T, \tau_2]$. A

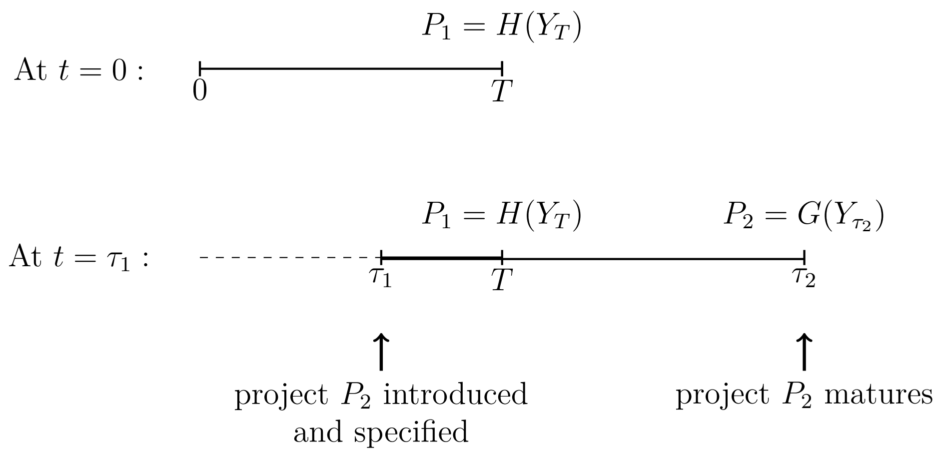

Figure 5.1: Model inputs for relative indifference valuation of the second real option.

summary of the model inputs is given in Figure 5.1.

To price the second project relative to the existing first project, we would regard the plain investor as the investor under the liability of first project. The writer then becomes the investor who holds both the first and second projects. Conditional on $\mathcal{F}_{\tau_1}$, we look for an extended forward performance criterion $U(x, \tau_2)$ under which the optimiality of the benchmark performance, i.e., the performance of the investor holding only the first project, can be preserved over $[T, \tau_2]$. Once such consistent forward evluation criterion is found, the relative indifference price of the second project over period $[\tau_1, \tau_2]$ then is the classical indifference price, conditional on $\mathcal{F}_{\tau_1}$, such that the writer is indifferent with and without the second project under $U(x, \tau_2)$. Throughout this work, we focus on the class of forward criteria that are in factor form, i.e.

$$U(x,\tau_2) = -e^{-\gamma x + F(Y_{\tau_2},\tau_2)}, \tag{5.38}$$

for some $F(y,\tau_2) \in \mathcal{F}_{\tau_1}$, with $x,\ y \in \mathbb{R}$. Conditional on $\mathcal{F}_{\tau_1}$, the optimiality of the plain investor should be preserved over $[T,\tau_2]$, indicating that

$$-\,e^{-\gamma(x-H(y))} = \operatorname{esssup} \mathbb{E}\left[U(X_{\tau_2},\tau_2)\Big|\mathcal{F}_{\tau_1}, X_T = x, Y_T = y\right],\ \text{a.s.}, \tag{5.39}$$

where the left hand side is the (benchmark) performance of the investor with exponential utility $U(x) = -e^{-\gamma x}$ at $t = T$, under the liability of the first project only. To determine the forward criterion $U(x,\tau_2)$, specifically to determine $F(Y_{\tau_2},\tau_2)$ in (5.38), we define the value function for $T \le t \le \tau_2$ as

$$V(x,t) = \operatorname{esssup} \mathbb{E}\left[U(X_{\tau_2},\tau_2)\Big|\mathcal{F}_{\tau_1}, X_t = x\right],\ \text{a.s.}, \tag{5.40}$$

and apply the distortion transformation as usual

$$V(x,t;\omega) = -e^{-\gamma x} v(Y_t,t;\omega)^{\frac{1}{1-\rho^2}}.$$

Standard argument (see [43]) implies that $V(x,t)$ in (5.40) solves a (random) HJB PDE with $v(y,t)$ being the solution to the (random) linear parabolic equation

$$v_t + \frac{1}{2}a^2(y)v_{yy} + \Big(b(y) - \rho\lambda_1 a(y)\Big)v_y = \frac{1}{2}(1-\rho^2)\lambda_1^2 v,\ \text{a.s.},\ T < t < \tau_2 \tag{5.41}$$

and the terminal condition $v(y,\tau_2) = e^{(1-\rho^2)F(y,\tau_2)} \in \mathcal{F}_{\tau_1}$. Conditional on $\mathcal{F}_{\tau_1}$, the Feynman-Kac yields that

$$v(y,t) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2) - \frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2 - t)}\Big|\mathcal{F}_{\tau_1},\ Y_t = y\right],\ \text{a.s.},$$

where the measure $\widetilde{\mathbb{Q}}$, conditional on $\mathcal{F}_{\tau_1}$, is defined on $\mathcal{F}_{\tau_2}$ as

