Synthesis of electrical and mechanical networks of restricted complexity

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Declaration

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Abstract

Title: Synthesis of electrical and mechanical networks of restricted complexity
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This dissertation is concerned with the synthesis of linear passive electrical and mechanical networks. The main objective is to gain a better understanding of minimal realisations within the simplest non-trivial class of networks of restricted complexity—the networks of the so-called "Ladenheim catalogue"—and thence establish more general results in the field of passive network synthesis. Practical motivation for this work stems from the recent invention of the inerter mechanical device, which completes the analogy between electrical and mechanical networks.

A full derivation of the Ladenheim catalogue is first presented, i.e. the set of all electrical networks with at most two energy storage elements (inductors or capacitors) and at most three resistors. Formal classification tools are introduced, which greatly simplify the task of analysing the networks in the catalogue and help make the procedure as systematic as possible.

Realisability conditions are thus derived for all the networks in the catalogue, i.e. a rigorous characterisation of the behaviours which are physically realisable by such networks. This allows the structure within the catalogue to be revealed and a number of outstanding questions to be settled, e.g. regarding the network equivalences which exist within the catalogue. A new definition of "generic" network is introduced, that is a network which fully exploits the degrees of freedom offered by the number of elements in the network itself. It is then formally proven that all the networks in the Ladenheim catalogue are generic, and that they form the complete set of generic electrical networks with at most two energy storage elements.

Finally, a necessary and sufficient condition is provided to efficiently test the genericity of any given network, and it is further shown that any positive-real function can be realised by a generic network.

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I would like to extend heartfelt thanks to my supervisor, Professor Malcolm Smith, for his support, advice and guidance throughout my research at the University of Cambridge. Many thanks also go to all other members of the Control Group, for numerous interesting discussions.

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Chapter 1

Introduction

This dissertation is concerned with the study of passive electrical and mechanical networks in the context of network synthesis. Network synthesis is a classical field which seeks to describe rigorously the behaviours which are physically realisable in a given domain, with certain specified components. The results in this dissertation will be presented in terms of linear passive electrical networks, with a particular focus on two-terminal networks, built from resistors, inductors and capacitors (RLC networks). Recently a new fundamental element for mechanical control, the *inerter* [74], was introduced, alongside the spring and the damper: it completes the so-called *force-current* analogy between the mechanical and electrical domain, thus allowing all the results presented here to be equivalently expressed in terms of passive mechanical networks, comprising springs, dampers and inerters. Given an arbitrary electrical or mechanical passive network, it is well-known how to characterise its driving-point behaviour (e.g. in terms of the driving-point impedance). Network synthesis can be thought of as the inverse problem, that is how to design a passive (RLC or spring-damper-inerter) network to realise a given driving-point behaviour.

Research in electrical network synthesis developed rapidly in the first half of the twentieth century, due to the broad scope it offered and due to the practical motivation of deriving useful results for analogue filter design, only to start petering out in the 1960s with the advent of integrated circuits. Some of the major results from the classical period include Foster's reactance theorem [23], which characterised the impedance of networks of inductors and capacitors only, Brune's concept of positive-real functions and his construction method for a general positive-real function using resistors, inductors, capacitors *and* transformers [8], and the Bott-Duffin theorem [7], which proved that transformers were unnecessary in the synthesis of positive-real impedances.

In recent years there has been renewed interest in network synthesis motivated in part by the introduction of the inerter, and independently due to the advocacy of R. Kalman [47]. These modern developments have highlighted the need for a better understanding of RLC synthesis, and represent the main motivation for the present work. Despite the wealth of classical results, a number of long-standing questions remain unanswered in fact. Notably, while the Bott-Duffin theorem provides a construction method for the synthesis of *any* positive-real function that makes use of resistors, inductors and capacitors only, one of its most striking features is that the number of reactive elements in the realisation (inductors and capacitors) appears excessive compared to the degree of the impedance function. Despite some progress in recent years [33, 38], very little is known on the question of obtaining a *minimal* realisation of a given positive-real function, since a great many positive-real functions can be realised in a much simpler manner than with the Bott-Duffin method.

It is worth noting that minimising network complexity is crucial in mechanical realisations of positive-real impedances. Numerous applications of the inerter have been recently researched and implemented in the field of mechanical control, ranging from vehicle suspension design [40, 59, 83] to vibration suppression [13, 72, 80, 82]. All these applications have highlighted the need for a better understanding of the most "economical" way to realise a given passive impedance, in order to obtain control mechanisms of limited volume and weight. More broadly, much remains to be discovered in the field, and the study of apparently simple classes of networks has given evidence of a deep complexity and structure within passive network realisations [43, 45]. This was highlighted by Kalman, whose advocacy of a renewed attack on the subject of network synthesis stemmed from his interest in obtaining further insight into a fundamental, classical discipline for which a general theory was missing [76], and which could have important and wider implications in other areas of science [47].

In this dissertation we seek to obtain further understanding on (1) the synthesis of low-complexity impedances and (2) non-minimality in RLC networks. The approach that we adopt is the enumerative approach, with the intent to uncover as much structure as possible within what can be seen as the simplest, yet non-trivial class of networks of restricted complexity. This class was first defined in the Master's thesis of E.L. Ladenheim [52], a student of Foster at the Polytechnic Institute of Brooklyn. Ladenheim determined the set of all essentially distinct two-terminal electrical networks comprising at most two reactive elements and at most three resistors—now known as the "Ladenheim catalogue". Until recent years Ladenheim's thesis appears to have been virtually unknown. A single citation in [25] independently led to two publications: one by Jiang and Smith [43], the other by Kalman [47]. The catalogue was subsequently a central part of discussions at the four workshops on Mathematical Aspects of Network Synthesis initiated by Uwe Helmke which were held alternately in Würzburg and Cambridge from 2010 to 2016. In the Ladenheim catalogue the impedance of each network is computed, and the (more challenging) inverse process is performed, i.e. given the impedance an expression for each element of the network is stated. There are, however, no derivations in Ladenheim's work and, more crucially, no conditions are given on the impedance coefficients which ensure positivity of the network parameters.

Some important networks of the Ladenheim catalogue were studied in [43] and realisability conditions for a "generating set" were obtained. A canonical form for biquadratics was also introduced which, through a graphical interpretation, helped better understand the realisation power of the class. A complete analysis of all the networks of the catalogue outside the generating set was however not attempted, and the underlying structure which relates the networks remained to be uncovered. In this work we present a formal derivation, analysis and classification of the complete Ladenheim catalogue, in order to uncover as much structure as possible and obtain more insight into the realisation of this (apparently simple) class within the biquadratic positive-real functions. This approach will lead to a number of more general results being established. Among these, a key outcome of the classification of the catalogue is a new definition of *generic* network, a concept that appears to be implicit in Ladenheim and Foster's work. This notion is developed here for networks of arbitrary size and is particularly useful in identifying networks which inevitably lead to non-minimal realisations.

1.1 Structure of the dissertation

Chapters 2 and 3 will present an extensive literature survey of fundamental topics of classical and modern network synthesis. Most of the content of these chapters, along with the results presented in Chapters 4, 5, 6, has been accepted for publication as a monograph in the *SIAM Advances in Design and Control* series [55]. The approach of Chapter 4 was included in a survey of recent work on electrical network synthesis, in collaboration with T.H. Hughes [34], while preliminary results in the study of the Ladenheim catalogue were presented as an extended abstract in [54]. Chapter 7 is the result of work carried out in collaboration with T.H. Hughes, which was presented in [35] and submitted for publication in [36].

Chapter 2 - Background on classical network synthesis
Fundamental notions of two-terminal electrical networks are reviewed in this chapter, and more detail is given on the main results from passive network synthesis mentioned in the Introduction, as well as other classical synthesis methods.

Chapter 3 - Recent developments in passive network synthesis
Modern developments in the field are summarised in this chapter, where relevant results from the literature on the classification of biquadratics are introduced. More details on the inerter device and on the mechanical-electrical analogy are given at the end of the chapter, where some of the applications of passive network synthesis to mechanical networks are mentioned.

Chapter 4 - The enumerative approach to network synthesis
One of the main contributions of this work is a complete, fresh analysis of the Ladenheim catalogue. The formal derivation of the catalogue is described in this chapter, and the notions of realisability set, equivalence and group action are introduced. These notions will provide the basis for an efficient classification and analysis of the catalogue.

Chapter 5 - Structure of the Ladenheim catalogue
For each equivalence class in the catalogue, the set of impedances that can be realised is derived in explicit form as a semi-algebraic set. Realisability conditions, expressed in terms of necessary and sufficient conditions, are then given in this chapter for each equivalence class, along with plots of the graphical representation of the realisability sets through the canonical form for biquadratics. The underlying structure that emerges from the catalogue is presented here in diagrammatic form, highlighting the interlacing partitions into orbits, equivalence classes and subfamilies.

Chapter 6 - Main results and discussion on the Ladenheim catalogue
With the knowledge of the realisability power of each network in the catalogue, the main results of our analysis of this class of networks are formally proven in this chapter. A new notion of generic network is introduced, and observations are made on the complete set of equivalences, on the smallest generating set for the class, and on Kalman's latest work. The class of six-element networks with four resistors is also analysed here, and two new equivalences are presented.

Chapter 7 - On a concept of genericity for RLC networks
The notion of generic network is further developed in this chapter, and a necessary and sufficient condition is provided to efficiently test this property without requiring the knowledge of the realisability set of the network. The result that any positive-real impedance can be realised as a generic network is also proven here.

Chapter 8 - Conclusions
The concluding chapter summarises the main contributions of the dissertation and suggests some directions for future research.

Appendices
A series of appendices which are useful in the study of the catalogue are provided. These include a table of all non-2-isomorphic planar graphs with at most five edges, the list of the 108 networks in the Ladenheim catalogue (in numerical and subfamily order) and proofs of the realisation theorems for all the five-element networks in the catalogue.

1.2 Notation

Throughout the dissertation we will adopt the following notation:

$\mathbb{R}$	real numbers
$\mathbb{R}_{>0}$	positive real numbers
$\mathbb{R}_{\geq 0}$	non-negative real numbers
$\mathbb{R}^n$	(column) vectors of real numbers
$(x_1, \ldots, x_n)$	column vector

Chapter 2

Background on classical network synthesis

A survey is undertaken in this chapter of general background and classical results of passive network synthesis to provide a broader context for the subsequent analysis. More recent developments and applications of passive network synthesis are presented in Chapter 3. Further details and material on the results discussed in this chapter can be found in [1], [11], [27], [78], [90].

2.1 Preliminaries of electrical networks

This dissertation is concerned with linear passive electrical networks, with a particular focus on one-ports. A one-port network (also know as a two-terminal network), as shown in Figure 2.1, has two external terminals (nodes, vertices), 1 and $1'$. The voltage v between the terminals of the port and the current i entering one terminal and leaving from the other are taken with the sign convention of Figure 2.1. One-ports may be connected to other one-ports, for instance another element, network or a driving source (voltage or current generating source). The *driving-point impedance* of N is defined by $Z(s) = \hat{v}(s)/\hat{i}(s)$, where $\hat{}$ denotes the Laplace transform, and $Y(s) = Z^{-1}(s)$ is the driving-point admittance.

Networks will comprise finite interconnections satisfying Kirchoff's laws that contain resistors, inductors and capacitors, and will be referred to as RLC networks. See Figure 2.2 for the standard symbols of the network elements. From time to time we shall refer to more general one-ports and multi-ports containing ideal transformers or coupled coils, though they do not form part of this study. The inductor and capacitor are termed

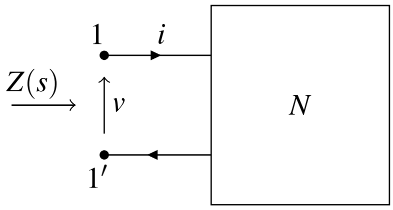


Figure 2.1: Electrical network N with two external terminals 1 and $1'$, terminal voltage v, terminal current i and driving-point impedance $Z(s)$.

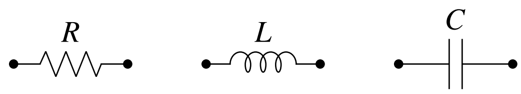


Figure 2.2: The three fundamental two-terminal electrical elements (one-ports): resistor, inductor and capacitor.

reactive elements and have impedance sL and $1/(Cs)$, respectively, where $L > 0$ and $C > 0$ denote inductance and capacitance. The resistor has impedance R, where $R > 0$ denotes the resistance.

Associated to each network is an undirected connected graph in which each edge (branch) corresponds to a network element and two special vertices (nodes) are identified, i.e. the external terminals. We restrict our attention to *planar* networks, whose graph can be embedded in a plane in such a way that no two edges intersect. For such an embedding of a planar graph, we define as *faces* of the graph each of the regions in which the plane is divided. A corollary of a famous theorem by Kuratowski establishes that graphs with fewer than five nodes or nine edges must be planar [69, Theorem 3-17]. In this study we deal with networks with fewer than nine edges therefore we do not need to worry about non-planarity.

A graph has a *dual* if and only if it is planar [69, Theorem 3-15]. Given a planar graph G, the vertices of its dual G' each correspond to a face of G, while faces of G' each correspond to a vertex of G. Two vertices in G' are connected by an edge if the corresponding faces in G have an edge in common. A graph and its dual always have the same number of edges, since there is a one-to-one correspondence between them [69, Section 3-3], while the number of vertices need not be the same. The dual of a planar graph is not necessarily unique, in the sense that the same graph can have non-isomorphic dual graphs (which can stem from distinct planar embeddings of the same graph). However, if G_1 and G_2 are dual graphs of the same planar graph then G_1

and G_2 are 2-isomorphic [69, Theorem 3-18] (see also Section 4.2 and [85, 86] regarding 2-isomorphism).

In the case of two-terminal electrical networks, the first step to obtain the dual of a network N is to consider a (voltage or current) driving source connected to the two external terminals: this step is needed in order to preserve the port in the duality process, since it would otherwise be replaced by a short circuit in the dual network. We then consider the graph G associated to the network and construct its dual G' according to the method outlined above. The dual network N' can then be obtained from G' by populating each edge in G' with the dual of the corresponding element in N: inductors are replaced by capacitors of equal value and vice versa, and resistors are replaced by resistors of reciprocal value [69, Section 6-6]. The voltage source is replaced by a current source and vice versa, which allows us to identify the two external terminals in the dual network [26, Chapter 10.9]. An example of a simple three-element RLC network and its dual is given in Figure 2.3, while Figure 2.4 illustrates the procedure which leads to the graph dual of the given network.

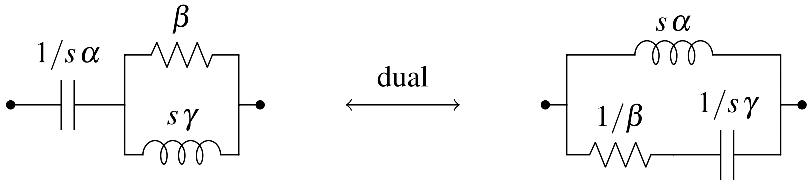


Figure 2.3: Two-terminal electrical network and its dual, with element values α, β, $\gamma > 0$. The impedance of each element in the network is the reciprocal of the corresponding element in the dual network.

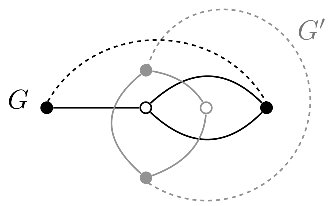


Figure 2.4: Graphical illustration of the procedure to obtain the dual of a given graph. The original graph G is drawn in black, with the dashed line indicating the additional edge corresponding to a voltage or current source connecting the driving-point terminals of G. The same holds for the dual graph G', which is drawn in grey. Note that G' has a vertex for every face of G, and that there is a one-to-one correspondence between edges of G and edges of G'.

Series-parallel (SP) networks are two-terminal networks which can be constructed inductively by combining other SP networks either in series or parallel, with single-element networks being SP. Networks that are not SP are termed *bridge networks*. Networks obtained through a series (parallel) connection of two SP networks are termed *essentially series* (*essentially parallel*) [64]. *Simple series-parallel* (SSP) networks are series-parallel networks which can be constructed inductively by combining a single element in series or in parallel with a SSP network, with single-element networks being SSP. Similar definitions can be given for the graphs associated to the networks.

2.2 Foster and Cauer canonical forms

Network synthesis in its modern sense originated in the famous Reactance Theorem of Foster [23]. The result completely characterises the one-ports that can be built with reactive elements only. The proof of the theorem is based on the solution of an analogous dynamical problem of the small oscillations of a mechanical system given by E.J. Routh [66]. The theorem takes the following form.

Theorem 2.1. *The most general driving-point impedance obtainable in a passive network without resistors (LC network) takes the form:*

$$Z(s) = k\left[\frac{(s^2+\omega_1^2)(s^2+\omega_3^2)\dots(s^2+\omega_{2n\pm1}^2)}{s(s^2+\omega_2^2)(s^2+\omega_4^2)\dots(s^2+\omega_{2n}^2)}\right]^{\pm1} \tag{2.1}$$

where $k \geq 0$, $0 < \omega_1 < \omega_2 \dots$ *and* $n \geq 0$. *Any such impedance may be physically realised in the form of Figure 2.5(a) or Figure 2.5(b) through a partial fraction expansion of* $Z(s)$ *or* $Z^{-1}(s)$ *in the form:*

$$k_0 s + \frac{k_\infty}{s} + \sum_{r=1}^{m} \frac{s k_r}{s^2 + p_r^2}\,. \tag{2.2}$$

where $k_r \geq 0$, $p_r > 0$ *(distinct) and* $m \geq 1$.

Two alternative realisations of purely reactive networks were introduced soon afterwards by W. Cauer [9] and arise from continued fraction expansions of $Z(s)$—see Figure 2.5(c) and Figure 2.5(d). Such expansions may be developed from $Z(s)$ by an alternating sequence of operations of the form "extract a pole" and "invert" where the simple pole extracted is at $s = 0$ or $s = \infty$. The continued fraction for Figure 2.5(c) takes the form

$$Z(s) = sL_1 + \cfrac{1}{sC_1 + \cfrac{1}{sL_2 + \cfrac{1}{sC_2 + \dots}}},$$

while that for Figure 2.5(d) takes the form

$$Z(s) = \frac{1}{sC_1} + \cfrac{1}{\cfrac{1}{sL_1} + \cfrac{1}{\cfrac{1}{sC_2} + \cfrac{1}{\cfrac{1}{sL_2} + \dots}}}.$$

Theorem 2.1 shows that the impedance of a purely reactive network has by necessity only simple poles and zeros which alternate on the imaginary axis, with $s = 0$ and $s = \infty$ always a simple pole or zero. Cauer [9] showed that a similar situation applies to other two-element-kind networks, for which canonical networks analogous to those of Figure 2.5 can be obtained. By analogy to the LC case, these networks are also termed Foster and Cauer canonical forms.

Theorem 2.2. *The most general driving-point impedance obtainable in a passive network which contains only resistors and capacitors (RC network) takes the form*

$$Z(s) = k\frac{(s+\lambda_1)(s+\lambda_3)\dots(s+\lambda_{2n\pm1})}{(s+\lambda_0)(s+\lambda_2)\dots(s+\lambda_{2n})}, \tag{2.3}$$

where $k \geq 0$, $0 \leq \lambda_0 < \lambda_1 < \lambda_2 \dots$ and $n \geq 0$. Any such impedance may be physically realised in the form of Figure 2.5(a) or Figure 2.5(b) (with inductors replaced by resistors) through a partial fraction expansion of $Z(s)$ or $Y(s)/s$, or in the form of Figure 2.5(c) or Figure 2.5(d) through a continued fraction expansion.

Theorem 2.3. *The most general driving-point impedance obtainable in a passive network which contains only resistors and inductors (RL network) takes the form*

$$Z(s) = k\frac{(s+\lambda_0)(s+\lambda_2)\dots(s+\lambda_{2n})}{(s+\lambda_1)(s+\lambda_3)\dots(s+\lambda_{2n\pm1})}, \tag{2.4}$$

where $k \geq 0$, $0 \leq \lambda_0 < \lambda_1 < \lambda_2 \dots$ and $n \geq 0$. Any such impedance may be physically realised in the form of Figure 2.5(a) or Figure 2.5(b) (with capacitors replaced by resistors) through a partial fraction expansion of $Z(s)/s$ or $Y(s)$, or in the form of Figure 2.5(c) or Figure 2.5(d) through a continued fraction expansion.

It is worth pointing out that the general form of the impedance in (2.1), (2.3) and

(2.4) is not altered if the networks contain transformers, and yet transformers are not needed in the canonical forms.

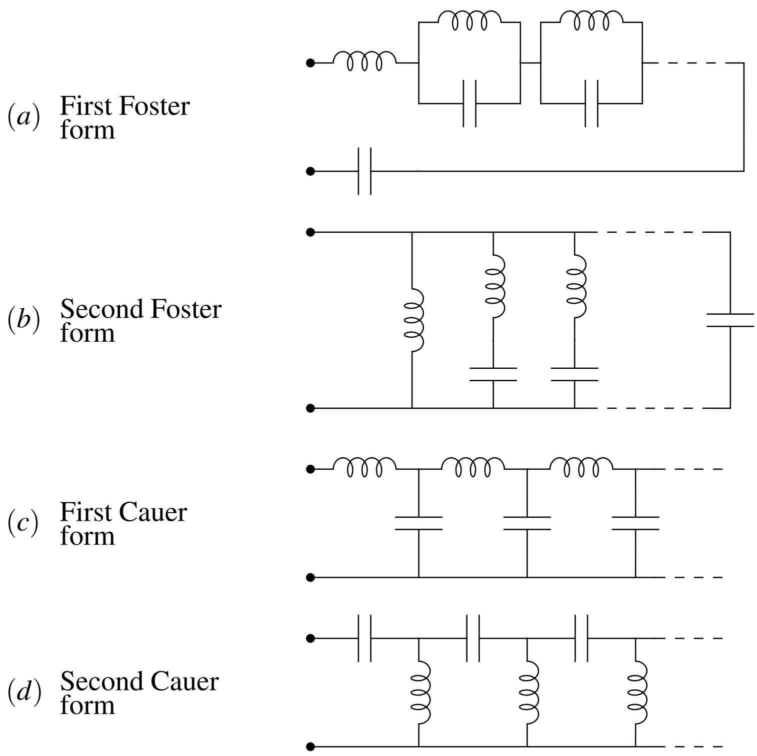


Figure 2.5: Foster and Cauer canonical forms for two-element-kind networks (LC case). For the RC case replace L by R; for the RL case, replace C by R.

2.3 Positive-real functions and passivity

A significant further step in the development of passive network synthesis was the paper of O. Brune [8]. His first important contribution was to introduce and give a complete characterisation of the class of positive-real (p.r.) functions, and further to show that the driving-point impedance of any passive one-port network must be positive-real.

Brune's derivation in [8] that the driving-point impedance is p.r. if and only if the network is passive is based on a physical argument which we now outline. Given the one-port passive network N in Figure 2.1 with driving-point impedance $Z(s)$, consider a voltage impulse applied to its terminals. For a passive network the resulting transient current will be bounded, hence it is necessary that all the zeros in $Z(s)$ have non-positive real part. By a similar argument, considering a current impulse at the port, it is necessary that all the poles in $Z(s)$ have non-positive real part. As we will see next, much stronger properties hold for the driving-point impedance of a passive network.

Consider now an applied voltage $v(t) = 0$ for $t < 0$ and $v(t) = e^{\gamma t}\cos(\omega t)$ for $t \geq 0$, with $\gamma > 0$. We can calculate that the forced response takes the form

$$i(t) = \frac{e^{\gamma t}}{|Z(\gamma + j\omega)|}\cos(\omega t - \theta)\,, \quad \text{where } \theta = \arg Z(\gamma + j\omega)\,,$$

and that this dominates any free response (transient) terms. Neglecting the transient terms, the energy delivered to the network can be computed as

$$W = \int_0^\tau v(t)i(t)\,dt = \frac{1}{4\,|\gamma + j\omega| \cdot |Z(\gamma + j\omega)|}\Bigg\{e^{2\gamma\tau}\cos(2\omega\tau - \theta - \chi) - \cos(\theta + \chi) + \frac{\cos\theta}{\cos\chi}(e^{2\gamma\tau} - 1)\Bigg\}\,,$$

where $\chi = \arg(\gamma + j\omega)$. If τ is large enough, the terms in $e^{2\gamma\tau}$ will be dominant, and we can conclude that $\cos\theta/\cos\chi \geq 1$ necessarily, where both numerator and denominator must be positive, since $\gamma > 0$. If $s = \gamma + j\omega$, this can be rewritten in the form of the following two conditions, which represent the base for Brune's definition of positive-real functions from which all other necessary conditions follow:

$$\mathrm{Re}(Z(s)) \geq 0 \quad \text{for} \;\; \mathrm{Re}(s) \geq 0\,, \tag{2.5}$$

$$|\arg Z(s)| \leq |\arg s| \quad \text{for} \;\; |\arg s| \leq \pi/2\,. \tag{2.6}$$

We note that (2.6) can be interpreted as a contraction property of such functions: the phase of the function is always smaller in absolute value than the phase of its argument. Condition (2.6) clearly implies (2.5), and Brune proved the remarkable result that the converse also holds using Pick's theorem, a generalisation of Schwarz's lemma [8, Theorem VII]. The two conditions can therefore be considered as equivalent definitions of positive-realness for a real-rational function $Z(s)$.

We now state two more commonly used definitions of positive-real functions, which are entirely equivalent to the definitions given in (2.5) and (2.6) [1, Section 2.7], [8, Theorem V]. We note that conditions similar to those in Theorem 2.2 can be given in terms of the zeros of $Z(s)$ [8, Theorem V, Corollary 1].

Definition 2.1. A rational function $Z(s)$ is defined to be positive-real if:

1. $Z(s)$ is real for real s;

2. $Z(s)$ is analytic for $Re(s) > 0$;
3. $Re(Z(s)) \geq 0$ for $Re(s) > 0$.

Definition 2.2. A rational function $Z(s)$ is defined to be positive-real if:

1. $Z(s)$ is real for real s;
2. $Z(s)$ is analytic for $Re(s) > 0$;
3. Poles on the imaginary axis are simple and have positive real residues;
4. $Re(Z(j\omega)) \geq 0$ for all ω.

We further state some useful properties of p.r. functions which appear in Brune's work and which all follow from the definition of positive realness.

Theorem 2.4. *If $Z(s)$ is a positive-real function then the following properties hold:*

1. $1/Z(s)$ *is positive-real;*
2. $Z(1/s)$ *is positive-real;*
3. *The degree of the numerator and denominator of $Z(s)$ can differ by at most one;*
4. *The real part of $Z(s)$ in the right half plane attains its minimum value on the imaginary axis;*
5. *Any poles on the extended imaginary axis can be extracted as in a partial fraction expansion, with the terms extracted and the remainder necessarily positive-real;*
6. $|\arg Z(s)| \leq |\arg s|$ *for* $|\arg s| \leq \pi/2$.

We conclude by stating a more formal definition of passivity [56, Definition 2.5], which can be shown to be equivalent to positive-realness [56, Theorem 4.3]. The definition formalises the notion that the total energy delivered to the network up to time τ is non-negative, meaning that no energy can be delivered to the environment.

Definition 2.3. A one-port network with driving-point voltage $v(t)$ and driving-point current $i(t)$ is passive if

$$\int_{-\infty}^{\tau} v(t)i(t)\,dt \geq 0$$

for all τ and for all compatible pairs $v(\cdot)$, $i(\cdot)$ which are square integrable on $(-\infty, \tau]$.

A similar definition with the lower limit replaced by t_0 is given in [1], with the additional assumption that the network is storing no energy at time t_0. A proof of the equivalence with positive-realness is given in [1, Theorem 2.7.3].

2.4 The Foster preamble and Brune cycle

The second important contribution of Brune [8] was to formulate a procedure to find a network that realises an arbitrary positive-real function. The procedure begins with a sequence of steps known as the *Foster preamble*. This involves the removal of any imaginary axis poles or zeros from $Z(s)$ and the reduction of its minimum real part to zero. For example, if $Z(s)$ has a pole at $s = \infty$ then we can write

$$Z(s) = sL + Z_1(s)$$

where $L > 0$ and $Z_1(s)$ is positive-real with no pole at $s = \infty$. In network terms this corresponds to the removal of a series inductor as shown in the top left figure in Table 2.1. Poles at the origin or on the imaginary axis are dealt with in a similar way. Zeros are similarly extracted by the corresponding operations on $Y(s) = 1/Z(s)$. At any point a constant equal to the minimum value of the real part of the function can be subtracted from $Z(s)$ or $Y(s) = 1/Z(s)$. A summary of the different operations which can be performed on either $Z(s)$ or $1/Z(s)$ is given in Table 2.1, in terms of network representations. The process is not unique.

If the process described above does not succeed in completely realising the function, then a p.r. function $Z(s)$ remains which has no poles or zeros on the extended imaginary axis and with the real part of $Z(j\omega)$ equal to zero at one or more finite, non-zero frequencies. Such functions are termed *minimum functions*. There then follows an operation known as a *Brune cycle* which extracts three inductors in a Y-configuration together with a capacitor. No matter how this extraction is performed, one of the inductors is required to have a negative inductance—which is not realisable passively. Brune's decisive step is to show that the Y-configuration can always be replaced by a pair of coupled coils of positive inductance and with a unity coupling coefficient (Figure 2.6). Such a transformer is in principle realisable passively, though difficult in practice.

We conclude by highlighting the main result from Section 2.3 and the present section in the following theorem, known as Brune's theorem:

Theorem 2.5.

1. *The driving-point impedance $Z(s)$ of any linear passive one-port network is positive-real;*
2. *If $Z(s)$ is positive-real then it is realisable by a network containing resistors, inductors, capacitors and ideal coupled coils.*

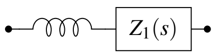
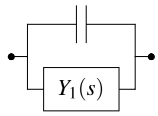
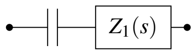
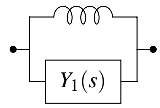
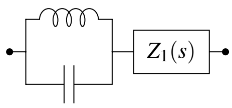
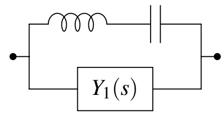
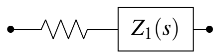
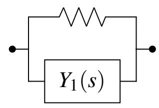
	from impedance function $Z(s)$	from admittance function $Y(s)$
Removal of a pole at infinity		
Removal of a pole at zero		
Removal of a pair of imaginary-axis poles		
Removal of a constant		

Table 2.1: Summary of the possible removal operations on an impedance function $Z(s)$ or admittance $Y(s)$. In each case the remainder of this extraction ($Z_1(s)$ or $Y_1(s)$) is p.r. by Theorem 2.4. The removal of zeros on the extended imaginary axis corresponds to the removal of poles from the reciprocal function. See [78] for a textbook explanation.

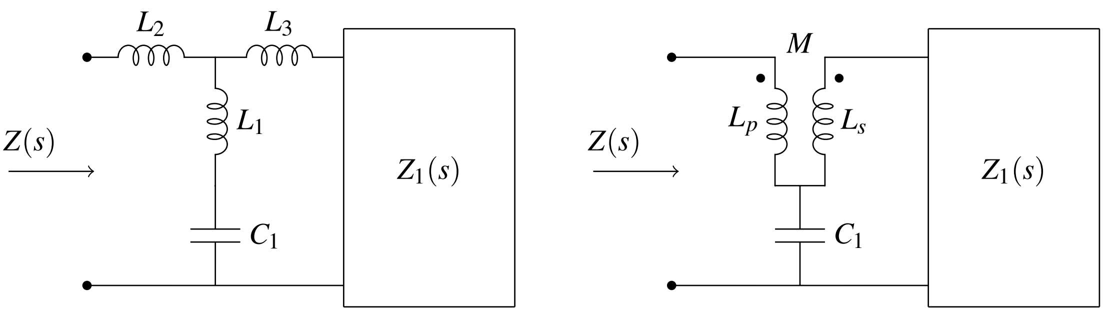


Figure 2.6: Replacement of the Y-configuration of inductors (where $L_1 > 0$, and one of L_2 and L_3 is negative) with coupled coils in a Brune cycle, with L_p, $L_s > 0$ and $M^2/(L_pL_s) = 1$ (unity coupling coefficient).

2.5 The Bott-Duffin construction and its simplifications

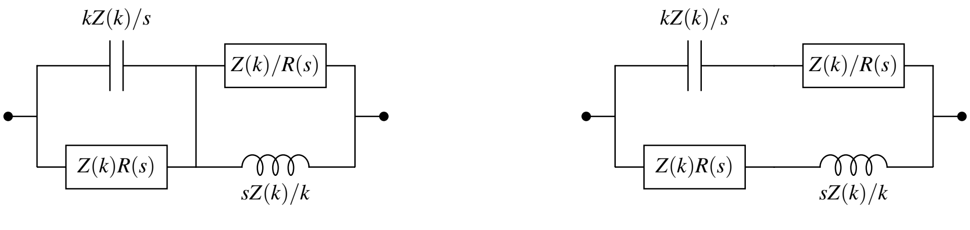
The remarkable fact that coupled coils or ideal transformers can be dispensed with in the realisation of positive-real functions was shown by R. Bott and R.J. Duffin [7]. Their construction begins in the same way as Brune's method, leading to a minimum function by the Foster preamble. The key step of the method is a replacement for the Brune cycle. This makes use of the Richards transformation [63] which states that, for any p.r. function $Z(s)$ and any $k > 0$,

$$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)}$$

is p.r. of degree no greater than $Z(s)$. Now suppose that $Z(s)$ is a minimum function with $Z(j\omega_1) = j\omega_1 X_1$ where $\omega_1 > 0$ and assume that $X_1 > 0$ (otherwise the argument is applied to $Z^{-1}(s)$). Then we can find a $k > 0$ so that $R(s)$ has a zero at $s = j\omega_1$, by choosing $Z(k)/k = X_1$. We now write:

$$\begin{aligned} Z(s) &= \frac{kZ(k)R(s) + Z(k)s}{k + sR(s)} \\ &= \frac{1}{\frac{1}{Z(k)R(s)} + \frac{s}{kZ(k)}} + \frac{1}{\frac{k}{Z(k)s} + \frac{R(s)}{Z(k)}} \qquad (2.7) \\ &= \frac{1}{\frac{1}{\frac{kZ(k)}{s} + \frac{Z(k)}{R(s)}} + \frac{1}{Z(k)R(s) + \frac{sZ(k)}{k}}} \qquad (2.8) \end{aligned}$$

and note that (2.7) and (2.8) correspond to the circuit diagrams of Figure 2.7(a)–(b).



(a) (b)

Figure 2.7: Networks realising the inverse of the Richards transformation.

We can then write:

$$\frac{1}{R(s)} = \frac{\gamma s}{s^2 + \omega_1^2} + \frac{1}{R_1(s)}$$

for some $\gamma > 0$ and $R_1(s)$ being p.r. of strictly lower degree than $R(s)$, which allows series or parallel resonant circuits to be extracted to obtain the networks shown in Figure 2.8. The Bott-Duffin method continues as necessary on the reduced degree impedances until resistors are obtained.

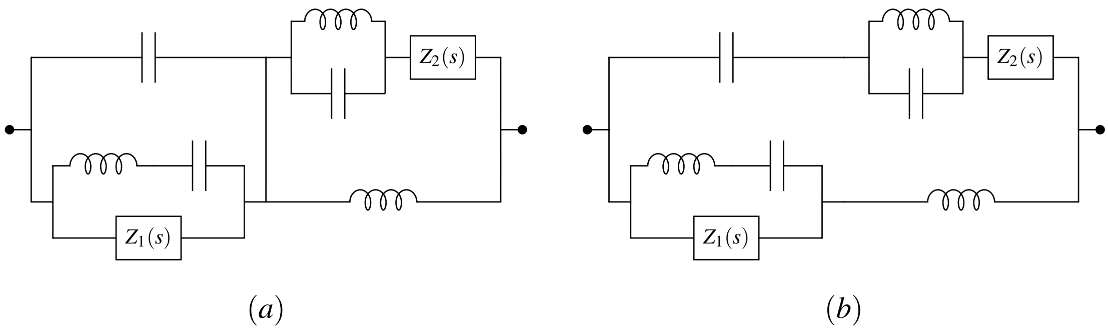


Figure 2.8: Bott-Duffin cycle for the minimum function $Z(s)$ in which $Z_1(s)$, $Z_2(s)$ have lower degree than $Z(s)$ in the case $X_1 > 0$.

Given the above construction method, we can now state the Bott-Duffin theorem:

Theorem 2.6. *Any positive-real function can be realised as the driving-point impedance of a network containing resistors, inductors and capacitors only.*

It should be noted that the networks of Figure 2.8(a)–(b) contain six reactive elements for a degree reduction of two from $Z(s)$ to $Z_1(s)$ and $Z_2(s)$. If $Z(s)$ is a biquadratic minimum function, then $Z_1(s)$ and $Z_2(s)$ are resistors and six reactive elements in total are used in the network realisation of $Z(s)$. This apparent extravagance prompted attempts to seek simpler realisations. Several authors independently found that the six reactive elements could be reduced to five if bridge networks were allowed [21,57,62]. The resulting network can be most easily derived by noting that the network in Figure 2.8(a) is a balanced bridge. Hence it must be entirely equivalent to the network of Figure 2.9. It turns out that by judicious choice of the additional inductance a Y-Δ transformation can be made which results in a pair of reactive elements being duplicated, and hence such a pair can be removed. The resulting network is shown in Figure 2.10. See [27] for a textbook explanation.

Despite the non-intuitive nature of these constructions, it has recently been shown that the Bott-Duffin construction is the simplest possible among series-parallel net-

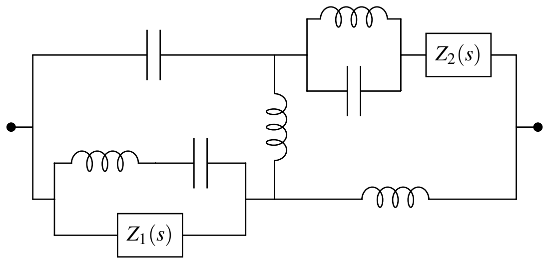


Figure 2.9: Bott-Duffin network with additional inductor.

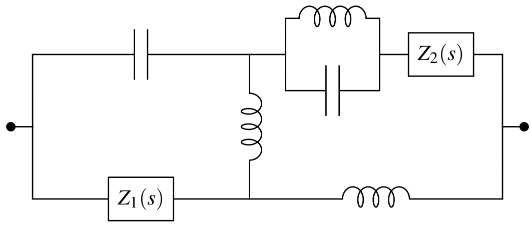


Figure 2.10: Pantell-Fialkow-Gerst-Reza simplification of the Bott-Duffin cycle.

works for the biquadratic minimum function [38] and that the Pantell-Fialkow-Gerst-Reza simplification cannot be improved upon in the generic case [33]. It follows that "non-minimality" is intrinsic to the RLC realisation of some driving-point impedances. This has prompted a fundamental treatment of this non-minimality using Willems' behavioural framework [32], [39]. See Section 3.4 for a discussion on the behavioural approach to passivity.

2.6 Darlington synthesis

Roughly midway between the appearance of the methods of Brune and Bott-Duffin a completely different procedure to realise driving-point impedances was devised by S. Darlington [17]. As in Brune's approach, ideal transformers are an integral part. Remarkably, only one resistor is needed, no matter how complex the positive-real function. In the present context this serves to emphasise the extra freedom that is obtained when transformers are allowed. The Darlington theorem states the following:

Theorem 2.7. *Given a positive-real impedance, it can always be realised as the driving-point impedance of a lossless (i.e. comprising no resistive elements) two-port network terminated in a single resistance, as shown in Figure 2.11.*

We now provide an outline of Darlington's realisation procedure, which proves the

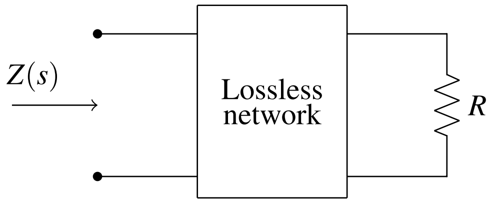


Figure 2.11: Network structure for the Darlington method.

above result. We note that the resistance R which is extracted is often chosen to be 1Ω, though it may be set to any positive value.

Darlington's procedure made use of a lossless two-port containing inductors, capacitors and transformers only. Such a two-port is "reciprocal", which means that the impedance matrix is symmetric [1]:

$$X(s) = \begin{pmatrix} x_{11}(s) & x_{12}(s) \\ x_{12}(s) & x_{22}(s) \end{pmatrix} \tag{2.9}$$

Assuming that the two-port lossless network in Figure 2.11 has impedance matrix (2.9), the driving-point impedance $Z(s)$ is given by

$$Z(s) = x_{11}\,\frac{R^{-1}(x_{11}x_{22} - x_{12}^2)/x_{11} + 1}{R^{-1}x_{22} + 1}. \tag{2.10}$$

We now write $Z(s)$ as a ratio of polynomials in the form

$$Z(s) = \frac{m_1 + n_1}{m_2 + n_2}\,, \tag{2.11}$$

where m_1, m_2 are polynomials of even powers of s and n_1, n_2 are polynomials of odd powers of s. If we factor m_1 out of the numerator and n_2 out of the denominator we get

$$Z(s) = \frac{m_1}{n_2}\,\frac{n_1/m_1 + 1}{m_2/n_2 + 1},$$

which, by comparison with (2.10), suggests the identification:

$$x_{11} = \frac{m_1}{n_2}, \quad x_{22} = R\,\frac{m_2}{n_2}, \quad x_{12} = \sqrt{R}\,\frac{\sqrt{m_1m_2 - n_1n_2}}{n_2},$$

providing $m_1m_2 - n_1n_2$ is a perfect square. The above corresponds to case A in [27, Chapter 9.6]. The alternative case B is obtained by factoring n_1 out of the numerator and m_2 out of the denominator in (2.11), and leads to the same expressions for x_{11}, x_{22}

and x_{12}, with the letters n and m interchanged. It is possible to achieve a real-rational function for x_{12} in one of the two cases by allowing, as necessary, the introduction of cancelling factors between numerator and denominator in (2.11).

It can finally be shown that the set of transfer impedances x_{11}, x_{22} and x_{12} which have been identified define an impedance matrix $X(s)$ which allows a realisation as a lossless two-port comprising inductors, capacitors and transformers only. This is achieved through an extension of Foster's synthesis method of Section 2.2 to the case of two-port networks, which was derived by Cauer [10]. In particular, a partial fraction expansion of $X(s)$, similar to (2.2), leads to an expression of the following type:

$$X(s) = sC_0 + \frac{1}{s}C_1 + \frac{s}{s^2 + p_2^2}\,C_2 + \frac{s}{s^2 + p_3^2}\,C_3 + \dots\,,$$

where $p_i > 0$ and C_0, C_1 etc are non-negative definite constant matrices. A typical term in this sum can always be realised in the form of a T-circuit as the one shown in Figure 2.12, and connected in series to the other two-ports, as shown in Figure 2.13 (see [27, Chapter 7] for further details). Note that if $Z(s)$ has poles or zeros on the imaginary axis then these may be extracted using the relevant steps in the Foster preamble (as outlined in Section 2.4) and included directly in the lossless two-port.

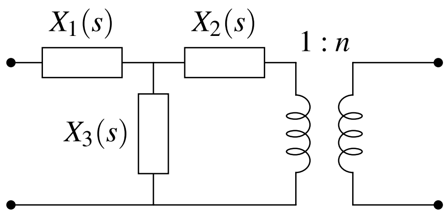


Figure 2.12: Lossless two-port realisation of a typical term in the Darlington synthesis, where $X_1(s)$, $X_2(s)$ and $X_3(s)$ are the impedances of an inductor, capacitor or parallel LC circuit.

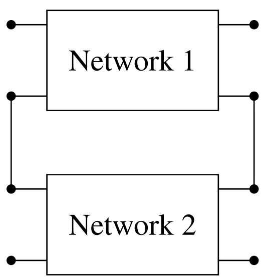


Figure 2.13: Series connection of two two-port networks corresponding to the sum of two terms in the Darlington synthesis.

2.7 Reactance extraction

A later approach to driving-point synthesis due to Youla and Tissi [91] is complementary to Darlington's. The framework is illustrated in Figure 2.14 and the approach is termed *reactance extraction*, in contrast to Darlington's approach of *resistance extraction.* Youla and Tissi's main result is the following:

Theorem 2.8. *Given a positive-real impedance, it can always be realised as the driving-point impedance of a non-dynamic (i.e. comprising only resistors and ideal transformers) multi-port network where all inductors and capacitors have been extracted, as shown in Figure 2.14.*

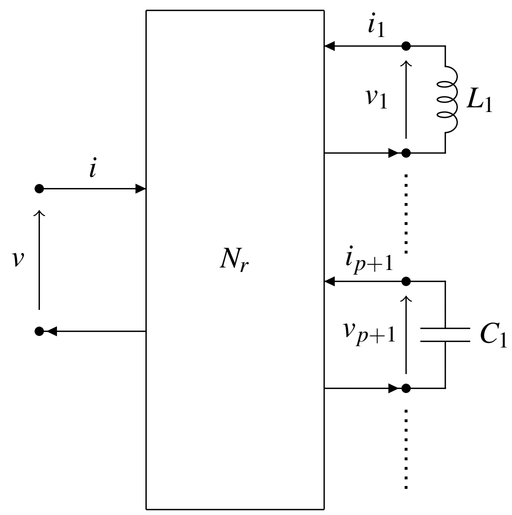


Figure 2.14: Network structure for the method of reactance extraction, where the network N_r contains only resistors and transformers.

The approach of [91] established a connection with the state-space approach to linear dynamical systems which allowed matrix methods to be applied to the synthesis problem (see [1], [90] for textbook treatments). If there are p inductors and q capacitors which are extracted as in Figure 2.14 and if $\mathbf{i}_a = [i_1, \ldots, i_p]^T$, $\mathbf{i}_b = [i_{p+1}, \ldots, i_{p+q}]^T$ denote the vectors of Laplace-transformed port currents (and similarly for the corresponding port voltage vectors $\mathbf{v}_a$ and $\mathbf{v}_b$) then, under mild conditions, there exists a constant matrix M such that the multi-port N_r is described by

$$\begin{pmatrix} v \\ \mathbf{v}_a \\ \mathbf{i}_b \end{pmatrix} = M \begin{pmatrix} i \\ \mathbf{i}_a \\ \mathbf{v}_b \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} i \\ \mathbf{i}_a \\ \mathbf{v}_b \end{pmatrix}$$

with $JM = M^T J$, $J = \text{diag}(I_{p+1}, -I_q)$ and M partitioned so that M_{11} is the $(1,1)$ element of M. Writing $\Lambda = \text{diag}(L_1, \ldots, L_p, C_1, \ldots, C_q)$ it follows that

$$Z(s) = M_{11} - M_{12} \left(sI + \Lambda^{-1} M_{22}\right)^{-1} \Lambda^{-1} M_{21}$$

which is an expression for the impedance in state-space form.

The contribution of [91] was to find all solutions of the multi-port reciprocal synthesis problem using a minimum number of inductors and capacitors, and making use of multi-port transformers in the realisation of N_r. The specific idea of extracting reactances in the form of Figure 2.14 had also been used earlier in [2], and subsequently in [61], to study the class of all biquadratic impedances that may be realised without transformers. This use relies on a condition known as *paramountcy* for the impedance matrix of a 3-port to be realisable using resistors only. In particular, if M is a symmetric $n \times n$ matrix, with $n \leq 3$, then a necessary and sufficient condition for M to be realisable as the impedance matrix of an n-port network comprising resistors only is that M is paramount, namely each principal minor of the matrix is not less than the absolute value of any minor built from the same rows [12,73]. We note that if $n > 3$ then this condition is only necessary. Also, if we allow transformers to be present in the realisation, then a general necessary and sufficient condition for M to be realisable is simply that it is non-negative definite (see e.g. [1, Chapter 9]).

The results of [91] have been exploited recently in [37] to establish algebraic criteria for the number of inductors and capacitors present in a realisation of an impedance function of a one-port network, as will be discussed in more detail in Section 3.3.

2.8 Summary

The central goal of network synthesis is to devise a network which realises a prescribed behaviour. Implicit in this task is the characterisation of the behaviours that are in principle realisable with certain specified components, and those that are not. In passive network synthesis the specified components are the standard (passive) electrical elements such as the resistor, capacitor, inductor or their mechanical equivalents—the latter being considered in the following chapter.

Network synthesis flourished as an active research topic in the first half of the twentieth century. By the 1960s a corpus of results had been established which is now considered classical. In this chapter we have introduced some important concepts and terminology of electrical networks and reviewed the most important results on classical network synthesis, which will provide a broader context for our subsequent analysis.

Chapter 3

Recent developments in passive network synthesis

Despite a golden period of advances starting in the 1920s and a wealth of elegant results, research in network synthesis slowly declined in the second half of the twentieth century, following the introduction of integrated circuits and the dwindling importance of analogue filters. Nevertheless, the basic results retained their fundamental importance, and the influence of circuit theory and network synthesis extended outside the electrical domain. Significant results in the systems and control community testify to an enduring importance and relevance of the subject: the Kalman-Yakubovich-Popov lemma (or "positive real lemma") relating passivity to positive-realness, dissipativity theory [88], behavioural modelling [87] etc.

Many questions in network synthesis were however still unanswered and some results not fully understood, an example being the apparent non-minimality of the Bott-Duffin networks and its simplifications [33], [38]. Interest in efficient realisations of passive mechanical networks, following the invention of the inerter mechanical device [74], prompted a fresh look at these questions. In this chapter we review the most important results in relation to these modern developments of network synthesis.

3.1 Regular positive-real functions and the Ladenheim catalogue

The class of all two-terminal electrical networks with at most five elements, of which at most two are reactive, represents the simplest, non-trivial class of networks of restricted complexity. It was first defined by E. Ladenheim, a student of R.M. Foster, in his

Master's thesis [52], which appears to be the first systematic attempt to study electrical networks by exhaustive enumeration. There are 108 networks in the class, which we refer to as the "Ladenheim catalogue", all of which realise impedances which are at most biquadratic. A formal derivation of the catalogue is given in Chapter 4.

The approach of Jiang and Smith [43], [45] was to study the realisation power of this class of enumerated networks using the notion of a *regular* p.r. function.

Definition 3.1. A positive-real function $Z(s)$ is defined to be regular if the smallest value of $\mathrm{Re}(Z(j\omega))$ or $\mathrm{Re}(Z^{-1}(j\omega))$ occurs at $\omega = 0$ or $\omega = \infty$.

For biquadratic functions regularity implies that the Foster preamble succeeds in reducing the function to a resistor, which means that a realisation is possible with at most two reactive elements and three resistors. It was shown in [43] that six such networks suffice to realise any regular biquadratic. A series of lemmas also showed that all but two of the 108 Ladenheim networks can realise only regular biquadratic impedances, and that the remaining two networks are capable of realising some but not all the non-regular biquadratics. The Ladenheim class was thus shown to possess a *generating set* comprising eight circuits (see also Section 6.3). Reichert's theorem, which is discussed in more detail in Section 3.2, shows that additional resistors beyond three do not extend the class of functions that are realised by the class, which establishes that not all p.r. functions have an RLC realisation with the total number of reactive elements being equal to the degree of the impedance.

The regularity concept has been further utilised to seek networks with more than two reactive elements that are capable of realising non-regular biquadratics. Building on previous work of Vasiliu, five-element structures with three reactive elements were investigated in [43] and series-parallel networks with six elements in [45]. The realisability region for all networks was characterised using a canonical form for biquadratics (which is reviewed in Section 3.1.2), and taken together, these networks were seen to be insufficient to cover the whole of the non-regular region for biquadratics.

An interesting survey by Kalman of the development of passive network synthesis from its early origins until the mid 1970s, when research on the topic gradually petered out, is given in [47]. Considerable attention is paid to Ladenheim's dissertation, and the possibilities for such an enumeration approach to provide a better understanding of transformerless synthesis. Such an approach is further outlined in [53] (see also [76]) and in [48, 49], where the potential role of algebraic invariant theory as a "natural and effective tool for the network synthesis problem" is stressed.

The present analysis of the complete Ladenheim catalogue can be seen as a continuation of several lines of thinking on the problem. Although [43] identified a generating set for the catalogue, a detailed analysis of the realisation power of all the networks was not undertaken. Recently Chen *et al.* [16] derived realisability conditions for the regular bridge networks of the Ladenheim class. This still left unknown the actual set of realisable impedance functions for many networks in the catalogue. Further, the multiplicity of solutions to the realisation problem was not known for most networks. Also, some networks are known to be equivalent to others, but the full set of equivalences had not been determined. Further, the smallest generating set for the catalogue had not been clearly established. More broadly, the amount of structure in the class was a matter of conjecture. The analysis carried out in Chapters 4, 5 and 6 is intended to answer, or improve understanding on, all these questions.

3.1.1 Positive-real and regular biquadratics

In this section we will review some relevant results from [43] on the classification of biquadratic impedances. As mentioned above, the concept of regularity greatly facilitates the classification of impedances, and Lemmas 1–8 in [43] provide useful properties of regular functions. We restrict our attention to *biquadratic* impedances of the form

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}\,, \tag{3.1}$$

where A, B, C, D, E, $F \geq 0$. This function is positive-real if and only if

$$\sigma = BE - (\sqrt{AF} - \sqrt{CD})^2 \geq 0 \tag{3.2}$$

(see [15, Corollary 11]). We now look for conditions under which the biquadratic (3.1) is regular. The resultant of the numerator and denominator in (3.1), that is

$$K = (AF - CD)^2 - (AE - BD)(BF - CE)\,, \tag{3.3}$$

plays an important role in answering this question. Its sign determines whether the reactive elements in a realisation of the biquadratic are of the same kind or of different kind (see Section 3.3). If $K < 0$, the reactive elements are of the same kind and the numerator and denominator in (3.1) have real, distinct roots which interlace each other; by Lemma 3 in [43] the impedance is regular. If $K = 0$, the numerator and denominator have a root in common and the biquadratic function reduces to a bilinear function or a constant; by Lemma 7 in [43] the impedance is regular. Finally, if $K > 0$, the network will have one inductor and one capacitor and need not be regular. By Lemma 5 in [43] the biquadratic impedance (3.1) is regular if and only if at least one of the following four cases is satisfied:

$$\text{Case 1)} \quad AF - CD \geq 0 \text{ and } \lambda_1 \geq 0\,, \tag{3.4}$$
$$\text{Case 2)} \quad AF - CD \geq 0 \text{ and } \lambda_2 \geq 0\,, \tag{3.5}$$
$$\text{Case 3)} \quad AF - CD \leq 0 \text{ and } \lambda_3 \geq 0\,, \tag{3.6}$$
$$\text{Case 4)} \quad AF - CD \leq 0 \text{ and } \lambda_4 \geq 0\,, \tag{3.7}$$

where

$$\lambda_1 = E(BF - CE) - F(AF - CD)\,, \tag{3.8}$$
$$\lambda_2 = B(AE - BD) - A(AF - CD)\,, \tag{3.9}$$
$$\lambda_3 = D(AF - CD) - E(AE - BD)\,, \tag{3.10}$$
$$\lambda_4 = C(AF - CD) - B(BF - CE)\,. \tag{3.11}$$

3.1.2 A canonical form for biquadratics

The analysis of the five-element networks in the Ladenheim catalogue is aided by a *canonical form* for biquadratics [43, 61]. For the impedance $Z(s)$ two simple transformations can be defined:

1. Multiplication by a constant α,
2. Frequency scaling: $s \to \beta s$.

It is easily seen that these transformations correspond to the following scalings of the network parameters: $R \to \alpha R$, $L \to \alpha\beta L$, $C \to \beta C/\alpha$. With such transformations the biquadratic (3.1) with $A, \ldots, F > 0$ can always be reduced to the canonical form

$$Z_c(s) = \frac{s^2 + 2U\sqrt{W}s + W}{s^2 + (2V/\sqrt{W})s + 1/W}, \quad (U, V, W > 0)\,, \tag{3.12}$$

where $\alpha = D/A$, $\beta = \sqrt[4]{CF/(AD)}$, $W = \sqrt{CD/(AF)}$, $U = B/(2\sqrt{AC})$ and $V = E/(2\sqrt{DF})$. We note that U corresponds to the damping ratio of the zeros of the biquadratic (3.1), and V is the damping ratio of the poles, while W is the ratio of the natural frequencies of zeros and poles.

