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The complex null string, Galilean conformal algebra and scattering equations

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Abstract: The scattering equation formalism for scattering amplitudes, and its stringy incarnation, the ambitwistor string, remains a mysterious construction. In this paper, we pursue the study a gauged-unfixed version of the ambitwistor string known as the null string. We explore the following three aspects in detail; its complexification, gauge fixing, and amplitudes. We first study the complexification of the string; the associated symmetries and moduli, and connection to the ambitwistor string. We then look in more details at the leftover symmetry algebra of the string, called Galilean conformal algebra; we study its local and global action and gauge-fixing. We finish by presenting an operator formalism, that we use to compute tree-level scattering amplitudes based on the scattering equations and a one-loop partition function. These results hopefully will open the way to understand conceptual questions related to the loop expansion in these twistor-like string models.

Keywords: Scattering Amplitudes, String theory and cosmic strings, Bosonic Strings

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1 Introduction

One of the most recent striking developments in the study of scattering amplitudes is the discovery of the Cachazo-He-Yuan (CHY) formalism [1, 2] for massless scattering in field theory. The CHY formalism recasts scattering amplitudes in terms of contour integrals in the complex plane based on the solutions to the *scattering equations*.

These contour integrals, reminiscent of the twistor string [3], were shown to originate from a new class of string theories dubbed 'ambitwistor strings' [4]. These allowed the extension of the original CHY formulae in many directions; loops [5–7], curved backgrounds [8–11], manifestly supersymmetric versions [12, 13], and even a string field theory [14, 15].

However, some basic aspects of this formalism remain unexplained, such as its gauged-unfixed form and the connection to standard string theory. Particularly at loop-level questions related to modular invariance and the integration domain are still not settled [16].

The extension of some recent developments at one and higher loops [17–19] may rely on a deeper understanding of these questions.

In [20], two of us argued that the ambitwistor string's origin is a theory partially characterized in the literature called null strings. This theory was initially introduced by Schild [21] as the *classical* tensionless limit of the usual string theory sigma-model.

The idea that ambitwistor strings, describing only massless field theory scattering, could be related to a tensionless limit of string theory is actually counter-intuitive, some evidence for it was present in [22, 23] but was not developed further. In [20] it was emphasized that this is only a classical statement. Quantum mechanically, it is a remarkable quantization ambiguity, already discovered in the 80's [24, 25], that truncates the spectrum of the string to a finite number of states, essentially the massless sector of the usual string (see also [26–29]).[1]

The goal of this paper is to build up on the work done in [20] in three directions, making more precise the relationship of this theory to the CHY formalism. In particular we hope that this should open the way to a deeper understanding of the loop expansion of these models. The main results we provide are:

- A study of the complexification of the null string, its symmetries and moduli. These we match with the ambitwistor string. Understanding the global structure of this moduli space will eventually lead to a proper determination of the integration cycle of the ambitwistor string at loop-level, along the lines of [6, 31].

- We use the representation theory of the constraint algebra of the string, called Galilean Conformal (GCA) [32–35], to show how the chirality of the string emerges due to decoupling of null states. We characterize its action on the moduli and the match the zero modes determinant with the ghost determinant from ambitwistor string. This gives a new perspective on the truncation of the spectrum and its chirality.

- We propose a new computation of tree-level amplitude and one-loop partition function using operator methods. The scattering equations emerge thanks to the integration of the original 'time' coordinate of the string, an idea originally due to [36]. We conjecture on modular transformations.

These three results are discussed in sections 2, 3 and 4, respectively. The sections are mostly self-contained and can be read independently.

2 The complex null string

2.1 From the null string to the ambitwistor string

The null string was originally obtained by Schild as a tensionless limit of the Nambu-Goto string [21]. The equivalent second order form of this action on which this work is based is

[1] Another choice of quantization yields a theory more compatible with what is expected from the high energy limit [30].

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the Lindström-Sundborg-Theodoridis (LST) action [37–40]:

$$S=\int d^2\sigma V^\alpha V^\beta \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \tag{2.1}$$

where G is the target space-time metric that we take to be flat $G_{\mu\nu}=\eta_{\mu\nu}$, $X^\mu(\sigma,\tau)$ are the coordinates of the string, and V^α, $\alpha=\{0,1\}$ is a vector field with density weight $(-1/2,-1/2)$.

Th light-cone gauge and BRST quantization of the null string was done in the seminal work [24]. To the best of our knowledge, it was observed there for the first time that a quantization ambiguity linked to the ordering of the operators leads to two very different quantum theories: a higher-spin type one, still poorly understood, and the one of interest for us, which is essentially the same as the ambitwistor string.

In this quantization the spectrum is truncated to the massless modes of string theory, and although the bosonic model has negative-norm states, the supersymmetric version is well-defined and its spectrum is the same as type II supergravity. For a more complete review of the null string, see [20], where the relation of the null string to the ambitwistor string was studied.

In this section, we come back on a geometrical aspect that was not discussed in this reference linked to the complexification of the model. Indeed, the LST action is a real one, while the ambitwistor string is a complex model. In a longer term perspective, understanding the complexified model, in loops for instance, will crucially rely on understanding the complexification from the real model itself [41, 42].

So let us describe step-by-step what we call the complex null string, its geometrical meaning and symmetries.

We first allow the target space to be a complex manifold $M^D_{\mathbb{C}}$ of (complex) dimension D, as well as allow the worldsheet field V to take complex values. That is, V takes values in the complexified tangent space to the worldsheet. At this point the worldsheet itself is still a two dimensional real manifold. This procedure gives a complexified version of the LST action where $X:\Sigma\mapsto M^D_{\mathbb{C}}$ and $V\in\left(\Omega^2(\Sigma)\right)^{\frac{1}{2}}\otimes T_{\mathbb{C}}\Sigma$ are respectively, a map from the worldsheet to complexified Minkowski space $M^D_{\mathbb{C}}\simeq\mathbb{C}^D$, and a complex vector field on the worldsheet with weight one half.

Because V is complexified, it generically defines a complex structure by requiring $V\in T^{(0,1)}\Sigma$, i.e.[2]

$$V\propto\bar\partial_{\bar z}. \tag{2.2}$$

Equivalently, it defines a conformal structure on Σ through

$$\tilde g^{\alpha\beta}:=V^{(\alpha}\overline{V}^{\beta)} \tag{2.3}$$

where $\overline{V}$ is the complex conjugate of V. In the real case, i.e. $V=\overline{V}$, this metric is degenerate as is usual in the null string.

[2] This $\bar\partial$ operator should be interpreted as a worldsheet field, depending on moduli, and not as a fixed background structure.

We can therefore think of a choice of V as a choice of complex structure together with a choice of "scaling". We now discuss the interpretation of this "scaling" part. Let,

$$V = \left(\frac{dzd\bar{z}}{e}\right)^{\frac{1}{2}} \otimes \bar{\partial}_{\bar{z}}. \tag{2.4}$$

Keeping V fixed while making a holomorphic change of coordinates $z \mapsto f(z)$ gives the following transformation law for "e":

$$e \mapsto e(\partial_z f)(\bar{\partial}_{\bar{z}}\bar{f})^{-1}. \tag{2.5}$$

This implies that for a given V we can think of the field e, as the coordinates of a Beltrami differential:

$$e\, d\bar{z} \otimes \partial_z. \tag{2.6}$$

As a consequence we have the following geometrical interpretation: if $\mathcal{M}$ is the space of complex structures on Σ then a choice of V is equivalent to choosing a point in $\Gamma := T\mathcal{M}$.

A quick look at the LST action, now written in terms of complex structure and Beltrami differential,

$$S\left[\bar{\partial}, e, X\right] = \int_\Sigma \frac{dzd\bar{z}}{e}(\bar{\partial}X)^2, \tag{2.7}$$

is enough to see that this is exactly the second order version of the ambitwistor action described in [4]:

$$S[\bar{\partial}, e, X, P] = \int dzd\bar{z}\left(P\cdot\bar{\partial}X - \frac{e}{2}P\cdot P\right). \tag{2.8}$$

Note that in this action, the complex structure is a field of the model, being integrated over, while the ambitwistor string is already gauge-fixed to conformal gauge.

2.2 Equations of motion and boundary term

To obtain the equations of motion we vary the action with respect to X,[3]

$$\delta S = 2\int_\Sigma d^2\sigma\,(\partial_\alpha \delta X_\mu)\,V^\alpha (VX)^\mu \tag{2.9}$$

and integrate by parts to obtain boundary term. This is done by rewriting (2.9) as

$$\delta S = 2\int_\Sigma d(\delta X_\mu)\wedge \epsilon_{\alpha\beta}V^\alpha (VX)^\mu d\sigma^\beta. \tag{2.10}$$

Then, the integration by part is straightforward

$$\delta S = 2\int_\Sigma d\left((\delta X_\mu)\,\epsilon_{\alpha\beta}V^\alpha (VX)^\mu d\sigma^\beta\right) - (\delta X_\mu) d\left(\epsilon_{\alpha\beta}V^\alpha (VX)^\mu d\sigma^\beta\right), \tag{2.11}$$

and we can extract equations of motions for the null string

$$\partial_\alpha\left(V^\alpha\,(VX)^\mu\right) = 0 \rightarrow \bar{\partial}\left(\frac{1}{e}\bar{\partial}X^\mu\right) = 0 \tag{2.12}$$

[3]Here and everywhere below $\sqrt{d\sigma^2}\ (VX)^\mu$ stands for $\sqrt{d\sigma^2}\ V^\alpha\partial_\alpha X^\mu$. These are D scalar fields on Σ with density weight one half.

