THE POLITICAL BUSINESS CYCLE: ENDOGENOUS ELECTION TIMING &

HYPERBOLIC MEMORY DISCOUNTING

by

Jake R. Cottle

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

In

Economics

Approved:

T. Scott Findley, Ph.D. Major Professor	Christopher Fawson, Ph.D. Committee Member
Lucas Rentschler, Ph.D. Outside Member	Richard S. Inouye, Ph.D. Vice Provost of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

2019

ABSTRACT

The Political Business Cycle: Endogenous Election Timing & Hyperbolic Memory Discounting

by

Jake R. Cottle, Master of Science

Utah State University, 2019

Major Professor: Dr. T. Scott Findley
Department: Economics and Finance

In the last 47 years there has been significant research on political determinants of macroeconomic cycles. The goal of this paper is to add to this line of research by extending the theoretical and empirical work of Nordhaus, D. Chappell, D.A. Peel, and Scott Findley. Nordhaus (1975) provided us with the first model depicting a rational politician who utilized monetary policy (i.e. inflation rate) to target levels of inflation and unemployment to maximize the number of votes needed for reelection. Chappel and Peel (1979) endogenized time in order to reflect governments such as England and Japan where the incumbent government selects the time of election that maximizes their total number of votes. Findley (2015), utilizing research from psychology, takes Nordhaus's original model and instead of voter's memories decaying exponentially he describes a voter whose memories decay hyperbolically. The idea of hyperbolic memory discounting is well documented in psychology literature and is described as a more accurate function of how individuals' memories decay over time. In this paper, I utilized Nordhaus's original model and added to it the endogenized time variable, Chappel and Peel (1979) and hyperbolic memory discount function, Findley (2015). I then compared the optimal date of election, E*, between the two models that differ only in how voters' weight past political-economic conditions. I then looked at the time paths of the inflation and unemployment functions to describe their behavior in relation to one another.

(38 pages)

PUBLIC ABSTRACT

The Political Business Cycle: Endogenous Election Timing & Hyperbolic Memory Discounting

Jake R. Cottle

In the models analyzed in this paper, there exists an incumbent politician with one objective, two choices, and voters who remember the past differently. The politician's primary goal is to get reelected, which is done by maximizing the number of votes on the day of election. The politician can increase their chances of reelection if they influence the state of the economy over time and ensure the economy is in its 'best' state on the days leading up to the election

In conducting this research, I wanted to study how different rates of memory decay influences the choices the politician makes during the course of their term. Also, I wanted to explore how long a politician would wait to have an election if that were a choice they could make. I found that voters who remember more of the past place a greater constraint on the incumbent leading to moderate fluctuations in the economy and frequent elections.

ACKNOWLEDGMENTS

I would like to thank my committee members, Dr. T. Scott Findley, Dr. Christopher Fawson, and Dr. Lucas Rentschler, for their support and assistance throughout the entire process.

I give special thanks to my family, friends, and colleagues for their encouragement, moral support, and patience as I worked my way from the initial proposal writing to this final document. I could not have done it without all of you.

Jake R. Cottle

CONTENTS

LIST OF TABLES

LIST OF FIGURES

Introduction

This paper explores how the predictions of the vintage political business cycle framework are altered when some key assumptions are relaxed in the vintage framework. Building on published work by Nordhaus (1975), Chappell and Peel (1979), and Findley (2015), I re-examine the standard framework by studying what happens when voters weight past political-economic conditions differently. Specifically, I study what happens to the predictions of the model for the optimal time paths of the unemployment rate, the inflation rate, in addition to the optimal timing of the election date, when voters weight the past hyperbolically (consistent with Jost's Second Law of Forgetting), instead of exponentially. As I report below, I find that hyperbolic memory discounting has first-order impacts on the key predictions of the benchmark political business cycle framework, especially on the prediction of when to select the timing of elections, in order to improve the chances of re-election.

1. "The Political Business Cycle" (Nordhaus 1975)

Nordhaus (1975) was a catalyst for what would become a long line of research dedicated to modeling the interaction between government and macro-economic conditions. Nordhaus was influenced by the earlier work of Kalecki (1943) who challenged the 'benevolent dictator' image of the government painted by early Keynesians, who believed that the government can and should provide economic stability through monetary and fiscal policies, implying that government wanted stability. Instead, Kalecki (1943) qualitatively described a world in which government (politicians) created recessions in order to weaken the bargaining power of workers on behalf of captains of industry.

Nordhaus (1975) was the first to quantitatively model macroeconomic conditions with these new political considerations brought to light by Kalecki (1943). The framework represents a world in which elected policymakers pull the levers of monetary and fiscal policy in order to influence the unemployment rate and the inflation rate, to make economic conditions favorable for reelection (maximize votes). Voters are assumed to prefer stable prices and low unemployment rates and they care about the past and present state of the economy when it comes to how they decide to vote. Periodically, the electorate is faced with a choice between two notional parties (economic ideologies/preferences), if the condition of the economy is better than some arbitrary expectation of 'usual' behavior than they vote for the incumbent, if not it’s a vote against the incumbent.

The solutions to the benchmark framework prescribe the policies that an incumbent politician should take over the course of the electoral regime in order to maximize the chances of reelection. Nordhaus's model results in policies that would lead to a politically induced business cycle with higher unemployment immediately after election, to combat high inflation, followed by a constant decrease in the unemployment rate over the entire election period. The model is formally described under Section 2 with an extension from Chappell and Peel (1979).

Post Nordhaus, there have been several papers examining the relationship between government and economic conditions. One of the main threads has been whether voters are forward-looking or backward-looking when deciding how to vote.

Retrospective voting behavior is surveyed extensively in Murakami (2008) and is reported to be more prevalent than previously reported. Retrospective voting is a referendum on the incumbent governments past performance relative to some arbitrary standard of performance, completely discounting future political promises (Hibbs 2006). Whereas, prospective voting is described as investing in the rationally formed expectations of future benefit discounting previous behavior (Hibbs 2006). Initially retrospective voting was considered to be a requirement to observe a cyclical pattern in macroeconomic outcomes; however, research since Nordhaus (1975) has identified that asymmetric information between the incumbent government and the voter is sufficient to provide the same cyclical effects (Snowden 1997). Given the large empirical evidence in both political science and psychology research, I will use the assumption of a retrospective voter.

2. Replication: "The political theory of the business cycle" Chappell and Peel (1979)

a. Overview

Chappell and Peel (1979) extended Nordhaus's original model by allowing for the timing of elections to be a choice of the incumbent, rather than being exogenously fixed. This implies that an incumbent government is free to call an election when it is in the incumbent's best interest, as compared to systems in which the timing of elections is prescribed by law, such as every four years in the United States. This model better represents the set of political opportunities available to incumbent governments in a country like the United Kingdom or Japan. In the United Kingdom, an incumbent government is free to call an election when they want but must do so no later than five years from the previous election. The model of Chappell and Peel (1979) models this political opportunity in the following way:

b. The Model

The Vote Function is

$$W_T = \int_0^T [-u^2 - \beta\rho] e^{rt}\, dt \qquad (1)$$

where

u = unemployment rate

ρ = rate of inflation

$\beta, \alpha_0, \alpha_1$ = positive constant

r = weighting factor

T = date of the next election

The Short-Run Phillips curve:

$$\rho = \alpha_0 - \alpha_1 u + \upsilon \qquad (2)$$

The time evolution of inflation expectations is governed by:

$$\frac{d\upsilon}{dt} = \phi(\rho - \upsilon) \qquad (3)$$

ϕ = positive constant

The government maximizes (1) subject to (2) and (3).

The Hamiltonian is:

$$H[u(t), \upsilon(t), \lambda(t), t] = -u^2 e^{rt} - \beta e^{rt}(\alpha_0 - \alpha_1 u + \upsilon) + \lambda\phi(\alpha_0 - \alpha_1 u) \qquad (4)$$

λ = continuous function of time (co-state variable)

Necessary conditions for an interior optimum are:

$$\frac{\partial H[\bullet]}{\partial u} = -2ue^{rt} + \beta\alpha_1 e^{rt} - \lambda\phi\alpha_1 \overset{Set}{=} 0$$

$$-2ue^{rt} = \lambda\phi\alpha_1 - \beta\alpha_1 e^{rt}$$

$$u = -\left(\frac{\lambda\phi\alpha_1 - \beta\alpha_1 e^{rt}}{2e^{rt}}\right)$$

$$u = \frac{\alpha_1}{2}\left(\beta - \lambda\phi e^{-rt}\right) \tag{5}$$

$$\frac{\partial H[\bullet]}{\partial \upsilon} = -\beta e^{rt} \overset{Set}{=} -\frac{d\lambda}{dt}$$

$$\frac{d\lambda}{dt} = \beta e^{rt} \tag{6}$$

c. Solving for the Optimal Time Paths for a Given Election Date

The first step in solving the system of differential equations, (3) and (6), is to find $\lambda(t)$:

$$\lambda(t) = \int \frac{d\lambda}{dt}\,dt = \int \beta e^{rt}\,dt = \frac{\beta}{r}e^{rt} + c_1 \tag{7}$$

where c_1 is a constant of integration. Apply the transversality condition $\lambda(T) = 0$ to find c_1:

$$\lambda(T) = \frac{\beta}{r}e^{rT} + c_1 = 0$$

$$c_1 = -\frac{\beta}{r}e^{rT}$$

Insert c_1 into (7) to obtain equation (8):

$$\lambda(t) = \frac{\beta}{r}e^{rt} - \frac{\beta}{r}e^{rT} = \frac{\beta}{r}(e^{rt} - e^{rT}) \tag{8}$$

