A lifting method for analyzing distributed synchronization on the unit sphere ⋆

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Abstract

This paper introduces a new lifting method for analyzing convergence of continuous-time distributed synchronization/consensus systems on the unit sphere. Points on the d-dimensional unit sphere are lifted to the $(d+1)$-dimensional Euclidean space. The consensus protocol on the unit sphere is the classical one, where agents move toward weighted averages of their neighbors in their respective tangent planes. Only local and relative state information is used. The directed interaction graph topologies are allowed to switch as a function of time. The dynamics of the lifted variables are governed by a nonlinear consensus protocol for which the weights contain ratios of the norms of state variables. We generalize previous convergence results for hemispheres. For a large class of consensus protocols defined for switching uniformly quasi-strongly connected time-varying graphs, we show that the consensus manifold is uniformly asymptotically stable relative to closed balls contained in a hemisphere. Compared to earlier projection based approaches used in this context such as the gnomonic projection, which is defined for hemispheres only, the lifting method applies globally. With that, the hope is that this method can be useful for future investigations on global convergence.

Key words: Multi-agent systems; consensus on the sphere; attitude synchronization; control of networks; control of constrained systems; asymptotic stabilization.

1 Introduction

This paper considers systems of agents continuously evolving on $\mathbb{S}^{d-1}$, where $d \geq 2$. The interactions between the agents are changing as a function of time. For such systems we are analyzing a large class of distributed synchronization/consensus control laws. The analysis tool is a lifting method, where an equivalent consensus protocol is analyzed in the ambient space that embeds the sphere. In comparison to projection methods that have been used in this context—e.g., the gnomonic projection—the proposed method is not locally but globally defined on the unit sphere. The control action is performed in the tangent plane. Only relative information between neighboring agents is used in the control laws. Under the assumption that the time-varying graph is uniformly quasi-strongly connected, we show that the consensus manifold is globally uniformly asymptotically stable relative to any closed ball on the sphere contained in an open hemisphere.

Synchronization on the circle, i.e., $d = 2$, is closely related to synchronization of oscillators (Dörfler & Bullo 2014) and it is equivalent to synchronization on $\mathsf{SO}(2)$, where several applications exist such as flocking in nature and alignment of multi-robot systems. Also for the two-dimensional sphere, i.e., $d = 3$, there are several applications such as formation flying and flocking of birds; consider for example a multi-robot system in 3D, where the relative directions between the robots are available and the goal is to align those. For higher dimensional spheres there are currently related problems such as distributed eigenvector computation, but concrete applications might arise in the future.

The control laws at hand—and slight variations or restrictions on the graph topologies, switchings of the graphs, dimensions of the sphere, and the nonlinear weights in the control laws etc.—have been studied from various perspectives (Scardovi et al. 2007, Sarlette 2009,

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Olfati-Saber 2006, Li & Spong 2014, Li 2015). There has recently been new developments (Pereira & Dimarogonas 2015, 2016, Markdahl & Goncalves 2016, Markdahl et al. 2016). In Markdahl et al. (2017), almost global consensus is shown by characterization of all equilibrium points when the graph is symmetric and constant (time-invariant). It is shown that the equlibria not in the consensus manifold are unstable and the equilibra in the consensus manifold are stable. A similar technique is used in Tron et al. (2012) to show that a consensus protocol on $\mathsf{SO}(3)$ is almost globally asymptotically stable. Now, the above-mentioned results about almost global convergence come at a price. Static undirected graph topologies are assumed as well as more restrictive classes of weights in the control protocols. Furthermore, compared to Markdahl et al. (2017), the right-hand sides of the system dynamics is not necessarily an intrinsic gradient and the linearization matrices at equilibriums are not necessarily symmetric. Hence, we cannot use the result due to Lojasiewicz (1982) about point convergence for gradient flows. This inspired us to take a closer look at methods that transform the consensus problem on the unit sphere (or a subset thereof) to an equivalent consensus problem in $\mathbb{R}^d$. Before we address the method—referred to as a lifting method—we briefly make some connections to the related problem of consensus on $\mathsf{SO}(3)$.

The problem of consensus on $\mathsf{SO}(3)$ has been extensively studied (Sarlette et al. 2009, Ren 2010, Sarlette et al. 2010, Tron et al. 2013, Tron & Vidal 2014, Deng et al. 2016, Thunberg et al. 2016). There is a connection between that problem and the problem of consensus on $\mathbb{S}^3$ when the unit quaternions are used to represent the rotations. For those, the gnomonic projection can be used to show consensus on the unit-quaternion sphere (Thunberg, Song, Hong & Hu 2014, Thunberg, Song, Montijano, Hong & Hu 2014). In another line of research, several methods have been introduced where control laws based on only relative information have been augmented with additional auxiliary (or estimation) variables, which are communicated between neighboring agents. By doing so, results about almost global convergence to the consensus manifold are achieved (Sarlette & Sepulchre 2009, Thunberg, Markdahl & Goncalves 2017). The latter of these two publications provides a control protocol for Stiefel manifolds, with the unit sphere and $\mathsf{SO}(d)$ as extreme cases. A similar technique had previously been used for the sphere (Scardovi et al. 2007). The idea of introducing auxiliary variables also extends to the related distributed optimization problem in Thunberg, Bernard & Goncalves (2017). In contrast to the mentioned works, in this paper we are not assuming additional communication between the agents by means of auxiliary variables. Instead only relative information is used in the protocols. In a practical setting (considering the case $d = 3$), such information can be measured by for example a vision sensor and requires no explicit communication between the agents.