The introduction of the canonical form reduces the number of coefficients from six in (3.1) to just three in (3.12) hence allowing an intuitive graphical interpretation of the realisable set for a given network. It is in fact possible, for a given value of W, to illustrate the set of values in the (U, V)-plane corresponding to real positive values of all inductances, capacitances and resistances in the network. We will call such a set the *realisability region* of a network for a given W (see also Section 4.3.1). Figure 3.1 shows the regions in the (U, V)-plane which correspond to a regular biquadratic for $W \leq 1$, as also shown in [43]. For $K \leq 0$ the biquadratic is always regular, while for $K > 0$ the two cases (3.4), (3.5) provide the conditions for regularity when $AF - CD > 0$ (i.e. $W < 1$ in canonical form).

We will adopt here the following notation, first introduced in [43]: for any rational function $\rho(A, B, C, D, E, F)$, the corresponding function for the canonical form is denoted by $\rho_c(U, V, W)$, where the expressions are obtained by replacing A, B, C, ... by 1, $2U\sqrt{W}$, W, ..., except for a multiplicative positive scaling. (See Table 5.9 for a list of the commonly used functions). Also, for any rational function $\rho_c(U, V, W)$, we define $\rho_c^*(U, V, W) = \rho_c(U, V, W^{-1})$ and $\rho_c^\dagger(U, V, W) = \rho(V, U, W)$. It is finally observed in [43] that $\sigma_c^* = \sigma_c^\dagger = \sigma_c$ and $K_c^* = K_c^\dagger = K_c$.

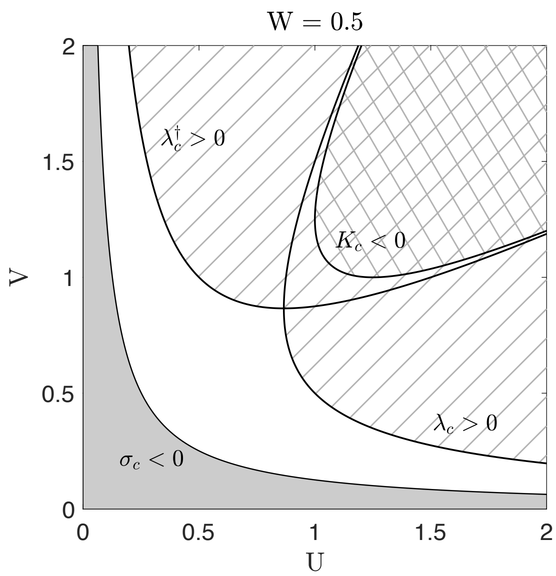


Figure 3.1: *Regular region* for $W = 0.5$. The expressions for K_c, σ_c, λ_c and $\lambda_c^\dagger$ can be found in Table 5.9. The shaded region ($\sigma_c < 0$) corresponds to non positive-real impedances, while the hatched region ($\lambda_c \geq 0$ or $\lambda_c^\dagger \geq 0$) is the regular region.

3.2 Reichert's theorem

Reichert's theorem [61], [44] establishes that the class of impedances which can be realised using two reactive elements is not increased by using more than three resistors. The theorem can be stated as follows:

Theorem 3.1. *Any biquadratic which can be realised using two reactive elements and an arbitrary number of resistors can be realised with two reactive elements and three resistors.*

An immediate consequence of this is that any impedance which can be realised with two reactive elements and an arbitrary number of resistors can also be realised by a network in the Ladenheim catalogue. Since the Ladenheim catalogue does not cover all the possible positive-real biquadratic impedances (realisability regions for all the networks in the catalogue will be illustrated in Chapter 5), a consequence of Reichert's theorem is that some p.r. functions will necessarily have an RLC realisation with more reactive elements than the degree of the impedance.

This result was first proven by Reichert in a German language publication [61], using a complicated topological argument. The proof was later reworked in [44] and new lemmas were provided to expand and clarify the main topological argument. More recently, an alternative proof based on a result in [14] was provided in [93]. We provide here an outline of the proof given in [44].

Proof outline. We first note the necessary and sufficient conditions of Section 3.1.1 for a biquadratic to be regular and the corresponding realisability region plotted in Figure 3.1 for $W = 0.5$. It was shown in [43] that any regular biquadratic can be realised by one of six series-parallel networks with two reactive elements and three resistors. It was also shown that, among all the networks with two reactive elements and three resistors, only two realise non-regular biquadratics. The overall realisability region of the class is shown in Figure 3.2 for $W = 0.6$. From the figure it is clear that the non-regular biquadratics corresponding to the region Γ are not realisable by a two-reactive, three-resistor network. The proof in [44] aims to show that the region Γ is not realisable even if an arbitrary number of resistors is allowed.

Given a network with one inductor, one capacitor and an arbitrary number of resistors, it can always be arranged in the form of Figure 3.3, following the reactance extraction method of Section 2.7. Networks with two inductors or two capacitors will always lead to a regular biquadratic (as already mentioned in Section 3.1.1) and are therefore not of interest for the proof.

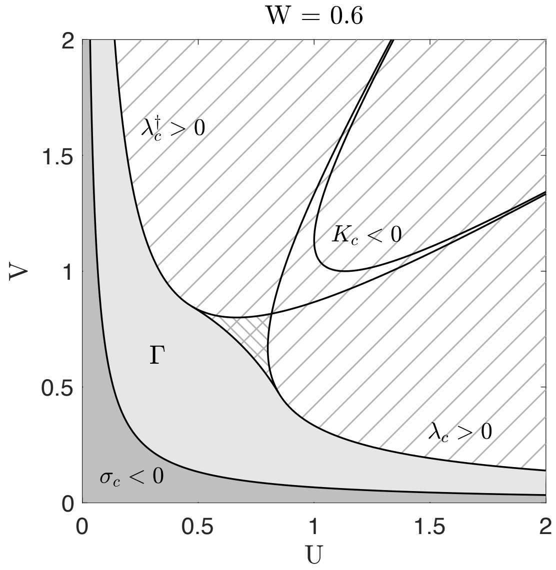


Figure 3.2: Overall realisability region (hatched) for networks in the Ladenheim catalogue, for $W = 0.6$. We note that the realisability region includes some non-regular biquadratics (crossed region). The dark grey region corresponds to non-p.r. biquadratics, while the light grey region Γ corresponds to non-regular p.r. biquadratics.

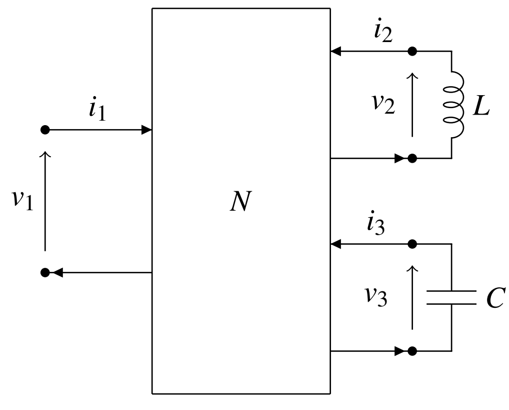


Figure 3.3: Network N with one inductor, one capacitor and an arbitrary number of resistors.

For the network in Figure 3.3 we can write under mild conditions:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} X_1 & X_4 & X_5 \\ X_4 & X_2 & X_6 \\ X_5 & X_6 & X_3 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} =: X \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}, \tag{3.13}$$

where X is a positive semidefinite, real matrix. In order for X to be realisable as a purely resistive network it is necessary that it is *paramount* (see Section 2.7). Eliminating i_2, i_3, v_2, v_3 from (3.13), and knowing that the driving-point impedance of the network is the ratio of the Laplace transforms of v_1 and i_1, we obtain

$$Z(s) = \frac{(X_1X_3 - X_5^2)s^2 + \left(\frac{X_1}{C} + \frac{\det(X)}{L}\right)s + \frac{X_1X_4 - X_4^2}{LC}}{X_3s^2 + \left(\frac{1}{C} + \frac{X_2X_3 - X_6^2}{L}\right)s + \frac{X_2}{LC}}. \tag{3.14}$$

Using an equivalent characterisation of Auth [2] for the impedance (3.14), and by equating this parametrisation to the biquadratic canonical form (3.12), the necessary condition on X is translated into a set of necessary conditions involving U, V and W, which can be interpreted in a topological way. The main part of the proof is based on this topological interpretation and is supported by a series of lemmas. In particular, it is shown in [44] that if it is postulated that there exists a region inside Γ which satisfies the necessary conditions for X to be realisable then this leads to a contradiction. Therefore, the non-regular region Γ in Figure 3.2 is not realisable even if an arbitrary number of resistors are added to the network. □

3.3 Algebraic criteria for circuit realisations

In [25] Foster stated the following fact for biquadratic impedances without proof: the sign of the resultant K of the numerator and denominator determines whether the reactive elements in a minimally reactive realisation are of the same type or of opposite type. This fact was highlighted in a more formal statement by Kalman [47], who suggested that a general proof was urgently needed. A proof was later provided in [37] together with a generalisation to impedances of any order.

The approach of [37] made use of Youla and Tissi's reactance extraction approach [90] together with classical results from matrix theory. A series of equivalent criteria are presented in [37], and are expressed in terms of the rank, signature or number of permanences/variations in the sign of the determinants of certain matrices. In particular, conditions are given on the Hankel matrix (whose entries are defined from the Laurent expansion of $Z(s)$), on an extended Cauchy index, and on the Sylvester and Bezoutian matrices (whose entries can be obtained from the coefficients of the numerator and denominator of $Z(s)$). We present here the conditions given in terms of the Sylvester matrices.

Given a p.r. impedance function

$$Z(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}\,, \tag{3.15}$$

let the associated *Sylvester matrices* be defined as

$$S_{2k} = \begin{pmatrix} b_n & b_{n-1} & \ldots & b_{n-k+1} & b_{n-k} & \ldots & b_{n-2k+1} \\ a_n & a_{n-1} & \ldots & a_{n-k+1} & a_{n-k} & \ldots & a_{n-2k+1} \\ 0 & b_n & \ldots & b_{n-k+2} & b_{n-k+1} & \ldots & b_{n-2k+2} \\ 0 & a_n & \ldots & a_{n-k+2} & a_{n-k+1} & \ldots & a_{n-2k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_n & b_{n-1} & \ldots & b_{n-k} \\ 0 & 0 & \ldots & a_n & a_{n-1} & \ldots & a_{n-k} \end{pmatrix},$$

for $k = 1, \ldots, n$, where $a_i = 0$ and $b_i = 0$ if $i < 0$. Then the following theorem holds:

Theorem 3.2. *Let $Z(s)$ in (3.15) be the impedance of an RLC network containing exactly p inductors and q capacitors (with $p + q = n$). Then $\det(S_{2n}) \neq 0$ and*

$$q = \text{permanences of sign in the sequence } \big(1, \det(S_2), \det(S_4), \ldots, \det(S_{2n})\big),$$
$$p = \text{variations of sign in the sequence } \big(1, \det(S_2), \det(S_4), \ldots, \det(S_{2n})\big).$$

Signs for any subsequence of zeros $\det\big(S_{2(k+1)}\big) = \det(S_{2(k+2)}) = \ldots = 0$, where $\det(S_{2k}) \neq 0$, are assigned as follows:

$$\text{sign}\big(\det(S_{2(k+j)}\big) = (-1)^{\frac{j(j-1)}{2}}\text{sign}\big(\det(S_{2k})\big).$$

We now apply Theorem 3.2 to a biquadratic impedance in the notation of (3.1). The associated Sylvester matrices are

$$S_4 = \begin{pmatrix} D & E & F & 0 \\ A & B & C & 0 \\ 0 & D & E & F \\ 0 & A & B & C \end{pmatrix}, \qquad S_2 = \begin{pmatrix} D & E \\ A & B \end{pmatrix},$$

and it is easily verified that the resultant $K = -\det(S_4)$ has the expression given in (3.3), while $\det(S_2) = BD - AE$. The number of inductors and capacitors in the network is therefore determined by the number of permanences and variations of sign in the sequence $\big(1, BD - AE, -K\big)$. When $K > 0$ there will therefore always be one inductor and one capacitor regardless of the sign of $BD - AE$, while when $K < 0$ one can differentiate between two different cases, as summarised in Table 3.1. We note that in Table 3.1 the sign of $-(AF - CD)$ is equivalently used instead of $\operatorname{sign}(BD - AE)$, as in [25].

	$AF - CD > 0$	$AF - CD < 0$	$AF - CD = 0$
$K > 0$	(1,1)	(1,1)	(1,1)
$K < 0$	(2,0)	(0,2)	–

Table 3.1: Number of reactive elements (# inductors, # capacitors) in a minimally reactive realisation of a biquadratic impedance. The case $K < 0$, $AF - CD = 0$ cannot occur, since the two conditions would imply $K = 0$.

3.4 The behavioural approach to passivity

The long-standing question of the apparent non-minimality of the Bott-Duffin networks, which was mentioned in Section 2.5, has recently prompted a fresh treatment of the driving-point behaviour of RLC networks [39] and, more generally, a new analysis of passive behaviours [32], using Willems' behavioural approach [60, 87].

Given an RLC n-port network, let $\mathbf{i}$ and $\mathbf{v}$ denote the vectors of length n of driving-point currents and voltages. Then the driving-point behaviour $\mathcal{B}$ of the network can be described as a linear time-invariant differential behaviour, i.e. the set of solutions to a system of differential equations of the form

$$P_0\mathbf{i} + P_1\frac{d}{dt}\mathbf{i} + \ldots + P_m\frac{d^m}{dt^m}\mathbf{i} = Q_0\mathbf{v} + Q_1\frac{d}{dt}\mathbf{v} + \ldots + Q_m\frac{d^m}{dt^m}\mathbf{v}, \qquad (3.16)$$

where $m \geq 0$, $P_0, \ldots, P_m$ and $Q_0, \ldots, Q_m$ are square real matrices of dimension n, and $\mathbf{i}$ and $\mathbf{v}$ are assumed to be locally integrable functions [31]. The system (3.16) can be written more compactly as

$$P\left(\frac{d}{dt}\right)\mathbf{i} = Q\left(\frac{d}{dt}\right)\mathbf{v}, \tag{3.17}$$

where $P(\xi) = P_0 + P_1\xi + \ldots + P_m\xi^m$, and similarly for $Q(\xi)$. The following definition of a *passive* system in terms of its behaviour was introduced in [88] and later adapted in [32].

Definition 3.2. The system described by the behaviour $\mathcal{B}$ in (3.17) is passive if for any given $(\mathbf{i}, \mathbf{v}) \in \mathcal{B}$ and $t_0 \in \mathbb{R}$ there exists a $K \in \mathbb{R}$ (dependent on $(\mathbf{i}, \mathbf{v})$ and t_0) such that if $(\hat{\mathbf{i}}, \hat{\mathbf{v}}) \in \mathcal{B}$ satisfies $(\hat{\mathbf{i}}(t), \hat{\mathbf{v}}(t)) = (\mathbf{i}(t), \mathbf{v}(t))$ for all $t < t_0$ then

$$\int_{t_0}^{t_1} \hat{\mathbf{i}}(t)^T \hat{\mathbf{v}}(t)\, dt > -K$$

for all $t_1 \geq t_0$.

In words the definition says that, given an element of the behaviour, i.e. a trajectory $(\mathbf{i}, \mathbf{v})$ which satisfies the representation (3.17), there is a bound K on the energy which can be extracted from the network from the present time t_0 onwards. This bound depends on the specific past trajectory up to t_0, but applies to any future trajectory after t_0. We note that this definition is different from the classic notion of passivity of a one-port given in Definition 2.3, but still formalises the underlying property that it is not possible to extract unlimited energy from the network.

In the case of one-port RLC networks, the behaviour of the system takes the form

$$p\left(\frac{d}{dt}\right) i = q\left(\frac{d}{dt}\right) v, \tag{3.18}$$

where $p(s)$ and $q(s)$ are polynomials in s with real coefficients. The driving-point impedance of the network is given by $Z(s) = p/q$ and we would expect the condition of positive-realness of Z to be equivalent to Definition 3.2, which would be in agreement with Sections 2.3 and 2.4. In general however, for a behaviour of the form (3.18) to be passive it is necessary but not sufficient that the function p/q is positive-real: this is due to the possibility of common roots between p and q, which arise when the behaviour is not controllable [31]. The following theorem provides a necessary *and sufficient* condition for the behaviour of a multi-port network to be passive [32].

Theorem 3.3. *Let $\bar{\lambda}$ denote the complex conjugate of λ and $\mathbb{R}^n[s]$ the vectors of dimension n whose entries are polynomials in s with real coefficients. The system* (3.17) *is passive if and only if the following three conditions hold:*

1. $P(\lambda)Q(\bar{\lambda})^T + Q(\lambda)P(\bar{\lambda})^T \geq 0$ *for all λ in the closed right half-plane;*
2. *rank*$\big([P \;\; -Q](\lambda)\big) = n$ *for all λ in the closed right half-plane;*
3. *If* $\mathbf{r} \in \mathbb{R}^n[s]$ *and* $\lambda \in \mathbb{C}$ *satisfy* $\mathbf{r}(s)^T\big(P(s)Q(-s)^T + Q(s)P(-s)^T\big) = 0$ *and* $\mathbf{r}(\lambda)^T[P \;\; -Q](\lambda) = 0$, *then* $\mathbf{r}(\lambda) = 0$.

In the case of a one-port characterised by the behaviour (3.18) with $Z(s) = p/q$ we have $n = 1$, and the first condition in Theorem 3.3 corresponds to positive-realness of Z, the second condition establishes that there are no pole-zero cancellations in the function p/q in the closed right half-plane, while the third condition implies that, in the lossless case (where $Z(s) + Z(-s) \equiv 0$), pole-zero cancellations are not allowed even in the left half-plane, meaning that p and q must be coprime [30]. It is therefore not sufficient that the function p/q is positive-real (condition 1) for the behaviour to be passive, since there might be pole-zero cancellations in p/q, which do not satisfy the other two conditions of the theorem.

3.5 Network analogies and the inerter

We conclude this chapter by describing mechanical applications of passive network synthesis. Many of the modern developments in the field of network synthesis were in fact motivated by the introduction of a new fundamental component for mechanical control, the inerter [74], alongside the spring and the damper. This new network element provided a way to realise passively any positive-real mechanical admittance or impedance and therefore to directly exploit the wealth of results from electrical network synthesis. Since its introduction in the early 2000s, the inerter has been successfully employed in passive suspensions in motorsport including Formula One cars [13] and is being extensively researched for a wide range of other applications.

We consider in this section *mechanical networks* consisting of a finite interconnection of mechanical elements. Analogous to the case of electrical networks, a *port* in a mechanical system is a pair of terminals to which an equal and opposite force F is applied with a relative velocity v between the terminals. The sign convention is shown in Figure 3.4.

There are two standard analogies between electrical and mechanical systems. Historically the first of these is the so-called *force-voltage* analogy (in which force is analogous

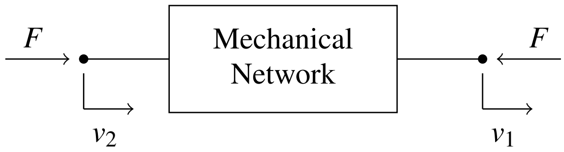


Figure 3.4: One-port (two-terminal) mechanical element or network, with the convention that a positive F gives a compressive force and a positive $v = v_2 - v_1$ corresponds to the terminals moving towards each other.

to voltage and velocity is analogous to current) as evidenced in the terminology electromotive force. The *force-current* (also known as *mobility*) analogy was subsequently introduced by Firestone [22] (see also [18, 28]), who also introduced the concepts of through and across variables. A *through variable* has the same value at the two terminals of the element (e.g. force and current) while an *across variable* is given by a difference of the value at the terminals (e.g. velocity, voltage). Insight on whether a variable is through or across can be gained by considering how measurements of such a variable are taken: through variables require a single measurement point (and typically require the system to be severed at that point) while across variables are measured as a difference between two measurement points (without having to break into the system). This framework allowed analogies to be extended to any dynamical system where through and across measurements can be obtained, such as thermal, fluid and acoustic systems [51, 71]. We mention that there is a corresponding analogy between electrical networks and mechanical systems in rotational form.

In the force-current analogy between mechanical and electrical networks, force (respectively velocity) corresponds to current (respectively voltage) and a fixed point in an inertial frame of reference corresponds to the electrical ground [71]. In this analogy the element correspondences are often stated in the following form:

$$\begin{aligned} \text{spring} &\longleftrightarrow \text{inductor} \\ \text{damper} &\longleftrightarrow \text{resistor} \\ \text{mass} &\longleftrightarrow \text{capacitor} \end{aligned}$$

The correspondence is perfect in the case of the spring and damper, but there is a restriction in the case of the mass due to the fact that it has only one independently movable terminal, the centre of mass. Since the force-velocity relationship relates the acceleration of the centre of mass to a fixed point in the inertial frame, the mass element is, in effect, analogous to a grounded capacitor. This means that, using the classical analogy described above, an RLC circuit may not have a direct mass-spring-damper mechanical analogue, given that in the electrical domain capacitors are not in general required to be grounded.

To complete the analogy, a new two-terminal device, the “inerter”, was introduced in [74], with the property that the applied force at its terminals is proportional to the *relative* acceleration between them. The constant of proportionality is called *inertance* and has the units of kilograms. A table of the circuit symbols of the six basic mechanical and electrical elements, with the inerter replacing the mass, is shown in Figure 3.5, along with their defining equations. In order to justify the introduction of the inerter as an ideal modelling element it should be possible to physically realise inerters which satisfy a number of practical requirements: it should be a two-terminal device which allows sufficient linear travel, which does not need to be attached to any fixed point, which works in any spatial orientation and motion, and which has a mass that is small compared to the elements to which it is connected and independent of the desired value of inertance. Many different physical embodiments of the inerter which satisfy these conditions to a sufficient degree of approximation were devised, ranging from mechanical devices like the rack and pinion inerter and the ballscrew inerter, to hydraulic mechanisms using a gear pump [58, 74, 75, 81] and the fluid inerter [77].

An embodiment of the inerter in rotational form was also given in [75], thus completing (along with the rotary spring and damper) the mechanical-electrical analogy in rotational form. In this case the two terminals of the device can independently rotate about a common axis and an equal and opposite torque is applied at the terminals. The relation between the torque at the terminals and their relative angular displacement, velocity and acceleration gives the defining equations of the rotary spring, damper and inerter, respectively.

In the force-current analogy, the *mechanical impedance* is taken to be the ratio between the Laplace transforms of velocity and force, i.e. between an across variable and a through variable (with the *admittance* being the reciprocal of the impedance). We also note that the force-current analogy respects the manner of interconnection, therefore in order to obtain the electrical or mechanical equivalent of a network it is sufficient to replace each element with the corresponding element in the other domain, while maintaining the same network topology [22].

The most significant consequence of the introduction of the inerter is the possibility to exploit the full freedom of passive network synthesis to synthesise mechanical impedances. The Bott-Duffin theorem (see Theorem 2.6) established that any positive-

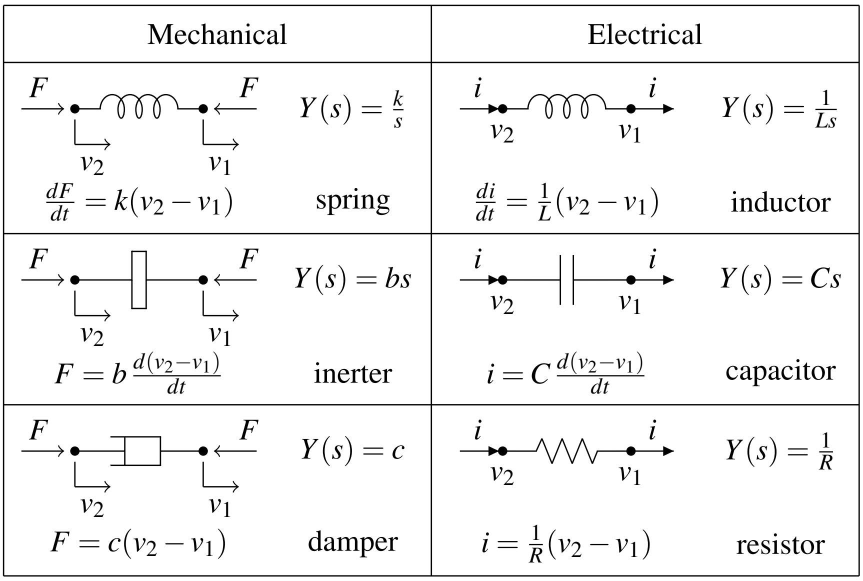


Figure 3.5: The basic mechanical and electrical circuit elements, with their symbols, admittance function and defining equations.

real function can be realised by an electrical network containing resistors, inductors and capacitors only; with the introduction of the inerter it is possible to find the mechanical equivalent of any given electrical network, and therefore the following result can be stated.

Theorem 3.4. *Any positive-real rational function can be realised as the impedance of a one-port mechanical network containing springs, dampers and inerters only.*

With the introduction of the inerter, problems of passive mechanical control can be split into two subproblems: (i) the design of a suitable positive-real mechanical admittance function $Q(s)$ (i.e. a control systems design problem, for example the optimisation of a given performance index) and (ii) the synthesis of a physical mechanical network with admittance $Q(s)$ (for which the passive network synthesis methods described in Chapter 2 and in the remainder of this dissertation can be employed, as explained in Section 5.5.1). Since Theorem 3.4 guarantees that any function can be realised as the impedance of a physical mechanism as long as it is positive-real, this design paradigm offers much more power and flexibility than traditional design methods.

The potential in being able to fully exploit passive network synthesis methods in the field of mechanical control has led to numerous applications of the inerter, which include vehicle suspension design [59,68,74], the control of motorcycle steering oscillations [19,20, 46], rail suspensions [40, 41, 83], building suspensions [80, 89], and vibration suppression for machine tools [82] or for support and isolation of structures [72]. The common feature of these and other novel applications is the relatively low complexity of the passive networks being considered—which is a motivation for the present work.

3.6 Summary

In this chapter we have reviewed relevant literature on modern developments of passive network synthesis. Special emphasis has been drawn on existing results on the realisation of biquadratic impedances, which represents the focus of the next chapters.

The recently renewed interest in electric circuit theory follows the introduction of a new element for mechanical control, the inerter, and the resulting analogy between electrical networks and passive mechanical networks, which was here reviewed. The issue of obtaining minimal realisations of general positive-real functions is crucial in the mechanical domain, and obtaining a better understanding of the minimal realisation of low complexity impedances has in fact been one of the motivations for the present work.

Chapter 4

The enumerative approach to network synthesis

We formally define and derive in this chapter the simplest yet non-trivial class of RLC networks of restricted complexity—the networks of the Ladenheim catalogue. We then introduce the main tools which allow a more systematic analysis and classification of the catalogue, i.e. the notions of realisability set, equivalence and group action.

4.1 Ladenheim's dissertation

In his dissertation [52], Ladenheim considers all two-terminal RLC networks with five elements or less, of which at most two are reactive (inductors or capacitors), and which do not simplify to networks with fewer elements by known network transformations. Considering networks with one reactive element is a trivial case, while the problem with three or more reactive elements is very complex. Ladenheim restricts his attention to networks with five elements (that is, networks with no more than three resistors) in virtue of the observation that the use of additional resistors beyond three does not change the biquadratic nature of the impedance. A later result, known as Reichert's theorem, proves that the class of impedances that can be realised by such networks is not increased by using more than three resistors, as outlined in Section 3.2, hence it is indeed not restrictive to consider networks of at most five elements.

The impedances realised by such networks are biquadratics of the form

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}\,, \tag{4.1}$$

where A, B, C, D, E, $F \geq 0$. Ladenheim's derivation and analysis of the canonical set involves the following steps:

- All possible basic graphs with at most five branches are listed and all the 148 essentially distinct networks which can be built from these graphs are found. Simple transformations allow some networks to be reduced to equivalent ones with fewer elements. In this way the set is reduced to 108 distinct networks.
- Ladenheim then computes the impedance of all 108 networks (starting from the one-element networks, up to the much more interesting five-element networks), i.e. the explicit form of coefficients A, B, C, D, E, F in (4.1) in terms of resistances, capacitances and inductances.
- An attempt is made on the inverse problem, namely expressions are stated for the inductances, capacitances and resistances for each network in terms of the coefficients A, B, C, D, E, F.
- A basic attempt at grouping some of the networks is then performed.

There are, however, no derivations in [52] and, more crucially, no attempt is made to find conditions on the coefficients A, B, C, D, E, F which guarantee that the expressions for the inductances, capacitances and resistances are real and positive. Deriving such necessary and sufficient conditions is one of the major tasks of the present work, which will allow the structure and inter-relationships within the catalogue to be illuminated. In preparation for this task, in the next section we will expand and rework the procedure to obtain the canonical set.

4.2 Definition and derivation of the Ladenheim catalogue

The first step in the derivation of the canonical set is to list all the connected graphs with at most five edges and two special vertices (the external terminals of Figure 2.1). These graphs are enumerated in Appendix B (see also [24,52,64]). They are first grouped based on the number of branches and, within each group, based on the number of vertices, and further according to the type of network (as defined in Section 2.1): for graphs $\mathtt{A} \ldots \mathtt{V}$ simple series-parallel (SSP) graphs appear before series-parallel (SP), which in turn appear before bridge graphs; for graph duals $\mathtt{A}^{\mathtt{d}} \ldots \mathtt{U}^{\mathtt{d}}$ the order is reversed; SSP or SP graphs with the same number of branches and vertices are further ordered, with essentially series graphs appearing before essentially parallel.

The next step is to populate each branch with a resistor or a reactive element to obtain all the *essentially distinct* RLC networks. Networks that are not essentially distinct are related by the operations of deformation, separation and series interchange [24]. The concept is formalised in graph theory as "2-isomorphism" [85, 86]. The networks that can be trivially simplified, namely those which contain a series or parallel connection of the same type of component, are excluded. This enumeration leads to a set of 148 essentially distinct RLC networks with at most five elements of which at most two are reactive.

Of these 148 networks, 40 networks are further eliminated as follows (see Sections 4.4 and 6.1 for the explicit formulae for the Zobel, Cauer-Foster and Y-Δ transformations):

- Eight networks with four resistors and one reactive element (four with graph structure `S` or $\mathtt{S}^{\mathtt{d}}$ and four with graph structure `V`) are eliminated since their impedance is a bilinear function which can be realised by simpler networks.

- Four networks with four elements (with graph structure `G` or $\mathtt{G}^{\mathtt{d}}$) can be reduced by a Zobel transformation to the three-element networks #15 and #17.

- Twenty series-parallel networks with five elements can be reduced by a Zobel transformation to networks with four elements or less. Specifically: four networks with graph structure `L` reduce to networks #20, #25, #28, #32; one with graph structure `M` reduces to network #72; five with graph structure `S` reduce to networks #22, #24, #30, #33, #73; five with graph structure $\mathtt{S}^{\mathtt{d}}$ reduce to networks #37, #40, #45, #48, #72; one with graph structure $\mathtt{M}^{\mathtt{d}}$ reduces to network #73; four with graph structure $\mathtt{L}^{\mathtt{d}}$ reduce to networks #35, #39, #43, #47.

- The four series-parallel networks (with graph structure `O` and $\mathtt{O}^{\mathtt{d}}$) shown in Figure 4.1, and thence also the four bridge networks, can be reduced to the four-element networks #21, #29, #36, #44 with a Cauer-Foster transformation. For reasons that will become clear in the analysis, this transformation is not considered as a true equivalence (see Section 6.1). However, for each of these networks one of the coefficients A, C, D or F in (4.1) is zero, and it is straightforward to show that any impedance realisable by such networks can hence also be realised by a network with fewer elements (e.g. by means of the observation that the impedance function is regular—see [43] and Section 6.2).

The 108 networks of the canonical set are shown in Appendix C. The numbering from the Ladenheim catalogue (although not entirely logical) has been preserved. The derivation of the canonical set will be discussed again in Section 6.2.

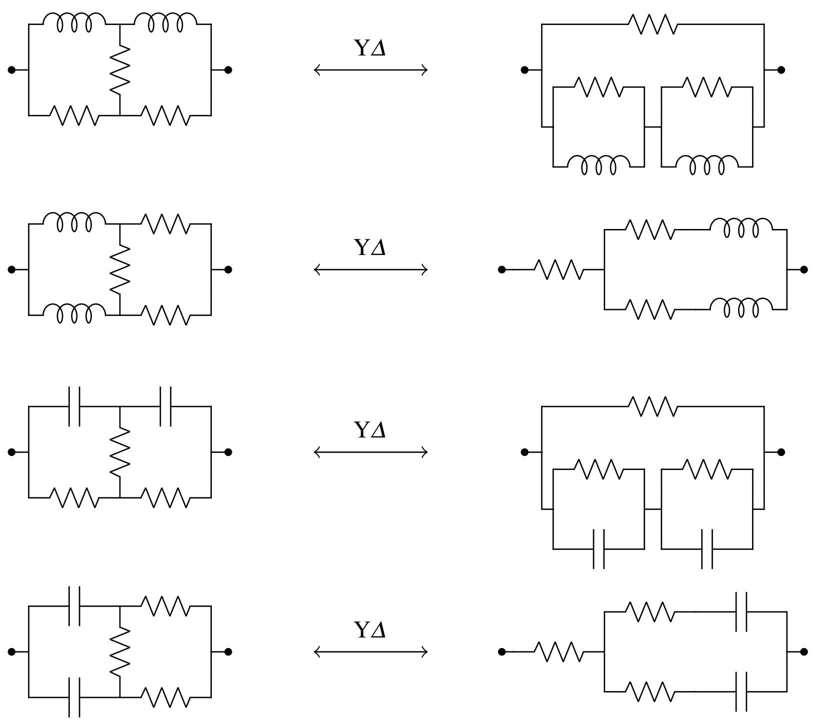


Figure 4.1: Eight networks which are not part of the catalogue since they realise impedances which can also be realised by four-element networks.

Abstractly we can therefore think of the Ladenheim catalogue as a set X_c containing 108 elements. The individual elements can be defined with varying levels of structure:

1. An oriented graph with two external terminals (the driving-point terminals) in which each branch consists of one of three types of elements (resistor, capacitor and inductor) with at most three resistors and the sum of the capacitors and inductors being no greater than two;

2. A set of oriented graphs with a fixed structure as described in 1., but with the branch element parameters varying over all real positive numbers.

Here, we are mostly considering an element to have the additional structure defined in 2., namely each element is actually a set. We refer to each element in the catalogue as a *network*, with network numbers according to the Ladenheim enumeration of Appendix C.

4.3 Approach to classification

To enhance our understanding of the Ladenheim catalogue we seek to uncover as much structure as possible. Our main tools for this purpose, which we describe in the following two sections, are: (1) the notion of equivalent network, (2) the use of group actions to formalise certain well-known transformations. The use of these tools in the classification of the networks in the catalogue will be described in Chapter 5.

4.3.1 Equivalence and realisability set

For each network in the catalogue we are interested to determine the set of impedance functions that a given network can realise when expressed in the form of (4.1). Our first claim is that the set of real numbers that the coefficients A, B, $\ldots$, F may assume for a given network is a *semi-algebraic set*. This can be seen as follows. Without loss of generality we will assume A, B, $\ldots$, $F \geq 0$. For a given network, the impedance $Z(s)$ can be computed as a biquadratic in s,

$$Z(s) = \frac{f_2 s^2 + f_1 s + f_0}{g_2 s^2 + g_1 s + g_0}, \tag{4.2}$$

with coefficients that are polynomial functions of the network parameters R_1, R_2, L_1, etc. Equating (4.2) with a candidate biquadratic impedance (4.1) leads to six polynomial equations of the form $kA = f_2(R_1, R_2, \ldots)$ etc for some positive constant k. In addition there are (up to) six inequalities: $k > 0$, $R_1 > 0$ etc. Taken together these comprise (up to) six polynomial equations and six polynomial inequalities in the (up to) 12 variables, which define a semi-algebraic set in the 12 parameters (variables). If we project this set onto the first six parameters A, B, $\ldots$, F then, using the Tarski-Seidenberg theorem [5], we again obtain a semi-algebraic set which is a subset of $\mathbb{R}^6_{\geq 0} = \{(x_1, x_2, \ldots, x_6) \text{ s.t. } x_i \geq 0 \text{ for } i = 1, \ldots, 6\}$. We will call this the *realisability set* of the network and we will denote it by $\mathcal{S}_n$ where n is the network number (according to the enumeration in Appendix C). Note that this set may also be defined abstractly within $\mathbb{P}^5$, the real projective space of dimension 5. It may sometimes be convenient to embed the realisability set in a higher dimensional space, as we have done for the Ladenheim catalogue, where all realisability sets are considered to belong to $\mathbb{R}^6_{\geq 0}$, even if the number of reactive elements is one or zero. In particular we use the notation of (4.1) for the candidate impedance with $A = D = 0$ when the driving-point impedance of the network is bilinear and $A = B = D = E = 0$ when it is a constant. A more formal definition of realisability set for an arbitrary two-terminal RLC network will be presented in Section 7.1.

We define two networks $\#p$ and $\#q$ to be *equivalent* if $\mathcal{S}_p = \mathcal{S}_q$. This equivalence relation induces a partition of the catalogue into *equivalence classes*. The objective of the present work is to determine all the semi-algebraic sets $\mathcal{S}_n$ for $n = 1, 2, \ldots, 108$. This allows the complete set of equivalences for the Ladenheim catalogue to be determined, and hence all equivalence classes. For those networks in which it is convenient to use the canonical form for biquadratics described in Section 3.1.2, the semi-algebraic set $\mathcal{S}_n$ can be further reduced to a semi-algebraic set $\mathcal{T}_n$ in the three variables U, V and W. We note that the realisability region defined in Section 3.1.2, corresponding to the set of realisable impedances for a fixed value of W, is also a semi-algebraic set, in the two variables U and V.

4.3.2 Group action

The classification of networks is further facilitated by the following transformations on the impedance $Z(s)$:

1. Frequency inversion: $s \to s^{-1}$,

2. Impedance inversion: $Z \to Z^{-1}$.

As noted in [43], the first transformation corresponds to replacing inductors with capacitors of reciprocal values (and vice versa), and the second to taking the network dual. We refer to these transformations as `s` and `d`. We further define a transformation which is the composition of the two: $\mathtt{p} = \mathtt{sd}$.

Defining in addition the identity element `e`, we see that $G = \{\mathtt{e}, \mathtt{s}, \mathtt{d}, \mathtt{p}\}$ is in fact the Klein 4-group, which has the following group table:

	e	d	s	p
e	e	d	s	p
d	d	e	p	s
s	s	p	e	d
p	p	s	d	e

We may then define a *group action* on the set of networks X_c by: $x \to gx$, where $x \in X_c$ and $g \in G$. This group action induces a partition of X_c into *orbits* [4]. In our case the orbits comprise one, two or four elements. In [43] these orbits were referred to as *quartets* and it was noted that sometimes quartets could reduce to two or one element(s).

It was also noted in [43] that frequency inversion corresponds to the transformation $W \leftrightarrow W^{-1}$ in canonical form, and duality corresponds to the transformation $U \leftrightarrow V$, $W \leftrightarrow W^{-1}$. It is easily seen that the transformation `p` corresponds to $U \leftrightarrow V$ in canonical form. Therefore, knowing the realisability conditions in canonical form for a given network in an orbit, the derivation of the conditions for the other networks in the orbit is immediate. The notation introduced in Section 3.1.2 is useful in writing the realisability conditions for all the networks in a given orbit, as frequency inversion corresponds to $*$ and the transformation `p` corresponds to $\dagger$.

In this work we depart from previous convention by depicting orbits in terms of the two actions `s` and `p`, rather than `s` and `d`. This is in part motivated by the fact that `s`-invariance can occur independently of `p`-invariance, while `d`-invariance always implies `s`-invariance within the catalogue—a matter that will be studied in more detail in Section 6.5. (We say that a network is `s`-invariant if the network remains the same after the `s` transformation, and similarly with `d` and `p`.) It is also the case that the `p` transformation takes a simpler form with respect to the canonical form than the `d` transformation.

4.4 Classical equivalences

We review here two well-known equivalences from linear network analysis. The first one, hereafter referred to as the "Zobel transformation", appears in explicit form in O.J. Zobel [94, Appendix III], together with other transformations, though it is clear from [94] that this transformation was common knowledge at the time. The well-known Y-Δ transformation, which follows, was first published by A.E. Kennelly [50].

4.4.1 Zobel transformation

For any two impedances Z_1 and Z_2, the networks in Figure 4.2 are equivalent in the sense defined in Section 4.3.1 when

$$a' = \frac{a(a+b)}{b}, \qquad b' = a+b, \qquad c' = c\left(\frac{a+b}{b}\right)^2$$

$$\left[\, a = \frac{a'b'}{a'+b'}, \qquad b = \frac{(b')^2}{a'+b'}, \qquad c = c'\left(\frac{b'}{a'+b'}\right)^2 \,\right],$$

for any real positive numbers a, b, etc. It is clear from the expressions above that, for any positive and finite value of a, b and c (respectively a', b', c') in the transformation, coefficients a', b' and c' (respectively a, b, c) are necessarily finite and strictly positive.

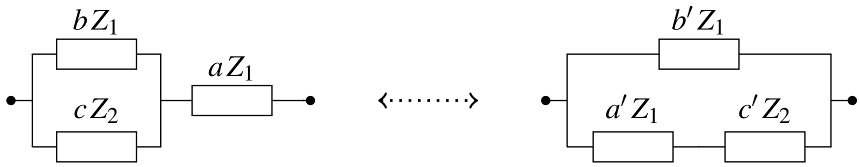


Figure 4.2: Zobel transformation.

4.4.2 Y-Δ transformation

For any real positive values R_1, R_2 etc the networks in Figure 4.3 are equivalent when

$$R_1 = \frac{R_b\,R_c}{R_a + R_b + R_c}, \qquad R_2 = \frac{R_a\,R_c}{R_a + R_b + R_c}, \qquad R_3 = \frac{R_a\,R_b}{R_a + R_b + R_c}$$

$$\left[\, R_a = \frac{R_P}{R_1}, \quad R_b = \frac{R_P}{R_2}, \quad R_c = \frac{R_P}{R_3}, \text{ where } R_P = R_1 R_2 + R_2 R_3 + R_1 R_3 \right].$$

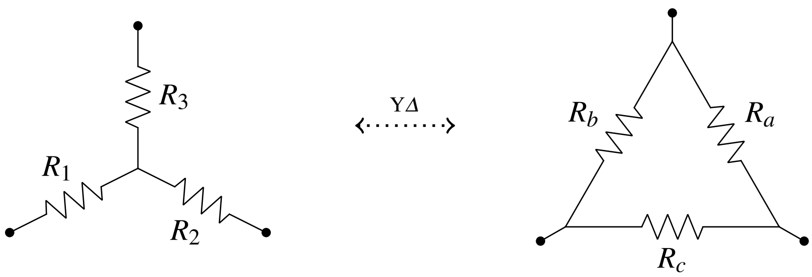


Figure 4.3: Y-Δ transformation.

4.5 Summary

In this chapter we have expanded and reworked the procedure for the derivation of the Ladenheim catalogue. This new derivation led to the same canonical set of 108 networks as in [52]. We then formalised the notions of realisability set and equivalence for RLC networks, and reviewed some simple, known network transformations: frequency and impedance inversion (which taken together with their composition and the identity allowed us to define a group action), and the Zobel and Y-Δ network equivalences.

The concepts that were reviewed and introduced in this chapter are at the base of our classification of the networks in the Ladenheim catalogue, and can more generally be used in the study of other classes of RLC networks (an example is outlined in Section 6.6, where the class of six-element networks with four resistors is studied).

Chapter 5

Structure of the Ladenheim catalogue

In this chapter we proceed to uncover the structure that underlies the Ladenheim catalogue using the notions introduced in Chapter 4. It is routine to verify that the group action defined in Section 4.3.2 induces a partition of the catalogue X_c into 35 orbits. A more difficult task is to identify the further structure that is revealed by the notion of equivalence introduced in Section 4.3.1. Our first step in this regard is to identify all the equivalences that result from the Zobel and the Y-Δ transformations described in Section 4.4. This results in a number of orbits "coalescing" through equivalence, and it is convenient to attach a numbering to the resulting "subfamilies", which are 24 in number. Subfamilies are numbered with Roman numerals, according to the number of elements in the networks, with subscript letters to distinguish the subfamilies according to types (e.g. subfamilies of four-element networks are numbered IV_{A}, IV_{B} etc).

At this point it is unclear whether there are further equivalences within the catalogue which cause some of these subfamilies to further coalesce. This turns out not to be the case with our notion of equivalence (as formalised in Theorem 6.3). To verify this, it is necessary to determine the realisability set $\mathcal{S}_n$ for one representative of each of the 24 subfamilies. This is one of the main contributions of this work, the results of which are summarised in Section 5.5. From our analysis it also turns out that some networks, which were classically thought to be equivalent through a Cauer-Foster transformation, are in fact not equivalent (see Section 6.1 for more detail).

Figures 5.1 and 5.2 in Section 5.1 show the subfamilies and their internal structure consisting of orbits and equivalence classes. These figures summarise the principal structure of the catalogue that has been identified. In abstract terms, the 24 subfamilies represent a partition of the catalogue into the "finest common coarsening" of two partitions generated by (i) orbits of the group action, (ii) equivalence classes due to network equivalence. This may be viewed as a main theorem of this work whose proof relies on identifying the realisability sets for every subfamily and showing that they are pairwise distinct (Theorem 6.4).

The chapter continues with Sections 5.2, 5.3 and 5.4 which show the mapping and inverse mapping between impedance coefficients and circuit parameters for one, two and three-element networks, four-element networks and five-element networks, respectively. A characterisation of the realisability set for one representative of each subfamily is derived, in terms of necessary and sufficient conditions. Such conditions are summarised in Section 5.5 for all 62 equivalence classes in the catalogue. We note that knowing the realisability set $\mathcal{S}_n$ for a network in a given equivalence class, it can also be easily determined for all other equivalence classes in the same subfamily, by an appropriate transformation of the conditions. More specifically, it is easily shown that the frequency inversion `s` transformation corresponds to replacing (A, B, C, D, E, F) in the realisability conditions by (C, B, A, F, E, D), while the `p` transformation corresponds to replacing (A, B, C, D, E, F) by (F, E, D, C, B, A) (the transformations in terms of the canonical form coefficients U, V and W have already been given in Section 4.3.2). These transformations greatly facilitated the derivation of the realisability conditions for all the networks in the catalogue, by allowing us to study a much smaller subset of networks. Finally, in Section 5.6, a graphical representation of the realisability region is provided for one equivalence class in each of the five-element subfamilies.

5.1 Catalogue subfamily structure with orbits and equivalences

A diagrammatic representation of the subfamilies, orbits and equivalence classes of the catalogue is shown in Figures 5.1 and 5.2. Network equivalences are represented through dashed arrows and define the equivalence classes shaded in grey (with one-network equivalence classes not shaded). Equivalence classes are identified by a superscript number (e.g. the two equivalence classes of subfamily $\mathrm{V_G}$ are $\mathrm{V_G^1}$ and $\mathrm{V_G^2}$). Frequency inversion (i.e. `s`) and the `p` transformation are indicated through arrows, while duality (i.e. `d`) and identity (i.e. `e`) are not shown. Appendix D shows the Ladenheim networks arranged corresponding to the structure of Figures 5.1 and 5.2.

The representative network for each subfamily is shown in Figure 5.3 and corresponds in most cases to the network in the upper-left position for each subfamily in the diagrammatic representations of Figures 5.1 and 5.2.

Table 5.1 shows the number of equivalence classes, orbits and networks in all 24 subfamilies, while in Table 5.2 the subfamilies are classified according to the graph topology of the networks they comprise.

	Subfamily	# Eq. classes	# Orbits	# Networks
1-element networks	I_A	1	1	1
	I_B	2	1	2
2-element networks	II_A	4	1	4
	II_B	2	1	2
3-element networks	III_A	2	1	4
	III_B	2	1	4
	III_C	4	1	4
	III_D	2	1	2
	III_E	2	1	2
4-element networks	IV_A	4	3	12
	IV_B	4	1	4
	IV_C	4	2	8
	IV_D	2	2	4
	IV_E	4	1	4
	IV_F	2	1	2
5-element networks	V_A	2	3	12
	V_B	4	2	8
	V_C	2	1	2
	V_D	2	1	2
	V_E	4	3	12
	V_F	2	2	6
	V_G	2	2	4
	V_H	2	1	2
	V_I	1	1	1
Total	24	62	35	108

Table 5.1: Number of equivalence classes, orbits and networks in each subfamily.

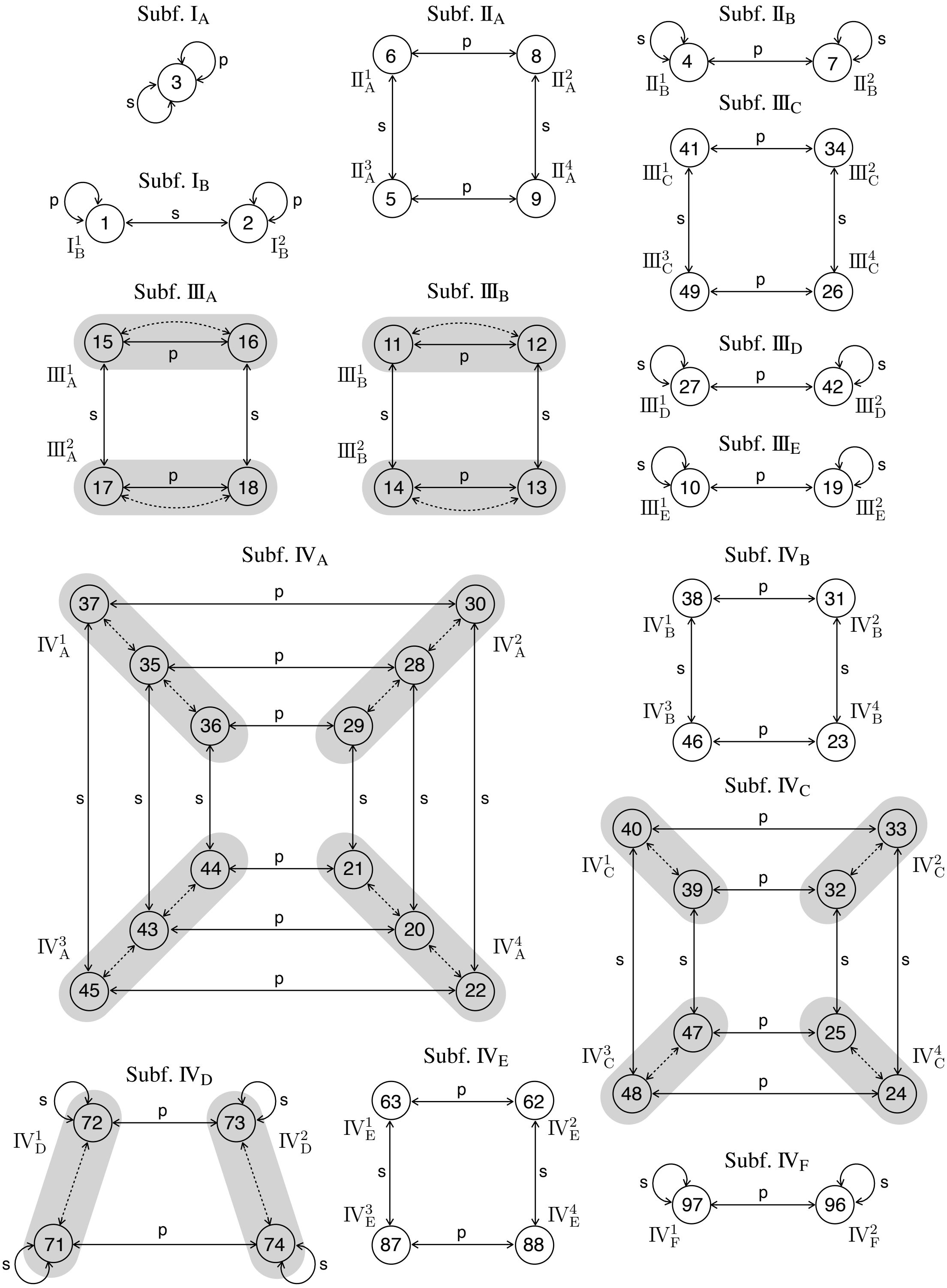


Figure 5.1: One-, two-, three- and four-element subfamilies, orbits and equivalence classes. All equivalences (dashed arrows) are the Zobel transformation defined in Section 4.4. One-network equivalence classes are not shaded.