Insert (8) into (5) in order to find $u^*(t)$, the optimal path of the unemployment rate, which is a function of a given election date, T.

$$u^*(t) = \frac{\alpha_1}{2}\left(\beta - \frac{\beta}{r}(e^{rt} - e^{rT})\phi e^{-rt}\right)$$
$$= \frac{\alpha_1\beta}{2r}[r - (e^{rt} - e^{rT})\phi e^{-rt}]$$
$$= \frac{\alpha_1\beta}{2r}[r - \phi e^{rt}e^{-rt} + \phi e^{rT}e^{-rt}]$$

$$u^*(t) = \frac{\alpha_1\beta}{2r}[r - \phi + \phi e^{r(T-t)}] \tag{9}$$

To solve for $\upsilon^*(t)$, the optimal path of the expected inflation rate, recall equation (3) and integrate both sides.

$$\frac{d\upsilon(t)}{dt} = \phi[\rho - \upsilon(t)]$$
$$\frac{d\upsilon(t)}{dt} = \phi[\alpha_0 - \alpha_1 u(t) + \upsilon(t) - \upsilon(t)]$$
$$\frac{d\upsilon(t)}{dt} = \phi[\alpha_0 - \alpha_1 u(t)]$$

$$\upsilon(t) = \int \frac{d\upsilon(t)}{dt}dt = \int^{t} \{\phi[\alpha_0 - \alpha_1 u(s)]\}\, ds + c_2 \tag{10}$$

where c_2 is a constant of integration.

Set $t = 0$ to solve for c_2:

$$\upsilon(0) = \int^{0} \{\phi[\alpha_0 - \alpha_1 u(s)]\}\, ds + c_2$$

$$c_2 = \upsilon_0 - \int^{0} \{\phi[\alpha_0 - \alpha_1 u(s)]\}\, ds$$

where $\upsilon(0) = \upsilon_0$. Insert c_2 into (10):

$$\upsilon(t) = \int \frac{d\upsilon(t)}{dt}\,dt = \int^{t} \{\phi[\alpha_0 - \alpha_1 u(s)]\}\,ds + \upsilon_0 - \int^{0} \{\phi[\alpha_0 - \alpha_1 u(s)]\}\,ds$$

$$\upsilon(t) = \upsilon_0 + \int_0^t \{\phi[\alpha_0 - \alpha_1 u(s)]\}\,ds$$

$$\upsilon(t) = \upsilon_0 + \int_0^t \phi\alpha_0\,ds - \int_0^t \phi\alpha_1 u(s)\,ds$$

$$\upsilon(t) = \upsilon_0 + \phi\alpha_0 t - \phi\alpha_1 \int_0^t u(s)\,ds \tag{11}$$

Side calculation for $\int_0^t u(s)\,ds$ utilizing (9):

$$\int_0^t u(s)\,ds = \int_0^t \frac{\alpha_1\beta}{2r}[r - \phi + \phi e^{r(T-s)}]\,ds$$

$$= \frac{\alpha_1\beta}{2r}[\int_0^t r\,ds - \int_0^t \phi\,ds + \int_0^t \phi e^{r(T-s)}\,ds]$$

$$= \frac{\alpha_1\beta}{2r}\{rt - \phi t + [\phi e^{r(T-s)}(-\frac{1}{r})]_0^t\}$$

$$= \frac{\alpha_1\beta}{2r}\{rt - \phi t + \phi e^{rT}(-\frac{1}{r}e^{-rt} + \frac{1}{r})\}$$

$$\int_0^t u(s)\,ds = \frac{\alpha_1\beta}{2r}\{rt - \phi t - \frac{\phi e^{rT}}{r}(e^{-rt} - 1)\} \tag{12}$$

Substitute (12) into (11) to obtain $\upsilon^*(t)$:

$$\upsilon(t) = \upsilon_0 + \phi\alpha_0 t - \phi\alpha_1[\frac{\alpha_1\beta}{2r}\{rt - \phi t - \frac{\phi e^{rT}}{r}(e^{-rt} - 1)\}]$$

$$\upsilon(t) = \upsilon_0 + \phi\alpha_0 t - \frac{\phi\alpha_1{}^2\beta}{2r}\{rt - \phi t + \frac{\phi e^{rT}}{r}(1 - e^{-rt})\}$$

$$\upsilon(t) = \upsilon_0 + \phi\alpha_0 t - \frac{r\phi\alpha_1{}^2\beta t}{2r} + \frac{\phi^2\alpha_1{}^2\beta t}{2r} - \frac{\phi^2\alpha_1{}^2\beta e^{rT}(1 - e^{-rt})}{2r^2}$$

$$\upsilon^*(t) = \upsilon_0 + \phi[\alpha_0 - \frac{\alpha_1{}^2\beta(r - \phi)}{2r}]t - \frac{\phi^2\alpha_1{}^2\beta e^{rT}(1 - e^{-rt})}{2r^2} \tag{13}$$

which determines the actual inflation rate via (2).

d. Solving the Optimal Election Date

Now that the optimal paths for the unemployment rate, the expected inflation rate, and the actual inflation rate have been derived, they can be inserted into the Hamiltonian to find the optimal date of re-election from the incumbent's perspective, T^*.

$$M(T)=\sup_{\{u\}} H[u^*(T),\upsilon^*(T),\lambda(T),T]=0 \tag{14}$$

$$M(T)=-u^*(T)^2e^{rT}-\beta e^{rT}[\alpha_0-\alpha_1u^*(T)+\upsilon^*(T)]+\lambda(T)\phi[\alpha_0-\alpha_1u^*(T)]=0$$

Remembering that the transversality condition is $\lambda(T)=0$, the optimality condition becomes:

$$M(T)=-u^*(T)^2e^{rT}-\beta e^{rT}[\alpha_0-\alpha_1u^*(T)+\upsilon^*(T)]=0 \tag{15}$$

Evaluate (9) and (13) at $t=T$:

$$u^*(T)=\frac{\alpha_1}{2}\left(\beta-\lambda(T)\phi e^{-rT}\right)=\frac{\alpha_1\beta}{2} \tag{16}$$

$$\upsilon^*(T)=\upsilon_0+\phi[\alpha_0-\frac{\alpha_1{}^2\beta(r-\phi)}{2r}]T-\frac{\phi^2\alpha_1{}^2\beta e^{rT}(1-e^{-rT})}{2r^2} \tag{17}$$

and then insert (16) and (17) into (15):

$$M(T)=-(\frac{\alpha_1\beta}{2})^2e^{rT}-\beta e^{rT}[\alpha_0-\alpha_1(\frac{\alpha_1\beta}{2})+\{\upsilon_0+\phi[\alpha_0-\frac{\alpha_1{}^2\beta(r-\phi)}{2r}]T-\frac{\phi^2\alpha_1{}^2\beta e^{rT}(1-e^{-rT})}{2r^2}\}]=0$$

$$\beta e^{rT}\{-\frac{\alpha_1{}^2\beta}{4}-\alpha_0+\frac{\alpha_1{}^2\beta}{2}-\upsilon_0-\phi[\alpha_0+\frac{\alpha_1{}^2\beta(\phi-r)}{2r}]T+\frac{\phi^2\alpha_1{}^2\beta(e^{rT}-1)}{2r^2}\}=0$$

Divide both sides by βe^{rT}:

$$-\frac{\alpha_1{}^2\beta}{4}-\alpha_0+\frac{\alpha_1{}^2\beta}{2}-\upsilon_0-\phi[\alpha_0+\frac{\alpha_1{}^2\beta(\phi-r)}{2r}]T+\frac{\phi^2\alpha_1{}^2\beta(e^{rT}-1)}{2r^2}=0$$

$$\frac{\alpha_1{}^2\beta}{2}-\frac{\alpha_1{}^2\beta}{4}-\alpha_0-\upsilon_0-\phi[\alpha_0+\frac{\alpha_1{}^2\beta(\phi-r)}{2r}]T+\frac{\phi^2\alpha_1{}^2\beta e^{rT}}{2r^2}-\frac{\phi^2\alpha_1{}^2\beta}{2r^2}=0$$

Rearrange algebraically further:

$$\frac{\alpha_1{}^2\beta}{2}-\frac{\alpha_1{}^2\beta}{4}-\alpha_0-\upsilon_0-\phi[\alpha_0+\frac{\alpha_1{}^2\beta(\phi-r)}{2r}]T+\frac{\phi^2\alpha_1{}^2\beta e^{rT}}{2r^2}+\{\frac{\phi r\alpha_1{}^2\beta e^{rT}}{2r^2}-\frac{\phi r\alpha_1{}^2\beta e^{rT}}{2r^2}\}-\frac{\phi^2\alpha_1{}^2\beta}{2r^2}=0$$

$$\frac{\alpha_1{}^2\beta}{2}-\frac{\alpha_1{}^2\beta}{4}-\alpha_0-\upsilon_0-\phi[\alpha_0+\frac{\alpha_1{}^2\beta(\phi-r)}{2r}]T+[\frac{\phi\alpha_1{}^2\beta(\phi-r)}{2r^2}]e^{rT}+\frac{\phi\alpha_1{}^2\beta e^{rT}}{2r}-\frac{\phi^2\alpha_1{}^2\beta}{2r^2}=0$$

Rearrange algebraically a second time:

$$[\frac{\phi\alpha_1{}^2\beta(\phi-r)}{2r^2}]e^{rT}-\phi[\alpha_0+\frac{\alpha_1{}^2\beta(\phi-r)}{2r}]T-\alpha_0-\upsilon_0+\frac{\phi\alpha_1{}^2\beta e^{rT}}{2r}+$$
$$\frac{\alpha_1{}^2\beta r}{2r}-\frac{\phi r\alpha_1{}^2\beta}{2r^2}-\frac{\alpha_1{}^2\beta}{4}-\frac{\phi^2\alpha_1{}^2\beta}{2r^2}+\frac{\phi r\alpha_1{}^2\beta}{2r^2}=0$$

Define the inherited rate of inflation, P_0, in the following way:

$$P_0=\alpha_0+\upsilon_0-\frac{\phi\alpha_1{}^2\beta e^{rT}}{2r}-\frac{\alpha_1{}^2\beta r}{2r}+\frac{\phi r\alpha_1{}^2\beta}{2r^2} \quad (18)$$

which implies

$$[\frac{\phi\alpha_1{}^2\beta(\phi-r)}{2r^2}]e^{rT}-\phi[\alpha_0+\frac{\alpha_1{}^2\beta(\phi-r)}{2r}]T-[P_0+\frac{\alpha_1{}^2\beta}{4}+\frac{\alpha_1{}^2\beta\phi(\phi-r)}{2r^2}]=0 \quad (19)$$

e. Numerical Examples

In order to numerically replicate the findings in Chappell and Peel (1979), values for observable parameters are taken directly from their paper. Moreover, equation (18) is solved in terms of υ_0, given that actual inflation rates are observable, while expectations are more difficult to observe empirically. This means that υ_0 is a function of the parameters of the model, including P_0 and T.