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together with a general expression for the boundary term:

$$\delta S_{\text{boundary}} = 2\int_{\partial\Sigma} (\delta X_\mu)\, \epsilon_{\alpha\beta} V^\alpha (VX)^\mu d\sigma^\beta. \tag{2.13}$$

Unfortunately there does not seem to be a set of boundary conditions which gives an interesting theory of open null strings nor null strings ending on branes. A contraction of the open string algebra can be done which has been claimed to describe a tensionless open string [43, 44], but it is not clear how to recover it from appropriate boundary conditions on the null string.

Therefore we continue we closed null strings. A clean way to understand the above integrands is as follows. Start with

$$(VX)^\mu V^\alpha\, d^2\sigma \otimes \partial_\alpha \in \Omega^2\left(\Sigma, T\Sigma\right), \tag{2.14}$$

which are D vector-valued two-forms on Σ, contracting this object with itself we obtain a 1-form on Σ. The resulting form is just the integrand of (2.13): $\epsilon_{\alpha\beta} V^\alpha (VX)^\mu d\sigma^\beta \in \Omega^1(\Sigma)$. The field equations (2.12) just state that this form is closed.

Finally, considering variations of the action with respect to an infinitesimal variation of V, we get two constraints:

$$V^\beta \partial_\beta X \cdot \partial_\alpha X = 0 \quad \forall \alpha \in 0,1 \qquad \rightarrow \qquad \bar\partial X \cdot \bar\partial X = 0, \quad \bar\partial X \cdot \partial X = 0. \tag{2.15}$$

These can be directly obtain by varying V in (2.1) or by using the parametrization (2.7) and considering variation of V as

$$\delta V = \delta\mu \left(\frac{dzd\bar z}{e}\right)^{\frac{1}{2}} \otimes \partial_z - \frac{\delta e}{2e}\left(\frac{dzd\bar z}{e}\right)^{\frac{1}{2}} \otimes \bar\partial_{\bar z}. \tag{2.16}$$

Here $\delta\mu$ is an infinitesimal variation of the almost complex structure $\delta\bar\partial_{\bar z} = \delta\mu\, \partial_z$.

Altogether, the constraints (2.15) are the usual null string statement that the pullback of the space-time metric on the worldsheet $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$ is degenerate, with the degeneracy direction given by V. Accordingly, integral lines of $(VX)^\mu$ in space-time are null lines and these null lines are orthogonal to each other.

2.3 Symmetries of the complexified null string action

From now on, we also consider the worldsheet variables to be complex. Accordingly $\Sigma_{\mathbb{C}}$ is now taken to be a two dimensional *complex* manifold with holomorphic coordinates z and $\tilde z$. In particular, $\tilde z$ is the complex conjugate of z anymore. Imposing $\tilde z = \bar z$ amounts to an embedding $\Sigma \hookrightarrow \Sigma_{\mathbb{C}}$ of a usual (one dimensional complex) worldsheet Σ into the complexified one. The interest of this procedure, of course, lies in the fact that $\tilde z = \bar z$ is not the only possible embedding, and we intend to make precise in a following work how the ambitwistor string can be seen as an alternative embedding of the null string. When referring to antiholomorphic functions we will mean holomorphic functions of $\tilde z$ unless explicitly stated otherwise. All fields are now holomorphic in $(z, \tilde z)$ and the worldsheet integral should be seen as a holomorphic two-form that must be integrated over a two-cycle. In particular, a choice of real worldsheet gives such a two-cycle.

Diffeomorphisms. We now consider the action of holomorphic transformations on $\Sigma_{\mathbb{C}}$

$$(z,\tilde{z}) \mapsto (f(z,\tilde{z}), g(z,\tilde{z})) . \tag{2.17}$$

We will refer to these transformations as diffeomorphisms of the complexified worldsheet. Infinitesimal diffeomorphisms are

$$(z,\tilde{z}) \mapsto (z+\epsilon(z,\tilde{z}), \tilde{z}+\tilde{\epsilon}(z,\tilde{z})) . \tag{2.18}$$

and can be thought of as the vector field $\boldsymbol{v} = \epsilon\partial_z + \tilde{\epsilon}\tilde{\partial}_{\tilde{z}}$ on $\Sigma_{\mathbb{C}}$. These infinitesimal diffeomorphisms act on the fields as

$$\begin{aligned} \mathcal{L}_{\boldsymbol{v}} X &= \epsilon\,\partial_z X + \tilde{\epsilon}\,\tilde{\partial}_{\tilde{z}} X, \\ \mathcal{L}_{\boldsymbol{v}} V &= \left(\frac{dzd\tilde{z}}{e}\right)^{\frac{1}{2}} \otimes \left(-\frac{1}{2e}\left(\epsilon\partial e - e\partial\epsilon + \tilde{\partial}_{\tilde{z}}(\tilde{\epsilon}e)\right)\bar{\partial}_{\tilde{z}} - \left(\tilde{\partial}_{\tilde{z}}\epsilon\right)\partial_z\right) . \end{aligned} \tag{2.19}$$

The Noether current for infinitesimal worldsheet diffeomorphisms is obtained by taking the integrand of the boundary term (2.13) with $\delta X = \mathcal{L}_{\boldsymbol{v}} X$:

$$J(\boldsymbol{v}) = v^{\alpha} T_{\alpha\beta} d\sigma^{\beta} = \epsilon\,\frac{1}{e}\partial X\cdot\bar{\partial}X\ dz + \bar{\epsilon}\,\frac{1}{e}\bar{\partial}X\cdot\bar{\partial}X\ dz \tag{2.20}$$

where the energy momentum tensor T is

$$T = (\partial_\alpha X)\cdot(VX)V^{\gamma}\epsilon_{\gamma\beta}\ d\sigma^{\alpha}\otimes d\sigma^{\beta} = \frac{1}{e}\partial X\cdot\bar{\partial}X\ dz\otimes dz + \frac{1}{e}\bar{\partial}X\cdot\bar{\partial}X\ d\bar{z}\otimes dz. \tag{2.21}$$

As expected, vanishing of the energy-momentum tensor is equivalent to the vanishing of the constraints (2.15).

Note that the left part of the energy-momentum tensor[4] $\iota_{\partial} T = \frac{1}{e}\partial X\cdot\bar{\partial}X\ dz$ is not simply related to the right part $\iota_{\bar{\partial}} T = \frac{1}{e}\bar{\partial}X\cdot\bar{\partial}X\ dz$. This is in contrast to the Poliakov string where one left and right movers contributions to the stress energy tensor are related by complex conjugation. This chirality of the null string can be traced back to the fact that V transforms differently under right (i.e. $\epsilon \neq 0, \tilde{\epsilon} = 0$) and left diffeomorphisms (i.e. $\epsilon = 0, \tilde{\epsilon} \neq 0$).

The extra null ray symmetry of the complex null string. The complex null string seems to enjoy one further local symmetry. Recasting it in the first order form (2.7), this symmetry corresponds to translations along null geodesics as discussed in [4]. This symmetry is also the origin of the interpretation of that model as living on ambitwistor space, since if we consider target space as parametrized by the fields $\{P, X\}$, then this extra symmetry implements the symplectic reduction by the constrain $P^2 = 0$. This reduced space is the space of null geodesics, also known as ambitwistor space. The infinitesimal version of the symmetry can be parametrized by a $(1,0)$-vector field $\boldsymbol{\alpha} = \alpha\partial_z$ on Σ and acts on the fields as follows:

$$\delta X = \frac{\alpha}{e}\,\bar{\partial}_{\bar{z}} X, \qquad \delta V = -\frac{\bar{\partial}_{\bar{z}}\alpha}{2e}\left(\frac{dzd\bar{z}}{e}\right)^{\frac{1}{2}}\otimes\bar{\partial}_{\bar{z}}. \tag{2.22}$$

[4]Here ι_v stands for the interior derivative.