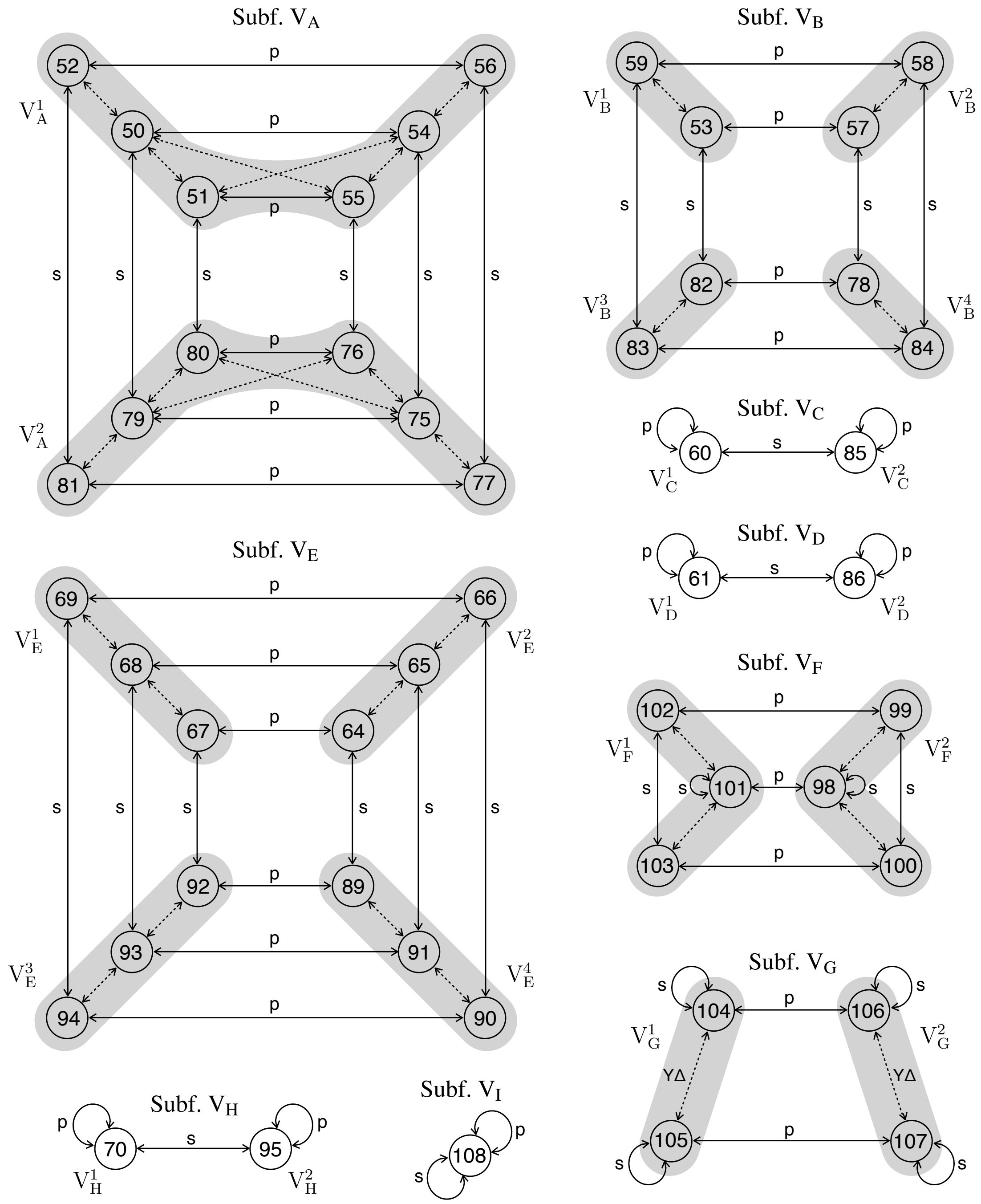


Figure 5.2: Five-element subfamilies, orbits and equivalence classes. Unless indicated otherwise, all equivalences (dashed arrows) are the Zobel transformation defined in Section 4.4. One-network equivalence classes are not shaded.

Subfamily	Network type
1, 2, 3-element subfamilies	SSP
$\mathrm{IV_A}$, $\mathrm{IV_C}$, $\mathrm{IV_D}$, $\mathrm{IV_E}$	SSP
$\mathrm{IV_B}$, $\mathrm{IV_F}$	SP
$\mathrm{V_A}$, $\mathrm{V_E}$	SSP
$\mathrm{V_B}$, $\mathrm{V_F}$	SP
$\mathrm{V_C}$, $\mathrm{V_D}$, $\mathrm{V_H}$, $\mathrm{V_I}$	Bridge
$\mathrm{V_G}$	SP / Bridge

Table 5.2: Classification of the subfamilies according to the type of networks they contain. Simple series-parallel networks are denoted by SSP and series-parallel networks by SP (see definitions in Section 2.1).

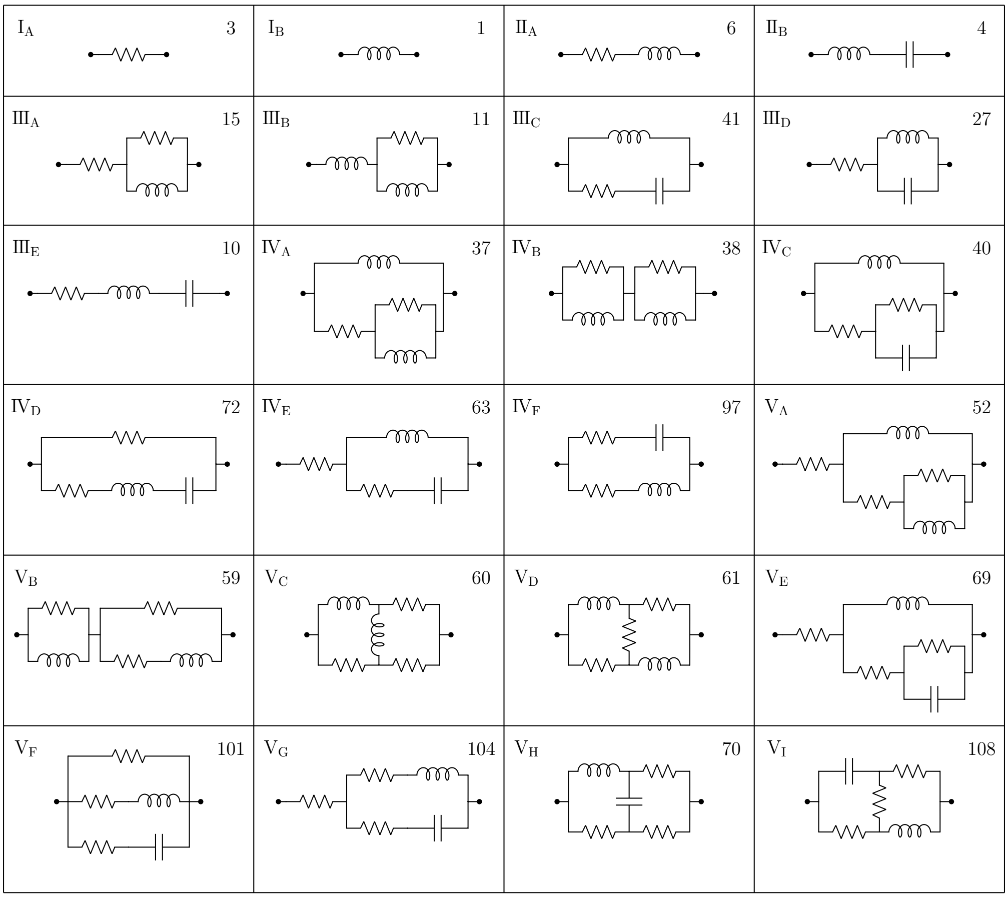


Figure 5.3: Representative networks for each of the 24 subfamilies in the catalogue, with subfamily name (top-left) and network number (top-right) indicated.

5.1.1 Minimal description of the Ladenheim catalogue

An important consequence of this new-found underlying structure is a minimal way to construct all the 108 networks of the Ladenheim catalogue, starting from a much smaller subset of 25 networks.

For 23 of the 24 subfamilies, the set of networks can be deduced uniquely from the information contained in Figures 5.1, 5.2 and 5.3. In fact, by applying the Y-Δ and Zobel transformations as illustrated in Figures 5.1 and 5.2, as well as the `s` and `p` network transformations, the remaining networks in each subfamily can be identified uniquely starting from the representative networks of Figure 5.3. The exceptional subfamily is $\mathrm{V_A}$. For the latter, a Zobel transformation can be applied to network #52 to uniquely obtain network #50. On network #50, however, a Zobel transformation can be further applied in two different ways to obtain networks #51 and #55 (see the corresponding networks in Appendix D). It is therefore necessary to provide the 24 networks shown in Figure 5.3 along with one of networks #51, #55, #80 or #76 to be able to uniquely derive the Ladenheim catalogue from the structure shown in Figures 5.1 and 5.2.

5.2 One-, two- and three-element networks

The derivation of the realisability set for a given network begins with the formulae for the impedance coefficients in terms of the network parameters, from which the inverse mapping may be studied. Table 5.3 contains expressions for coefficients A, B, ..., F for the representative network in each subfamily of one-, two- and three-element networks. Such expressions can be easily found by computing the impedance of each network. Table 5.4 shows the result of the inverse problem, i.e. expressions for the inductances, capacitances and resistances in each network are shown in terms of coefficients A, B, ..., F. The inverse problem is quite straightforward for networks with at most three elements, hence the elimination procedure which leads to the expressions in Table 5.4 is not shown.

Realisability conditions are summarised in Section 5.5 for all equivalence classes of one-, two- and three-element networks. These conditions are easily deduced for the representative equivalence classes from the requirement of positivity of the expressions in Tables 5.3 and 5.4. Realisability conditions for all other equivalence classes in the subfamilies can be easily found by an appropriate transformation of the polynomials appearing in the conditions, as described at the beginning of this chapter.

Network No.	Equiv. class	A	B	C	D	E	F
3	I^1_A	0	0	R_1	0	0	1
1	I^1_B	0	L_1	0	0	0	1
6	II^1_A	0	L_1	R_1	0	0	1
4	II^1_B	L_1C_1	0	1	0	C_1	0
15	$\mathrm{III}^1_\mathrm{A}$	0	$L_1(R_1+R_2)$	R_1R_2	0	L_1	R_1
11	$\mathrm{III}^1_\mathrm{B}$	L_1L_2	$R_1(L_1+L_2)$	0	0	L_1	R_1
41	$\mathrm{III}^1_\mathrm{C}$	$R_1L_1C_1$	L_1	0	L_1C_1	R_1C_1	1
27	$\mathrm{III}^1_\mathrm{D}$	$R_1L_1C_1$	L_1	R_1	L_1C_1	0	1
10	$\mathrm{III}^1_\mathrm{E}$	L_1C_1	R_1C_1	1	0	C_1	0

Table 5.3: Expressions for A, B, ..., F in terms of resistances, inductances and capacitances for one-, two- and three-element networks.

Network No.	Equiv. class	R_1	R_2	L_1	L_2	C_1
3	I^1_A	C/F	–	–	–	–
1	I^1_B	–	–	B/F	–	–
6	II^1_A	C/F	–	B/F	–	–
4	II^1_B	–	–	A/E	–	E/C
15	$\mathrm{III}^1_\mathrm{A}$	$\frac{BF-CE}{EF}$	C/F	$\frac{BF-CE}{F^2}$	–	–
11	$\mathrm{III}^1_\mathrm{B}$	$\frac{-(AF-BE)}{E^2}$	–	$\frac{-(AF-BE)}{EF}$	A/E	–
41	$\mathrm{III}^1_\mathrm{C}$	A/D	–	B/F	–	D/B
27	$\mathrm{III}^1_\mathrm{D}$	C/F	–	B/F	–	D/B
10	$\mathrm{III}^1_\mathrm{E}$	B/E	–	A/E	–	E/C

Table 5.4: Expressions for inductances, capacitances and resistances in terms of coefficients A, B, ..., F for one-, two- and three-element networks.

5.3 Four-element networks

The analysis of four-element networks is slightly more complicated than the case with three or fewer elements. Table 5.5 shows the expressions for coefficients A, B, $\ldots$, F for the representative network in each subfamily of four-element networks, which again can be easily found by computing the impedance of the network. Table 5.6 contains expressions for inductances, capacitances and resistances for each network in terms of coefficients A, B, $\ldots$, F. For all subfamilies other than $\mathrm{IV_B}$ the derivation of such expressions in explicit form is quite straightforward.

The following remarks should be made regarding the derivation of the realisability conditions given in Table 5.6 for four-element networks:

- For network #37 in equivalence class $\mathrm{IV_A^1}$ and for network #40 in equivalence class $\mathrm{IV_C^1}$ it is easily seen that

$$K|_{C=0} = F\big[A(AF - BE) + B^2 D\,\big]\,, \tag{5.1}$$

  where the expression for K can be found in Table 5.9. From (5.1), $K < 0$ implies $AF - BE < 0$, hence the condition on $AF - BE$ can be omitted from the realisability conditions for equivalence class $\mathrm{IV_A^1}$ if the condition on K is included.

- For network #72 in equivalence class $\mathrm{IV_D^1}$, condition $AF - CD = 0$ follows from the expressions for A, F, C and D in Table 5.5 and implies that $R_2 = A/D = C/F$.

- For network #63 in equivalence class $\mathrm{IV_E^1}$, the condition $\lambda_1 = 0$ follows from the expressions for A, B, $\ldots$, F in Table 5.5.

- For network #97 in equivalence class $\mathrm{IV_F^1}$, τ_1 is defined in Table 5.9, and condition $\tau_1 = 0$ once again follows from the expressions for A, B, $\ldots$, F in Table 5.5.

The realisability conditions summarised in Section 5.5 follow. For subfamily $\mathrm{IV_B}$ a more thorough analysis of the realisability conditions is needed, and Theorem A.1 provides a full characterisation of the realisability set.

Network	A	B	C	D	E	F
#37 (IV_A^1)	$L_1L_2(R_1+R_2)$	$L_2R_1R_2$	0	L_1L_2	$L_1(R_1+R_2)+L_2R_1$	R_1R_2
#38 (IV_B^1)	$L_1L_2(R_1+R_2)$	$R_1R_2(L_1+L_2)$	0	L_1L_2	$L_1R_2+L_2R_1$	R_1R_2
#40 (IV_C^1)	$R_1R_2L_1C_1$	$L_1(R_1+R_2)$	0	$R_1L_1C_1$	$L_1+R_1R_2C_1$	R_1+R_2
#72 (IV_D^1)	$R_2L_1C_1$	$R_1R_2C_1$	R_2	L_1C_1	$(R_1+R_2)C_1$	1
#63 (IV_E^1)	$L_1C_1(R_1+R_2)$	$L_1+R_1R_2C_1$	R_2	L_1C_1	R_1C_1	1
#97 (IV_F^1)	$R_1L_1C_1$	$R_1R_2C_1+L_1$	R_2	L_1C_1	$(R_1+R_2)C_1$	1

Table 5.5: Expressions for A, B, ..., F in terms of resistances, inductances and capacitances for four-element networks.

Network	R_1	R_2	L_1	L_2	C_1
#37 (IV_A^1)	$\frac{K}{DF(AF-BE)}$	$\frac{-B^2}{AF-BE}$	$\frac{-B\,K}{F(AF-BE)^2}$	B/F	–
#38 (IV_B^1)	See Theorem A.1				
#40 (IV_C^1)	$\frac{-K}{DF(AF-BE)}$	A/D	B/F	–	$\frac{BD^2F}{K}$
#72 (IV_D^1)	$\frac{-BC}{BF-CE}$	A/D	$\frac{-DC^2}{F(BF-CE)}$	–	$\frac{-(BF-CE)}{C^2}$
#63 (IV_E^1)	$\frac{AF-CD}{DF}$	C/F	$\frac{AF-CD}{EF}$	–	$\frac{DE}{AF-CD}$
#97 (IV_F^1)	A/D	C/F	$\frac{AF+CD}{EF}$	–	$\frac{DE}{AF+CD}$

Table 5.6: Expressions for inductances, capacitances and resistances in terms of coefficients A, B, ..., F for four-element networks.

5.4 Five-element networks

We finally consider the most interesting case of five-element networks. Table 5.8 shows the expressions for coefficients A, B, ..., F for the representative network in each of the nine five-element subfamilies, which can be found by computing the impedance of the network (see e.g. [69, Section 7.2]). Table 5.7 contains expressions for inductances, capacitances and resistances for subfamilies V_A and V_E; for all the other subfamilies,

Network	R_1	R_2	R_3	L_1	L_2	C_1
#52 ($\mathrm{V}_{\mathrm{A}}^1$)	$\frac{-K}{D\,\lambda_1}$	$\frac{(BF-CE)^2}{F\,\lambda_1}$	$\frac{C}{F}$	$\frac{-K(BF-CE)}{\lambda_1^2}$	$\frac{BF-CE}{F^2}$	–
#59 ($\mathrm{V}_{\mathrm{B}}^1$)	See Theorem A.3					
#60 ($\mathrm{V}_{\mathrm{C}}^1$)	See Theorem A.4					
#61 ($\mathrm{V}_{\mathrm{D}}^1$)	See Theorem A.5					
#69 ($\mathrm{V}_{\mathrm{E}}^1$)	$\frac{K}{D\,\lambda_1}$	$\frac{AF-CD}{DF}$	$\frac{C}{F}$	$\frac{BF-CE}{F^2}$	–	$\frac{D^2(BF-CE)}{K}$
#101 ($\mathrm{V}_{\mathrm{F}}^1$)	See Theorem A.7					
#104 ($\mathrm{V}_{\mathrm{G}}^1$)	See Theorem A.8					
#70 ($\mathrm{V}_{\mathrm{H}}^1$)	See Theorem A.9					
#108 ($\mathrm{V}_{\mathrm{I}}^1$)	See Theorem A.10					

Table 5.7: Expressions for inductances, capacitances and resistances in terms of coefficients A, B, ..., F for five-element networks.

expressions for the network elements can be found in the theorems referenced in the table.

The characterisation of the realisability set $\mathcal{S}_n$ for each representative network in terms of necessary and sufficient conditions has been derived in this programme of work in Theorems A.2–A.10.

We note that realisability conditions for the regular bridge networks in the Ladenheim catalogue have also been independently derived in Chen *et al.* 2016 [16], while the realisability conditions for subfamilies V_{A}, V_{E} and V_{H} are already know from [43]. In [16], the multiplicity of solutions is not taken into account, and some conditions are expressed in a different form. Some considerations on the smallest generating set are made in Section 6.3 in relation to results found in [16].

Table 5.9 lists the expressions for all the polynomials which appear in the realisability conditions summarised in Table 5.10, both in terms of the biquadratic impedance coefficients A, B, ..., F of (4.1) and in terms of the coefficients U, V and W of the canonical form (3.12). The notation for the polynomials in canonical form is described in Section 3.1.2.

Network	A	B	C
#52 (V^1_A)	$L_1L_2(R_1+R_2+R_3)$	$L_1R_3(R_1+R_2)+L_2R_1(R_2+R_3)$	$R_1R_2R_3$
#59 (V^1_B)	$L_1L_2(R_2+R_3)$	$L_1R_2R_3+L_2(R_1R_2+R_1R_3+R_2R_3)$	$R_1R_2R_3$
#60 (V^1_C)	$L_1L_2(R_1+R_3)$	$L_2(R_1R_2+R_1R_3+R_2R_3)+R_2L_1(R_1+R_3)$	$R_1R_2R_3$
#61 (V^1_D)	$L_1L_2(R_1+R_2+R_3)$	$L_1R_1(R_2+R_3)+L_2R_2(R_1+R_3)$	$R_1R_2R_3$
#69 (V^1_E)	$R_1L_1C_1(R_2+R_3)$	$L_1(R_1+R_2+R_3)+R_1R_2R_3C_1$	$R_3(R_1+R_2)$
#101 (V^1_F)	$L_1C_1R_1R_3$	$R_3(R_1R_2C_1+L_1)$	R_2R_3
#104 (V^1_G)	$L_1C_1(R_1+R_3)$	$L_1+C_1(R_1R_2+R_1R_3+R_2R_3)$	R_2+R_3
#70 (V^1_H)	$L_1C_1(R_1R_2+R_1R_3+R_2R_3)$	$R_1R_2R_3C_1+L_1(R_1+R_3)$	$R_2(R_1+R_3)$
#108 (V^1_H)	$L_1C_1R_3(R_1+R_2)$	$C_1R_1R_2R_3+L_1(R_1+R_2+R_3)$	$R_1(R_2+R_3)$

Network	D	E	F
#52 (V^1_A)	L_1L_2	$L_2R_1+L_1(R_1+R_2)$	R_1R_2
#59 (V^1_B)	L_1L_2	$L_1R_2+L_2(R_1+R_3)$	$R_2(R_1+R_3)$
#60 (V^1_C)	L_1L_2	$L_2(R_1+R_2)+L_1(R_1+R_2+R_3)$	$R_3(R_1+R_2)$
#61 (V^1_D)	L_1L_2	$L_1(R_2+R_3)+L_2(R_1+R_3)$	$R_1R_2+R_1R_3+R_2R_3$
#69 (V^1_E)	$R_1L_1C_1$	$L_1+R_1R_2C_1$	R_1+R_2
#101 (V^1_F)	$L_1C_1(R_1+R_3)$	$C_1(R_1R_2+R_1R_3+R_2R_3)+L_1$	R_2+R_3
#104 (V^1_G)	L_1C_1	$C_1(R_1+R_2)$	1
#70 (V^1_H)	$L_1C_1(R_2+R_3)$	$C_1R_1(R_2+R_3)+L_1$	$R_1+R_2+R_3$
#108 (V^1_H)	$L_1C_1(R_1+R_2)$	$C_1(R_1R_2+R_1R_3+R_2R_3)+L_1$	R_2+R_3

Table 5.8: Expressions for A, B, …, F in terms of resistances, inductances and capacitances for five-element networks.

Special polynomials in terms of A, B, ..., F	Reduced expressions in terms of U, V, W
$K = (AF-CD)^2-(AE-BD)(BF-CE)$	$K_c = 4(U^2+V^2) - 4UV(W^{-1}+W)$ $+(W^{-1}-W)^2$
$\sigma = BE - (\sqrt{AF} - \sqrt{CD})^2$	$\sigma_c = 4UV + 2 - (W^{-1}+W)$
$\lambda_1 = E(BF-CE) - F(AF-CD)$	$\lambda_c = 4UV - 4V^2W - (W^{-1}-W)$
$\lambda_2 = B(AE-BD) - A(AF-CD)$	$\lambda_c^\dagger = 4UV - 4U^2W - (W^{-1}-W)$
$\lambda_3 = D(AF-CD) - E(AE-BD)$	$\lambda_c^* = 4UV - 4V^2W^{-1} - (W - W^{-1})$
$\lambda_4 = C(AF-CD) - B(BF-CE)$	$\lambda_c^{*\dagger} = 4UV - 4U^2W^{-1} - (W - W^{-1})$
$\eta = (AF+CD)^2-(AE-BD)(BF-CE)$	$\eta_c = 4(U^2+V^2) - 4UV(W+W^{-1})$ $+(W^{-1}+W)^2$
$\mu_1 = K - 4CD(2AF - 2CD - BE)$	$\mu_c = 4(U^2+V^2) + 4UV(3W - W^{-1})$ $+(1-W^{-2})(9W^2-1)$
$\mu_2 = K - 4AF(2CD - 2AF - BE)$	$\mu_c^* = 4(U^2+V^2) + 4UV(3W^{-1} - W)$ $+(1-W^2)(9W^{-2}-1)$
$\tau_1 = K - DF(B^2 - 4AC)$	$\tau_c = 4V^2 - 4UV(W^{-1}+W)$ $+(W^{-1}+W)^2$
$\tau_2 = K - AC(E^2 - 4DF)$	$\tau_c^\dagger = 4U^2 - 4UV(W^{-1}+W)$ $+(W^{-1}+W)^2$
$\delta = BE - 2(AF+CD)$	$\delta_c = 4UV - 2(W^{-1}+W)$
$\zeta_1 = -E(BF-CE) + 2F(AF-CD)$	$\zeta_c = 4V(V - UW^{-1}) + 2(W^{-2}-1)$
$\zeta_2 = -B(AE-BD) + 2A(AF-CD)$	$\zeta_c^\dagger = 4U(U - VW^{-1}) + 2(W^{-2}-1)$
$\zeta_3 = E(AE-BD) - 2D(AF-CD)$	$\zeta_c^* = 4V(V-UW) + 2(W^2-1)$
$\zeta_4 = B(BF-CE) - 2C(AF-CD)$	$\zeta_c^{*\dagger} = 4U(U-VW) + 2(W^2-1)$
$\psi = (AF+CD)(K+4ACDF)$ $-2ABCDEF$	$\psi_c = 4(W^{-1}+W)(U^2+V^2) -$ $4UV(W^2+4+W^{-2}) + (W^{-1}+W)^3$
$\rho_1 = -K + 2CD(AF-CD)$	$\rho_c = -4(U^2+V^2) + 4UV(W+W^{-1})$ $-(1-W^{-2})(3W^2-1)$
$\rho_2 = -K + 2AF(CD-AF)$	$\rho_c^* = -4(U^2+V^2) + 4UV(W+W^{-1})$ $-(1-W^2)(3W^{-2}-1)$
$AF - CD$	$W^{-1} - W$
$E^2 - 4DF$	$4W^{-1}(V^2-1)$
$B^2 - 4AC$	$4W(U^2-1)$

Table 5.9: Polynomials appearing in the realisability conditions, expressed in terms of both A, B, ..., F and U, V, W. The expressions in U, V, W are obtained by replacing A, B, C, ... by 1, $2U\sqrt{W}$, W, ... (from (3.12)), except for a multiplicative positive scaling.

5.5 Summary of realisability conditions

Table 5.10 summarises the realisability conditions for all equivalence classes in the catalogue. Expressions for the symbols appearing in the conditions can be found in Table 5.9. Unless indicated otherwise, we assume A, B, $\ldots$, $F > 0$. The notation regarding the multiplicity of solutions has the following meaning:

1/2 Depending on the orbit, there can be one or two solutions.

∞ There are infinitely many solutions, since one of the network elements can take any value within a certain interval (while the other elements can be computed according to the formulae in the realisation theorems).

* When any of the quantities in the conditions is zero, there is only one solution.

In networks with one, two or three elements, the cases in which $C = F = 0$ have not been considered, since they would lead to a trivial cancellation of the frequency variable s at the numerator and denominator.

Subf.	Eq. class	Networks	Realisability conditions	# sol.
$\mathrm{I_A}$	$\mathrm{I_A^1}$	3	$A = B = D = E = 0$	1
$\mathrm{I_B}$	$\mathrm{I_B^1}$	1	$A = C = D = E = 0$	1
	$\mathrm{I_B^2}$	2	$A = B = D = F = 0$	1
$\mathrm{II_A}$	$\mathrm{II_A^1}$	6	$A = D = E = 0$	1
	$\mathrm{II_A^2}$	8	$A = C = D = 0$	1
	$\mathrm{II_A^3}$	5	$A = D = F = 0$	1
	$\mathrm{II_A^4}$	9	$A = B = D = 0$	1
$\mathrm{II_B}$	$\mathrm{II_B^1}$	4	$B = D = F = 0$	1
	$\mathrm{II_B^2}$	7	$A = C = E = 0$	1
$\mathrm{III_A}$	$\mathrm{III_A^1}$	15, 16	$A = D = 0,\ BF - CE > 0$	1
	$\mathrm{III_A^2}$	17, 18	$A = D = 0,\ BF - CE < 0$	1
$\mathrm{III_B}$	$\mathrm{III_B^1}$	11, 12	$C = D = 0,\ AF - BE < 0$	1
	$\mathrm{III_B^2}$	13, 14	$A = F = 0,\ BE - CD > 0$	1

Subf.	Eq. class	Networks	Realisability conditions	# sol.
$\mathrm{III_C}$	$\mathrm{III_C^1}$	41	$C=0,\ AF-BE=0$	1
	$\mathrm{III_C^2}$	34	$D=0,\ AF-BE=0$	1
	$\mathrm{III_C^3}$	49	$A=0,\ BE-CD=0$	1
	$\mathrm{III_C^4}$	26	$F=0,\ BE-CD=0$	1
$\mathrm{III_D}$	$\mathrm{III_D^1}$	27	$E=0,\ AF-CD=0$	1
	$\mathrm{III_D^2}$	42	$B=0,\ AF-CD=0$	1
$\mathrm{III_E}$	$\mathrm{III_E^1}$	10	$D=F=0$	1
	$\mathrm{III_E^2}$	19	$A=C=0$	1
$\mathrm{IV_A}$	$\mathrm{IV_A^1}$	35, 36, 37	$K<0,\ C=0$	1
	$\mathrm{IV_A^2}$	28, 29, 30	$K<0,\ D=0$	1
	$\mathrm{IV_A^3}$	43, 44, 45	$K<0,\ A=0$	1
	$\mathrm{IV_A^4}$	20, 21, 22	$K<0,\ F=0$	1
$\mathrm{IV_B}$	$\mathrm{IV_B^1}$	38	1. $C=0,\ K<0$ 2. $C=0,\ K=0,\ E^2-4DF=0$	1 ∞
	$\mathrm{IV_B^2}$	31	1. $D=0,\ K<0$ 2. $D=0,\ K=0,\ B^2-4AC=0$	1 ∞
	$\mathrm{IV_B^3}$	46	1. $A=0,\ K<0$ 2. $A=0,\ K=0,\ E^2-4DF=0$	1 ∞
	$\mathrm{IV_B^4}$	23	1. $F=0,\ K<0$ 2. $F=0,\ K=0,\ B^2-4AC=0$	1 ∞
$\mathrm{IV_C}$	$\mathrm{IV_C^1}$	39, 40	$K>0,\ C=0,\ AF-BE<0$	1
	$\mathrm{IV_C^2}$	32, 33	$K>0,\ D=0,\ AF-BE<0$	1
	$\mathrm{IV_C^3}$	47, 48	$K>0,\ A=0,\ BE-CD>0$	1
	$\mathrm{IV_C^4}$	24, 25	$K>0,\ F=0,\ BE-CD>0$	1
$\mathrm{IV_D}$	$\mathrm{IV_D^1}$	71, 72	$AF-CD=0,\ BF-CE<0$	1
	$\mathrm{IV_D^2}$	73, 74	$AF-CD=0,\ BF-CE>0$	1

Subf.	Eq. class	Networks	Realisability conditions	# sol.
IV_E	IV_E^1	63	$\lambda_1 = 0,\ AF - CD > 0$	1
	IV_E^2	62	$\lambda_2 = 0,\ AF - CD > 0$	1
	IV_E^3	87	$\lambda_3 = 0,\ AF - CD < 0$	1
	IV_E^4	88	$\lambda_4 = 0,\ AF - CD < 0$	1
IV_F	IV_F^1	97	$\tau_1 = 0$	1
	IV_F^2	96	$\tau_2 = 0$	1
V_A	V_A^1	50, 51, 52, 54, 55, 56	$K < 0,\ AF - CD > 0$	1
	V_A^2	75, 76, 77, 79, 80, 81	$K < 0,\ AF - CD < 0$	1
V_B	V_B^1	53, 59	1. $AF - CD > 0,\ K < 0$ 2. $AF - CD > 0,\ K = 0,\ E^2 - 4DF = 0$	1/2 ∞
	V_B^2	57, 58	1. $AF - CD > 0,\ K < 0$ 2. $AF - CD > 0,\ K = 0,\ B^2 - 4AC = 0$	1/2 ∞
	V_B^3	82, 83	1. $AF - CD < 0,\ K < 0$ 2. $AF - CD < 0,\ K = 0,\ E^2 - 4DF = 0$	1/2 ∞
	V_B^4	78, 84	1. $AF - CD < 0,\ K < 0$ 2. $AF - CD < 0,\ K = 0,\ B^2 - 4AC = 0$	1/2 ∞
V_C	V_C^1	60	$\eta \leq 0,\ AF - CD > 0$	2^*
	V_C^2	85	$\eta \leq 0,\ AF - CD < 0$	2^*
V_D	V_D^1	61	$K \leq 0,\ \mu_1 \leq 0,\ AF - CD > 0$	2^*
	V_D^2	86	$K \leq 0,\ \mu_2 \leq 0,\ AF - CD < 0$	2^*
V_E	V_E^1	67, 68 ,69	$K > 0,\ \lambda_1 > 0,\ AF - CD > 0$	1
	V_E^2	64, 65, 66	$K > 0,\ \lambda_2 > 0,\ AF - CD > 0$	1
	V_E^3	92, 93, 94	$K > 0,\ \lambda_3 > 0,\ AF - CD < 0$	1
	V_E^4	89, 90, 91	$K > 0,\ \lambda_4 > 0,\ AF - CD < 0$	1

Subf.	Eq. class	Networks	Realisability conditions	# sol.
$\mathrm{V_F}$	$\mathrm{V_F^1}$	101, 102, 103	1. $K > 0$, $\tau_1 < 0$	1
			2. $K = \tau_1 = 0$	∞
	$\mathrm{V_F^2}$	98, 99, 100	1. $K > 0$, $\tau_2 < 0$	1
			2. $K = \tau_2 = 0$	∞
$\mathrm{V_G}$	$\mathrm{V_G^1}$	104, 105	1. $AF - CD > 0$	
			a. $\tau_1 < 0$, $\lambda_1 = 0$	1
			b. $\lambda_1 > 0$, $\tau_1 = 0$, $\delta > 0$	1
			c. $\tau_1 < 0$, $\lambda_1 > 0$, $K \geq 0$, $\delta > 0$, $\zeta_1 > 0$	2*
			2. $AF - CD \geq 0$, $\tau_1\lambda_1 > 0$	1
			3. $AF - CD = 0$, $K = 0$	∞
			4. $AF - CD < 0$ and one of	
			a. $\tau_1 < 0$, $\lambda_3 = 0$	1
			b. $\lambda_3 > 0$, $\tau_1 = 0$, $\delta > 0$	1
			c. $\tau_1 < 0$, $\lambda_3 > 0$, $K \geq 0$, $\delta > 0$, $\zeta_3 > 0$	2*
			5. $AF - CD \leq 0$, $\tau_1\lambda_3 > 0$	1
$\mathrm{V_G}$	$\mathrm{V_G^2}$	106, 107	1. $AF - CD > 0$	
			a. $\tau_2 < 0$, $\lambda_2 = 0$	1
			b. $\lambda_2 > 0$, $\tau_2 = 0$, $\delta > 0$	1
			c. $\tau_2 < 0$, $\lambda_2 > 0$, $K \geq 0$, $\delta > 0$, $\zeta_2 > 0$	2*
			2. $AF - CD \geq 0$, $\tau_2\lambda_2 > 0$	1
			3. $AF - CD = 0$, $K = 0$	∞
			4. $AF - CD < 0$ and one of	
			a. $\tau_2 < 0$, $\lambda_4 = 0$	1
			b. $\lambda_4 > 0$, $\tau_2 = 0$, $\delta > 0$	1
			c. $\tau_2 < 0$, $\lambda_4 > 0$, $K \geq 0$, $\delta > 0$, $\zeta_4 > 0$	2*
			5. $AF - CD \leq 0$, $\tau_2\lambda_4 > 0$	1

Subf.	Eq. class	Networks	Realisability conditions	# sol.
V_H	V_H^1	70	$\mu_1 \geq 0$, $AF - CD > 0$ and one of	
			1. signs of λ_1, λ_2, ρ_1 not all the same	2^*
			2. $\lambda_1 = \lambda_2 = 0$, $\rho_1 = 0$	∞
	V_H^2	95	$\mu_2 \geq 0$, $AF - CD < 0$ and one of	
			1. signs of λ_3, λ_4, ρ_2 not all the same	2^*
			2. $\lambda_3 = \lambda_4 = 0$, $\rho_2 = 0$	∞
V_I	V_I	108	$K \geq 0$ and one of	
			1. $\tau_1\tau_2 < 0$	1
			2. $\tau_1 = 0$, $\tau_2 < 0$, $\psi > 0$	1
			3. $\tau_2 = 0$, $\tau_1 < 0$, $\psi > 0$	1
			4. $\tau_1 < 0$, $\tau_2 < 0$, $\psi > 0$	2^*
			5. $\tau_1 = \tau_2 = 0$, $\psi = 0$	∞

Table 5.10: Realisability conditions and multiplicity of solutions for all equivalence classes in the catalogue.

5.5.1 Realisation procedure for a biquadratic impedance

We illustrate here a possible approach for the synthesis of a candidate biquadratic impedance which makes use of the information summarised in Table 5.10. Given a p.r. impedance in the form (4.1), the first step is to verify whether it is realisable by a network in the Ladenheim catalogue. We recall that networks in the catalogue can realise all the regular biquadratics, and a subset of the non-regular biquadratics (namely the non-regular impedances realised by equivalence classes V_H^1 and V_H^2). Using the results summarised in Section 3.1.1 we can easily check whether the impedance is regular, from which these three cases follow:

1. The impedance is regular, hence realisable by one or more networks in the catalogue. By computing some of the polynomial quantities which appear in the realisability conditions in Table 5.10 (e.g. K, $AF - CD$, τ_1, τ_2 etc) it is possible to find all the equivalence classes within the catalogue that realise the given impedance, as well as the multiplicity of the solutions.

2. The impedance is non-regular, but the realisability conditions of either V_H^1 or V_H^2

hold, hence the impedance is realisable by either network #70 or network #95.

3. The impedance is non-regular and not realisable by a network in subfamily V_H. A realisation of the impedance with five elements or less is therefore not possible, and an alternative synthesis method among those described in Chapters 2 and 3 must be used, for example a possible realisation with six elements as in [45] or the Bott-Duffin method.

By way of illustration, we consider the synthesis of the following biquadratic impedance

$$Z(s) = \frac{3212.9s^2 + 99696s + 13.9226}{s^2 + 7.6735s + 52.6273}, \tag{5.2}$$

which is the mechanical impedance of a train suspension system, obtained in [84] as the result of the optimisation of a passenger comfort index over the class of all second-order positive-real impedances. The Bott-Duffin method is used in [84] to realise (5.2), which leads to the nine-element network shown in Figure 5.4 (without loss of generality we consider here only the electrical equivalent of the mechanical networks in question).

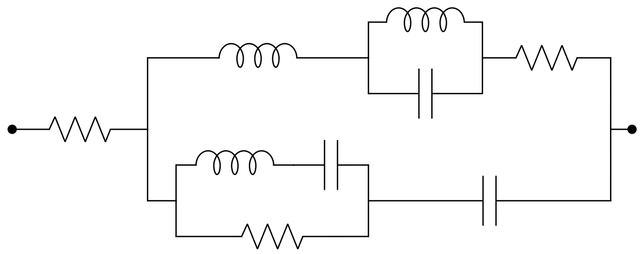


Figure 5.4: Realisation of the biquadratic impedance (5.2) using the Bott-Duffin method.

By applying instead the procedure described above, we can verify that $K > 0$, $AF - CD > 0$ and $\lambda_1 > 0$, hence Case 1 in (3.4) is satisfied and the impedance is regular. It can then be easily verified that $\lambda_2 < 0$, $\tau_1 < 0$ and $\tau_2 > 0$ (from the expressions in Table 5.9). Therefore, from Table 5.10, the impedance is realisable by equivalence classes V_E^1, V_F^1, V_H^1 and V_I. More specifically, networks #67, #68, #69, #101, #102, #103, #108 realise the impedance with multiplicity one, while for network #70 two distinct combinations of the element values exist which lead to the same impedance—that is, a total of nine solutions to the realisation problem exist within the Ladenheim catalogue. The values of the network elements in each realisation can be found in the corresponding realisation theorems for networks #69, #70, #101, #108, and can be obtained following a similar approach for the remaining networks.

We note that the eight five-element networks which were here found to realise impedance (5.2) represent the full set of networks which realise *minimally* the given impedance.

5.6 Realisability regions for five-element networks

We conclude this chapter by showing a graphical representation of the realisability region for one equivalence class in each of the nine five-element subfamilies. The realisability regions are obtained from the conditions summarised in Table 5.10 expressed in canonical form (see Table 5.9 for expressions for all the polynomials appearing in the realisability conditions in terms of U, V and W). The regions are then plotted in the (U, V)-plane for significant values of W, as shown in Figures 5.5–5.13 (hatched regions). We recall that in every figure the grey region corresponds to $\sigma_c < 0$ and represents non positive-real biquadratics, whereas the region corresponding to $\lambda_c > 0$ and/or $\lambda_c^\dagger > 0$ represents regular biquadratics for $W \leq 1$: it can be seen that all but one subfamily (i.e. subfamily $\mathrm{V_H}$) realise regular biquadratics, as pointed out in [43].

Figures 5.14 and 5.15 show the number of distinct networks which can realise impedances in a given region of the (U, V)-plane, again for significant values of W, as well as the equivalence classes such networks belong to. If a network can realise a given impedance with two distinct combinations of values of resistances, inductances and capacitances, we consider that there are two distinct solutions and the network is counted twice.

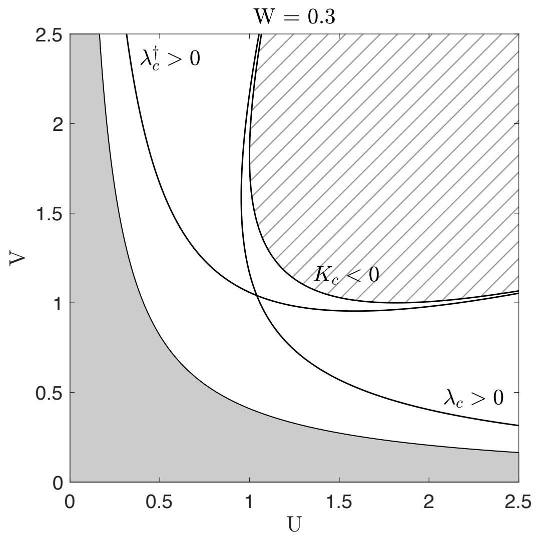


Figure 5.5: Equivalence class $\mathrm{V_A^1}$. The interior of the hatched region is realisable, with multiplicity of solution equal to one.

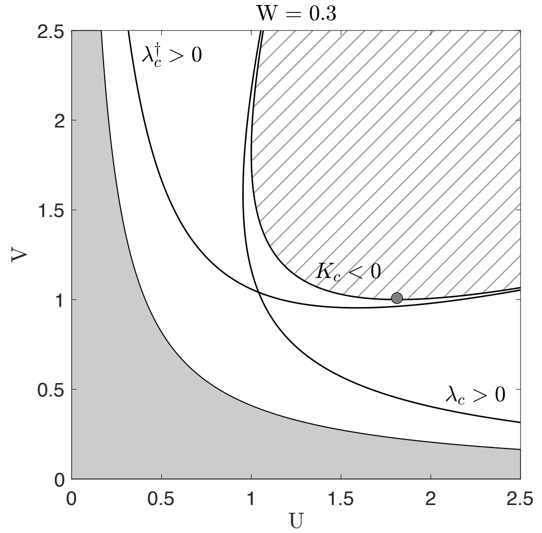


Figure 5.6: Equivalence class V^1_B. We note that the realisability region is the same as equivalence class V^1_A, with the addition of the point on the boundary of K_c in which $V = 1$ (i.e. $E^2 - 4DF = 0$), which corresponds to infinitely many solutions. In the interior of the realisability region there are always two solutions, which may coincide depending on which orbit is considered.

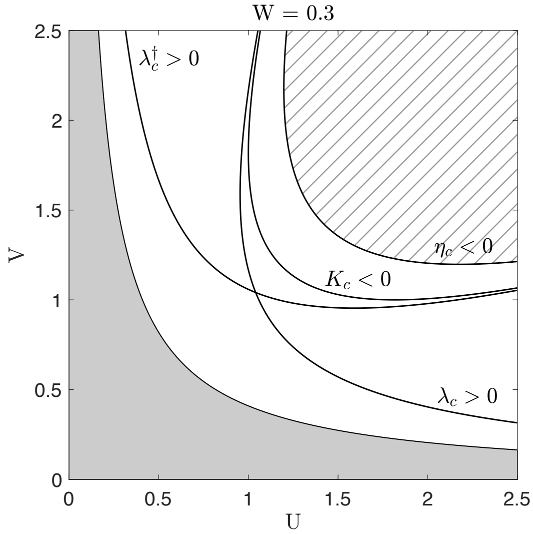


Figure 5.7: Equivalence class V^1_C. There are always two solutions for this subfamily, which coincide on the boundary of the realisability region.

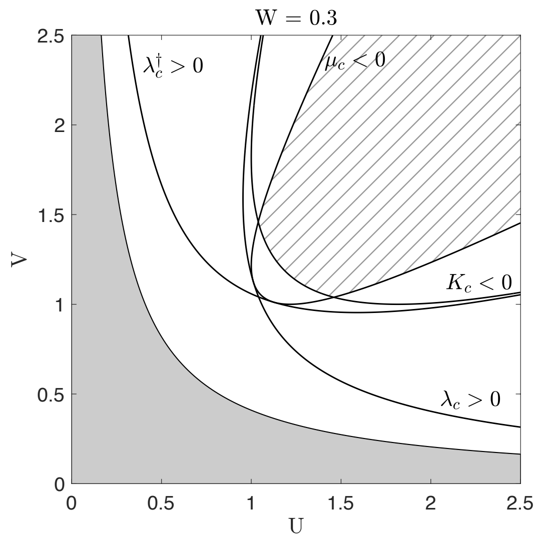


Figure 5.8: Equivalence class V^1_D. There are always two solutions for this subfamily, which coincide on the boundary of the realisability region.

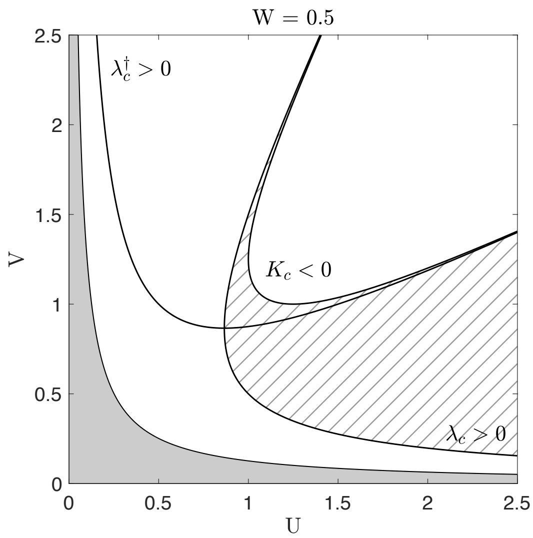


Figure 5.9: Equivalence class V^1_E. The interior of the hatched region is realisable with multiplicity of solution equal to one.

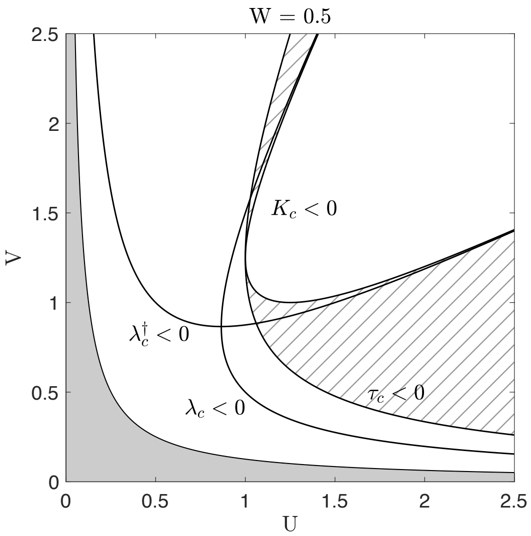


Figure 5.10: Equivalence class $\mathrm{V}^1_{\mathrm{F}}$. The realisability region is given by the interior of the hatched region, where there is one solution, with the addition of the point where $K_c = \tau_c$, which corresponds to infinitely many solutions.

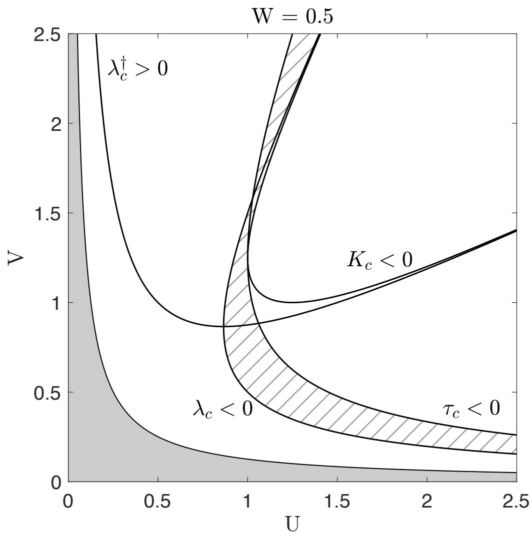


Figure 5.11: Equivalence class $\mathrm{V}^1_{\mathrm{G}}$. Only the active boundaries for the realisability region have been plotted. See Figure A.10 for more details on the boundaries and on the multiplicity of solutions.

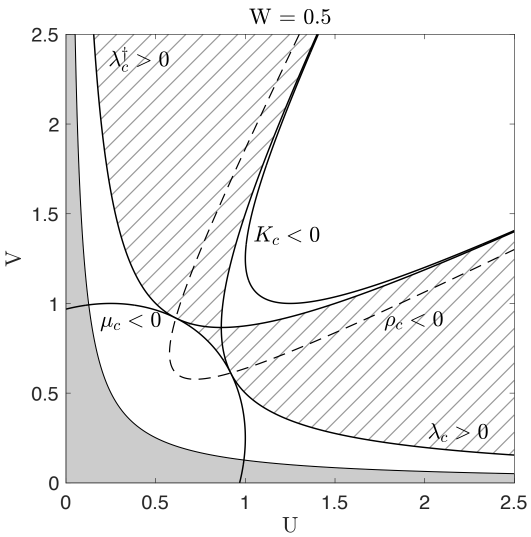


Figure 5.12: Equivalence class $\mathrm{V}^1_{\mathrm{H}}$. See Figures A.12, A.13 and Theorem A.9 for more details on the boundaries and on the multiplicity of solutions.

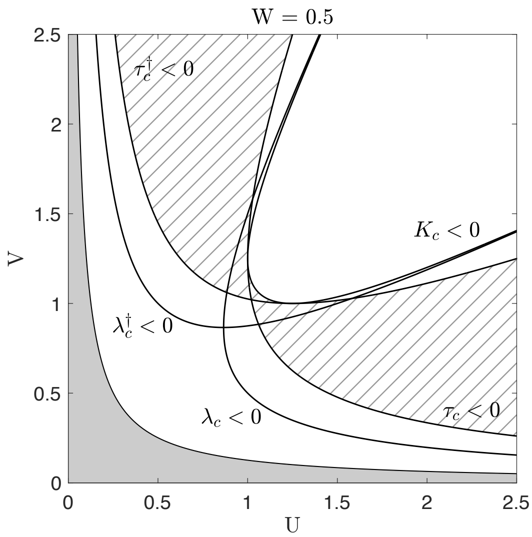


Figure 5.13: Subfamily V_{I}. Only the active boundaries for the realisability region have been plotted. See Figure A.15 for more details on the boundaries and on the multiplicity of solutions.

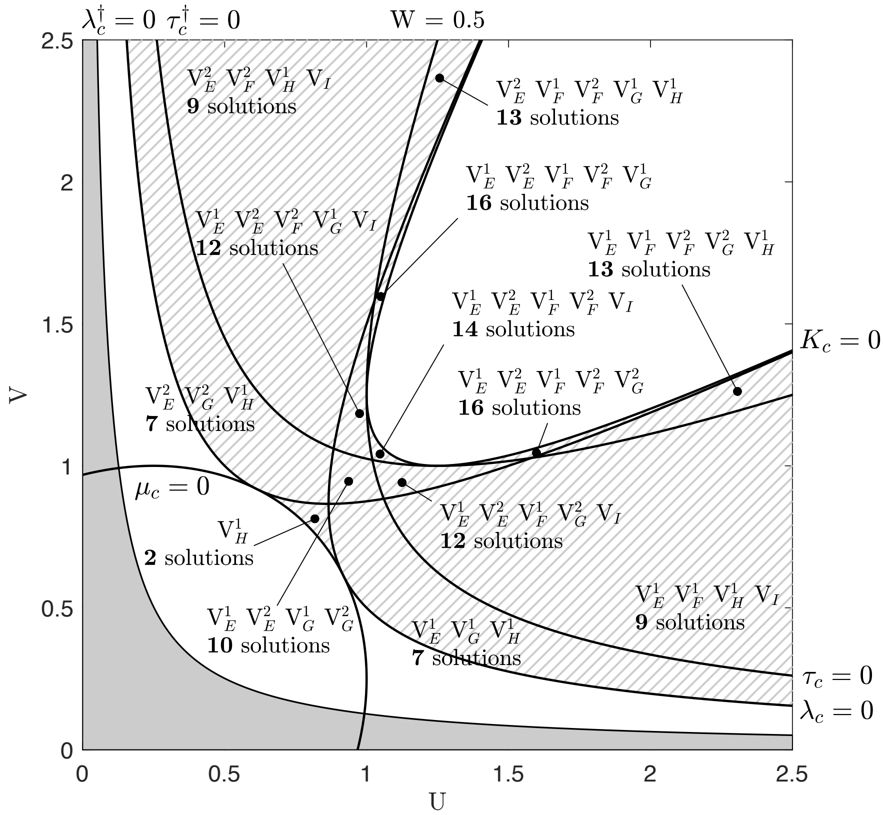


Figure 5.14: Number of distinct networks (and name of the corresponding equivalence classes) which can realise impedances in all realisable regions with $K_c > 0$, for $W = 0.5$.

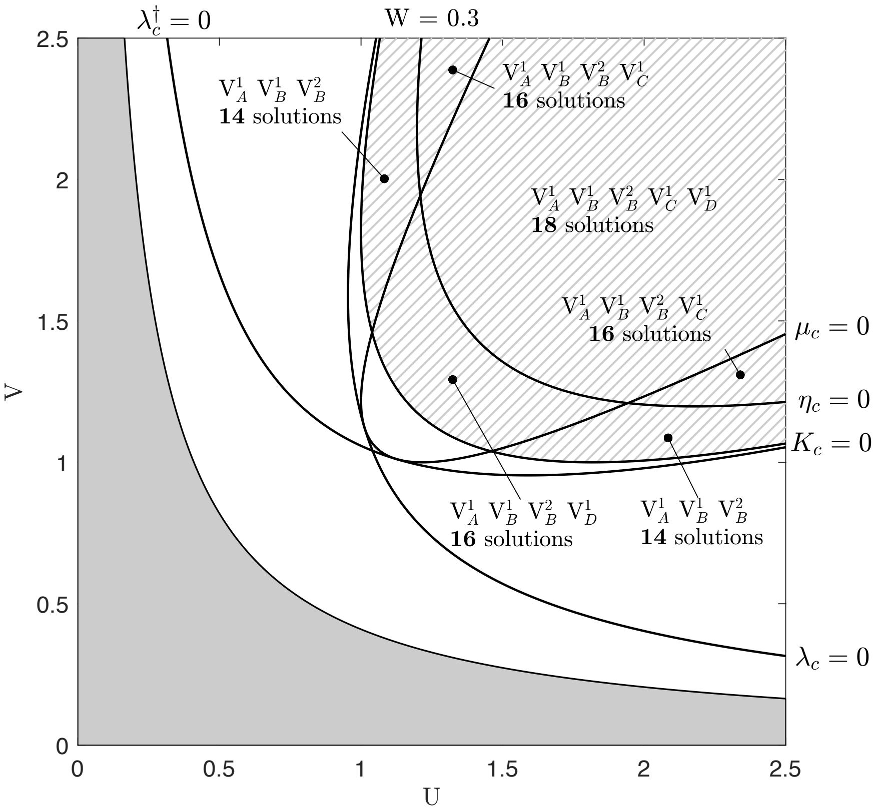


Figure 5.15: Number of distinct networks (and name of the corresponding equivalence classes) which can realise impedances in all realisable regions with $K_c < 0$, for $W = 0.3$.