Chappell and Peel (1979) obtained a value of $T^*=14.66$ years for their first set of parameter values. Utilizing similar analytical methods and parameter values, I obtain a value of $T^*=13.43$ years. It appears that the discrepancy is the result of the fact that computational abilities have progressed significantly between 1979 and the present. To further investigate this conjecture, upon consultation, I numerically depict in Figure 1 how (15)-(17), or alternatively, how (19) behaves over a reasonable range of T values. Figure 1 shows the values of the Hamiltonian from $T = 0$ to $T = 15$. The values of the Hamiltonian range from -0.0043 to 0.0023. The value of the Hamiltonian at $T = 14.66$ is $.00181$, whereas the $T = 13.43$ that I get has a Hamiltonian value of $-.0000033$. It is therefore likely that improvements in computing have made it possible to more precisely approximate the point at which the Hamiltonian is exactly equal to 0. As such, I now proceed with the main analysis of the thesis.

3. Hyperbolic Memory Discounting and the Timing of Elections

a. Overview

In another extension of Nordhaus's seminal work, Findley (2015) evaluated an alternative form of the memory discount function. Taking a lesson from the field of psychology, Findley (2015) built on a body of research findings that given two memories of the same strength, the older will decay at a lower rate than the more recent memory. This phenomenon is known in psychology as Jost's Second Law of Forgetting. The hyperbolic discount function is the way to mathematically represent Jost's Second Law, and thus is utilized as a comparison to the exponential function where memories decay at a constant rate over time. Findley (2015) presents a very focused research objective, which is to examine how the predictions (u^* and π^*) of the vintage political business cycle model are affected when voters forget the past hyperbolically, holding all else constant.

The main finding of Findley (2015) is that the amplitudes of the unemployment and inflation rates are moderated for the hyperbolic case as compared to the case of exponential memory discounting. The lesson implied by the findings of Findley (2015) is that an incumbent is hindered in their ability to manipulate the economy at the early part of the electoral term. This result stems from the fact that hyperbolic voters retain more of their older memories relative to exponential voters.

The following sub-sections will formally present the two political business cycle models with endogenous dates of election timing and differing memory discount functions, exponential (sub-section b) and hyperbolic (sub-section c). I will utilize the model and notation of Findley (2015) hereafter.

b. The Model

The model follows Findley (2015) and parallels the solution technique of Chappell and Peel (1979). The objective of an incumbent government is to maximize the vote function at time E, the date of the next election, by choosing values of u, the unemployment rate, while taking into account how well voters remember the past. The assumption of a vote-maximizing politician is just one of potentially many motives that politicians might have and provides a tool kit and a set of properties that allow for the following analysis.

$$\max \quad V = \int_0^E \upsilon\left[u(t), \pi(t)\right] M\left(E-t\right) dt \tag{20}$$

where $M\left(E-t\right)$ is the memory discount function in general form.

The incumbent government is constrained only by an expectations-augmented short-run Phillips Curve relationship

$$\pi(t) = \alpha - \zeta u(t) + \psi \varepsilon(t) \tag{21}$$

where ψ is the proportion by which expected inflation materializes into actual inflation, which for our case will be set equal to one, suggesting that the long-run Phillips Curve is vertical at the natural rate of unemployment. As in Nordhaus (1975), the expected rate of inflation, $\varepsilon(t)$, evolves adaptively

$$\frac{d\varepsilon(t)}{dt} = \gamma\left[\pi(t) - \varepsilon(t)\right]$$

where

u = unemployment rate

π = rate of inflation

ζ = slope coefficient of the short-run Phillips Curve

ρ = rate of exponential memory decay

β = hyperbolic discount function parameter

θ, α = positive constants

E = time between elections

Following Findley (2015), the optimal solutions $\left[u^*(t), \pi^*(t)\right]$ in general form, as obtained from Findley (2015), are

$$u^*(t) = \frac{\theta\zeta}{2}\left[1 + \frac{\gamma}{M(E-t)}\int_t^E M(E-s)\,ds\right] \tag{22}$$

$$\pi^*(t) = \alpha - \zeta u^*(t) + \left[\varepsilon(0) + \int_0^t \left[\alpha\gamma - \gamma\zeta u^*(j)\right]dj\right] \tag{23}$$

c. Exponential Memory Discounting: Derivation of $u_e^*(E)$ and $\pi_e^*(E)$

Evaluate (22) and (23) at E:

$$u^*(E) = \frac{\theta\zeta}{2}\left[1 + \frac{\gamma}{M(E-E)}\int_E^E M(E-s)\,ds\right] = \frac{\theta\zeta}{2} \tag{24}$$

$$\pi^*(E) = \alpha - \frac{\theta\zeta^2}{2} + \varepsilon(0) + \int_0^E \left\{ \alpha\gamma - \frac{\gamma\theta\zeta^2}{2}\left[1 + \frac{\gamma}{M(E-j)}\int_j^E M(E-s)\,ds\right]\right\} dj$$
(25)

Assuming that the memory discount function takes the exponential form:

$$\pi_e^*(E) = z + \int_0^E \left\{ \alpha\gamma - \frac{\gamma\theta\zeta^2}{2}\left[1 + \frac{\gamma}{e^{-\rho(E-j)}}\int_j^E e^{-\rho(E-s)}ds\right]\right\} dj$$
$$= z + \int_0^E \left\{ \alpha\gamma - \frac{\gamma\theta\zeta^2}{2}\left[1 + \frac{\gamma e^{\rho(E-j)}}{\rho} e^{-\rho(E-s)}\right]_j^E\right\} dj$$
$$= z + \int_0^E \left\{ \alpha\gamma - \frac{\gamma\theta\zeta^2}{2}\left[1 + \frac{\gamma e^{\rho(E-j)}}{\rho} - \frac{\gamma}{\rho}\right]\right\} dj$$

$$\pi_e^*(E) = z + \alpha\gamma E - \frac{\gamma\theta\zeta^2 E}{2} + \frac{\gamma^2\theta\zeta^2 E}{2\rho^2}\left(1 - e^{\rho E}\right) + \frac{\gamma^2\theta\zeta^2 E}{2\rho}$$
$$z = \alpha - \frac{\theta\zeta^2}{2} + \varepsilon(0) \qquad (26)$$

d. Hyperbolic Memory Discounting: Derivation of $u_h^*(E)$ and $\pi_h^*(E)$

From (25), assuming that the memory discount function takes the hyperbolic form:

$$\pi_h^*(E) = \alpha - \frac{\theta\zeta^2}{2} + \varepsilon(0) + \int_0^E \left\{ \alpha\gamma - \frac{\gamma\theta\zeta^2}{2}\left[1 + \frac{\gamma}{\left[1+\beta(E-j)\right]^{-1}}\int_j^E \left[1+\beta(E-s)\right]^{-1} ds\right]\right\} dj$$

Replace: $z = \alpha - \frac{\theta\zeta^2}{2} + \varepsilon(0)$:

$$\pi_h^*(E) = z + \int_0^E \left\{ \alpha\gamma - \frac{\gamma\theta\zeta^2}{2}\left[1 + \frac{\gamma}{\left[1+\beta(E-j)\right]^{-1}}\left[\frac{\ln(1+\beta E-\beta s)}{-\beta}\right]_j^E\right]\right\} dj$$
$$= z + \int_0^E \left\{ \alpha\gamma - \frac{\gamma\theta\zeta^2}{2}\left[1 + \frac{\gamma\left[1+\beta(E-j)\right]}{\beta}\ln(1+\beta E-\beta j)\right]\right\} dj$$

$$\pi_h{*}(E) = z + \alpha\gamma E - \frac{\gamma\theta\zeta^2 E}{2} - \frac{\gamma^2\theta\zeta^2}{2\beta}\int_0^E \left(1+\beta E-\beta j\right)\ln(1+\beta E-\beta j)\,dj \quad (27)$$

Side calculation for $\int\left(1+\beta E-\beta j\right)\ln(1+\beta E-\beta j)dj$:

Substitute $w = 1+\beta E-\beta j$:

$$\begin{aligned} w &= 1+\beta E-\beta j \\ \frac{dw}{dj} &= -\beta j \\ -\frac{1}{\beta j}dw &= dj \\ -\frac{1}{\beta}&\int w\ln(w)dw \end{aligned}$$