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\bigg|_{\tau_2} = e^{-\lambda_1(W^1_{\tau_2}-W^1_T)-\frac{1}{2}\lambda_1^2(\tau_2-T)}.$$

Under the measure $\widetilde{\mathbb{Q}}$, the process $\widetilde{W}_s = W_s - W_T + \rho\lambda_1(s-T)$, for $T \le s \le \tau_2$, is a standard Brownian motion with $\widetilde{W}_T = 0$, and, it follows that the process Y, conditional on $\mathcal{F}_{\tau_1}$, has the dynamics

$$dY_s = \Big(b(Y_s, s) - \rho\lambda_1 a(Y_s, s)\Big)ds + a(Y_s, s)d\widetilde{W}_s, \tag{5.42}$$

with $Y_t = y \in \mathbb{R}$, for $T \le t \le s \le \tau_2$ under the measure $\widetilde{\mathbb{Q}}$. Then the forward consistency condition (5.39) implies

$$-e^{-\gamma(x-H(y))} = -e^{-\gamma x}\left(\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-t)}\Big|\mathcal{F}_{\tau_1},\ Y_T = y\right]\right)^{\frac{1}{1-\rho^2}}, \text{ a.s.,}$$

which yields

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)}\Big|\mathcal{F}_{\tau_1},\ Y_T = y\right] = e^{\gamma(1-\rho^2)H(y)+\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-T)}, \text{ a.s..} \tag{5.43}$$

It in turn leads to that the random function

$$h(y,t;\omega) := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)}\Big|\mathcal{F}_{\tau_1},\ Y_t = y\right]$$

is a nonnegative solution to the random linear parabolic equation

$$h_t + \frac{1}{2}a^2(y)h_{yy} + \Big(b(y) - \rho\lambda_1 a(y)\Big)h_y = 0,\ T < t < \tau_2, \tag{5.44}$$

with initial condition $h(y,T;\omega) = e^{\gamma(1-\rho^2)H(y)+\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-T)} \in \mathcal{F}_{\tau_1}$, where we have applied the fact that Y, conditional on $\mathcal{F}_{\tau_1}$, has dynamics given by (5.42) under the measure $\widetilde{\mathbb{Q}}$. Equation (5.44) is ill-posed, and we refer to the work

[47] for more detailed discussions of the nonnegative solution to the (random) ill-posed parabolic equation. Now once we find the nonnegative solution to (5.44), it is straightforward to get

$$F(y, \tau_2; \omega) = \frac{1}{1-\rho^2} \ln h(y, \tau_2; \omega), \tag{5.45}$$

for $y \in \mathbb{R}$, and we are able to define the value functions for the plain investor and the writer over $[T, \tau_2]$, respectively. Indeed, the former is given by (5.40) as

$$V(x,t) = -e^{-\gamma x} \left(\mathbb{E}_{\widetilde{\mathbb{Q}}} \left[e^{(1-\rho^2)F(Y_{\tau_2}, \tau_2) - \frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2 - t)} \middle| \mathcal{F}_{\tau_1},\ Y_t \right] \right)^{\frac{1}{1-\rho^2}}, \tag{5.46}$$

whereas the latter is defined in a similar way, but with the liability of the second project taken into account,

$$V^W(x,t) := \operatorname{esssup} \mathbb{E} \left[U(X_{\tau_2} - G(Y_{\tau_2}), \tau_2) \middle| \mathcal{F}_{\tau_1}, X_t = x, Y_t \right] \text{ a.s.}. \tag{5.47}$$

As before, the standard argument and the (conditional) distortion transformation give rise to, for $T \le t \le \tau_2$,

$$V^W(x,t) =$$
$$- e^{-\gamma x} \left(\mathbb{E}_{\widetilde{\mathbb{Q}}} \left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2}) + F(Y_{\tau_2}, \tau_2)\right) - \frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2 - t)} \middle| \mathcal{F}_{\tau_1},\ Y_t \right] \right)^{\frac{1}{1-\rho^2}}. \tag{5.48}$$