With the associated Noether Current

$$J(\alpha) = \alpha\, \frac{1}{e^2}\bar{\partial}X \cdot \bar{\partial}X\, dz. \tag{2.23}$$

Note that this extra symmetry is not be present in the real case since it does not respect the reality condition $X = \overline{X}$. Even more remarkable is that this symmetry mimics the action of antiholomorphic diffeomorphisms (2.19) but is parametrized by a holomorphic vector field. The dictionary between them is as simple as setting

$$\tilde{\epsilon} = \frac{\alpha}{e}. \tag{2.24}$$

It is also easy to see using a Hamiltonian formalism that these two gauge redundancies are the same on-shell, at least infinitesimally. This is analogous to what happens in the case of the Hamiltonian action of the worldline formalism for a massless particle [45]. There the worldline diffeomorphisms and translations along null geodesics give the same gauge redundancy of the action on-shell. It is clear now that to reach the ambitwistor string from the null string one needs to complexify the latter. This allows us to access this equivalent parametrization of the antiholomorphic diffeomorphisms by a holomorphic vector field, and gives a completely chiral theory, the ambitwistor string.

2.4 Moduli

In this section we study the moduli of the complexification of the vector field V. Using the equations for the variation of V, (2.19) we define operators P and Q whose zero modes correspond to automorphisms of the string. Using the natural pairing, the zero modes of their adjoints $P^\dagger$ and $Q^\dagger$ are the moduli of the null string. We shall see that after trading the antiholomorphic diffeomorphisms by the holomorphic scaling symmetry from the previous section the results found in [6] for the ambitwistor string are reproduced.

As already explained the moduli $\mathcal{M}$ of the vector field V can be parametrized by a complex structure $d\bar{z}\partial_{\tilde{z}}$ and a Beltrami differential $ed\bar{z}\partial_z$, see (2.4). Then a variation δV is an element of the tangent space $T\mathcal{M}$ to the moduli and can be parametrized by a doublet $(\delta\mu\; d\bar{z}\partial_z, \delta e\; d\bar{z}\partial_z)$ of Beltrami differentials, see (2.16).

The infinitesimal gauge transformations of the null string are infinitesimal diffeomorphisms given by $\epsilon\partial_z$ (left diffeomorphisms) and $\tilde{\epsilon}\tilde{\partial}_{\tilde{z}}$ (right diffeomorphism).[5] A generic variation of V under infinitesimal diffeomorphisms defines a map $\Gamma\left[T_{\mathbb{C}}\Sigma\right] \to T\mathcal{M}$ given by two operators

$$\begin{aligned} P : \Gamma\left[T_{\mathbb{C}}^{(1,0)}\Sigma\right] &\to T\mathcal{M}, & \qquad Q : \Gamma\left[T_{\mathbb{C}}^{(0,1)}\Sigma\right] &\to T\mathcal{M}, \\ \epsilon\partial &\mapsto \mathcal{L}_{\epsilon\partial}V & \tilde{\epsilon}\tilde{\partial} &\mapsto \mathcal{L}_{\tilde{\epsilon}\tilde{\partial}}V \end{aligned} \tag{2.25}$$

comparing (2.16) with (2.19) we obtain

$$P\left(\epsilon\,\partial_z\right) = \left(P_\mu\,,\, P_e\right) = \left(-\tilde{\partial}_{\tilde{z}}\epsilon\; d\tilde{z}\,\partial_z\,,\, \left(\epsilon\,\partial_z e - e\,\partial_z\epsilon\right) d\tilde{z}\,\partial_z\right), \tag{2.26}$$

$$Q\left(\tilde{\epsilon}\,\bar{\partial}_{\bar{z}}\right) = \left(Q_\mu, Q_e\right) = \left(0\,,\, -\tilde{\partial}_{\tilde{z}}\left(\tilde{\epsilon}e\right)\; d\bar{z}\,\partial_z\right). \tag{2.27}$$

[5] Recall that z and $\tilde{z}$ are considered independent complex variables.

By construction, $G = \mathrm{Im}(P) \cup \mathrm{Im}(Q)$ is the subspace of $T\mathcal{M}$ spanned by the gauge transformations. We are looking for variations of the gauge parameters that cannot be the result of a gauge transformation. By picking a metric on $T\mathcal{M}$, these non-gauge variations can be taken to be $G^\perp$. One can easily get such a metric by making a choice of hermitian metric on $\Sigma_\mathbb{C}$, $ds^2 = dz \odot d\bar{z} + d\tilde{z} \odot d\bar{\tilde{z}}$. If u and v are any tensor of same type, we note $\overline{u}v$ the hermitian pairing induced by the above metric.

We can define $G^\perp$ to be the subset of $T\mathcal{M}$ such that for all $\epsilon, \tilde{\epsilon}$:

$$\begin{aligned} &\int_\Sigma dz d\tilde{z}\, (\delta\bar{\mu}\ P_\mu(\epsilon) + \delta\bar{e} P_e(\epsilon)) = 0, \\ &\int_\Sigma dz d\tilde{z}\, (\delta\bar{\mu}\ Q_\mu(\tilde{\epsilon}) + \delta\bar{e} Q_e(\tilde{\epsilon})) = 0. \end{aligned} \tag{2.28}$$

We are therefore looking for $\mathrm{Ker}(P^\dagger) \cap \mathrm{Ker}(Q^\dagger)$, with $P^\dagger : T\mathcal{M} \to \Gamma[T^{(1,0)}_\mathbb{C}\Sigma]$, $Q^\dagger : T\mathcal{M} \to \Gamma[T^{0,1}_\mathbb{C}\Sigma]$ defined by

$$\int_\Sigma dz d\tilde{z}\, \left(\bar{\delta\mu}\ P_\mu(\epsilon) + \delta e P_e(\epsilon)\right) = \int_\Sigma dz d\tilde{z}\ \overline{P^\dagger(\delta\mu, \delta e)}\ \epsilon \tag{2.29}$$

$$\int_\Sigma dz d\tilde{z}\, (\delta\bar{\mu}\ Q_\mu(\tilde{\epsilon}) + \delta\bar{e}\ Q_e(\tilde{\epsilon})) = \int_\Sigma dz d\tilde{z}\ \overline{Q^\dagger(\delta\mu, \delta e)}\ \tilde{\epsilon}. \tag{2.30}$$

We can obtain these operators explicitly by an integration by parts:

$$\begin{aligned} &\int_\Sigma dz d\tilde{z}\, (\delta\bar{\mu}\, (P_\mu(\epsilon) + Q_\mu(\tilde{\epsilon})) + \delta\bar{e}\, (P_e(\epsilon) + Q_e(\tilde{\epsilon}))) \\ &\quad = \int_\Sigma dz d\tilde{z} \left(\delta\bar{\mu}\ \tilde{\partial}\epsilon + \delta\bar{e}\ (\epsilon\partial e - e\partial\epsilon) + \delta\bar{e}\ \tilde{\partial}(\tilde{\epsilon} e)\right), \\ &\quad = \int_\Sigma dz d\tilde{z} \left(\epsilon \left(-\tilde{\partial}\ \delta\bar{\mu} + \partial e\, \delta\bar{e} + \partial\,(e\ \delta\bar{e})\right) - (\tilde{\epsilon} e)\, \tilde{\partial}\delta\bar{e}\right). \end{aligned} \tag{2.31}$$

Requiring that this holds for any ϵ and any $\tilde{\epsilon}$ gives the equations

$$\begin{aligned} \bar{\partial}\delta\bar{\mu} - \partial e\delta\bar{e} - \partial(e\delta\bar{e}) &= 0 \\ e\,\bar{\partial}\delta\bar{e} &= 0. \end{aligned} \tag{2.32}$$

If we now parametrize the diffeomorphisms in $\bar{z}$ by a holomorphic field $\alpha = e\tilde{\epsilon}$ using the equivalence of these diffeomorphisms with the scaling symmetry, we recover the same equations for the moduli as [6]. Here too it was essential that the null string be complexified in order to match the ambitwistor string.

3 Symmetry algebra

In this section we come back on the symmetry algebra of the null string. Following recent terminology [33–35] is called a 2-dimensional Galilean Conformal Algebra, $\mathfrak{gca}_2$. This algebra is isomorphic to the 3-dimensional Bondi-Metzner-Sachs $\mathfrak{bms}_3$ algebra — the symmetry algebra of the null boundary of 3-dimensional Minkowski spacetime. This isomorphism is at the root of various conjectures concerning flat space holography [46–48] which have triggered interest for $\mathfrak{gca}_2$ representations and supersymmetric extensions thereof [32, 49–62]. See also [63] for a connection with Carrollian ultra-relativistic physics.[6] In the text, we frequently use the BMS terminology, and call the GCA transformations superrotations and supertranslations.

Our motivation in studying GCA's comes from wanting to set up a vertex operator formalism for the null string where the loop-momentum zero modes are already integrated. In addition, the symmetry algebra of the null string is a GCA and not just the traditional Virasoro algebra of string theory, it would appear necessary to start from scratch and work out the equivalent of the basic tools that we have in ordinary CFTs; state-operator map and vertex operator formalism.

In this section, we will show that the representation theory of the $\mathfrak{gca}_2$ for the null string actually forces the representations to truncate down to the usual Virasoro representations. To do so, we will mostly use of the analysis of null states of the $\mathfrak{gca}_2$ presented in [32].

