Linear Constraints in Optimal Transport

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ABSTRACT

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This thesis studies the problem of optimal mass transportation with linear constraints – supermartingale and martingale transport in discrete and continuous time. Appropriate versions of corresponding dual problems are introduced and shown to satisfy fundamental properties: weak duality, absence of a duality gap, and the existence of a dual optimal element. We show how the existence of a dual optimizer implies that primal optimizers can be characterized geometrically through their support – an infinite dimensional analogue of complementary slackness. In discrete time martingale and supermartingale transport problems, we utilize this result to establish the existence of canonical transport plans, that is joint optimizers for large families of reward functions. To this end, we show that the optimal support coincides for these families. We additionally characterize these transport plans through order-theoretic minimality properties, with respect to second stochastic order and convex order, respectively, in the supermartingale and the martingale case. This characterization further shows that the canonical transport plan is unique.

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Contents

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Gewidmet meinen Eltern Sabina und Franz.

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— 1 —

Introduction

This chapter provides an overview of optimal transport with constraints, and an outline of the body of the thesis.

1.1 Optimal Transport and Linear Constraints

We will first give a short overview about the common questions in optimal transport. Let μ and ν be probability measures on the real line and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a reward function. A Monge transport is a map $T : \mathbb{R} \to \mathbb{R}$ that satisfies $\nu = \mu \circ T^{-1}$. The Monge optimal transport problem is to find a Monge transport that maximizes the total reward,

$$\int f(x, T(x))\mu(dx)$$

among all such maps.

As this is a difficult optimization problem with a non-convex domain, we consider the Kantorovich relaxation. A Kantorovich transport of μ and ν is a measure P on $\mathbb{R}^2$ whose first and second marginals are μ and ν, respectively. We will denote the set of all such measures by $\Pi(\mu, \nu)$. We assume throughout that μ and ν have finite first moment and observe that $\Pi(\mu, \nu)$ is compact, with respect to the weak topology.

The Kantorovich transport problem is to find $P \in \Pi(\mu, \nu)$ that maximizes the total reward, given by

$$P(f) = E^P[f(X, Y)],$$

the expectation of f under P. Observe that $P \in \Pi(\mu, \nu)$ is a Monge transport if it is of the form $P = \mu \otimes \delta_{T(x)}$.

When f satisfies the Spence-Mirrlees condition $f_{xy} > 0$, then the optimal P is unique and given by the so called Hoeffding-Frechet Coupling $P = ((F_\mu)^{-1}, (F_\nu)^{-1}) \circ \lambda_{[0,1]}$. It can be characterized through its support: If $(x, y), (x', y') \in \text{supp}(P)$ and $x < x'$, then $y \leq y'$. This is called monotonicity.

The Kantorovich transport problem has an associated dual problem which we can derive as follows:

Let $\phi \in L^1(\mu)$ and $\psi \in L^1(\nu)$ such that $f(x, y) \leq \phi(x) + \psi(y)$ on $\mathbb{R}^2$. We denote the collection of all such pairs (ϕ, ψ) by $\mathcal{D}_{\mu,\nu}(f)$. Then we have

$$P(f) = E^P[f(X, Y)] \leq E^P[\phi(X) + \psi(Y)] = \mu(\phi) + \nu(\psi).$$

It can be shown that strong duality also holds:

Theorem 1.1.1 (Kellerer)**.** *For any measurable $f \geq 0$,*

$$\sup_{P \in \Pi(\mu,\nu)} P(f) = \inf_{\phi,\psi} \mu(\phi) + \nu(\psi)$$

and dual optimizers $\hat{\phi}, \hat{\psi}$ exist.

A linear equality constraint in this setup is defined through a family of measurable functions $\mathcal{G}$ on $\mathbb{R}^2$. We set $\mathcal{P} := \{P \in \Pi(\mu, \nu) : P(g) = 0 \forall g \in \mathcal{G}\}$ and consider the objective

$$P(f) = E^P[f(X, Y)].$$

The classical example for constrained transport problems is Martingale transport, which is characterized through the set of functions $\mathcal{G} = \{h(x)(y-x) : h \in C_b(\mathbb{R})\}$. For this constraint we have $\mathcal{P} = \{P \in \Pi(\mu, \nu) : P \text{ is a martingale}\}$. We will denote the set of martingale transports by $\mathcal{M}(\mu, \nu)$. For the primal problem to be well-defined, we need $\mathcal{M}(\mu, \nu)$ to be non-empty, which is equivalent to μ and ν being in convex order, that is $\mu(g) \leq \nu(g)$ for all convex functions g.

When f satisfies a Martingale version of the Spence-Mirrlees condition, $f_{xyy} > 0$, then the optimal P is unique and can be characterized through its support: If $(x, y_1), (x, y_2), (x', y') \in \operatorname{supp}(P)$, and $x < x'$, $y_1 < y_2$, then $y' \notin (y_1, y_2)$.

This constraint on the optimization domain formally appears as a Lagrange multiplier in the dual problem. That is, we consider functions $\phi(x) + \psi(y) + h(x)(y-x) \geq f(x, y)$. Assuming that these functions are sufficiently integrable (in a sense to be made precise), we have for $P \in \mathcal{M}(\mu, \nu)$

$$P(f) = E^P[f(X, Y)] \leq E^P[\phi(X) + \psi(Y) + h(X)(Y - X)] = \mu(\phi) + \nu(\psi).$$

An analogue of Theorem 1.1.1 can then be established.

Generalizations of the Martingale transport problem allow for inequality constraints, multiple fixed marginal measures or even continuous time marginals. Establishing solutions to the above described problems in these settings is the content of this thesis.

1.2 Thesis Outline

The results in this thesis are separated into three chapters that correspond to the articles [71, 72, 53]. The chapters are generally self-contained and introduce their respective prerequisites. The following paragraphs give abstracts of all chapters:

Canonical Supermartingale Couplings. Two probability distributions μ and ν in second stochastic order can be coupled by a supermartingale, and in fact by many. We aim to characterize a canonical choice. To this end, we construct and investigate two couplings which arise as optimizers for constrained Monge-Kantorovich optimal transport problems where only supermartingales are allowed as transports. Much like the Hoeffding-Fréchet coupling of classical transport and its symmetric counterpart, the antitone coupling, these can be characterized by order-theoretic minimality properties, as simultaneous optimal transports for certain classes of reward (or cost) functions, and through no-crossing conditions on their supports; however, our two couplings have asymmetric geometries. Remarkably, supermartingale optimal transport decomposes into classical and martingale transport in several ways.

Multiperiod Martingale Transport. We consider a multiperiod optimal transport problem where distributions $\mu_0, \dots, \mu_n$ are prescribed and a transport corresponds to a scalar martingale X with marginals $X_t \sim \mu_t$. We introduce particular couplings called left-monotone transports; they are characterized equivalently by a no-crossing property of their support, as simultaneous optimizers for a class of bivariate transport cost functions with a Spence–Mirrlees property, and by an order-theoretic minimality property. Left-monotone transports are unique if μ_0 is atomless, but not in general. In the one-period case $n = 1$, these transports reduce to the Left-Curtain coupling of Beiglböck and Juillet. In the multiperiod case, the bivariate marginals for dates $(0, t)$ are of Left-Curtain type, if and only if $\mu_0, \ldots, \mu_n$ have a specific order property. The general analysis of the transport problem also gives rise to a strong duality result and a description of its polar sets. Finally, we study a variant where the intermediate marginals $\mu_1, \ldots, \mu_{n-1}$ are not prescribed.

Robust Pricing and Hedging around the Globe. We consider the martingale optimal transport duality for càdlàg processes with given initial and terminal laws. Strong duality and existence of dual optimizers (robust semi-static superhedging strategies) are proved for a class of payoffs that includes American, Asian, Bermudan, and European options with intermediate maturity. We exhibit an optimal superhedging strategy for which the static part solves an auxiliary problem and the dynamic part is given explicitly in terms of the static part. In the case of finitely supported marginal laws, solving for the static part reduces to a semi-infinite linear program.

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— 2 —

Canonical Supermartingale Couplings

This chapter is based on the article [71] of the same title, authored by Marcel Nutz and Florian Stebegg. It is published in Annals of Probability.

2.1 Introduction

Let μ and ν be probability measures on the real line. A measure P on $\mathbb{R}^2$ whose first and second marginals are μ and ν, respectively, is called a coupling (or transport) of μ and ν, and the set of all such measures is denoted by $\Pi(\mu, \nu)$. We shall be interested in couplings that are supermartingales; that is, if (X, Y) denotes the identity on $\mathbb{R}^2$, then $E^P[Y|X] \leq X$ P-a.s. Thus, we assume throughout that μ and ν have a finite first moment, and denote by $\mathcal{S}(\mu, \nu)$ the set of supermartingale couplings. A classical result of Strassen (cf. Proposition 2.2.1) shows that $\mathcal{S}(\mu, \nu)$ is nonempty if and only if μ and ν are in convex-decreasing (or second stochastic) order, denoted $\mu \leq_{cd} \nu$ and defined by the requirement that $\mu(\phi) \leq \nu(\phi)$ for any convex and decreasing function ϕ, where $\mu(\phi) := \int \phi \, d\mu$. Given $\mu \leq_{cd} \nu$, there are typically infinitely many supermartingale couplings. Our question: are there some special, "canonical" choices? The aim of this paper is to introduce and describe two such couplings, called the *Increasing* and the *Decreasing Supermartingale Transport* and denoted $\vec{P}$ and $\overleftarrow{P}$, respectively. They have remarkable properties that are, in several ways, analogous to the Hoeffding–Fréchet and Antitone couplings which can be considered canonical choices in $\Pi(\mu, \nu)$ but typically are not supermartingales. As will be apparent from the structure of the analysis, the study undertaken can also be seen as a model problem of *optimal transport under inequality constraints.*

Synopsis

The couplings $\vec{P}$ and $\overleftarrow{P}$ will be characterized in three different ways: an order-theoretic minimality property, optimality for a specific class of transport reward (or cost) functions, and a geometric property of the support stating that certain paths do or do not intersect.

Let us begin with the order-theoretic characterization. To explain the idea, suppose that μ consists of finitely many atoms at $x_1, \ldots, x_n \in \mathbb{R}$, then a coupling of μ and ν can be defined by specifying a "destination" measure for each atom. We know from Strassen's result that the convex-decreasing order plays a special role, so it is natural to rank all possible destination measures for the first atom (as allowed by the given marginal ν and the supermartingale constraint) according to that order. A minimal element $\mathcal{S}^{\nu}(\mu|_{x_1})$ called the *shadow* will be shown to exist; essentially, it maximizes the barycenter of the destination measure and minimizes the variance.

The procedure can be iterated after subtracting $\mathcal{S}^\nu(\mu|_{x_1})$ from ν, and that determines a supermartingale coupling of μ and ν. Depending on the order in which the atoms are processed, the coupling will have a very different structure. Two obvious choices are the increasing and the decreasing order of the x_k, and that gives rise to $\vec{P}$ and $\overleftarrow{P}$ (the arrows representing the order of processing). In the general, continuum version of the construction, we instead specify the destination of $\mu|_{(-\infty,x]}$ and $\mu|_{[x,\infty)}$ for each $x \in \mathbb{R}$. The following is taken from Theorem 2.6.7 in the body of the paper; the precise definition of the shadow can be found in Lemma 2.6.2.