The relative indifference price of the second project over $[T, \tau_2]$ is then the conditional "break-even" price between the value functions $V(x,t)$ and $V^W(x,t)$. Indeed, it is the process $H_t^{2|1}$ that satisfies, conditional on $\mathcal{F}_{\tau_1}$,

$$V(X_t - H_t^{2|1}, t) = V^W(X_t, t), \text{ a.s., } T \le t \le \tau_2.$$

Further computation yields that, for $T \le t \le \tau_2$,

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)}\Big|\mathcal{F}_{\tau_1},\ Y_t\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)}\Big|\mathcal{F}_{\tau_1},\ Y_t\right]}, \tag{5.49}$$

with $F(y,\tau_2) \in \mathcal{F}_{\tau_1}$ given by (5.45), being the nonnegative solution to the ill-posed problem (5.44). To calculate the relative indifference price $H_t^{2|1}$ of the second project over period $[\tau_1, T]$, we still need to compare the optimal performance of the (benchmark) plain investor who holds only the first project and that of the writer who holds both the first and the second project. At $t = T$, the plain investor pays liability $H(Y_T)$ under the exponential utility $U(x) = -e^{-\gamma x}$, whereas the writer pays both $H(Y_T)$ and $H_T^{2|1}$, with the latter given by (5.49). The price $H_T^{2|1}$ can be seen as the time $t = T$ analogue of the terminal liability $G(Y_{\tau_2})$ under the extended forward criterion $U(x, \tau_2)$ that has been found. Denote

$$\widehat{G}(Y_T) := \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)}\Big|\mathcal{F}_{\tau_1},\ Y_T\right],$$

then from (5.49) we have $H_T^{2|1} = \widehat{G}(Y_T) - H(Y_T) - \frac{\lambda_1^2}{2\gamma}(\tau_2 - T)$. The value function of the plain investor who is holding a single liability $H(Y_T)$ over $[\tau_1, T]$ udner the exponential utility at $t = T$ follows from the classical result (see [43])

$$V(x,t) = -e^{-\gamma x}\left(\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-t)}\Big|Y_t\right]\right)^{\frac{1}{1-\rho^2}},$$

where the measure $\mathbb{Q}$ is defined on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_T = e^{-\lambda W_T^1 - \frac{1}{2}\lambda^2 T}.$$

The writer's optimization problem is similar but with the liability $H(Y_T)+H_T^{2|1}$ at $t=T$; the value function is given by

$$V^W(x,t) = -e^{-\gamma x}\left(\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)\widehat{G}(Y_T)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-T)-\frac{1}{2}(1-\rho^2)\lambda^2(T-t)}\Big|Y_t\right]\right)^{\frac{1}{1-\rho^2}}.$$

The relative indifference price of the second project is again defined as the "break-even" price between the two value functions, i.e.,

$$V(X_t - H_t^{2|1}, t) = V^W(X_t, t),$$

for $\tau_1 \le t \le T$. A further computation leads to

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)}\Big|Y_T\right]\Big|Y_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{(1-\rho^2)\gamma H(Y_T)}\Big|Y_t\right]} - \frac{\lambda_1^2}{2\gamma}(\tau_2 - T). \tag{5.50}$$

We summarize the result in the following proposition.

Proposition 5.3.1. *Suppose that the ill-posed (random) parabolic equation*

$$h_t + \frac{1}{2}a^2(y)h_{yy} + \Big(b(y) - \rho\lambda_1 a(y)\Big)h_y = 0,\ \ T < t < \tau_2,$$

with $h(y,T;\omega) = e^{\gamma(1-\rho^2)H(y)+\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-T)} \in \mathcal{F}_{\tau_1}$*, has a nonnegative classical solution* $h(y,t;\omega)$*,* $T \le t \le \tau_2$*, almost surely. Then, conditional on* $\mathcal{F}_{\tau_1}$*, the relative forward indifference price of the second project given the first project is*

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)}\Big|Y_T\right]\Big|Y_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{(1-\rho^2)\gamma H(Y_T)}\Big|Y_t\right]}$$

$$-\frac{\lambda_1^2}{2\gamma}(\tau_2 - T),$$

for $\tau_1 \leq t \leq T$, *and*

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2}) + F(Y_{\tau_2},\tau_2)\right)}\Big| Y_t\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)F(Y_{\tau_2},\tau_2)}\Big| Y_t\right]},$$

for $T < t \leq \tau_2$, *where the function* $F(y,\tau_2;\omega) = \frac{1}{1-\rho^2}\ln h(y,\tau_2;\omega)$, *for* $y \in \mathbb{R}$.