This surprising fact justifies intuitively why it has been possible to use standard CFT tools so far both in the ambitwistor and null string (see in particular the recent work on one-loop null string amplitudes of [64]). It will also shed a new light on the remarkable chirality of the ambitwistor and null strings and the truncation of their spectrum.

Before starting, we would like to briefly comment on conformal non-relativistic symmetries, mostly to disambiguate the terminology. The algebra studied here is not the Schrödinger algebra of [65–68] but the algebra obtained by an Inönü-Wigner contraction of the usual Poincaré algebra. It exists in any dimensions, and only in two dimensions it has the infinite dimensional extension which also makes it a contraction of a product of two Virasoro algebras [32, 69, 70]. We refer to [32, 71] for further details and references.

3.1 Gauge fixing and residual symmetries

Consider partially gauge-fixing the null string action (2.1) by making a choice of complex structure. Looking at the variations of V, (2.19), we see that the "right-diffeomorphisms" ($\epsilon = 0$) preserve this gauge choice. However, requiring that "left-diffeomorphisms" ($\tilde{\epsilon} = 0$) preserve this complex structure imposes $\tilde{\partial}\epsilon = 0$, that is, it is only a function of z.

$$\begin{aligned} &\mathcal{L}_{\epsilon} X = \epsilon\, \partial_z X, && \mathcal{L}_{\tilde{\epsilon}} X = \tilde{\epsilon}\, \tilde{\partial}_{\tilde{z}} X, \\ &\mathcal{L}_{\epsilon} V = \left(\frac{dz d\tilde{z}}{e}\right)^{\frac{1}{2}} \otimes \left(-\frac{1}{2e}\left(\epsilon\partial e - e\partial\epsilon\right)\, \tilde{\partial}_{\tilde{z}}\right), && \mathcal{L}_{\tilde{\epsilon}} V = \left(\frac{dz d\tilde{z}}{e}\right)^{\frac{1}{2}} \otimes -\frac{\tilde{\partial}_{\tilde{z}}\left(\tilde{\epsilon} e\right)}{2e}\, \tilde{\partial}_{\tilde{z}}. \end{aligned} \tag{3.1}$$

We can further gauge-fix by choosing a particular value for the Lagrange multiplier scale field e. Residual symmetries then have to satisfy

$$\epsilon\partial_z e - e\partial_z \epsilon + \tilde{\partial}_{\tilde{z}}(\tilde{\epsilon} e) = 0, \qquad \tilde{\partial}_{\tilde{z}}\epsilon = 0. \tag{3.2}$$

Taking e to be constant, these symmetries are generated by vector fields of the form

$$\epsilon\partial_z + \tilde{\epsilon}\tilde{\partial}_{\tilde{z}} = f(z)\partial_z + \left(\tilde{z}\partial_z f(z) + g(z)\right)\tilde{\partial}_{\tilde{z}} \tag{3.3}$$

[6] It is known that in two dimensions, ultra- and non-relativistic physics are classically equivalent, essentially because there are as many space and time dimensions.

where f, g are any holomorphic functions. The associated Noether currents are

$$J_f = f(z)\left(\tilde{\partial}X \cdot \partial X - \tilde{z}\partial\left(\tilde{\partial}X \cdot \tilde{\partial}X\right)\right)d\tilde{z} \qquad \text{and} \qquad \tilde{J}_g = g(z)\left(\tilde{\partial}X \cdot \tilde{\partial}X\right)d\tilde{z}. \tag{3.4}$$

These vector fields form a GCA which play the same role in the null string as the Virasoro algebra does in the usual string. Note that the GCA contains a single copy of the Virasoro algebra as a subalgebra giving the null string its chiral character.

Put differently, the transformation (3.3) defines two operators that we can call $L(F)$ and $M(g)$ whose mode expansion are given by

$$L(f) = \sum_{n\in\mathbb{Z}} f_n L_n, \qquad M(g) = \sum_{n\in\mathbb{Z}} M_n g_n \tag{3.5}$$

with

$$L_n = -z^n(z\partial_z + (n+1)\tilde{z}\partial_{\tilde{z}}), \qquad M_n = z^{n+1}\partial_{\tilde{z}}. \tag{3.6}$$

In the BMS language, L_n and M_n are the generators of superrotations and supertranslations, respectively.[7] They obey the following commutation relations

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad [L_n, M_m] = (n-m)M_{n+m}, \qquad [M_n, M_m] = 0. \tag{3.7}$$

At the quantum level, central extensions are admissible. The centrally-extended algebra is

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0} \\
[L_n, M_m] &= (n-m)M_{n+m} + \frac{c_M}{12}m(m^2-1)\delta_{m+n,0} \\
[M_n, M_m] &= 0.
\end{aligned} \tag{3.8}$$

For the ambitwistor string, $c_M = 0$ and $c_L = d-2$ is canceled by the inclusion of the b-c and $\tilde{b}$-$\tilde{c}$ ghost systems. The vacuum chosen to study the representations of the GCA is the same as the one used in the ambitwistor quantization and is defined by[8]

$$L_n|0\rangle = 0, \qquad M_n|0\rangle = 0, \qquad \forall n \geq 0. \tag{3.9}$$

3.2 GCA Hilbert space and null states

We now proceed to investigate the GCA representations. We will use the analysis of [32] and argue that they simply truncate down to a chiral Virasoro representation.

The upshot is that due to how the $P^2 = 0$ constraint is imposed, the GCA action automatically descends to a chiral CFT action at the level of the spectrum.

---

[7] The combination $L'_n = L_n - i(n+1)\frac{\tilde{z}}{z}M_n = -z^{n+1}\partial_z$ generates exactly chiral conformal transformations we are after. However, since the change of generators involves the variables themselves, it is not clear what can be made of this observation.

[8] In the other quantization, supposed to produce a higher-spin theory [24, 72], the operator ordering stipulates that all the modes of P annihilate the vacuum. Therefore, all the modes of the constraint annihilate the vacuum $\forall n \in \mathbb{Z}$, $L_n|0\rangle = M_n|0\rangle = 0$, and it is not clear how to build non-trivial representations. This may reflect that the theory is likely to be free, as expected from the Coleman-Mandula theorem.

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We would like to conjecture that for this reason we can have a well-defined state-operator map for the chiral CFT as well as a standard vertex operator formalism. It still remains an important question to understand these issues in full generality in the GCA and may open the way towards massive theories for instance, where the constraint $P^2 = 0$ should not be applied.

We start by reviewing some elements of the analysis of [32] on the representations of the $\mathfrak{gca}_2$ algebra. We look at states with well-defined scaling properties

$$L_0|\Delta\rangle = \Delta|\Delta\rangle. \tag{3.10}$$

Then, since $[L_0, M_0] = 0$ the representations are actually indexed by another quantum number ξ called "rapidity" [32, 73]

$$L_0|\Delta,\xi\rangle = \Delta|\Delta,\xi\rangle, \qquad M_0|\Delta,\xi\rangle = \xi|\Delta,\xi\rangle. \tag{3.11}$$

Descendant states are then built out by the successive action of the operators L_{-n}, M_{-m}, $n, m > 0$.

We now follow the analysis of [32, section 5] on the GCA null states. Here $c_M = 0$ and the physical state conditions impose $\Delta = 2$, and, importantly, $\xi = 0$. The first condition states that physical states are primaries of conformal weight two. The second condition is on-shellness of the state, i.e. $k^\mu k_\mu = 0$ for a state with momentum k^μ.

This is this last condition that actually implies that the null string does not use of the full GCA symmetry. We will see that it implies that the M_{-n} descendants decouple. The argument adapted from [32], goes as follows.

At level one, there are two descendant states $L_{-1}|\Delta,0\rangle$ and $M_{-1}|\Delta,0\rangle$. It is immediate to see that the second one, $M_{-1}|\Delta,0\rangle$, is orthogonal to all other states in the Hilbert space. Therefore $M_{-1}|\Delta,0\rangle = 0$. At level two, descendants made of powers of M_{-1} and M_{-2} are the following states

$$(M_{-1})^2|\Delta,0\rangle, \qquad L_{-1}M_{-1}|\Delta,0\rangle\,, \qquad M_{-2}|\Delta,0\rangle\,. \tag{3.12}$$

The first two states vanish immediately, because $M_{-1}|\Delta,0\rangle = 0$. The second state is, again, orthogonal to all other states, precisely because $M_0|\Delta,0\rangle = 0$. The whole sector of the Hilbert space made of M_{-n}'s is therefore null and decouples from the physical Hilbert space. We are then left with a chiral Virasoro module. This is the reason why it is possible to treat the null string and ambitwistor string as a chiral CFT, and intuitively, is the origin of the holomorphicity of all twistor string models.

3.3 Gauge-fixing the global GCA

After the gauge-fixing, there is still a residual gauge symmetry which is given by the global part of the GCA. Below we explain how this residual gauge redundancy is removed by fixing the positions of 3 operators, in analogy with the similar string-theoretic version.