Theorem 2.1.1. *There exists a unique measure $\vec{P}$ on $\mathbb{R}^2$ which transports $\mu|_{(-\infty,x]}$ to its shadow $\mathcal{S}^\nu(\mu|_{(-\infty,x]})$ for all $x \in \mathbb{R}$. Similarly, there exists a unique measure $\overleftarrow{P}$ which transports $\mu|_{[x,\infty)}$ to $\mathcal{S}^\nu(\mu|_{[x,\infty)})$ for all $x \in \mathbb{R}$. Moreover, these two measures are elements of $\mathcal{S}(\mu,\nu)$.*

While the shadow construction illuminates the local order-theoretic nature of the couplings, it does not reveal the global geometric structure that is apparent in Figures 2.1 and 2.2 (rendered on page 10). The figures show simulations of $\vec{P}$ and $\overleftarrow{P}$ for piecewise uniform marginals and discrete marginals; the mass is transported from the x-axis (top) to the y-axis (bottom).

The Monge–Kantorovich optimal transport problem is a framework that enables a geometric description for its optimal transports, and thus it is desirable to represent $\vec{P}$ and $\overleftarrow{P}$ as corresponding solutions. More precisely, we shall introduce the

Increasing Supermartingale Transport $\overrightarrow{P}$

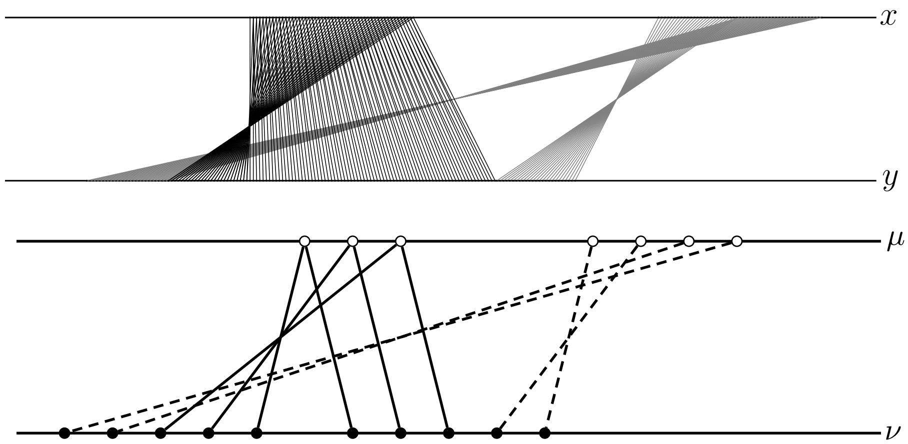

Figure 2.1: Simulations of the Increasing Supermartingale Transport. We observe an interval of Left-Curtain kernels (black/continuous) on the left and an interval of Antitone kernels (gray/dashed) on the right. The destinations of the right interval are on both sides of the destinations of the left one.

Decreasing Supermartingale Transport $\overleftarrow{P}$

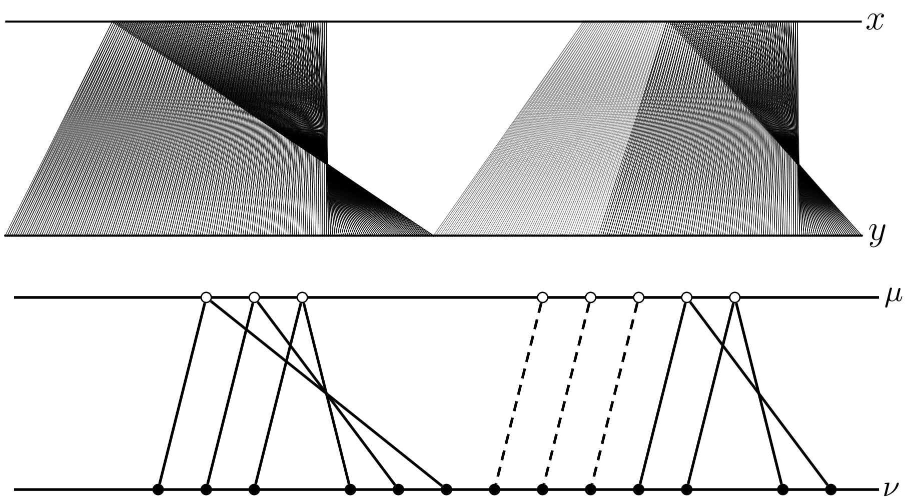

Figure 2.2: Simulations of the Decreasing Supermartingale Transport. We observe an interval of Right-Curtain kernels on the left, followed by an interval of Hoeffding–Fréchet kernels and another interval of Right-Curtain kernels. The destinations of these intervals preserve the original order.

supermartingale optimal transport problem

$$\sup_{P \in \mathcal{S}(\mu,\nu)} P(f) \tag{2.1.1}$$

where transports are required to be supermartingales, and then $\vec{P}, \overleftarrow{P}$ will be optimizers for reward functions f satisfying certain geometric properties. To make the connection with other texts on optimal transport, notice that $P(f) = E^P[f(X,Y)]$ in our notation, and that f can be seen as a cost function by a change of sign. We shall say that $f : \mathbb{R}^2 \to \mathbb{R}$ is *supermartingale Spence–Mirrlees* if

$$f(x_2,\cdot) - f(x_1,\cdot) \text{ is strictly decreasing and strictly convex for all } x_1 < x_2. \tag{2.1.2}$$

If f is smooth, this can be expressed through the cross-derivatives conditions $f_{xy} < 0$ and $f_{xyy} > 0$; the first one is the negative of the classical Spence–Mirrlees condition and the second is the so-called martingale Spence–Mirrlees condition. The following is a slightly simplified statement of Corollary 2.9.4.

Theorem 2.1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel, supermartingale Spence–Mirrlees and suppose that there exist $a \in L^1(\mu)$, $b \in L^1(\nu)$ such that*

$$|f(x,y)| \leq a(x) + b(y), \quad x, y \in \mathbb{R}.$$

Then, $\vec{P}$ is the unique solution of the supermartingale optimal transport problem (2.1.1)*. Similarly, $\overleftarrow{P}$ is the unique solution of $\inf_{P \in \mathcal{S}(\mu,\nu)} P(f)$, or equivalently of* (2.1.1) *if* *instead $-f$ is supermartingale Spence–Mirrlees.*

Since $\vec{P}$ and $\overleftarrow{P}$ correspond to the combinations $f_{xy} < 0, f_{xyy} > 0$ and $f_{xy} > 0, f_{xyy} < 0$ of known conditions, it is natural to ask for the remaining two combinations, $f_{xy} > 0, f_{xyy} > 0$ and $f_{xy} < 0, f_{xyy} < 0$. While the associated optimal transports also have interesting features, they turn out to depend on the function f within that class and hence, cannot be called canonical; cf. Section 2.10.

The third characterization of $\vec{P}$ and $\overleftarrow{P}$ is through their supports. A point (x, y) in the support can be thought of as a path that the transport is using, and the conditions are expressed as crossing or no-crossing conditions between the paths of the transport. While this characterization is an incarnation of the *c-cyclical monotonicity* of classical transport, the supermartingale constraint requires a novel distinction of the origins x into a set M of "martingale points" and their complement. Intuitively, the supermartingale constraint is binding at points of M and absent elsewhere—this will be made precise later on (Corollary 2.5.3). Thus, we work with a Borel set $\Gamma \in \mathcal{B}(\mathbb{R}^2)$ that should be thought of as a support, and a second set $M \in \mathcal{B}(\mathbb{R})$. We call the pair (Γ, M)

(i) first-order left-monotone if $y_1 \leq y_2$ whenever $x_2 \notin M$,

(ii) first-order right-monotone if $y_2 \leq y_1$ whenever $x_1 \notin M$,

for all paths $(x_1, y_1), (x_2, y_2) \in \Gamma$ with $x_1 < x_2$. We also need the following properties of Γ alone: considering three paths $(x, y_1), (x, y_2), (x', y') \in \Gamma$ with $y_1 < y_2$, the set Γ is second-order left-monotone (right-monotone) if $y' \notin (y_1, y_2)$ whenever $x < x'$

$(x > x')$. The latter two properties are taken from [13] where they are simply called left- and right-monotonicity, and all four properties are summarized in Figure 2.3.

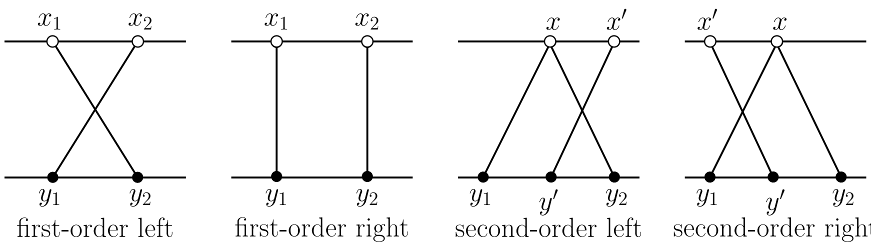

Figure 2.3: Forbidden configurations in the monotonicity properties

The following result is the summary of Theorem 2.8.1 and Corollary 2.9.6 in the body of the paper.

Theorem 2.1.3. *There exist nondegenerate*[1] $(\Gamma, M) \in \mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R})$ *which are first-order right-monotone and second-order left-monotone such that* $\vec{P}$ *is concentrated on* Γ *and* $\vec{P}|_{M\times\mathbb{R}}$ *is a martingale. Conversely, if* (Γ, M) *have those properties and* $P \in \mathcal{S}(\mu, \nu)$ *is a transport which is concentrated on* Γ *and* $P|_{M\times\mathbb{R}}$ *is a martingale, then* $P = \vec{P}$.

The analogous statement, interchanging left and right, holds for $\overleftarrow{P}$.

With some additional work, these theorems will allow us to explain the geometric features apparent in Figures 2.1 and 2.2. To that end, let us first recall two pairs of related couplings.

Our characterizations highlight the analogies between $\vec{P}, \overleftarrow{P}$ and the classical Hoeffding–Fréchet and Antitone couplings $P_{HF}, P_{AT} \in \Pi(\mu, \nu)$; see, e.g., [75, Section 3.1]. Indeed, the latter have a minimality property similar to Theorem 2.1.1, but

[1] This is a minor notion detailed in Definition 2.7.5.

for the first stochastic order instead of the convex-decreasing one. Moreover, they are optimal transports for reward functions satisfying the classical Spence–Mirrlees condition $f_{xy} > 0$ and its reverse, and they are characterized by what we called the first-order left- and right-monotonicity of their supports Γ (with $M = \mathbb{R}$).

The second pair of related couplings is given by the Left- and Right-Curtain couplings P_{LC}, P_{RC} introduced in [13] where martingale transport is studied; that is, the given marginals are in *convex* order and the transports are martingales. Indeed, these couplings are special cases of $\vec{P}$ and $\overleftarrow{P}$ that arise when the marginals $\mu \leq_{cd} \nu$ have the same barycenter—this corresponds to the fact that a supermartingale with constant mean is a martingale and vice versa. In that case, the first-order properties turn out to be irrelevant: in the shadow construction, the barycenter is constant and hence only the variance needs to be minimized; the condition for the reward functions is $f_{xyy} > 0$ (or < 0), and the second-order monotonicity property of Γ alone describes the support. As we shall see, it is the interaction between the first and second-order properties as well as the set M that generates the rich structure of $\vec{P}$ and $\overleftarrow{P}$.