5.7 Summary

In this chapter the set of 108 networks which forms the Ladenheim catalogue was partitioned into 24 subfamilies, each comprising a number of equivalence classes and orbits, thus uncovering the structure which is intrinsic to this class of networks. The realisability set for one representative network in each subfamily was derived, and the corresponding realisability regions were plotted for all five-element subfamilies. The main results which emerged from this classification are more formally stated and proven in Chapter 6.

Chapter 6

Main results and discussion on the Ladenheim catalogue

Following the realisability analysis for the 108 networks of the Ladenheim catalogue of Chapter 5 we are ready to assess the structure that has been revealed in the catalogue. Our first task is to consider in more depth the classical Cauer and Foster canonical forms in Section 6.1. We then develop formally some of the main results of our analysis of the catalogue in Section 6.2. A discussion is then given on the smallest set of networks needed to realise any regular biquadratic in Section 6.3, and some remarks are made on conjectures contained in Kalman's latest work in Section 6.4. We conclude the chapter by studying in Section 6.5 properties of invariance to duality in RLC networks, and presenting in Section 6.6 two new equivalences which were found by analysing the class of six-element networks with four resistors.

6.1 Cauer-Foster transformation

Below are the transformations between the so-called Cauer canonical form and Foster first and second canonical forms (cf. Section 2.2 and [9, 23]). By applying the Zobel transformation of Figure 4.2 to the networks on the left-hand side of Figures 6.1 and 6.2, one can define a number of additional "quasi-equivalences" (a concept that will become clearer at the end of this section). A number of these quasi-equivalences can also be found in [94, Appendix III].

The networks in Figure 6.1 are related by the following transformation:

$$a' = \frac{ac(b+d)^2}{ad^2+b^2c}, \quad b' = b+d, \quad c' = \frac{(ad-bc)^2}{ad^2+b^2c}, \quad d' = \frac{bd(b+d)(ad-bc)^2}{(ad^2+b^2c)^2}$$

$$\Big[\ a = \frac{(b'c')^2 - (a'd' + c'd' - M)^2}{4d'M}, \quad b = \frac{-b'(a'd' + c'd' - b'c' - M)}{2M},$$
$$c = \frac{-(b'c')^2 + (a'd' + c'd' + M)^2}{4d'M}, \quad d = \frac{b'(a'd' + c'd' - b'c' + M)}{2M},$$
$$\text{where} \quad M = \sqrt{(a'd' - b'c')^2 + c'd'(2a'd' + 2b'c' + c'd')}\ \Big],$$

for any real positive numbers a, b, etc, where $Z_1(s)$ and $Z_2(s)$ are arbitrary impedances.

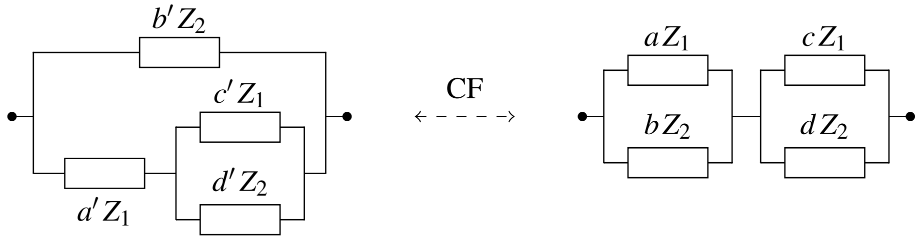


Figure 6.1: Cauer-Foster quasi-equivalence.

The networks in Figure 6.2 are related by the following transformation:

$$a' = \frac{bd}{b+d}, \quad b' = \frac{ad^2 + b^2c}{(b+d)^2}, \quad c' = \frac{ac(ad^2 + b^2c)}{(ad - bc)^2}, \quad d' = \frac{(ad^2 + b^2c)^2}{(b+d)(ad - bc)^2}$$

$$\Big[\ a = \frac{(a'b' + a'c' + b'd' + N)\, N}{d'(-a'b' - a'c' + b'd' + N)}, \quad b = \frac{2a'\, N}{-a'b' - a'c' + b'd' + N},$$
$$c = \frac{(a'b' + a'c' + b'd' - N)\, N}{d'(a'b' + a'c' - b'd' + N)}, \quad d = \frac{2a'\, N}{a'b' + a'c' - b'd' + N},$$
$$\text{where} \quad N = \sqrt{(a'b' + a'c')^2 + b'd'(2a'b' - 2a'c' + b'd')}\ \Big],$$

for any real positive numbers a, b, etc, where $Z_1(s)$ and $Z_2(s)$ are arbitrary impedances.

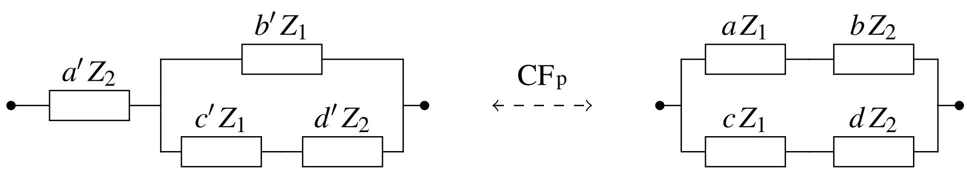


Figure 6.2: p-transformation of Cauer-Foster quasi-equivalence.

From the formulae for the transformation in Figure 6.1 it follows that if $ad - bc = 0$, with a, b, c and d finite and positive, then $c' = d' = 0$, and the network reduces to a two-block structure (i.e. the parallel connection of $a'Z_1$ and $b'Z_2$). This means that the network on the right-hand side can be reduced to a two-block structure with a suitable choice of *strictly positive* and finite coefficients a, b, c and d, while a similar reduction for the network on the left-hand side requires the coefficients c' and d' to be zero. Similar considerations hold for the transformation in Figure 6.2, which is the dual form of the transformation in Figure 6.1, in the sense of duality of the graph but not of the network elements (i.e. the p transformation).

Subfamilies $\mathrm{IV_A}$–$\mathrm{IV_B}$ and $\mathrm{V_A}$–$\mathrm{V_B}$ (which are, respectively, Cauer forms and Foster forms) are related by the transformations above, as shown in Figures 6.3 and 6.4. In the classical development of the subject (see e.g. [27], [78]), the networks of these two pairs of subfamilies were considered to be equivalent. However, the derivation of the realisability conditions for such networks (which are summarised in Section 5.5) led to the following observations:

- For the Cauer forms, the resultant of the numerator and denominator K is strictly negative, while in the Foster forms we can have $K = 0$ (i.e. a pole-zero cancellation).
- There is only one solution to the realisation problem for the Cauer forms, while there are two solutions for the Foster forms. These solutions are identical for the networks in subfamily $\mathrm{IV_B}$ and in the inner orbit of subfamily $\mathrm{V_B}$, due to the symmetry of such networks, while for the networks in the outer orbit of $\mathrm{V_B}$ the two solutions are distinct.

The two forms therefore define different realisability sets (as can be also seen from Figures 5.5 and 5.6), hence the networks are not truly equivalent according to our definition. We can define a "weaker" type of equivalence, or *quasi-equivalence*, compared to the transformations of Section 4.4 if the realisability sets of two networks differ only on a subset of lower dimension (cf. Definition 6.1). We also note that, by considering all the networks of the above-mentioned four subfamilies (which are related through the Zobel transformation and p-transformation/frequency inversion as illustrated in Figures 6.3 and 6.4), we obtain the complete set of quasi-equivalences for the catalogue.

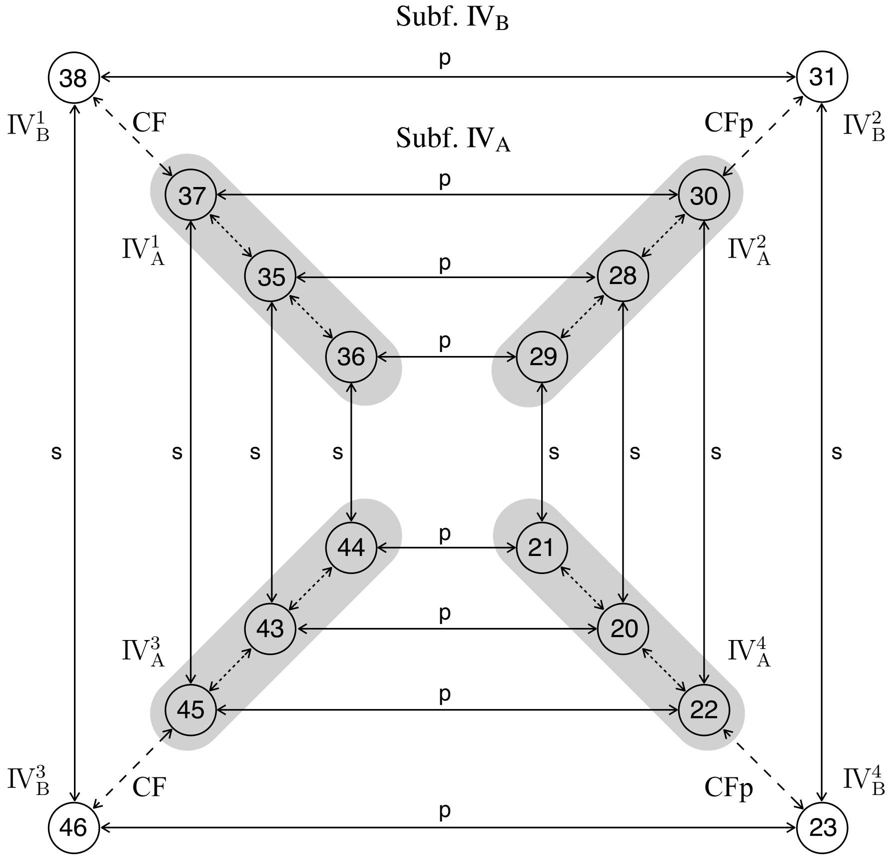


Figure 6.3: Relation between subfamilies IV_A–IV_B through Cauer-Foster transformations. CF indicates the Cauer-Foster transformation of Figure 6.1, while CFp indicates the transformation of Figure 6.2.

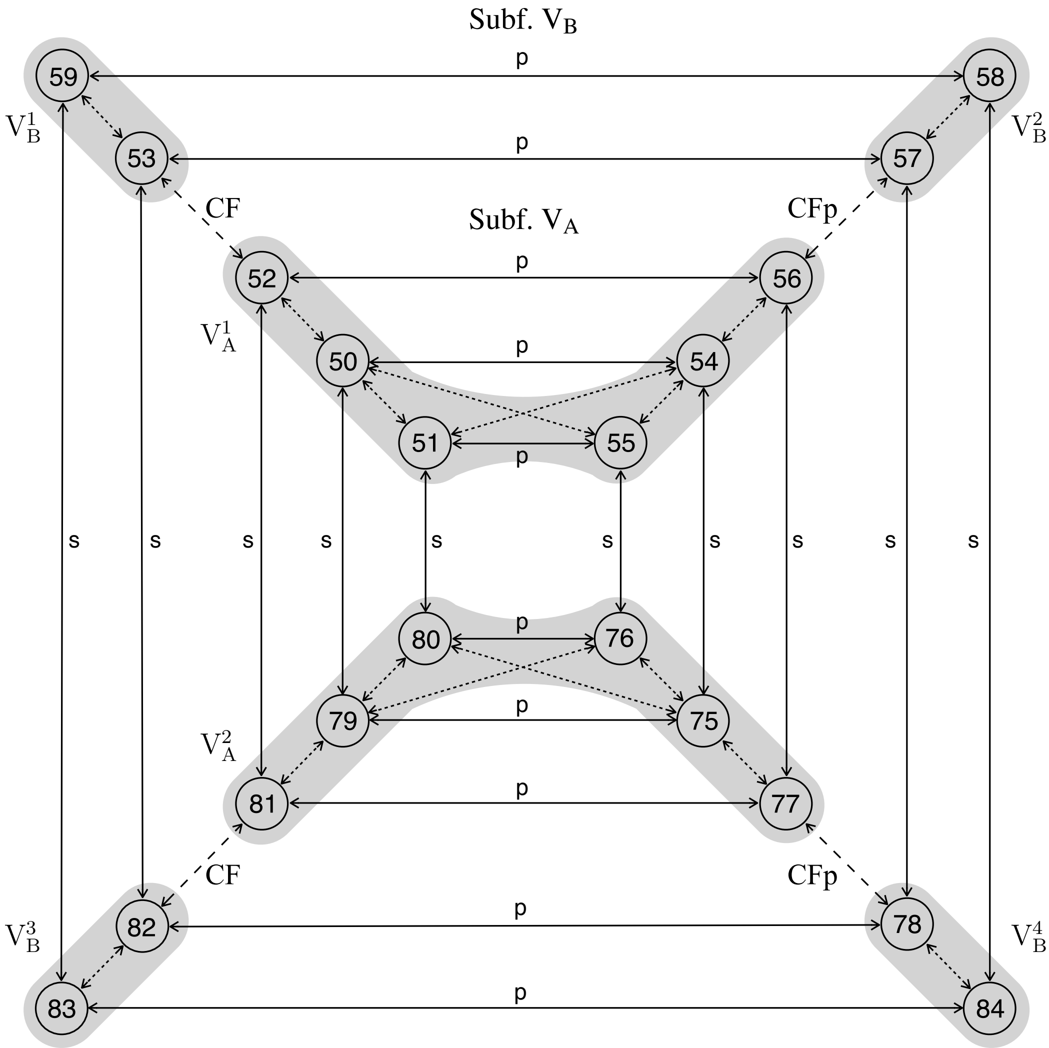


Figure 6.4: Relation between subfamilies V_A–V_B through Cauer-Foster transformations. CF indicates the Cauer-Foster transformation of Figure 6.1, while CFp indicates the transformation of Figure 6.2.

6.2 Formal results on the Ladenheim catalogue

Table 5.10 in Section 5.5 summarises the realisability conditions for the networks of the catalogue as polynomial equations and inequalities in the variables A, B, $\dots$, F. These implicitly define the semi-algebraic sets $\mathcal{S}_n$ of Section 4.3.1 as subsets of $\mathbb{R}^6_+$. We state a standard definition for the dimension of a semi-algebraic set [3] and propose a definition of a generic network (further general results on this new concept of genericity for RLC networks will be discussed in Chapter 7). We note that a similar notion of "non-redundant" system appears in [70].

Definition 6.1. The *dimension* $\dim(\mathcal{S})$ of a semi-algebraic set $\mathcal{S}$ is defined as the largest d such that there exists a one-to-one smooth map from the open cube $(-1,1)^d \subset \mathbb{R}^d$ into $\mathcal{S}$.

Definition 6.2. An RLC two-terminal network containing m elements is *generic* if $\dim(\mathcal{S}) = m+1$ where $\mathcal{S}$ is the realisability set of the network.

Theorem 6.1. *All 108 networks of the Ladenheim catalogue are generic.*

Proof. We begin with the five-element networks. For each network we observe that there exists a point $(A_0, B_0, \dots, F_0) \in \mathcal{S}_n$ and $\bar{\epsilon} > 0$ such that $(A_0+\epsilon_1, B_0+\epsilon_2, \dots, F_0+\epsilon_6) \in \mathcal{S}_n$ providing $|\epsilon_i| < \bar{\epsilon}$ for $i = 1, \dots, 6$. (All that is required is to find a point whose feasibility is determined only by polynomial strict inequalities. For example, for $n = 104$, it is sufficient to find a point satisfying 2. in the corresponding entry in Table 5.10 with $AF - CD > 0$.) For such a point we have the smooth mapping

$$(x_1, x_2, \dots, x_6) \in (-1,1)^6 \to (A_0 + x_1\bar{\epsilon}, B_0 + x_2\bar{\epsilon}, \dots, F_0 + x_6\bar{\epsilon}) \in \mathcal{S}_n$$

which is one-to-one, hence the network is generic.

For the four-element networks we will take as example network #71 for which the realisability conditions are: $AF - CD = 0, BF - CE < 0$. Consider any point $(A_0, B_0, \dots, F_0) \in \mathcal{S}_{71}$ and let $A = A_0 + \epsilon_1$, $B = B_0 + \epsilon_2, \dots, E = E_0 + \epsilon_5$, $F = CD/A$. Then there exists an $\bar{\epsilon} > 0$ such that $(A, B, \dots, F) \in \mathcal{S}_{71}$ providing $|\epsilon_i| < \bar{\epsilon}$ for $i = 1, \dots, 5$. Now define a mapping

$$(x_1, x_2, \dots, x_5) \in (-1,1)^5 \to (A_0 + x_1\bar{\epsilon}, \dots, E_0 + x_5\bar{\epsilon}, F_1) \in \mathcal{S}_{71}$$

where $F_1 = (C_0 + x_3\bar{\epsilon})(D_0 + x_4\bar{\epsilon})/(A_0 + x_1\bar{\epsilon})$. This mapping is smooth, one-to-one and onto a neighbourhood of $(A_0, B_0, \dots, F_0) \in \mathcal{S}_{71}$. We have thus found an open neighbourhood of a general point in $\mathcal{S}_{71}$ which is homeomorphic (having a bi-continuous invertible mapping) to the open cube $(-1,1)^5$, and hence the unit sphere, in $\mathbb{R}^5$. It is not possible for such a neighbourhood to be homeomorphic to a sphere of different dimension [29, Theorem 2.26] hence the network is generic. The proof for other four-element networks is similar.

For the three-element networks we will take as example network #31 for which the realisability conditions are: $C = 0, AF - BE = 0$. Consider any point

$$(A_0, B_0, 0, D_0, E_0, F_0) \in \mathcal{S}_{31}$$

and let $A = A_0 + \epsilon_1$, $B = B_0 + \epsilon_2$, $C = 0$, $D = D_0 + \epsilon_3$, $E = E_0 + \epsilon_4$, $F = BE/A$. Then there exists an $\bar{\epsilon} > 0$ such that $(A, B, \ldots, F) \in \mathcal{S}_{31}$ providing $|\epsilon_i| < \bar{\epsilon}$ for $i = 1, \ldots, 4$. Now define a mapping

$$(x_1, x_2, x_3, x_4) \in (-1,1)^4 \to (A_0 + x_1\bar{\epsilon}, B_0 + x_2\bar{\epsilon}, 0, D_0 + x_3\bar{\epsilon}, E_0 + x_4\bar{\epsilon}, F_1) \in \mathcal{S}_{31}$$

where $F_1 = (B_0 + x_2\bar{\epsilon})(E_0 + x_4\bar{\epsilon})/(A_0 + x_1\bar{\epsilon})$. This mapping is smooth, one-to-one and onto a neighbourhood of $(A_0, B_0, \ldots, F_0) \in \mathcal{S}_{31}$. We have thus found an open neighbourhood of a general point in $\mathcal{S}_{31}$ which is homeomorphic to the open cube $(-1,1)^4$, and hence the network is generic. The proof for other three-element networks is similar. For the two- and one-element networks the proof is elementary. □

Theorem 6.2. *The 108 networks of the Ladenheim catalogue form the complete set of all essentially distinct (up to 2-isomorphism), generic, two-terminal RLC networks comprising at most two reactive elements.*

Proof. The enumeration procedure to determine the 148 essentially distinct networks is as described in Section 4.2. The 40 networks that were eliminated to produce the canonical set are now easily seen to be non-generic (see also Example 7.2 in Chapter 7): eight networks with four resistors and one reactive element have a realisability set of dimension 4; four networks with four elements which can be reduced by a Zobel transformation to three-element networks have a realisability set of dimension 4; twenty series-parallel networks with five elements which can be reduced by a Zobel transformation to networks with four elements or less have a realisability set of dimension at most 5; and finally, the eight networks shown in Figure 4.1 have one of the coefficients A, C, D or F in (4.1) equal to zero and hence have a realisability set of dimension at most 5. □

Theorem 6.3. *The Ladenheim catalogue comprises 62 equivalence classes as listed in Table 5.10 according to the definition of equivalence in Section 4.3.1.*

Proof. The networks within each class in Table 5.10 have already been seen to be equivalent by a Zobel or Y-Δ transformation. It remains to show that each class is distinct, i.e. the corresponding realisability sets are distinct. From Theorem 6.1 we can immediately conclude that networks with a different number of elements are distinct, since they define realisability sets of different dimension. To complete the proof we must show that any two equivalence classes of networks with the same number of elements are distinct, i.e. there is a point in one realisability set that is not in the other. This is easily seen for equivalence classes within the same subfamily, and is trivial for the one-, two- and three-element cases. For any pair of five-element equivalence classes it is straightforward to find points in one realisability region which are not in the other through the graphical representations in Figures 5.5–5.13, in Section 5.6. Finally, for the four-element networks, we first observe that the equivalence classes of subfamilies IV_A and IV_B (for which $K < 0$) are necessarily distinct from the remaining subfamilies (for which $K > 0$, which is implied by the realisability conditions); subfamilies IV_A and IV_B are distinct from each other, since there is always an extra point in the realisability set for the latter; in IV_C one coefficient is always zero, hence it is different from IV_D, IV_E and IV_F; IV_D is necessarily distinct from IV_E (due to the condition on $AF - CD$) and IV_F (since we can have $\tau_1 = 0$ or $\tau_2 = 0$ with $AF - CD \neq 0$ in the latter); finally, by plotting the corresponding curves of the realisability region, we easily find distinct points in the realisability sets for IV_E and IV_F. □

It should be remarked that the above theorem shows that, within the Ladenheim catalogue, there are no new equivalences among the circuits that are not derived through the Zobel or Y-Δ transformations.

We now proceed to show that the 24 subfamilies of the Ladenheim catalogue comprise the *finest common coarsening* of the partitions induced by the equivalence relations of (1) group action and (2) network equivalence. This is also known as the *join* of the two equivalence relations [6, Definition 7, §18]. We will be content to state this formally in terms of the notion of transitive closure of a relation. A relation on a set X is a subset R of $X \times X$, and when $(a, b) \in R$ we write aRb and say that a and b are related by R. The union $T = R \cup S$ of two relations R and S on X is the union of the corresponding subsets of $X \times X$, hence aTb iff aRb or aSb (we note that the union of two equivalence relations is not in general an equivalence relation). Similarly, we say that S contains R and write $R \subseteq S$ iff $aRb \Rightarrow aSb, \ \forall a, b \in X$ [65, pp. 573–581]. The *transitive closure* of a relation R is the smallest transitive relation containing R.

Theorem 6.4. *The 24 subfamilies of the Ladenheim catalogue comprise the partition induced by the transitive closure of the union of the two equivalence relations given by (1) group action and (2) network equivalence.*

Proof. The two equivalence relations (which we will refer to as R_1 and R_2) generate two partitions of the catalogue X, π_1 and π_2, into 35 orbits and 62 equivalence classes, as shown in Figures 5.1 and 5.2. The 24 subfamilies also form a partition π of the catalogue, which is generated by an equivalence relation W. If we define the relation $T = R_1 \cup R_2$, we observe that within each block of π there is a finite path in T between any two networks. Therefore W satisfies the property of being the "connectivity relation" of $R_1 \cup R_2$ [65, p. 600], which is the same as the transitive closure of $R_1 \cup R_2$ [65, Section 9.4, Theorem 2]. □

6.3 Smallest generating set of the catalogue

In [43] it was shown that only two networks (one from each of the two equivalence classes of subfamily V_A) are needed to realise any positive-real biquadratic impedance with resultant $K \leq 0$, while four networks (one from each equivalence class of subfamily V_E) are needed to realise all regular biquadratics with $K \geq 0$. These six networks, taken together with the only two networks of the catalogue which can realise non-regular impedances (i.e. the networks in subfamily V_H), represent a *generating set* for the Ladenheim catalogue: any impedance which can be realised by a network in the catalogue can also be realised by one of these eight networks. It should be pointed out that the case $K = 0$ involves some element values being taken to be zero or infinity. If one maintains the condition that all element values are finite and non-zero, then the $K = 0$ cases should be covered by an appropriate set of simpler networks with fewer elements.

In [16, Corollary 1] it is shown that the number of networks required to realise all regular biquadratics with $K > 0$ can be reduced by one compared to [43] if one considers subfamily V_I and one network from each of the two equivalence classes of subfamily V_G (i.e. three networks in total). Hence, the generating set can be reduced by one compared to [43], at the expense of covering the following special cases with four-element networks, since biquadratics which satisfy the following conditions cannot be realised by networks in subfamilies V_G and V_I:

1. either λ_1 or λ_2 negative and one of τ_1 or τ_2 zero, (6.1)
2. either λ_3 or λ_4 negative and one of τ_1 or τ_2 zero. (6.2)

(See Table 5.9 for the expressions for the polynomials appearing in the conditions). The conditions in (6.1) can be deduced by combining the realisability regions of equivalence classes V_G^1, V_G^2 and V_I (plotted in Figures 5.11 and 5.13 for $AF-CD > 0$), and similarly for (6.2) (when $AF - CD < 0$). We note that (6.1) and (6.2) are boundary cases for the individual networks but end up being in the interior of the region which is realisable by the whole generating set, as one can see from Figure 5.14. As in [43] certain other boundary cases such as where $K = 0$ also require some element values to be zero or infinity, and these also would need to be covered by simpler networks if one maintains the condition that all element values are finite and non-zero.

We remark that the new generating set identified in [16], despite having one fewer network, is not particularly useful in practice, since it introduces more bridge networks and series-parallel networks, as opposed to the generating set in [43] of mostly simple series-parallel networks which are easier to realise.

6.4 Remarks on Kalman's 2011 Berkeley seminar

In this section we will review some of the concepts found in the notes on a talk given by Kalman in Berkeley (26 October, 2011) on electrical network synthesis [53] and draw a connection with the results obtained in our analysis of the Ladenheim catalogue.

The following definitions can be found in [53]. Given a network $\Gamma \in X$ defined by an undirected connected graph G_Γ, the impedance of the network is expressed as

$$Z_\Gamma = \frac{a_\Gamma(s)}{b_\Gamma(s)}\,,$$

where $a_\Gamma(s)$ and $b_\Gamma(s)$ are relatively prime polynomials in s, with degree α_Γ and β_Γ, respectively. The following polynomial map is defined, which takes the network parameters to the impedance Z_Γ:

$$\psi_\Gamma : \mathbb{R}^r \to \mathbb{P}^{\alpha_\Gamma+\beta_\Gamma+1}\,.$$

The domain of the map ψ_Γ is the space of parameters for the network (where r is the number of network elements), while the codomain is the projective space $\mathbb{P}^{\alpha_\Gamma+\beta_\Gamma+1}$ derived from the coefficients of the numerator $a_\Gamma(s)$ and denominator $b_\Gamma(s)$. Abstractly, the two problems of analysis and synthesis/realisation can be defined through the map ψ_Γ and the inverse map ψ_Γ^{-1}.

In [53], a network Γ is defined to be *minimal* (or *generic*) if the map ψ_Γ is finite-to-one. We have seen that, for networks in the catalogue, when the map ψ_Γ is finite-to-one it can be either one-to-one or two-to-one. In the latter case, there are two distinct solutions to the synthesis problem, i.e. given a network $\Gamma \in X$ one can find two distinct combinations of edge weights (network parameters) which lead to the same impedance. Also, given a network Γ, the multiplicity of solutions can vary with the image point, from two to one or vice versa. This is not unexpected, given the conditions of positivity and realness of the solutions.

We note that this notion of generic network is different from the one we proposed in Definition 6.2. It is interesting to point out that at the image points corresponding to the resultant $K = 0$ and/or one (or both) discriminants of the numerator or denominator equal to zero (i.e. $B^2 - 4AC = 0$ and/or $E^2 - 4DF = 0$) the map ψ_Γ can become infinite-to-one, as can be seen from Table 5.10[1]. This means that for such image points there is an infinite number of combinations of network parameters which lead to the same given impedance (image point in $\mathbb{P}^5$), When the resultant K is equal to zero, there is a pole-zero cancellation in the impedance (or two, if $B^2 - 4AC = E^2 - 4DF = 0$), which results in the impedance becoming bilinear (or constant). It can also be observed that in some of the subfamilies in which this occurs, the above conditions lead to a bridge balancing, for some network in the subfamily. We note however, based on our analysis of the catalogue, that this occurs only on a subset of the realisability set of dimension $d - 2$ (or less), if the realisability set has dimension d. Therefore we can say that for all networks in the catalogue (which are generic according to Definition 6.2) the map ψ_Γ is finite-to-one almost everywhere.

The following claim can also be found in [53]:

Claim. *The following are equivalent for a network $\Gamma \in X$ with associated graph G_Γ:*

1. Γ *is simple series-parallel,*
2. *The resultant of $a_\Gamma(s)$ and $b_\Gamma(s)$ is a product of monomials in the parameters,*
3. *Each coordinate of the inverse to ψ is expressible as a ratio of "invariants" (entries of the adjoint of the Sylvester matrix).*

The equivalence of 1. and 2. does not hold, since it can be easily calculated that the orbit of networks #69, #66, #90, #94 in subfamily $\mathrm{V_E}$ and of networks #40, #33, #24, #48 in subfamily $\mathrm{IV_C}$ both consist of networks which are simple series-parallel but whose resultants are not the product of monomials. Specifically, it can be checked that the

[1]For network #70 in equivalence class $\mathrm{V_H^1}$ it can be easily verified that when conditions $\lambda_1 = \lambda_2 = 0$, $\rho_1 = 0$ hold, then $B^2 - 4AC = E^2 - 4DF = 0$ and $K \neq 0$. Similar considerations can be made for network #95 in equivalence class $\mathrm{V_H^2}$.

resultant for networks #69 and #40 equals $R_1^2 L_1^3 C_1 (R_1 + R_2)^2$ in both cases. On the other hand, the equivalence of 1. and 3. does appear to hold, in the light of the analysis carried out in this work, with a few remarks. In the notation of the present analysis, the biquadratic impedance

$$Z(s) = \frac{As^2 + Bs + C}{Ds^2 + Es + F}$$

has the following associated Sylvester matrix, which can be obtained from the definition of Sylvester matrix given in Section 3.3:

$$S = \begin{bmatrix} D & E & F & 0 \\ A & B & C & 0 \\ 0 & D & E & F \\ 0 & A & B & C \end{bmatrix}.$$

The adjoint (or adjugate) matrix of S is

$$\mathrm{adj}(S) = \begin{bmatrix} \lambda_4 & \lambda_1 & -C(BF-CE) & F(BF-CE) \\ A(BF-CE) & -D(BF-CE) & C(AF-CD) & -F(AF-CD) \\ -A(AF-CD) & D(AF-CD) & -C(AE-BD) & F(AE-BD) \\ A(AE-BD) & -D(AE-BD) & \lambda_2 & \lambda_3 \end{bmatrix},$$

and one can verify from Tables 5.4, 5.6 and 5.7 and from Theorems A.1–A.10 that, for all simple series-parallel networks in the catalogue[2], the network elements can all be expressed as a ratio of entries of $\mathrm{adj}(S)$, while this is not the case for networks which are not simple series-parallel. Note from Tables 5.4 and 5.6 that, for equivalence classes $\mathrm{III}_{\mathrm{B}}^{1}$, $\mathrm{IV}_{\mathrm{A}}^{2}$ and $\mathrm{IV}_{\mathrm{C}}^{2}$, we have $\lambda_2|_{D=0} = A(BE - AF)$ and the result still holds. Similar considerations hold for the other equivalence classes in these three subfamilies. Also note that the equivalence of 1. and 3. holds only if we allow multiplication by coefficients A, B, $\ldots$, F and by the resultant $K = -\det(S)$.

6.5 A note on d-invariance of RLC networks

As mentioned in Section 4.3.2, it is the case that, within the catalogue, all networks which are `d`-invariant (i.e. left unchanged under duality, namely networks #3 and #108) are also invariant under the actions `s` and `p`. In other words, any network in the catalogue which is 2-isomorphic to its dual is also 2-isomorphic to the networks obtained through

[2]See Table 5.2 for a list of subfamilies which contain simple series-parallel networks.

the s and p transformations. Here we show that this property does not hold in general.

In Figure 6.5 a first example is given of an orbit of networks which are d-invariant but not s- and p-invariant. By using a Y-Δ transformation in two ways on each network, a resistor can be eliminated to produce the orbit of four non-isomorphic networks in Figure 6.6. We observe that in the latter there is a Y-Δ transformation connecting two pairs of networks, which are therefore equivalent. We now show that there is a further equivalence which allows us to conclude that all four networks in Figure 6.6 are equivalent.

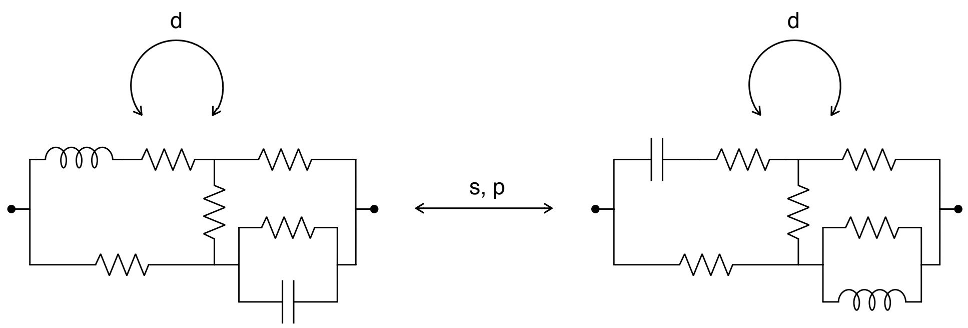


Figure 6.5: Example of d-invariant networks which are not s- or p-invariant.

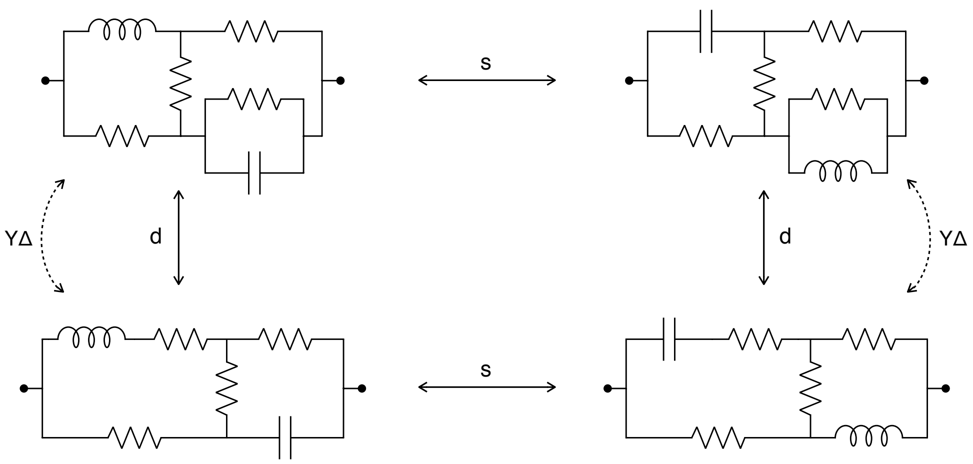


Figure 6.6: Orbit of non-isomorphic, equivalent networks which realise a biquadratic impedance.

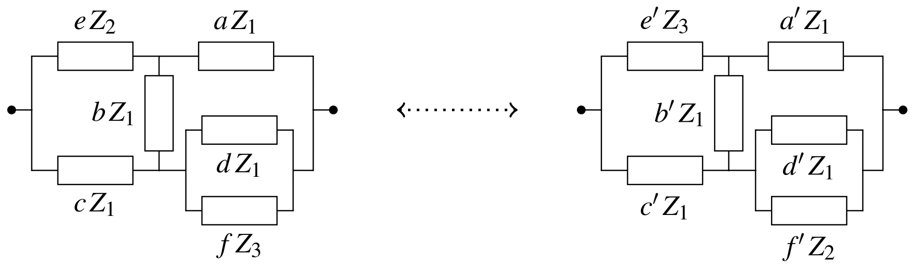


Figure 6.7: New equivalence between RLC networks.

It can be shown that the networks in Figure 6.7 are related by the following transformation:

$$a' = \frac{c(ac+P)}{d(b+c)}, \quad b' = \frac{Pb\,(ac+P)}{d(b+c)\big[a(b+c)+P\big]}, \quad c' = \frac{aP}{a(b+c)+P}$$
$$d' = \frac{(ac+P)^2}{(a+b)\big[a(b+c)+P\big]}, \quad e' = \frac{f\,P^2}{d^2(b+c)^2}, \quad f' = \frac{e(ac+P)^2}{\big[a(b+c)+P\big]^2},$$
$$\text{where} \quad P = bc+bd+cd,$$

for any real positive numbers a, b, $\dots$, f, where $Z_1(s)$, $Z_2(s)$ and $Z_3(s)$ are arbitrary impedances. The inverse transformation is given by the same expressions above, by replacing a, b, $\dots$, f with a', b', $\dots$, f'.

Using this new equivalence on the networks of Figure 6.6 we can conclude that the four networks, although not 2-isomorphic, are all equivalent. This suggests that a weaker property might hold in general, namely that a network being equivalent to its dual always implies that it is also equivalent to the networks obtained through frequency inversion and the p transformation. It was not possible to find a counterexample to this among the networks with up to six elements. The higher-order networks shown in Figure 6.8, however, turned out to be an example of networks which are d-invariant but not s- or p-invariant and *not* equivalent, as we will now show, thus refuting this last conjecture.

In order to prove that the networks in Figure 6.8 are not equivalent it is sufficient to find an impedance in the realisability set of one network which is not realisable by the other network. This is the case for the impedance obtained by setting $r_1 = r_4 = 2$ and $r_2 = r_3 = r_5 = l_1 = l_2 = c_1 = c_2 = 1$ in the first network, which gives the biquartic impedance

$$Z_1(s) = \frac{n_1(s)}{d_1(s)} = \frac{8s^4+36s^3+57s^2+36s+9}{8s^4+36s^3+51s^2+36s+9}, \tag{6.3}$$

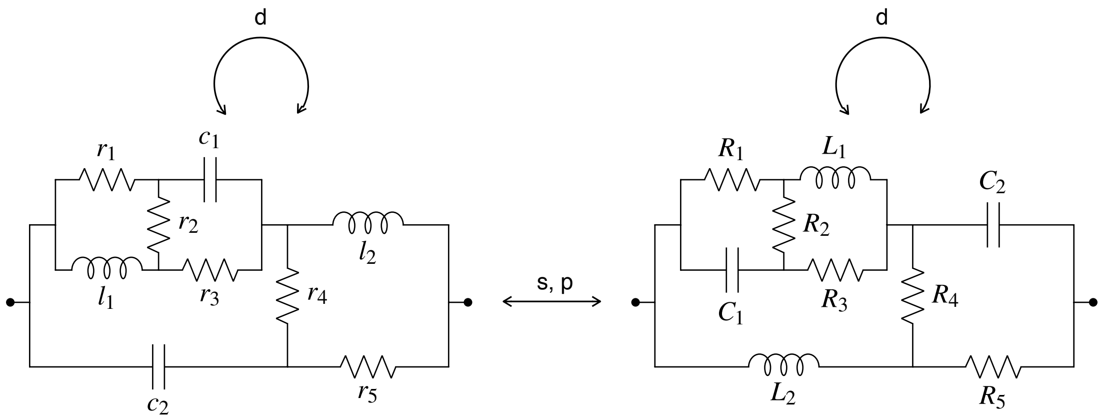


Figure 6.8: Example of d-invariant networks in the same orbit which are not equivalent.

where $n_1(s)$ and $d_1(s)$ have no common factor. An expression $Z_2(s) = n_2(s)/d_2(s)$ can be obtained for the impedance of the second network, where $n_2(s)$ and $d_2(s)$ are fourth-order polynomials in s whose coefficients depend on the network parameters R_1, R_2, L_1 etc. By setting $Z_1(s) = Z_2(s)$ we obtain the eighth-order polynomial equation

$$n_1(s)d_2(s) - n_2(s)d_1(s) = 0\,,$$

which is satisfied if the coefficients of each power of s are equal to zero. This leads to nine polynomial equations in the nine network parameters R_1, R_2, L_1 etc. Through an elimination procedure we obtain $R_3 = R_5 = 1$, $L_2 = C_2$, $C_1 = L_1$ and the following expressions for R_1, C_2, R_4 which are rational functions of L_1 and R_2 only

$$\begin{aligned} R_1 &= -\frac{3L_1^2R_2 + 4R_2 + 4}{3L_1^2 - 4R_2 - 4}, \\ C_2 &= \frac{-8L_1(R_2+1)}{3L_1^2(R_2-1) - 18L_1(R_2+1) + 8(R_2+1)}, \\ R_4 &= \frac{3C_2^2(1-R_1) + 2R_1}{3C_2^2(R_1-1) - 2}\,, \end{aligned}$$

where L_1 and R_2 still have to satisfy the following pair of polynomial equations

$$\begin{aligned} &27(R_2-1)^2\,L_1^4 - 108(R_2^2-1)\,L_1^3 + 12(R_2+1)(19R_2-1)\,L_1^2 \\ &- 288(R_2+1)^2\,L_1 + 128(R_2+1)^2 = 0\,, \end{aligned} \tag{6.4}$$

$$
\begin{aligned}
&- 9(R_2-1)^3 L_1^5 + 108(R_2+1)(R_2-1)^2 L_1^4 - 12(R_2^2-1)(31R_2+23) L_1^3 \\
&+ 48(17R_2-1)(R_2+1)^2 L_1^2 - 32(29R_2+25)(R_2+1)^2 L_1 \\
&+ 384(R_2+1)^3 = 0 . \qquad (6.5)
\end{aligned}
$$

For a given value of R_2, the two equations will have a common root if and only if their resultant (which is a function of R_2 only) is zero. The resultant of (6.4) and (6.5) is a constant multiple of

$$(1861R_2^3 + 13371R_2^2 + 32475R_2 + 17093)(R_2-1)^6(R_2+1)^{13} ,$$

and it is easily seen that its only real, positive solution is $R_2 = 1$. For $R_2 = 1$ equations (6.4) and (6.5) reduce to the quadratics

$$
\begin{aligned}
432\, L_1^2 - 1152\, L_1 + 512 = 0 , \\
3072\, L_1^2 - 6912\, L_1 + 3072 = 0 ,
\end{aligned}
$$

which have no common root. We can therefore conclude that no real, positive values exist for R_1, R_2, L_1 etc which make $Z_2(s)$ equal the candidate impedance (6.3), hence the two networks of Figure 6.8 are not equivalent.

6.6 Six-element networks with four resistors

In the light of the discussion in Section 6.5 and the introduction of the new equivalence shown in Figure 6.7, we now consider the entire class of networks containing two reactive elements and four resistors. Although it was shown that additional resistors beyond three do not expand the class of functions that are realised by the Ladenheim catalogue (see Section 3.2 and [61], [44]), it is still interesting to explore the structure of this class and possibly uncover further equivalences.

Within the Ladenheim catalogue there are 25 basic graph structures with five edges, of which 24 are series-parallel graphs and only one is a bridge graph (graph `V` in Appendix B). We consider now all the distinct graph structures with six edges. There are 72 such graphs, of which 66 are series-parallel [64] and six are bridge graphs. Half of the 66 series-parallel graphs are presented in [79] (the other half being obtained through duality), while the six bridge graphs are shown in Figure 6.9, and can be obtained from [24].

Considering all the essentially distinct RLC networks which can be obtained by

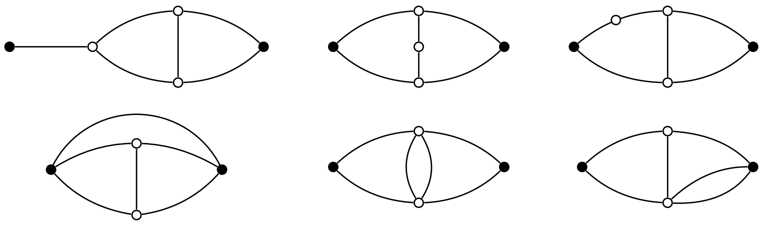


Figure 6.9: Bridge graphs with six edges. The bottom three graphs are the duals of the top three graphs.

populating the edges with four resistors and two reactive elements, and eliminating the networks which further reduce through a Zobel or a Y-Δ transformation, leads to a class of 52 networks, of which 44 are bridge networks and eight are series-parallel networks. These 52 networks can be analysed and grouped according to the classification tools presented in Chapters 4 and 5. Specifically, we can partition the set into 15 orbits and, identifying all the equivalences that result from Zobel and Y-Δ transformations, 28 disjoint sets of equivalent networks. Considering the finest common coarsening of these two partitions leads to nine subfamilies. At this point it is not clear whether new equivalences might further reduce the number of subfamilies. From Section 6.5 we already know of one further equivalence. We now investigate if there are any additional equivalences, by comparing the realisability regions of such networks.

The realisability regions were computed numerically and plotted for one network in each subfamily. We note that letting one of the four resistances go to zero or infinity leads to networks of the Ladenheim catalogue. It was possible to verify that for all the networks with one inductor and one capacitor the realisability region is obtained as the union of the realisability regions of the five-element networks which the network can reduce to. An example is given in Figure 6.11.

Looking at the realisability region of the networks in the orbit shown in Figure 6.12, and noting that it is symmetric with respect to the $U = V$ bisector, suggests that these networks are all equivalent to their $\mathtt{p}$-transform. In fact, it can be shown that the networks in Figure 6.10 are equivalent. In particular, they are related by the transformation:

$$a' = \frac{T(a+e+f)}{a(c+f)}, \quad b' = \frac{(a+e+f)^2\, b}{(e+f)^2}, \quad c' = \frac{(a+e+f)\big[T + e(a+c)\big]}{(e+f)^2}$$

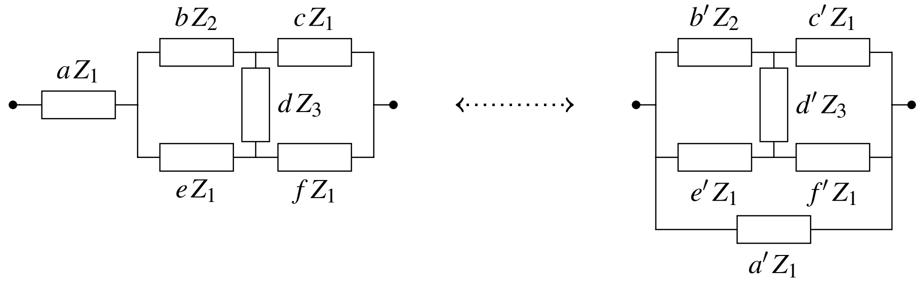


Figure 6.10: New equivalence between RLC networks.

$$d' = \frac{d(a+e+f)^2\left[T+e(a+c)\right]^2}{(T+ce)^2\,(e+f)^2}, \quad e' = \frac{(a+e+f)^2\,Te}{f(e+f)(T+ce)},$$
$$f' = \frac{T(a+e+f)\left[T+e(a+c)\right]}{c(e+f)(T+ce)}, \quad \text{where} \quad T = ac+af+cf$$

$$\Bigg[\ a = \frac{a'c'f'(e'+f')}{(a'+e'+f')\,T')}, \quad b = \frac{(a')^2\,b'(c'+f')^2}{(T')^2}, \quad c = \frac{c'(c'+f')(T'+c'e')(a')^2}{(T')^2\,(a'+e'+f')},$$
$$d = \frac{(a')^2\,d'(c'+f')^2(T'+c'e')^2}{(T')^2\left[T'+e'(c'+f')\right]^2}, \quad e = \frac{(a')^2\,e'(c'+f')^2}{T'\left[T'+e'(c'+f')\right]},$$
$$f = \frac{(a')^2\,f'(c'+f')(T'+c'e')^2}{(a'+e'+f')T'\left[T'+e'(c'+f')\right]}, \quad \text{where} \quad T' = a'c'+a'f'+c'f'\ \Bigg],$$

for any real positive numbers a, b, etc, where $Z_1(s)$, $Z_2(s)$ and $Z_3(s)$ are arbitrary impedances.

Therefore two new equivalences among RLC networks surfaced in this analysis: the equivalence shown in Figure 6.7 and the one in Figure 6.10. These led to the new equivalences shown in Figures 6.6 and 6.12 in the class of two-reactive, four-resistor networks. Using numerical analysis it was shown that there are no further equivalences in this class. Furthermore the two new equivalences did not cause any of the initial nine subfamilies to coalesce. Hence we can state the following proposition.

Proposition 6.1. *The 52 networks in the class of two-reactive, four-resistor networks can be partitioned into 23 equivalence classes, which form nine subfamilies. (Table 6.1 provides more detail on the structure of the subfamilies.)*

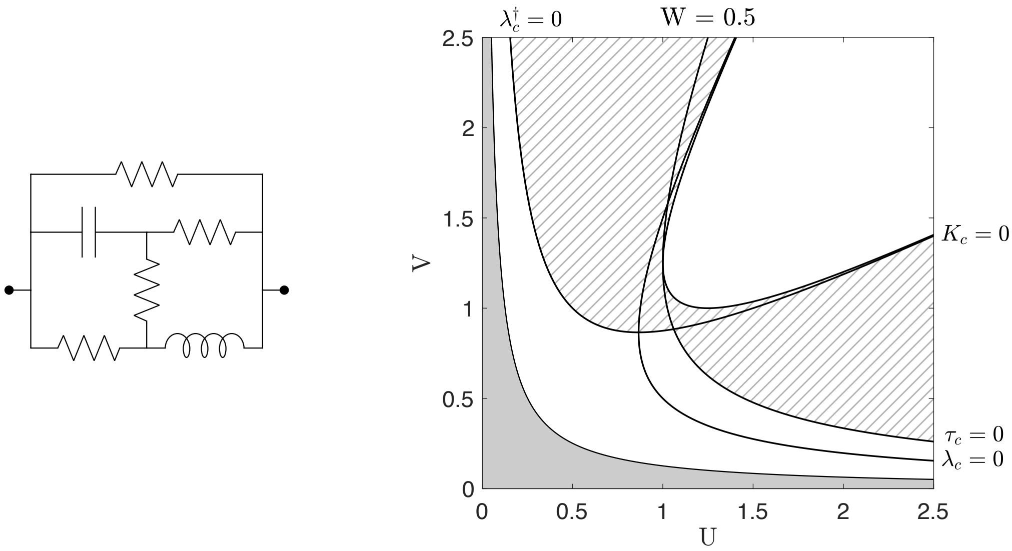


Figure 6.11: Two-reactive, six-element network and corresponding realisability region, for $W = 0.5$. Since the network is s-invariant, the region is the same for $W = 2$. Allowing one of the four resistances to be zero or infinity leads to networks of equivalence classes $\mathrm{V_I}$, $\mathrm{V_E^2}$, $\mathrm{V_E^3}$, $\mathrm{V_F^1}$ and $\mathrm{V_G^2}$. It can be verified from the plots in Section 5.6 that the realisability region shown above is the union of the realisability regions of the above-mentioned five-element equivalence classes.

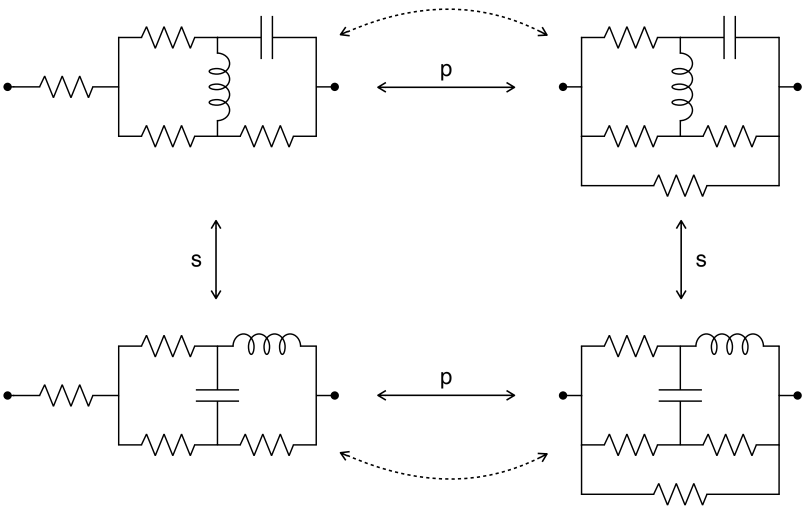


Figure 6.12: Orbit of six-element networks related by a new equivalence.

		# Eq. classes	# Orbits	# Networks
Subfamilies of RL/RC networks	subf. 1	2	2	8
	subf. 2	4	1	4
	subf. 3	4	1	4
	subf. 4	2	1	4
Subfamilies of RLC networks	subf. 5	2	5	14
	subf. 6	2	2	8
	subf. 7	2	1	2
	subf. 8	4	1	4
	subf. 9	1	1	4
	Total	23	15	52

Table 6.1: Composition of each of the nine subfamilies within the class of two-reactive, six-element networks.

6.7 Summary

In this chapter the main results on the classification of the Ladenheim catalogue were presented and proven in a formal way. One of the key outcomes of this work was the introduction of a new definition of generic network (which will be further explored in Chapter 7) in terms of the dimension of the realisability set it defines, which allowed us to prove that the Ladenheim catalogue forms the complete set of all essentially-distinct generic networks with at most two reactive elements.

Other important results were presented here, including remarks regarding the well-known Cauer-Foster transformation and on Kalman's approach to the problem of passive network synthesis, and the discovery of two new network equivalences.

Chapter 7

On a concept of genericity for RLC networks

In this chapter we further develop the notion of generic network, introduced in Definition 6.2, which is of particular interest and importance in relation to the concept of minimality. Notably, it is in general not known how to find an electric circuit to realise an arbitrary given impedance function minimally (i.e. using the least possible number of elements) [14, 34, 47]. Surprisingly, well-known networks which are apparently non-minimal, such as the Bott-Duffin realisation and its simplifications, have in fact recently been shown to be minimal for certain impedance functions [33, 38].

In this context, the concept of genericity is useful in identifying networks which do not fully exploit the degrees of freedom offered by the number of elements in the network, and which will therefore inevitably lead to non-minimal realisations. In other words, the realisability set of a non-generic network has a smaller dimension than the dimension that the realisability set of a network with the same number of elements could in principle have.