5.3.3 Decomposition of risk under relative forward indifference valuation

In this section, we demonstrate the decomposition for the relative indifference price, the residual optimal wealth process and the residual risk process. The discussion is carried out for the case of the relative forward indifference valuation of the second project given the first project. To this end, we first introduce

$$Z_{\tau_2} := \frac{1}{\gamma}\left(F(Y_{\tau_2},\tau_2) - \frac{\lambda_1^2}{2}(\tau_2 - T)\right), \tag{5.51}$$

where $F(y,\tau_2) \in \mathcal{F}_{\tau_1}$, for $y \in \mathbb{R}$, is given by (5.45). Then the forward consistency equation (5.39) can rewrite as

$$-e^{-\gamma(x-H(y))} = \operatorname{esssup} \mathbb{E}\left[-e^{-\gamma(X_{\tau_2}-Z_{\tau_2})+\frac{\lambda_1^2}{2}(\tau_2-T)}\Big|\mathcal{F}_{\tau_1}, X_T = x, Y_T = y\right], \text{ a.s..} \tag{5.52}$$

We can now regard the quantity Z_{τ_2} as the future reincarnation of the first project after its expiry $t = T$, in the sense that the optimality of the performance of the investor who only holds the first project can be maintained through $[T, \tau_2]$, if she pays the virtual payoff Z_{τ_2} at $t = \tau_2$ instead of paying the actual payoff $H(Y_T)$ at expiry $t = T$. Note that the payoff Z_{τ_2} in equation (5.52) is evaluated under the forward criterion $U^0(x, \tau_2) = -e^{-\gamma x + \frac{\lambda_1^2}{2}(\tau_2 - T)}$ which is consistent with the criterion $U^0(x, T) = -e^{-\gamma x}$ at $t = T$. In fact, it is easy to recognize that the criterion $U^0(x, \tau_2)$ is the extended forward criterion from $U^0(x, T)$ for an investor who only invests in the stock and Bond markets without taking any liability from the first and the second projects (i.e., the genuine plain investor). We can also rewrite the relative forward indifference pricing formula in Proposition 5.3.1 using the introduced virtual payoff Z_{τ_2}, i.e., conditional on $\mathcal{F}_{\tau_1}$,

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2}) + Z_{\tau_2}\right)}\Big| Y_T\right]\Big| Y_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T)}\Big| Y_t\right]},$$

for $\tau_1 \leq t \leq T$, and

$$H_t^{2|1} = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2}) + Z_{\tau_2}\right)}\Big|\ Y_t\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\Big|\ Y_t\right]},$$

for $T < t \leq \tau_2$.

We next define the optimal wealth processes for the writer who values the second project in relation to the first project, and the benchmark investor who holds only the first project. Let $\Pi^{2|1,W^*}$ and $\Pi^{2|1,*}$ be their respective optimal control processes by following the relative forward indifference valuation procedure. Then, conditional on $\mathcal{F}_{\tau_1}$, the writer's optimal wealth satisfies

$$dX_s^{2|1,W^*} = \mu_s \Pi_s^{2|1,W^*} ds + \sigma_s \Pi_s^{2|1,W^*} dW_s^1, \quad t \leq s \leq \tau_2, \tag{5.53}$$

with initial condition $X_t^{2|1,W^*} = x + h^{2|1}(y,t;\omega)$, for $\tau_1 \le t \le \tau_2$. Similarly, the optimal wealth for the benchmark investor follows

$$dX_s^{2|1,*} = \mu_s \Pi_s^{2|1,*} ds + \sigma_s \Pi_s^{2|1,*} dW_s^1, \quad t \le s \le \tau_2, \tag{5.54}$$

with initial condition $X_t^{2|1,*} = x$, for $\tau_1 \le t \le \tau_2$. Here, $\mu_s = \mu$, $\sigma_s = \sigma$ for $\tau_1 \le s \le T$ and $\mu_s = \mu_1 \in \mathcal{F}_{\tau_1}$, $\sigma_s = \sigma_1 \in \mathcal{F}_{\tau_1}$ for $T < s \le \tau_2$. The random function $h(y,t;\omega)$, conditional on $\mathcal{F}_{\tau_2}$, is the relative indifference price

$$h^{2|1}(y,t;\omega) = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\Big| Y_T\right]\Big| Y_t = y\right]}{\mathbb{E}_{\mathbb{Q}}\left[e^{\gamma(1-\rho^2)H(Y_T)}\Big| Y_t = y\right]}, \tag{5.55}$$

for $\tau_1 \le t \le T$ and

$$h^{2|1}(y,t;\omega) = \frac{1}{\gamma(1-\rho^2)} \ln \frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\Big|\ Y_t = y\right]}{\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\Big|\ Y_t = y\right]}, \tag{5.56}$$

for $T < t \le \tau_2$. Motivated by the similar definition for the single project indifference valuation in [43], we then introduce the residual optimal wealth process and the residual risk process associated to the relative forward indifference valuation of the second project given the first project

Definition 5.3.1. Let the relative forward indifference price be given by $H_t^{2|1}$ for $\tau_1 \le t \le \tau_2$, and the optimal wealth processes for the writer and the benchmark investor be, respectively, (5.53) and (5.54). We define the residual optimal wealth process for the relative indifference valuation of the second project given the first as

$$L_s^{2|1} = X_s^{2|1,W^*} - X_s^{2|1,*}, \quad t \le s \le \tau_2, \quad L_t^{2|1} = h(y,t;\omega),$$

for $\tau_1 \le t \le \tau_2$, and the residual risk process as

$$R_s^{2|1} = L_s^{2|1} - H_s^{2|1}, \qquad t \le s \le \tau_2, \quad R_t^{2|1} = 0,$$

for $\tau_1 \le t \le \tau_2$.