In the proposed lifting method, we lift the states from the $(d-1)$-dimensional sphere into $\mathbb{R}^d$. The non-negative weights in the consensus protocol for the states in the lifting space are nonlinear functions. Each agent moves in a direction that is a weighted combination of the directions to the neighbors. The weights contain rational functions of the norms of the states of the agents. Since these rational functions are not well-defined at the origin, fundamental questions arise about existence, uniqueness, and invariance of sets. Those questions are answered with positive answers. The hope is that this lifting method will serve as a stepping-stone to future analysis on (almost) global convergence to the consensus manifold on the unit sphere. Compared to the approach in Markdahl et al. (2017) where all the "bad" equlibria on $\mathbb{S}^{d-1}$ were characterized, we only need to characterize one point, which is the origin in the "lifted space". If we were to show that this point has a region of attraction that is of measure zero, we would have equivalently shown the desired result about almost global convergence on the unit sphere (assuming $d \geq 3$). However, the non-differentiability of this point remains an additional challenge.

An extended version of this paper is available at arXiv, see Thunberg et al. (2018).

2 Preliminaries

We begin this section with some *set-definitions*. The $(d-1)$-dimensional unit sphere is

$$\mathbb{S}^{d-1} = \{y \in \mathbb{R}^d : \|y\|_2 = 1\}.$$

The special orthogonal group in dimension d is

$$\mathsf{SO}(d) = \{Q \in \mathbb{R}^{d\times d} : Q^T = Q^{-1}, \det(Q) = 1\}.$$

The set of skew symmetric matrices in dimension d is

$$\mathsf{so}(d) = \{\Omega \in \mathbb{R}^{d\times d} : \Omega^T = -\Omega\}.$$

The set $\mathcal{H} \subset \mathbb{S}^{d-1}$ is an open *hemisphere* if there is $v \in \mathbb{S}^{d-1}$ such that $\mathcal{H} = \{w \in \mathbb{S}^{d-1} : w^T v > 0\}$.

We consider a multi-agent system with n agents. Each agent has a corresponding state $x_i(t) \in \mathbb{S}^{d-1}$ for $t \in [0, \infty)$. The initial state of each agent i at time 0 is $x_{i0} \in \mathbb{S}^{d-1}$. Another way to represent the states of the agents is to use rotation matrices. Let $R_i(t) \in \mathsf{SO}(d)$ satisfy $R_i(t)p = x_i(t)$ for all i and $t \in [0, \infty)$, where $p = [1, 0, \ldots, 0]^T$ is the *north pole*; we also define $-p$ as the *south pole*. Let $R_{i0}p = x_{i0}$ for all i, where R_{i0} is the initial R_i-matrix at time 0. The R_i-matrices can be interpreted as transformations from body coordinate frames—denoted by $\mathcal{F}_i$'s—of the agents to a world coordinate frame $\mathcal{F}_W$. They are transforming the unit vector

p in the body frames to the corresponding unit vector (or point on the unit sphere) in the world coordinate frame. The R_i's and their dynamics are not uniquely defined, but this is not of importance for the analysis. We choose to define the dynamics of the R_i's according to (2) below.

The dynamics of the x_i-vectors are given by

$$\dot{x}_i = (I - x_i x_i^T) R_i [0, v_i^T]^T = R_i [0, v_i^T]^T, \qquad (1)$$

where $v_i(t) \in \mathbb{R}^{d-1}$ for all t. The v_i-vectors are the controllers for the agents and those are defined in the body coordinate frames, i.e., the $\mathcal{F}_i$'s. For the R_i-matrices the dynamics is

$$\dot{R}_i = R_i \begin{bmatrix} 0 & -v_i^T \\ v_i & 0 \end{bmatrix}. \qquad (2)$$

The matrix on the right-hand side of R_i in (2) is an element of $\mathsf{so}(d)$. The control is performed in the tangent space of the sphere, which means that there are $d-1$ degrees of freedom for the control. This is the reason why the v_i-vectors are $(d-1)$-dimensional. Before we proceed, we provide some additional explanation for the expression in the right-hand side of (2). According to its definition, the first column of R_i is equal to x_i and by multiplying $\dot{x}_i$ by R_i^T from the right we obtain—due to (1)—the following expression

$$R_i^T \dot{x}_i = [0, v_i^T]^T.$$

This means that

$$R_i^T \dot{R}_i = \begin{bmatrix} 0 & \star \\ v_i & \star \end{bmatrix},$$

where we the $\star$-parts are left to be chosen. We know that the matrix in the right-hand side above needs to be skew symmetric, since R_i is a rotation matrix. We also know that the first column of it must be equal to $[0, v_i^T]^T$. The matrix of minimum Euclidean norm that fulfills these two requirements is equal to

$$\begin{bmatrix} 0 & -v_i^T \\ v_i & 0 \end{bmatrix},$$

i.e., the one we chose in the right-hand side of (2).