We provide here a necessary and sufficient condition for genericity of an RLC network which can be efficiently tested in practice and which does not require the knowledge of the realisability set of the network. The genericity concept is illustrated with several examples throughout this chapter, and a series of useful lemmas and corollaries are presented. We conclude the chapter by proving that the Bott-Duffin networks are generic, from which it follows that any positive-real impedance can be realised by a generic RLC network.

7.1 Preliminaries

We generalise here the notion of realisability set for an arbitrary two-terminal RLC network. This notion was introduced in Section 4.3.1 for networks of the Ladenheim catalogue, which all realise (at most) biquadratic impedances.

Consider an RLC two-terminal network $\mathcal{N}$ with $m \geq 1$ elements (resistors, capacitors or inductors) and corresponding parameters $E_1, \ldots, E_m \in \mathbb{R}_{>0}$. It follows from Kirchhoff's tree theorem [69, Section 7.2] that the driving-point impedance of $\mathcal{N}$ takes the form

$$Z(s) = \frac{f_k s^k + f_{k-1} s^{k-1} + \cdots + f_0}{g_k s^k + g_{k-1} s^{k-1} + \cdots + g_0} \tag{7.1}$$

where $f_i = f_i(E_1, \ldots, E_m)$, $g_j = g_j(E_1, \ldots, E_m)$ for $0 \leq i, j \leq k$ are polynomials in $E_1, \ldots, E_m$ with non-negative integer coefficients, at least one g_j is not identically zero, and not both of f_k and g_k are identically zero. We refer to the integer k as the *order* of the impedance, which cannot exceed the number of reactive elements in the network. Consider also the candidate impedance function

$$Z(s) = \frac{a_k s^k + a_{k-1} s^{k-1} + \cdots + a_0}{b_k s^k + b_{k-1} s^{k-1} + \cdots + b_0}, \tag{7.2}$$

where $a_i, b_j \in \mathbb{R}_{\geq 0}$ for $0 \leq i, j \leq k$. For the equality of (7.1) and (7.2) it is necessary and sufficient that

$$\left.\begin{array}{rcl} a_0 &=& c f_0(E_1, \ldots, E_m), \\ &\vdots& \\ a_k &=& c f_k(E_1, \ldots, E_m), \\ b_0 &=& c g_0(E_1, \ldots, E_m), \\ &\vdots& \\ b_k &=& c g_k(E_1, \ldots, E_m) \end{array}\right\} \tag{7.3}$$

for some $c > 0$. We define the *realisability set* of $\mathcal{N}$ to be the set

$$\mathcal{S} = \Big\{ (a_0, \ldots, a_k, b_0, \ldots, b_k) \text{ such that (7.3) holds,} \\ E_1, \ldots, E_m \in \mathbb{R}_{>0} \text{ and } c \in \mathbb{R}_{>0} \Big\}.$$

Let $\mathbf{x} = (x_1, \ldots, x_{m+1}) = (E_1, \ldots, E_m, c) \in \mathbb{R}^{m+1}_{>0}$ and define the function $\mathbf{h} : \mathbb{R}^{m+1}_{>0} \to \mathbb{R}^{2k+2}_{\geq 0}$ as follows:

$$\mathbf{h}(\mathbf{x}) = c\,(f_0, \ldots, f_k, g_0, \ldots, g_k)$$

Then $\mathcal{S}$ is the image of $\mathbb{R}_{>0}^{m+1}$ under $\mathbf{h}$.

The set $\mathcal{S}$ can also be seen to be the projection onto the first $2k+2$ components of the real semi-algebraic set

$$\mathcal{S}_f = \Big\{(a_0, \ldots, a_k, b_0, \ldots, b_k, E_1, \ldots, E_m, c) \text{ such that (7.3) holds,} \\ E_1, \ldots, E_m \in \mathbb{R}_{>0} \text{ and } c \in \mathbb{R}_{>0}\Big\}$$

in $\mathbb{R}_{\geq 0}^{2k+m+3}$. Hence $\mathcal{S}$ is a real semi-algebraic set using the Tarski-Seidenberg theorem [5]. We use the notation $\pi_{\{r_1,\ldots,r_p\}}(\cdot)$ to denote the projection of a real semi-algebraic set onto the components with indices $r_1, \ldots, r_p$. Thus, $\mathcal{S} = \pi_{\{1,\ldots,2k+2\}}(\mathcal{S}_f)$.

7.2 A necessary and sufficient condition for genericity

Considering the definition of dimension $\dim(\mathcal{S})$ of a semi-algebraic set $\mathcal{S}$ given in Section 6.2, the following lemmas hold.

Lemma 7.1. *For a semi-algebraic set $\mathcal{S} \subset \mathbb{R}^n$ let $\pi = \pi_{\{r_1,\ldots,r_p\}}$ for some indices $r_1, \ldots, r_p$ with $p < n$. Then $dim(\pi(\mathcal{S})) \leq dim(\mathcal{S})$ [3, Lemma 5.30].*

Lemma 7.2. *Let $\mathcal{N}$ be an RLC two-terminal network with $m \geq 1$ elements and realisability set $\mathcal{S}$. Then $dim(\mathcal{S}) \leq m+1$.*

Proof. Given $E_{i,0} > 0$ for $1 \leq i \leq m$ and $c_0 > 0$ there exists $\epsilon > 0$ such that $E_i = E_{i,0} + \epsilon x_i > 0$ and $c = c_0 + \epsilon x_{m+1} > 0$ for $(x_1, \ldots, x_{m+1}) \in (-1,1)^{m+1}$. Hence there is a smooth one-to-one mapping from $(-1,1)^{m+1}$ into some neighbourhood of any point in $\mathcal{S}_f$, which means that $\dim(\mathcal{S}_f) = m+1$. Note that this neighbourhood contains all points in $\mathcal{S}_f$ that are sufficiently close to the given point in the Euclidean metric. Such a neighbourhood in $\mathcal{S}_f$ is homeomorphic to the unit cube in $\mathbb{R}^{m+1}$, hence to the unit sphere in $\mathbb{R}^{m+1}$, hence not homeomorphic to a unit sphere in any other dimension [29, Theorem 2.26]. The result now follows from Lemma 7.1. □

Given an m-element network with network parameters $E_1, \ldots, E_m \in \mathbb{R}_{>0}$ whose impedance takes the form (7.1), we now introduce the matrix $D(E_1, \ldots, E_m)$ defined by

$$D = \begin{pmatrix} \frac{\partial f_0}{\partial E_1} & \cdots & \frac{\partial f_0}{\partial E_m} & f_0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial f_k}{\partial E_1} & \cdots & \frac{\partial f_k}{\partial E_m} & f_k \\ \frac{\partial g_0}{\partial E_1} & \cdots & \frac{\partial g_0}{\partial E_m} & g_0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_k}{\partial E_1} & \cdots & \frac{\partial g_k}{\partial E_m} & g_k \end{pmatrix} \tag{7.4}$$

and note that the derivative of $\mathbf{h}$ is given by $\mathbf{h}' = D\,\mathrm{diag}(c, \ldots, c, 1)$. We now prove a necessary and sufficient condition for a network to be generic, according to the definition of genericity introduced in Section 6.2.

Theorem 7.1. *Let $\mathcal{N}$ be an RLC two-terminal network with $m \geq 1$ elements and realisability set $\mathcal{S}$. Then $\mathcal{N}$ is generic if and only if there exists $\mathbf{E}_0 = (E_{1,0}, \ldots, E_{m,0}) \in \mathbb{R}^m_{>0}$ such that rank$(D(\mathbf{E}_0)) = m + 1$.*

Proof. Assume that there exists $\mathbf{E}_0 \in \mathbb{R}^m_{>0}$ such that $\mathrm{rank}(D(\mathbf{E}_0)) = m+1$ and note that $\mathrm{rank}(\mathbf{h}'(\mathbf{x}_0)) = m + 1$ for $\mathbf{x}_0 = (\mathbf{E}_0, c)$ for any $c > 0$. Let A be a square submatrix of $\mathbf{h}'(\mathbf{x}_0)$ consisting of rows $l_1, \ldots, l_{m+1}$ for which $\det(A) \neq 0$. Let $\hat{\mathbf{h}}(\mathbf{x})$ be the restriction of $\mathbf{h}(\mathbf{x})$ to the components $l_1, \ldots, l_{m+1}$. Then, by the inverse function theorem [67, Theorem 9.24], $\hat{\mathbf{h}}(\mathbf{x})$ is a one-to-one mapping from a neighbourhood of $\mathbf{x}_0$ into $\mathbb{R}^{m+1}_{>0}$, which means that $\mathbf{h}(\mathbf{x})$ is a smooth one-to-one mapping from a neighbourhood of $\mathbf{x}_0$ into $\mathcal{S}$. Hence $\dim(\mathcal{S}) = m + 1$ which means that $\mathcal{N}$ is generic.

Conversely, assume that $\dim(\mathcal{S}) = m+1$. Then there exists $\mathbf{x}_0 = (E_{1,0}, \ldots, E_{m,0}, c_0) \in \mathbb{R}^{m+1}_{>0}$ such that $\mathbf{h}(\mathbf{x})$ is a smooth one-to-one mapping from a neighbourhood of $\mathbf{x}_0$ into $\mathcal{S}$. Then there exists a smooth inverse $\mathbf{w}(\mathbf{y})$ from a neighbourhood of $\mathbf{y}_0 = \mathbf{h}(\mathbf{x}_0)$ within $\mathcal{S}$ into a neighbourhood of $\mathbf{x}_0$. In particular $\mathbf{w}(\mathbf{h}(\mathbf{x})) = \mathbf{x}$ in a neighbourhood of $\mathbf{x}_0$. Using the chain rule [67, Theorem 9.15] $\mathbf{w}'(\mathbf{y}_0)\mathbf{h}'(\mathbf{x}_0) = I$, so $\mathrm{rank}(\mathbf{h}'(\mathbf{x}_0)) = m + 1$. Writing $\mathbf{x}_0 = (\mathbf{E}_0, c)$ then $\mathrm{rank}(D(\mathbf{E}_0)) = m + 1$, which completes the proof. □

Corollary 7.1. *If an RLC two-terminal network $\mathcal{N}$ contains elements $E_1, \ldots, E_m \in \mathbb{R}_{>0}$ and has impedance $f(s)/g(s)$, then $\mathcal{N}$ is generic if and only if there exist $\mathbf{E}_0 = (E_{1,0}, \ldots, E_{m,0}) \in \mathbb{R}^m_{>0}$ such that, for $\mathbf{x} \in \mathbb{R}^{m+1}$,*

$$\begin{pmatrix} \frac{\partial f}{\partial E_1} & \frac{\partial f}{\partial E_2} & \cdots & \frac{\partial f}{\partial E_m} & f \\ \frac{\partial g}{\partial E_1} & \frac{\partial g}{\partial E_2} & \cdots & \frac{\partial g}{\partial E_m} & g \end{pmatrix}_{\mathbf{E}_0} \mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}. \tag{7.5}$$

Proof. It can be easily verified that the left-hand side of (7.5) yields two polynomials in s whose coefficients are given by the rows of the vector $D\mathbf{x}$, where D is defined in (7.4). In order for both polynomials to be zero, each coefficient of each power of s has to be zero, from which we can conclude that the left-hand side of (7.5) is equivalent to $D\mathbf{x} = \mathbf{0}$. By Theorem 7.1, the network $\mathcal{N}$ is generic if and only if the matrix D in (7.4) is full column rank for some $E_1, \ldots, E_m \in \mathbb{R}_{>0}$. This is equivalent to

$$\mathbf{x} \in \mathbb{R}^{m+1} \text{ and } D\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}\,.$$

Therefore $\mathcal{N}$ is generic if and only if (7.5) holds for $\mathbf{x} \in \mathbb{R}^{m+1}$. □

Corollary 7.2. *Let $\mathcal{N}$ be a generic RLC network whose impedance takes the form of* (7.1)*. Then the number of resistors in $\mathcal{N}$ is less than or equal to $k+1$.*

Proof. Let n be the number of resistors in $\mathcal{N}$ and m be the total number of elements. Then in order that $\text{rank}(\mathbf{h}'(\mathbf{x}_0)) = m+1$ it is necessary that $2k+2 \geq m+1$. Given that $k \leq m-n$, the result follows. □

7.3 Examples

The necessary and sufficient condition in Theorem 7.1, together with the necessary condition in Corollary 7.2, provides an efficient way of verifying genericity of any given RLC network which does not rely on obtaining the realisability set of the network. Throughout this section we will say that $\text{rank}(D) = p$, where the general expression for D is given in (7.4), if $p = \max_{E_1,\ldots,E_m \in \mathbb{R}_{>0}}(\text{rank}(D(E_1,\ldots,E_m)))$.

Example 7.1. The network in Figure 7.1 is a first trivial example of a non-generic network. This can be verified through Corollary 7.2 or by considering that the network can be reduced to a network consisting of a single resistor, which defines a realisability set of dimension two.

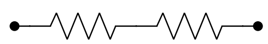


Figure 7.1: A simple non-generic network.

Example 7.2. It was shown in Theorem 6.1 that the 108 networks in the Ladenheim catalogue are all generic. We show in this example how this result can be verified using the necessary and sufficient condition of Theorem 7.1.

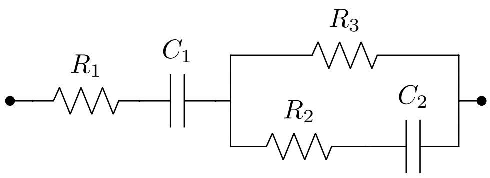


Figure 7.2: Non-generic network.

In the derivation of the Ladenheim catalogue, a number of networks are discarded because non-generic (see also Theorem 6.2). These comprise networks which contain a series or parallel connection of the same type of component (which can each be shown to be non-generic in a similar way to Example 7.1) and another 40 networks which also turn out to be non-generic. An example of one of these 40 networks is shown in Figure 7.2. The impedance of this network is a biquadratic, with

$$\begin{aligned} f_2 &= C_1C_2(R_1R_2 + R_1R_3 + R_2R_3)\,, \\ f_1 &= C_1(R_1 + R_3) + C_2(R_2 + R_3)\,, \\ f_0 &= 1\,, \\ g_2 &= C_1C_2(R_2 + R_3)\,, \\ g_1 &= C_1\,, \\ g_0 &= 0\,. \end{aligned}$$

Since $g_0 = 0$, it follows that one row in the matrix $D \in \mathbb{R}^{6\times6}_{\geq 0}$ is identically zero. Therefore $\text{rank}(D) \leq 5$ and from Theorem 7.1 the network is non-generic. It can also be seen, through a Zobel transformation, that the network reduces to a generic four-element network. An example of one of the remaining 108 generic networks in the canonical set (network #95) is shown in Figure 7.3(a). The impedance of this network is a biquadratic and it can be easily computed that the determinant of the matrix $D \in \mathbb{R}^{6\times6}_{\geq 0}$ is equal to

$$\begin{aligned} &- C_1L_1\big(C_1R_1R_2(R_1R_2 + R_2R_3) + L_1R_3(R_2 + R_3)\big) \\ &\times \big(C_1R_1R_2(R_2 + 2R_3)(R_1 + R_3) - L_1(R_2 + R_3)(2R_1 + R_3)\big), \end{aligned}$$

which is not identically zero, hence $\text{rank}(D) = 6$. Therefore, the network is generic by Theorem 7.1 and defines a realisability set $\mathcal{S}$ of dimension six.

The four-element network in Figure 7.3(b) is another generic network from the Ladenheim catalogue which realises a biquadratic impedance. The determinant of the 5×5

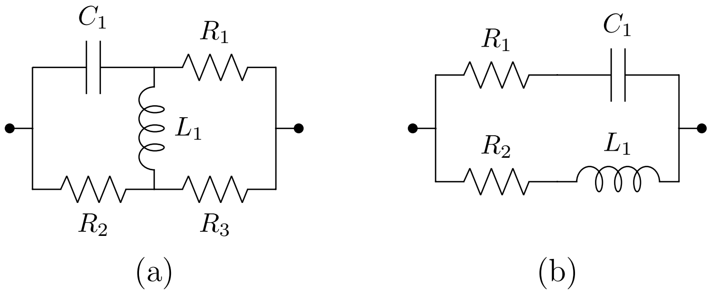


Figure 7.3: Two generic networks (networks #95 and #97, respectively) from the Ladenheim catalogue.

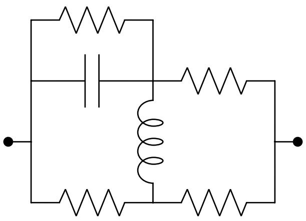


Figure 7.4: Non-generic network.

submatrix obtained from $D \in \mathbb{R}^{6\times 5}_{\geq 0}$ by removing the last row is equal to

$$R_2C_1(R_1R_2C_1 - L_1)\,,$$

which is not identically zero, hence $\mathrm{rank}(D) = 5$. Therefore, the network is generic by Theorem 7.1 and defines a realisability set $\mathcal{S}$ of dimension five. Since all six impedance coefficients are non-zero, this means that they must be interdependent. In fact we can show that

$$(f_2g_0 + f_0g_2)(f_2g_0 + f_0g_2 - f_1g_1) + f_0f_2g_1^2 = 0\,,$$

as also derived in Table 5.10.

Example 7.3. By considering an additional resistor in the generic network of Figure 7.3(a) we obtain the network of Figure 7.4. This network is no longer generic, by Corollary 7.2. In fact, it can be computed that the impedance is a biquadratic and that $D \in \mathbb{R}^{6\times 7}_{\geq 0}$, hence $\mathrm{rank}(D) \leq 6$. This network has been considered in [42], [93].

Example 7.4. The impedance of the three-reactive five-element network in Figure 7.5 (which has been analysed in [43]) is a bicubic, and $D \in \mathbb{R}^{8\times 6}_{\geq 0}$ in this case. The determi-

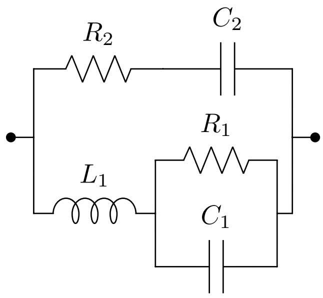


Figure 7.5: Three-reactive five-element generic network.

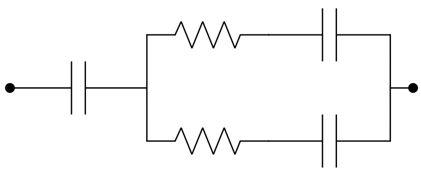


Figure 7.6: Three-reactive five-element non-generic network.

nant of the submatrix obtained by removing the last two rows of D is equal to

$$R_1^3 L_1^2 C_1^2 C_2^3 (R_1 R_2 C_1 - R_2^2 C_2 - L_1)\,,$$

which is not identically zero, hence $\text{rank}(D) = 6$. Therefore, the network is generic by Theorem 7.1 and defines a realisability set of dimension six.

Example 7.5. The impedance of the three-reactive five-element network in Figure 7.6 is a biquadratic, with $g_0 = 0$. This is an example where the order of the impedance $k = 2$ is strictly less than the number of reactive elements. It can be computed that $D \in \mathbb{R}_{\geq 0}^{5\times 6}$, hence $\text{rank}(D) \leq 5$ necessarily and the network is non-generic by Theorem 7.1.

Example 7.6. The seven-element network in Figure 7.7 (see Figure 3 in [33]) is another example where the order of the impedance ($k = 4$) is strictly less than the number of reactive elements in the network, as pointed out in [33]. This order reduction can be seen from Kirchoff's tree theorem (see [69, Section 7.2]) since there can be no spanning tree of the network which contains all three capacitors. In this case $D \in \mathbb{R}_{\geq 0}^{10\times 8}$ and it can be computed that the determinant of any square submatrix of D formed by deleting any two rows is non-zero. Hence the network is generic and defines a realisability set of dimension eight. Note that this means that an impedance of lower order than the number of reactive elements in the network does not imply that the network is non-generic.

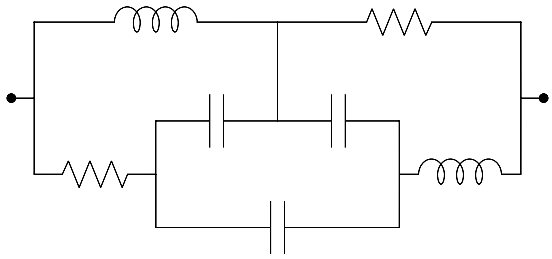


Figure 7.7: Five-reactive element generic network from [33] of fourth order.

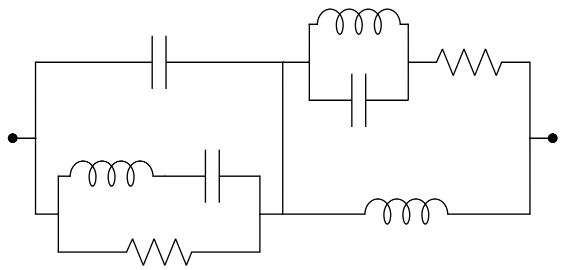


Figure 7.8: Bott-Duffin network for the realisation of a biquadratic.

Example 7.7. The network in Figure 7.8 has the same structure as the Bott-Duffin construction for the biquadratic minimum function $Z(s)$ with $Z(j\omega_1) = j\omega_1 X_1$, where $\omega_1 > 0$ and $X_1 > 0$ (see Section 2.5). Assuming that all network elements can vary freely, it is interesting to see whether the network is generic. The network has eight elements and its impedance is of order six, hence $D \in \mathbb{R}^{14\times 9}_{\geq 0}$. It can be computed that $\text{rank}(D) = 9$, hence the network is generic and defines a realisability set of dimension nine. It can also be verified that by adding a resistor in series or in parallel to the network in Figure 7.8 the resulting network is still generic (with a realisability set of dimension ten).

7.4 Interconnection of generic networks

In this section we look at the genericity of interconnections of networks, and prove the result that a non-generic subnetwork embedded within a network leads to non-genericity of the overall network. A series of lemmas are also proven here, which help to show the main result that the Bott-Duffin networks are generic, and hence any positive-real impedance can be realised generically.

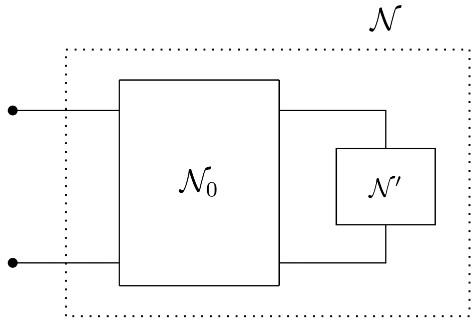


Figure 7.9: Two-terminal network $\mathcal{N}$ with a two-terminal subnetwork $\mathcal{N}'$.

Lemma 7.3. *Consider an RLC two-terminal network $\mathcal{N}$ with the structure shown in Figure 7.9, in which the network $\mathcal{N}_0$ comprises $m \geq 1$ elements with parameters $E_1, \ldots, E_m \in \mathbb{R}_{>0}$ and the network $\mathcal{N}'$ comprises $n \geq 1$ elements with parameters $E_{m+1}, \ldots, E_{m+n} \in \mathbb{R}_{>0}$. If the driving-point impedance of $\mathcal{N}'$ is $f(s)/g(s)$, then the impedance of $\mathcal{N}$ takes the form*

$$Z(s) = \frac{u(s)f(s) + v(s)g(s)}{w(s)f(s) + x(s)g(s)}\,, \tag{7.6}$$

where $u(s)$, $v(s)$, $w(s)$ and $x(s)$ are polynomials in s whose coefficients are polynomials in $E_1, \ldots, E_m$, while $f(s)$ and $g(s)$ are polynomials whose coefficients are polynomials in $E_{m+1}, \ldots, E_{m+n}$.

Proof. Let G be the undirected graph with edges corresponding to the network elements $E_1, \ldots, E_m$ of $\mathcal{N}$ and one edge corresponding to network $\mathcal{N}'$. Let $\widetilde{G}$ be the graph obtained by connecting together the vertices corresponding to the driving-point terminals in G. Denote by $f_G(s)$ the Laurent polynomial given by the sum over all spanning trees in G of the product of the admittances of all edges in each spanning tree, and similarly for $f_{\widetilde{G}}(s)$. Then, by Kirchhoff's matrix tree theorem, the impedance of $\mathcal{N}$ is equal to $f_{\widetilde{G}}(s)/f_G(s)$ (see [69, Section 7.2]). Given that the admittance of one of the edges in G and $\widetilde{G}$ is $g(s)/f(s)$, it follows that the impedance of $\mathcal{N}$ takes the form (7.6). □

Theorem 7.2. *Let $\mathcal{N}$ and $\mathcal{N}'$ be as in Lemma 7.3. If the subnetwork $\mathcal{N}'$ is non-generic then $\mathcal{N}$ is non-generic.*

Proof. Let $f(s)$, $g(s)$, $u(s)$, $v(s)$, $w(s)$ and $x(s)$ be as in Lemma 7.3. Then the impedance

$Z(s) = a(s)/b(s)$ of $\mathcal{N}$ takes the form (7.6), and we can write

$$\begin{pmatrix} a(s) \\ b(s) \end{pmatrix} = M \begin{pmatrix} f(s) \\ g(s) \end{pmatrix}, \tag{7.7}$$

where

$$M = \begin{pmatrix} u(s) & v(s) \\ w(s) & x(s) \end{pmatrix}$$

is a matrix of polynomials whose coefficients depend only on $\mathbf{E} = (E_1, \ldots, E_m)$, while $f(s)$ and $g(s)$ are polynomials whose coefficients depend on $\mathbf{E}' = (E_{m+1}, \ldots, E_{m+n})$. By Corollary 7.1, the network $\mathcal{N}$ is generic if and only if

$$\mathbf{x} \in \mathbb{R}^{m+n+1} \text{ and } D\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}, \tag{7.8}$$

where

$$D = \begin{pmatrix} \frac{\partial a}{\partial E_1} & \cdots & \frac{\partial a}{\partial E_{m+n}} & a \\ \frac{\partial b}{\partial E_1} & \cdots & \frac{\partial b}{\partial E_{m+n}} & b \end{pmatrix}_{\bar{\mathbf{E}}},$$

for some $\bar{\mathbf{E}} = (\bar{E}_1, \ldots, \bar{E}_{m+n}) \in \mathbb{R}^{m+n}_{>0}$. Since M is independent of $\mathbf{E}'$, it follows from (7.7) that

$$D = \begin{pmatrix} * & \big| & MD' \end{pmatrix}_{\bar{\mathbf{E}}},$$

where the first block of the matrix corresponds to the partial derivatives of $a(s)$ up to E_m and

$$D' = \begin{pmatrix} \frac{\partial f}{\partial E_{m+1}} & \cdots & \frac{\partial f}{\partial E_{m+n}} & f \\ \frac{\partial g}{\partial E_{m+1}} & \cdots & \frac{\partial g}{\partial E_{m+n}} & g \end{pmatrix}.$$

Since $\mathcal{N}'$ is non-generic, given any $\mathbf{E}' \in \mathbb{R}^n_{>0}$ there exists a real vector $\mathbf{y} \neq \mathbf{0}$ such that $D'_{\mathbf{E}'}\mathbf{y} = \mathbf{0}$. It follows that, for any given $\bar{\mathbf{E}} \in \mathbb{R}^{m+n}_{>0}$, there exists $\mathbf{0} \neq \mathbf{y} \in \mathbb{R}^{n+1}$ such that

$$D \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} * & \big| & MD' \end{pmatrix}_{\bar{\mathbf{E}}} \begin{pmatrix} \mathbf{0} \\ \mathbf{y} \end{pmatrix} = \mathbf{0},$$

which contradicts (7.8). Therefore $\mathcal{N}$ is non-generic. □

Corollary 7.3. *A necessary condition for the series or parallel connection of two networks $\mathcal{N}_1$ and $\mathcal{N}_2$ to be generic is that $\mathcal{N}_1$ and $\mathcal{N}_2$ are generic.*

Proof. This follows from Theorem 7.2. □

Remark 7.1. It is worth noting that the necessary condition in Corollary 7.3 is not sufficient for a series connection of two networks to be generic. The networks in Figures 7.1 and 7.2 are simple examples of non-generic networks consisting of a series connection of two generic networks.

Remark 7.2. By Theorem 7.2, we can conclude that any network containing a series or parallel connection of the same type of component is non-generic. This allows us to discard any such network from the canonical set in the Ladenheim catalogue, as discussed in Example 7.2.

Lemma 7.4. *Consider an RLC two-terminal network $\mathcal{N}$ with the structure shown in Figure 7.10, where the subnetwork $\mathcal{N}_1$ is generic and does not have an impedance zero at the origin. Then $\mathcal{N}$ is generic.*

Proof. Let network $\mathcal{N}_1$ have impedance $f(s)/g(s)$ of order n and network elements $\mathbf{E} = (E_1, \ldots, E_m) \in \mathbb{R}^m_{>0}$. Then the impedance $Z(s) = a(s)/b(s)$ of $\mathcal{N}$ is given by

$$Z(s) = \frac{R(f(s) + sLg(s)) + sLf(s)}{f(s) + sLg(s)} \,.$$

Since $\mathcal{N}_1$ is generic, it follows from Corollary 7.1 that

$$\mathbf{y} \in \mathbb{R}^{m+1} \text{ and } D_1\mathbf{y} = \mathbf{0} \quad \Rightarrow \quad \mathbf{y} = \mathbf{0}\,, \tag{7.9}$$

where

$$D_1 = \begin{pmatrix} \frac{\partial f}{\partial E_1} & \cdots & \frac{\partial f}{\partial E_m} & f \\ \frac{\partial g}{\partial E_1} & \cdots & \frac{\partial g}{\partial E_m} & g \end{pmatrix}_{\mathbf{E}_0}$$

for some $\mathbf{E_0} = (E_{1,0}, \ldots, E_{m,0}) \in \mathbb{R}^m_{>0}$. To prove that $\mathcal{N}$ is generic we need to show that, for $\mathbf{x} \in \mathbb{R}^{m+3}$,

$$\begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial E_1} & \cdots & \frac{\partial a}{\partial E_m} & a \\ \frac{\partial b}{\partial R} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial E_1} & \cdots & \frac{\partial b}{\partial E_m} & b \end{pmatrix}_{\bar{\mathbf{E}}} \mathbf{x} = \mathbf{0} \;\Rightarrow\; \mathbf{x} = \mathbf{0}\,, \tag{7.10}$$

for some $\bar{\mathbf{E}} = (\bar{R}, \bar{L}, \bar{E}_1, \ldots, \bar{E}_m) \in \mathbb{R}^{m+2}_{>0}$. To show this, we let $\bar{E}_i = E_{i,0}$ $(i = 1, \ldots, m)$ and we pick $\bar{R}, \bar{L} \in \mathbb{R}_{>0}$ arbitrarily. Then, since $a(s)$ and $b(s)$ depend on $E_1, \ldots, E_m$ through $f(s)$ and $g(s)$, by the chain rule (7.10) is equivalent to

$$\begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial f} & \frac{\partial a}{\partial g} \\ \frac{\partial b}{\partial R} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial f} & \frac{\partial b}{\partial g} \end{pmatrix}_{\mathbf{E}_0} \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & D_1 \end{array} \right) \mathbf{x} = \mathbf{0} \;\Rightarrow\; \mathbf{x} = \mathbf{0} \tag{7.11}$$

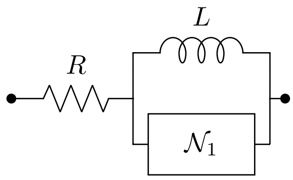


Figure 7.10: Two-terminal network with a generic subnetwork $\mathcal{N}_1$.

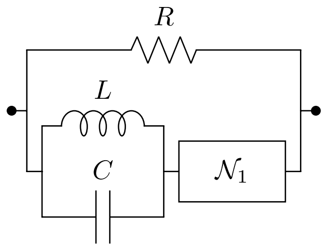


Figure 7.11: Two-terminal network with a generic subnetwork $\mathcal{N}_1$.

where I_2 is the two-by-two identity matrix. Since (7.9) holds, it then suffices to show that

$$\begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial f} & \frac{\partial a}{\partial g} \\ \frac{\partial b}{\partial R} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial f} & \frac{\partial b}{\partial g} \end{pmatrix} \begin{pmatrix} u \\ v \\ w(s) \\ z(s) \end{pmatrix} = \mathbf{0} \;\Rightarrow\; \begin{pmatrix} u \\ v \\ w(s) \\ z(s) \end{pmatrix} = \mathbf{0} \tag{7.12}$$

for any given real scalars u, v and polynomials $w(s)$, $z(s)$ of degree less than or equal to n. The left-hand side of (7.12) yields the following two polynomial equations:

$$u(sLg(s) + f(s)) + v(Rg(s) + f(s))s + w(s)(sL + R) + sRLz(s) = 0 \tag{7.13}$$

$$svg(s) + w(s) + sLz(s) = 0\,. \tag{7.14}$$

Subtracting (7.14) multiplied by R from (7.13) we obtain

$$u(sLg(s) + f(s)) + vsf(s) + sLw(s) = 0\,. \tag{7.15}$$

We let $s = 0$ in (7.14) and (7.15) to conclude that $w(0) = 0$ and $u = 0$ (since $f(0) \neq 0$). Equation (7.15) now reduces to $vf(s) + Lw(s) = 0$, and again by setting $s = 0$ we can conclude that $v = 0$. Finally, $w(s) = z(s) = 0$ easily follows from (7.14) and (7.15). We have therefore shown that (7.12) holds, hence $\mathcal{N}$ is generic. □

Lemma 7.5. *Consider an RLC two-terminal network $\mathcal{N}$ with the structure shown in Figure 7.11, where the subnetwork $\mathcal{N}_1$ is generic. Then $\mathcal{N}$ is generic.*

Proof. Let network $\mathcal{N}_1$ have impedance $f(s)/g(s)$ of order n and network elements $\mathbf{E} = (E_1, \ldots, E_m) \in \mathbb{R}^m_{>0}$. Then the impedance $Z(s) = a(s)/b(s)$ of $\mathcal{N}$ is given by

$$Z(s) = \frac{Lsg(s) + (1 + \alpha s^2)f(s)}{GLsg(s) + (1 + \alpha s^2)(Gf(s) + g(s))},$$

where $\alpha = LC$ and $G = 1/R$. By Corollary 7.1, $\mathcal{N}$ is generic if and only if

$$\mathbf{x} \in \mathbb{R}^{m+4} \text{ and } D\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}, \tag{7.16}$$

where

$$D = \begin{pmatrix} \frac{\partial a}{\partial G} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial \alpha} & \frac{\partial a}{\partial E_1} & \cdots & \frac{\partial a}{\partial E_m} & a \\ \frac{\partial b}{\partial G} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial \alpha} & \frac{\partial b}{\partial E_1} & \cdots & \frac{\partial b}{\partial E_m} & b \end{pmatrix}_{\mathbf{E}_0},$$

for some $\mathbf{E_0} = (G_0, L_0, \alpha_0, E_{1,0}, \ldots, E_{m,0}) \in \mathbb{R}^{m+3}_{>0}$. By the same argument as Lemma 7.4, applying the chain rule we can conclude that (7.16) holds if, for any given $(E_1, \ldots, E_m) \in \mathbb{R}^m_{>0}$, there exist $G, L \in \mathbb{R}_{>0}$ such that the following holds

$$\begin{pmatrix} \frac{\partial a}{\partial G} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial \alpha} & \frac{\partial a}{\partial f} & \frac{\partial a}{\partial g} \\ \frac{\partial b}{\partial G} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial \alpha} & \frac{\partial b}{\partial f} & \frac{\partial b}{\partial g} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y(s) \\ z(s) \end{pmatrix} = \mathbf{0} \;\Rightarrow\; \begin{pmatrix} u \\ v \\ w \\ y(s) \\ z(s) \end{pmatrix} = \mathbf{0} \tag{7.17}$$

for any given real scalars u, v, w and polynomials $y(s)$, $z(s)$ of degree less than or equal to n. The left-hand side of (7.17) yields the following two polynomial equations:

$$sg(s)v + s^2 f(s)w + (1 + \alpha s^2)y(s) + Lsz(s) = 0, \tag{7.18}$$

$$\begin{aligned} &sg(s)(Lu + Gv + sw) + sG(sf(s)w + Lz(s)) \\ &\quad + (1 + \alpha s^2)(f(s)u + Gy(s) + z(s)) = 0. \end{aligned} \tag{7.19}$$

Subtracting (7.18) multiplied by G from (7.19) we obtain

$$Lsg(s)u + (1 + \alpha s^2)(f(s)u + z(s)) + s^2 g(s)w = 0. \tag{7.20}$$

Since $g(s)$ cannot vanish identically on the imaginary axis, then we can pick $\alpha > 0$ such

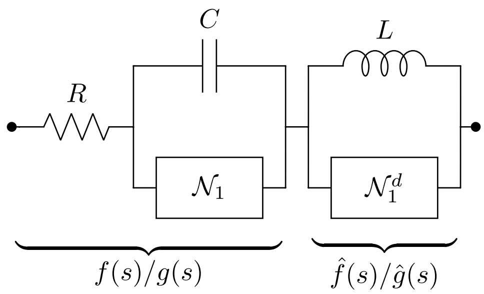


Figure 7.12: Two-terminal network with generic subnetworks $\mathcal{N}_1$ and $\mathcal{N}_1^d$.

that $g(j/\sqrt{\alpha}) \neq 0$. Substituting $s = j/\sqrt{\alpha}$ in (7.20) we obtain $g(j/\sqrt{\alpha})(Lju - w/\sqrt{\alpha}) = 0$, the only real solution of which is $u = w = 0$. From (7.20) it now follows that $z(s) = 0$. Equation (7.18) now reduces to

$$sg(s)v + (1 + \alpha s^2)y(s) = 0$$

from which we conclude, by substituting $s = j/\sqrt{\alpha}$, that $v = 0$. From the same equation we then conclude that $y(s) = 0$. We have therefore shown that (7.17) holds, hence $\mathcal{N}$ is generic.

□

Lemma 7.6. *Let $\mathcal{N}$ be an RLC network and $\mathcal{N}^d$ its dual network. If $\mathcal{N}$ is generic then so is $\mathcal{N}^d$.*

Proof. Let $\mathcal{N}$ have impedance $f(s)/g(s)$ and network elements $E_1, \dots, E_m$. Then $\mathcal{N}^d$ will have impedance $\hat{f}(s)/\hat{g}(s) = g(s)/f(s)$ and network elements $\hat{E}_1, \dots, \hat{E}_m$ such that $f(s, E_1, \dots, E_m) = \hat{g}(s, \hat{E}_1, \dots, \hat{E}_m)$ and $g(s, E_1, \dots, E_m) = \hat{f}(s, \hat{E}_1, \dots, \hat{E}_m)$. We can then easily conclude by applying Corollary 7.1 that if $\mathcal{N}$ is generic then also $\mathcal{N}^d$ is generic. □

Lemma 7.7. *Let $\mathcal{N}$ be an RLC two-terminal network with the structure shown in Figure 7.12, where the network $\mathcal{N}_1$ is generic and has no impedance pole at the origin, and $\mathcal{N}_1^d$ denotes its dual. Then $\mathcal{N}$ is generic.*

Proof. Let network $\mathcal{N}_1$ have element values $\mathbf{E} = (E_1, \dots, E_m) \in \mathbb{R}^m_{>0}$, and let its dual $\mathcal{N}_1^d$ have element values $\hat{\mathbf{E}} = (\hat{E}_1, \dots, \hat{E}_m) \in \mathbb{R}^m_{>0}$. Since $\mathcal{N}_1$ is generic, by Corollary 7.1 we can find element values $\mathbf{E}_0 = (E_{1,0}, \dots, E_{m,0})$ such that

$$\mathbf{t}_1 \in \mathbb{R}^{m+1} \text{ and } D_1\mathbf{t}_1 = \mathbf{0} \quad \Rightarrow \quad \mathbf{t}_1 = \mathbf{0}\,, \tag{7.21}$$

where

$$D_1 = \begin{pmatrix} \frac{\partial q}{\partial E_1} & \cdots & \frac{\partial q}{\partial E_m} & q \\ \frac{\partial d}{\partial E_1} & \cdots & \frac{\partial d}{\partial E_m} & d \end{pmatrix}_{\mathbf{E}_0}, \tag{7.22}$$

and such that, if the impedance of $\mathcal{N}_1$ is $q(s)/d(s)$, $q(s)$ and $d(s)$ are coprime. By taking the network dual of $\mathcal{N}_1$, we then obtain element values $\hat{\mathbf{E}}_{\mathbf{0}} = (\hat{E}_{1,0}, \ldots, \hat{E}_{m,0})$ for $\mathcal{N}_1^d$ such that

$$\mathbf{t}_2 \in \mathbb{R}^{m+1} \text{ and } D_2\mathbf{t}_2 = \mathbf{0} \quad \Rightarrow \quad \mathbf{t}_2 = \mathbf{0}\,, \tag{7.23}$$

where

$$D_2 = \begin{pmatrix} \frac{\partial d}{\partial \hat{E}_1} & \cdots & \frac{\partial d}{\partial \hat{E}_m} & d \\ \frac{\partial q}{\partial \hat{E}_1} & \cdots & \frac{\partial q}{\partial \hat{E}_m} & q \end{pmatrix}_{\hat{\mathbf{E}}_0}, \tag{7.24}$$

with the resulting impedance of $\mathcal{N}_1^d$ being $d(s)/q(s)$.

Let $Z(s) = a(s)/b(s)$ be the impedance of the network in Figure 7.12. Then $Z(s)$ may be written as

$$\begin{aligned} Z(s) = \frac{a(s)}{b(s)} &= \frac{f(s)}{g(s)} + \frac{\hat{f}(s)}{\hat{g}(s)} \\ &= \frac{d(s)R + q(s)(1+sRC)}{d(s)+sCq(s)} + \frac{sLd(s)}{d(s)+sLq(s)}\,, \end{aligned} \tag{7.25}$$

where $f(s)/g(s)$ and $\hat{f}(s)/\hat{g}(s)$ are the impedances of the two subnetworks indicated in Figure 7.12. From the expressions in (7.25) we see that, if $L \neq C$, then $g(s)$ and $\hat{g}(s)$ are necessarily coprime, since $d(0) \neq 0$ by assumption and $q(s)$ and $d(s)$ are coprime. We can also easily see from (7.25) that $\hat{f}(0) = 0$, $\hat{g}(0) \neq 0$, $f(0) \neq 0$, $g(0) \neq 0$. Furthermore, denoting $\deg(g(s))$ by n, then $\deg(\hat{g}(s)) = n$ and $\deg(f(s))$, $\deg(\hat{f}(s)) \leq n$.

We will now show that,

$$\mathbf{x} \in \mathbb{R}^{2m+4} \text{ and } D\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}\,, \tag{7.26}$$

where

$$D = \begin{pmatrix} \frac{\partial a}{\partial R} & \frac{\partial a}{\partial C} & \frac{\partial a}{\partial E_1} & \cdots & \frac{\partial a}{\partial E_m} & \frac{\partial a}{\partial L} & \frac{\partial a}{\partial \hat{E}_1} & \cdots & \frac{\partial a}{\partial \hat{E}_m} & a \\ \frac{\partial b}{\partial R} & \frac{\partial b}{\partial C} & \frac{\partial b}{\partial E_1} & \cdots & \frac{\partial b}{\partial E_m} & \frac{\partial b}{\partial L} & \frac{\partial b}{\partial \hat{E}_1} & \cdots & \frac{\partial b}{\partial \hat{E}_m} & b \end{pmatrix}, \tag{7.27}$$

for element values $\mathbf{E}_0$, $\hat{\mathbf{E}}_0$, R_0, L_0, C_0, where $L_0 \neq C_0$. By the chain rule, D may be expressed as

$$D = \underbrace{\begin{pmatrix} \frac{\partial a}{\partial f} & \frac{\partial a}{\partial g} & \frac{\partial a}{\partial \hat{f}} & \frac{\partial a}{\partial \hat{g}} \\ \frac{\partial b}{\partial f} & \frac{\partial b}{\partial g} & \frac{\partial b}{\partial \hat{f}} & \frac{\partial b}{\partial \hat{g}} \end{pmatrix}_{\mathbf{E}_0, R_0, C_0, \hat{\mathbf{E}}_0, L_0}}_{M} \underbrace{\left(\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_2 \end{array}\right)}_{N}, \tag{7.28}$$

where

$$Q_1 = \begin{pmatrix} \frac{\partial f}{\partial R} & \frac{\partial f}{\partial C} & \frac{\partial f}{\partial E_1} & \cdots & \frac{\partial f}{\partial E_m} \\ \frac{\partial g}{\partial R} & \frac{\partial g}{\partial C} & \frac{\partial g}{\partial E_1} & \cdots & \frac{\partial g}{\partial E_m} \end{pmatrix}_{\mathbf{E}_0, R_0, C_0},$$

$$Q_2 = \begin{pmatrix} \frac{\partial \hat{f}}{\partial L} & \frac{\partial \hat{f}}{\partial \hat{E}_1} & \cdots & \frac{\partial \hat{f}}{\partial \hat{E}_m} & \hat{f} \\ \frac{\partial \hat{g}}{\partial L} & \frac{\partial \hat{g}}{\partial \hat{E}_1} & \cdots & \frac{\partial \hat{g}}{\partial \hat{E}_m} & \hat{g} \end{pmatrix}_{\hat{\mathbf{E}}_0, L_0}. \tag{7.29}$$

We therefore need to show that

$$\mathbf{x} \in \mathbb{R}^{2m+4} \text{ and } D\mathbf{x} = MN\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{0}\,. \tag{7.30}$$

Consider a fixed but arbitrary $\mathbf{x} \in \mathbb{R}^{2m+4}$, let $\mathbf{y} = N\mathbf{x}$, and note that $\mathbf{y}$ takes the form $(u(s), v(s), w(s), z(s))$, where $u(s)$, $v(s)$, $w(s)$, $z(s)$ are polynomials of degree less than or equal to n and $w(0) = 0$ (since $\hat{f}(0) = 0$). We will first show that if $M\mathbf{y} = \mathbf{0}$ then $\mathbf{y} = \alpha(f(s), g(s), -\hat{f}(s), -\hat{g}(s))$ for some real constant α. The matrix equation $M\mathbf{y} = \mathbf{0}$ yields the following two polynomial equations:

$$\hat{g}(s)u(s) + \hat{f}(s)v(s) + g(s)w(s) + f(s)z(s) = 0\,, \tag{7.31}$$

$$\hat{g}(s)v(s) + g(s)z(s) = 0\,. \tag{7.32}$$

Equation (7.32) can be written as $z(s)/v(s) = -\hat{g}(s)/g(s)$, from which we conclude that $v(s) = \alpha g(s)$ for some real constant α, since $g(s)$ and $\hat{g}(s)$ are coprime polynomials with $\deg(g(s)) = \deg(\hat{g}(s)) = n$, while $\deg(v(s))$, $\deg(z(s)) \leq n$. From (7.32) it then follows that $z(s) = -\alpha\hat{g}(s)$. Equation (7.31) now reduces to

$$\hat{g}(s)(u(s) - \alpha f(s)) + g(s)(w(s) + \alpha\hat{f}(s)) = 0\,. \tag{7.33}$$

We recall that $w(0) = \hat{f}(0) = 0$ and $\hat{g}(0) \neq 0$. Therefore, for $s = 0$, (7.33) yields $\hat{g}(0)(u(0) - \alpha f(0)) = 0$, from which we conclude that $u(s) - \alpha f(s)$ is divisible by s. But by writing (7.33) as

$$\frac{w(s) + \alpha\hat{f}(s)}{u(s) - \alpha f(s)} = -\frac{\hat{g}(s)}{g(s)}$$

we can conclude that $u(s) - \alpha f(s)$ is also divisible by $g(s)$, since $g(s)$ and $\hat{g}(s)$ are coprime and $\deg(u(s) - \alpha f(s)) \leq n$. Therefore $u(s) - \alpha f(s)$ is divisible by $sg(s)$ (which has degree $n+1$), from which it follows that $u(s) = \alpha f(s)$ necessarily. Equation (7.31) finally gives $w(s) = -\alpha \hat{f}(s)$.

At this point we have shown that

$$\mathbf{x} \in \mathbb{R}^{2m+4} \text{ and } MN\mathbf{x} = \mathbf{0} \;\Rightarrow\; N\mathbf{x} = \alpha \begin{pmatrix} f(s) \\ g(s) \\ -\hat{f}(s) \\ -\hat{g}(s) \end{pmatrix}. \tag{7.34}$$

If we partition $\mathbf{x}$ into two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ each of dimension $m+2$, the right-hand side of (7.34) may be written as

$$\left(\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_2 \end{array}\right) \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} - \alpha \begin{pmatrix} f(s) \\ g(s) \\ -\hat{f}(s) \\ -\hat{g}(s) \end{pmatrix} = \mathbf{0},$$

which is equivalent to

$$\left(\begin{array}{c|c} Q_1 & \begin{matrix} f(s) \\ g(s) \end{matrix} \end{array}\right) \begin{pmatrix} \mathbf{x}_1 \\ -\alpha \end{pmatrix} = \mathbf{0}, \tag{7.35}$$

$$\left(\begin{array}{c|c} Q_2 & \begin{matrix} -\hat{f}(s) \\ -\hat{g}(s) \end{matrix} \end{array}\right) \begin{pmatrix} \mathbf{x}_2 \\ -\alpha \end{pmatrix} = \mathbf{0}. \tag{7.36}$$

It follows from (7.21)–(7.24) and the proof of Lemma 7.4 that

$$\mathbf{t}_1 \in \mathbb{R}^{m+3} \text{ and } \left(\begin{array}{c|c} Q_1 & \begin{matrix} f(s) \\ g(s) \end{matrix} \end{array}\right) \mathbf{t}_1 = \mathbf{0} \quad\Rightarrow\quad \mathbf{t}_1 = \mathbf{0},$$
$$\text{and} \qquad \mathbf{t}_2 \in \mathbb{R}^{m+2} \text{ and } Q_2\mathbf{t}_2 = \mathbf{0} \quad\Rightarrow\quad \mathbf{t}_2 = \mathbf{0}.$$

Therefore we can conclude from (7.35) that $\mathbf{x}_1 = \mathbf{0}$, $\alpha = 0$ and from (7.36) that $\mathbf{x}_2 = \mathbf{0}$. Therefore (7.30) holds and, by Corollary 7.1, the network $\mathcal{N}$ is generic. □

Remark 7.3. Lemma 7.7 may be generalised to the series connection of two RLC two-terminal networks $\mathcal{N}_1$ and $\mathcal{N}_2$. Namely, under the following assumptions we may conclude that the series connection of $\mathcal{N}_1$ and $\mathcal{N}_2$ is generic:

- The two networks are generic;
- One of the two networks has an impedance zero at the origin, and the other does not;
- The two networks do not have any coincident impedance poles for almost all element values.

We now have all the ingredients to present a proof of the genericity of the Bott-Duffin networks. We note that, if the impedance function $Z(s)$ is a biquadratic, then the Bott-Duffin method leads to a generic network with the structure of Figure 7.8, as already discussed in Example 7.7. However, it remains to consider the cases for which the impedance is not biquadratic.

Theorem 7.3. *Any positive-real impedance can be realised by a generic RLC network.*

Proof. The Bott-Duffin theorem states that any positive-real impedance function can be realised by an RLC network (see Section 2.5). It therefore suffices to show that each of the steps involved in the construction of such a network $\mathcal{N}$ preserves genericity (see [27] for a textbook explanation of the Bott-Duffin procedure).

To obtain a network $\mathcal{N}$ to realise an arbitrary given positive-real function $Z(s)$, the steps in the Bott-Duffin procedure (coupled with the so-called Foster preamble) are as follows:

1. Subtract any imaginary axis impedance poles (resulting in an impedance of lower order).
2. Subtract a constant equal to the smallest value of $\text{Re}(Z(j\omega))$ for $\omega \in \mathbb{R} \cup \infty$, resulting in a network whose impedance $\hat{Z}(s)$ has no imaginary axis impedance poles and satisfies one of the following properties:
   (a) $\hat{Z}(s)$ has an admittance pole at the origin or at infinity;
   (b) $\hat{Z}(s)$ has an admittance pole elsewhere on the imaginary axis;
   (c) $\hat{Z}(s)$ is a minimum function.

   In each case, the impedance can then be reduced to one of lower order.

The network realisations corresponding to cases 1, 2a, 2b and 2c each take the form of one of the networks in Figures 7.10–7.12, or can be obtained from such networks through a combination of frequency inversion and duality transformations (in certain cases it is necessary to replace the resistor by a short or open circuit). That genericity is preserved in each case can be shown using Lemmas 7.4–7.7 and minor modifications thereof. The Bott-Duffin procedure continues inductively until the resulting impedance has order zero. This final impedance can be realised by a single resistor, which itself is generic. This establishes the genericity of all of the other networks in the inductive procedure, whereupon we conclude that $\mathcal{N}$ is generic. □

7.5 Summary

In this chapter we have further developed the notion of a generic network, that is a network which realises a set of impedance parameters of dimension one more than the number of elements in the network. A necessary and sufficient condition was provided to test the genericity of any given network without requiring the knowledge of its realisability set, and was applied to a series of illustrative examples. This test can prove to be particularly useful in the analysis of high-order networks, for which obtaining realisability conditions expressed in a meaningful way is a virtually impossible task—for example, a new equivalence might simplify a given network, thus making it not suitable as a candidate for a network synthesis problem. Finally, we proved that a network with a non-generic subnetwork is itself non-generic, and that any positive-real impedance can be realised by a generic network.

This chapter is the result of work carried out in collaboration with T.H. Hughes. This work was presented in [35] and submitted for publication in [36].

Chapter 8

Conclusions

The main focus of this dissertation has been on obtaining a complete understanding of the enumerative approach to passive network synthesis in the simplest non-trivial case, which led to further, more general results being obtained. Some useful notions were first introduced for the formal classification of the class of networks comprising at most two reactive elements and at most three resistors—the networks of the Ladenheim catalogue. Based on the analysis of this fundamental class of networks, a number of results were proven and a new notion of genericity was introduced, which proves to be useful in addressing questions of minimality in RLC networks.

We summarise below the main contributions of the dissertation.

8.1 Contributions of the dissertation

- Chapter 4 described a formal derivation of the Ladenheim catalogue. Before this fresh derivation, no summary of the procedure to obtain the set of 108 networks exsisted in the literature, and no guarantee that the enumeration carried out in [52] was error-free. A formal notion of "realisability set" was introduced as a semi-algebraic set in the space of impedance parameters in order to compare the realisation power of all the networks. A previous notion of "network quartet" was replaced by the orbits induced by the group actions `s` (frequency inversion) and `p` (circuit dual without element dual), together with `d` (dual) and `e` (identity). The use of `s` and `p` as the primary representatives is new in the present work and was suggested by the more convenient grouping of the equivalence classes within the subfamilies.