Partial Fraction Decomposition:

$$\begin{aligned} v &= \frac{1}{2}w^2 \\ dv &= w\,dw \\ u &= \ln(w) \\ du &= \frac{1}{w}dw \end{aligned}$$

$$-\frac{1}{\beta}\int w\ln(w)dw = -\frac{1}{\beta}\left[\frac{1}{2}w^2\ln(w) - \frac{1}{2}\int w\,dw\right]$$

$$= -\frac{1}{\beta}\left[\frac{1}{2}w^2\ln(w) - \frac{1}{2}\left(\frac{1}{2}w^2\right)\right]$$

$$\int_0^E \left(1+\beta E-\beta j\right)\ln(1+\beta E-\beta j)dj == -\frac{1}{\beta}\left[\frac{1}{2}w^2\ln(w) - \frac{1}{2}\left(\frac{1}{2}w^2\right)\right]_0^E$$

$$= \frac{1}{4\beta}\left[2(1+\beta E)^2 \ln(1+\beta E) + 1 - (1+\beta E)^2\right] \tag{28}$$

Insert (28) into (27):

$$\pi_h{}^*(E) = z + \alpha\gamma E - \frac{\gamma\theta\zeta^2 E}{2} - \frac{\gamma^2\theta\zeta^2}{2\beta}\left[\frac{1}{4\beta}\left[2(1+\beta E)^2 \ln(1+\beta E) + 1 - (1+\beta E)^2\right]\right]$$

$$\begin{gathered}\pi_h{}^*(E) = z + \alpha\gamma E - \frac{\gamma\theta\zeta^2 E}{2} - \frac{\gamma^2\theta\zeta^2}{8\beta^2}\left[1 + (1+\beta E)^2\left[2\ln(1+\beta E) - 1\right]\right] \\ z = \alpha - \frac{\theta\zeta^2}{2} + \varepsilon(0)\end{gathered} \tag{29}$$

4. Analysis

There are two primary objectives of this section: (1) understand how the optimal timing of the election date, E^*, differs between exponential and hyperbolic memory discounting; and, (2) examine how the optimal paths of unemployment rate and inflation rate are affected by changes in the unobserved parameter γ and the discount function parameters.

Baseline parameter values were selected from Findley (2015) in order to stay consistent with estimates found from empirical studies and to stay close to the original model. The slope coefficient on the short-run Phillips curve, ζ, is set to 0.6 which Findley (2015) reports is the midpoint of the range of common estimates. As noted by (24), the unemployment rate observed at the time of an election is $\theta\zeta / 2$. Findley (2015) utilizes this to calibrate θ, so the unemployment rate immediately following an election is 4%. A value of $\theta = 0.133$ and $\zeta = 0.6$ ensure the target of 4% is achieved. A value of $\alpha = 0.03$ is chosen to generate a 5% natural rate of unemployment in the model.

In order to be able to compare the models in a meaningful way, I apply the technique outlined in Findley (2015) that controls for differences in total memories by equalizing the areas under alternative discount functions. This is necessary because differences attributed to the different slopes of the alternative discount functions could just be the result of differences in total memories (level effects) possessed by a voter who forgets exponentially versus a voter who forgets hyperbolically. The exercise of equating the areas under the curve is synonymous with equating the total number of memories remembered between the dates of elections. This exercise controls for level effects when the optimal date of election is the same for both discount functions, but the exercise of holding memories constant can be an imperfect control when the dates of election timing are allowed to vary with differences in the discount functions.

The following steps outline the exercise of equalizing the areas under the alternative discount functions. First, $E_{\exp}*$ is numerically approximated by performing a grid search to approximately determine where the value of the Hamiltonian is equal to zero, for a given value of the exponential memory discount rate, ρ . Then the discount rate of the hyperbolic function, β, is chosen in a way that minimizes the sum of the squared difference in the areas under the discount functions between $t=0$ to $t=E_{\exp}*$. Next, with β calibrated, $E_{hyp}*$ is numerically approximated by performing a numerical grid search. This exercise of holding total memories constant does not hold exactly because $E_{\exp}*$ does not equal $E_{hyp}*$. The difference in most of the sensitivity analysis is relatively small and still provides some interesting observations, recognizing that some of the differences in the predictions are due to level effects, and not just due to the slope effects. Figure 12 illustrates a circumstance with larger level effects after controlling for slope effects.

Table 1 reports differences in values of $E*$ for both exponential and hyperbolic memory discount functions given various combinations of values for γ and ρ when $\pi(0) = 0.05$ (inherited rate of inflation of 5% from Chappel and Peel (1979)). One straightforward observation is that the values of $E^*_{\exp}$ and E^*_{hyp} are very close for low values of ρ. As γ increases, given any value of ρ, the difference in $E*$ and the difference in the area under the discount functions approach zero. This would imply that an incumbent politician is less able to manipulate economic conditions when voters respond more quickly to changes in prices. Likewise, if voters retain more of their memories, then politicians are less able to manipulate economic conditions. Another observation is that the closer the value of the speed of adjustment in inflation expectations, γ, and the rate of exponential memory decay, ρ, the more likely the model will generate a value greater than 100% for the unemployment rate at the start of the electoral regime. An unemployment rate greater than 100% invalidates the solution technique of the optimal control problem.

Table 2 is similar to Table 1; however, I have set the inherited rate of inflation equal to 2%. It is identified in Chappell and Peel (1979) that a one percentage point increase in the inherited rate of inflation will lead to an increase in the optimal date of the next election. Table 2 reports a decrease in $E*$ with the 3% reduction in the inherited rate of inflation, which supports the comparative statics analysis in Chappell and Peel (1979). All previous analysis of Table 1 holds in Table 2. An additional observation is that on average the difference in the areas under the alternative discount functions is greater across all combinations of parameter values.

As observed in Figures 2-6, holding all else equal, an increase in voters' adjustments to prices moderates the amplitude of the unemployment rate, with the exponential function more sensitive to changes in values than the hyperbolic function. Also observed in Figures 2-6, an increase in the rate of memory decay increases the amplitude of the unemployment rate, again affecting the unemployment rate corresponding to the exponential function disproportionately. Rates of exponential memory decays entertained did not exceed 50% because higher values generated unemployment rates that exceeded 100% for the exponential case and could only be moderated with a nearly immediate voter response in price changes. Findley (2015) discusses empirical estimates in memory discount rates that range from 50%-144%. It should be noted that the hyperbolic function continues to generate valid solutions even when the exponential function did not. The highest value of the exponential discount rate at which I was able to obtain valid solutions and remains within realistic unemployment rates was $\rho = 0.7$, but it was accompanied by a fast voter response in prices ($\gamma = 0.95$). The common finding across all values of the memory discount rate and the speed at which voters adjust to inflation expectations is that the optimal policy for an incumbent politician is to target a high unemployment rate post-election and continuously decrease the unemployment rate up to the date of the next election.

Figures 7-11 depict the optimal path of inflation rates. It is observed that the amplitude is moderated for the hyperbolic function, as also seen with the unemployment rate. The moderated amplitude is much more apparent with higher values of memory discount rates. This observation is to be expected given our analysis of results reported in Table 1 and 2, where the exponential and hyperbolic functions have different optimal dates of the next election. The paths observed in all combinations of the memory discount rate and the speed at which voters adjust to inflation expectations is that prices will decline quickly and around the midpoint of the electoral regime will then begin to rise until the date of the next election.

The model is hindered in its ability to entertain higher discount rates reported in the psychology research because of the condition observed by Chappell and Peel (1979) that requires the speed at which voters adjust to prices be greater than the rate at which their memories are discounted ($\gamma > \rho, \beta > 0$). Chappell and Peel (1979) identify this condition as being necessary, but in a numerical exercise a unique solution was still sometimes found in instances when this condition did not hold. Therefore, it is likely that this condition is sufficient rather than necessary.

5. Conclusion

Section 1 discussed the impact of the classic-framework model of Nordhaus (1977) that was a catalyst for future research on the links between incumbent governments and macro-economic conditions. Nordhaus (1977) observed a cyclical pattern in both the unemployment and inflation rates. Moreover, the assumption of a retrospective voter has been backed up extensively by published work in the fields of political science and psychology and the assumption of an opportunistic incumbent politician has been observed in a model illustrating an incumbent who faces a lower probability of reelection. Section 2 replicates the methodologies used in Chappell and

Peel (1979) to find the optimal date of election. Section 3 provides an overview of Findley (2015) and the steps to solving the optimal paths of unemployment and inflation rates are presented. Section 4 provides an analytical comparison of the effects on the optimal timing of elections and the optimal times paths of unemployment and inflation rates given alternative forms of memory discounting.

I observe cyclical patterns in the unemployment and inflation rates. The unemployment rate starts high but then falls continuously up to the date of the next election. Prices have a period of deflation at the start of the electoral regime, followed by inflationary pressure leading up to the date of the next election. Also, observed is a moderated amplitude in the hyperbolic case similar to what is observed in Findley (2015). Given changes in the speed of price adaptation and memory discount rates, the model with hyperbolic memory discounting provides much more stability in unemployment and inflation rates given a wide range of parameter values as observed in the size of the amplitudes in the predicted business cycle.

The model could be applied to further analysis that might account for higher discount rates and a constrained date of the next election. The model is currently unable to be studied at parameterizations with high discount rates because of the 'necessary condition' observed by Chappell and Peel (1979). A different functional form of the vote function may be needed in order to work around this condition in further analysis. Incumbent politicians are unconstrained in the date of election in the model. But in the case of the United Kingdom, an incumbent government is required to call an election no later than five years from the last. It would be a modelling innovation to add this additional facet of reality. It is my goal to incorporate this modelling innovation in future research.