We will study a class of distributed synchronization/consensus control laws on the unit sphere, where the agents are moving in directions comprising conical combinations of directions to neighbors. In this protocol only local and relative information is used. Before we provide these control laws we need to introduce directed graphs and time-varying directed graphs.

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the node-set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge-set. Each node in the node set corresponds to a unique agent. The set $\mathcal{N}_i \in \mathcal{V}$ is the neighbor set or neighborhood of agent i, where $j \in \mathcal{N}_i$ if and only if $(i,j) \in \mathcal{E}$. We continue with the following definitions addressing connectivity of directed graphs.

In a directed graph $\mathcal{G}$, a *directed path* is a sequence of distinct nodes, such that any consecutive pair of nodes in the sequence comprises an edge in the graph. We say that i is *connected to* j if there is a directed path from i to j. We say that the graph is *quasi-strongly connected* if there is at least one node that is a center or a root node in the sense that all the other nodes are connected to it. We say that the graph is *strongly connected* if for all $(i,j) \in \mathcal{V} \times \mathcal{V}$ it holds that i is connected to j.

Now we define *time-varying graphs*. We define those by first defining *time-varying neighborhoods*. The time-varying neighborhood $\mathcal{N}_i(t)$ of agent i is a piece-wise constant right-continuous set-valued function that maps from $\mathbb{R}$ to $2^{\mathcal{V}}$. We assume that there is $\tau_D > 0$ such that $\inf_k(\tau_{i(k+1)} - \tau_{ik}) > \tau_D$ for all i, where $\{\tau_{ik}\}_{k=-\infty}^{\infty}$ is the set of time points of discontinuity of $\mathcal{N}_i(t)$. The constant τ_D is as a lower bound on the dwell-time between any two consecutive switches of the topology. We define the *time-varying graph* $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ as

$$\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) = (\mathcal{V}, \bigcup_i \bigcup_{j \in \mathcal{N}_i(t)} \{(i,j)\}).$$

Furthermore, the *union graph* of $\mathcal{G}(t)$ during the time interval $[t_1, t_2)$ is defined by

$$\mathcal{G}([t_1, t_2)) = \bigcup_{t \in [t_1, t_2)} \mathcal{G}(t) = (\mathcal{V}, \bigcup_{t \in [t_1, t_2)} \mathcal{E}(t)),$$

where $t_1 < t_2 \leq \infty$. We say that the graph $\mathcal{G}(t)$ is *uniformly (quasi-) strongly connected* if there exists a constant $T > 0$ such that the union graph $\mathcal{G}([t, t+T))$ is (quasi-) strongly connected for all t.

Now we provide the synchronization protocol that is to be studied. For each agent i, the controller v_i is defined by

$$v_i = \begin{bmatrix} 0 & 0 \\ 0 & I_{d-1} \end{bmatrix} \sum_{j \in \mathcal{N}_i(t)} f_{ij}(\|x_{ij} - p\|) x_{ij}, \qquad (3)$$

where $x_{ij} = R_i^T x_j = R_i^T R_j p$, which is x_j represented in the frame $\mathcal{F}_i$. The x_{ij}'s are what we refer to as relative information and the control law (3) is constructed by only such information. For each (i,j), it holds that $f_{ij} : \mathbb{R} \rightarrow$

$\mathbb{R}$. The f_{ij}-functions are assumed to be Lipschitz, and attain positive values for positive arguments. The $\mathcal{N}_i(t)$'s are neighborhoods of a time-varying directed graph $\mathcal{G}(t)$, whose connectivity is at least uniformly quasi-strong. These control laws will be analyzed in the paper.

The expressions in (3) are more easily understood if they are expressed in the world frame $\mathcal{F}_W$. We define

$$u_i = (I - x_i x_i^T) \sum_{j \in \mathcal{N}_i(t)} f_{ij}(\|x_j - x_i\|)(x_j - x_i), \quad (4)$$

for all i, which is v_i expressed in the frame $\mathcal{F}_W$. The vector u_i is the sum of the positively weighted directions to the neighbors of agent i, projected onto the tangent space at the point x_i. Also for analysis purposes, (4) is easier to work with than (3). The closed loop system is

$$\begin{aligned} \dot{x}_i &= (I - x_i x_i^T) \sum_{j \in \mathcal{N}_i(t)} f_{ij}(\|x_j - x_i\|)(x_j - x_i), \quad (5) \\ &= (I - x_i x_i^T) \sum_{j \in \mathcal{N}_i(t)} f_{ij}(\|x_j - x_i\|)x_j, \end{aligned}$$

for all i.