The method previously used in [32, 73] was to consider the $\mathfrak{gca}_2$ as a contraction of the usual Vir $\times$ $\overline{\text{Vir}}$ algebra, under which the coordinates $z, \tilde{z}$ are scaled as

$$\begin{aligned} z &= t + \epsilon x \\ \tilde{z} &= t - \epsilon x \end{aligned} \tag{3.13}$$

with $\epsilon \to 0$. An SL$(2,\mathbb{C})$ transformation then induces the following transformation

$$t + \epsilon x \to \frac{a(t+\epsilon x)+b}{c(t+\epsilon x)+d} = \frac{at+b}{ct+d} + \epsilon \frac{x}{(ct+d)^2} \,. \tag{3.14}$$

Here, again following our wish to work out the details of the model, we will derive these relations from the explicit form of the global GCA transformations.

We start from the representation of eq. (3.6). The generators L_0, L_1, L_{-1} and M_0, M_1, M_{-1} constitute the global part of the gauge group. Their expressions read

$$\begin{aligned} L_{-1} &= -\partial_t \,, \qquad & L_0 &= x\partial x - t\partial_t \,, \qquad & L_1 &= -2tx\partial_x - t^2\partial_t \,, \\ M_{-1} &= \partial_x \,, \qquad & M_0 &= t\partial_x \,, \qquad & M_1 &= t^2\partial_x \,. \end{aligned} \tag{3.15}$$

We claimed that these generators are globally defined, but there is a subtlety here. Due to the term $-2tx\partial_x$, L_1 is not well defined for $t \to \infty$, unless $x = 0$. We shall see later that it is always possible to fix $x = 0$, and moreover that these terms produce only off-diagonal terms in the determinant of the zero modes which anyway do not contribute to the total determinant. It is also intriguing to see that, at fixed t, all the M_n's for all $n \in \mathbb{Z}$ are well defined, but only M_{-1} is for all values of t. Infinitesimal transformations associated to these six generators can be written easily, an read for the L_{-1}, L_0, L_1 with parameters $\delta a_{-1}, \delta a_0, \delta a_1$:

$$\begin{aligned} \delta t &= \delta a_{-1} + (\delta a_0)t - (\delta a_1)t^2 \\ \delta x &= -\delta a_0 x - 2(\delta a_1)tx \end{aligned} \tag{3.16}$$

and for the M_i's with parameters $\delta b_{-1}, \delta b_0, \delta b_1$

$$\begin{aligned} \delta t &= 0 \\ \delta x &= \delta b_{-1} + (\delta b_0)t + (\delta b_1)t^2 \,. \end{aligned} \tag{3.17}$$

To integrate to the finite form, in principle one has to solve a differential equation. Take the special conformal transformation of the conformal group, generated by $\delta z = -(\delta\alpha)z^2$. It is solved by writing $\frac{\delta z}{z(\alpha)^2} = \delta\alpha$ which gives $1/\tilde{z} - 1/z = \alpha$, i.e. $\tilde{z} = \frac{z}{1+\alpha z}$. In the case of the GCA transformations, only the L_1 requires a little care. Calling $s = a_1$, it reads

$$\frac{\delta t}{t(s)^2} = -\delta c, \qquad \frac{\delta x}{x(s)} = -2\delta s \times t(s) \tag{3.18}$$

where we have made the dependence on c explicit in the functions t, x. Integrating t gives $t(s) = t(0)/(1 + st(0))$, which can be plugged into $\delta x/x$ to give $x(s) = x(0)/(1 + st)^2$. Combining with L_0 and L_{-1} we obtain the following finite transformations:

$$t \to \tilde{t} = \frac{at+b}{ct+d}, \qquad x \to \tilde{x} = \frac{x}{(ct+d)^2} \tag{3.19}$$

for the L_i's and

$$t \to \tilde{t} = t, \qquad x \to \tilde{x} = x + e + ft + gt^2 \tag{3.20}$$

for the M_i's.

Given 3 points (t_i, x_i) on $\mathbb{C}^2$ we apply the finite transformations above to perform the usual gauge fixing of the t's to $0, 1, \infty$ and fix x_1, x_2, x_3 to zero. For four points, we have determined explicitly that this produces the two GCA-independent quantities found in [32] using the previously described squeeze limit:

$$t = \frac{t_{23}t_{14}}{t_{12}t_{34}}, \qquad \frac{x}{t} = \frac{x_{12}}{t_{12}} - \frac{x_{14}}{t_{14}} - \frac{x_{23}}{t_{23}} + \frac{x_{34}}{t_{34}}. \tag{3.21}$$

This means that for $x_1 = x_2 = x_3 = 0, t_1 = 0, t_2 = 1$ and $t_3 = \infty$, we just have $t_4 = t$ and $x_4 = x$. The finite BMS_3 transformations have been computed in [74], it would be interesting to understand if they have any geometrical interpretation in the $\mathfrak{gca}_2$ side.

Lastly we compute the Jacobian for gauge-fixing the global GCA. In a BRST framework this comes from integrating out the zero modes of the ghosts associated to the constraints (2.15). There are six ghosts, one for each global generator of the GCA (3.15). Therefore there are six global sections which we can fix by picking three points on the worldsheet $\{(t_1, x_1), (t_3, x_3), (t_3, x_3)\}$ and calculating the determinant of the matrix of zero mode sections evaluated at these points

$$M = \begin{pmatrix} A & 0 \\ B & -A \end{pmatrix}, \qquad \text{where} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ {t_1}^2 & {t_2}^2 & {t_3}^2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & x_2 & x_3 \\ -2x_1t_1 & -2x_2t_2 & -2x_3t_3 \end{pmatrix}. \tag{3.22}$$

This matrix has an off-diagonal part because the L_n and M_n generators do not commute, in contrast to the left- and right-handed Virasoro algebra in usual 2D CFTs. However, the off-diagonal does not contribute to the determinant which is

$$\det(M) = -((t_1 - t_2)(t_2 - t_3)(t_3 - t_1))^2, \tag{3.23}$$

which is precisely the same as found in the ambitwistor string. Note that since the x coordinates decouple from the determinant their fixed values are immaterial to the correlation function, effectively all that is needed to fix the global GCA is choosing three points in the t coordinate. This is precisely what happens in the ambitwistor string, where one only fixes three holomorphic coordinates to fix the global GCA at tree-level.

4 Operator formalism and scattering equations

4.1 Formalism

As we mentioned, the ambitwistor complexified gauge is an elegant way to reproduce the CHY formulae. However, subtleties show up at loop-level which render this power somewhat useless, in particular when discussing questions related to modular invariance and the role of the loop momentum for instance. In this section, we set up an operator formalism[9] which will remain somewhat agnostic about the complexification since the manipulations are purely algebraic. We then use it to gain insights into the appearance of the scattering equations in the ambitwistor string by comparing the amplitude computed in these two

[9]Here done somewhat heuristically since we neglect the ghosts for the most part.

different ways. We also make connection with an interesting one-loop computation using CFT methods presented in [64].

The formalism will essentially follow the analogous operator construction in string theory, presented in the classic reference [75]. To set up the formalism, we consider the canonical quantization of the null string in Schild's gauge [24]:

$$V \sim \partial_\tau \,. \tag{4.1}$$

Note that this is a Lorentzian gauge fixing condition and should be contrasted with the more Euclidean condition chosen earlier in (2.2).

Even though the amplitude calculation is only well-defined in the model with two supersymmetries, we work in the purely bosonic model since it has all the important features without the added combinatorial complexity of having the fermions. This feature will prove sufficient to exhibit the essential properties of the model, the scattering equations in particular.

The relevant field to quantize is X, for which the equation of motion $\partial_\tau^2 X = 0$ gives the following classical solutions

$$X(\tau, \sigma) = Y(\sigma) + \tau P(\sigma) \tag{4.2}$$

which we expand in modes

$$Y(\sigma) = \sum_{n\in\mathbb{Z}} y_n e^{-i\sigma n}, \qquad P(\sigma) = \sum_{n\in\mathbb{Z}} p_n e^{-i\sigma n}\,, \tag{4.3}$$

with canonical commutation relations $[y_n, p_m] = i\delta_{m+n,0}$. Here and below we omit Lorentz indices for convenience. In this gauge, the two constraints are given by $\partial_\tau X \cdot \partial_\sigma X = 0$ and $\partial_\tau X \cdot \partial_\tau X = 0$. The mode expansion of these in terms of the corresponding L and M generators and their commutation relations can be found in [20, 34]. For what follows, we only need the zero modes of these operators: L_0 generates rotations along the circle and M_0 is the worldsheet Hamiltonian (we provide their explicit expression below).