We can similarly define the processes $L^{1,2}$ and $R^{1,2}$ for the total payoff $G(Y_{\tau_2}) + Z_{\tau_2}$, under the classical forward criterion $U^0(x, \tau_2) = -e^{-\gamma + \frac{\lambda_1^2}{2}(\tau_2 - T)}$, and regard this problem as the problem for determining the (non-relative) indifference price of the two projects together, with the payoff of the first project being replaced by its future reincarnation Z_{τ_2} at $t = \tau_2$. Also, under the same extended forward criterion $U^0(x, \tau_2)$, we define the processes L^1 and R^1 associated to the problem of pricing only the first project under its future virtual payoff Z_{τ_2}, without the liability of the second project. Then the following proposition claims that a desirable decomposition among the risks processes exists. Simply speaking, the residual risk due to the hedging for both projects under the (non-relative) criterion $U^0(x, \tau_2)$ can be decomposed into the risk due to the hedging for only the first project under $U^0(x, \tau_2)$ and the risk due to the hedging for the second project in relation to the first project under the relative forward criterion $U(x, \tau_2)$.

Proposition 5.3.2. *Let the incremental optimal hedging strategy for the relative indifference valuation under the forward criterion $U(x, \tau_2)$ be*

$$\Delta\Pi_t^{2|1,*} = \Pi_t^{2|1,W^*} - \Pi_t^{2|1,*}, \quad \tau_1 \le t \le \tau_2,$$

and similarly define $\Delta\Pi^{1,2,*}$, $\Delta\Pi^{1,*}$ *for the problems under the (non-relative) forward criterion* $U^0(x,\tau_2)$*, respectively. Let also the associated indifference prices be given by* $H^{2|1}$, $H^{1,2}$ *and* H^1*. Then, conditional on* $\mathcal{F}_{\tau_1}$*,*

$$\Delta\Pi_t^{1,2,*} = \Delta\Pi_t^{1,*} + \Delta\Pi_t^{2|1,*}, \ \ a.s.,$$

$$H_t^{1,2} = H_t^1 + H_t^{2|1}, \ \ a.s.,$$

$$L_t^{1,2} = L_t^1 + L_t^{2|1}, \ \ a.s.,$$

$$R_t^{1,2} = R_t^1 + R_t^{2|1}, \ \ a.s.,$$

for $\tau_1 \leq t \leq \tau_2$*.*

Proof. We first focus on the valuation problems over $[T,\tau_2]$. Recall that the value function of the writer under the relative forward criterion $U(x,\tau_2)$ is $V^W(x,t)$ given by (5.48). Conditional on $\mathcal{F}_{\tau_1}$, the associated HJB equation yields the optimal control policy

$$\begin{aligned}\pi^{2|1,W^*}(x,y,t;\omega) &= \rho\frac{a(y)}{\sigma_1}\frac{1}{\gamma(1-\rho^2)}\frac{\partial}{\partial y}\Big(\ln v^{1,2,W^*}(y,t;\omega)\Big) + \frac{\mu_1}{\gamma\sigma_1^2}\\ &= \rho\frac{a(y)}{\sigma_1}h_y^{1,2}(y,t;\omega) + \frac{\mu_1}{\gamma\sigma_1^2},\end{aligned}$$

for $T \leq t \leq \tau_2$, where

$$v^{1,2,W^*}(y,t;\omega) := \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{(1-\rho^2)\left(\gamma G(Y_{\tau_2})+F(Y_{\tau_2},\tau_2)\right)-\frac{1}{2}(1-\rho^2)\lambda_1^2(\tau_2-t)}\Big|\mathcal{F}_{\tau_1},\ Y_t=y\right]$$

and

$$h^{1,2}(y,t;\omega) := \frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\left(G(Y_{\tau_2})+Z_{\tau_2}\right)}\Big|\mathcal{F}_{\tau_1},\ Y_t=y\right].$$

We also have the benchmark investor's value function given by (5.46) with the associated optimal control policy given by

$$\pi^{2|1,*}(x,y,t;\omega) = \rho\frac{a(y)}{\sigma_1}h_y^1(y,t;\omega) + \frac{\mu_1}{\gamma\sigma_1^2},$$

and

$$h^1(y,t;\omega) := \frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\middle|\mathcal{F}_{\tau_1},\ Y_t = y\right],$$

for $T \le t \le \tau_2$. It hence follows that

$$\Delta\Pi_t^{2|1,*} = \rho\frac{a(Y_t)}{\sigma_1}\Big(h_y^{1,2}(Y_t,t;\omega) - h_y^1(Y_t,t;\omega)\Big).$$