REFERENCES

Chappell, D., Peel, D.A., 1979. On the political theory of the business cycle. Econ. Lett. 2, 327-332.
Findley, T., 2015. Hyperbolic memory discounting and the political business cycle. Eur. J. Polit. Econ., 345-359.
Hibbs Jr., D.A., 1977. Political parties and macroeconomic policies. Am. Polit. Sci. Rev. 71, 1467-1487.
Hibbs, D., 2006. Voting and the Macroeconomy. In (Ed.), The Oxford Handbook of Political Economy. Oxford University Press, 565-586.
Kalecki, M., 1943. Political aspects of full employment. Polit. Q. 14, 322-330.
Murakami, M.H., 2008. Paradoxes of democratic accountability: polarized parties, hard decisions, and no despot to veto. Crit. Rev. J. Polit. Soc. 20, 619-635.
Nordhaus, W.D., 1975. The political business cycle. Rev. Econ. Stud. 42, 169-190.
Snowden, B., 1997. Politics and the business cycle. Polit. W. 68, 255-265.

Table 1:
Changes in Values of E*
Exponential/Hyperbolic $\{\pi(0) = 0.05\}$
(Differential Areas of Discount Functions)

		γ					
		0.3	0.4	0.5	0.6	0.7	0.8
ρ	0.03	11.34/11.30 (-0.30%)	8.52/8.50 (-0.08%)	6.82/6.81 (-0.13)	5.69/5.69 (-0.17%)	4.88/4.88 (-0.21%)	4.27/4.27 (0.01%)
	0.10	11.35/10.94 (-2.35%)	8.48/8.30 (-1.53%)	6.78/6.69 (-0.97%)	5.65/5.60 (-0.88%)	4.85/4.81 (-0.45%)	4.25/4.22 (-0.35%)
	0.20	12.32/10.47 (-7.53%)	8.73/8.02 (-4.46%)	6.86/6.50 (-3.06%)	5.67/5.47 (-2.48%)	4.85/4.72 (-1.81%)	4.24/4.15 (-1.53%)
	0.35		11.10/7.65 (-14.29%)	7.53/6.26 (-8.36%)	5.95/5.29 (-5.75%)	4.99/4.58 (-4.45%)	4.31/4.04 (-3.40%)
	0.50				6.99/5.13 (-12.42%)	5.42/4.46 (-8.64%)	4.54/3.95 (-6.77%)

Red: Invalid, u(E*) > 100%

Table 2:
Changes in Values of E*
Exponential/Hyperbolic $\{\pi(0) = 0.02\}$
(Differential Areas of Discount Functions)

		γ					
		0.3	0.4	0.5	0.6	0.7	0.8
ρ	0.03	9.26/9.22 (-0.33%)	6.92/6.91 (-0.27%)	5.53/5.52 (0.03%)	4.6/4.6 (-0.22%)	3.94/3.94 (-0.26%)	3.45/3.45 (0.01%)
	0.10	9.66/9.21 (-3.08%)	7.11/6.92 (-1.81%)	5.63/5.54 (-1.42%)	4.67/4.61 (-1.07%)	3.99/3.95 (-0.57%)	3.48/3.46 (-0.79%)
	0.20	11.12/9.23 (-9.88%)	7.64/6.85 (-6.14%)	5.90/5.52 (-4.31%)	4.83/4.60 (-3.07%)	4.09/3.95 (-2.44%)	3.55/3.46 (-1.80%)
	0.35		10.45/6.71 (-17.28%)	6.83/5.40 (-10.68%)	5.29/4.55 (-7.99%)	4.37/3.92 (-5.87%)	3.73/3.44 (-4.79%)
	0.50				6.52/4.49 (-15.34%)	4.94/3.88 (-11.17%)	4.06/3.41 (-8.65%)

Red: Invalid, u(E*) > 100%

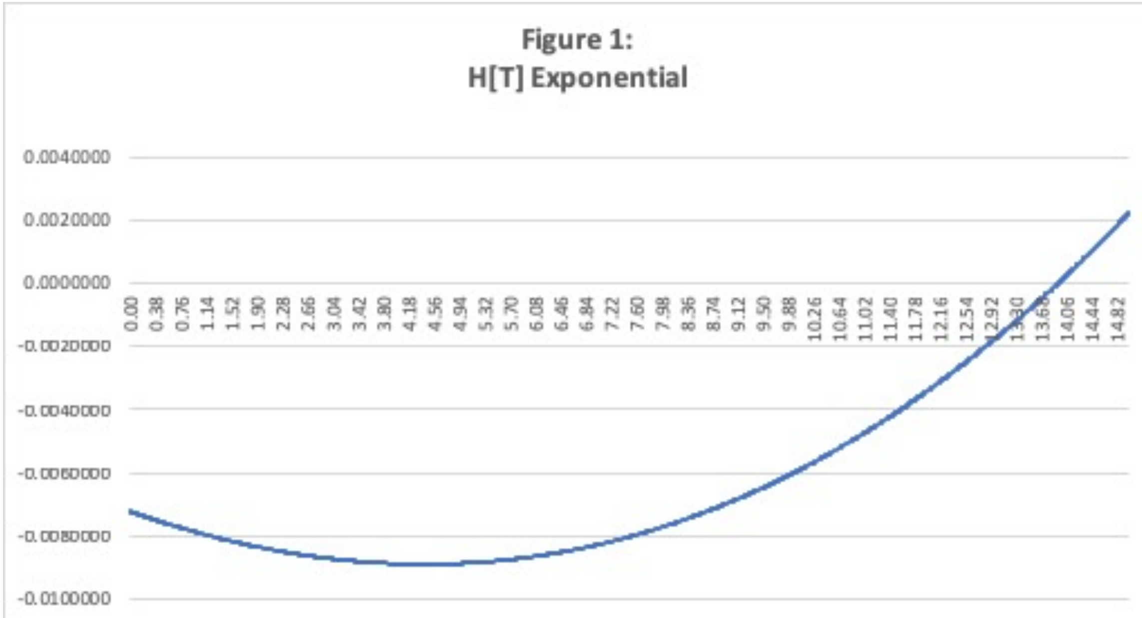
Figure 1:
H[T] Exponential
0.0040000
0.0020000
0.0000000
-0.0020000
-0.0040000
-0.0060000
-0.0080000
-0.0100000

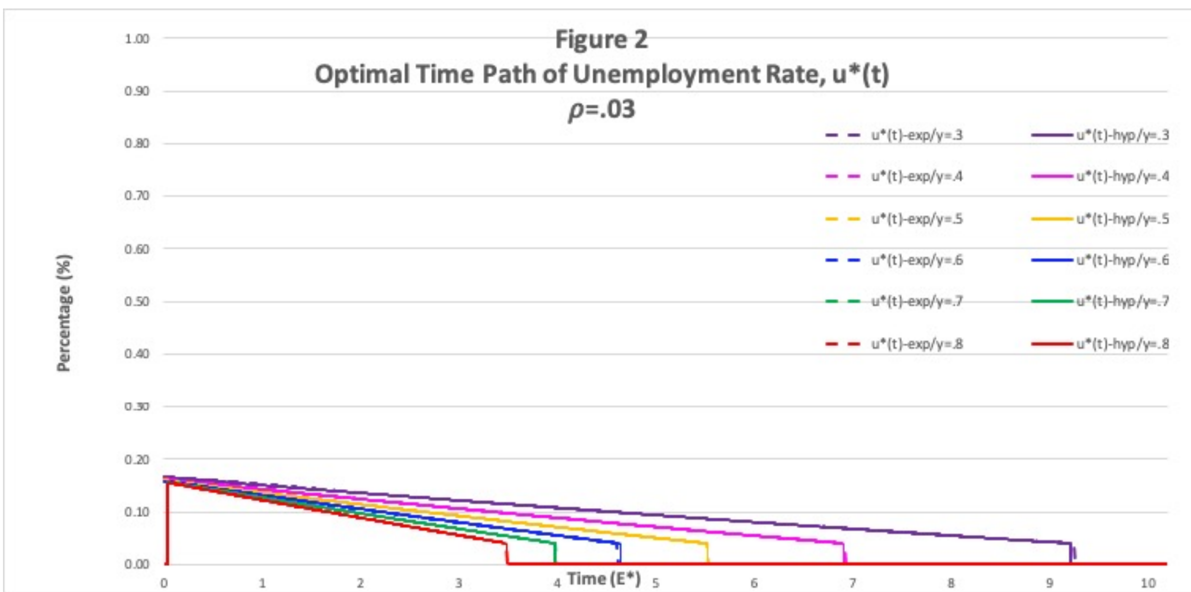
Figure 2
Optimal Time Path of Unemployment Rate, u*(t)
ρ=.03
Percentage (%)
Time (E*)
u*(t)-exp/y=.3
u*(t)-hyp/y=.3
u*(t)-exp/y=.4
u*(t)-hyp/y=.4
u*(t)-exp/y=.5
u*(t)-hyp/y=.5
u*(t)-exp/y=.6
u*(t)-hyp/y=.6
u*(t)-exp/y=.7
u*(t)-hyp/y=.7
u*(t)-exp/y=.8
u*(t)-hyp/y=.8