Turning to $\vec{P}$ in Figure 2.1, the first observation is that there are only two types of transport kernels. On the left, $\vec{P}$ uses martingale kernels of the Left-Curtain type: each point on the x-axis is mapped to two points on the y-axis, and any two points x, x' satisfy the condition of second-order left-monotonicity. On the right, the transport is of Monge-type (each point x is mapped to a single point y) and has the structure of an Antitone coupling: any two paths intersect, which is the first-order right-monotonicity property. On the strength of the same property, points x in the portion to the right (thus not in M) can further be divided into two groups—the left group is mapped to points y to the right of the destinations of the martingale points, and vice versa. These facts about $\vec{P}$ are true not only in our example, but for arbitrary atomless marginals $\mu \leq_{cd} \nu$; see Remark 2.9.7.

Let us now turn to $\overleftarrow{P}$ in Figure 2.2. Similarly as before, we observe two types of paths; the Right-Curtain and the Hoeffding–Fréchet kernels. However, the intervals of martingale and deterministic transport alternate twice—there is no longer a unique phase transition; in general, there can be countably many transitions. On the other hand, the order of the intervals is now preserved by the transport—this corresponds to the combination of the first- and second-order properties. These two differences show that the geometries of $\overleftarrow{P}$ and $\vec{P}$ differ fundamentally and suggest that one cannot obtain one coupling from the other by a transformation of the marginals. By contrast, it is well known that P_{AT} can be constructed from P_{HF} via the transformation $(x, y) \mapsto (x, -y)$, whereas P_{RC} can be obtained from P_{LC} via $(x, y) \mapsto (-x, -y)$.

One common feature of $\vec{P}$ and $\overleftarrow{P}$ is that each consists of an optimal martingale transport and an optimal (unconstrained) Monge–Kantorovich transport. That turns out to be a general fact: a result that we call Extremal Decomposition (Corollary 2.5.3) states that given an optimal supermartingale transport P for an arbitrary reward function f, the restriction of P to $M \times \mathbb{R}$ is an optimal martingale transport and the restriction to $M^c \times \mathbb{R}$ is an optimal Monge–Kantorovich transport between its own marginals. (These marginals, however, are not easily determined a priori.)

Finally, let us mention two descriptions of $\vec{P}$ and $\overleftarrow{P}$ that suggest themselves (at least under additional conditions) but are not discussed in this paper. First, one may expect a purely analytic construction based on ordinary differential equations involving the distribution functions of the marginals. Second, one may represent the couplings as solutions to Skorokhod embedding problems with μ and ν as initial and target distributions. These descriptions will be provided in future contributions.

Methodology and Literature

Most of our results are based on the study of the optimal transport problem (2.1.1). We analyze this problem for general, Borel-measurable reward functions f, formulate a corresponding dual problem and establish strong duality; i.e., absence of a duality gap and existence of dual optimizers. A formal application of Lagrange duality suggests to consider triplets $\varphi \in L^1(\mu)$, $\psi \in L^1(\mu)$, $h : \mathbb{R} \to \mathbb{R}_+$ such that

$$\varphi(x) + \psi(y) + h(x)(y - x) \geq f(x, y), \quad (x, y) \in \mathbb{R}^2 \tag{2.1.3}$$

and define the dual value as $\inf\{\mu(\varphi) + \nu(\psi)\}$, where the infimum is taken over all triplets. Indeed, φ and ψ are Lagrange multipliers for the constraints μ and ν, whereas $h(x)(y - x)$ with $h \geq 0$ represents the supermartingale constraint $E^P[Y|X] \leq X$. While the corresponding duality for standard transport (without h) is valid by the celebrated result of [66], the dual problem for the supermartingale case needs to be relaxed in three ways to avoid a duality gap and ensure dual existence (Theorem 2.4.11). Namely, the range of h needs to be widened on parts of the state space, the integrability of φ and ψ needs to be loosened, and the inequality (2.1.3) needs to be relaxed on paths (x, y) that are not used by any transport (see Section 2.10 for pertinent counterexamples). In particular, it is important to classify all obstructions to supermartingale couplings; i.e., "barriers" that cannot be crossed (Proposition 2.3.2). Remarkably, there are no barriers beyond a specific point as soon as the barycenters of the marginals are not identical.

For the martingale transport, a related duality theory was provided in [16]. In that case, the barycenters of the marginals agree and the compactness arguments underlying the duality focus on controlling the convexity of certain functions. While we shall greatly benefit from those ideas, the supermartingale case requires us to control simultaneously first and second order properties (slope and convexity) which gives rise to substantial differences on the technical side; in fact, it turns out that controlling the slope necessitates a nontrivial increment between the barycenters of μ and ν.

Strong duality results in a monotonicity principle (Theorem 2.5.2) along the lines of the c-cyclical monotonicity condition of classical transport (e.g., [4, Theorem 2.13]): a variational result linking the optimality of a transport to the pointwise properties of its support. This principle is our main tool to study the couplings $\vec{P}$ and $\overleftarrow{P}$, parallel to the celebrated variational principle for the martingale case in [13] which has pioneered the idea that concepts similar to cyclical monotonicity can be developed beyond the classical transport setting. In the supermartingale transport problem, the monotonicity principle has a novel form describing a pair (Γ, M) of sets as in Theorem 2.1.3 rather than the support Γ alone. The set M enters the variational formulation by determining the class of competitors, much like it determines which paths are subject to the first-order monotonicity condition, and turns out to be fundamental in determining the geometries of $\vec{P}$ and $\overleftarrow{P}$.

As a variational result, the monotonicity principle necessitates knowing a priori that an optimal transport exists. We show that a supermartingale Spence–Mirrlees function f is automatically continuous (Proposition 2.9.2) in a tailor-made topology that is coarse enough to preserve weak compactness of $\mathcal{S}(\mu, \nu)$, and that yields the required existence. This result, together with the purely geometric formulation of the Spence–Mirrlees conditions (Definition 2.7.1), also improves the literature on martingale transport [13, 51, 62] where a range of assumptions is imposed on f both to ensure existence and to express the Spence–Mirrlees condition in terms of partial derivatives or a specific functional form; cf. Corollary 2.9.5. A second generalization is that Theorem 2.1.2 remains true if the Spence–Mirrlees condition (2.1.2) is satisfied in the non-strict sense, except that the optimizer need not be unique.

With the appropriate definitions in place, the proofs of Theorems 2.1.2 and 2.1.3 are then applications of the monotonicity principle that are based on the interplay between the first- and second-order monotonicity and Spence–Mirrlees conditions, and the structure of the set M. The construction of $\vec{P}$ and $\overleftarrow{P}$ with the minimality property of Theorem 2.1.1 rests on the precise understanding of the shadow of a single atom (Lemma 2.6.3) and compactness arguments. This construction is independent of the other results; it draws from a similar construction in [13] for the convex order. Technical steps aside, the main difference is that the barycenter of the shadow is part of the optimization rather than being fixed a priori by a martingale condition.

To the best of our knowledge, supermartingale couplings have not been specifically studied in the extant literature. However, as indicated above, martingale optimal transport has received considerable attention since it was introduced in [9] and [43].

In particular, [13, 51, 58, 59, 62] study optimal martingale transports between two marginals for specific cost functions; the martingale Spence–Mirrlees condition in the form $f_{xyy} > 0$ appears for the first time in [51], generalizing the functional form used in [13].

Martingale optimal transport is motivated by considerations of model uncertainty in financial mathematics. If, in the financial context, dynamic hedging is restricted by a no-shorting constraint, the dual problem is supermartingale transport. Thus, it can be seen as a special case of the dual problem in [41] where general portfolio constraints are studied. For background on Monge–Kantorovich transport, we refer to [4, 75, 76, 86, 85]. Recently, a rich literature has emerged around martingale transport and model uncertainty; see [57, 73, 84] for surveys and, e.g., [3, 15, 22, 23, 24, 25, 34, 41, 44, 45, 46, 70, 87] for models in discrete time, [19, 21, 27, 28, 37, 35, 36, 52, 50, 56, 68, 69, 81, 83] for continuous time, and [7, 10, 11, 26, 29, 47, 48, 49, 54, 55, 63, 74] for related Skorokhod embedding and mimicking problems.

The remainder of this paper is organized as follows. While Section 2.2 recalls basic facts related to the convex-decreasing order, Section 2.3 contains a complete description of the barriers to supermartingale couplings and more precisely, the structure of $\mathcal{S}(\mu,\nu)$-polar sets. After these preparations, Section 2.4 presents a complete duality theory for Borel reward functions, and Section 2.5 formulates the resulting monotonicity principle. Section 2.6 introduces the couplings $\vec{P}$ and $\overleftarrow{P}$ via the shadow construction. In Section 2.7, we propose the Spence–Mirrlees conditions for reward functions and show via the monotonicity principle that the associated optimal transports are supported on sets (Γ, M) satisfying corresponding monotonicity properties.

Section 2.8 continues the analysis by showing that any coupling supported on such sets must coincide with $\vec{P}$ or $\overleftarrow{P}$, respectively. In Section 2.9, we close the circle: Spence–Mirrlees functions are shown to admit optimal transports and on the strength of the duality theory, that allows us to conclude the existence of suitable sets (Γ, M). The main theorems stated in the Introduction then follow. The concluding Section 2.10 collects a number of counterexamples.

2.2 Preliminaries

It will be useful to consider finite measures, not necessarily normalized to be probabilities. Let μ, ν be finite measures on $\mathbb{R}$ with finite first moment. Extending the notation from the Introduction, we write $\Pi(\mu, \nu)$ for the set of all couplings; i.e., measures P on $\mathbb{R}^2$ such that $P \circ X^{-1} = \mu$ and $P \circ Y^{-1} = \nu$, where $(X, Y) : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity. Moreover, $\mathcal{S}(\mu, \nu)$ is the subset of all $P \in \Pi(\mu, \nu)$ which are supermartingales; i.e., $\int Y \mathbf{1}_A(X)\, dP \leq \int X \mathbf{1}_A(X)\, dP$ for all $A \in \mathcal{B}(\mathbb{R})$, and finally $\mathcal{M}(\mu, \nu)$ consist of all $P \in \Pi(\mu, \nu)$ satisfying this condition with equality.