On the other hand, the (non-relative) indifference valuation of the two projects with the payoff $G(Y_{\tau_2})+Z_{\tau_2}$ under the (non-relative) forward criterion $U^0(x,\tau_2)$ can be solved following the standard argument, and we obtain

$$H_t^{1,2} = \frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)\big(G(Y_{\tau_2})+Z_{\tau_2}\big)}\middle|\mathcal{F}_{\tau_1},\ Y_t\right],$$

as well as the hedging policy $\pi^{1,2,W^*} = \pi^{2|1,W^*}$ for the writer with both projects under $U^0(x,\tau_2)$. The benchmark Merton investor under the criterion $U^0(x,\tau_2)$ has the optimal policy given by $\pi^{1,2,*}(x,y,t;\omega) = \frac{\mu_1}{\gamma\sigma_1^2}$, for $T \le \tau_1 \le \tau_2$. It hence yields

$$\Delta\Pi_t^{1,2,*} = \rho\frac{a(Y_t)}{\sigma_1}h_y^{1,2}(Y_t,t;\omega).$$

Finally, we can compute the (non-relative) indifference price of the first project under the extended forward criterion $U^0(x,\tau_2)$, again, following the standard argument to get

$$H_t^1 = \frac{1}{\gamma(1-\rho^2)}\ln\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[e^{\gamma(1-\rho^2)Z_{\tau_2}}\middle|\mathcal{F}_{\tau_1},\ Y_t\right],$$

and the incremental hedging strategy

$$\Delta\Pi_t^{1*} = \rho \frac{a(Y_t)}{\sigma_1} h_y^1(Y_t, t; \omega).$$

It hence follows directly that $\Delta\Pi_t^{1,2,*} = \Delta\Pi_t^{1,*} + \Delta\Pi_t^{2|1,*}$, a.s., and $H_t^{1,2} = H_t^1 + H_t^{2|1}$, a.s., in regard of (5.56). By the definition of the residual optimal wealth processes $L^{2|1}$, $L^{1,2}$, L^1, the linearity of the wealth dynamics and the additive property $\Delta\Pi_t^{1,2,*} = \Delta\Pi_t^{1,*} + \Delta\Pi_t^{2|1,*}$, we have $dL_s^{1,2} = dL_s^1 + dL_s^{2|1}$, for $t \leq s \leq \tau_2$, with the initial condition $L_t^{1,2} = L_t^1 + L_t^{2|1}$, due to the additive property $h^{1,2}(y,t) = h^1(y,t) + h^{2|1}(y,t)$. This proves that $L_t^{1,2} = L_t^1 + L_t^{2|1}$, a.s., for $T \leq t \leq \tau_2$. The additivity of the residual risk processes follows from that of the residual optimal wealth processes and that of the indifference price processes. The analysis over the interval $[\tau_1, T]$ is similar. $\square$

Appendices

Appendix A

Appendix for Chapter 3

A.1 Properties of functions F, G

If $m \in (0,1)$, then

$$G\left(\sqrt{2\lambda}\ln\sqrt{\frac{1+m}{1-m}};m,\lambda\right)=\cosh\left(\ln\sqrt{\frac{1+m}{1-m}}\right)-\frac{1}{m}\sinh\left(\ln\sqrt{\frac{1+m}{1-m}}\right)$$

$$=\frac{1}{2}\left(\sqrt{\frac{1+m}{1-m}}+\sqrt{\frac{1-m}{1+m}}\right)-\frac{1}{2m}\left(\sqrt{\frac{1+m}{1-m}}-\sqrt{\frac{1-m}{1+m}}\right)=0.$$

Similarly, $F\left(\sqrt{2\lambda}\ln\sqrt{\frac{1+m}{1-m}};m,\lambda\right) = \cosh\frac{\sqrt{2\lambda}\ln\sqrt{\frac{1+m}{1-m}}}{\sqrt{2\lambda}} - m\sinh\frac{\sqrt{2\lambda}\ln\sqrt{\frac{1+m}{1-m}}}{\sqrt{2\lambda}} = \cosh\left(\ln\sqrt{\frac{1+m}{1-m}}\right)-m\sinh\left(\ln\sqrt{\frac{1+m}{1-m}}\right)$