  Together with the notion of equivalence, a systematic analysis of the class of networks was made possible. This approach was presented in as formal a way as possible, in order for it to be generalisable to other classes of networks of restricted complexity. A possible generalisation of this approach was in fact outlined in Chapter 6, where the class of six-element networks with four resistors was studied.

- The notions of equivalence and group action allowed an initial partition of the catalogue into 24 subfamilies, outlined in Chapter 5. Expressions for the network parameters for one representative network in each subfamily were obtained, and necessary and sufficient conditions on the impedance coefficients were derived which guarantee positivity of the network parameters. These conditions were obtained from "realisation theorems" for the five-element networks, and were proven in Appendix A. All algebraic manipulations in the proofs were verified in Maple.

  Despite realisability conditions having already been derived for some networks in the catalogue in [43] and [16], realisability conditions for most of the networks were still not known, and the issue of the multiplicity of solutions had not been addressed in some of the existing derivations.

  The structure that emerged from the realisability analysis was summarised in Figures 5.1 and 5.2, which provide a useful diagrammatic representation of the various relations which connect the 108 networks.

- Knowing the realisability conditions for all the networks in the Ladenheim catalogue, it was possible to settle a number of outstanding questions in Chapter 6, and to provide formal proofs to a series of results. Namely, it was shown that no new equivalences exist within the catalogue that were not known classically, and the question of the smallest generating set for the class was settled. It was also shown that the classical Cauer and Foster forms are almost but not completely equivalent, and may be termed quasi-equivalent. It was further shown that the catalogue comprises 62 equivalence classes, arranged in subfamilies which represent the "join" of the two equivalence relations given by the group action and network equivalence.

  A new notion of generic network was introduced, based on the dimension of the realisability set that the network defines. It was shown that the Ladenheim catalogue forms the complete set of generic, two-terminal RLC networks with at most two reactive elements.

  Other useful results stemmed from the analysis of the catalogue, namely it was proven that the property of invariance to duality in networks can occur independently of invariance to frequency inversion and $\mathtt{p}$ transformation, and two new equivalences between RLC networks were identified in the study of six-element, four-resistive networks.

- A necessary and sufficient condition for the genericity of an arbitrary two-terminal RLC network was proven in Chapter 7. This provided an efficient way to test whether a network can only lead to non-minimal realisations which does not require deriving the realisability set of the network itself (which can be a virtually impossible task for higher-order networks). It was proven that any network with a non-generic subnetwork is necessarily non-generic, and that any positive-real impedance has a generic realisation, which is obtained through the Bott-Duffin method. The contributions of this chapter were the result of joint work with T.H. Hughes.

8.2 Directions for future research

- In Section 6.5 it was proven that the networks in Figure 6.8 are not equivalent, by searching for an impedance which is realisable by one network and not by the other. It would be useful to obtain a more formal procedure that can be automated to check the equivalence of any two given networks. A conjecture in this regard is that a necessary condition for two generic networks to be equivalent is that they have the same total number of elements, and more specifically the same number of each type of element.

- The so-called "structure-imittance" approach was introduced in [92], in the context of mechanical network synthesis. This approach seeks an optimal network configuration and element values among all networks with a certain number of springs, dampers and inerters, while also allowing constraints on the element values. The case of networks with one damper, one inerter and at most two springs is analysed in [92]. This approach represents a different way to enumerate circuits, and it would be interesting to see how it relates to the results that were obtained for the Ladenheim catalogue (i.e. considering networks with one spring, one inerter and at most three dampers, in mechanical terms).

- The next class of networks that one could analyse in terms of complexity would be the set of all generic networks with three reactive elements and at most four resistors. A complete enumeration of the networks in this class has never been attempted, and very few studies exist in the literature on impedance functions of order three (bicubics), with the exception of some special subclasses. We note that such a classification would prove to be considerably harder than the biquadratic case, due to both the much larger number of networks in the class and the increased complexity in deriving realisability conditions. A classification of this class might lead, however, to new network equivalences being discovered and other general results in passive network synthesis being established.

Appendix A

Realisation theorems

A.1 Equivalence class $\mathbf{IV_B^1}$

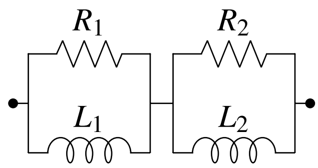


Figure A.1: Network #38, from subfamily $\mathrm{IV_B}$. By Lemma 8 in [43] it can only realise regular impedances.

Theorem A.1. *Let A, B, C, D, E, $F \geq 0$. The positive-real biquadratic impedance* (4.1) *can be realised as in Figure A.1, with R_1, R_2, L_1 and L_2 positive and finite, if and only if*

$$A, B, D, E, F > 0 \tag{A.1}$$

$$C = 0\,, \tag{A.2}$$

and either

$$K < 0 \tag{A.3}$$

or

$$K = 0 \;\; \textit{and} \;\; E^2 - 4DF = 0\,, \tag{A.4}$$

where K is defined in (3.3)*. If condition* (A.3) *is satisfied, then R_1 and R_2 are the two* *roots of the quadratic equation in* x

$$-D(E^2-4DF)\,x^2+A(E^2-4DF)\,x+(A^2F-ABE+B^2D)=0\,, \tag{A.5}$$

while, if (A.4) *holds, then* R_1 *and* R_2 *are any two positive values such that* $R_1+R_2=A/D$*. The two inductances can be obtained as*

$$L_1=\frac{R_1(B-ER_2)}{F(R_1-R_2)}\,, \tag{A.6}$$

$$L_2=\frac{R_2(B-ER_1)}{F(R_2-R_1)}\,, \tag{A.7}$$

if $R_1\neq R_2$*, and as the two roots of the quadratic equation in* y

$$2EF\,y^2-2BE\,y+AB=0 \tag{A.8}$$

if $R_1=R_2$*. Due to the symmetry of the network, the two solutions of the quadratic do not lead to two properly distinct solutions.*

Proof. Necessity. The impedance of the network shown in Figure A.1 is a biquadratic, which can be computed as

$$Z(s)=\frac{n(s)}{d(s)}\,, \tag{A.9}$$

where

$$n(s)=L_1L_2(R_1+R_2)s^2+R_1R_2(L_1+L_2)s\,,$$
$$d(s)=L_1L_2s^2+(L_1R_2+L_2R_1)s+R_1R_2\,.$$

Equating impedance (A.9) with (4.1) we obtain, for a positive constant k,

$$L_1L_2(R_1+R_2)=kA\,, \tag{A.10}$$

$$R_1R_2(L_1+L_2)=kB\,, \tag{A.11}$$

$$0=kC\,, \tag{A.12}$$

$$L_1L_2=kD\,, \tag{A.13}$$

$$L_1R_2+L_2R_1=kE\,, \tag{A.14}$$

$$R_1R_2=kF\,, \tag{A.15}$$

which are a set of necessary and sufficient conditions for (4.1) to be realised as in Figure A.1. It can be calculated that

$$K = -k^{-4}\, L_1 L_2 R_1^2 R_2^2 (L_1 R_2 - L_2 R_1)^2 \le 0\,, \tag{A.16}$$

$$E^2 - 4DF = k^{-2}\, (L_1 R_2 - L_2 R_1)^2 \ge 0\,, \tag{A.17}$$

from which we can conclude that

$$K = 0 \;\Leftrightarrow\; E^2 - 4DF = 0 \;\Leftrightarrow\; L_1 R_2 = L_2 R_1\,, \tag{A.18}$$

hence (A.1)–(A.4) are necessary.

Sufficiency. We now assume that (A.1)–(A.2) hold and either (A.3) or (A.4). We show that we can find R_1, R_2, L_1, L_2 positive which satisfy (A.10)–(A.15) with $k > 0$.

Eliminating $R_1 R_2$ from (A.11) and (A.15) we obtain

$$L_2 = B/F - L_1\,, \tag{A.19}$$

and from (A.14) and (A.19) we then obtain

$$k = \frac{L_1 R_2 + L_2 R_1}{E} = \frac{F L_1 (R_2 - R_1) + B R_1}{EF}\,. \tag{A.20}$$

Substituting (A.20) into (A.15) and solving for L_1 we obtain (A.6), and from (A.19) and (A.6) we obtain (A.7), providing $R_1 \neq R_2$ (which we assume for the time being). Eliminating $L_1 L_2$ from (A.10) and (A.13) we obtain

$$R_2 = A/D - R_1\,. \tag{A.21}$$

Substituting the values thus obtained for L_1, L_2, R_2 and k into (A.13) we get the quadratic (A.5) in R_1, and we note from the expression in Table 5.9 that

$$K|_{C=0} = F(A^2 F - ABE + B^2 D)\,, \tag{A.22}$$

which is the third coefficient in (A.5). It is easily seen that the sum of the two roots of (A.5) is A/D, hence R_2 can be obtained as the other root of the same quadratic, as stated in the theorem.

It can be easily verified that

$$A^2(E^2 - 4DF) = (AE - 2BD)^2 - 4DK/F\,, \tag{A.23}$$

hence condition (A.3) implies $E^2 - 4DF > 0$. We can therefore conclude that, if (A.3) holds, the first and third coefficients in (A.5) are strictly negative, while the second coefficient is strictly positive. It also follows from $E^2 - 4DF > 0$ that the discriminant of (A.5),

$$\Delta_x = (E^2 - 4DF)(AE - 2BD)^2 \,,$$

is greater than or equal to zero, hence the quadratic has two real positive roots. Since we are assuming $R_1 \neq R_2$, the discriminant will necessarily be non-zero, hence $AE-2BD \neq 0$ in this case. We finally verify that the values obtained from (A.6) and (A.7) for L_1 and L_2 are always positive. Since R_1 and R_2 are the two distinct roots of the quadratic (A.5), L_1 and L_2 will be positive if B/E lies strictly between the two roots of (A.5) (from the expressions in (A.6) and (A.7)). This is true providing

$$-D(E^2 - 4DF)\,x^2 + A(E^2 - 4DF)\,x + (A^2F - ABE + B^2D) > 0 \tag{A.24}$$

for $x = B/E$. After some manipulation, inequality (A.24) can be reduced to $F(AE - 2BD)^2/E > 0$, which holds in this case.

If $R_1 \neq R_2$ and conditions (A.4) hold, then all the coefficients in (A.5) are zero and any value of x solves the quadratic. Therefore R_1 can be chosen arbitrarily, providing $R_2 = A/D - R_1 > 0$. From (A.23), conditions (A.4) imply $AE - 2BD = 0$. It was observed above that L_1 and L_2 are positive if B/E lies strictly between the two resistances. Without loss of generality we can assume $R_1 < R_2$ and, since $R_1 + R_2 = A/D$, it follows that $R_1 < A/(2D) < R_2$. Since $A/(2D) = B/E$ it then follows that $R_1 < B/E < R_2$, hence L_1 and L_2 are positive. We have therefore shown that (A.10)–(A.15) can be satisfied for R_1, R_2, L_1, $L_2 > 0$ (under the assumption that $R_1 \neq R_2$ and either (A.3) or (A.4) hold).

We now turn to the case that $R_1 = R_2$. Equations (A.19) and (A.20) still hold, and eliminating L_1L_2 from (A.10) and (A.13) we obtain $R_1 = R_2 = A/(2D)$. Equation (A.15), for $R_1 = R_2$, results in the identity

$$AE - 2BD = 0\,. \tag{A.25}$$

Substituting the values thus obtained for R_1, R_2, L_2 and k into (A.13) we obtain the quadratic (A.8) in L_1. It can be easily seen that the sum of the two roots of (A.8) is B/F, hence L_2 can also be obtained as a root of (A.8)[1]. It can also be easily verified

[1] Again we note that, since $R_1 = R_2$, swapping the order of the inductors does not lead to two properly distinct solutions.

that

$$A(2AF - BE) = 2K/F + B(AE - 2BD)$$

which, by virtue of (A.25), reduces to $A(2AF - BE) = 2K/F$. The discriminant of (A.8),

$$\Delta_y = -4BE(2AF - BE) = -8BEK/(AF)\,,$$

is therefore greater than or equal to zero, if either (A.3) or (A.4) hold. Given the signs of the coefficients in (A.8), we can therefore conclude that the two roots of the quadratic are positive and real. We finally note that replacing $x = A/(2D)$ in the quadratic (A.5) we obtain $(AE - 2BD)^2/(4D)$, which by virtue of (A.25) is zero. Hence, even in this case R_1 and R_2 can be obtained as the two roots of (A.5). □

A.2 Equivalence class $\mathbf{V_A^1}$

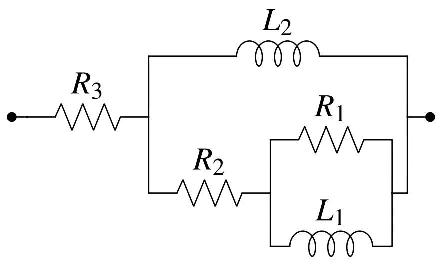


Figure A.2: Network #52, from subfamily V_A. By Theorem 1 in [43] it can only realise regular impedances.

Theorem A.2. *The positive-real biquadratic impedance* (4.1) *with* A, B, C, D, E, $F > 0$ *can be realised as in Figure A.2, with* R_1, R_2, R_3, L_1 *and* L_2 *positive and finite, if and only if*

$$AF - CD > 0\,, \tag{A.26}$$

$$K < 0\,. \tag{A.27}$$

If conditions (A.26) *and* (A.27) *are satisfied, then*

$$R_1 = \frac{-K}{D\lambda_1}\,, \tag{A.28}$$

$$R_2 = \frac{(BF - CE)^2}{F\lambda_1}\,, \tag{A.29}$$

$$R_3 = \frac{C}{F}\,, \tag{A.30}$$

$$L_1 = \frac{-K(BF - CE)}{\lambda_1^2}\,, \tag{A.31}$$

$$L_2 = \frac{BF - CE}{F^2}\,. \tag{A.32}$$

where λ_1 *and* K *are defined in Table 5.9.*

Proof. Note this result is given without proof in [43, Appendix A], with R_1 and R_3 interchanged. We provide here a proof for convenience.

Necessity. The impedance of the network shown in Figure A.2 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.33}$$

where

$$n(s) = L_1L_2(R_1 + R_2 + R_3)s^2 + \big(L_1R_3(R_1 + R_2) + L_2R_1(R_2 + R_3)\big)s + R_1R_2R_3\,,$$
$$d(s) = L_1L_2s^2 + \big(L_2R_1 + L_1(R_1 + R_2)\big)s + R_1R_2\,.$$

Equating impedance (A.33) with (4.1), we obtain, for a positive constant k,

$$L_1L_2(R_1 + R_2 + R_3) = kA\,, \tag{A.34}$$

$$L_1R_3(R_1 + R_2) + L_2R_1(R_2 + R_3) = kB\,, \tag{A.35}$$

$$R_1R_2R_3 = kC\,, \tag{A.36}$$

$$L_1L_2 = kD\,, \tag{A.37}$$

$$L_2R_1 + L_1(R_1 + R_2) = kE\,, \tag{A.38}$$

$$R_1R_2 = kF\,, \tag{A.39}$$

which are a set of necessary and sufficient conditions for (4.1) to be realised as in Figure A.2. It can be calculated that

$$AF - CD = k^{-2}\, R_1R_2L_1L_2\,(R_1 + R_2) > 0\,, \tag{A.40}$$

$$K = -k^{-4}\, R_1^4R_2^2L_1L_2^3 < 0\,, \tag{A.41}$$

hence (A.26) and (A.27) are necessary.

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$ satisfying conditions (A.26) and (A.27) we now show that we can find R_1, R_2, R_3, L_1, L_2 positive which satisfy (A.34)–(A.39) with $k > 0$.

From (A.39) we obtain $k = R_1R_2/F$ and eliminating the term R_1R_2 from (A.36) and (A.39) we get (A.30). Eliminating L_1 from (A.35) and (A.38) we obtain (A.32), while eliminating the term L_1L_2 from (A.34) and (A.37) and solving for R_2 we get

$$R_2 = \frac{AF - CD - DFR_1}{DF}\,. \tag{A.42}$$

Solving for L_1 from (A.37) using expressions obtained we find:

$$L_1 = \frac{R_1\,(AF - CD - DFR_1)}{BF - CE}\,. \tag{A.43}$$

We now have expressions for R_2, R_3, L_1, L_2 and k which only contain R_1 together with A, B, C, D, E, F. Substituting such expressions into (A.38) and solving for R_1 we obtain (A.28) and, substituting the latter into (A.42) and (A.43), we get (A.29) and (A.31), respectively.

Conditions (A.26) and (A.27) imply $BF - CE > 0$, from which it can be shown, using again (A.26) and (A.27), that $\lambda_1 > 0$ (note this also follows from [43, Lemma 7]). Hence all network elements are positive. □

A.3 Equivalence class $\mathbf{V_B^1}$

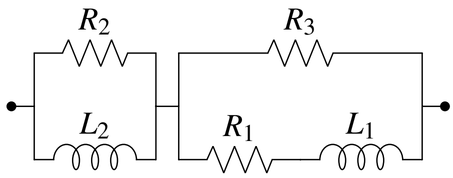


Figure A.3: Network #59, from subfamily V_B. By Theorem 1 in [43] it can only realise regular impedances.

Theorem A.3. *The positive-real biquadratic impedance* (4.1) *with A, B, C, D, E, $F > 0$ can be realised as in Figure A.3, with R_1, R_2, R_3, L_1 and L_2 positive and finite,* *if and only if*

$$AF - CD > 0 \tag{A.44}$$

and either

$$K < 0 \tag{A.45}$$

or

$$K = 0 \quad and \quad E^2 - 4DF = 0\,, \tag{A.46}$$

where K *is defined in* (3.3)*. In case* (A.45)*,* R_2 *is either of the two positive roots of the quadratic equation in* x

$$c_1\, x^2 + c_2\, x + c_3 = 0\,, \tag{A.47}$$

where

$$\begin{aligned} c_1 &= -DF(E^2 - 4DF)\,,\\ c_2 &= (E^2 - 4DF)(AF - CD)\,,\\ c_3 &= K\,, \end{aligned}$$

while, in case (A.46)*,* R_2 *may take any positive value strictly less than* $(AF - CD)/(DF)$*. The other network elements are given by*

$$R_3 = A/D - R_2\,, \tag{A.48}$$

$$R_1 = \frac{CR_3}{FR_3 - C}\,, \tag{A.49}$$

$$L_1 = \frac{D(R_1R_2 + R_2R_3 - R_3^2)}{B - ER_3}\,, \tag{A.50}$$

$$L_2 = \frac{DR_2(R_1 + R_3)}{FL_1}\,, \tag{A.51}$$

unless $R_2 = (AE - BD)/(DE)$*, in which case* L_1 *is either of the two positive roots of the quadratic equation in* y

$$d_1\, y^2 + d_2\, y + d_3 = 0\,, \tag{A.52}$$

where

$$\begin{aligned} d_1 &= E^2F(BF - CE)^2\,,\\ d_2 &= -B^2E^2F(BF - CE)\,,\\ d_3 &= B^4DF\,, \end{aligned}$$

and the other elements are still given by (A.48)–(A.49) *and* (A.51)*.*

Proof. Necessity. The impedance of the network shown in Figure A.3 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.53}$$

where

$$n(s) = L_1L_2(R_2+R_3)s^2 + \big(L_1R_2R_3 + L_2(R_1R_2+R_1R_3+R_2R_3)\big)s + R_1R_2R_3\,,$$
$$d(s) = L_1L_2s^2 + \big(L_1R_2 + L_2(R_1+R_3)\big)s + R_2(R_1+R_3)\,.$$

Equating impedance (A.53) with (4.1), we obtain, for a positive constant k,

$$L_1L_2(R_2+R_3) = kA\,, \tag{A.54}$$
$$L_1R_2R_3 + L_2(R_1R_2+R_1R_3+R_2R_3) = kB\,, \tag{A.55}$$
$$R_1R_2R_3 = kC\,, \tag{A.56}$$
$$L_1L_2 = kD\,, \tag{A.57}$$
$$L_1R_2 + L_2(R_1+R_3) = kE\,, \tag{A.58}$$
$$R_2(R_1+R_3) = kF\,, \tag{A.59}$$

which are a set of necessary and sufficient conditions for (4.1) to be realised as in Figure A.3. It can be calculated that

$$AF - CD = k^{-2}\, L_1L_2R_2(R_1R_2+R_2R_3+R_3^2) > 0\,, \tag{A.60}$$
$$K = -k^{-4}\, L_1L_2R_2^2R_3^2(L_1R_2 - L_2(R_1+R_3))^2 \le 0\,, \tag{A.61}$$
$$E^2 - 4DF = k^{-2}\,(L_1R_2 - L_2(R_1+R_3))^2 \ge 0\,, \tag{A.62}$$

and from (A.61) and (A.62) we can conclude that

$$K = 0 \;\Leftrightarrow\; E^2 - 4DF = 0 \;\Leftrightarrow\; L_1R_2 = L_2(R_1+R_3)\,,$$

hence (A.44)–(A.46) are necessary.

Sufficiency. We now assume that (A.44) and either (A.45) or (A.46) hold. We show that we can find R_1, R_2, R_3, L_1, L_2 positive which satisfy (A.54)–(A.59) with $k > 0$. From (A.44) and $K \le 0$ it follows that $AE - BD > 0$ and $BF - CE > 0$, which we will assume for the rest of the proof.

From (A.59) we obtain

$$k = R_2(R_1+R_3)/F\,, \tag{A.63}$$

and from (A.57) and (A.63) we get (A.51). Substituting (A.63) into (A.56) we obtain (A.49) (and it will follow below that $R_3 \neq C/F$, using (A.67) and (A.69)). Eliminating L_1L_2 from (A.54) and (A.57) we get (A.48). Using (A.63) and (A.51), equations (A.55) and (A.58) reduce to

$$FR_2R_3\,L_1^2 - BR_2(R_1+R_3)\,L_1 + DR_2(R_1+R_3)(R_1R_2+R_1R_3+R_2R_3) = 0\,,$$
$$FR_2\,L_1^2 - ER_2(R_1+R_3)\,L_1 + DR_2(R_1+R_3)^2 = 0\,.$$

Eliminating the L_1^2 term from the first equation we obtain

$$L_1(B-ER_3) - D(R_1R_2+R_2R_3-R_3^2) = 0\,. \tag{A.64}$$

Assuming $R_3 \neq B/E$ and solving for L_1 we obtain (A.50). We now have expressions for R_1, R_3, L_1, L_2 and k which only contain R_2, together with A, B, C, D, E, F. Substituting such expressions into (A.58) we get the quadratic equation (A.47) in R_2. We note that $R_3 \neq B/E$ is equivalent to $R_2 \neq (AE-BD)/(DE)$, from (A.48).

We now assume that conditions (A.44)–(A.45) hold with $R_2 \neq (AE-BD)/(DE)$. The discriminant of (A.47) is given by

$$\Delta_c = (E^2-4DF)\,\theta_2^2\,,$$

where $\theta_2 = AEF - 2BDF + CDE$, and it can be verified that the following identity always holds:

$$(BF-CE)(AE-BD)(E^2-4DF) = \theta_2^2 - E^2\,K\,. \tag{A.65}$$

From (A.65) we can conclude that, if (A.44)–(A.45) hold, then $E^2-4DF>0$ necessarily, hence the first and third coefficient in (A.47) are negative while the second is positive. If $\theta_2 = 0$, then the discriminant of (A.47) is zero, and the two coincident roots of (A.47) are $R_2 = (AF-CD)/(2DF)$. The following identity always holds:

$$\frac{AE-BD}{DE} - \frac{AF-CD}{2DF} = \frac{\theta_2}{2DEF}\,, \tag{A.66}$$

from which it follows, if $\theta_2 = 0$, that

$$R_2 = \frac{AF-CD}{2DF} = \frac{AE-BD}{DE}\,,$$

which is a contradiction in this case. We can therefore conclude that $\theta_2 \neq 0$, i.e. the discriminant of (A.47) is strictly greater than zero. Therefore (A.47) has two distinct positive roots, which we denote as R_{2_A} and R_{2_B}, with $R_{2_A} < R_{2_B}$. From (A.48), R_3 is positive for both sets of solutions if $R_{2_B} < A/D$. From (A.49), making use of (A.48), and from (A.50), making use of the identities (A.48) and (A.49), we obtain:

$$R_1 = \frac{CDR_3}{-DF\,R_2 + AF - CD}\,, \tag{A.67}$$

$$L_1 = \frac{DR_1R_3(-2D\,FR_2 + AF - CD)}{C(-DE\,R_2 + AE - BD)}\,. \tag{A.68}$$

We now show that

$$\begin{aligned} &R_{2_A} < \min\left\{\frac{AE-BD}{DE}, \frac{AF-CD}{2DF}\right\}, \\ &\max\left\{\frac{AE-BD}{DE}, \frac{AF-CD}{2DF}\right\} < R_{2_B} < \frac{AF-CD}{DF} < \frac{A}{D}\,, \end{aligned} \tag{A.69}$$

from which it will follow from (A.48), (A.67) and (A.68) that $R_3 > 0$, $R_1 > 0$ and $L_1 > 0$ for both solutions (we note, incidentally, from (A.66) that

$$\frac{AE-BD}{DE} \gtrless \frac{AF-CD}{2DF}\,,$$

depending on the sign of θ_2). We first show that $(AE - BD)/(DE)$ and $(AF - CD)/(2DF)$ always lie between the two roots of (A.47). Since (A.47) represents a parabola that opens down, the latter is true if and only if

$$c_1\left(\frac{AE-BD}{DE}\right)^2 + c_2\left(\frac{AE-BD}{DE}\right) + c_3 > 0\,, \tag{A.70}$$

$$c_1\left(\frac{AF-CD}{2DF}\right)^2 + c_2\left(\frac{AF-CD}{2DF}\right) + c_3 > 0\,. \tag{A.71}$$

After some manipulation, inequalities (A.70) and (A.71) reduce to $\theta_2^2/E^2 > 0$ and $\theta_2^2/(4DF) > 0$, respectively, which both hold in this case. We finally show that

$$R_{2_B} = \frac{-c_2 - \sqrt{\Delta_c}}{2c_1} < \frac{AF-CD}{DF}\,, \tag{A.72}$$

which completes the proof of (A.69). Inequality (A.72) reduces to

$$\sqrt{\Delta_c} < (E^2 - 4DF)(AF - CD)\,, \tag{A.73}$$

where both sides of (A.73) are positive. The inequality can therefore be squared and reduced to

$$-4DF\,(E^2-4DF)\,K>0\,,$$

which holds in this case. Therefore, if (A.44) and (A.45) hold with $R_2 \neq (AE - BD)/(DE)$, all network elements are positive, for both sets of solutions.

We now assume conditions (A.44) and (A.46) hold with $R_2 \neq (AE-BD)/(DE)$. From (A.46) it follows that $c_1 = c_2 = c_3 = 0$ in (A.47) and any value of x solves (A.47). Any value of $R_2 \neq (AE-BD)/(DE)$ strictly less than $(AF-CD)/(DF)$ still yields positive R_3 and R_1 (from (A.48) and (A.67), respectively). From (A.65) it follows that $\theta_2 = 0$ which, using (A.66), implies that

$$\frac{AE-BD}{DE} = \frac{AF-CD}{2DF}\,.$$

Therefore $R_2 \neq (AF-CD)/(2DF)$ and the expression for L_1 in (A.68), which is still valid in this case, thus reduces to

$$L_1 = \frac{2DF\,R_1R_3}{CE}\cdot\frac{\left(R_2-\frac{AF-CD}{2DF}\right)}{\left(R_2-\frac{AE-BD}{DE}\right)} = \frac{2DF\,R_1R_3}{CE}\,, \tag{A.74}$$

which ensures positivity of L_1, and hence L_2.

We now check positivity when conditions (A.44) and (A.45) hold in the case that $R_3 = B/E$ which, as noted before, is equivalent to

$$R_2 = \frac{AE-BD}{DE}\,, \tag{A.75}$$

from (A.48). In this case equation (A.64) cannot be used to obtain L_1. However, we now have expressions for R_1, R_2, R_3, L_2 and k which only contain L_1, together with A, B, C, D, E, F. Substituting such expressions into (A.58) we get the quadratic equation (A.52) in L_1, from which we can conclude that L_1 is finite. Since (A.55) also needs to be satisfied (equivalently (A.64)), it follows that $R_1R_2 + R_2R_3 - R_3^2 = 0$. Considering the values obtained for R_1, R_2 and R_3 we get

$$R_1R_2 + R_2R_3 - R_3^2 = \frac{B^2\,\theta_2}{DE^2\,(BF-CE)} = 0\,, \tag{A.76}$$

from which we can conclude that $\theta_2 = 0$ in this case. From (A.49), $R_1 = BC/(BF-CE)$, which is always positive. It is easily seen that the first and third coefficients in (A.52)

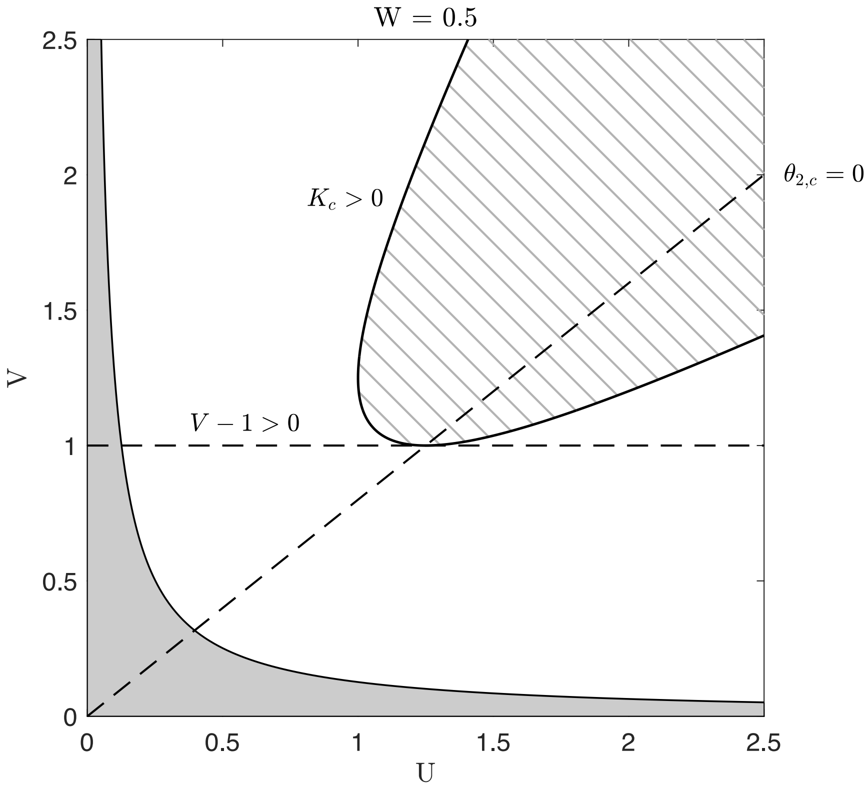


Figure A.4: Realisability region for the network on the (U, V)-plane, for $W = 0.5$. The interior of the hatched region corresponds to case (A.45) (with $R_2 = (AE - BD)/(DE)$ being a repeated root in (A.47) when $\theta_{2,c} = 0$), while the point where the curves K_c and $V - 1$ intersect corresponds to case (A.46).

are positive, while the second is negative. The discriminant of (A.52),

$$\Delta_d = B^4 E^2 F^2 (BF - CE)^2 (E^2 - 4DF),$$

is greater than zero hence the quadratic has two distinct positive roots. We finally note that substituting the expression for R_2 given in (A.75) into the left-hand side of (A.47), we obtain θ_2^2/E^2, which equals zero in this case. Therefore R_2 can still be computed as a root of (A.47).

The remaining case to be dealt with is when (A.44) and (A.46) hold and $R_2 = (AE - BD)/(DE)$. In this case the reasoning of the previous paragraph remains valid, and we notice that the discriminant of (A.52) $\Delta_d = 0$, hence there are two coincident solutions for L_1. □

Figure A.4 shows the realisability region for the network, plotted on the (U, V)-plane for $W = 0.5$ (i.e. $AF - CD > 0$). The polynomial θ_2 expressed in terms of U, V, W is $\theta_{2,c} = V(W^2+1) - 2UW$, while the expressions in terms of U, V, W for K and $E^2 - 4DF$ can be found in Table 5.9. We note that in the interior of the hatched region there are always two solutions to the realisation problem, with no solutions on the boundary $K_c = 0$ except when $V = 1$ (i.e. $E^2 - 4DF = 0$), when there are infinitely many solutions. It can be shown that $V = 1$ and $K_c = 0$ imply $\theta_{2,c} = 0$.

A.4 Equivalence class $\mathbf{V}_{\mathbf{C}}^{\mathbf{1}}$

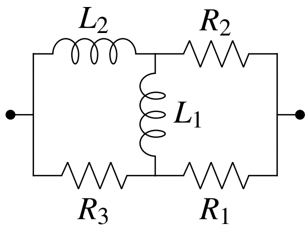


Figure A.5: Network #60, from subfamily $\mathrm{V_C}$. By Lemma 3 in [43] it can only realise regular impedances.

Theorem A.4. *The positive-real biquadratic impedance* (4.1) *with* A, B, C, D, E, $F > 0$ *can be realised as in Figure A.5, with* R_1, R_2, R_3, L_1 *and* L_2 *positive and finite, if and only if*

$$AF - CD > 0\,, \tag{A.77}$$

$$\eta \leq 0\,, \tag{A.78}$$

where η *is defined in Table 5.9. If conditions* (A.77)–(A.78) *are satisfied, then* R_1 *is either of the two positive roots of the quadratic equation in* x

$$DF\,\tau_1\,x^2 - \psi\,x + AC\,\tau_2 = 0\,, \tag{A.79}$$

where τ_1, τ_2 *and* ψ *are defined in Table 5.9, and*

$$R_2 = \frac{CR_1}{FR_1 - C}\,, \tag{A.80}$$

$$R_3 = \frac{A - DR_1}{D}\,, \tag{A.81}$$

$$L_1 = \frac{R_1^2\,(-DEF\,R_1 + \gamma_2)}{(AF + CD)(FR_1 - C)}\,, \tag{A.82}$$

$$L_2 = \frac{R_1\,\gamma_1 - ABC}{(AF + CD)(FR_1 - C)}\,, \tag{A.83}$$

where

$$\gamma_1 = A(BF - CE) + BCD\,, \tag{A.84}$$

$$\gamma_2 = F(AE - BD) + CDE\,. \tag{A.85}$$

Proof. Necessity. The impedance of the network shown in Figure A.5 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.86}$$

where

$$n(s) = L_1L_2(R_1 + R_3)s^2 + \big(L_2(R_1R_2 + R_1R_3 + R_2R_3) + R_2L_1(R_1 + R_3)\big)s + R_1R_2R_3\,,$$
$$d(s) = L_1L_2s^2 + \big(L_2(R_1 + R_2) + L_1(R_1 + R_2 + R_3)\big)s + R_3(R_1 + R_2)\,.$$

Equating impedance (A.86) with (4.1), we obtain

$$L_1L_2(R_1 + R_3) = kA\,, \tag{A.87}$$

$$L_2(R_1R_2 + R_1R_3 + R_2R_3) + R_2L_1(R_1 + R_3) = kB\,, \tag{A.88}$$

$$R_1R_2R_3 = kC\,, \tag{A.89}$$

$$L_1L_2 = kD\,, \tag{A.90}$$

$$L_2(R_1 + R_2) + L_1(R_1 + R_2 + R_3) = kE\,, \tag{A.91}$$

$$R_3(R_1 + R_2) = kF\,, \tag{A.92}$$

where k is a positive constant. It can be calculated that

$$AF - CD = k^{-2}\,L_1L_2R_3(R_1^2 + R_1R_3 + R_2R_3)\,,$$
$$\eta = -k^{-4}\,L_1L_2R_3^2[L_1R_2(R_1 + R_3) - L_2R_1(R_1 + R_2)]^2\,,$$

hence (A.77)–(A.78) are necessary.

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$ satisfying conditions (A.77)–(A.78), we now show that we can find R_1, R_2, R_3, L_1, L_2

positive which satisfy (A.87)–(A.92) with $k > 0$. From (A.89) we obtain

$$k = \frac{R_1 R_2 R_3}{C}\,. \tag{A.93}$$

Eliminating $L_1 L_2$ from (A.87) and (A.90) we obtain (A.81), and substituting (A.93) into (A.92) we get (A.80). Substituting (A.93), (A.80) and (A.81) into (A.88) and (A.91), and solving the equations for L_1 and L_2, we obtain (A.82) and (A.83). We now have expressions for elements R_2, R_3, L_1, L_2 and k which only contain R_1, together with A, B, C, D, E and F. Substituting such expressions into (A.87) we obtain the quadratic equation (A.79). The discriminant of (A.79),

$$\Delta = (AF + CD)^2(\eta - 4ACDF)\,\eta\,, \tag{A.94}$$

is greater than or equal to zero, by virtue of (A.78). It can be easily seen that the first and third coefficients in (A.79) are negative, since $\tau_1 = \eta - B^2DF$ and $\tau_2 = \eta - ACE^2$, while the second coefficient is positive, since $-\psi = -(AF+CD)\eta+2ABCDEF$. Therefore the equation has two real positive roots, which are distinct if the discriminant Δ is strictly greater than zero.

It can be easily seen from the expressions in Table 5.9 that $K = \eta - 4ACDF$, hence (A.78) implies $K < 0$. From $K < 0$ and (A.77) it follows that

$$BF - CE > 0\,, \tag{A.95}$$

$$AE - BD > 0\,, \tag{A.96}$$

therefore γ_1 and γ_2 in (A.84)–(A.85) are positive. We denote the two solutions of (A.79) as R_{1_A} and R_{1_B}, with $R_{1_A} \le R_{1_B}$. From (A.80) and (A.81), R_2 and R_3 are positive, for both solutions, providing

$$C/F < R_{1_A} \le R_{1_B} < A/D \tag{A.97}$$

and, from (A.82) and (A.83) (assuming (A.97) holds), L_1 and L_2 are positive if

$$ABC/\gamma_1 < R_{1_A} \le R_{1_B} < \gamma_2/(DEF)\,. \tag{A.98}$$

It can be easily verified that $C/F < ABC/\gamma_1$ and $\gamma_2/(DEF) < A/D$, hence if inequality (A.98) holds, so does (A.97). Since R_{1_A} and R_{1_B} are the solutions to (A.79), which represents a parabola that opens down, inequality (A.98) holds if and only if ABC/γ_1 and $\gamma_2/(DEF)$ are located, respectively, to the left and to the right of the parabola's axis of symmetry and (A.79) evaluated at ABC/γ_1 and $\gamma_2/(DEF)$ gives negative values, that is

$$\frac{ABC}{\gamma_1} < \frac{\psi}{2DF\,\tau_1} < \frac{\gamma_2}{DEF}\,, \tag{A.99}$$

and

$$DF\,\tau_1\left(\frac{ABC}{\gamma_1}\right)^2 - \psi\left(\frac{ABC}{\gamma_1}\right) + AC\,\tau_2 < 0\,, \tag{A.100}$$

$$DF\,\tau_1\left(\frac{\gamma_2}{DEF}\right)^2 - \psi\left(\frac{\gamma_2}{DEF}\right) + AC\,\tau_2 < 0\,. \tag{A.101}$$

The left and right inequalities in (A.99) can be rewritten as

$$\begin{aligned} 2ABCDF\,\tau_1 - \psi\gamma_1 &= (AF+CD)(2ABCDF(AF+CD) - \eta\gamma_1) > 0\,, \\ DEF\,\psi - 2DF\,\tau_1\gamma_2 &= DF(AF+CD)(2BDF(AF+CD) - E\,\eta) > 0\,, \end{aligned}$$

which both hold. After some manipulation, inequalities (A.100) and (A.101), respectively, reduce to

$$\begin{aligned} -A^2C^2(AE-BD)(BF-CE)(AF+CD)^2/\gamma_1^2 < 0\,, \\ -(AE-BD)(BF-CE)(AF+CD)^2/E^2 < 0\,, \end{aligned}$$

which always hold. Therefore the values of the five elements and the constant k are all positive, for both sets of solutions. □

A.5 Equivalence class $\mathbf{V_D^1}$

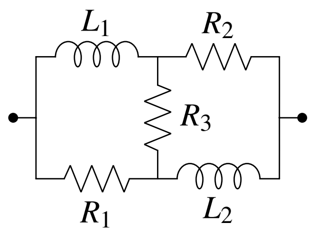


Figure A.6: Network #61, from subfamily V_D. By Lemma 3 in [43] it can only realise regular impedances.

Theorem A.5. *The positive-real biquadratic impedance* (4.1) *with* A, B, C, D, E, $F > 0$ *can be realised as in Figure A.6, with* R_1, R_2, R_3, L_1 *and* L_2 *positive and finite,* *if and only if*

$$K \leq 0\,, \tag{A.102}$$

$$\mu_1 \leq 0\,, \tag{A.103}$$

$$AF - CD > 0\,, \tag{A.104}$$

where K is defined in (3.3) *and μ_1 in Table 5.9.*[2]

If conditions (A.102)–(A.104) *are satisfied, then R_3 is either of the two positive roots of the quadratic equation in x*

$$-D\,\lambda_1\,x^2 + \rho_1\,x - C\,\lambda_2 = 0\,, \tag{A.105}$$

where λ_1, λ_2 and ρ_1 are defined in Table 5.9, R_2 is either of the two positive roots of the quadratic equation in y

$$d_1\,y^2 + d_2\,y + d_3 = 0\,, \tag{A.106}$$

where

$$d_1 = D(C - FR_3)\,, \tag{A.107}$$

$$d_2 = -(C - FR_3)(A - DR_3)\,, \tag{A.108}$$

$$d_3 = -CR_3(A - DR_3), \tag{A.109}$$

and R_1 is the other root. The values of the inductors are

$$L_1 = \frac{-D(R_1 + R_3)(R_1 - R_2)}{B - ER_1}\,, \tag{A.110}$$

$$L_2 = \frac{D(R_2 + R_3)(R_1 - R_2)}{B - ER_2}\,, \tag{A.111}$$

if $R_1 \neq R_2$. If $R_1 = R_2$ then L_1 and L_2 are the two positive roots of the quadratic equation in z

$$p_1\,z^2 + p_2\,z + p_3 = 0\,, \tag{A.112}$$

[2]Conditions (A.102)–(A.104) could be rewritten with $AF - CD > 0$ replaced by $AF - 9CD \geq 0$. Even though $AF - CD > 0$ is apparently a weaker condition, it is actually equivalent in the theorem statement since conditions (A.102)–(A.103) imply $AF - 9CD \geq 0$ when $AF - CD > 0$, as shown in [16, Theorem 2].

where

$$p_1 = E^2 F(AE - BD)\,, \tag{A.113}$$
$$p_2 = -BE^2(2AE - 3BD)\,, \tag{A.114}$$
$$p_3 = B(2AE - 3BD)(AE - BD)\,. \tag{A.115}$$

Proof. Necessity. The impedance of the network shown in Figure A.6 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.116}$$

where

$$n(s) = L_1L_2(R_1+R_2+R_3)\,s^2 + \big(L_1R_1(R_2+R_3) + L_2R_2(R_1+R_3)\big)s + R_1R_2R_3\,,$$
$$d(s) = L_1L_2\,s^2 + \big(L_1(R_2+R_3) + L_2(R_1+R_3)\big)s + R_1R_2 + R_1R_3 + R_2R_3\,.$$

Equating impedance (A.116) with (4.1), we obtain

$$L_1L_2(R_1+R_2+R_3) = kA\,, \tag{A.117}$$
$$L_1R_1(R_2+R_3) + L_2R_2(R_1+R_3) = kB\,, \tag{A.118}$$
$$R_1R_2R_3 = kC\,, \tag{A.119}$$
$$L_1L_2 = kD\,, \tag{A.120}$$
$$L_1(R_2+R_3) + L_2(R_1+R_3) = kE\,, \tag{A.121}$$
$$R_1R_2 + R_1R_3 + R_2R_3 = kF\,, \tag{A.122}$$

where k is a positive constant. It can be calculated that

$$K = -k^{-4}\,L_1L_2\big[L_1R_1(R_2+R_3)^2 - L_2R_2(R_1+R_3)^2\big]^2\,, \tag{A.123}$$
$$\mu_1 = -k^{-4}\,L_1L_2\big[L_1R_1(R_2^2-R_3^2) - L_2R_2(R_1^2-R_3^2)\big]^2\,, \tag{A.124}$$
$$AF - CD = k^{-2}\,L_1L_2(R_1+R_2)(R_1R_2 + R_1R_3 + R_2R_3 + R_3^2)\,, \tag{A.125}$$

hence (A.102)–(A.104) are necessary.

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$ satisfying conditions (A.102)–(A.104), we now show that we can find R_1, R_2, R_3, L_1, L_2 positive which satisfy (A.117)–(A.122) with $k > 0$. Solving (A.118) and (A.121) for L_1

and L_2, we obtain

$$L_1 = \frac{k(B - ER_2)}{(R_2 + R_3)(R_1 - R_2)}\,, \tag{A.126}$$

$$L_2 = \frac{-k(B - ER_1)}{(R_1 + R_3)(R_1 - R_2)}\,, \tag{A.127}$$

assuming $R_1 \neq R_2$ and R_1, $R_2 \neq B/E$, and from (A.120), (A.126)–(A.127) we obtain the expressions for L_1 and L_2 in (A.110) and (A.111). Eliminating L_1L_2 from (A.117) and (A.120) we obtain

$$R_1 = A/D - R_2 - R_3\,, \tag{A.128}$$

and from (A.122) and (A.128) we get

$$k = \frac{(A/D - R_2 - R_3)(R_2 + R_3) + R_2R_3}{F}\,. \tag{A.129}$$

Substituting (A.128) and (A.129) into (A.119), we get the quadratic equation (A.106), with $y = R_2$, from which an expression for R_2^2 can be found, as follows:

$$R_2^2 = \frac{(A - DR_3)(C(R_2 + R_3) - FR_2R_3)}{D(C - FR_3)}\,. \tag{A.130}$$

We now have expressions for elements R_1, L_1, L_2 and for the constant k which only contain R_2 and R_3, together with A, B, C, D, E and F, and an expression for R_2^2 which contains R_3, together with the same six coefficients, and is linear in R_2. Substituting the expressions for L_1, L_2 and k into (A.120) we obtain a polynomial in R_2 and R_3 of fourth order. Then, substituting for R_2^2 from (A.130) twice to reduce the power of R_2, after further manipulation all terms in R_2 cancel out, and we obtain the quadratic equation (A.105) in R_3. The discriminant of (A.105),

$$\Delta_c = K\mu_1\,, \tag{A.131}$$

is greater than or equal to zero, by virtue of (A.102)–(A.103). Conditions (A.102) and (A.104) imply that $\rho_1 > 0$ and

$$BF - CE > 0\,,$$
$$AE - BD > 0\,.$$

Hence, using again (A.102) and (A.104), λ_1 and λ_2 are both positive (note this also follows from [43, Lemma 7]). Therefore (A.105) has two real positive roots

$$R_{3_A} = \frac{\rho_1 - \sqrt{\Delta_c}}{2D\lambda_1}\,, \tag{A.132}$$

$$R_{3_B} = \frac{\rho_1 + \sqrt{\Delta_c}}{2D\lambda_1}\,, \tag{A.133}$$

where $R_{3_A} \leq R_{3_B}$, which coincide if and only if $\Delta_c = 0$, i.e. if K and/or μ_1 are zero.

If $R_1 = R_2$ then (A.110) and (A.111) do not apply. In this case the left hand sides of (A.118) and (A.121) are multiples of $L_1 + L_2$, which can be eliminated to yield $R_1 = R_2 = B/E$. As before, (A.128) holds and it follows that

$$R_3 = \frac{AE - 2BD}{DE}\,. \tag{A.134}$$

Solving (A.122) yields $k = B(2AE - 3BD)/(DE^2F)$, which implies that

$$\beta := BF(AE - 2BD) - CE(2AE - 3BD) \tag{A.135}$$

is zero, using (A.119). It can then be verified that R_3 still satisfies (A.105) (and is therefore positive) and B/E is a double root of (A.106). From (A.134) we can therefore conclude that, in this case,

$$AE - 2BD > 0\,. \tag{A.136}$$

Solving for L_1 or L_2 from (A.120) and (A.121) shows that they are both solutions of the quadratic (A.112).

Positivity of R_1 and R_2

From equations (A.117) and (A.120), and from (A.119) and (A.122), respectively, it can be easily shown that

$$R_1 + R_2 + R_3 = A/D\,,$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = C/F\,,$$

hence each R_i is smaller than A/D and greater than C/F. It follows that coefficients d_1 and d_3 in (A.106) are negative, and d_2 is positive. Hence, equation (A.106) has two positive solutions if and only if the discriminant

$$\Delta_d = (A - DR_3)(C - FR_3)\big[DF\,R_3^2 - (AF - 3CD)\,R_3 + AC\big] \tag{A.137}$$

is non-negative, i.e. if and only if

$$DF\,R_3^2 - (AF - 3CD)\,R_3 + AC \leq 0\,. \tag{A.138}$$

Substituting (A.132) into (A.138), after some manipulation (A.138) is equivalent to

$$2D^2(BF - CE)(BF - 3CE)\sqrt{\Delta_c} \leq -2D^2\xi\,, \tag{A.139}$$

where

$$\xi = K(BF - 3CE)^2 - 2C\lambda_1\beta \tag{A.140}$$

$$= \mu_1\,(BF - CE)^2 + 2C\lambda_1\beta\,, \tag{A.141}$$

with β defined as in (A.135). From (A.140) it follows that, if $\beta \geq 0$, then $\xi \leq 0$ and from (A.141) it follows that, if $\beta < 0$, then $\xi < 0$. Therefore we can conclude that $\xi \leq 0$. Hence, if $BF - 3CE \leq 0$, inequality (A.139) holds. If $BF - 3CE > 0$, both sides of (A.139) are non-negative and the inequality can be squared, and reduced to

$$-16C^2D^4\beta^2\lambda_1^2 \leq 0\,, \tag{A.142}$$

which holds. A similar argument holds if we substitute (A.133) into (A.138). Therefore, considering either R_{3_A} or R_{3_B}, all solutions to (A.106) are positive. The two positive values of R_2 thus obtained from R_{3_A} are

$$R'_{2_A},\ R''_{2_A} = \left.\frac{-d_2 \pm \sqrt{\Delta_d}}{2d_1}\right|_{R_3=R_{3_A}}.$$

The same expression holds for the other two solutions R'_{2_B} and R''_{2_B} with R_{3_A} replaced by R_{3_B}. From (A.106) the sum of the two roots is

$$R'_{2_A} + R''_{2_A} = A/D - R_{3_A}\,,$$

hence, from (A.128) it follows that

$$R'_{1_A} = A/D - R'_{2_A} - R_{3_A} = R''_{2_A}\,, \tag{A.143}$$

$$R''_{1_A} = A/D - R''_{2_A} - R_{3_A} = R'_{2_A}\,. \tag{A.144}$$

Similar equalities hold for R'_{1_B} and R''_{1_B}. This is due to the symmetry of the network, and can be seen from equations (A.117)–(A.122): if the resistors R_1, R_2 and the inductors L_1, L_2 are swapped, the impedance is unchanged. Note we have shown that R_1 and R_2 are the two positive solutions of (A.106) for either choice of R_3.

Positivity of L_1 and L_2

When $R_1 \neq R_2$, it follows from (A.110)–(A.111) that the two following cases result in positive values for both inductances:

$$R_2 < B/E < R_1\,, \tag{A.145}$$

$$R_1 < B/E < R_2\,. \tag{A.146}$$

Since R_1 and R_2 are the two roots of the same quadratic equation, only one of the two cases can occur depending on whether $R_1 \gtrless R_2$, which is a matter of how the two roots of (A.106) are labelled. We will now show that, considering the set of solutions R_{3_A}, R'_{2_A} and R'_{1_A}, with $R'_{2_A} < R''_{2_A}$, (A.145) always holds if conditions (A.102)–(A.104) hold.

Since R'_{2_A} and $R'_{1_A} = R''_{2_A}$ are the solutions to the quadratic equation (A.106), which represents a parabola that opens down, inequality (A.145) holds if and only if (A.106) evaluated at B/E gives a positive value, that is:

$$d_1\,B^2/E^2 + d_2\,B/E + d_3 > 0\,. \tag{A.147}$$

After some manipulation, inequality (A.147) can be reduced to:

$$\gamma_1 R_{3_A}^2 + \gamma_2 R_{3_A} + \gamma_3 < 0\,, \tag{A.148}$$

where

$$\begin{aligned}
\gamma_1 &= DE(BF - CE)\,,\\
\gamma_2 &= -(AE - BD)(BF - CE)\,,\\
\gamma_3 &= BC(AE - BD)\,.
\end{aligned}$$

Substituting (A.132) into (A.148), after some manipulation the inequality reduces to

$$2D^2(BF - CE)(BF - 3CE)(AF - CD)\sqrt{\Delta_c} < -2D^2(AF - CD)\,\xi\,,$$

which always holds, since (A.139) holds with strict inequality.

For the case $R_1 = R_2$, as shown by (A.136), $AE - 2BD > 0$, hence also $2AE - 3BD >$

0, which means that p_1 and p_3 in (A.112) are positive while p_2 is negative. Using the fact that $\beta = 0$, the discriminant of (A.112) simplifies to the following expression:

$$\Delta_p = -BE^3 (2AE - 3BD)^2 \left[2(AF - CD) - BE\right].$$

It can be verified that the following identity always holds

$$2\,K = (AF - CD)\big[2(AF - CD) - BE\big] - \beta\,, \tag{A.149}$$

which implies that $2(AF - CD) - BE \leq 0$, from (A.102), (A.104) and $\beta = 0$. It follows that $\Delta_p \geq 0$ and that both solutions of (A.112) are positive. □

A.6 Equivalence class $\mathbf{V_E^1}$

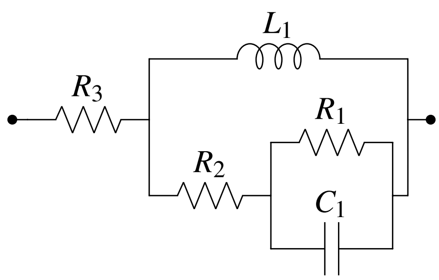


Figure A.7: Network #69, from subfamily $\mathrm{V_E}$. By Theorem 1 in [43] it can only realise regular impedances.