Let $x = [x_1^T, x_2^T, \ldots, x_n^T]^T$ and $x_0 = [x_{10}^T, x_{20}^T, \ldots, x_{n0}^T]^T$. We define the set

$$\mathcal{A} = \{x : x_i = x_j \text{ for all } i, j\},$$

which is the synchronization/consensus set. Throughout the paper we assume that the closed-loop dynamics of the system is given by (5). We study the convergence of $x(t)$ to the consensus set $\mathcal{A}$. When we talk about convergence we refer to the concepts below.

For the system (5), we say that the set $\mathcal{A}$ is *attractive* relative to a forward invariant set $\mathcal{S} \subset (\mathbb{S}^{d-1})^n$ if

$$(x_0 \in \mathcal{S}) \Longrightarrow (\text{dist}(x(t), \mathcal{A}) \to 0 \text{ as } t \to \infty)$$

where $\text{dist}(v, \mathcal{A}) = \inf_{w \in \mathcal{A}} \|w - v\|$. Furthermore, we say that the set $\mathcal{A}$ is *globally uniformly asymptotically stable* relative to a forward invariant compact set $\mathcal{S} \subset (\mathbb{S}^{d-1})^n$ if

(1) for every $\epsilon > 0$ there is $T(\epsilon) > 0$ such that $(x_0 \in \mathcal{S}) \Longrightarrow (\text{dist}(x(T(\epsilon)), \mathcal{A}) \leq \epsilon)$;

(2) for every $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that $(x_0 \in \mathcal{S}$ and $\text{dist}(x_0, \mathcal{A}) \leq \delta) \Longrightarrow (\text{dist}(x(t), \mathcal{A}) \leq \epsilon$ for all $t \geq 0)$.

The equivalent definitions to the above will also be used (after changing the sets $\mathcal{S}$ and $\mathbb{S}^{d-1}$) for other systems evolving in $(\mathbb{R}^d)^n$ or linear subspaces thereof. *Forward invariance*, or simply *invariance*, of a set means that if the initial state is contained in the set, then the state is contained in the set for all future times.

The two concepts of global convergence respective almost global convergence relative to a forward invariant set $\mathcal{S}$ refer to, respectively, the situations where convergence occur for all initial points in $\mathcal{S}$ and convergence occur for all initial points in a set $\mathcal{B}$ where $\mathcal{S} - \mathcal{B}$ has measure zero.

3 The lifting method

In this section we propose a method where the x_i's are relaxed to be elements in $\mathbb{R}^d$. Those elements, we call them z_i's, are then projected down onto the sphere $\mathbb{S}^{d-1}$ to create the y_i's (which are shown to be equivalent to the x_i's). Provided $z_i \neq 0$, the projection is given by

$$y_i = \frac{z_i}{\|z_i\|}. \quad (6)$$

This projection as well as the lifting is illustrated in Figure 1. Points in $\mathbb{R}^d$ are projected down onto the sphere in the sense of minimizing the least squares distance.

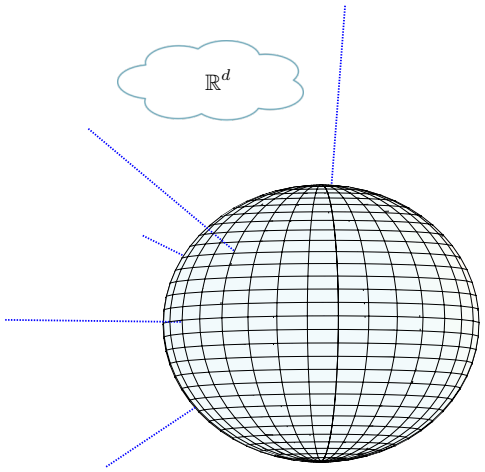

Fig. 1. Illustration of the lifting method.

We let $z_i(t) \in \mathbb{R}^d$ be governed by the following dynamical system

$$\dot{z}_i = \sum_{j \in \mathcal{N}_i(t)} f_{ij}\left(\left\| \frac{z_j}{\|z_j\|} - \frac{z_i}{\|z_i\|} \right\|\right) \frac{\|z_i\|}{\|z_j\|}(z_j - z_i), \quad (7)$$

for all i. Let the initial state of the system be $z_0 = [z_{10}^T, z_{20}^T, \ldots, z_{n0}^T]^T$. Equation (7) describes a consensus protocol with nonlinear weights that contain rational functions of the norms of the states. The question is how this dynamical system is related to (5). The following proposition provides the answer.

Proposition 1 *Suppose none of the z_{i0}'s are equal to zero. On the time interval $[0,\infty)$ the dynamics for the y_i's is given by*

$$\dot{y}_i = (I - y_i y_i^T) \sum_{j\in\mathcal{N}_i(t)} f_{ij}(\|y_j - y_i\|)(y_j - y_i), \qquad (8)$$

i.e., it is the same as (5).