We say that μ and ν are in *convex-decreasing order*, or *second stochastic order*, denoted $\mu \leq_{cd} \nu$, if $\mu(\phi) \leq \nu(\phi)$ for any convex, nonincreasing function $\phi : \mathbb{R} \to \mathbb{R}$. It then follows that μ and ν have the same total mass; moreover, we shall use repeatedly that it suffices to check the inequality for functions ϕ of linear growth. An alternative characterization of this order refers to the put (price) function, defined by

$$p_\mu : \mathbb{R} \to \mathbb{R}, \quad p_\mu(t) := \int (t - s)^+ \, \mu(ds).$$

Writing $\text{bary}(\mu) := (\int x\, d\mu)/\mu(\mathbb{R})$ for the barycenter (with $\text{bary}(\mu) := 0$ if $\mu = 0$) and $\partial^{\pm} p_\mu$ for the right and left derivatives, the following properties are easily verified:

(i) p_μ is nonnegative, increasing[2] and convex,

(ii) $\partial^+ p_\mu(t) - \partial^- p_\mu(t) = \mu(\{t\})$,

(iii) $\lim_{t\to-\infty} p_\mu(t) = 0$ and $\lim_{t\to\infty} p_\mu(t) = \infty \mathbf{1}_{\mu\neq 0}$,

(iv) $\lim_{t\to\infty}\{p_\mu(t) - \mu(\mathbb{R})[t - \text{bary}(\mu)]\} = 0$.

In particular, we may extend p_μ continuously to $\overline{\mathbb{R}} = [-\infty, \infty]$. The following result is classical; see, e.g., [42, Theorem 2.58].

Proposition 2.2.1. *Let μ, ν be finite measures on $\mathbb{R}$ with finite first moment and $\mu(\mathbb{R}) = \nu(\mathbb{R})$. The following are equivalent:*

(i) $\mu \leq_{cd} \nu$,

(ii) $p_\mu \leq p_\nu$,

(iii) $\mathcal{S}(\mu, \nu) \neq \emptyset$,

(iv) there exists a stochastic kernel $\kappa(x, dy)$ with finite mean such that $\int y\, \kappa(x, dy) \leq x$ for all $x \in \mathbb{R}$ and $\nu = (\mu \otimes \kappa) \circ Y^{-1}$.

In all that follows, the statement $\mu \leq_{cd} \nu$ implicitly means that μ, ν are finite measures on $\mathbb{R}$ with finite first moment. Moreover, such a pair and the corresponding supermartingale optimal transport problem will be called *proper* if the barycenters of μ and ν do *not* coincide. In the improper case, the problem degenerates to a martingale optimal transport problem because any supermartingale with constant

[2]Throughout this paper, increasing means nondecreasing.

mean is a martingale. Indeed, let us convene that μ and ν are in *convex order*, denoted $\mu \leq_c \nu$, if $\mu(\phi) \leq \nu(\phi)$ for any convex function $\phi : \mathbb{R} \to \mathbb{R}$, and introduce the symmetric potential function

$$u_\mu : \mathbb{R} \to \mathbb{R}, \quad u_\mu(t) := \int |t-s|\, \mu(ds).$$

Given $\mu \leq_{cd} \nu$, the following are then equivalent:

(a) $\mathrm{bary}(\mu) = \mathrm{bary}(\nu)$,

(b) $\mu \leq_c \nu$,

(c) $u_\mu \leq u_\nu$,

(d) $\mathcal{M}(\mu, \nu) \neq \emptyset$,

(e) the kernel κ in (iv) can be chosen with $\int y\, \kappa(x, dy) = x$ for all $x \in \mathbb{R}$.

2.3 Barriers and Polar Sets

We fix $\mu \leq_{cd} \nu$ throughout this section. Our first aim is to characterize all points $x \in \overline{\mathbb{R}}$ which cannot be crossed by any supermartingale transport $P \in \mathcal{S}(\mu, \nu)$.

Definition 2.3.1. A point $x \in \overline{\mathbb{R}}$ is called a *barrier* if $Y \leq x$ P-a.s. on $\{X \leq x\}$ and $Y \geq x$ P-a.s. on $\{X \geq x\}$, for all $P \in \mathcal{S}(\mu, \nu)$.

We may note that $\pm\infty$ are always barriers. The following result not only shows how barriers can be described as points where the put functions touch, but also introduces a particular barrier x^* which divides the real line into two parts: To the left of x^*, the supermartingale transport problem is in fact just a martingale transport problem. To the right of x^*, we have a proper supermartingale transport problem and there are no non-trivial barriers. The convention $\sup \emptyset = -\infty$ is used.

Proposition 2.3.2. *Define* $x^* := \sup\{x \in \mathbb{R} : p_\mu(x) = p_\nu(x)\} \in \overline{\mathbb{R}}$. *Then*

(i) x^* *is a barrier and* $p_\mu(x^*) = p_\nu(x^*)$*,*

(ii) a point $x \in [-\infty, x^*)$ *is a barrier if and only if* $p_\mu(x) = p_\nu(x)$*,*

(iii) if $x \in (x^*, \infty]$ *is a barrier then* $\mu(x, \infty) = \nu(x, \infty) = 0$*.*

Moreover, x^* *is the maximal barrier* $x \in \overline{\mathbb{R}}$ *such that* $P|_{\{X<x\}}$ *is a martingale transport for some (and then all)* $P \in \mathcal{S}(\mu, \nu)$*.*

The reverse implication in (iii) is almost true: a point x with $\mu(x, \infty) = \nu(x, \infty) = 0$ is not crossed by any transport. However, if μ has an atom at x, this mass may be transported to $(-\infty, x)$ and then x does not satisfy our definition of a barrier which is chosen so that any mass at the barrier remains invariant.

Before reporting the proof in Section 2.3, we use the above result to characterize the polar sets and the irreducible components of the supermartingale transport problem.

Definition 2.3.3. The pair $\mu \leq_{cd} \nu$ is *irreducible* if the set $I = \{p_\mu < p_\nu\}$ is connected and $\mu(I) = \mu(\mathbb{R})$. In this situation, let J be the union of I and any endpoints of I that are atoms of ν; then (I, J) is the *domain* of (μ, ν).

This definition coincides with the notion of [13, 16] in the context of martingale transport. More precisely, for $x < x^*$, we have $p_\mu(x) = p_\nu(x)$ if and only if $u_\mu(x) = u_\nu(x)$.

In the general case, the supermartingale transport problem will be decomposed into at most countably many irreducible components. There is a barrier between any two components so that they do not interact. Moreover, the components are minimal in that they do not contain (non-trivial) barriers. In the case where μ and ν are discrete measures, this amounts to saying that within a component, any atom of μ is connected to any atom of ν by some supermartingale transport. In the general case, this is made precise by saying that the polar sets of $\mathcal{S}(\mu,\nu)$ within a component are precisely the ones of $\Pi(\mu,\nu)$. We recall that a set is called polar for a family $\mathcal{P}$ of measures if it is P-null for all $P \in \mathcal{P}$.

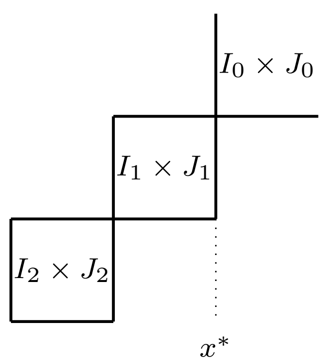

Figure 2.4: Illustration of two martingale components, separated from the supermartingale component $I_0 \times J_0$ by the maximal barrier x^*.

Proposition 2.3.4. *Let $\mu \leq_{cd} \nu$, let $I_0 = (x^*, \infty)$ and let $(I_k)_{1\leq k\leq N}$ be the (open) components of $\{p_\mu < p_\nu\} \cap (-\infty, x^*)$, where $N \in \{0, 1, \ldots, \infty\}$.*

(i) Set $I_{-1} = \mathbb{R} \setminus \cup_{k\geq 0} I_k$ and $\mu_k = \mu|_{I_k}$ for $k \geq -1$, so that $\mu = \sum_{k\geq -1} \mu_k$. Then,

there exists a unique decomposition $\nu = \sum_{k \geq -1} \nu_k$ *such that*

$$\mu_{-1} = \nu_{-1} \quad \text{and} \quad \mu_0 \leq_{cd} \nu_0 \quad \text{and} \quad \mu_k \leq_c \nu_k \quad \text{for all} \quad k \geq 1.$$

Moreover, this decomposition satisfies $I_k = \{p_{\mu_k} < p_{\nu_k}\}$ *for all* $k \geq 0$*; i.e., each such pair* (μ_k, ν_k) *is irreducible. Finally, any* $P \in \mathcal{S}(\mu, \nu)$ *admits a unique decomposition* $P = \sum_{k \geq -1} P_k$ *such that* $P_0 \in \mathcal{S}(\mu_0, \nu_0)$ *and* $P_k \in \mathcal{M}(\mu_k, \nu_k)$ *for all* $k \neq 0$.

(ii) Let $B \subseteq \mathbb{R}^2$ *be a Borel set. Then* B *is* $\mathcal{S}(\mu, \nu)$*-polar if and only if there exist a* μ*-nullset* N_μ *and a* ν*-nullset* N_ν *such that*

$$B \subseteq (N_\mu \times \mathbb{R}) \cup (\mathbb{R} \times N_\nu) \cup \left(\Delta \cup \bigcup_{k \geq 0} I_k \times J_k\right)^c,$$

where $\Delta = \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ *is the diagonal.*

Proofs of Propositions 2.3.2 and 2.3.4

We begin with the proof of Proposition 2.3.2, stated through a sequence of lemmas. We may assume that μ and ν are probability measures.

Lemma 2.3.5. *Let* $x \in \overline{\mathbb{R}}$*. If* $p_\mu(x) = p_\nu(x)$*, then* x *is a barrier.*

Proof. Let $p_\mu(x) = p_\nu(x)$ and let $E[\,\cdot\,]$ be the expectation associated with an arbitrary $P \in \mathcal{S}(\mu, \nu)$. Using $E[Y|X] \leq X$ and Jensen's inequality,

$$(x - X)^+ \leq (x - E[Y|X])^+ \leq E[(x - Y)^+|X],$$

and since $p_\mu(x) = p_\nu(x)$ means that $E[(x-X)^+] = E[(x-Y)^+]$, it follows that

$$(x-X)^+ = E[(x-Y)^+|X].$$

As a first consequence, we have

$$E[(x-Y)^+\mathbf{1}_{X\geq x}] = E[(x-X)^+\mathbf{1}_{X\geq x}] = 0$$

and hence $Y \geq x$ P-a.s. on $\{X \geq x\}$. A second consequence is that

$$E[(x-Y)\mathbf{1}_{X\leq x}] \leq E[(x-Y)^+\mathbf{1}_{X\leq x}] = E[(x-X)^+\mathbf{1}_{X\leq x}].$$

Since $E[Y|X] \leq X$ implies

$$E[(x-Y)\mathbf{1}_{X\leq x}] \geq E[(x-X)\mathbf{1}_{X\leq x}] = E[(x-X)^+\mathbf{1}_{X\leq x}],$$

it follows that $E[(x-Y)\mathbf{1}_{X\leq x}] = E[(x-Y)^+\mathbf{1}_{X\leq x}]$ and thus $Y \leq x$ P-a.s. on $\{X \leq x\}$. □

Lemma 2.3.6. *Let $x \in \overline{\mathbb{R}}$ be a barrier. The following are equivalent:*

(i) $E^P[X\mathbf{1}_{X<x}] = E^P[Y\mathbf{1}_{X<x}]$ for some (and then all) $P \in \mathcal{S}(\mu,\nu)$.

(ii) $P|_{\{X<x\}}$ is a martingale transport for some (and then all) $P \in \mathcal{S}(\mu,\nu)$.