Theorem A.6. *The positive-real biquadratic impedance* (4.1) *with* A, B, C, D, E, $F > 0$ *can be realised as in Figure A.7, with* R_1, R_2, R_3, C_1 *and* L_1 *positive and finite, if and only if*

$$AF - CD > 0\,, \tag{A.150}$$

$$\lambda_1 > 0\,, \tag{A.151}$$

$$K > 0\,, \tag{A.152}$$

where λ_1 *and* K *are defined in Table 5.9. If conditions* (A.150)–(A.152) *are satisfied, then*

$$R_1 = \frac{K}{D\lambda_1}\,, \tag{A.153}$$

$$R_2 = \frac{AF - CD}{DF}\,, \tag{A.154}$$

$$R_3 = \frac{C}{F}\,, \tag{A.155}$$

$$C_1 = \frac{D^2(BF - CE)}{K}\,, \tag{A.156}$$

$$L_1 = \frac{BF - CE}{F^2}\,. \tag{A.157}$$

Proof. Note this result is given without proof in [43, Appendix A], with R_1 and R_3 interchanged. We provide here a proof for convenience.

Necessity. The impedance of the network shown in Figure A.7 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.158}$$

where

$$n(s) = R_1L_1C_1(R_2 + R_3)s^2 + \big(L_1(R_1 + R_2 + R_3) + R_1R_2R_3C_1\big)s + R_3(R_1 + R_2)\,,$$
$$d(s) = R_1L_1C_1s^2 + \big(L_1 + R_1R_2C_1\big)s + R_1 + R_2\,.$$

Equating impedance (A.33) with (4.1), we obtain, for a positive constant k,

$$R_1L_1C_1(R_2 + R_3) = kA\,, \tag{A.159}$$

$$L_1(R_1 + R_2 + R_3) + R_1R_2R_3C_1 = kB\,, \tag{A.160}$$

$$R_3(R_1 + R_2) = kC\,, \tag{A.161}$$

$$R_1L_1C_1 = kD\,, \tag{A.162}$$

$$L_1 + R_1R_2C_1 = kE\,, \tag{A.163}$$

$$R_1 + R_2 = kF\,, \tag{A.164}$$

which are a set of necessary and sufficient conditions for (4.1) to be realised as in Figure A.7. It can be calculated that

$$AF - CD = k^{-2}\, R_1R_2L_1C_1\,(R_1 + R_2) > 0\,, \tag{A.165}$$

$$\lambda_1 = k^{-3}\, L_1^2\,(R_1 + R_2)^2 > 0\,, \tag{A.166}$$

$$K = k^{-4}\, R_1^2L_1^3C_1\,(R_1 + R_2)^2 > 0\,, \tag{A.167}$$

hence (A.150)–(A.152) are necessary.

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$ satisfying conditions (A.150)–(A.152) we now show that we can find R_1, R_2, R_3, L_1, C_1 positive which satisfy (A.159)–(A.164) with $k > 0$.

From (A.164) we obtain $k = (R_1 + R_2)/F$ and eliminating the term $R_1 + R_2$ from (A.161) and (A.164) we get (A.155). Eliminating the term $R_1 L_1 C_1$ from (A.159) and (A.162) and solving for R_2 we obtain (A.154), while eliminating C_1 from (A.160) and (A.163) we get (A.157). Solving for C_1 from (A.162) using expressions obtained we find:

$$C_1 = \frac{AF - CD + DFR_1}{R_1\,(BF - CE)}\,. \tag{A.168}$$

We now have expressions for R_2, R_3, L_1, C_1 and k which only contain R_1 together with A, B, C, D, E, F. Substituting such expressions into (A.163) and solving for R_1 we obtain (A.153) and, substituting the latter into (A.168), we get (A.156).

Conditions (A.150) and (A.151) imply $BF - CE > 0$, from the expression for λ_1. Hence all network elements are positive. □

A.7 Equivalence class $\mathbf{V_F^1}$

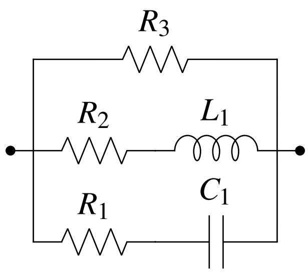


Figure A.8: Network #101, from subfamily $\mathrm{V_F}$. By Theorem 1 in [43] it can only realise regular impedances.

Theorem A.7. *The positive-real biquadratic impedance* (4.1) *with* A, B, C, D, E, $F > 0$ *can be realised as in Figure A.8, with* R_1, R_2, R_3, L_1 *and* C_1 *positive and finite, if and only if*

$$K > 0\,, \tag{A.169}$$

$$\tau_1 < 0\,, \tag{A.170}$$

where K is defined in (3.3) *and τ_1 in Table 5.9, or*

$$K = \tau_1 = 0\,. \tag{A.171}$$

If (A.169)–(A.170) *hold, R_3 is the largest positive root of the quadratic equation in x*

$$\tau_1\, x^2 + (AF + CD)(B^2 - 4AC)\, x - AC(B^2 - 4AC) = 0\,, \tag{A.172}$$

while, if (A.171) *holds, then R_3 may take any value such that*

$$R_3 > max\left\{\frac{A}{D}, \frac{B}{E}, \frac{C}{F}\right\}\,. \tag{A.173}$$

The other network elements are given by:

$$R_1 = \frac{AR_3}{DR_3 - A}\,, \tag{A.174}$$

$$R_2 = \frac{CR_3}{FR_3 - C}\,, \tag{A.175}$$

$$L_1 = \frac{DR_3^2(R_1 + R_2)}{(R_1 + R_3)(ER_3 - B)}\,, \tag{A.176}$$

$$C_1 = \frac{D(R_2 + R_3)}{FL_1(R_1 + R_3)}\,. \tag{A.177}$$

Proof. Necessity. The impedance of the network shown in Figure A.8 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.178}$$

where

$$n(s) = R_1R_3L_1C_1\, s^2 + R_3(R_1R_2C_1 + L_1)s + R_2R_3\,,$$
$$d(s) = L_1C_1(R_1 + R_3)\, s^2 + \big(C_1(R_1R_2 + R_1R_3 + R_2R_3) + L_1\big)s + R_2 + R_3\,.$$

Equating impedance (A.178) with (4.1) we obtain, for a positive constant k,

$$R_1R_3L_1C_1 = kA\,, \tag{A.179}$$

$$R_3(R_1R_2C_1 + L_1) = kB\,, \tag{A.180}$$

$$R_2R_3 = kC\,, \tag{A.181}$$

$$L_1C_1(R_1 + R_3) = kD\,, \tag{A.182}$$

$$C_1(R_1R_2 + R_1R_3 + R_2R_3) + L_1 = kE\,, \tag{A.183}$$
$$R_2 + R_3 = kF\,, \tag{A.184}$$

which are a set of necessary and sufficient conditions for (4.1) to be realised as in Figure A.8. It can be calculated that

$$K = k^{-4}\, C_1L_1R_3^4\,(C_1R_1R_2 - L_1)^2 \geq 0\,, \tag{A.185}$$
$$\tau_1 = -k^{-4}\, C_1L_1R_3^2\,(R_1R_2 + R_1R_3 + R_2R_3)\,(C_1R_1R_2 - L_1)^2 \leq 0\,, \tag{A.186}$$

hence

$$K = 0 \;\;\Leftrightarrow\;\; \tau_1 = 0 \;\;\Leftrightarrow\;\; L_1 = C_1R_1R_2$$

and (A.169)–(A.171) are necessary.

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$, either satisfying conditions (A.169)–(A.170) or condition (A.171), we now show that we can find R_1, R_2, R_3, L_1, C_1 positive which satisfy (A.179)–(A.184) with $k > 0$.

From (A.184) we obtain

$$k = \frac{R_2 + R_3}{F}\,, \tag{A.187}$$

and substituting (A.187) into (A.182) we get (A.177). Equations (A.180) and (A.183) now reduce to

$$FR_3(R_1 + R_3)\,L_1^2 - B(R_1 + R_3)(R_2 + R_3)\,L_1 + DR_1R_2R_3(R_2 + R_3) = 0\,,$$
$$F(R_1 + R_3)\,L_1^2 - E(R_1 + R_3)(R_2 + R_3)\,L_1 + D(R_2 + R_3)(R_1R_2 + R_1R_3 + R_2R_3) = 0\,.$$

Eliminating the term in L_1^2 from the two equations and solving for L_1 we obtain (A.176). Eliminating L_1C_1 from (A.179) and (A.182) and solving for R_1 we obtain (A.174), while from (A.181) and (A.184) we obtain (A.175). We now have expressions for R_1, R_2, L_1, C_1 and k which only contain R_3 together with A, B, C, D, E, F. Substituting such expressions into (A.180) we obtain the quadratic equation (A.172).

If conditions (A.169)–(A.170) hold, then from the expression for τ_1 we have $DF(B^2 - 4AC) > K > 0$, hence the first and third coefficients in (A.172) are negative while the second is positive. The discriminant of (A.172) is

$$\Delta = (B^2 - 4AC)(ABF - 2ACE + BCD)^2\,.$$

It can be verified that the following identity always holds

$$(ABF - 2ACE + BCD)^2 = 4AC\,K + (B^2 - 4AC)(AF - CD)^2\,,$$

hence $ABF - 2ACE + BCD \neq 0$ in this case. Therefore $\Delta > 0$ and the quadratic has two distinct positive roots.

From (A.174), (A.175) and (A.176), respectively, we have that R_1, R_2 and L_1 are positive if

$$R_3 > A/D\,, \tag{A.188}$$

$$R_3 > C/F\,, \tag{A.189}$$

$$R_3 > B/E\,. \tag{A.190}$$

Since (A.172) represents a parabola that opens down, (A.188), (A.189) and (A.190) hold for the largest solution of (A.172) if (A.172) evaluated at A/D, C/F and B/E gives positive values, that is:

$$\tau_1\,A^2/D^2 + (AF + CD)(B^2 - 4AC)\,A/D - AC(B^2 - 4AC) > 0\,, \tag{A.191}$$

$$\tau_1\,C^2/F^2 + (AF + CD)(B^2 - 4AC)\,C/F - AC(B^2 - 4AC) > 0\,, \tag{A.192}$$

$$\tau_1\,B^2/E^2 + (AF + CD)(B^2 - 4AC)\,B/E - AC(B^2 - 4AC) > 0\,. \tag{A.193}$$

After some manipulation, inequalities (A.191)–(A.193) reduce to

$$\frac{A^2\,K}{D^2} > 0\,, \qquad \frac{C^2\,K}{F^2} > 0\,, \qquad \frac{(ABF - 2ACE + BCD)^2}{E^2} > 0\,,$$

which all hold in this case. This means that A/D, C/F and B/E always lie between the two positive roots of (A.172), so (A.188)–(A.190) are satisfied by the largest root only.

Finally, if condition (A.171) holds then $B^2 - 4AC = 0$ (from the expression for τ_1) and any value of x solves (A.172). Any positive value of R_3 which satisfies (A.173) guarantees positivity of R_1, R_2 and L_1, from (A.174), (A.175) and (A.176), and the theorem statement follows. □

A.8 Equivalence class $\mathrm{V}_{\mathrm{G}}^{1}$

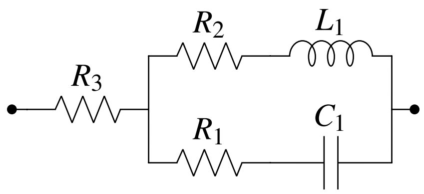


Figure A.9: Network #104, from subfamily V_{G}. By Theorem 1 in [43] it can only realise regular impedances.

Theorem A.8. *The positive-real biquadratic impedance* (4.1) *with A, B, C, D, E, $F > 0$ can be realised as in Figure A.9, with R_1, R_2, R_3, L_1 and C_1 positive and finite, if and only if the conditions of one of the following five cases are satisfied*

Case 1) $AF - CD > 0$ *and one of*

(a) $\tau_1 < 0,\ \lambda_1 = 0$ (A.194)

(b) $\lambda_1 > 0,\ \tau_1 = 0,\ \delta > 0$ (A.195)

(c) $\tau_1 < 0,\ \lambda_1 > 0,\ K \geq 0,\ \delta > 0,\ \zeta_1 > 0$ (A.196)

Case 2) $AF - CD \geq 0$ *and* $\tau_1\lambda_1 > 0$ (A.197)

Case 3) $AF - CD = 0$ *and* $K = 0$ (A.198)

Case 4) $AF - CD < 0$ *and one of*

(a) $\tau_1 < 0,\ \lambda_3 = 0$ (A.199)

(b) $\lambda_3 > 0,\ \tau_1 = 0,\ \delta > 0$ (A.200)

(c) $\tau_1 < 0,\ \lambda_3 > 0,\ K \geq 0,\ \delta > 0,\ \zeta_3 > 0$ (A.201)

Case 5) $AF - CD \leq 0$ *and* $\tau_1\lambda_3 > 0$ (A.202)

Polynomials K, λ_1, λ_3, ζ_1, ζ_3, δ and τ_1 are defined in Table 5.9.

When the conditions in cases (1a), (1b), (2), (4a), (4b) or (5) hold, R_3 is the smallest, positive root of the quadratic equation in x

$$\gamma\, x^2 + 2DF\,\delta\, x + \tau_1 = 0\,, \tag{A.203}$$

where

$$\gamma = -DF\,\left(E^2 - 4DF\right)\,. \tag{A.204}$$

When the conditions in cases (1c) or (4c) hold, R_3 is either of the two positive roots of the same quadratic equation, and when the conditions of case (3) hold, R_3 may take any value such that $R_3 < A/D = C/F$. The other elements of the network can be computed as:

$$R_1 = -R_3 + A/D\,, \tag{A.205}$$

$$R_2 = -R_3 + C/F\,, \tag{A.206}$$

$$L_1 = D(R_1 + R_2)/E\,, \tag{A.207}$$

$$C_1 = \frac{E}{F(R_1 + R_2)}\,. \tag{A.208}$$

Proof. Necessity. The impedance of the network shown in Figure A.9 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.209}$$

where

$$n(s) = L_1C_1(R_1 + R_3)s^2 + \big(L_1 + C_1(R_1R_2 + R_1R_3 + R_2R_3)\big)s + R_2 + R_3\,,$$
$$d(s) = L_1C_1s^2 + C_1(R_1 + R_2)s + 1\,.$$

Equating impedance (A.209) with (4.1), we obtain

$$L_1C_1(R_1 + R_3) = kA\,, \tag{A.210}$$

$$L_1 + C_1(R_1R_2 + R_1R_3 + R_2R_3) = kB\,, \tag{A.211}$$

$$R_2 + R_3 = kC\,, \tag{A.212}$$

$$L_1C_1 = kD\,, \tag{A.213}$$

$$C_1(R_1 + R_2) = kE\,, \tag{A.214}$$

$$1 = kF\,, \tag{A.215}$$

where k is a positive constant. It can be calculated that

$$K = k^{-4}\, L_1C_1(C_1R_1R_2 - L_1)^2\,, \tag{A.216}$$

$$\lambda_1 = -k^{-3}\, C_1R_2[C_1R_2(R_1 + R_2) - 2L_1]\,, \tag{A.217}$$

$$\delta = k^{-2}\, C_1[C_1(R_1 + R_2)(R_1R_2 + R_2R_3 + R_1R_3) - L_1(R_1 + R_2 + 4R_3)]\,, \tag{A.218}$$

$$\tau_1 = -k^{-4}\, C1^2 L_1 R_3 [C_1(R_1 + R_2)(2R_1 R_2 + R_1 R_3 + R_2 R_3) - 2L_1(R_1 + R_2 + 2R_3)]\,, \tag{A.219}$$

$$\gamma = -k^{-4}\, L_1 C_1^2 [C_1(R_1 + R_2)^2 - 4L_1]\,, \tag{A.220}$$

$$\zeta_1 = k^{-3}\, C_1 [C_1 R_2^2 (R_1 + R_2) + L_1(R_1 - 3R_2)]\,, \tag{A.221}$$

$$AF - CD = k^{-2}\, L_1 C_1 (R_1 - R_2)\,, \tag{A.222}$$

hence $K \geq 0$ is necessary. From (A.215) we obtain

$$k = \frac{1}{F}\,. \tag{A.223}$$

Substituting (A.223) into (A.214), we obtain (A.208), which can be used in (A.213) to obtain (A.207). Eliminating $L_1 C_1$ from (A.210) and (A.213) we obtain (A.205), and substituting (A.223) into (A.212) we get (A.206). We now have expressions for network elements R_1, R_2, L_1, C_1 and k which only contain R_3, together with A, B, C, D, E and F. Substituting such expressions into (A.211) we obtain the quadratic equation (A.203). The discriminant of (A.203), $\Delta = 4DE^2FK$, is always greater than or equal to zero, and the equation therefore has two real solutions, which are coincident if $K = 0$.

In considering the roots of (A.203) we subdivide according to whether $\gamma\tau_1 > 0$, $\gamma\tau_1 < 0$ or $\gamma\tau_1 = 0$. We further subdivide into the following seven cases in which there is at least one positive root of (A.203):

(i) $\gamma > 0, \quad \tau_1 > 0, \quad \delta < 0$

(ii) $\gamma < 0, \quad \tau_1 < 0, \quad \delta > 0$

(iii) $\gamma > 0, \quad \tau_1 < 0$

(iv) $\gamma < 0, \quad \tau_1 > 0$

(v) $\gamma = 0, \quad \tau_1 \delta < 0$

(vi) $\tau_1 = 0, \quad \gamma\,\delta < 0$

(vii) $\gamma = 0, \quad \delta = 0, \quad \tau_1 = 0$

The first two cases correspond to two positive roots of (A.203), (iii)–(vi) to one positive root, and (vii) to infinitely many. We will now show that cases (i)–(vii) are equivalent to (A.194)–(A.202).

Step 1

We will first show that $\gamma > 0$ implies $\tau_1 > 0$, hence case (iii) cannot occur. From (A.220),

$\gamma > 0$ is equivalent to $L_1 > C_1(R_1 + R_2)^2/4$, and from (A.219) $\tau_1 > 0$ is equivalent to $2L_1(R_1 + R_2 + 2R_3) - C_1(R_1 + R_2)(2R_1R_2 + R_1R_3 + R_2R_3) > 0$. Replacing L_1 in the latter expression by its strict lower bound (implied by $\gamma > 0$) gives

$$\begin{aligned} &C_1(R_1 + R_2)^2(R_1 + R_2 + 2R_3)/2 - C_1(R_1 + R_2)(2R_1R_2 + R_1R_3 + R_2R_3) \\ &= C_1(R_1 - R_2)^2(R_1 + R_2)/2 \geq 0\,. \end{aligned}$$

Hence $\gamma > 0$ implies $\tau_1 > 0$.

Step 2

We now show that $\gamma \geq 0$, $\tau_1 > 0$ implies $\delta < 0$, hence the latter condition can be removed from (i) and from the subcase in which $\tau_1 > 0$ in (v). It can be verified that the following identity holds

$$(AF + CD)DF\,\delta = -DF(AF - CD)^2 - DF\tau_1 - AC\gamma \qquad \text{(A.224)}$$

from which the result follows.

Step 3

We will now show that $\tau_1 < 0$, $\delta > 0$ is impossible when $\gamma = 0$, hence case (v) can be reduced to $\gamma = 0$, $\tau_1 > 0$ (having already shown in step 2 that the condition $\delta < 0$ may be omitted). From (A.220), $\gamma = 0$ if $L_1 = C_1(R_1 + R_2)^2/4$. Substituting this expression for L_1 into (A.218) and (A.219) yields

$$\tau_1 = k^{-4}\,\frac{C_1^4(R_1 + R_2)^3(R_1 - R_2)^2R_3}{8} = -2DF\,R_3\,\delta\,, \qquad \text{(A.225)}$$

hence $\tau_1 \geq 0$ and $\delta \leq 0$, and the result follows. It is then easily seen that cases (i), (iv) and (v) taken together correspond to the single condition $\tau_1 > 0$.

Step 4

We now show that $\tau_1 > 0$ implies $\lambda_1 > 0$ if $AF - CD \geq 0$ (respectively $\lambda_3 > 0$ if $AF - CD \leq 0$). From (A.219) $\tau_1 > 0$ is equivalent to

$$L_1 > \frac{C_1(R_1 + R_2)(2R_1R_2 + R_1R_3 + R_2R_3)}{2(R_1 + R_2 + 2R_3)}$$

and from (A.217) $\lambda_1 > 0$ is equivalent to $2L_1 - C_1R_2(R_1 + R_2) > 0$. Replacing L_1 in the latter expression by its strict lower bound (implied by $\tau_1 > 0$) gives

$$\begin{aligned} &\frac{C_1(R_1+R_2)(2R_1R_2+R_1R_3+R_2R_3)}{R_1+R_2+2R_3} - C_1R_2(R_1+R_2) \\ &= \frac{C_1(R_1^2-R_2^2)(R_2+R_3)}{R_1+R_2+2R_3}\,, \end{aligned}$$

which is greater than or equal to zero if $AF - CD \geq 0$ (see (A.222)).

Hence, we can conclude that cases (i), (iv) and (v) are equivalent to the subcase of (A.197) in which $\tau_1 > 0$, $\lambda_1 > 0$ if $AF - CD \geq 0$ (or, if $AF - CD \leq 0$, to the subcase of (A.202) in which $\tau_1 > 0$, $\lambda_3 > 0$).

Step 5

We will now show that $\tau_1 = 0$ implies $\delta \geq 0$ and $\gamma \leq 0$. From (A.219), if $\tau_1 = 0$ then

$$L_1 = \frac{C_1(R_1+R_2)(2R_1R_2+R_1R_3+R_2R_3)}{R_1+R_2+2R_3}\,.$$

Substituting this value for L_1 into (A.218) and (A.220) yields

$$\delta = k^{-2}\,\frac{C_1^2(R_1+R_2)(R_1-R_2)^2R_3}{2(R_1+R_2+2R_3)} = \frac{-R_3}{2DF}\gamma$$

and the result follows. From (A.224) it also follows that $\tau_1 = 0$ and $\delta > 0$ always imply $\gamma < 0$, hence the latter condition can be omitted in this case. Therefore, case (vi) reduces to $\tau_1 = 0$, $\delta > 0$.

Step 6

It can be shown that condition $\tau_1 = 0$ also implies $\lambda_1 > 0$ if $AF - CD > 0$, and $\lambda_3 > 0$ if $AF - CD < 0$. This is clear from the proof in step 4 if we consider $\tau_1 \geq 0$ and $AF - CD \neq 0$. The case $AF - CD = 0$ (which corresponds to $R_1 = R_2$) cannot happen when $\tau_1 = 0$ and $\delta > 0$, since

$$\tau_1 = -4k^{-4}\,C_1^2L_1R_3(R_1+R_3)(C_1R_1^2-L_1)\,, \tag{A.226}$$

$$\delta = 2k^{-2}\,(R_1+2R_3)(C_1R_1^2-L_1) \tag{A.227}$$

when $R_1 = R_2$. Hence, case (vi) is equivalent to (A.195) and (A.200).

Step 7

We now turn to case (ii) and prove that $\zeta_1 > 0$ is necessary if $AF - CD \geq 0$ (respectively $\zeta_3 > 0$ is necessary if $AF - CD \leq 0$). From (A.220) condition $\gamma < 0$ is equivalent to $C_1 > 4L_1/(R_1 + R_2)^2$ and from (A.221) $\zeta_1 > 0$ is equivalent to $C_1 R_2^2(R_1 + R_2) + L_1(R_1 - 3R_2) > 0$. Replacing C_1 in the latter expression by its strict lower bound (implied by $\gamma < 0$) gives

$$4L_1 R_2^2/(R_1 + R_2) + L_1(R_1 - 3R_2) = L_1(R_1 - R_2)^2/(R_1 + R_2) \geq 0\,,$$

hence $\zeta_1 > 0$.

Step 8

We now show that conditions $\delta > 0$ and $\zeta_1 > 0$ always imply $\gamma < 0$ if $AF - CD \geq 0$ (the same result holds if $\delta > 0$, $\zeta_3 > 0$ and $AF - CD \leq 0$). Condition $\delta > 0$ is equivalent to $BE - 2(AF + CD) > 0$, while $\zeta_1 > 0$ can be written as $-BEF + CE^2 + 2F(AF - CD) > 0$. Using these two inequalities we get

$$C(E^2 - 4DF) > BEF - 2F(AF + CD) > 0\,,$$

hence $\gamma = -DF(E^2 - 4DF) < 0$. Therefore, combining with step 7, we see that the condition on γ may be replaced by $\zeta_1 > 0$ (respectively $\zeta_3 > 0$) in case (ii).

Step 9

We now consider case (ii) with the condition $AF - CD \geq 0$ and subdivide it into three cases: $\lambda_1 > 0$, $\lambda_1 < 0$ and $\lambda_1 = 0$ (the case $AF - CD \leq 0$ is analogous with λ_1, ζ_1 replaced by λ_3, ζ_3). In this step we consider the case $\lambda_1 > 0$. It can be proven that, in this case, $AF - CD > 0$ necessarily, as follows. If $AF - CD = 0$ (i.e. $R_1 = R_2$) then τ_1 is given by (A.226) and from (A.217) it follows that

$$\lambda_1 = -2k^{-3}\, C_1 R_1 (C_1 R_1^2 - L_1)\,. \qquad \text{(A.228)}$$

Therefore λ_1 and τ_1 necessarily have the same sign when $AF - CD = 0$, which is a contradiction in this case. Hence, if $\lambda_1 > 0$ case (ii) is equivalent to (A.196) (or if $\lambda_3 > 0$ when $AF - CD \leq 0$, case (ii) is equivalent to (A.201)). We will need to retain the condition $K \geq 0$ in these cases since it is not implied by other conditions.

Step 10

If $\lambda_1 < 0$ in case (ii), we now show that $\delta > 0$ holds automatically (so it can be omitted) when $AF - CD \geq 0$ (the same is true if $\lambda_3 < 0$ when $AF - CD \leq 0$). Condition $\tau_1 < 0$ can be written as

$$BE(AF + CD) > ACE^2 + (AF + CD)^2\,,$$

while $\lambda_1 < 0$ is equivalent to $A\lambda_1 + BE(AF + CD) - BE(AF + CD) < 0$, which can be written as

$$BE(AF + CD) < ACE^2 + BCDE + AF(AF - CD)\,.$$

Comparing the lower and upper bounds of $BE(AF + CD)$ above implies $CD(BE - 3AF - CD) > 0$. It follows that

$$\delta = BE - 2(AF + CD) > 3AF + CD - 2(AF + CD) = AF - CD \geq 0\,,$$

hence $\delta > 0$. We also note, from the expressions for ζ_1, ζ_3 and λ_1, λ_3 in Table 5.9, that

$$\zeta_1 = -\lambda_1 + F(AF - CD)\,, \tag{A.229}$$

$$\zeta_3 = -\lambda_3 - D(AF - CD)\,, \tag{A.230}$$

hence $\lambda_1 < 0$ implies $\zeta_1 > 0$ and $\lambda_3 < 0$ implies $\zeta_3 > 0$, hence the condition on ζ_1 (respectively ζ_3) can be omitted if the condition $\lambda_1 < 0$ (respectively $\lambda_3 < 0$) is included. Therefore, if $\lambda_1 < 0$ and $AF - CD \geq 0$, case (ii) is equivalent to the subcase of (A.197) in which $\tau_1 < 0$, $\lambda_1 < 0$ (the other subcase being covered by step 4). Similarly, if $\lambda_3 < 0$ and $AF - CD \leq 0$, case (ii) is equivalent to the subcase of (A.202) in which $\tau_1 < 0$, $\lambda_3 < 0$.

Step 11

Finally, if $\lambda_1 = 0$ in case (ii) with $AF - CD \geq 0$, we observe that $AF - CD > 0$ necessarily. This follows from step 9, where it was shown that τ_1 and λ_1 have the same sign when $AF - CD = 0$, hence $\tau_1 < 0$, $\lambda_1 = 0$, $AF - CD = 0$ cannot occur in this case.

It now follows as in step 10 that $\delta > 0$ holds automatically, with the relaxed condition $\lambda_1 \leq 0$, so it can be omitted in this case. Therefore, as in step 10, from the identities (A.229) and (A.230) it is clear that if $\lambda_1 = 0$ then $\zeta_1 > 0$ (and if $\lambda_3 = 0$ then $\zeta_3 > 0$), hence the condition on ζ_1 (respectively ζ_3) can be omitted if the condition $\lambda_1 = 0$ (respectively $\lambda_3 = 0$) is included.

In summary, if $\lambda_1 = 0$ when $AF - CD \geq 0$, case (ii) is equivalent to (A.194) (or if $\lambda_3 = 0$ when $AF - CD \leq 0$, case (ii) is equivalent to (A.199)).

Step 12

We show next that case (vii) is equivalent to (A.198), i.e.

$$\gamma = 0,\ \ \delta = 0,\ \ \tau_1 = 0 \quad \Leftrightarrow \quad AF - CD = 0,\ \ K = 0\,.$$

If $AF - CD = 0$ (i.e. $R_1 = R_2$, from (A.222)) then τ_1 and δ have the expressions shown in (A.226) and (A.227), respectively, and

$$\gamma = -4k^{-4}\, C_1^2 L_1 (C_1 R_1^2 - L_1)\,. \qquad \text{(A.231)}$$

It follows that if $K = 0$ (i.e. $L_1 = C_1 R_1 R_2$, from (A.216)) then $\gamma = 0$, $\delta = 0$ and $\tau_1 = 0$. We now show that $\gamma = 0$, $\delta = 0$, $\tau_1 = 0 \Rightarrow AF - CD = 0$, $K = 0$. From (A.220), $\gamma = 0$ if and only if

$$L_1 = C_1 (R_1 + R_2)^2 / 4\,. \qquad \text{(A.232)}$$

It was shown in step 3 that using this value for L_1 we obtain the expressions for τ_1 and δ in (A.225), from which we see that $R_1 = R_2$. It follows from (A.222) that $AF - CD = 0$, and from (A.232) that $L_1 = C_1 R_1^2$, hence $K = 0$ (from (A.216)).

Step 13

We finally turn to the necessity condition $K \geq 0$ and show that it can be omitted in the conditions of the theorem (i.e. it is implied by the other conditions) in all but two cases, namely (A.196) and (A.201), other than case (A.198) which is already dealt with in step 12.

If $AF - CD \geq 0$ we first show that $\lambda_1 \leq 0$ implies $K \geq 0$ (the same result holds if $AF - CD \leq 0$ and $\lambda_3 \leq 0$). It can be verified from the expressions in Table 5.9 that

$$FK = -\lambda_1 (AF - CD) + D(BF - CE)^2\,,$$

from which the result follows. Therefore, $K \geq 0$ can be omitted from (A.194) and from the subcase of (A.197) in which $\lambda_1 < 0$ (respectively from (A.199) and from the subcase of (A.202) in which $\lambda_3 < 0$).

We next show that $\tau_1 \geq 0$ implies $K \geq 0$. From the expression in Table 5.9, $\tau_1 \geq 0$ implies $K \geq DF(B^2 - 4AC)$. If $B^2 - 4AC \geq 0$ we can immediately conclude that $K \geq 0$.

Otherwise, if $B^2 - 4AC < 0$, it follows that

$$\begin{aligned}4ACK &= 4AC[(AF-CD)^2 - (AE-BD)(BF-CE)]\\ &> B^2(AF-CD)^2 - 4AC(AE-BD)(BF-CE)\\ &= (ABF-2ACE+BCD)^2 \geq 0\,.\end{aligned}$$

Therefore, $K \geq 0$ can be omitted from (A.195), (A.200) and from the remaining subcases in (A.197), (A.202).

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$, we can calculate one or two sets of solutions (depending on the signs of τ_1 and γ) for R_1, R_2, R_3, L_1, C_1 and k in cases (A.194)–(A.197) and (A.199)–(A.202), using (A.203), (A.205)–(A.208), and (A.223), and an infinite number of solutions in case (A.198). It remains to show that the necessary conditions derived are sufficient to ensure positivity of R_1 and R_2, and thence L_1 and C_1 from (A.207) and (A.208).

From (A.205), R_1 is positive if and only if $R_3 < A/D$, and, from (A.206), R_2 is positive if and only if $R_3 < C/F$. Therefore, in order for both R_1 and R_2 to be positive, we need

$$R_3 < \frac{C}{F} \quad \text{if } AF - CD \geq 0\,, \tag{A.233}$$

$$R_3 < \frac{A}{D} \quad \text{if } AF - CD \leq 0\,. \tag{A.234}$$

We consider the case that $AF - CD \geq 0$ (if $AF - CD \leq 0$ the proof is analogous, with λ_1 replaced by λ_3 and ζ_1 by ζ_3). We denote the two roots of (A.203) as

$$R_{3_A} = \frac{-2DF\,\delta - \sqrt{\Delta}}{2\gamma}\,, \qquad R_{3_B} = \frac{-2DF\,\delta + \sqrt{\Delta}}{2\gamma}\,, \tag{A.235}$$

when $\gamma \neq 0$.

If conditions (A.194)–(A.196) or (A.197) hold (with $\tau_1 < 0$ and $\lambda_1 < 0$ in the latter case), then $\gamma < 0$ (see steps 5 and 8) and $R_{3_A} \geq R_{3_B}$. We note that in case (A.195) $R_{3_B} = 0$, while in the other cases both R_{3_A} and R_{3_B} are strictly positive. Inequality (A.233) reduces to

$$\sqrt{\Delta} < 2D\zeta_1 \qquad (R_3 = R_{3_A})\,, \tag{A.236}$$

$$\sqrt{\Delta} > -2D\zeta_1 \qquad (R_3 = R_{3_B})\,. \tag{A.237}$$

Both sides of (A.236) are positive, since $\zeta_1 > 0$ if $\gamma < 0$ (this was proven in step 7), and the inequality can therefore be squared and reduced to

$$4F\,\gamma\,(AF - CD)\,\lambda_1 < 0\,, \tag{A.238}$$

which holds for cases (A.195) and (A.196) but not for (A.194) and the subcase of (A.197). Inequality (A.237) need not be verified for case (A.195) (since $R_{3_B} = 0$ in this case), and is immediately satisfied for the other cases, since $\zeta_1 > 0$. Therefore, in conclusion, both roots of (A.203) yield positive values for R_1 and R_2 in case (A.196), and only the smallest positive root in cases (A.194), (A.195), and (A.197) in the subcase with $\tau_1 < 0$ and $\lambda_1 < 0$.

We now turn to the subcase of (A.197) with $\tau_1 > 0$ and $\lambda_1 > 0$. In this case nothing can be concluded on the sign of γ, so we examine three cases on the sign of γ separately.

If $\gamma < 0$ then we see directly from (A.203) that R_{3_A} is the only positive root of (A.203), and inequality (A.233) reduces to (A.236). We will now show that in this case $\zeta_1 > 0$. Condition $\tau_1 > 0$ can be written as

$$-BE(AF + CD) > -ACE^2 - (AF - CD)^2 - 4ACDF\,.$$

It follows that

$$\begin{aligned}\zeta_1(AF + CD) &= -BEF(AF + CD) + CE^2(AF + CD) + 2F(A^2F^2 - C^2D^2)\\ &> -F(AF - CD)^2 - 4ACDF^2 + C^2DE^2 + 2F(A^2F^2 - C^2D^2)\,.\end{aligned}$$

Knowing that $\gamma < 0$ (i.e. $E^2 > 4DF$), if E^2 is replaced by $4DF$, the lower bound becomes

$$\begin{aligned}&- F(AF - CD)^2 - 4ACDF^2 + 4C^2D^2F + 2F(A^2F^2 - C^2D^2)\\ &= F(AF - CD)^2 \geq 0\,,\end{aligned}$$

hence $\zeta_1 > 0$. Therefore, both sides of (A.236) are positive and the inequality can be squared and reduced to (A.238), which holds if $AF - CD \neq 0$. If $AF - CD = 0$ it can be verified that the following identity holds:

$$C^2\,\gamma - F^2\tau_1 = F(AF + CD)\,\lambda_1\,. \tag{A.239}$$

From (A.239) it follows that the case $AF - CD = 0$, $\gamma < 0$, $\tau_1 > 0$, $\lambda_1 > 0$ cannot occur.

If $\gamma > 0$, the quadratic (A.203) has two positive roots $R_{3_B} \geq R_{3_A}$ since $\delta < 0$ (see step 2), and inequality (A.233) reduces to

$$\sqrt{\Delta} > 2D\zeta_1 \qquad (R_3 = R_{3_A})\,, \tag{A.240}$$

$$\sqrt{\Delta} < -2D\zeta_1 \qquad (R_3 = R_{3_B})\,. \tag{A.241}$$

If $\zeta_1 \geq 0$ inequality (A.241) does not hold, and if $\zeta_1 < 0$ the same inequality can be squared and reduced to (A.238), which again does not hold. We now turn to (A.240). If $\zeta_1 < 0$ inequality (A.240) holds, and if $\zeta_1 \geq 0$ the inequality can be squared and reduced to

$$4F\,\gamma\,(AF - CD)\,\lambda_1 > 0\,,$$

which again holds, providing $AF - CD \neq 0$. We therefore need to check the case $AF - CD = 0$ directly. If $AF - CD = 0$ then $\zeta_1 = -\lambda_1$, from the expression in Table 5.9. Therefore, if the conditions of case (A.197) hold with $\tau_1 > 0$, $\lambda_1 > 0$, $\gamma > 0$ and $AF - CD = 0$ then $\zeta_1 < 0$, and inequality (A.240) is immediately satisfied.

If $\gamma = 0$, the only root of (A.203) is $R_3 = -\tau_1/(2DF\,\delta) > 0$ since $\delta < 0$ (see step 2), and inequality (A.233) reduces to $\tau_1 + 2CD\,\delta < 0$. It can be verified that

$$\tau_1 + 2CD\,\delta = -C^2\,\gamma/F^2 - \lambda_1\,(AF - CD)/F\,,$$

hence, for $\gamma = 0$, the inequality reduces to

$$\tau_1 + 2CD\,\delta = -\lambda_1(AF - CD)/F < 0\,,$$

which holds if $AF - CD \neq 0$, since $\lambda_1 > 0$. It can also be verified that the following identity always holds for $AF - CD = 0$ and $\gamma = 0$:

$$\tau_1 = -A\,\lambda_1 - CE^2\,K/\lambda_1\,. \tag{A.242}$$

From (A.242) it follows that the case $AF - CD = 0$, $\tau_1 > 0$, $\lambda_1 > 0$ cannot occur. We can therefore conclude from the analysis of the three cases $\gamma < 0$, $\gamma > 0$ and $\gamma = 0$ above that if the conditions of case (A.197) hold with $\tau_1 > 0$ and $\lambda_1 > 0$, then solution R_{3_A} yields positive values for R_1 and R_2 if $\gamma \neq 0$, while if $\gamma = 0$ the only root of (A.203) yields positive values for R_1 and R_2. This establishes that R_3 should be the smallest positive root in case (A.197).

In case (A.198) we have $\gamma = 0$, $\delta = 0$ and $\tau_1 = 0$ (see step 12), hence any value of x satisfies (A.203). Since $A/D = C/F$ in this case, inequalities (A.233) and (A.234) are equivalent, and the theorem statement follows (i.e. R_3 can be chosen arbitrarily provided that $R_3 < C/F$). □

Figure A.10 shows the realisability region for the network, plotted on the (U, V)-plane for $W = 0.8$ (i.e. $AF - CD > 0$). The expressions in terms of U, V, W for all the symbols appearing in the figure can be found in Table 5.9. It is clear from the figure that the curve γ_c does not act as a boundary for the realisability region, and it was in fact possible in the proof to eliminate γ from the realisability conditions. Curves δ_c and ζ_c are not active boundaries either, but are still needed to properly define the region corresponding to cases (A.195) and (A.196). We note that it may be possible to write the conditions in different ways or to further simplify such conditions.

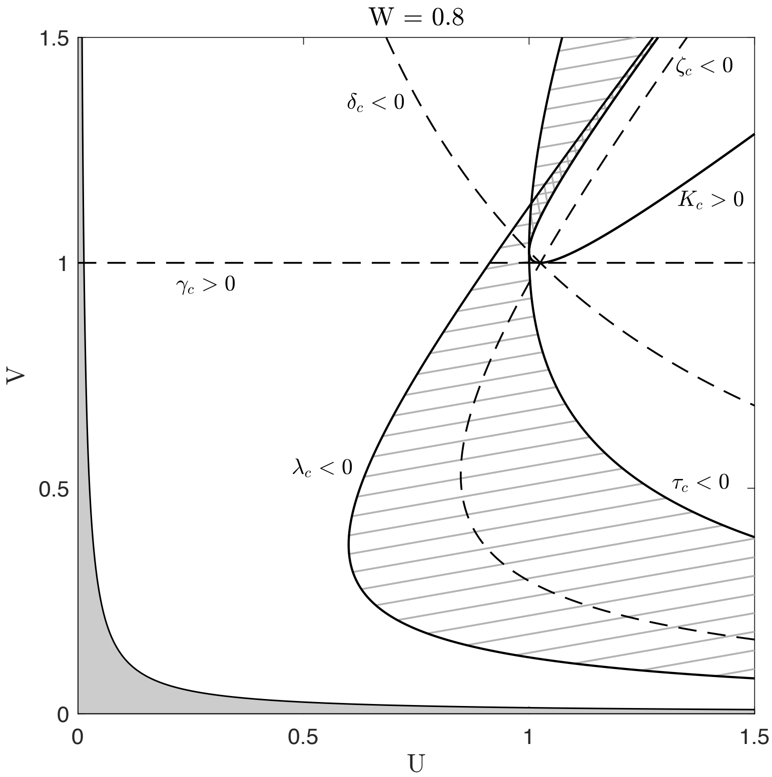


Figure A.10: Realisability region for the network on the (U, V)-plane, for $W = 0.8$. The hatched regions correspond to case (A.197), while the crossed region to case (A.196). Cases (A.194) and (A.195) correspond to the boundaries of the crossed region, with $\lambda_c = 0$ and $\tau_c = 0$, respectively. The dashed curves, namely γ_c, δ_c and ζ_c, are not active boundaries for the realisability region.

A.9 Equivalence class $\mathbf{V_H^1}$

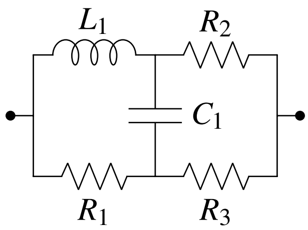


Figure A.11: Network #70, from subfamily V_H. By Theorem 3 in [43] it can only realise regular impedances.

Theorem A.9. *The positive-real biquadratic impedance* (4.1) *with* A, B, C, D, E, $F > 0$ *can be realised as in Figure A.11, with* R_1, R_2, R_3, L_1 *and* C_1 *positive and finite, if and only if*

$$AF - CD > 0\,, \tag{A.243}$$

$$\mu_1 \geq 0\,, \tag{A.244}$$

and

$$\textit{signs of } \lambda_1, \lambda_2, \rho_1 \textit{ not all the same}\,, \tag{A.245}$$

where λ_1, λ_2, μ_1 *and* ρ_1 *are defined in Table 5.9. When one of* λ_1, λ_2, ρ_1 *equals zero, the other two must have different algebraic signs or both equal zero, i.e.*

$$\lambda_1 = \lambda_2 = 0, \quad \rho_1 = 0\,. \tag{A.246}$$

If conditions (A.243)–(A.245) *are satisfied, then* R_3 *is any positive root of the quadratic equation in* x

$$D\lambda_1 x^2 + \rho_1 x + C\lambda_2 = 0\,, \tag{A.247}$$

while if conditions (A.243), (A.244) *and* (A.246) *are satisfied* R_3 *is any positive value.* R_2 *is the positive root of the quadratic equation in* y

$$c_1\, y^2 + c_2\, y + c_3 = 0 \tag{A.248}$$

where

$$c_1 = AF - CD\,,$$
$$c_2 = (A + DR_3)(FR_3 - C)\,,$$
$$c_3 = -CR_3(A + DR_3)\,,$$

and

$$R_1 = \frac{R_2(A - DR_3) + AR_3}{D(R_2 + R_3)}\,, \tag{A.249}$$

$$L_1 = \frac{DR_1(R_1R_2 + R_1R_3 + R_3^2)}{(R_2 + R_3)\big[E(R_1 + R_3) - B\big]}\,, \tag{A.250}$$

$$C_1 = \frac{D(R_1 + R_2 + R_3)}{F(R_2 + R_3)L_1}\,. \tag{A.251}$$

Proof. Note this result was also proven in [43, Appendix B], where μ_1 and ρ_1 are called η_1 and η_2, respectively. We provide here an independent proof.

Necessity. The impedance of the network shown in Figure A.11 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.252}$$

where

$$n(s) = L_1C_1(R_1R_2 + R_1R_3 + R_2R_3)s^2 + \big(C_1R_1R_2R_3 + L_1(R_1 + R_3)\big)s + R_2(R_1 + R_3)\,,$$
$$d(s) = L_1C_1(R_2 + R_3)s^2 + \big(C_1R_1(R_2 + R_3) + L_1\big)s + R_1 + R_2 + R_3\,.$$

Equating impedance (A.252) with (4.1) we obtain, for a positive constant k,

$$L_1C_1(R_1R_2 + R_1R_3 + R_2R_3) = kA\,, \tag{A.253}$$
$$C_1R_1R_2R_3 + L_1(R_1 + R_3) = kB\,, \tag{A.254}$$
$$R_2(R_1 + R_3) = kC\,, \tag{A.255}$$
$$L_1C_1(R_2 + R_3) = kD\,, \tag{A.256}$$
$$C_1R_1(R_2 + R_3) + L_1 = kE\,, \tag{A.257}$$
$$R_1 + R_2 + R_3 = kF\,, \tag{A.258}$$

which are a set of necessary and sufficient conditions for (4.1) to be realised as in Figure A.11. It can be calculated that

$$AF - CD = k^{-2}\, R_1 L_1 C_1 (R_1 R_2 + R_1 R_3 + 2R_2 R_3 + R_3^2) > 0\,, \tag{A.259}$$

$$\begin{aligned} \mu_1 = k^{-4}\, L_1 C_1 \big[& R_1 R_2 C_1 (R_1 + 2R_3)(R_2 + R_3) \\ & - L_1 (R_1 + R_3)(2R_2 + R_3)\big]^2 \geq 0\,, \end{aligned} \tag{A.260}$$

$$K = k^{-4}\, L_1 C_1 \big[R_1^2 R_2 C_1 (R_2 + R_3) + L_1 R_3 (R_1 + R_3)\big]^2 > 0\,, \tag{A.261}$$

hence (A.243) and (A.244) are necessary. From (A.258) we obtain

$$k = (R_1 + R_2 + R_3)/F\,, \tag{A.262}$$

and from (A.256) we get (A.251). Equations (A.254) and (A.257) now reduce to

$$\begin{aligned} F(R_1 + R_3)(R_2 + R_3)\, L_1^2 - B(R_1 + R_2 + R_3)(R_2 + R_3)\, L_1 & \\ + D R_1 R_2 R_3 (R_1 + R_2 + R_3) &= 0\,, \\ F L_1^2 - E(R_1 + R_2 + R_3)\, L_1 + D R_1 (R_1 + R_2 + R_3) &= 0\,. \end{aligned}$$

Eliminating the term in FL_1^2 from the two equations and solving for L_1 gives (A.250). Eliminating $L_1 C_1$ from (A.253) and (A.256) and solving for R_1 gives (A.249), and substituting (A.249) and (A.262) into (A.255) we get the quadratic equation (A.248), with $y = R_2$. From (A.248) an expression for R_2^2 can be found, as follows:

$$R_2^2 = \frac{(A + DR_3)\big[(C - FR_3)R_2 + CR_3\big]}{AF - CD}\,. \tag{A.263}$$

We now have expressions for elements R_1, L_1, C_1 and for the constant k which only contain R_2 and R_3, together with A, B, C, D, E and F, and an expression for R_2^2 which contains R_3, together with the same coefficients, and is linear in R_2. Substituting the expressions for R_1, L_1, C_1 and k into (A.257) we obtain a polynomial of fourth order in R_2 and fifth order in R_3. Then, substituting for R_2^2 from (A.263) twice to reduce the power of R_2, after further manipulation all terms in R_2 cancel out and the higher powers in R_3 are eliminated, and we obtain the quadratic equation (A.247) in R_3.

When λ_1 and λ_2 in (A.247) are of opposite sign, the quadratic has one real positive root. When λ_1 and λ_2 are of the same sign, the quadratic has two real positive roots if and only if ρ_1 is of opposite sign and the discriminant is non-negative. It can be calculated that the discriminant of (A.247) is $K\mu_1$, and is therefore non-negative by (A.260) and (A.261). It is easily seen that when one of the three coefficients in (A.247) is zero it is necessary that the other two have different sign, in order for R_3 to be positive. Similarly, when two of the three coefficients in (A.247) are zero it is necessary that the other one also equals zero, in order for R_3 to be non-zero. Therefore conditions (A.245) and (A.246) are necessary.

We finally show that the condition $K > 0$ is implied by (A.243) and (A.245)–(A.246), and can therefore be omitted from the theorem statement. It can be verified that the following identities always hold

$$E^2F\,K = -\lambda_1 E^2(AF - CD) + D\big(\lambda_1 + F(AF - CD)\big)^2\,,$$
$$AB^2\,K = -\lambda_2 B^2(AF - CD) + C\big(\lambda_2 + A(AF - CD)\big)^2\,,$$

therefore when $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$ it follows that $K > 0$. If $\lambda_1 > 0$ and $\lambda_2 > 0$ it follows from (A.245) that $\rho_1 < 0$, hence $K = -\rho_1 + 2CD(AF - CD) > 0$.

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$ satisfying (A.243)–(A.246), we can calculate from (A.247) one, two or infinitely many positive solutions for R_3, depending on the signs of λ_1, λ_2 and ρ_1. We now show that any positive value of R_3 leads to positive values of the other network elements.

It is easily seen that coefficients c_1 and c_3 in (A.248) are of opposite sign, therefore the quadratic always has one real positive root $R_2 > 0$. From (A.249), R_1 is positive if

$$R_2(A - DR_3) > -AR_3\,. \tag{A.264}$$

If $R_3 \leq A/D$ then (A.264) is satisfied, otherwise it can be rewritten as

$$R_2 < \frac{-AR_3}{A - DR_3}\,. \tag{A.265}$$

Since R_2 is the only positive solution to the quadratic equation (A.248), which represents a parabola that opens up, inequality (A.265) holds if and only if (A.248) evaluated at $-AR_3/(A - DR_3)$ gives a positive value, that is:

$$c_1\left(\frac{-AR_3}{A - DR_3}\right)^2 + c_2\left(\frac{-AR_3}{A - DR_3}\right) + c_3 > 0\,. \tag{A.266}$$

After some manipulation, inequality (A.266) can be reduced to

$$\frac{R_3^4\,(AF - CD)\,D^2}{(A - DR_3)^2} > 0\,,$$

which always holds.

Hence R_1 is positive and, from (A.262), also k is positive. From (A.251), L_1 and C_1 have the same sign, and we can therefore conclude from (A.254) that they are both positive. □

Figure A.12 shows the realisability region for the network, plotted on the (U, V)-plane, for $W = 0.7$ while Figure A.13 shows the realisability region for $W = 0.25$. The expressions in terms of U, V, W for all the symbols appearing in the figures can be found in Table 5.9. It is clear from both figures that the curve ρ_c is not an active boundary, but is still needed to properly define the realisability regions.

The interior of the hatched region is realisable with two distinct solutions, unless either λ_c, $\lambda_c^\dagger$ or ρ_c is zero, in which case there is only one solution. The boundary is realisable only for $\mu_c = 0$, where there are two coincident solutions. It can finally be verified that case (A.246), where there are infinite solutions, corresponds to the case $W = 1/3$. As pointed out in [43] and [61], there exists a realisable region with $\lambda_c < 0$ and $\lambda_c^\dagger < 0$ (corresponding to non-regular impedances) only for $W \in (1/3, 1)$, as can be seen from Figures A.12 and A.13.

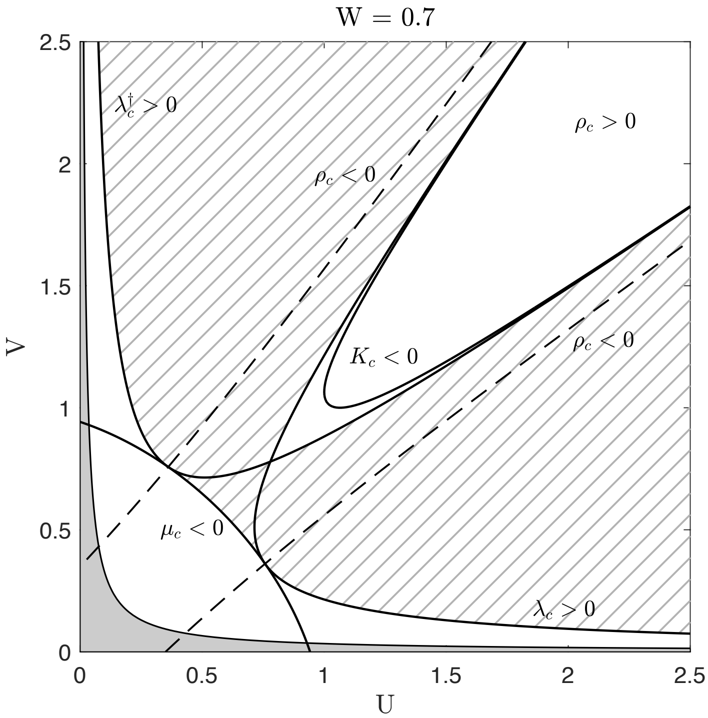


Figure A.12: Realisability region for the network on the (U, V)-plane, for $W = 0.7$. The region defined by $\lambda_c < 0$, $\lambda_c^\dagger < 0$ and $\mu_c \geq 0$ corresponds to non-regular impedances.

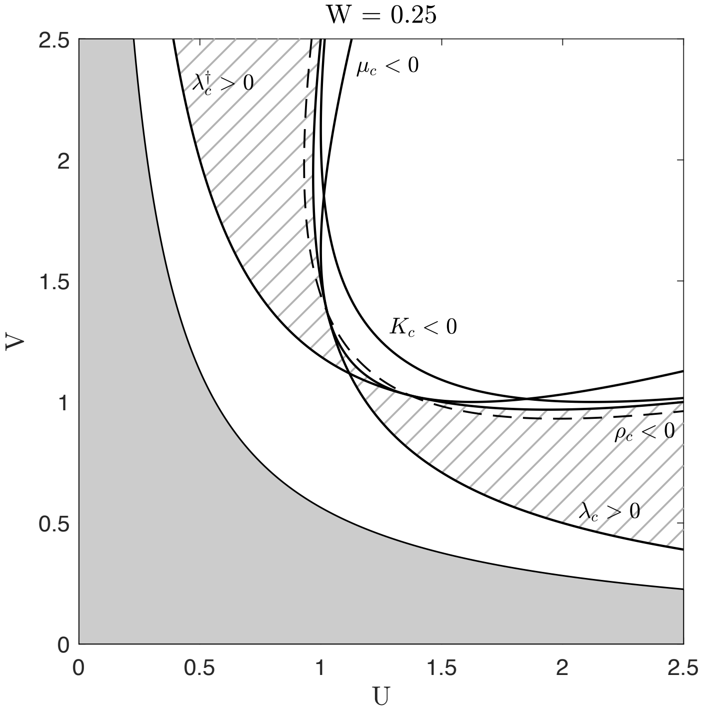


Figure A.13: Realisability region for the network on the (U, V)-plane, for $W = 0.25$.