Proof: Proposition 2 below provides the result that the solution to (7) is well-defined on the interval $[0,\infty)$ and $(\mathbb{R}^d\backslash\{0\})^n$ is forward invariant. Given that result, the y_i's and their derivatives are well defined. Now,

$$\begin{aligned}\dot{y}_i &= \frac{1}{\|z_i\|}(I - y_i y_i^T)\dot{z}_i \\ &= (I - y_i y_i^T) \sum_{j\in\mathcal{N}_i(t)} f_{ij}(\|y_j - y_i\|)\left(y_j - y_i\frac{\|z_i\|}{\|z_j\|}\right) \\ &= (I - y_i y_i^T) \sum_{j\in\mathcal{N}_i(t)} f_{ij}(\|y_j - y_i\|)(y_j - y_i). \qquad \blacksquare\end{aligned}$$

Proposition 2 *Suppose the dynamics for the z_i's is governed by* (7)*. Suppose there is no i such that $z_{i0} = 0$. Let $H(t, z_0)$ be the convex hull of the $z_i(t)$'s during $[0, t_f)$ when the initial condition is z_0. Let t_f be such that the solution exists during $[0, t_f)$. Then the solution to* (7) *exists and is unique for all times $t > 0$, $(H(t, z_0))^n$ is forward invariant, and the set $(\mathbb{R}^d\backslash\{0\})^n$ is forward invariant.*

Proof: We first address the claim that $(H(t, z_0))^n$ is forward invariant. It suffices to verify that for each i, the right-hand side of (7) is either inward-pointing relative to $H(t, z_0)$, or equal to 0. Now, due to the structure of (7), this is true.

Now we address the invariance of $(\mathbb{R}^d\backslash\{0\})^n$, and, by doing that, obtain the existence and uniqueness result for the solution during $[0,\infty)$ for free, since the right-hand side of (7) is locally Lipschitz on $(\mathbb{R}^d\backslash\{0\})^n$. Now, suppose there is $i_1 \in \mathcal{V}$ and a finite time $t_1 > 0$ such that $\lim_{t\uparrow t_1} z_{i_1}(t) = 0$ and there is no j and $t_0 \in [0, t_1)$ such that $\lim_{t\uparrow t_0} z_j(t) = 0$. This means that there is a first finite time t_1 for which at least one state, z_{i_1} that is, attains the value 0. The assumption is equivalent to assuming that $(\mathbb{R}^d\backslash\{0\})^n$ is not forward invariant.

For $i \in \mathcal{V}$ and $t \in [0, t_1)$ it holds that

$$\frac{d}{dt}\|z_i\| = \sum_{j\in\mathcal{N}_i(t)} g(z_i, z_j)(\|z_i\|\theta_{ij} - \frac{\|z_i\|^2}{\|z_j\|}), \text{ for all } i,$$

where $g(z_i, z_j) = f(\|\frac{z_j}{\|z_j\|} - \frac{z_i}{\|z_i\|}\|)$ and $\theta_{ij} = \frac{z_i^T z_j}{\|z_i\|\|z_j\|}$.

For $z_i, z_j \neq 0$, it also holds that

$$\begin{aligned}\frac{d}{dt}\frac{\|z_i\|}{\|z_j\|} = &\sum_{k\in\mathcal{N}_i(t)} g_{ik}(z_i, z_k)\left(\frac{\|z_i\|}{\|z_j\|}\theta_{ik} - \frac{\|z_i\|^2}{\|z_j\|\|z_k\|}\right) \\ &- \sum_{l\in\mathcal{N}_j(t)} g_{jl}(z_j, z_l)\left(\frac{\|z_i\|}{\|z_j\|}\theta_{jl} - \frac{\|z_i\|}{\|z_l\|}\right).\end{aligned}$$

We define $v_{kl} = \frac{\|z_k\|}{\|z_l\|}$ for all k, l and write the equation above as

$$\begin{aligned}\dot{v}_{ij} = &\sum_{k\in\mathcal{N}_i(t)} g_{ik}(z_i, z_k)(v_{ij}\theta_{ik} - v_{ij}v_{ik}) \\ &- \sum_{l\in\mathcal{N}_j(t)} g_{jl}(z_j, z_l)(v_{ij}\theta_{jl} - v_{il}).\end{aligned}$$

Let $\alpha > 0$ be an upper bound for the f_{ij}'s, which is equivalent to an upper bound for the g_{ij}'s. Such a bound must exist (since the set $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ is compact and the function $(f_{ij} \circ \text{dist})$ is continuous on $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$, where $\text{dist}(\cdot,\cdot)$ is the function that returns the Euclidean distance between two points in $\mathbb{R}^d$).

Let $V(t) = \max_{(i,j)\in\mathcal{V}\times\mathcal{V}} v_{ij}(t)$. On $[0, t_1)$ it holds that

$$D^+V \leq 3\alpha n V, \qquad (9)$$

where D^+ is the upper Dini-derivative. By using the Comparison Lemma for (9), we can conclude that V is bounded from above by $e^{3\alpha n t_1}V(0)$ on $[0, t_1)$.