We postulate that a vertex operator with momentum k placed at the $(\sigma, \tau) = (0, 0)$ assumes the following form

$$V(0,0) := (\epsilon \cdot \dot{X}(0,0))^2 e^{ik\cdot X(0,0)} = (\epsilon \cdot P(0))^2 e^{ik\cdot Y(0)}. \tag{4.4}$$

where $\varepsilon_{\mu\nu} = \epsilon_{(\mu}\epsilon_{\nu)}$ is the graviton's polarization. The amplitude is obtained from a correlator of local insertions of these operators. First we apply a vertex operator to the incoming vacuum, propagate this state using the worldsheet propagator Δ, act with another vertex operator, and so on, until we contract with the outgoing vacuum. That is, at four points,

$$\langle \epsilon_1; k_1 | V_2(0,0) \Delta V_3(0,0) | \epsilon_4; k_4 \rangle. \tag{4.5}$$

The full amplitude is obtained by summing over permutations of the external particles. We use the following expression for the worldsheet propagator;

$$\Delta = \frac{\delta(L_0 - 2)}{M_0} = \int d\rho d\phi e^{-\rho M_0} e^{-i\phi(L_0-2)}. \tag{4.6}$$

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This formulation is closely related to one used in [76] for the HSZ string and has its origin in the descent procedure from [5]. It would be interesting to compare this expression with the expression derived rigorously in [6].

The zero point energy contribution for L_0 occurs when one picks the ambitwistor vacuum (3.9), also defined in terms of the p_n, y_m modes:

$$p_n|0\rangle = 0; \qquad y_n|0\rangle = 0 \qquad \forall n > 0, \tag{4.7}$$

and equivalently the following operator ordering

$$: y_n p_m := \begin{cases} y_n p_m & \text{if} \quad m > 0 \\ p_m y_n & \text{if} \quad n > 0 \end{cases} \tag{4.8}$$

which is the appropriate one here. These operators are responsible for moving vertex operators along the worldsheet as

$$e^{-\rho M_0} e^{2i\pi\phi L_0} V(0,0) e^{\rho M_0} e^{i\phi L_0} = V(\rho,\phi) \tag{4.9}$$

and are given by

$$L_0 = \sum_{n\in\mathbb{Z}} n : p_{-n} \cdot y_n : , \qquad M_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} p_{-n} \cdot p_n. \tag{4.10}$$

The correlator (4.5) becomes

$$\int d\rho d\phi \langle \epsilon_1; k_1 | V_2(0,0) V_3(\rho,\phi) | \epsilon_4; k_4 \rangle \,. \tag{4.11}$$

The only place where Y appear is in the exponentials, so the commutator between them and polynomials of P are easy to evaluate and will not have any dependence on ρ. The only term with non-trivial dependence on the modulus ρ is given by commuting the exponential parts of V_2 through the other vertex operators, for example

$$e^{ik_2\cdot Y_-(0)} e^{ik_3\cdot Y_+(\phi) + i\rho k\cdot P_+(\phi)} = e^{ik_3\cdot Y_+(\phi)+i\rho k\cdot P_+(\phi)} e^{ik_2\cdot Y_-(0)} e^{-i\rho k_2\cdot k_3 G(0,\phi)} \tag{4.12}$$

where the $Y_\pm(\phi) = \sum_{\pm n\geq 0} y_n e^{-in\phi}$ and the same for P. The function $G(\phi_1,\phi_2) = (1 - e^{-i(\phi_1-\phi_2)})^{-1}$ is the propagator on the cylinder. We give more details on its computation in the next section.

The full computation of the correlator for an n point scattering is actually done using the Baker-Campbell-Hausdorff formula. Its full dependence of on the moduli ρ comes in the exponential

$$\exp\left(i\rho\left(k_3\cdot k_1 + \frac{k_3\cdot k_2}{1-\frac{1}{z}}\right)\right) \tag{4.13}$$

with $z = e^{-i\phi}$. Here is where the complexification comes in. Complexifying the moduli and changing the integration contour of ρ,[10] such that the above exponential integrates to

[10]Together with a change of variable $\rho \to \frac{\rho}{z}$.

a delta-function, its argument coincides with the four point scattering equation

$$k_3 \cdot P(z) = \sum_{i \neq 3} \frac{k_3 \cdot k_i}{z - z_i} = 0. \tag{4.14}$$

Here the gauge $\{z_1, z_2, z_4\} = \{0, 1, \infty\}$ appears naturally. In the original coordinates this corresponds to picking $\{\sigma_1, \sigma_2, \sigma_4\} = \{i\infty, 0, -i\infty\}$, which can only be achieved with complex moduli.

The inclusion of fermions does not change the above calculation of the exponential factors, the same is true if more vertex operators are included. The dependence on the moduli associated to the Hamiltonian M_0 is always exponential and, by picking the right contour, can be integrated into the delta functions imposing the scattering equations. This way of obtaining the scattering equations is reminiscent of the descent procedure described in [5], but here we made no use of the CFT description. To recover actual gravity amplitudes we simply use the $\mathcal{N} = 2$ version of the null string and consider correlators of the form

$$\langle \epsilon_1; k_1 | V_2 \Delta V_3 \Delta \cdots \Delta V_{n-1} | \epsilon_n; k_n \rangle \tag{4.15}$$

and sum over permutations. The vertex operators have the form

$$V(0,0) = (\epsilon \cdot P + \epsilon \cdot \psi k \cdot \psi)^2 e^{ik \cdot Y}(0,0). \tag{4.16}$$

After expressing all the propagators in terms of moduli and commuting them through to the vacuum the calculation is essentially the same as in the ambitwistor string up to change of coordinates in the moduli space.

4.2 Cylinder propagator and n-point scattering equations

Here we give more details on the computation of the propagator $\langle XX \rangle$ on the cylinder using the operator formulation. A similar computation was performed proposed in [64] using a operator and path integral methods — we find agreement with these results. With this propagator we see how the scattering equations in the operator formalism arise from a contour deformation of the time variable τ. Similar observations were made in [36, 64]. It is important for us to revisit these analyses because it allows us to constrain further the complexification of the null string. Using the definitions of the previous section, the correlator is given by:

$$\langle X(\tau_1, \sigma_1) X(\tau_2, \sigma_2) \rangle = T\left(X(\tau_1, \sigma_1) X(\tau_2, \sigma_2)\right) - :X(\tau_1, \sigma_1) X(\tau_2, \sigma_2): \tag{4.17}$$

where $T(\ldots)$ and $:\ldots:$ denote time and normal ordering, respectively. The usual ordering would be τ-ordering, however the computation does not change if we use a σ-ordering. The reason why we make this comment is because there is an intuitive sense in which the ambitwistor normal ordering amounts to exchanging space and time on the worldsheet, as described by Siegel in [36].

Suppose $\tau_1 > \tau_2$, (or $\sigma_1 > \sigma_2$):

$$\begin{aligned}
\langle X(\tau_1,\sigma_1)X(\tau_2,\sigma_2)\rangle &= \sum_{n,m\in\mathbb{Z}} (y_n+\tau_1 p_n)(y_m+\tau_2 p_m) - :(y_n+\tau_1 p_n)(y_m+\tau_2 p_m): \\
&= \sum_{n>0,m<0} \big(\tau_1(p_n y_m - y_m p_n) + \tau_2(y_n p_m - p_m y_n)\big) e^{in\sigma_1 + im\sigma_2} \\
&= -i(\tau_1-\tau_2)\sum_{n>0} e^{in(\sigma_1-\sigma_2)}
\end{aligned} \tag{4.18}$$

finally giving

$$\langle X(\tau_1,\sigma_1)X(\tau_2,\sigma_2)\rangle = -i(\tau_1-\tau_2)\frac{z_1}{z_1-z_2} \tag{4.19}$$

where we put $z_i = \exp(i\sigma_i)$. In terms of σ and τ this can be rewritten $\langle X(\tau_1,\sigma_1)X(\tau_2,\sigma_2)\rangle = (\tau_1-\tau_2)\left(\cot\left(\frac{\sigma_1-\sigma_2}{2}\right)+1\right)/2$ where the invariance by translation symmetry is now obvious. The constant piece will drop out of the propagator by $1 \leftrightarrow 2$ symmetry, so we can as well remove it from the start. This amount to replace the previously derived propagator by

$$\langle X(\tau_1,\sigma_1)X(\tau_2,\sigma_2)\rangle = -\frac{i}{2}(\tau_1-\tau_2)\frac{z_1+z_2}{z_1-z_2}\,. \tag{4.20}$$

The null-string's Koba-Nielsen factor, abbreviated $\sum k_i\cdot k_j\langle X_i X_j\rangle$, then reduces to

$$\begin{aligned}
\sum k_i\cdot k_j\langle X_i X_j\rangle &= -\frac{i}{2}\sum_{i,j} k_i\cdot k_j \tau_{ij}\frac{z_i+z_j}{z_i-z_j} \\
&= -\frac{i}{4}\sum_{i<j} k_i\cdot k_j \tau_i\frac{z_i+z_j}{z_i-z_j} \\
&= -\frac{i}{2}\sum_{i=1}^{n}\tau_i z_i\left(\sum_{j=1}^{n} k_i\cdot k_j\frac{1+z_i/z_j}{z_i-z_j}\right)
\end{aligned} \tag{4.21}$$

where to go from the first to second line we used momentum conservation.