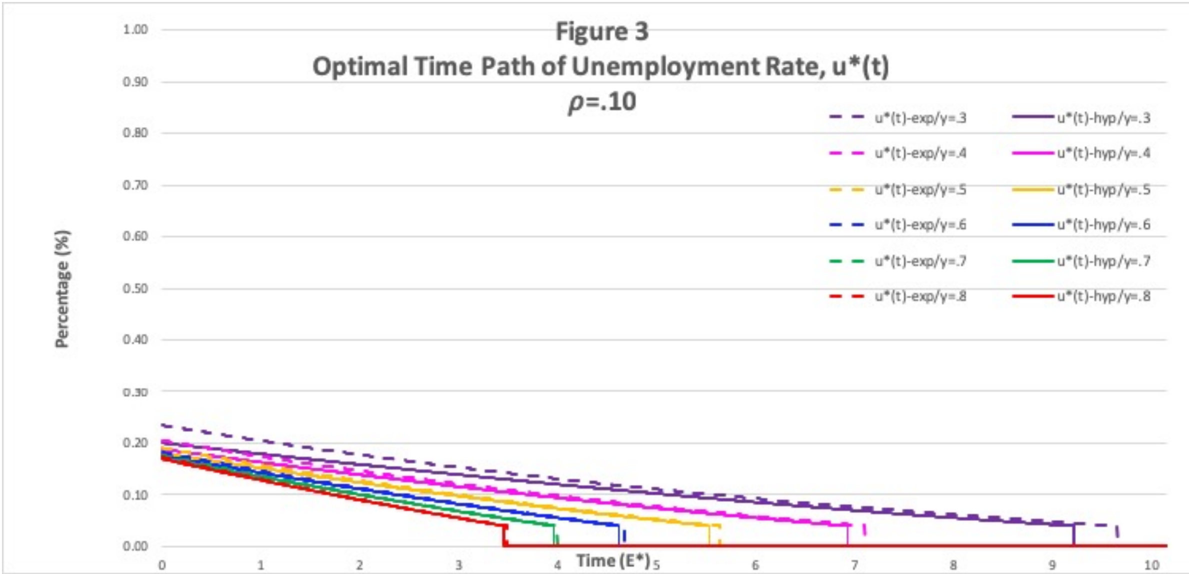
Figure 3
Optimal Time Path of Unemployment Rate, u*(t)
ρ=.10
u*(t)-exp/y=.3
u*(t)-hyp/y=.3
u*(t)-exp/y=.4
u*(t)-hyp/y=.4
u*(t)-exp/y=.5
u*(t)-hyp/y=.5
u*(t)-exp/y=.6
u*(t)-hyp/y=.6
u*(t)-exp/y=.7
u*(t)-hyp/y=.7
u*(t)-exp/y=.8
u*(t)-hyp/y=.8
Percentage (%)
Time (E*)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0
1
2
3
4
5
6
7
8
9
10

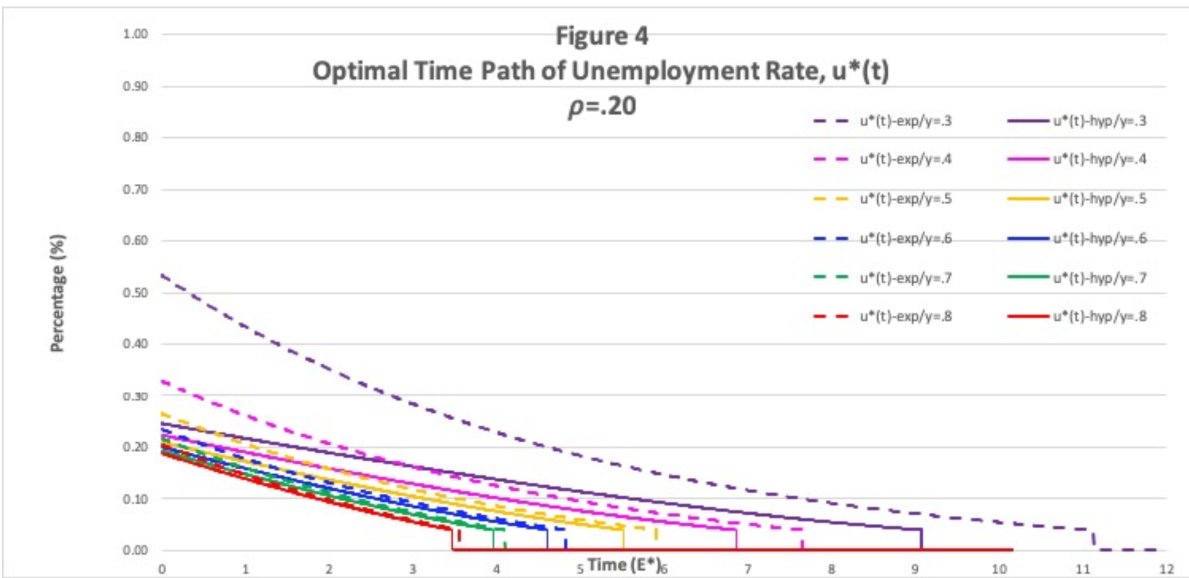
Figure 4
Optimal Time Path of Unemployment Rate, u*(t)
ρ=.20
u*(t)-exp/y=.3
u*(t)-hyp/y=.3
u*(t)-exp/y=.4
u*(t)-hyp/y=.4
u*(t)-exp/y=.5
u*(t)-hyp/y=.5
u*(t)-exp/y=.6
u*(t)-hyp/y=.6
u*(t)-exp/y=.7
u*(t)-hyp/y=.7
u*(t)-exp/y=.8
u*(t)-hyp/y=.8
Percentage (%)
Time (E*)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0
1
2
3
4
5
6
7
8
9
10
11
12

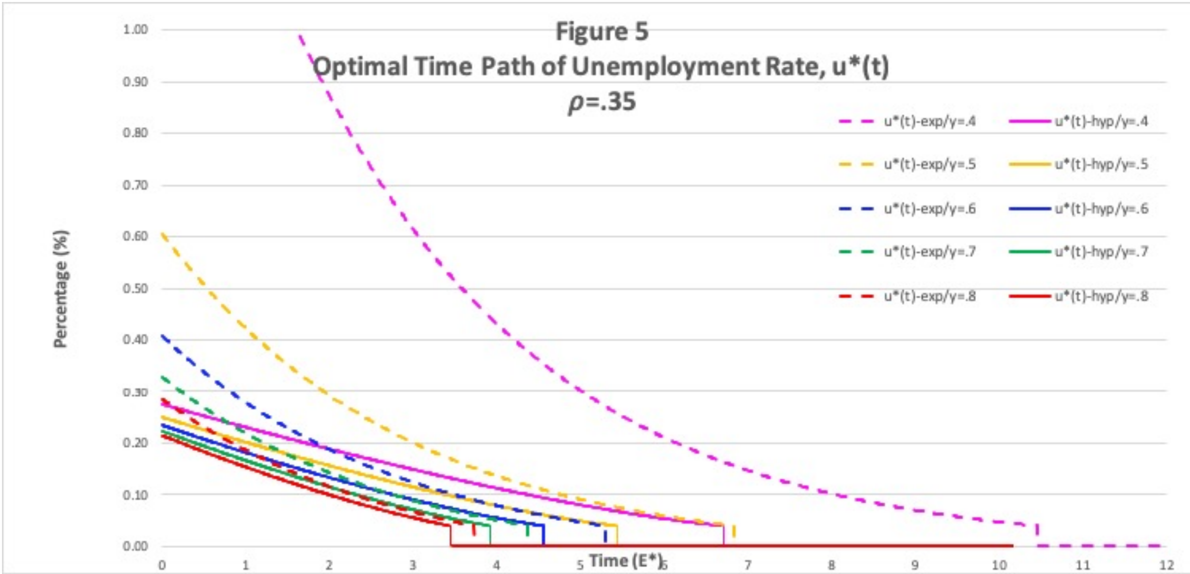
Figure 5
Optimal Time Path of Unemployment Rate, u*(t)
ρ=.35
u*(t)-exp/y=.4
u*(t)-hyp/y=.4
u*(t)-exp/y=.5
u*(t)-hyp/y=.5
u*(t)-exp/y=.6
u*(t)-hyp/y=.6
u*(t)-exp/y=.7
u*(t)-hyp/y=.7
u*(t)-exp/y=.8
u*(t)-hyp/y=.8
Percentage (%)
Time (E*)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0
1
2
3
4
5
6
7
8
9
10
11
12

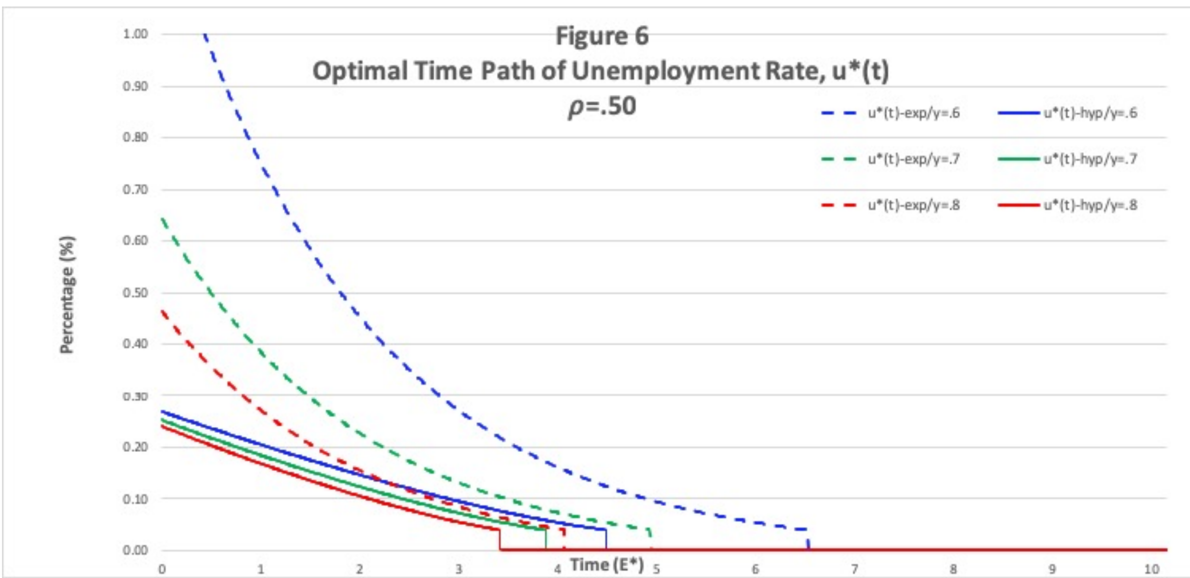
Figure 6
Optimal Time Path of Unemployment Rate, u*(t)
ρ=.50
u*(t)-exp/y=.6
u*(t)-hyp/y=.6
u*(t)-exp/y=.7
u*(t)-hyp/y=.7
u*(t)-exp/y=.8
u*(t)-hyp/y=.8
Percentage (%)
Time (E*)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0
1
2
3
4
5
6
7
8
9
10