Now, for i_1 and $t \in [0, t_1)$ it holds that

$$\begin{aligned}\frac{d}{dt}\|z_{i_1}\| &= \sum_{j\in\mathcal{N}_{i_1}(t)} g(z_{i_1}, z_j)(\|z_{i_1}\|\theta_{ij} - \frac{\|z_{i_1}\|^2}{\|z_j\|}) \\ &\geq -n\alpha(e^{3\alpha n t_1}V(0) + 1)\|z_{i_1}\|.\end{aligned}$$

By using the Comparison Lemma, we can conclude that $\|z_{i_1}(t)\| \geq \|z_{i_1 0}\|e^{-n\alpha(e^{3\alpha n t_1}V(0)+1)}$. But this, in turn, means that $\lim_{t\uparrow t_1}\|z_{i_1}(t)\| > 0$, which is a contradiction. ■

In the following proposition we make use of $H(t, z_0)$, which was defined in Proposition 2.

Proposition 3 *Suppose the dynamics for the z_i's are governed by* (7) *and $\mathcal{G}(t)$ is uniformly quasi-strongly connected. Suppose $0 \in \mathbb{R}^d$ is not contained in the convex hull $H(t, w)$ for w in $\mathbb{R}^{nd}$. Then the consensus set $\mathcal{A}_z$—defined as the set where all the z_i's are equal in $(H(0, w))^n$—is globally uniformly asymptotically stable relative to $(H(0, w))^n$. Furthermore, there is a point $\bar{z} \in \mathbb{R}^d$ that all the z_i's converge to.*

Proof: Invariance of $(H(t,z_0))^n$ is an indirect consequence of the fact that (7) is a consensus protocol. On this set the right-hand side of (7) is Lipschitz continuous in z and piece-wise continuous in t. Since the right-hand side of (7) is Lipschitz continuous in z and piece-wise continuous in t, we can use Theorem 2 in Thunberg, Hu & Goncalves (2017) to find a continuously differentiable function $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ such that 1) $\max_{(i,j)\in\mathcal{V}\times\mathcal{V}} W(z_i(t), z_j(t))$ is decreasing as a function of t; and 2) $\dot{W}(z_i(t), z_j(t))$ is strictly negative if $(i,j) \in \arg\max_{(i,j)\in\mathcal{V}\times\mathcal{V}} W(z_i(t), z_j(t))$ and there is $k \in \mathcal{N}_i(t)$ such that $y_i \neq y_k$ or there is $l \in \mathcal{N}_j(t)$ such that $z_j \neq z_l$. The existence of such a function guarantees that the consensus set $\mathcal{A}_z$ is globally uniformly asymptotically stable relative to $(H(0,z_0))^n$. It holds that the function $\|z_i - z_j\|^2$ is such a W-function. Convergence to a point for all the z_i's can be shown by using the facts that $(H(t,z_0))^n$ is forward invariant for all t and z converges to $\mathcal{A}_z$. ■

As a remark to the previous proposition, we should say that more restrictive results about attractivity of $\mathcal{A}$ can be shown by using the results in Shi & Hong (2009), Lin et al. (2007).

Proposition 4 *Suppose the graph $\mathcal{G}(t)$ is uniformly quasi-strongly connected. Then for any closed ball B contained in the hemisphere the consensus set $\mathcal{A}$ is globally uniformly asymptotically stable relative to B^n under* (5).

Proof: Forward invariance of B^n holds due to the structure of (5). Due to Proposition 3 we know that the consensus set $\mathcal{A}_z$ is globally uniformly asymptotically stable relative to $(H(0,z_0))^n$ and that there is a point $\bar{z} \in \mathbb{R}^d$ that all the z_i's converge to. We also know that the projected y_i-variables follow the protocol (8), which is the same as (5). The norms of the z_i's are uniformly bounded on $(H(0,z_0))^n$ and $(H(0,z(T)))^n$ is forward invariant for all $T > 0$. Thus the desired result readily follows. ■

Corollary 5 *Suppose the dynamics for the z_i's is governed by* (7) *and suppose the graph $\mathcal{G}(t)$ is quasi-strongly connected. If the z_i's converge to a point $\bar{z} \in \mathbb{R}^d$ that is not equal to zero, then the y_i's converge to a point $\bar{y} \in \mathbb{S}^{d-1}$. Furthermore, if the convex hull of the z_{i0}'s does not contain the point zero, then the y_i's converge to a point $\bar{y} \in \mathbb{S}^{d-1}$.*

Proof: Straightforward application of Proposition 1, Proposition 2, and Proposition 4. ■

Variations of Proposition 4 have appeared in the literature before. The idea of using the gnomonic projection to show consensus on the hemisphere was used in Thunberg, Song, Hong & Hu (2014), Thunberg et al. (2016) where restricted versions of Proposition 4 were given for the dimension $d = 4$ in the context of attitude synchronization. Recently the attractivity of $\mathcal{A}$ relative to open hemispheres was established under quasi-strong graph connectivity (Lageman & Sun 2016) using the gnomonic projection. The graph was not time-varying in that context.