Then, as argued above, the τ_i integration should be complexified in such a way as to give rise to the scattering equations, (this last fact was originally proposed by Siegel in [36])

$$\int d\tau_i e^{E_i\tau_i} \sim \delta(E_i)\,. \tag{4.22}$$

with E_i the term in the parenthesis in eq. (4.21). Note that due to global GCA invariance, there are only $n-3$ independent GCA cross ratios and hence $n-3$ scattering equations. In our present case, with the conformal mapping $\sigma \to \exp(i\sigma)$ used here, the scattering equations appear first as

$$\forall i = 1,\dots,n-3\,,\qquad E_i = \sum_j k_i\cdot k_j\left(\frac{1}{z_{ij}}+\frac{z_j}{z_i z_{ij}}\right) = 0\,. \tag{4.23}$$

Using the partial fraction identity $\frac{z_j}{z_i z_{ij}} = \frac{1}{z_{ij}} - \frac{1}{z_i}$ and momentum conservation they reduce to the CHY scattering equations. The extra factor of z_i with τ_i in the exponential finally ensures that the measure is invariant. When $\tau_i \to \tilde{\tau}_i = z_i\tau_i$ and $\sigma_i \to z_i = \exp(i\sigma_i)$:

$$d\sigma_i d\tau_i \to dz_i d\tilde{\tau}_i \tag{4.24}$$

up to numerical factors of $2i\pi$.

4.3 Partition function

The operator formalism can also be used to give a tentative calculation of the partition function. Consider the trace

$$\mathcal{Z}(\rho,\phi)=\mathrm{Tr}(\exp(2\pi i\phi P-2\pi\rho H)). \tag{4.25}$$

Here $P=L_0-\frac{c}{24}$ is the generator of translations in space along ϕ, and $H=M_0$ is the Hamiltonian generating time evolution along ρ. Here we have Wick rotated to Euclidean signature, hence the absence of a factor of i in front of the Hamiltonian. A generic state in the Hilbert space is given by polynomials of the negative modes y_n and p_n

$$|\phi_I\rangle=x_{a_1}^{\mu_1}\cdots x_{a_n}^{\mu_n}p_{b_1}^{\nu_1}\cdots p_{b_j}^{\nu_m}|k\rangle \tag{4.26}$$

where $|k\rangle=\exp(x_0\cdot k)|0\rangle$ is the vacuum with momenta k and I is a multi-index.

Acting with these translation operators on a generic state and tracing over gives

$$\mathcal{Z}(\rho,\phi)=\int\frac{dk}{(2\pi)}e^{-\pi\rho k^2}e^{2i\pi\phi c/24}\prod_{a=1}^{\infty}\prod_{b=1}^{\infty}\sum_{N_a=0}^{\infty}\sum_{N_b=0}^{\infty}e^{2\pi iaN_a\phi}e^{2\pi ibN_b\phi}. \tag{4.27}$$

Performing the Gaussian integral and the sum we arrive at

$$\mathcal{Z}_1(\rho,\phi)=(4\pi^2\rho)^{-1/2}q^{c/24}\prod_{b=1}^{\infty}(1-q^b)^{-2} \tag{4.28}$$

where $q=e^{2\pi i\phi}$ is in principle a complex number of unit modulus. In the above we neglected the spacetime indices of the oscillators,[11] so in D dimensions the partition function is

$$\mathcal{Z}(\rho,\phi)=(4\pi^2\rho)^{-D/2}q^{c/24}\left(\prod_{b=1}^{\infty}(1-q^b)^{-2}\right)^D. \tag{4.29}$$

Note how similar it is to the partition function of a (non-chiral) single boson

$$\mathcal{Z}_X=(4\pi^2\tau_2)^{-1/2}\left|q^{1/24}\prod_{n=1}^{\infty}(1-q^n)^{-1}\right|^2 \tag{4.30}$$

but in this case q is the modular parameter of the torus, not a unit norm complex number as in the null string.

Comparing to the partition function of the ambitwistor string found in [5] we see that there is an extra modulus, ρ, in the null string. Furthermore, the modulus q is the modular parameter of the torus in [5] while in the null string it is a complex number of unit norm. The ambitwistor string also has an explicit integration over the zero mode of P, leading to a loop-momentum integration. In the case of the null string the loop-momentum integral is exchanged for an integral over the extra modulus. We expect that it is this modulus ρ

[11] We also threw out a dimension dependent overall constant which is basically the volume of a $D-1$ sphere.

which controls the UV behaviour of the theory. From the previous sections we know that the moduli space of the complexified null string is the cotangent to the moduli space of Riemann surfaces. So it is natural to conjecture that the moduli space of the real null string is a some real cycle in this space. In fact, recent work in one-loop amplitudes in the null string [64] seems to support this hypothesis. The partition function computed in this paper by different methods seems to be the same as ours with a specific choice of contour.[12]

4.4 Comment on modular invariance

After complexifying we can imagine that the null string is a Galilean conformal field theory obtained by contracting some CFT. Then the parameters (ρ, ϕ) should inherit modular transformations from the parent theory, see [77–79]. With respect to the parent CFT Virasoro, the GCA zero mode operators are

$$L_0 = \mathcal{L}_0 - \bar{\mathcal{L}}_0$$
$$M_0 = \epsilon(\mathcal{L}_0 + \bar{\mathcal{L}}) \,.$$

Here, ϵ is a parameter that we will take to zero to perform the algebra contraction. Call ζ, $\bar{\zeta}$ the parameters associated[13] with $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$, respectively, then the GCA parameters are $2\phi = \zeta + \bar{\zeta}$ and $2\rho = \zeta - \bar{\zeta}$, associated to L_0 and M_0, respectively. The parameter ζ and its complex conjugate are the modular parameters of the torus carrying an action of the modular group $\mathrm{SL}(2,\mathbb{Z})$

$$\zeta \to \frac{a\zeta + b}{c\zeta + d}, \qquad a, b, c, d \in \mathbb{Z}, \qquad ab - dc = 1 \,. \tag{4.31}$$

When taking the limit, ρ scales as ϵ since it is associated with M_0. Making this explicit in the above and expanding to first order in ϵ gives

$$\phi + \epsilon\rho \to \frac{a(\phi + \epsilon\rho) + b}{c(\phi + \epsilon\rho) + d} = \frac{a\phi + b}{c\phi + d} + \epsilon \frac{\rho}{(c\phi + d)^2} \,. \tag{4.32}$$

The claim is then that the modular transformations for the null string are generated by

$$(\phi, \rho) \to (\phi + 1, \rho) \tag{4.33}$$

$$(\phi, \rho) \to \left(\frac{-1}{\phi}, \frac{\rho}{\phi^2} \right) . \tag{4.34}$$

With these transformations in hand we can examine how the partition function behaves under them. Rewriting it in terms of the eta-function $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ gives

$$Z_{XP} = (4\pi^2\rho)^{-D/2}(\eta(\phi))^{-2D}. \tag{4.35}$$

Under modular transformations the eta-function behaves as

$$\eta(\tau + 1) = \exp(i\pi/12)\eta(\tau)$$
$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \,.$$

[12] The part that was computed there was the matter part; it matches our expression, up to numerical factors. The integration contour was, there also, conjectured.

[13] In the sense of defining the partition function as above.

It's clear that under these transformations Z_{XP} picks up a phase. But all is not lost yet, so far we haven't included the ghost sector. Naively the partition function for the ghosts is just η^4. This is even worse since it picks up factors of ϕ under modular transformations. But the ghost partition function should not be taken into account without the anti-ghost insertions which builds the measure in the moduli space. Instead of deriving this measure we will assume modular invariance and show that it uniquely fixes the ghost partition function and the measure on the moduli space. The claim is that the ghost partition function is $\rho\eta(\phi)^4$ since this picks up a phase independent of ϕ under modular transformations. Combining these partition functions gives

$$Z_{XP}Z_g = (4\pi^2\rho)^{-D/2}(\eta(\phi))^{-2D}\rho(\eta(\phi))^4. \tag{4.36}$$

Its easy to see that the relative phases cancel when $D = 26$.[14] There is also a unique modular invariant measure in the space of (ϕ, ρ) which combines with the partition function to give

$$\int \frac{d\phi d\rho}{(\rho)^2}(4\pi^2\rho)^{-13}(\eta(\phi))^{-52}\rho(\eta(\phi))^4\,. \tag{4.37}$$

Note that in the above formula there was no need of assigning a modular transformation to the field P to get a modular invariant function like in [5]. As expected, its role has been taken over by the factor $(\rho)^{-13}$. An integrand that goes with it also will not depend on the zero mode of P, but will depend on a new modulus. Like the tree-level amplitude we expect this dependence to be exponential which might allow for new loop-level scattering equations without an explicit loop momentum.

As we mentioned, a one-loop amplitude in the bosonic null string has been proposed in [64]. Given a particular choice of contour the authors recovered scalar boxes in Schwinger parametrization. It would be very interesting to compute the one-loop amplitude using the above operator formalism and compare with their results.