Proof. If (i) holds for some $P \in \mathcal{S}(\mu,\nu)$, then (ii) holds for the same P since a supermartingale with constant mean is a martingale, and the converse holds as any martingale has constant mean. We complete the proof by showing that if (i) holds for one $P \in \mathcal{S}(\mu,\nu)$, it necessarily holds for all elements of $\mathcal{S}(\mu,\nu)$. The cases $x = \pm\infty$ are clear, so let $x \in \mathbb{R}$.

Let $P \in \mathcal{S}(\mu,\nu)$ and let ν' be the second marginal of $P' := P|_{\{X<x\}}$. As x is a barrier, we have $\nu' = \nu$ on $(-\infty, x)$. If $\bar{P} \in \mathcal{S}(\mu,\nu)$ is arbitrary and $\bar{P}', \bar{\nu}'$ are defined analogously, we have $\bar{\nu}' = \nu = \nu'$ on $(-\infty, x)$ by the same reasoning. But then also $\nu'(\{x\}) = \bar{\nu}'(\{x\})$, since this is the remaining mass transported from $(-\infty, x)$:

$$\nu'(\{x\}) = \mu(-\infty, x) - \nu'(-\infty, x) = \mu(-\infty, x) - \bar{\nu}'(-\infty, x) = \bar{\nu}'(\{x\}).$$

As a result, $\bar{\nu}' = \nu'$ on $(-\infty, x]$. In particular, $\bar{P}'$ satisfies (i) whenever P does. $\square$

Next, we define $x_* \in \overline{\mathbb{R}}$ by

$$x_* := \sup \big\{ x \in \mathbb{R} : \, p_\mu(x) = p_\nu(x), \ \ E[X\mathbf{1}_{X<x}] = E[Y\mathbf{1}_{X<x}] \big\},$$

where the expectation is taken under an arbitrary $P \in \mathcal{S}(\mu,\nu)$. Indeed, Lemmas 2.3.5 and 2.3.6 show that the definition is independent of the choice of P. Only a posteriori shall we see that $x_* = x^*$ is the quantity defined in Proposition 2.3.2; i.e., that the second condition in the definition of x_* is actually redundant.

Lemma 2.3.7. *We have $p_\mu(x_*) = p_\nu(x_*)$ and $E[X\mathbf{1}_{X<x_*}] = E[Y\mathbf{1}_{X<x_*}]$.*

Proof. The claim is trivial if $x_* = -\infty$. Otherwise, it suffices to observe that $p_\mu(x)$, $p_\nu(x)$, $E[X\mathbf{1}_{X<x}]$, $E[Y\mathbf{1}_{X<x}]$ are continuous in x along increasing sequences. This

follows by monotone/dominated convergence; recall that the elements of $\mathcal{S}(\mu,\nu)$ have a finite first moment. $\square$

Lemma 2.3.8. *Let $x \in \overline{\mathbb{R}}$ be a barrier such that $P|_{\{X<x\}}$ is a martingale transport for some $P \in \mathcal{S}(\mu,\nu)$. Then $p_\mu(x) = p_\nu(x)$.*

Proof. The cases $x = \pm\infty$ are clear, so let $x \in \mathbb{R}$. The martingale property yields that

$$p_\mu(x) = E[(x-X)\mathbf{1}_{X<x}] = E[(x-Y)\mathbf{1}_{X<x}].$$

Since $Y \leq x$ P-a.s. on $\{X < x\}$ and $\{Y < x\} \subseteq \{X < x\}$ P-a.s.,

$$E[(x-Y)\mathbf{1}_{X<x}] = E[(x-Y)^+\mathbf{1}_{X<x}] \geq E[(x-Y)^+\mathbf{1}_{Y<x}] = p_\nu(x).$$

Thus, $p_\mu(x) \geq p_\nu(x)$. As the converse inequality is always true, we deduce that $p_\mu(x) = p_\nu(x)$. $\square$

The following completes the proof of Proposition 2.3.2(ii) modulo the identity $x_* = x^*$.

Corollary 2.3.9. *Let $x \in [-\infty, x_*]$ be a barrier. Then $p_\mu(x) = p_\nu(x)$.*

Proof. We may assume that $x \in \mathbb{R}$, which entails $x_* > -\infty$. Lemmas 2.3.6 and 2.3.7 show that the restriction of any $P \in \mathcal{S}(\mu,\nu)$ to $\{X < x_*\}$ is a martingale transport. As $x \leq x_*$, the same holds for the restriction to $\{X < x\}$, and now Lemma 2.3.8 applies. $\square$

Lemma 2.3.10. *If $\bar{x} \in (x_*, \infty]$ is a barrier, then $\mu(\bar{x},\infty) = \nu(\bar{x},\infty) = 0$.*

Proof. The case $\bar{x} = \infty$ is clear. Let $\bar{x} \in (x_*, \infty)$ be a barrier and suppose for contradiction that $\mu(\bar{x}, \infty) > 0$ or $\nu(\bar{x}, \infty) > 0$.

Case 1: $\nu(\bar{x}, \infty) > 0$. We shall contradict the barrier property by constructing an element of $\mathcal{S}(\mu, \nu)$ which transports mass from $(-\infty, \bar{x})$ to $(\bar{x}, \infty)$, and vice versa.

Let $P \in \mathcal{S}(\mu, \nu)$ be arbitrary and let $P = \mu \otimes \kappa$ be a disintegration such that for all $x < \bar{x}$, we have $\mathrm{bary}(\kappa(x)) \leq x$ and $\kappa(x, dy)$ is concentrated on $(-\infty, \bar{x}]$ but not on $\{\bar{x}\}$; these choices are possible due to the barrier and the supermartingale property.

For each $x \in (-\infty, \bar{x})$, let $\varepsilon(x) \in [0, 1]$ be the largest number such that

$$\kappa'(x) := (1 - \varepsilon(x))\kappa(x)|_{(-\infty,\bar{x})} + \tilde{\varepsilon}(x)\nu|_{(\bar{x},\infty)} + \kappa(x)|_{\{\bar{x}\}}$$

satisfies $\mathrm{bary}(\kappa'(x)) \leq x$; here $\tilde{\varepsilon}(x)$ is the unique constant such that $\kappa'(x)$ is a probability measure. This defines a stochastic kernel with the properties

$$\kappa'(x)\{\bar{x}\} = \kappa(x)\{\bar{x}\} \quad \text{for all } x, \quad \kappa'(x)[\bar{x}, \infty) > \kappa(x)[\bar{x}, \infty) \quad \text{if } \varepsilon(x) > 0.$$

Moreover, $\varepsilon > 0$ on a set of positive μ-measure, as otherwise $P|_{\{X<\bar{x}\}}$ is a martingale transport which would contradict $\bar{x} > x_*$ (Lemma 2.3.8). Let ν_2 be the restriction to $(\bar{x}, \infty)$ of the second marginal of

$$\mu|_{(-\infty,\bar{x})} \otimes \kappa'.$$

By truncating the above function $\varepsilon(\cdot)$ at some positive constant $\bar{\varepsilon}$, we may assume that $\nu_2 \leq \nu$ while retaining the other properties. Thus, we can define a measure $\mu_2 \leq \mu$ by taking the preimage of ν_2 under P (obtained by disintegrating $P = \nu(dy) \otimes \hat{\kappa}(y, dx)$ and taking μ_2 to be the first marginal of $\nu_2(dy) \otimes \hat{\kappa}(y, dx)$). Moreover, let ν_1 be the restriction to $(-\infty, \bar{x})$ of the second marginal of

$$\mu|_{(-\infty,\bar{x})} \otimes \kappa \quad - \quad \mu|_{(-\infty,\bar{x})} \otimes \kappa'.$$

Then $c := \nu_1(\mathbb{R}) = \mu_2(\mathbb{R})$ and by construction,

$$\mu|_{(-\infty,\bar{x})} \otimes \kappa' \quad + \quad (\mu|_{[\bar{x},\infty)} - \mu_2) \otimes \kappa \quad + \quad c^{-1}\mu_2 \otimes \nu_1$$

is an element of $\mathcal{S}(\mu, \nu)$. Since $\mu(-\infty, \bar{x}) > 0$ and $\nu(\bar{x}, \infty) > 0$, it transports mass across $\bar{x}$, contradicting that $\bar{x}$ is a barrier.

Case 2: $\mu(\bar{x}, \infty) > 0$ and $\nu(\bar{x}, \infty) = 0$. Note that in this case, $\nu|_{[\bar{x},\infty)}$ is concentrated at $\bar{x}$ and the entire mass $\mu(\bar{x}, \infty) > 0$ is transported to that atom by any $P \in \mathcal{S}(\mu, \nu)$, in addition to any mass coming from $(-\infty, \bar{x}]$. We shall contradict the barrier property by constructing an element of $\mathcal{S}(\mu, \nu)$ which transports mass from $(\bar{x}, \infty)$ to $(-\infty, \bar{x})$; this will be balanced by moving appropriate mass from $(-\infty, \bar{x})$ to $\{\bar{x}\}$.

Let $P \in \mathcal{S}(\mu, \nu)$ be arbitrary and let κ be as above. For each $x \in (-\infty, \bar{x})$, let $\varepsilon(x) \in [0, 1]$ be the largest number such that

$$\kappa'(x) := (1 - \varepsilon(x))\kappa(x)|_{(-\infty,\bar{x})} + \tilde{\varepsilon}(x)\nu|_{\{\bar{x}\}}$$

satisfies $\mathrm{bary}(\kappa'(x)) \leq x$; again, $\tilde{\varepsilon}(x)$ is the unique constant such that $\kappa'(x)$ is a probability measure. This defines a stochastic kernel with

$$\kappa'(x)\{\bar{x}\} \geq \kappa(x)\{\bar{x}\} \quad \text{for all } x, \quad \kappa'(x)\{\bar{x}\} > \kappa(x)\{\bar{x}\} \quad \text{if } \varepsilon(x) > 0,$$

and again, $\varepsilon > 0$ on a set of positive μ-measure. Let ν_2 be the restriction to $\{\bar{x}\}$ of the second marginal of

$$\mu|_{(-\infty,\bar{x})} \otimes \kappa' \;\; - \;\; \mu|_{(-\infty,\bar{x})} \otimes \kappa.$$

After truncating $\varepsilon(\cdot)$ we again have $\nu_2 \leq \nu$; recall that P transports the mass $\mu(\bar{x},\infty) > 0$ to $\bar{x}$. Continuing the construction as above, the latter property shows that $\mu_2(\bar{x},\infty) > 0$, and thus the barrier property is again contradicted. $\square$

Corollary 2.3.11. *We have $p_\mu < p_\nu$ on (x_*,∞) and hence*

$$x_* = \sup\{x \in \mathbb{R} : \; p_\mu(x) = p_\nu(x)\} \equiv x^*.$$

Proof. Let $x \in (x_*,\infty)$ and suppose that $p_\mu(x) = p_\nu(x)$. In view of Lemma 2.3.5, x is a barrier, and now Lemma 2.3.10 yields that

$$E[x - X] = E[(x - X)\mathbf{1}_{X\leq x}] = p_\mu(x) = p_\nu(x) = E[(x - Y)\mathbf{1}_{Y\leq x}] = E[x - Y]$$

and thus $E[X] = E[Y]$. It follows that $E[X\mathbf{1}_{X<x}] = E[Y\mathbf{1}_{X<x}]$ and hence $x \leq x_*$ by the definition of x_*, a contradiction. $\square$

Proof of Proposition 2.3.2. In view of Corollary 2.3.11, Proposition 2.3.2 is a consequence of Lemmas 2.3.5, 2.3.7, 2.3.10 and Corollary 2.3.9. □

Proof of Proposition 2.3.4(i). According to Proposition 2.3.2, we face a pure martingale transport problem on $(-\infty, x^*]$; in particular, we may apply the decomposition result of [13, Theorem 8.4] on this part of the state space to obtain ν_k and P_k for $k \geq 1$. Since x^* is itself a barrier by Proposition 2.3.2, the only possible choice for ν_0 is

$$\nu_0 = \nu|_{(x^*,\infty)} + [\mu(x^*, \infty) - \nu(x^*, \infty)]\delta_{x^*},$$

and this measure satisfies $\mu_0 \leq_{cd} \nu_0$. □

The following is the main step towards the proof of the second part of Proposition 2.3.4.