To get a better understanding of Proposition 4, a numerical example is provided by Fig 2. In this example there are five agents with a uniformly quasi-strongly connected interaction graph. The agents were initially uniformly distributed on a hemisphere and the f_{ij}-functions were chosen as constants; either equal to 1 or 2. In the figure, the red discs denote the initial positions and the yellow disc denote the final consensus point. We have also denoted two points on the trajectories where the graph switches.

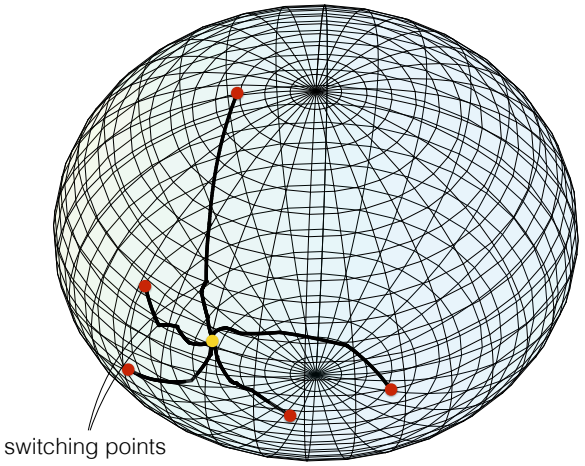

Fig. 2. Convergence on a hemisphere.

Now, the case when $0 \in \mathbb{R}^d$ is contained in the convex hull of the z_i's is more intriguing. We provide the following result.

Proposition 6 *Suppose the dynamics for the z_i's is governed by* (7) *and $\mathcal{G}(t)$ is uniformly strongly connected. Suppose $0 \in \mathbb{R}^d$ is contained in the convex hull of the z_{i0}'s, i.e., in $H(0,z_0)$, and there is no i such that $z_{i0} = 0$. Furthermore, suppose that the z_{i0}'s are contained in a compact set where the f_{ij}'s are bounded from below by a positive constant K_d. Then the set $\mathcal{A}_z$—defined as the set where all the z_i's are equal in the convex hull of the z_{i0}'s—is attractive. Furthermore, there is a fixed point $\bar{z} \in \mathbb{R}^d$ that all the z_i's converge to.*

Proof: The proof has been relegated to Thunberg et al. (2018). ■

The point $0 \in \mathbb{R}^d$, plays a crucial role for this lifting method. If the z_i's converge to a point that is not equal to 0, then the y_i's converge to a consensus formation. On the other hand, we do not know when convergence

to 0 for the z_i's imply non-convergence to a consensus formation for the y_i's.

It has recently been shown that if the graph $\mathcal{G}(t)$ is static (or time-invariant) and symmetric, the f_{ij}'s fulfill certain differentiability assumptions, and $f_{ij} = f_{ji}$ for all i, j, then $\mathcal{A}$ is almost globally attractive under (5) (Markdahl et al. 2017) (a simple choice of such f_{ij}'s is when $f_{ij} = f_{ji} = a_{ij} = a_{ji} > 0$, i.e., the a_{ij}'s are positive scalars). On the other hand, the result does not hold for dimension $d = 2$. This means that under the same conditions on the graph and the f_{ij}'s as in Markdahl et al. (2017), the region of attraction of the point 0 has measure zero when $d \geq 3$ and has positive measure when $d = 2$. Linking those results to the lifting method—e.g., by means of a geometric interpretation—remains an open problem.

4 Conclusions

This paper addresses distributed synchronization or consensus on the unit sphere. A large class of consensus control laws is considered in which only relative information is used between neighbors for a time-switching interaction graph. We investigate how a lifting method can be used in the convergence analysis for these control laws. The proposed method is new in this context. It lifts the states from $\mathbb{S}^{d-1}$ to $\mathbb{R}^d$. In the higher-dimensional space, the dynamics of the states is described by a consensus protocol, where each agent is moving in a conical combination of the directions to its neighbors. The weights in the conical combination contain rational functions of the norms of the agents' states.

The paper provides more general convergence results than has been reported before for hemispheres and furthermore provides convergence results for a consensus protocol with rational weights of the norms of the states. However, an additional purpose of the paper was—by introducing the lifting method—to hopefully serve as a stepping-stone towards future research on global convergence results.

References

Deng, J., Liu, Z., Wang, L. & Baras, J. S. (2016), Attitude synchronization of multiple rigid bodies in SE(3) over proximity networks, *in* 'Proceedings of the IEEE's 35th Chinese Control Conference (CCC)', pp. 8275–8280.

Dörfler, F. & Bullo, F. (2014), 'Synchronization in complex networks of phase oscillators: A survey', *Automatica* **50**(6), 1539–1564.

Lageman, C. & Sun, Z. (2016), Consensus on spheres: Convergence analysis and perturbation theory, *in* 'Proceedings of the IEEE's 55th Conference on Decision and Control (CDC)', pp. 19–24.

Li, W. (2015), 'Collective motion of swarming agents evolving on a sphere manifold: A fundamental framework and characterization', *Scientific reports* **5**.