5 Discussion

Summary. In this paper, we pushed the study of the null string into three different but related directions. First we complexified the worldsheet and target space where we noticed an emergent symmetry which does not preserve the original real contour. This symmetry is on-shell gauge equivalent to holomorphic diffeomorphisms and corresponds to translations along null geodesics which is the same as one of the gauge symmetries of the ambitwistor string. In the same section we also studied the moduli space of the null string and concluded it is the same as the ambitwistor string when viewed through the lens of this emergent symmetry.

Next we studied the role of the Galilean conformal algebra in the structure of the null string. We showed how the constraints of the null string restrict the state space to be the same as a chiral CFT. This motivates why one can use the usual state-operator

[14] That is sufficient, not necessary.

correspondence in these models. Then we showed how the residual symmetry acts locally and globally, and how to gauge fix it gives rise to a Jacobian which matches with the ambitwistor string ghosts correlator. In doing this we showed how the chiral gauge-fixing of the ambitwistor string translates into the gauge fixing of the nulls string and vice versa.

Lastly we looked at tree-level amplitudes using an operator formalism. There we showed explicitly at four points how the extra moduli of the null string can be used to obtain the tree-level amplitudes in the CHY form, that is, localized to the scattering equations. Next we calculated the cylinder propagator and gave an n-point argument for how the scattering equations appear at tree-level. We closed the section by calculating the partition function from operator methods, pointing out its differences and similarities with other ambitwistor partition functions in the literature and showed that our partition function is invariant under a conjectured action of modular transformations in the moduli space of the null string.

Perspectives. Going forward, there are many directions of research which this work opens.

First, a full treatment of the path integral in the real setting, if it makes sense, would be illuminating and might follow the lines advocated in [80]. The idea would be then to determine the complex integration cycle (that are known as Lefchetz thimbles [6, 31, 81]) by computing the intersection between the real and complex case.

It would also be very interesting to understand the details of the procedure sketched in section 4.2. In particular, it seems that there could be a choice in the order of integration, τ or z first. Even at tree-level doing so is difficult but could lead to a new representation of the CHY formulae. At loop-level, an interesting possibility arises, the loop momenta would naturally arise within the scattering equations instead of being an explicit variable of integration. If it is possible to do the z integral first, then the τ integral seems to reduce to a Schwinger proper-time parametrization. Evidence for this was proposed in [64]. However, we already mentioned that a lot of subtleties are present at loop-level, and it is not at all obvious that such a thing is possible. For this reason it will be necessary to understand further the moduli space of the null string at loop-level.

Recently another proposal for a gauge-unfixed version of the ambitwistor models in a first order setting was put forward in [82]. It argued that the resulting models are essentially topological, and the BRST localization [6] of the ambitwistor string on the scattering equations is essentially a kind of topological localization. It would be interesting to connect the two approaches and put in perspective the earlier results of [80].

Concerning supersymmetry, we mentioned that the analysis presented here can be carried straightforwardly in the RNS model of Mason-Skinner [4], or in the pure spinor version of the formalism [7, 12, 13].

In our previous paper [20] we noticed that there are chiral models in which the tension is still present as a free parameter, these were later studied in [27, 28, 36, 83]. It would be interesting to see if the methods developed in this paper can be applied to these models and how they relate to the null string and the usual string.

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Finally, and in relation with the comment at the beginning of section 3 on non-relativistic symmetries, it would be interesting to see if there exist other type of string models which could be quantized following the methods exposed in this paper. In particular, as recalled in [84], Kar claimed in [85] that Schild's strings (by opposition to our LST strings) enjoy a larger set of reparametrisations, spanning the full Newman-Unti group. They are given by $(\tau, \sigma) \to (f(\tau,\sigma), g(\sigma))$. It would be interesting to study the quantization of these strings and see if they can be related to LST strings.

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A Comments on worldline symmetries

The equivalence between the antiholomorphic diffeomorphisms and translations along null geodesics in the null string closely resembles a similar phenomenon of the particle action. Here we review the this equivalence in the worldline in order to illustrate what happens in the null string.

The worldline action for a massless particle is written in second order form as

$$S = \int \sqrt{-g} g^{\tau\tau} (\partial_\tau x)^2 . \tag{A.1}$$

This action is invariant under diffeomorphism. Under $\tau \to \epsilon(\tau)$, the metric transforms as

$$\delta g_{\tau\tau} = \epsilon \partial_\tau g_{\tau\tau} + 2 g_{\tau\tau} \partial_\tau \epsilon, \qquad \delta(\sqrt{g}) = \partial_\tau(\epsilon\sqrt{g}) \, . \tag{A.2}$$

We write the action in the first order formalism by introducing the canonical momenta p

$$S_{(p,x)} = \int \left(p\partial_t x - \frac{e}{2} p^2 \right) .$$

The corresponding equation of motion for p is $e^{-1}\partial_t x = p$. Importantly, since the above action is equivalent to (A.1) it still is diffeomorphism invariant. Under $\tau \to \tau + \epsilon(\tau)$ the fields transform as

$$\delta x = \epsilon \partial_t x, \qquad \delta p = \epsilon \partial_t p, \qquad \delta e = \partial_t(\epsilon e) \tag{A.3}$$

where we identify $e = \sqrt{g_{\tau\tau}}$. However, it is also the case that the gauge symmetries are generated by the constraints. In this case the constraint p^2 generates the gauge symmetry of the system

$$\delta x = \alpha p, \qquad \delta p = 0, \qquad \delta e = \partial_t \alpha \, . \tag{A.4}$$

These two symmetries should be somehow equivalent, except that in the Hamiltonian form we usually discard time parametrization, as these are produced by changing the values of the Lagrange multipliers in the extended Hamiltonian.

Henneaux and Teitelboim describe this phenomenon in their book ([45, chapter 3.1.5, *Trivial gauge transformations*]) in some details. The important fact to notice here is that the two symmetries just differ by a trivial "equation of motion symmetry". In other words, an α transformation is equal, on-shell, ($\partial_t p = 0$) to a diffeomorphism, with parameter $\epsilon = e\alpha$. The gauge transformation that is obtained from the difference between these two is a trivial gauge transformation. These trivial transformations that vanish on-shell can always be written as [45, theorem 3.1],

$$\delta' y^i = \epsilon^{ij} \frac{\delta S}{\delta y^j} \tag{A.5}$$

for canonical variables y^i with action S and, crucially, ϵ^{ij} some antisymmetric variable. In our case, (A.3)–(A.4) gives

$$\begin{aligned} \delta' x &= \epsilon(\partial_t x - ep) = \frac{\delta S_{(p,x)}}{\delta p} \\ \delta' p &= \epsilon \partial_t p = -\frac{\delta S_{(p,x)}}{\delta x} \,. \end{aligned} \tag{A.6}$$

These transformations form an ideal within the set of gauge transformation (their commutator with other always give another equation of motion symmetry). They should be disregarded, and a way to see this is that the associated charge is a function that vanishes identically.

Something very similar happens in the complexified null string. The antiholomorphic or τ diffeomorphisms are equivalent on-shell to the scaling symmetry present in the ambitwistor string generated by the P^2 constraint.

It would be interesting to revisit this analysis using the light-front formalism developed in [86] to understand more conceptually the constraint analysis presented here.

B Electrostatic equilibrium

It was observed long ago that the scattering equations actually describe an electrostatics equilibrium on the sphere [30, 87]. We comment on this observation from the point of view of the real null string.

Starting from the real LST action, the insertion of plane wave vertex operators in the path integral, induces the addition of source terms to the action, which play the role of boundary conditions in the path integral:

$$\int d^2\sigma \left(V^\alpha V^\beta \partial_\alpha X \cdot \partial_\beta X + i \sum_{j=1}^{n} k_j \cdot X(\sigma,\tau) \delta^{(2)}(\sigma - \sigma_j, \tau - \tau_j) \right) . \tag{B.1}$$

The corresponding X equations of motion read

$$\partial_\alpha (V^\alpha V^\beta \partial_\beta X^\mu) + i \sum_{j=1}^{n} k_j^\mu \delta^{(2)}(\sigma - \sigma_j, \tau - \tau_j) = 0 \,. \tag{B.2}$$

We want to interpret the following vector field as our electric field (or rather a collection of electric fields, for $\mu = 0, \ldots, D-1$)

$$\tilde{E}_\mu^\alpha = V^\alpha V^\beta \partial_\beta X_\mu \,. \tag{B.3}$$

This vector field has a density weight, which we can compensate by introducing an auxiliary metric g on the worldsheet, so we should

$$\sqrt{-g} E_\mu^\alpha = V^\alpha V^\beta \partial_\beta X_\mu \tag{B.4}$$

so that E is then a proper vector field. The equation of motion (B.2) then gives straight away Gauss's law in the presence of sources. It would be interesting to work out the similar configuration at loop level, pushing further the analysis of [16].

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