Lemma 2.3.12. *If $\mu \leq_{cd} \nu$ is irreducible, the $\Pi(\mu, \nu)$-polar sets and the $\mathcal{S}(\mu, \nu)$-polar sets coincide.*

Proof. If μ and ν have the same barycenter, then $\mathcal{S}(\mu, \nu) = \mathcal{M}(\mu, \nu)$ and this is the result of [16, Corollary 3.4]. Thus, we may assume that (μ, ν) is proper. By Proposition 2.3.2, the associated domain (I, J) satisfies $I = (x^*, \infty)$ for some $x^* \in [-\infty, \infty)$, while $J = I$ if $\nu(\{x^*\}) = 0$ (including the case $x^* = -\infty$) and $J = [x^*, \infty)$ if $\nu(\{x^*\}) > 0$.

Since $\mathcal{S}(\mu, \nu) \subseteq \Pi(\mu, \nu)$, it suffices to show that for any $\pi \in \Pi(\mu, \nu)$ there exists $P \in \mathcal{S}(\mu, \nu)$ such that $P \gg \pi$. Let us show more generally that

for any measure π on $\mathbb{R}^2$ with marginals $\pi_1 \leq \mu$ and $\pi_2 \leq \nu$

$$\text{there exists } P \in \mathcal{S}(\mu, \nu) \text{ such that } P \gg \pi.$$

While π is necessarily supported by $I \times J$, we shall prove the claim under the additional condition that π is concentrated on a compact rectangle $K \times L \subset I \times J$. This entails no loss of generality: a general π may be decomposed into a sum $\pi = \sum_n \pi^n$ of measures satisfying this condition, and if P^n are the corresponding supermartingale measures, $P = \sum_n 2^{-n} P^n$ satisfies the claim.

The definition of (I, J) implies that ν assigns positive mass to any neighborhood of the lower endpoint x^* of J. More precisely, we can find a compact set $B \subset J$, located entirely to the left of $K \subset I$, such that $\nu(B) > 0$. (If $\nu(\{x^*\}) > 0$ we can simply take $B = \{x^*\}$.) Consider a disintegration $\pi = \pi_1 \otimes \kappa$ where $\kappa(x, dy)$ is concentrated on L for all $x \in K$. We introduce another stochastic kernel κ' of the form

$$\kappa'(x, dy) = \frac{\kappa(x, dy) + \varepsilon(x)\nu(dy)|_B}{c(x)}.$$

Here $c(x) \geq 1$ is the normalizing constant such that $\kappa'(x, dy)$ is a stochastic kernel. Moreover, $\varepsilon(x) := 0$ for x such that $\mathrm{bary}(\kappa(x)) \leq x$, whereas for x with $\mathrm{bary}(\kappa(x)) > x$ we let $\varepsilon(x)$ be the unique positive number such that $\mathrm{bary}(\kappa'(x)) = x$—this number exists by the intermediate value theorem; note that B is located to the left of $x \in K$. By construction,

$$\pi' := \nu(B)\pi_1 \otimes \kappa'$$

is a supermartingale measure with $\pi' \gg \pi$ and its marginals satisfy $\pi'_1 \leq \pi_1 \leq \mu$ as well as $\pi'_2 \leq \nu$; the latter is due to $\pi_1(\mathbb{R}) \leq \mu(\mathbb{R}) = 1$ and $\kappa'(x) \leq \nu(B)^{-1}\, \nu|_B + \kappa(x)$

and $\kappa(x)$ being concentrated on B^c. We also note that

$$\pi' \text{ is concentrated on a quadrant of the form } [k,\infty)^2 \tag{2.3.1}$$

with $[k,\infty) \subseteq J$; here $k \in \mathbb{R}$ is determined by the lower bound of the set B. We shall complete the proof by constructing $P \in \mathcal{S}(\mu,\nu)$ such that $P \gg \pi'$.

(i) We first consider the case where $\nu(\{x^*\}) = 0$ and hence $I = J = (x^*,\infty)$ and $k > x^*$. Using that $p_\nu - p_\mu$ is continuous, strictly positive on I and

$$\lim_{t\to\infty} p_\nu(t) - p_\mu(t) = \mu(\mathbb{R})[\mathrm{bary}(\nu) - \mathrm{bary}(\mu)] > 0,$$

we see that $p_\nu - p_\mu$ is uniformly bounded away from zero on $[k,\infty)$. On the other hand, $p_{\pi'_2} - p_{\pi'_1}$ is uniformly bounded on $[k,\infty)$ since

$$\lim_{t\to\infty} p_{\pi'_2}(t) - p_{\pi'_1}(t) = \pi'_1(\mathbb{R})[\mathrm{bary}(\pi'_1) - \mathrm{bary}(\pi'_2)] < \infty.$$

As a result, there exists $\epsilon > 0$ such that

$$p_\mu - \epsilon p_{\pi'_1} \leq p_\nu - \epsilon p_{\pi'_2}$$

on $[k,\infty)$, but then also on $\mathbb{R}$ because $p_{\pi'_1} = p_{\pi'_2} = 0$ outside of $[k,\infty)$ due to (2.3.1). Noting that this inequality may also be stated as

$$p_{\mu-\epsilon\pi'_1} \leq p_{\nu-\epsilon\pi'_2},$$

Proposition 2.2.1 shows that there exists some $P' \in \mathcal{S}(\mu - \epsilon\pi_1', \nu - \epsilon\pi_2')$, and we complete the proof by setting $P := P' + \epsilon\pi_1'(\mathbb{R})^{-1}\pi'$.

(ii) In the case $\nu(\{x^*\}) > 0$ we need to argue differently that there exists $\epsilon > 0$ such that $p_\mu - \epsilon p_{\pi_1'} \leq p_\nu - \epsilon p_{\pi_2'}$ on $[k, \infty)$. By enlarging $[k, \infty)$, we may assume that $k = x^*$ is the left endpoint of J. As $\mu(I) = \mu(\mathbb{R}) = \nu(J)$,

$$\partial^+ p_\mu(x^*) = \partial^+ p_\mu(x^*) - \partial^- p_\mu(x^*) = \mu(\{x^*\}) = 0$$

and similarly

$$\partial^+ p_{\pi_1'}(x^*) = 0, \quad \partial^+ p_{\pi_2'}(x^*) = \pi_2'(\{x^*\}), \quad \partial^+ p_\nu(x^*) = \nu(\{x^*\}) > 0.$$

Since $\nu(\{x^*\}) \geq \pi_2'(\{x^*\})$, it follows that

$$0 \neq \partial^+(p_\nu - p_\mu)(x^*) \geq \partial^+(p_{\pi_2'} - p_{\pi_1'})(x^*).$$

The existence of the desired $\epsilon > 0$ then follows and the rest of the argument is as in (i). □

Proof of Proposition 2.3.4(ii). By the decomposition in Proposition 2.3.4(i) and Lemma 2.3.12, a Borel set $B \subseteq \mathbb{R}^2$ is $\mathcal{S}(\mu, \nu)$-polar if and only if $B \cap (I_k \times J_k)$ is $\Pi(\mu_k, \nu_k)$-polar for all $k \geq 0$ and $B \cap \Delta$ is P_{-1}-null. It remains to apply the result of [18, Proposition 2.1] for each $k \geq 0$: a Borel set B_k is $\Pi(\mu_k, \nu_k)$-polar if and only if $B_k \subseteq (N_{\mu_k} \times \mathbb{R}) \cup (\mathbb{R} \times N_{\nu_k})$ for corresponding nullsets N_{μ_k} and N_{ν_k}. □

2.4 Duality Theory

In this section, we introduce and analyze a dual problem for supermartingale optimal transport. We shall prove that this problem admits an optimizer and that there is no duality gap.

Integration on a Proper Irreducible Component

We first introduce the notion of integrability that will be used for the dual elements. Let $\mu \leq_{cd} \nu$ be proper and irreducible with domain (I, J), and let $\chi : J \to \mathbb{R}$ be a concave increasing function. Since χ^+ has linear growth, $\mu(\chi)$ and $\nu(\chi)$ are well defined in $[-\infty, \infty)$. In what follows, we give a meaningful definition of the difference $\mu(\chi) - \nu(\chi)$ in cases where both terms are infinite. We write χ' for the *left* derivative of χ, with the convention that $\chi'(\infty) := \lim_{t\to\infty} \chi'(t) = \inf_{t\in I} \chi'(t)$, and $-\chi''$ for the second derivative measure of the convex function $-\chi$ on I. Finally, recall that $I = (x^*, \infty)$. If ν has an atom at x^*, then χ may have a jump at x^* and we denote its magnitude by

$$\Delta\chi(x^*) := \chi(x^*+) - \chi(x^*) \in \mathbb{R}_+.$$

Lemma 2.4.1. *Let $\mu \leq_{cd} \nu$ be proper and irreducible with domain (I, J), let $\chi : J \to \mathbb{R}$ be a concave increasing function, and let $P = \mu \otimes \kappa$ be an arbitrary element of*

$\mathcal{S}(\mu,\nu)$. *Then*

$$(\mu-\nu)(\chi) := \int_I \left[\chi(x) - \int_J \chi(y)\kappa(x,dy)\right] \mu(dx)$$
$$= \chi'(\infty)[\mathrm{bary}(\mu)-\mathrm{bary}(\nu)] + \int_I (p_\mu - p_\nu)\, d\chi'' + \Delta\chi(x^*)\nu(\{x^*\}).$$

In particular, the definition of $(\mu-\nu)(\chi) \in [0,\infty]$ *does not depend on* P.

Proof. We first note that the above integrals are well-defined with values in $[0,\infty]$; indeed, $\chi(x) - \int_J \chi(y)\kappa(x,dy)$ is nonnegative by Jensen's inequality and $p_\mu - p_\nu$ is nonpositive by Proposition 2.2.1. Moreover, by linearity of the ν-integral, it suffices to show the claimed identity for continuous χ. Indeed, χ can only have a discontinuity at x^*, and in that case we may decompose

$$\chi = \bar\chi - \Delta\chi(x^*)\mathbf{1}_{\{x^*\}}, \quad \text{where} \quad \bar\chi := \chi\mathbf{1}_I + \chi(x^*+)\mathbf{1}_{\{x^*\}}$$

with $\bar\chi$ being continuous, concave and increasing.