Li, W. & Spong, M. W. (2014), 'Unified cooperative control of multiple agents on a sphere for different spherical patterns', *IEEE Transactions on Automatic Control* **59**(5), 1283–1289.

Lin, Z., Francis, B. & Maggiore, M. (2007), 'State agreement for continuous-time coupled nonlinear systems', *SIAM Journal on Control and Optimization* **46**(1), 288–307.

Lojasiewicz, S. (1982), 'Sur les trajectoires du gradient dune fonction analytique', *Seminari di geometria* **1983**, 115–117.

Markdahl, J. & Goncalves, J. (2016), Global converegence properties of a consensus protocol on the n-sphere, *in* 'Decision and Control (CDC), 2016 IEEE 55th Conference on', IEEE, pp. 3487–3492.

Markdahl, J., Thunberg, J. & Gonçalves, J. (2017), 'Almost global consensus on the n-sphere', *IEEE Transactions on Automatic Control* .

Markdahl, J., Wenjun, S., Hu, X., Hong, Y. & Goncalves, J. (2016), Global and invariant aspects of consensus on the n-sphere, *in* 'the 22nd International Symposium on Mathematical Theory of Networks and Systems (MTNS)'.

Olfati-Saber, R. (2006), Swarms on Sphere: A Programmable Swarm with Synchronous Behaviors like Oscillator Networks, *in* 'Proceedings of the 45th IEEE Conference on Decision and Control', pp. 5060–5066.

Pereira, P. O. & Dimarogonas, D. V. (2015), Family of controllers for attitude synchronization in s^2, *in* '54th IEEE Conference on Decision and Control'.

Pereira, P. O. Dimitris, B. & Dimarogonas, D. (2016), A common framework for attitude synchronization of unit vectors in networks with switching topology, *in* 'Proceedings of the 55th Conference on Decision and Control (CDC)'.

Ren, W. (2010), 'Distributed cooperative attitude synchronization and tracking for multiple rigid bodies', *IEEE Transactions on Control Systems Technology* **18**(2), 383–392.

Sarlette, A. (2009), Geometry and Symmetries in Coordination Control, PhD thesis, Université de Liège.

Sarlette, A., Bonnabel, S. & Sepulchre, R. (2010), 'Coordinated motion design on lie groups', *IEEE Transactions on Automatic Control* **55**(5), 1047–1058.

Sarlette, A. & Sepulchre, R. (2009), 'Consensus optimization on manifolds', *SIAM Journal on Control and Optimization* **48**(1), 56–76.

Sarlette, A., Sepulchre, R. & Leonard, N. E. (2009), 'Autonomous rigid body attitude synchronization', *Automatica* **45**(2), 572–577.

Scardovi, L., Sarlette, A. & Sepulchre, R. (2007), 'Synchronization and balancing on the n-torus', *Systems & Control Letters* **56**(5), 335–341.

Shi, G. & Hong, Y. (2009), 'Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies', *Automatica* **45**(5), 1165–

1175.

Thunberg, J., Bernard, F. & Goncalves, J. (2017), 'Distributed methods for synchronization of orthogonal matrices over graphs', *Automatica* **80**, 243–252.

Thunberg, J., Goncalves, J. & Hu, X. (2016), 'Consensus and formation control on SE (3) for switching topologies', *Automatica* **66**, 109–121.

Thunberg, J., Hu, X. & Goncalves, J. (2017), 'Local lyapunov functions for consensus in switching nonlinear systems'. To appear in IEEE's Transactions on Automatic Control.

Thunberg, J., Markdahl, J., Bernard, F. & Goncalves, J. (2018), 'A lifting method for analyzing distributed synchronization on the unit sphere', *arXiv preprint arXiv:1805.02528* .

Thunberg, J., Markdahl, J. & Goncalves, J. (2017), 'Dynamic controllers for column synchronization of rotation matrices: a QR-factorization approach', *Automatica* **93**, 20–25.

Thunberg, J., Song, W., Hong, Y. & Hu, X. (2014), 'Distributed attitude synchronization using backstepping and sliding mode control', *Journal of Control Theory and Applications* **12**(1), 48–55.

Thunberg, J., Song, W., Montijano, E., Hong, Y. & Hu, X. (2014), 'Distributed attitude synchronization control of multi-agent systems with switching topologies', *Automatica* **50**(3), 832–840.

Tron, R., Afsari, B. & Vidal, R. (2012), 'Intrinsic consensus on SO(3) with almost-global convergence', *Proceedings of the IEEE's 51st IEEE Conference on Decision and Control (CDC)* (3), 2052–2058.

Tron, R., Afsari, B. & Vidal, R. (2013), 'Riemannian consensus for manifolds with bounded curvature', *IEEE Transactions on Automatic Control* **58**(4), 921–934.

Tron, R. & Vidal, R. (2014), 'Distributed 3-D localization of camera sensor networks from 2-D image measurements', *IEEE Transactions on Automatic Control* **59**(12), 3325–3340.