$$=\frac{1}{2}\left(\sqrt{\frac{1+m}{1-m}}+\sqrt{\frac{1-m}{1+m}}\right)-\frac{m}{2}\left(\sqrt{\frac{1+m}{1-m}}-\sqrt{\frac{1-m}{1+m}}\right),$$

$$=\frac{1}{2}\frac{1-m^2}{\sqrt{1-m^2}}=\frac{1}{2}\sqrt{1-m^2}>0.$$

Appendix B

Appendix for Chapter 4

B.1 Proposition B.1.1

Proposition B.1.1. *Assume that the function g satisfies the assumption in Theorem 4.3.1. Then, for any $\varepsilon > 0$*

$$\sup_{x>0} \frac{g'(x)}{x+\varepsilon} < \infty. \tag{B.1}$$

Proof. We first show that under the assumption in Theorem 4.3.1, $g(0) = \lim_{x\downarrow 0} g(x) = \inf_{x>0} g(x) > -\infty$. Since $\inf_{x>0} \frac{g'(x)}{x} = b > 0$, then $g'(x) \geq bx > 0$, $\forall x > 0$ implies $g(0) = \lim_{x\downarrow 0} g(x) = \inf_{x>0} g(x) < \infty$ exists. Also, since $\sup_{x>0} g''(x) \leq a$, then for $0 < s < t$,

$$-\infty < g(t) \leq g(s) + g'(s)(t-s) + \frac{a}{2}(t-s)^2.$$

Under the fact $0 \leq g'(0) < \infty$, we obtain $g(0) > -\infty$ as $s \to 0$. We can now, without loss of generality, assume $g(0) = 0$ (the initial criterion $U(x, r, 0)$ and the forward problem do not change except for a positive multiplicative constant). To show (B.1), we consider three non-overlapping cases as follows.

1. $-\infty < g''(0) \le a$ and $g'(0) = 0$.

   It is easy to see $g'(x) \le g'(0) + ax = ax$ for $x > 0$, and hence for any $\varepsilon > 0$,

$$\sup_{x>0} \frac{g'(x)}{x+\varepsilon} \le \sup_{x>0} \frac{g'(x)}{x} \le a,$$

   giving $d_\varepsilon = a$ in (4.17) and (4.18) in Theorem 4.3.1, for any $\varepsilon > 0$.

2. $0 < g''(0) \le a$ and $0 < g'(0) < \infty$.

   For this case, we consider the following extension of the current function $g(x) \in C^2(\mathbb{R}^+)$ to a nonnegative $C^2(\mathbb{R})$ function with bounded second order derivative. Define the constant $c_0 = -\frac{(g'(0))^2}{2g''(0)}$ and the function $\tilde{g}(x) := g(x) - c_0$ for $x \ge 0$ and $\tilde{g}(x) = g'(0)x + \frac{g''(0)}{2}x^2 - c_0$ for $x < 0$. Then it is easy to see that $0 \le \tilde{g}(x) \in C^2(\mathbb{R})$, and $\tilde{g}''(x) \le a, \forall\ x \in \mathbb{R}$. It then follows from [27] that $\tilde{g}'(x) \le \sqrt{2a}\tilde{g}(x)$, $\forall\ x \in \mathbb{R}$. In particular, we have

$$g'(x) \le \sqrt{2a}\sqrt{g(x) - c_0},\ \forall\ x > 0. \tag{B.2}$$

   Moreover, condition (4.7) yields $g(x) - c_0 \le g'(0)x + \frac{a}{2}x^2 - c_0$. We next want to find $K > 0$, such that $g'(0)x + \frac{a}{2}x^2 - c_0 \le K(x+\varepsilon)^2$, $\forall x > 0$. Direct computation shows

$$\begin{aligned} &K(x+\varepsilon)^2 - \left(g'(0)x + \frac{a}{2}x^2 - c_0\right) \\ =& \left(K - \frac{a}{2}\right)x^2 + \left(2K\varepsilon - g'(0)\right)x + K\varepsilon^2 + c_0. \end{aligned}$$

   Therefore, if $K > \max\{\frac{a}{2}, \frac{g'(0)}{2\varepsilon}, -\frac{c_0}{\varepsilon^2}\}$, then the above quadratic function is strictly increasing for $x > 0$, with an initial value $K\varepsilon^2 + c_0 > 0$ at

$x = 0$. Combining this result with (B.2), we obtain the desired result

$$g'(x) \leq \sqrt{2a}\sqrt{g(x) - c_0} \leq \sqrt{2aK}(x + \varepsilon), \ \forall \ x > 0.$$

3. $-\infty < g''(0) \leq 0$ and $0 < g'(0) < \infty$.