We first consider $\chi \in L^1(\mu)\cap L^1(\nu)$. After fixing an arbitrary point $a \in I = (x^*,\infty)$; we may assume without loss of generality that $\chi(a)=0$ by shifting χ. Then, by the definition of the second derivative measure,

$$\chi(y) = \chi'(a)(y-a) + \int_{(x^*,a)} (t-y)^+\,\chi''(dt) + \int_{[a,\infty)} (y-t)^+\,\chi''(dt)$$

for $y \in I$. Using monotone convergence on the right hand side, this extends to $y \in J$ since χ is continuous. Writing $B := \mathrm{bary}(\mu) - \mathrm{bary}(\nu)$, integrating this formula over

J shows that $\int_J \chi(s)(\mu-\nu)(ds)$ equals

$$\begin{aligned}
&\chi'(a)\int_J s\,(\mu-\nu)(ds) + \int_{[x^*,a)}\int_{(x^*,a)} (t-s)^+\,\chi''(dt)\,(\mu-\nu)(ds)\\
&+\int_{[a,\infty)}\int_{[a,\infty)} (s-t)^+\,\chi''(dt)\,(\mu-\nu)(ds)\\
&= B\chi'(a) + \int_{(x^*,a)}\int_{[x^*,a)} (t-s)^+\,(\mu-\nu)(ds)\,\chi''(dt)\\
&\quad +\int_{[a,\infty)}\int_{[a,\infty)} (s-t)^+\,(\mu-\nu)(ds)\,\chi''(dt)\\
&= B\chi'(a) + \int_{(x^*,a)}\int_J (t-s)^+\,(\mu-\nu)(ds)\,\chi''(dt)\\
&\quad +\int_{[a,\infty)}\int_J (s-t)^+\,(\mu-\nu)(ds)\,\chi''(dt).
\end{aligned}$$

Using $(s-t)^+ = (t-s)^+ + s - t$ in the last integral, this can be rewritten as

$$\begin{aligned}
&B\chi'(a) + \int_I (p_\mu - p_\nu)(t)\,\chi''(dt) + \int_{[a,\infty)}\int_J s\,(\mu-\nu)(ds)\,\chi''(dt)\\
&= B\chi'(a) + \int_I (p_\mu - p_\nu)(t)\,\chi''(dt) + B\chi''[a,\infty)\\
&= B\chi'(\infty) + \int_I (p_\mu - p_\nu)(t)\,\chi''(dt),
\end{aligned}$$

which completes the proof for $\chi \in L^1(\mu)\cap L^1(\nu)$.

For general (continuous) χ, let $\chi_n = \chi$ on $[x^*+1/n,\infty)$ and extend χ_n to J by an affine function with smooth fit. Then χ_n is concave, increasing and of linear growth, thus in $L^1(\mu)\cap L^1(\nu)$, while χ_n decreases to χ stationarily and $\chi_{n+1}-\chi_n$ is concave and increasing. Using monotone convergence, we then see that

$$\int_I \left[\chi_n(x) - \int_J \chi_n(y)\kappa(x,dy)\right] \mu(dx) \nearrow \int_I \left[\chi(x) - \int_J \chi(y)\kappa(x,dy)\right] \mu(dx),$$

$$\int_I (p_\mu - p_\nu)(t)\chi_n''(dt) \nearrow \int_I (p_\mu - p_\nu)\, d\chi'',$$

and now the result follows since $\chi_n'(\infty) = \chi'(\infty)$. □

Our next aim is to define expressions of the form $\mu(\varphi)+\nu(\psi)$ in a situation where the individual integrals are not necessarily finite. We continue to assume that $\mu \leq_{cd} \nu$ is proper and irreducible with domain (I,J).

Definition 2.4.2. Let $\varphi : I \to \overline{\mathbb{R}}$ and $\psi : J \to \overline{\mathbb{R}}$ be Borel functions. If there exists a concave increasing function $\chi : J \to \mathbb{R}$ such that $\varphi - \chi \in L^1(\mu)$ and $\psi + \chi \in L^1(\nu)$, we say that χ is a *moderator* for (φ,ψ) and set

$$\mu(\varphi) + \nu(\psi) := \mu(\varphi - \chi) + \nu(\psi + \chi) + (\mu - \nu)(\chi) \ \in (-\infty, \infty];$$

this value is independent of the choice of χ. We denote by $L^{ci}(\mu,\nu)$ the space of all pairs (φ,ψ) which admit a moderator χ such that $(\mu-\nu)(\chi) < \infty$.

Closedness on a Proper Irreducible Component

In this section, we introduce the dual problem for a proper and irreducible pair $\mu \leq_{cd} \nu$ with domain (I,J). It will be convenient to work with a nonnegative reward function f and alleviate this restriction later on (Remark 2.4.12).

Definition 2.4.3. Let $f : I \times J \to [0, \infty]$. We denote by $\mathcal{D}^{ci,pw}_{\mu,\nu}(f)$ the set of all Borel functions $(\varphi, \psi, h) : \mathbb{R} \to \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \mathbb{R}_+$ such that $(\varphi, \psi) \in L^{ci}(\mu, \nu)$ and

$$\varphi(x) + \psi(y) + h(x)(y - x) \geq f(x, y), \quad (x, y) \in I \times J.$$

We emphasize that in this definition, the inequality is stated in the pointwise ("pw") sense.

The following is the key result of this section. We mention that its assertion fails if (μ, ν) is not proper; cf. Section 2.10 for a counterexample.

Proposition 2.4.4. *Let $f, f_n : I \times J \to [0, \infty]$ be such that $f_n \to f$ pointwise and let $(\varphi_n, \psi_n, h_n) \in \mathcal{D}^{ci,pw}_{\mu,\nu}(f_n)$ satisfy $\sup_n \mu(\varphi_n) + \nu(\psi_n) < \infty$. Then, there exist*

$$(\varphi, \psi, h) \in \mathcal{D}^{ci,pw}_{\mu,\nu}(f) \quad \text{such that} \quad \mu(\varphi) + \nu(\psi) \leq \liminf_n \mu(\varphi_n) + \nu(\psi_n).$$

For the course of the proof, we abbreviate $\mathcal{D}^{ci}(f) := \mathcal{D}^{ci,pw}_{\mu,\nu}(f)$.

Lemma 2.4.5. *Let $(\varphi, \psi, h) \in \mathcal{D}^{ci}(0)$. There exists a moderator $\chi : J \to \mathbb{R}$ for (φ, ψ) such that $\chi \leq \varphi$ on I and $-\chi \leq \psi$ on J. In particular, we have $(\mu - \nu)(\chi) \leq \mu(\varphi) + \nu(\psi)$.*

Proof. Let $P = \mu \otimes \kappa$ be a disintegration of some $P \in \mathcal{S}(\mu, \nu)$. For later use, we first argue that $(\varphi, \psi, h) \in \mathcal{D}^{ci}(0)$ implies

$$\iint h(x)(y - x)\, \kappa(x, dy)\, \mu(dx) = \int h(x)(\operatorname{bary}(\kappa(x)) - x)\, \mu(dx) > -\infty. \quad (2.4.1)$$

(This is quite different from the property that $h(X)(Y-X) \in L^1(P)$ which may fail.) As $h \geq 0$ and $\mathrm{bary}(\kappa(x)) - x \leq 0$ μ-a.s., the integrals are well-defined. Moreover, the stated identity is clear. By the assumption $(\phi, \psi) \in L^{ci}(\mu, \nu)$ and therefore a moderator $\tilde{\chi}$ exists so that $\tilde{\varphi} := \varphi - \tilde{\chi} \in L^1(\mu)$ and $\tilde{\psi} := \psi + \tilde{\chi} \in L^1(\nu)$ and

$$(\mu - \nu)(\tilde{\chi}) = \iint [\tilde{\chi}(x) - \tilde{\chi}(y)]\, \kappa(x, dy)\, \mu(dx) \in \mathbb{R}_+.$$

We have

$$\tilde{\varphi}(x) + \tilde{\psi}(y) + [\tilde{\chi}(x) - \tilde{\chi}(y)] \geq -h(x)(y - x), \quad (x, y) \in I \times J.$$

For μ-a.e. $x \in I$, the integral with respect to $\kappa(x, dy)$ can be computed term-by-term and that yields

$$\tilde{\varphi}(x) + \int \tilde{\psi}(y)\, \kappa(x, dy) + \int [\tilde{\chi}(x) - \tilde{\chi}(y)]\, \kappa(x, dy) \geq -h(x)(\mathrm{bary}(\kappa(x)) - x).$$

In view of the stated integrability properties, we may again integrate term-by-term with respect to μ and obtain

$$\mu(\tilde{\varphi}) + \nu(\tilde{\psi}) + (\mu - \nu)(\tilde{\chi}) \geq -\int h(x)(\mathrm{bary}(\kappa(x)) - x)\, \mu(dx).$$

The left-hand side is finite by the assumption that $(\phi, \psi) \in L^{ci}(\mu, \nu)$, so the claim (2.4.1) follows. Summarizing the above for later reference, we have

$$
\begin{aligned}
&P[\varphi(X) + \psi(Y) + h(X)(Y - X)] \\
&\quad = P[\tilde{\varphi}(X) + \tilde{\psi}(Y) + (\tilde{\chi}(x) - \tilde{\chi}(y)) + h(X)(Y - X)] \\
&\quad = \mu(\varphi) + \nu(\psi) + \iint h(x)(y - x)\, \kappa(x, dy)\, \mu(dx) \in \mathbb{R}, \qquad (2.4.2)
\end{aligned}
$$

where the application of Fubini's theorem in the second equality is justified by the nonnegativity of the integrand.

We now move on to the main part of the proof. The function

$$\chi(y) := \inf_{x \in I} [\varphi(x) + h(x)(y - x)], \quad y \in J$$

is concave and increasing as an infimum of affine and increasing functions. As $(\varphi, \psi) \in L^{ci}(\mu, \nu)$, we have $\varphi < \infty$ on a nonempty set, thus $\chi < \infty$ everywhere on J. Moreover, $\chi \leq \varphi$ on I by the definition of χ. Our assumption that

$$\varphi(x) + \psi(y) + h(x)(y - x) \geq 0, \quad (x, y) \in I \times J \qquad (2.4.3)$$

shows that $\chi \geq -\psi$ on J. Since $(\varphi, \psi) \in L^{ci}(\mu, \nu)$, the set $\{\psi < \infty\}$ is dense in $\mathrm{supp}(\nu)$, and by concavity it follows that $\chi > -\infty$ on the interior of the convex hull of $\mathrm{supp}(\nu)$. As χ is increasing, it follows that $\chi > -\infty$ on I. Moreover, $\{\psi < \infty\}$ must contain any atom of ν and in particular $J \setminus I$, so that $\chi > -\infty$ on J.