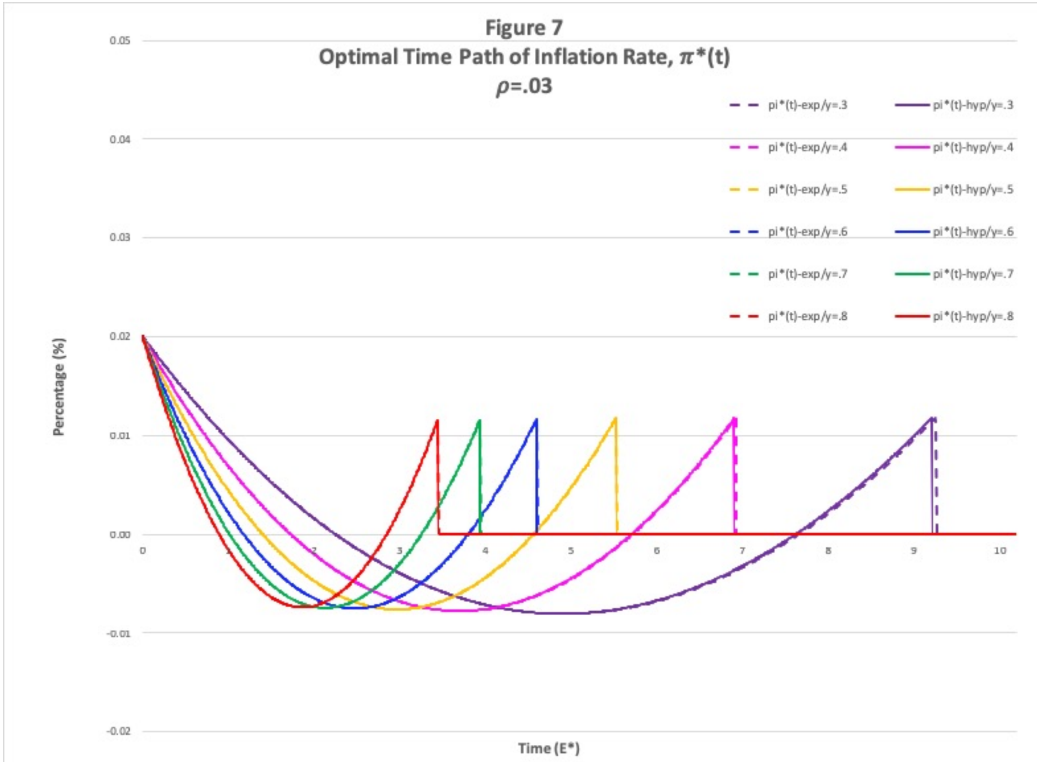
Figure 7
Optimal Time Path of Inflation Rate, π*(t)
ρ=.03
pi*(t)-exp/γ=.3
pi*(t)-hyp/γ=.3
pi*(t)-exp/γ=.4
pi*(t)-hyp/γ=.4
pi*(t)-exp/γ=.5
pi*(t)-hyp/γ=.5
pi*(t)-exp/γ=.6
pi*(t)-hyp/γ=.6
pi*(t)-exp/γ=.7
pi*(t)-hyp/γ=.7
pi*(t)-exp/γ=.8
pi*(t)-hyp/γ=.8
Percentage (%)
Time (E*)

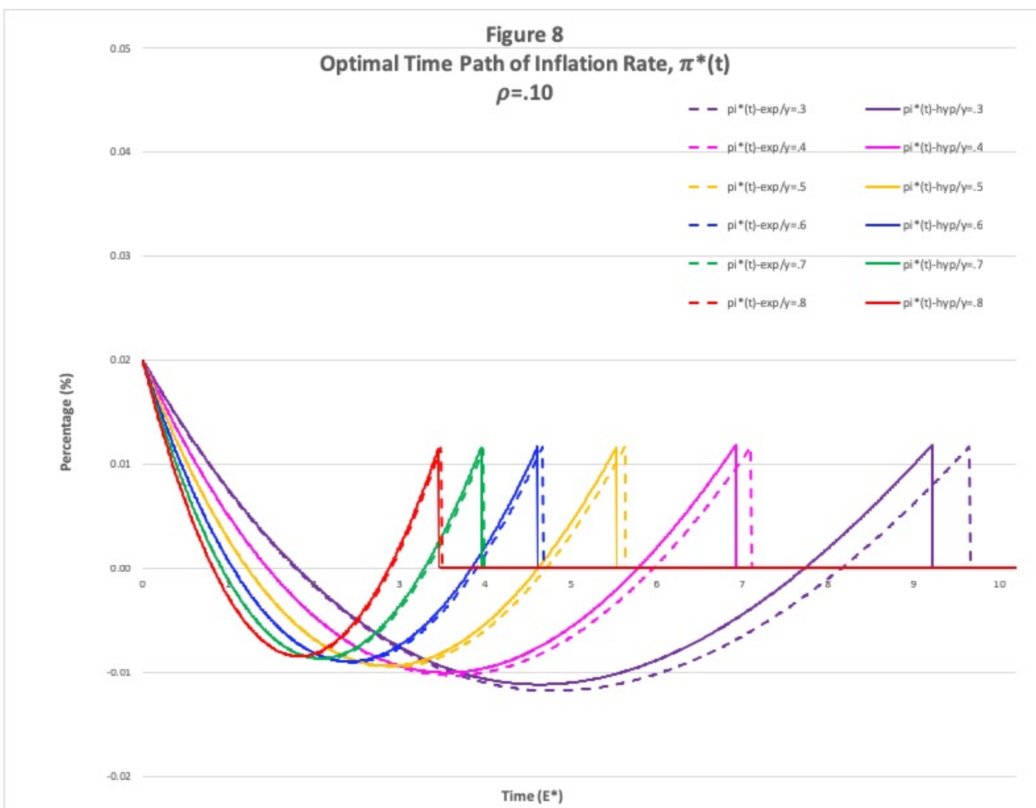
Figure 8
Optimal Time Path of Inflation Rate, π*(t)
ρ=.10
pi*(t)-exp/γ=.3
pi*(t)-hyp/γ=.3
pi*(t)-exp/γ=.4
pi*(t)-hyp/γ=.4
pi*(t)-exp/γ=.5
pi*(t)-hyp/γ=.5
pi*(t)-exp/γ=.6
pi*(t)-hyp/γ=.6
pi*(t)-exp/γ=.7
pi*(t)-hyp/γ=.7
pi*(t)-exp/γ=.8
pi*(t)-hyp/γ=.8
Percentage (%)
Time (E*)

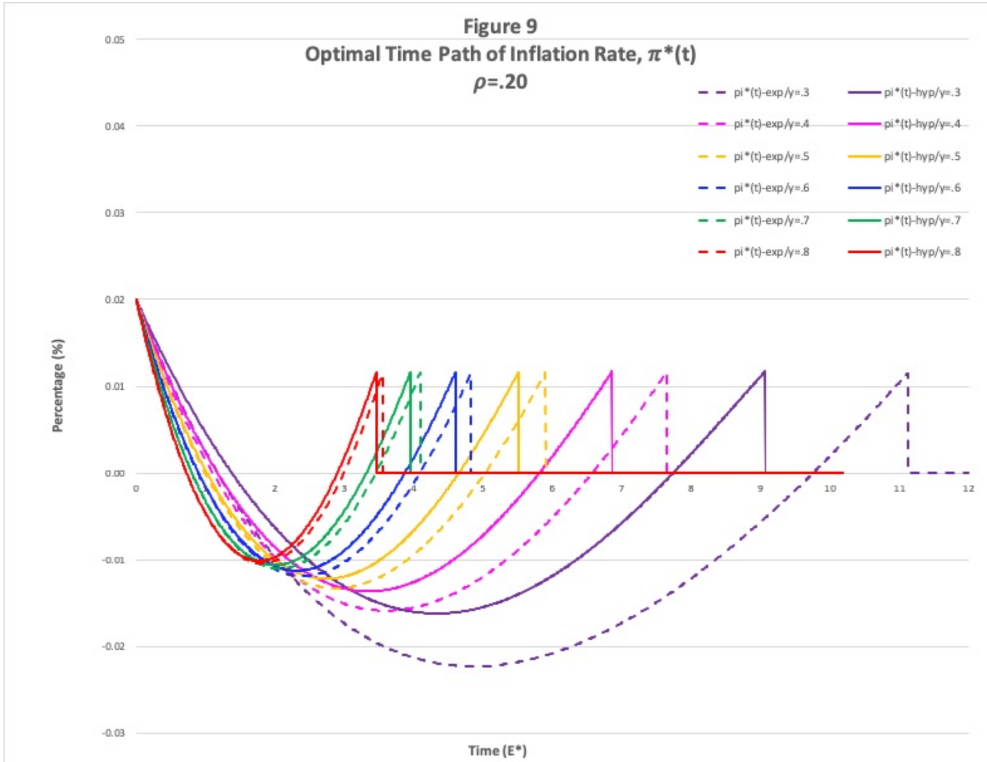
Figure 9
Optimal Time Path of Inflation Rate, π*(t)
ρ=.20
pi*(t)-exp/y=.3
pi*(t)-hyp/y=.3
pi*(t)-exp/y=.4
pi*(t)-hyp/y=.4
pi*(t)-exp/y=.5
pi*(t)-hyp/y=.5
pi*(t)-exp/y=.6
pi*(t)-hyp/y=.6
pi*(t)-exp/y=.7
pi*(t)-hyp/y=.7
pi*(t)-exp/y=.8
pi*(t)-hyp/y=.8
Percentage (%)
Time (E*)