   In this case, we also aim to give an extension of the function $g(x) \in C^2(\mathbb{R})$, maintaining the nonnegative property and the property that the second order derivative is bounded by $a > 0$, to yield a similar result as (B.2). First we consider the scenario $-a \leq g''(0) \leq 0$ and introduce the function $\tilde{g}(x) = A\arctan(Bx + \theta) + C$, $\forall x \leq 0$ with constants $A,\ B,\ C$ and θ to be determined. Direct computation yields

$$\tilde{g}'(x) = \frac{AB}{(Bx+\theta)^2 + 1}, \quad \text{and} \quad \tilde{g}''(x) = \frac{-2AB^2\,(Bx+\theta)}{\left((Bx+\theta)^2+1\right)^2}.$$

   Then the continuity condition of the first and second order derivatives at 0 imply $\tilde{g}'(0) = g'(0) > 0$ and $\tilde{g}''(0) = g''(0)$, i.e.,

$$g'(0) = \frac{AB}{\theta^2+1} > 0, \quad \text{and} \quad g''(0) = -\frac{2\theta AB^2}{(\theta^2+1)^2},$$

   respectively. Moreover, one can show the second order derivative $\tilde{g}''(x) \leq \frac{3\sqrt{3}}{8}AB^2$, $\forall x \in \mathbb{R}$, if $AB^2 > 0$. Therefore, to have the extended function $\tilde{g}''(x) \leq a$, we impose a third condition on the parameters $0 < \frac{3\sqrt{3}}{8}AB^2 \leq a$. The last step is to show the three conditions

$$g'(0) = \frac{AB}{\theta^2+1} > 0, \quad g''(0) = -\frac{2\theta AB^2}{(\theta^2+1)^2}, \quad \text{and} \quad 0 < \frac{3\sqrt{3}}{8}AB^2 \leq a \tag{B.3}$$

are indeed compatible and lead to solutions. Precisely, since $-\frac{3\sqrt{3}}{8} \leq -\frac{2\theta}{(\theta^2+1)^2} \leq \frac{3\sqrt{3}}{8}$, and $0 < \frac{3\sqrt{3}}{8}AB^2 \leq a$, it is then clear that

$$-a \leq -\frac{2\theta AB^2}{(\theta^2+1)^2} \leq a,$$

indicating that the second condition of (B.3) compatible, due to the fact $-a \leq g''(0) \leq 0$. This shows there exist properly selected constants AB^2 and θ such that the last two conditions of (B.3) are satisfied. Once these two constants are given, we can use the first condition to completely determine A, B, and θ. The constant C is simply determined by $\tilde{g}(0) = g(0) = 0$. Hence, we have a function $\tilde{g}(x) = A\arctan(Bx + \theta) + C$, $\forall x \leq 0$, that is a C^2 extension of the original function $g(x)$ to negative real line with $\tilde{g}''(x) \leq a$, and bounded from below by a finite constant $c_0 := \inf_{x\leq 0} \tilde{g}(x)$. Since $\tilde{g}(0) = g(0) = 0$ and $\tilde{g}'(0) = g'(0) > 0$, we have $c_0 < 0$. Then similar to the second case, we shift both $g(x)$, $\forall x \geq 0$ and $\tilde{g}(x)$, $\forall x < 0$ upwards by $-c_0$, and hence arrive at the same result as (B.2) together with the same estimate following it as in the second case. Next, for the other scenario $-\infty < g''(0) < -a$, before the concatenation with the arctan function as depicted above, it is necessary to shift the second order derivative $g''(0)$ back to the region $[-a, 0]$. Precisely, we achieve this by introducing

$$\tilde{g}(x) = -\frac{a + 2g''(0)}{24}x^4 + \frac{1}{2}g''(0)x^2 + g'(0)x, \quad \forall x \in [-1, 0].$$

It is then easy to verify that $\tilde{g}(x)$, $\forall x \in [-1, 0]$ is a C^2 extension of the original function $g(x)$, $\forall x \geq 0$ with $\tilde{g}''(x) = -\left(\frac{a}{2} + g''(0)\right)x^2 +$

$g''(0) \leq -\frac{a}{2}$, $\forall x \in [-1, 0]$. Now at $x = -1$, we are back in the scenario discussed above, since $\tilde{g}''(-1) = -\frac{a}{2} \in [-a, 0]$. Moreover, $\tilde{g}'(-1) = \frac{a}{6} - \frac{2}{3}g''(0) + g'(0) > 0$. Hence, we can construct the function $\tilde{g}(x) = A \arctan(B(x+1)+\theta)+C$, $\forall x \leq -1$ exactly as in the previous scenario, after replacing $g'(0)$ and $g''(0)$ by $\tilde{g}'(-1)$ and $\tilde{g}''(-1)$, respectively in conditions (B.3). It then yields the extended function $\tilde{g}(x)$ that is defined for all $x \leq 0$ with desired properties.

□

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