A.10 Equivalence class $\mathbf{V_I}$

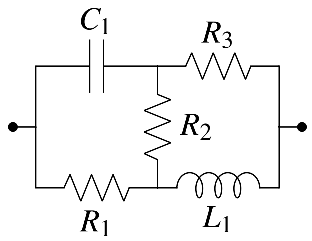


Figure A.14: Network #108, from subfamily $\mathrm{V_I}$. By Lemma 12 in [43] it can only realise regular impedances.

Theorem A.10. *The positive-real biquadratic impedance* (4.1) *with A, B, C, D, E, $F > 0$ can be realised as in Figure A.14, with R_1, R_2, R_3, L_1 and C_1 positive and finite, if and only if*

$$K \geq 0\,, \tag{A.267}$$

where K is defined in (3.3)*, and the conditions of one of the following cases hold:*

$$\begin{aligned} &\textit{Case 1)} \quad \tau_1\tau_2 < 0\,, &\text{(A.268)}\\ &\textit{Case 2)} \quad \tau_1 = 0,\ \ \tau_2 < 0,\ \ \psi > 0\,, &\text{(A.269)}\\ &\textit{Case 3)} \quad \tau_2 = 0,\ \ \tau_1 < 0,\ \ \psi > 0\,, &\text{(A.270)}\\ &\textit{Case 4)} \quad \tau_1 < 0,\ \ \tau_2 < 0,\ \ \psi > 0\,, &\text{(A.271)}\\ &\textit{Case 5)} \quad \tau_1 = 0,\ \ \tau_2 = 0,\ \ \psi = 0\,, &\text{(A.272)} \end{aligned}$$

where τ_1, τ_2 and ψ are defined in Table 5.9. In cases (A.268)–(A.270)*, R_2 is the only positive root of the quadratic equation in x*

$$DF\,\tau_1\,x^2 + \psi\,x + AC\,\tau_2 = 0\,, \tag{A.273}$$

while if (A.271) *holds then R_2 is either of the two positive roots of the same quadratic. If* (A.272) *holds then R_2 is any positive value. The other elements of the network are given by:*

$$\begin{aligned} R_1 &= C/F\,, &\text{(A.274)}\\ R_3 &= A/D\,, &\text{(A.275)}\\ L_1 &= \frac{(DR_2 + A)(AF + CD)}{DEF\,R_2 + \gamma_2}\,, &\text{(A.276)}\\ C_1 &= \frac{DR_2 + A}{L_1(FR_2 + C)}\,, &\text{(A.277)} \end{aligned}$$

where $\gamma_2 = AEF - BDF + CDE$.

Proof. Necessity. The impedance of the network shown in Figure A.14 is a biquadratic, which can be computed as

$$Z(s) = \frac{n(s)}{d(s)}\,, \tag{A.278}$$

where

$$\begin{aligned} n(s) &= L_1C_1R_3(R_1 + R_2)s^2 + \big(C_1R_1R_2R_3 + L_1(R_1 + R_2 + R_3)\big)s + R_1(R_2 + R_3)\,,\\ d(s) &= L_1C_1(R_1 + R_2)s^2 + \big(C_1(R_1R_2 + R_1R_3 + R_2R_3) + L_1\big)s + R_2 + R_3\,. \end{aligned}$$

Equating impedance (A.278) with (4.1), we obtain

$$L_1C_1R_3(R_1 + R_2) = kA\,, \tag{A.279}$$

$$C_1R_1R_2R_3 + L_1(R_1 + R_2 + R_3) = kB\,, \tag{A.280}$$

$$R_1(R_2 + R_3) = kC\,, \tag{A.281}$$

$$L_1C_1(R_1 + R_2) = kD\,, \tag{A.282}$$

$$C_1(R_1R_2 + R_1R_3 + R_2R_3) + L_1 = kE\,, \tag{A.283}$$

$$R_2 + R_3 = kF\,, \tag{A.284}$$

where k is a positive constant. From (A.284) we obtain

$$k = (R_2 + R_3)/F\,. \tag{A.285}$$

Eliminating $L_1C_1(R_1 + R_2)$ from (A.279) and (A.282) we obtain (A.275), while eliminating $R_2 + R_3$ from (A.281) and (A.284) we obtain (A.274). Solving (A.282) for C_1 then gives (A.277). Equations (A.280) and (A.283) now reduce to

$$\begin{aligned}(DFR_2 + AF + CD)(FR_2 + C)\,L_1^2 - B(DR_2 + A)(FR_2 + C)\,L_1 \\ +ACR_2(DR_2 + A) = 0\,, \\ DF(FR_2 + C)\,L_1^2 - E(DR_2 + A)(FR_2 + C)\,L_1 \\ +(DR_2 + A)\big((AF + CD)R_2 + AC\big) = 0\,.\end{aligned}$$

Eliminating the term in $(FR_2 + C)L_1^2$ from the two equations and solving for L_1 gives (A.276). We now have expressions for R_1, R_3, L_1, C_1 and k which only contain R_2, together with A, B, C, D, E, F. Substituting such expressions into (A.283) we obtain the quadratic equation (A.273) in R_2. It can be calculated that

$$K = k^{-4}\, L_1C_1(R_1 + R_2)^2(R_2 + R_3)^2(L_1 - R_1R_3C_1)^2\,,$$

from which (A.267) is necessary. The discriminant of (A.273),

$$\Delta = (AF + CD)^2\,(K + 4ACDF)\,K\,, \tag{A.286}$$

is therefore non-negative, from (A.267). If $\tau_1\tau_2 < 0$ then (A.273) has one real positive root (case (A.268)), while if $\tau_1\tau_2 > 0$ then there are two positive roots if and only if ψ is of the opposite sign to τ_1 and τ_2. It can be verified that the following identity always holds

$$\psi = AF\,\tau_1 + CD\,\tau_2 + AD(BF - CE)^2\,. \tag{A.287}$$

Hence $\tau_1 > 0$, $\tau_2 > 0$ implies $\psi > 0$, from which it follows that ψ being of opposite sign to τ_1 and τ_2 corresponds only to (A.271). If $\tau_1 = 0$ then we need $\tau_2\psi < 0$ in order for R_2 to be positive. It follows from (A.287) that the case $\tau_2 > 0$, $\psi < 0$ cannot occur hence the case $\tau_1 = 0$, $\tau_2\psi < 0$ reduces to (A.269). A similar argument holds for case (A.270), when $\tau_2 = 0$. Finally, if $\tau_1 = 0$, $\tau_2 = 0$, $\psi = 0$ (i.e. case (A.272)) then any value of x solves (A.273), so no restriction is placed on R_2.

Sufficiency. Given a positive-real impedance (4.1) with A, B, C, D, E, $F > 0$ satisfying (A.267), we can calculate from (A.273) one positive solution for R_2 in cases (A.268)–(A.270), two positive solutions in case (A.271) and infinitely many in case (A.272). We now show that in all cases the values obtained for R_2 lead to a positive L_1, and thence C_1 (from (A.277)).

From (A.276), L_1 will be positive if

$$R_2 > -\gamma_2/(DEF)\,, \tag{A.288}$$

hence if $\gamma_2 \geq 0$ then (A.288) is immediately satisfied. It can be verified that the following identity always holds

$$B\,\gamma_2 = (AF + CD)^2 - \tau_2\,, \tag{A.289}$$

hence $\gamma_2 > 0$ (and therefore $L_1 > 0$) in all the cases in which $\tau_2 \leq 0$, i.e. in (A.269)–(A.272) and the subcase $\tau_1 > 0$, $\tau_2 < 0$ of (A.268). In the other subcase of (A.268), i.e. $\tau_1 < 0$, $\tau_2 > 0$, the only positive root of the quadratic (A.273) is

$$R_2 = \frac{-\psi - \sqrt{\Delta}}{2DF\,\tau_1}\,. \tag{A.290}$$

Substituting (A.290) into (A.288) we obtain the inequality

$$(AF + CD)\left[E(K + 4ACDF) - 2BDF(AF + CD)\right] < E\sqrt{\Delta}\,. \tag{A.291}$$

For (A.291) to hold it is sufficient that the inequality holds when both sides are squared, and after some manipulation the latter inequality reduces to

$$-4DF(BF - CE)(AE - BD)(AF + CD)^2\,\tau_1 < 0\,. \tag{A.292}$$

As mentioned above, if $\gamma_2 \geq 0$ inequality (A.288) is immediately satisfied. If $\gamma_2 < 0$ then $AE - BD < 0$ and $BF - CE > 0$ (directly from the expression for γ_2) hence inequality (A.292) holds in this case. Therefore L_1 is positive for all sets of solutions.

□

Figure A.15 shows the realisability region for the network on the (U, V)-plane for $W = 0.5$ (i.e. $AF - CD > 0$). The expressions in terms of U, V, W for all the symbols appearing in the figure can be found in Table 5.9. It is clear from the figure that the curve ψ_c is not an active boundary, but is still needed to properly define the realisability region.

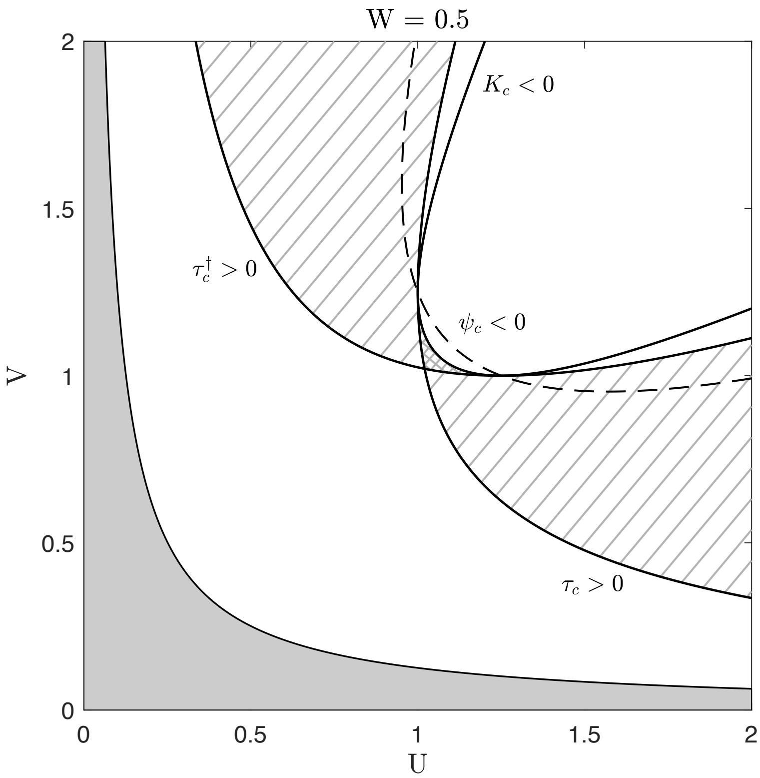


Figure A.15: Realisability region for the network on the (U, V)-plane, for $W = 0.5$. The hatched regions correspond to case (A.268), while the crossed region to case (A.271). Cases (A.269) and (A.270) correspond to the boundaries of the crossed region, with $\tau_c = 0$ and $\tau_c^\dagger = 0$, respectively. It can be verified that the conditions of case (A.272) imply $AF - CD = 0$ (i.e. $W = 1$), hence case (A.272) cannot be represented in this plot.

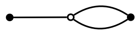
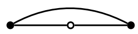
Appendix B

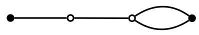
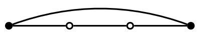
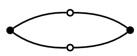
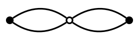
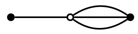
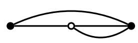
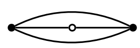
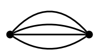
Basic graphs

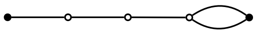
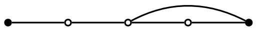
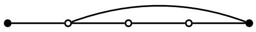
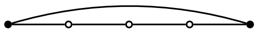
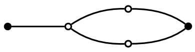
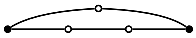
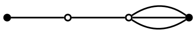
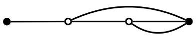
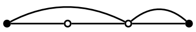
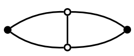
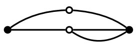
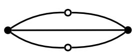
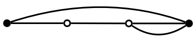
Below is the enumeration of all the basic graphs with at most five edges, with the corresponding network numbers from the Ladenheim canonical set. The superscript `d` indicates the graph dual, while values in brackets indicate the 40 networks which are eliminated from the canonical total (see Section 4.2).

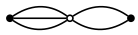
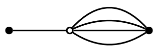
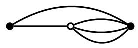
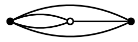
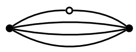
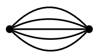
Graph	One-element networks	R	L	C	Tot.	Canonical tot.	Network number
`A`		1	1	1	3	3	1, 2, 3
				Total	3	3	

Graph	Two-element networks	$2R$	R–L R–C	LC	Tot.	Canonical tot.	Network number
`B`		0	2	1	3	3	4, 5, 6
`B`$^{\mathtt{d}}$		0	2	1	3	3	7, 8, 9
				Total	6	6	

Graph	Three-element networks	$3R$	$2R$–L $2R$–C	$2L$–R $2C$–R	LRC	Tot.	Canonical tot.	Network number
C		0	0	0	1	1	1	10
D		0	2	2	3	7	7	11, 14, 15, 17, 26, 27, 34
D^{d}		0	2	2	3	7	7	12, 13, 16, 18, 41, 42, 49
C^{d}		0	0	0	1	1	1	19
					Total	16	16	

Graph	Four-element networks	$3R$–L $3R$–C	$2R$–$2L$ $2R$–$2C$	$2R$–LC	Tot.	Canonical tot.	Network number
E		0	0	0	0	0	-
F		0	2	2	4	4	20, 25, 28, 32
G		(2)	4	5	11	9	21, 22, 24, 29, 30, 33, 63, 71, 87
H		0	0	1	1	1	72
I		0	2	1	3	3	23, 31, 97
I^{d}		0	2	1	3	3	38, 46, 96
H^{d}		0	0	1	1	1	73
G^{d}		(2)	4	5	11	9	36, 37, 40, 44, 45, 48, 62, 74, 88
F^{d}		0	2	2	4	4	35, 39, 43, 47
E^{d}		0	0	0	0	0	-
				Total	38	34	

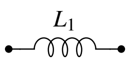
Graph	Five-element networks	4*R*–*L* 4*R*–*C*	3*R*–2*L* 3*R*–2*C*	3*R*–*LC*	Tot.	Canonical tot.	Network number
J		0	0	0	0	0	-
K		0	0	0	0	0	-
L		0	(2)	(2)	4	0	-
M		0	0	(1)	1	0	-
N		0	0	0	0	0	-
O		0	(2)	1	3	1	104
P		0	0	0	0	0	-
Q		0	0	0	0	0	-
R		0	2	2	4	4	50, 68, 79, 93
S		(2)	(2) + 4	(3) + 4	15	8	51, 52, 64, 69, 80, 81, 89, 94
T		0	(2)	1	3	3	53, 82, 98
U		0	2	2	4	4	59, 83, 99, 100
V		(4)	(4) + 4	5	17	9	60, 61, 70, 85, 86, 95, 105, 107, 108
U$^{\mathrm{d}}$		0	2	2	4	4	58, 84, 102, 103
T$^{\mathrm{d}}$		0	2	1	3	3	57, 78, 101
S$^{\mathrm{d}}$		(2)	(2) + 4	(3) + 4	15	8	55, 56, 66, 67, 76, 77, 90, 92
R$^{\mathrm{d}}$		0	2	2	4	4	54, 65, 75, 91
Q$^{\mathrm{d}}$		0	0	0	0	0	-

Graph	Five-element networks	$4R$–L $4R$–C	$3R$–$2L$ $3R$–$2C$	$3R$–LC	Tot.	Canonical tot.	Network number
$\mathtt{P}^{\mathtt{d}}$		0	0	0	0	0	-
$\mathtt{O}^{\mathtt{d}}$		0	(2)	1	3	1	106
$\mathtt{N}^{\mathtt{d}}$		0	0	0	0	0	-
$\mathtt{M}^{\mathtt{d}}$		0	0	(1)	1	0	-
$\mathtt{L}^{\mathtt{d}}$		0	(2)	(2)	4	0	-
$\mathtt{K}^{\mathtt{d}}$		0	0	0	0	0	-
$\mathtt{J}^{\mathtt{d}}$		0	0	0	0	0	-
				Total:	85	49	
				Total:	148	108	

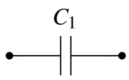
Appendix C

The Ladenheim networks (numerical order)

Network #1

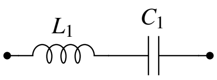


Network #2

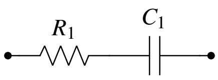


Network #3

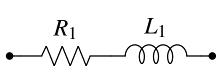
Network #4



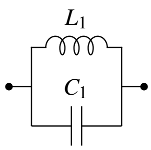
Network #5



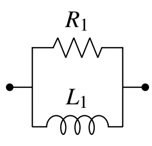
Network #6



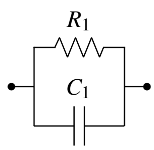
Network #7



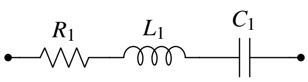
Network #8



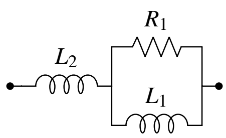
Network #9



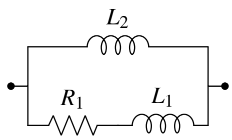
Network #10



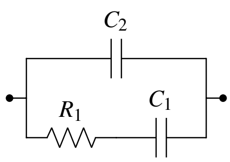
Network #11



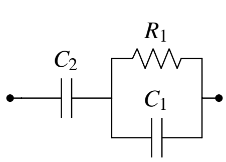
Network #12



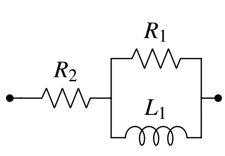
Network #13



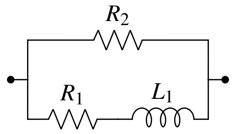
Network #14



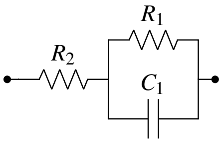
Network #15



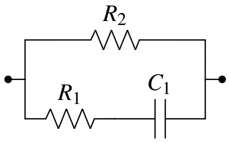
Network #16



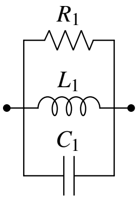
Network #17



Network #18



Network #19



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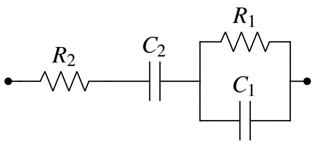

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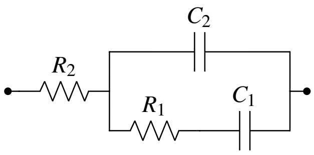

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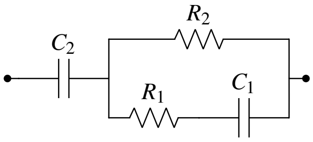

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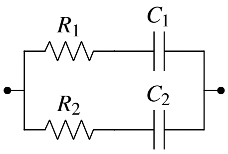

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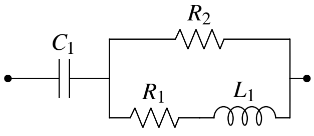

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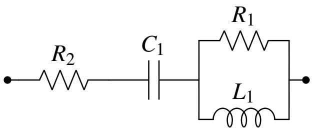

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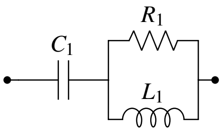

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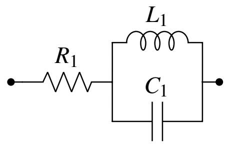

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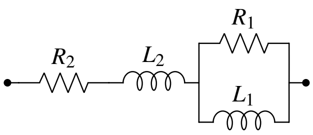

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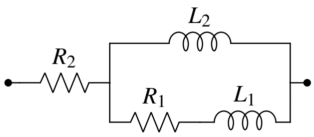

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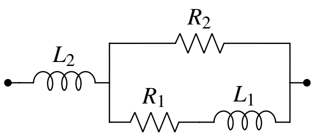

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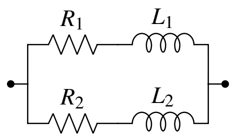

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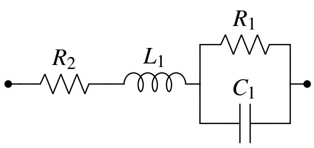

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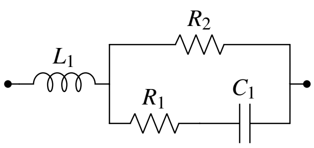

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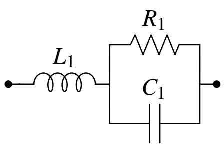

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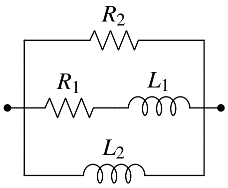

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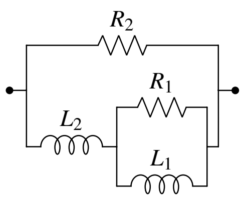

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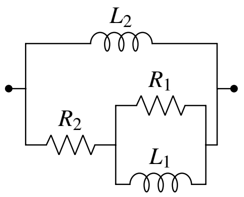

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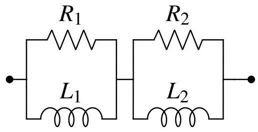

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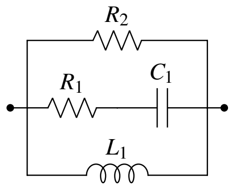

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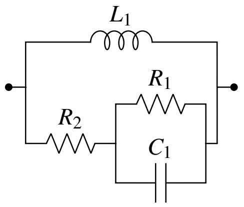

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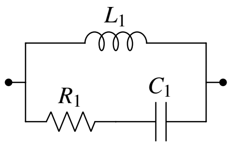

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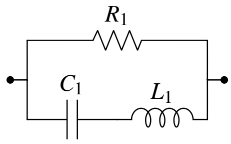

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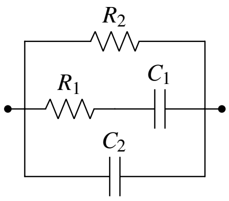

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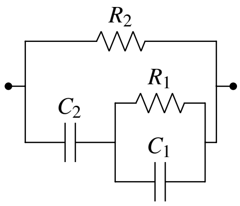

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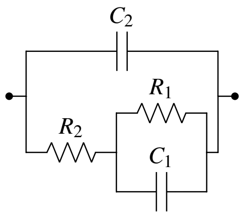

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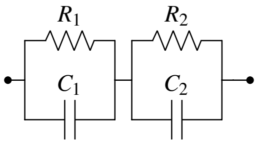

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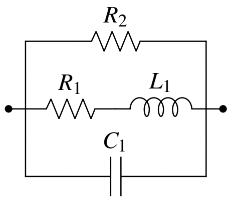

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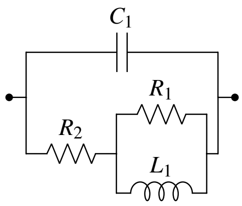

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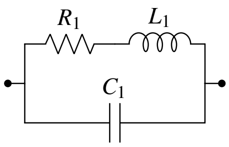

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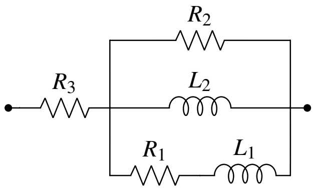

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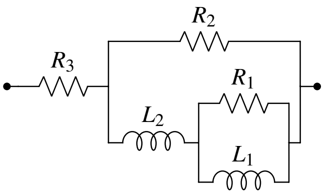

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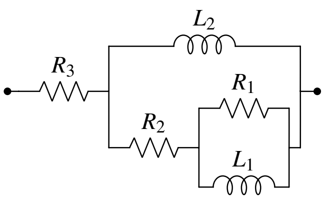

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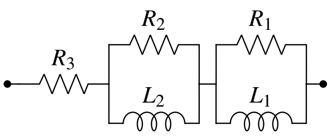

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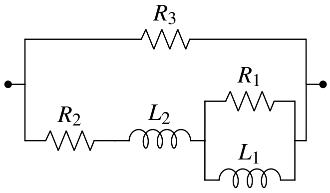

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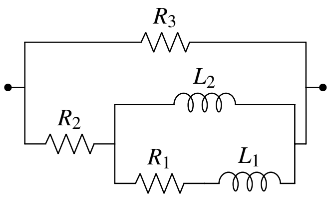

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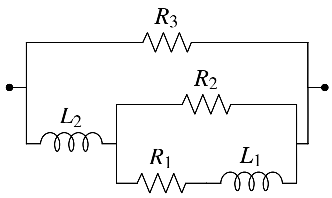

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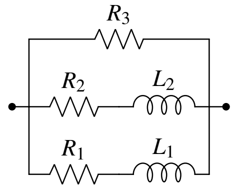

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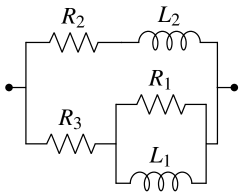

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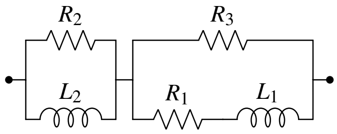

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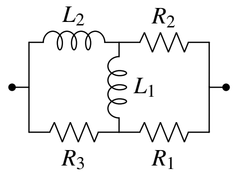

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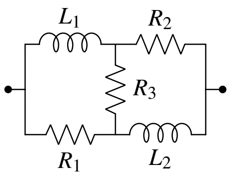

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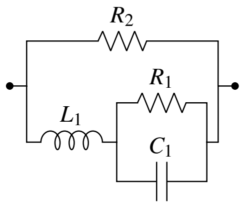

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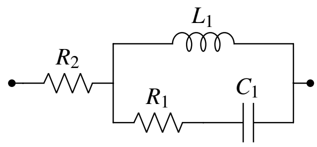

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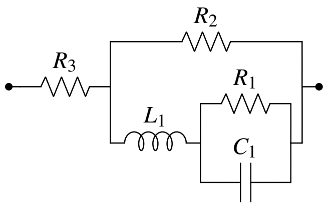

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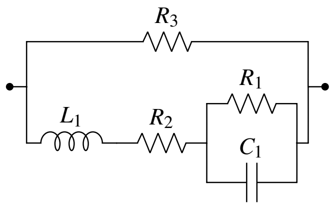

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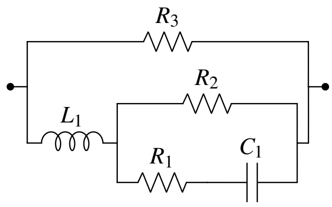

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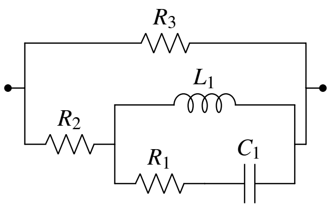

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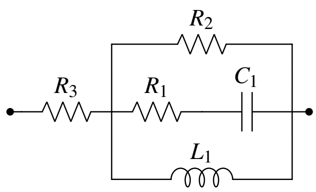

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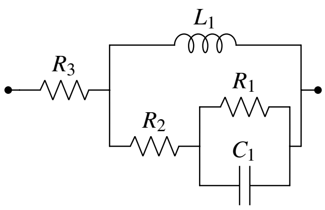

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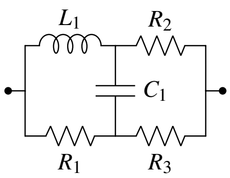

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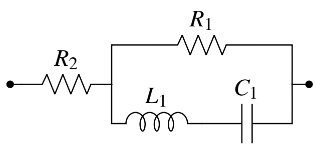

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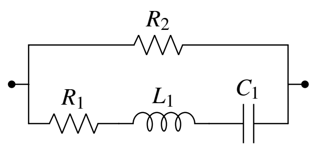

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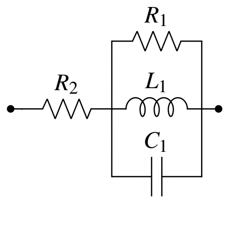

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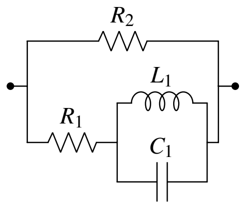

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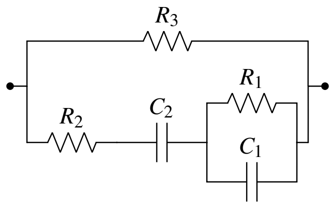

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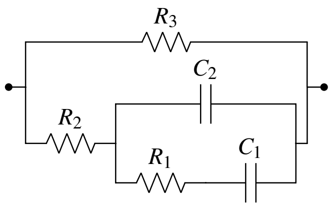

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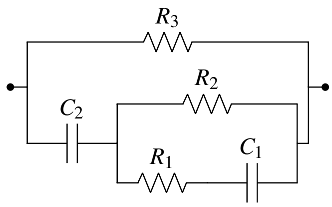

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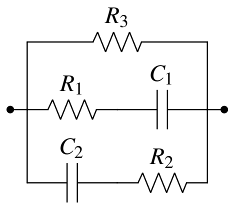

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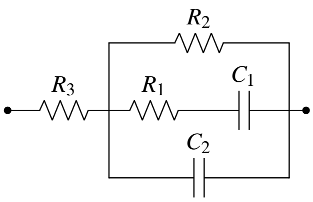

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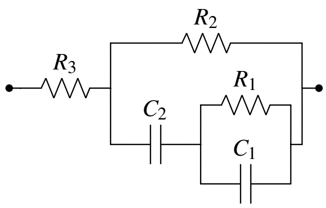

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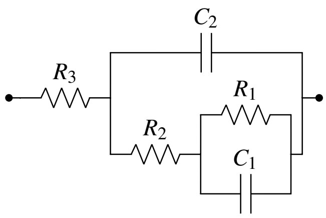

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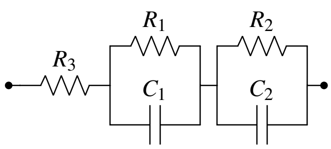

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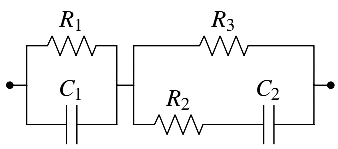

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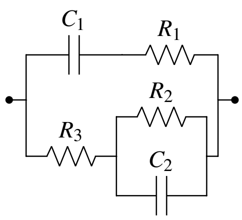

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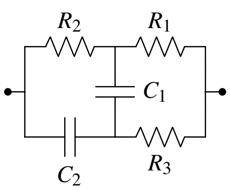

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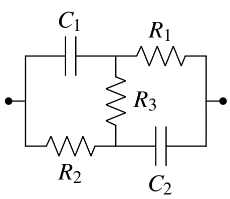

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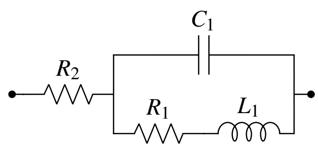

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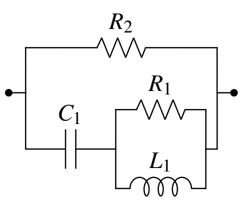

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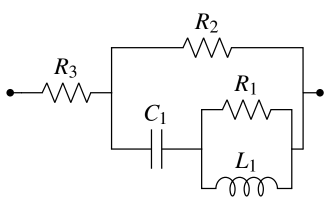

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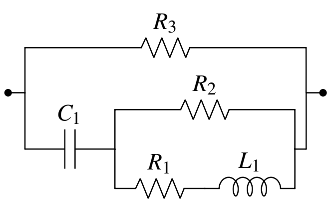

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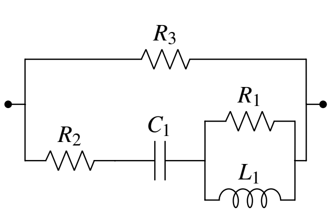

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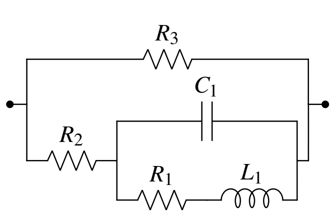
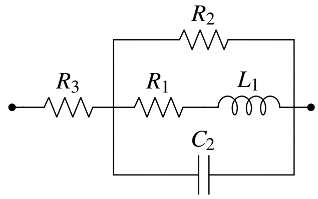
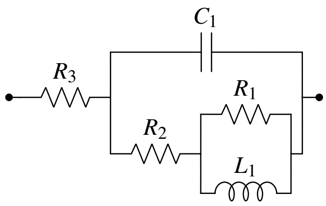


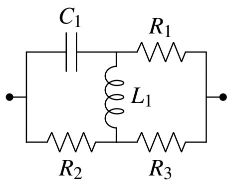
Network #93



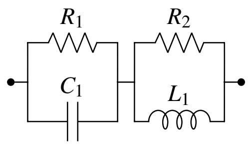
Network #94



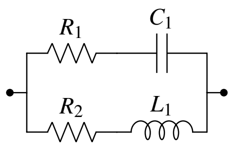
Network #95



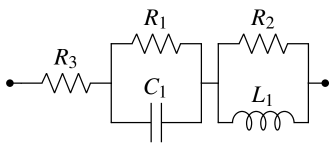
Network #96



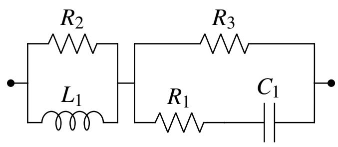
Network #97



Network #98



Network #99



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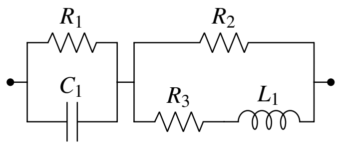

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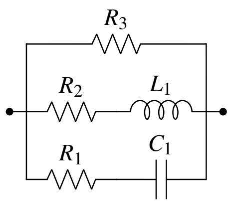

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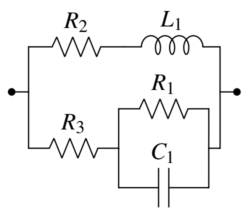
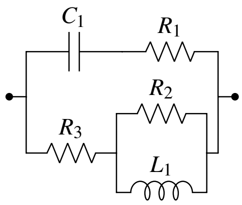


Network #103



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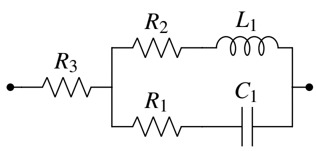

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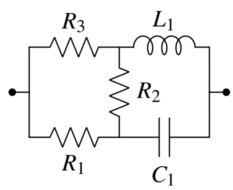

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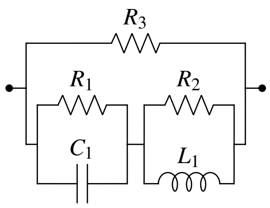
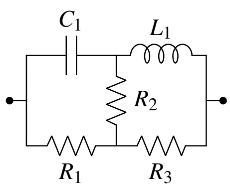
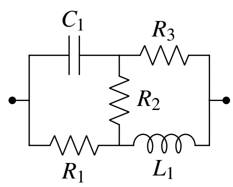


Network #107



Network #108



Appendix D

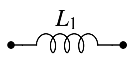
The Ladenheim networks (subfamily order)

Subf. I_A

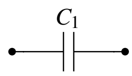
Network #3

Subf. I_B

Network #1

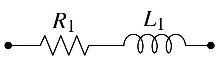


Network #2

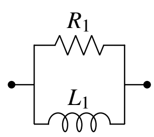


Subf. II_A

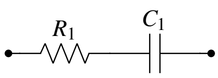
Network #6



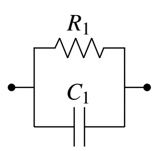
Network #8



Network #5

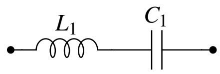


Network #9

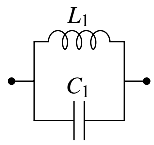


Subf. $\mathrm{II_B}$

Network #4

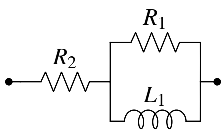


Network #7

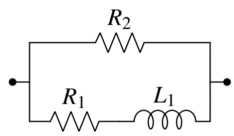


Subf. $\mathrm{III_A}$

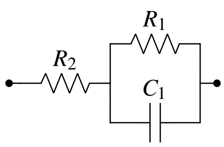
Network #15



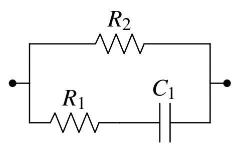
Network #16



Network #17

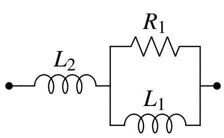


Network #18

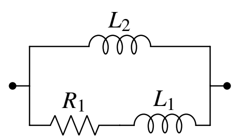


Subf. $\mathrm{III_B}$

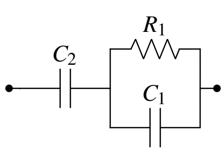
Network #11



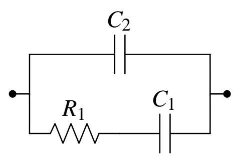
Network #12



Network #14

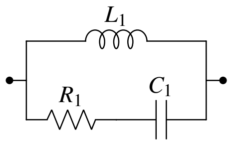


Network #13

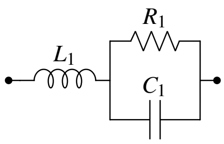


Subf. III_C

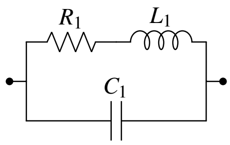
Network #41



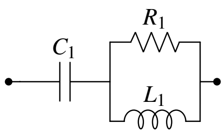
Network #34



Network #49

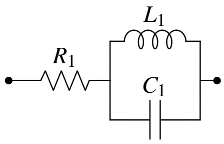


Network #26

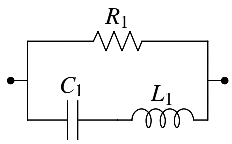


Subf. III_D

Network #27

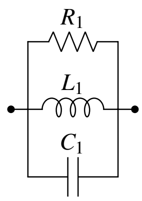


Network #42

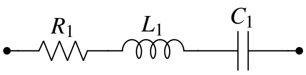


Subf. III_E

Network #19

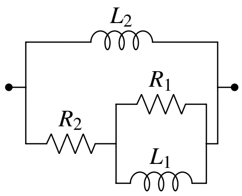


Network #10

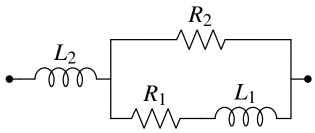


Subf. IV_A

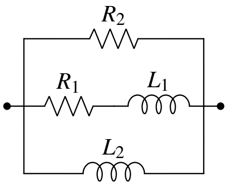
Network #37



Network #30



Network #35



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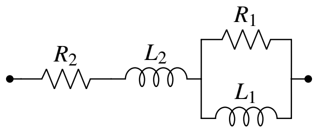

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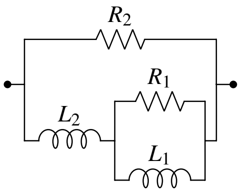
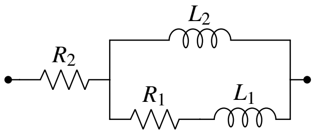


Network #29



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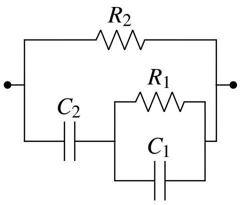

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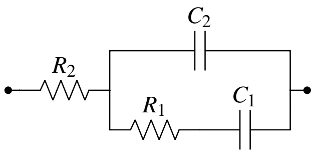

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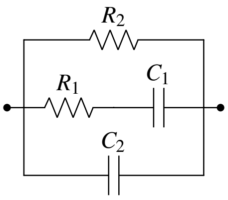
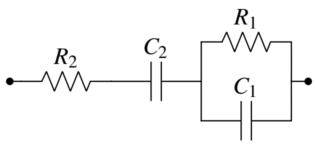
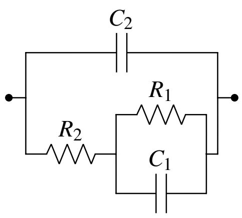


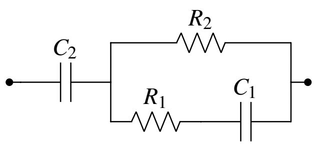
Network #20



Network #45

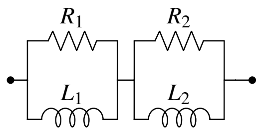


Network #22

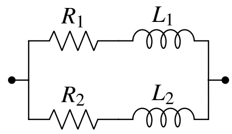


Subf. IV_B

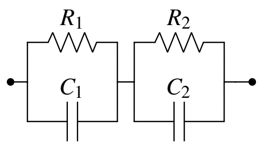
Network #38



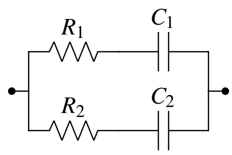
Network #31



Network #46

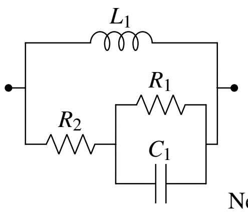


Network #23

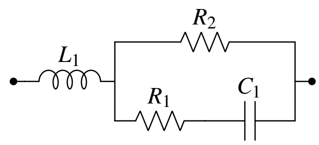


Subf. IV_C

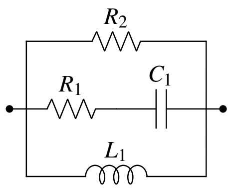
Network #40



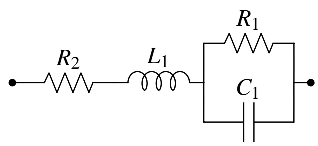
Network #33



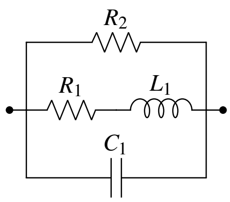
Network #39



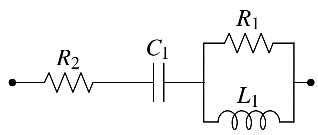
Network #32



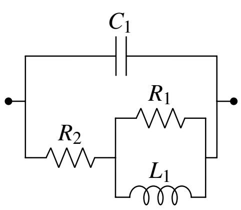
Network #47



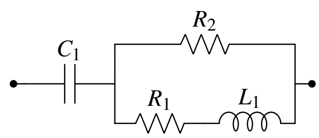
Network #25



Network #48

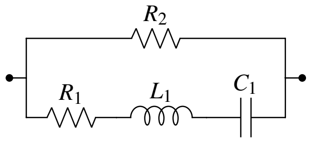


Network #24

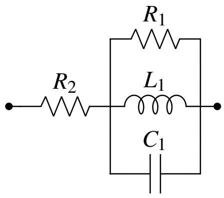


Subf. IV_D

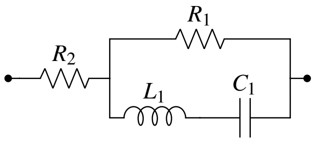
Network #72



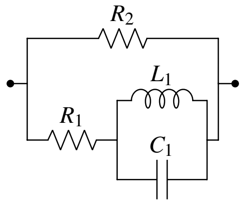
Network #73



Network #71

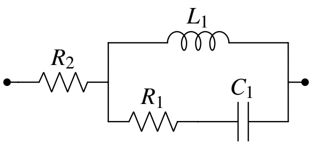


Network #74

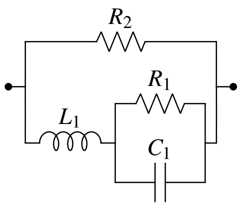


Subf. IV_E

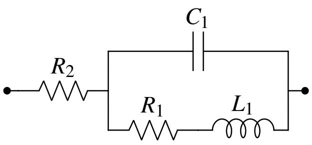
Network #63



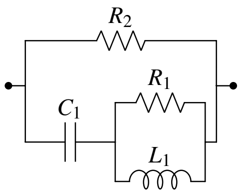
Network #62



Network #87

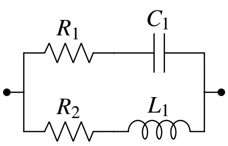


Network #88

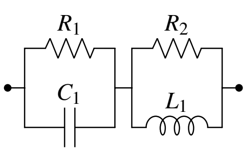


Subf. IV_F

Network #97

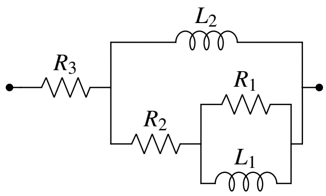


Network #96



Subf. V_A

Network #52



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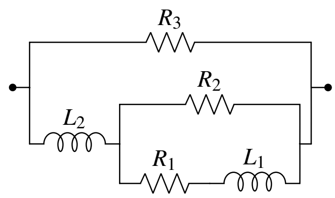

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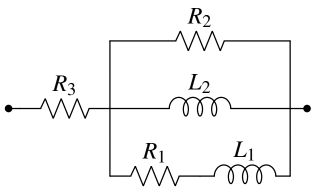

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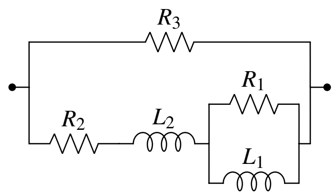

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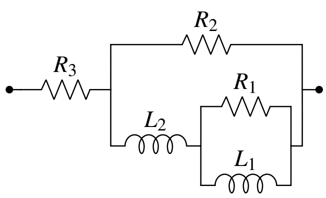

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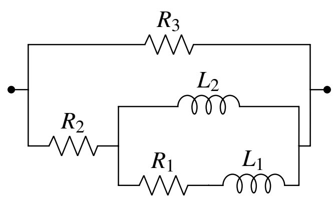

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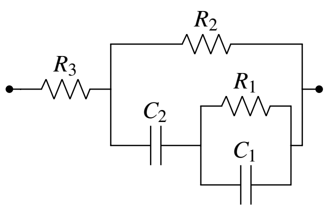

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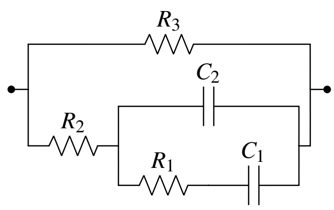

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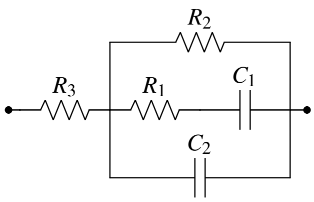

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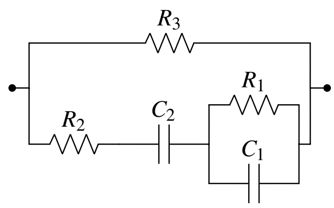

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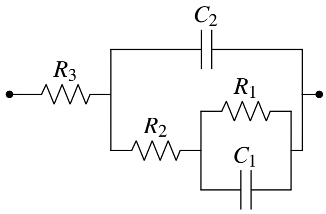

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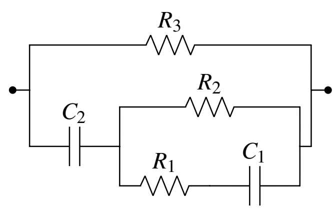
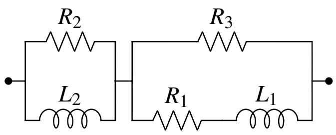


Subf. V_B

Network #59



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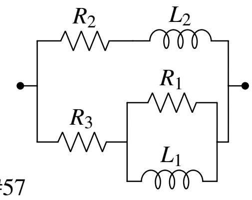

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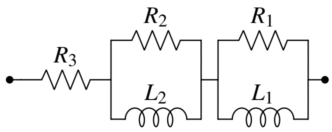

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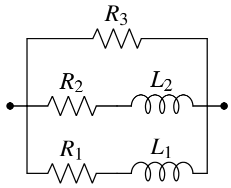

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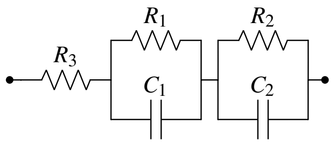

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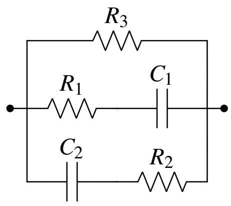

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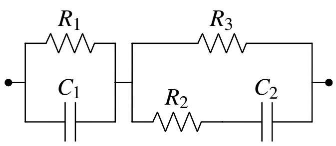

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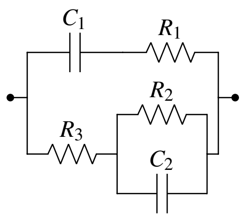
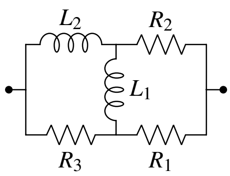
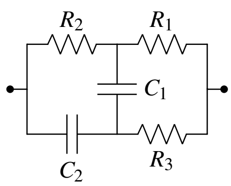


Subf. V_C

Network #60

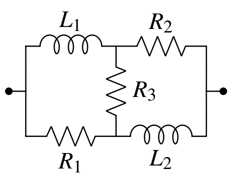


Network #85

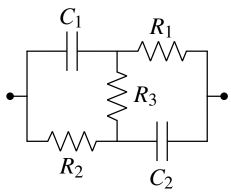


Subf. V_D

Network #61

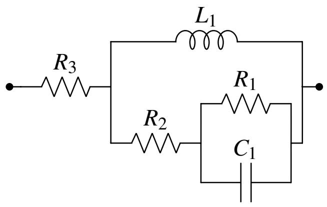


Network #86



Subf. V_E

Network #69



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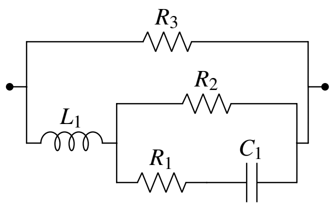

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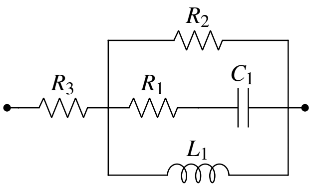

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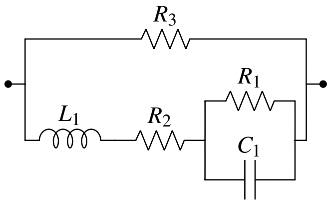

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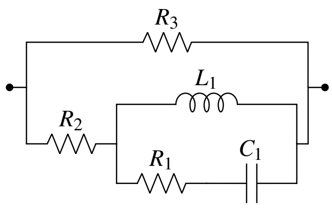

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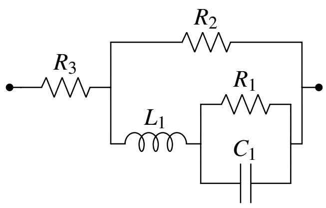

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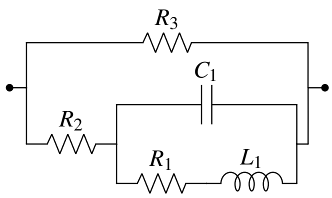

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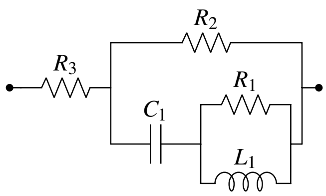

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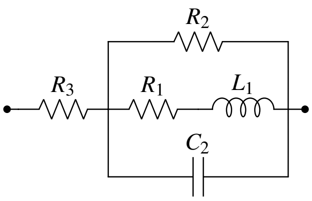

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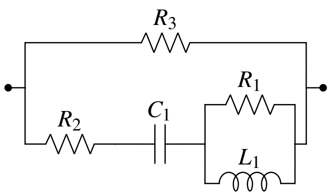

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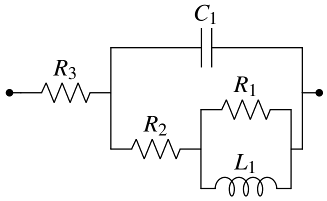

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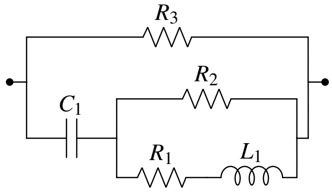
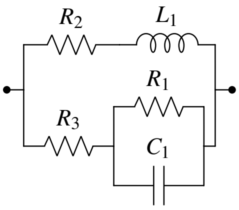
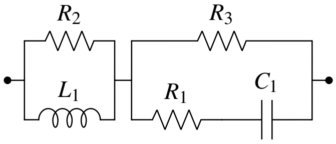


Subf. V_F

Network #102



Network #99



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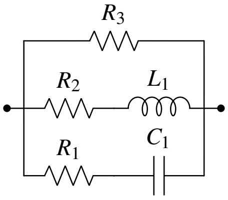

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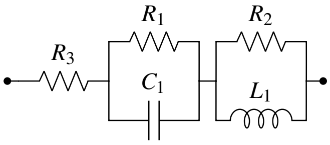
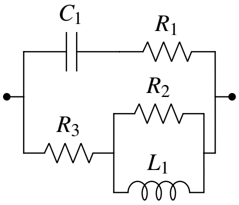


Network #103



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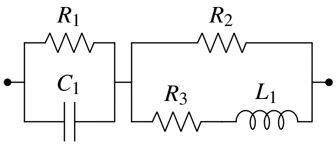
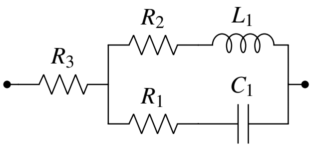
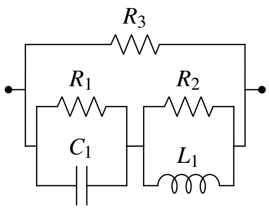


Subf. V_G

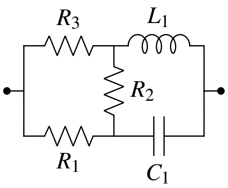
Network #104



Network #106



Network #105



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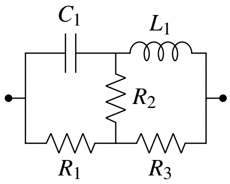
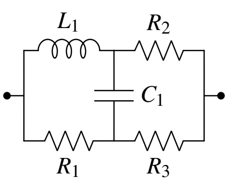


Subf. V_H

Network #70



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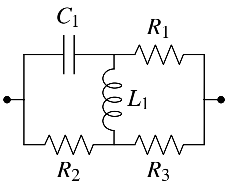
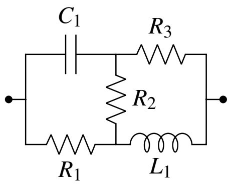


Subf. V_I

Network #108



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