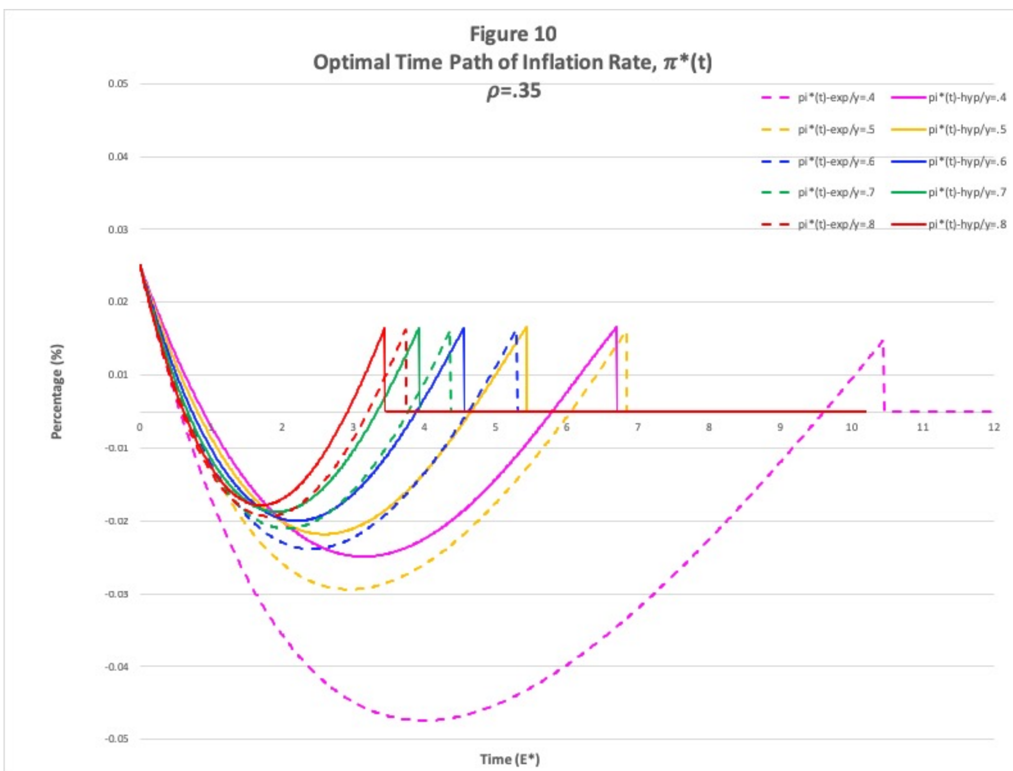
Figure 10
Optimal Time Path of Inflation Rate, π*(t)
ρ=.35
pi*(t)-exp/y=.4
pi*(t)-hyp/y=.4
pi*(t)-exp/y=.5
pi*(t)-hyp/y=.5
pi*(t)-exp/y=.6
pi*(t)-hyp/y=.6
pi*(t)-exp/y=.7
pi*(t)-hyp/y=.7
pi*(t)-exp/y=.8
pi*(t)-hyp/y=.8
Percentage (%)
Time (E*)

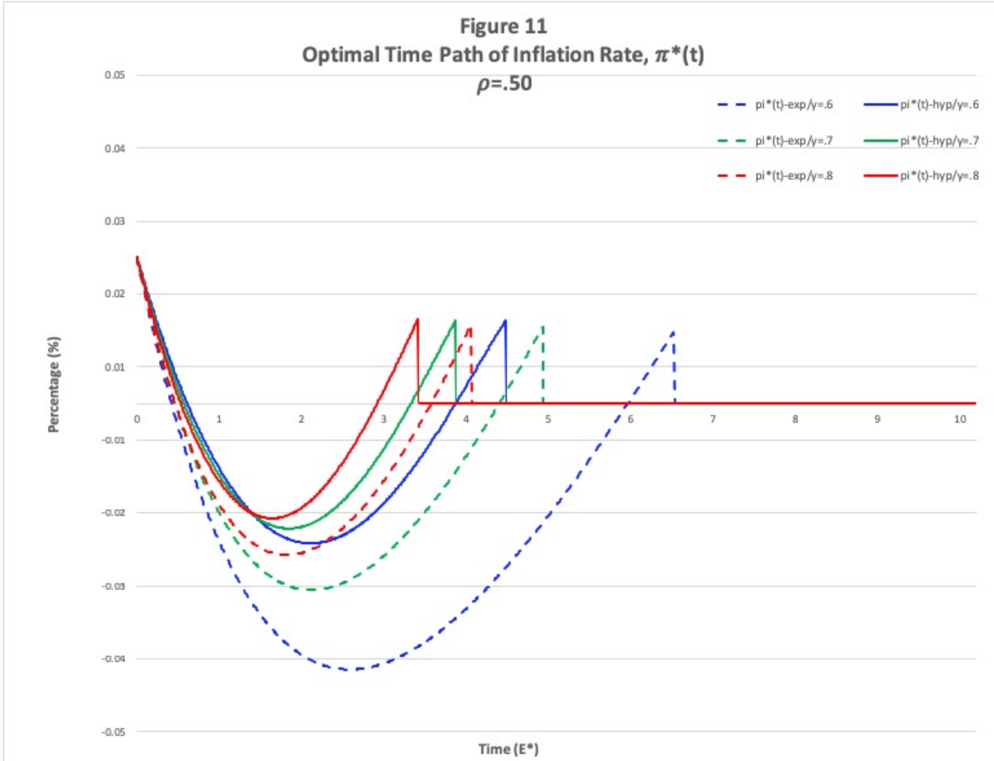
Figure 11
Optimal Time Path of Inflation Rate, π*(t)
ρ=.50
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
Percentage (%)
0 1 2 3 4 5 6 7 8 9 10
Time (E*)
pi*(t)-exp/γ=.6
pi*(t)-hyp/γ=.6
pi*(t)-exp/γ=.7
pi*(t)-hyp/γ=.7
pi*(t)-exp/γ=.8
pi*(t)-hyp/γ=.8

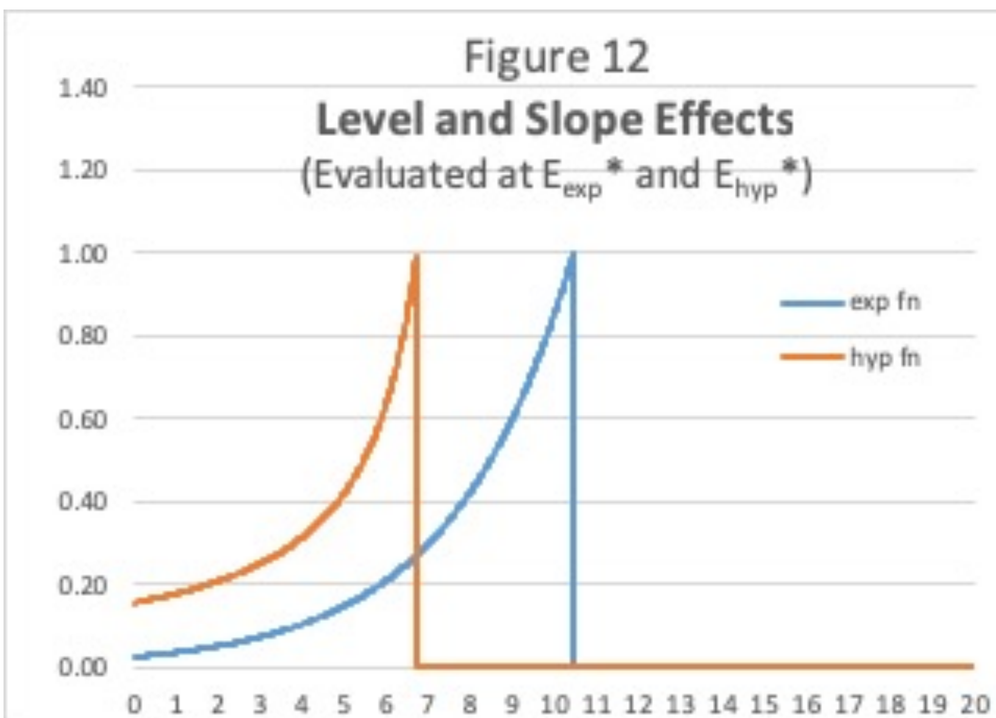
Figure 12
Level and Slope Effects
(Evaluated at E_exp* and E_hyp*)
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
exp fn
hyp fn