Set $\bar{\varphi} := \varphi - \chi \geq 0$ and $\bar{\psi} := \psi + \chi \geq 0$. By the first part of the proof, the iterated integral with respect to κ and μ of the function

$$\varphi(x) + \psi(y) + h(x)(y - x) = \tilde{\varphi}(x) + \tilde{\psi}(y) + [\tilde{\chi}(x) - \tilde{\chi}(y)] + h(x)(y - x)$$

is finite. The function

$$\bar{\varphi}(x) + \bar{\psi}(y) + [\chi(x) - \chi(y)] + h(x)(y - x) \tag{2.4.4}$$

is identical to the above; therefore, the iterated integral of (2.4.4) is again finite. For fixed $x \in I$, all four terms in (2.4.4) are bounded from below by linearly growing functions. It follows that for μ-a.e. $x \in I$, the integral with respect to $\kappa(x, dy)$ can be computed term-by-term, which yields

$$\bar{\varphi}(x) + \int \bar{\psi}(y)\,\kappa(x, dy) + \int [\chi(x) - \chi(y)]\,\kappa(x, dy) + h(x)(\mathrm{bary}(\kappa(x)) - x).$$

The first three terms are nonnegative, and the last term is known to be μ-integrable by the first part of the proof. Thus, we may again integrate term-by-term with respect to μ. In conclusion, the iterated integral of (2.4.4), which was already determined to be finite, may also be computed term-by-term. In particular, we deduce that

$$\mu(\bar{\varphi}) < \infty, \quad \nu(\bar{\psi}) < \infty, \quad (\mu - \nu)(\chi) < \infty,$$

showing that $(\bar{\varphi}, \bar{\psi}) \in L^{ci}(\bar{\mu}, \bar{\nu})$ with concave moderator χ, and

$$\mu(\varphi) + \nu(\psi) = \mu(\bar{\varphi}) + \nu(\bar{\psi}) + (\mu - \nu)(\chi) \geq (\mu - \nu)(\chi)$$

as desired. □

Our last tool for the proof of Proposition 2.4.4 is a compactness principle for concave increasing functions. We mention that the conclusion fails if the pair $\mu \leq_{cd} \nu$ is not proper: a nontrivial difference between the barycenters is needed to control the first derivatives.

Proposition 2.4.6. *Let $a = \mathrm{bary}(\mu)$ and let $\chi_n : J \to \mathbb{R}$ be concave increasing functions such that*

$$\chi_n(a) = 0 \quad \textit{and} \quad \sup_{n \geq 1} (\mu - \nu)(\chi_n) < \infty.$$

There exists a subsequence χ_{n_k} which converges pointwise on J to a concave increasing function $\chi : J \to \mathbb{R}$ such that $(\mu - \nu)(\chi) \leq \liminf_k (\mu - \nu)(\chi_{n_k})$.

Proof. By our assumption, $(\mu - \nu)(\chi_n)$ is bounded uniformly in n. Since $\mathrm{bary}(\mu) >$ $\mathrm{bary}(\nu)$, the second representation in Lemma 2.4.1 shows that there exists a constant $C > 0$ such that

$$0 \leq \chi_n'(\infty) \leq C \quad \text{and} \quad 0 \leq \int_I (p_\mu - p_\nu)\, d\chi_n'' \leq C, \quad \text{and} \quad 0 \leq \Delta\chi_n(x^*) \leq C$$

in the case where $\nu(\{x^*\}) > 0$. For a suitable subsequence χ_{n_k}, we have

$$\lim_k \chi'_{n_k}(\infty) = \liminf_n \chi'_n(\infty) \tag{2.4.5}$$

and similarly

$$\lim_k \Delta\chi_{n_k}(x^*) = \liminf_n \Delta\chi_n(x^*) \quad \text{if } \nu(\{x^*\}) > 0. \tag{2.4.6}$$

Without loss of generality we assume that $n_k = k$. Given $y_0 \in I$, we recall from the proof of Lemma 2.3.12 that $p_\mu - p_\nu$ is strictly negative and uniformly bounded away from zero on $[y_0, \infty) \subseteq (x^*, \infty) = I$, and deduce that

$$0 \leq -\chi''_n[y_0, \infty) \leq C'$$

for a constant C'. Since the (left) derivative χ'_n is decreasing, it follows that

$$\chi'_n(y) = -\chi''_n[y, \infty) + \chi'_n(\infty) \leq C' + C \quad \text{for all} \quad y \in [y_0, \infty).$$

Thus, the Lipschitz constant of χ_n is bounded on compact subsets of I, uniformly in n. Recalling that $\chi_n(a) = 0$, the Arzela–Ascoli theorem then yields a function $\chi : I \to \mathbb{R}$ such that $\chi_n \to \chi$ locally uniformly, after passing to a subsequence. Clearly χ is concave and increasing, and integration by parts shows that $-\chi''_n$ converges to the second derivative measure $-\chi''$ associated with χ, in the sense of weak convergence relative to the compactly supported continuous functions on I. Approximating $p_\mu - p_\nu$

from above with compactly supported continuous functions, we then see that

$$\int_I (p_\mu - p_\nu)\, d\chi'' \le \liminf_{n\to\infty} \int_I (p_\mu - p_\nu)\, d\chi_n''.$$

Using also (2.4.5), (2.4.6) and the representation in Lemma 2.4.1, we conclude that $(\mu-\nu)(\chi) \le \liminf_{n\to\infty}(\mu-\nu)(\chi_n)$ as desired. □

We can now derive the closedness result.

Proof of Proposition 2.4.4. We may assume that $\liminf_n \mu(\varphi_n)+\nu(\psi_n) = \limsup_n \mu(\varphi_n)+\nu(\psi_n)$, by passing to a subsequence. Since $(\varphi_n,\psi_n,h_n) \in \mathcal{D}^{ci}(f_n)$ and $f_n \ge 0$, we can introduce the associated moderators χ_n as in Lemma 2.4.5. We may assume that $\chi_n(a) = 0$, where $a := \mathrm{bary}(\mu) \in I$, by shifting φ_n and ψ_n appropriately. After passing to a subsequence, Proposition 2.4.6 then yields a pointwise limit $\chi : J \to \mathbb{R}$.

Since $\varphi_n \ge \chi_n \to \chi$, Komlos' lemma (in the form of [33, Lemma A1.1] and its subsequent remark) yields $\bar\varphi_n \in \mathrm{conv}\{\varphi_n, \varphi_{n+1}, \dots\}$ which converge μ-a.s., and similarly for ψ_n. Without loss of generality, we may assume that $\bar\varphi_n = \varphi_n$, and similarly for ψ_n. Thus, $\varphi := \limsup \varphi_n$ and $\psi := \limsup \psi_n$ satisfy

$$\varphi_n \to \varphi \quad \mu\text{-a.s.}, \quad \varphi - \chi \ge 0 \quad \text{and} \quad \psi_n \to \psi \quad \nu\text{-a.s.}, \quad \psi + \chi \ge 0.$$

Fatou's lemma and Proposition 2.4.6 then show that

$$\begin{aligned}
\mu(\varphi - \chi) &+ \nu(\psi + \chi) + (\mu - \nu)(\chi) \\
&\leq \liminf \mu(\varphi_n - \chi_n) + \liminf \nu(\psi_n + \chi_n) + \liminf(\mu - \nu)(\chi_n) \\
&\leq \liminf[\mu(\varphi_n - \chi_n) + \nu(\psi_n + \chi_n) + (\mu - \nu)(\chi_n)] \\
&= \liminf[\mu(\varphi_n) + \nu(\psi_n)] < \infty.
\end{aligned}$$

In particular, $(\varphi, \psi) \in L^{ci}(\mu, \nu)$ with moderator χ, and then the above can be stated as $\mu(\varphi) + \nu(\psi) \leq \liminf \mu(\varphi_n) + \nu(\psi_n)$.

It remains to find a corresponding h. For any function $g : J \to \overline{\mathbb{R}}$, let $g^{ci} : J \to \overline{\mathbb{R}}$ denote the concave-increasing upper envelope. Fix $x \in I$. Our assumption that

$$\varphi_n(x) + h_n(x)(y - x) \geq f_n(x, y) - \psi_n(y), \quad (x, y) \in I \times J$$

implies that

$$\varphi_n(x) + h_n(x)(y - x) \geq [f_n(x, \cdot) - \psi_n]^{ci}(y), \quad (x, y) \in I \times J.$$

Using also the general inequality $\liminf(g_n^{ci}) \geq (\liminf g_n)^{ci}$, we deduce that for all

$y \leq x$,

$$\begin{aligned}
\varphi(x) &\geq \liminf \varphi_n(x) \\
&\geq \liminf \varphi_n(x) + h_n(x)(y - x) \\
&\geq \liminf [f_n(x, \cdot) - \psi_n]^{ci}(y) \\
&\geq [\liminf (f_n(x, \cdot) - \psi_n)]^{ci}(y) \\
&= [f(x, \cdot) - \psi]^{ci}(y) \\
&=: \hat{\varphi}(x, y).
\end{aligned}$$

As $\nu\{\psi = \infty\} = 0$ and $f > -\infty$, we have $\hat{\varphi}(x, y) > -\infty$ for all $y \in J$. If $x \notin N := \{\varphi = \infty\}$, choosing $y := x$ in the above inequalities also shows $\hat{\varphi}(x, x) \leq \varphi(x) < \infty$. As a result, the concave and increasing function $\hat{\varphi}(x, \cdot)$ is finite and therefore admits a finite left derivative

$$\partial^- \hat{\varphi}(x, \cdot)(y) \geq 0 \quad \text{for} \quad y \in I.$$

We define $h(x) := \partial^- \hat{\varphi}(x, \cdot)(x)$. By concavity, it follows that

$$\varphi(x) + h(x)(y - x) \geq \hat{\varphi}(x, x) + \partial^- \hat{\varphi}(x, \cdot)(x)(y - x) \geq \hat{\varphi}(x, y) \geq f(x, y) - \psi(y)$$

for all $y \in J$, for $x \in I \setminus N$. We may extend h to I by setting $h = 0$ on N. As $\varphi = \infty$ on N, the desired inequality

$$\varphi(x) + \psi(y) + h(x)(y - x) \geq f(x, y)$$

then extends to $x \in I$, and thus $(\varphi, \psi, h) \in \mathcal{D}^{ci}(f)$. $\square$

Duality on a Proper Irreducible Component

Recall that the pair $\mu \leq_{cd} \nu$ is proper and irreducible. We define the primal and dual values as follows.

Definition 2.4.7. Let $f : \mathbb{R}^2 \to [0, \infty]$. The *primal* and *dual* problems are respectively given by

$$\mathbf{S}_{\mu,\nu}(f) := \sup_{P \in \mathcal{S}(\mu,\nu)} P(f) \ \in [0, \infty],$$

$$\mathbf{I}^{pw}_{\mu,\nu}(f) := \inf_{(\varphi,\psi,h) \in \mathcal{D}^{ci,pw}_{\mu,\nu}(f)} \mu(\varphi) + \nu(\psi) \ \in [0, \infty],$$

where $P(f)$ is the outer integral if f is not measurable.

A function $f : \mathbb{R}^2 \to [0, \infty]$ is *upper semianalytic* if the sets $\{f \geq c\}$ are analytic for all $c \in \mathbb{R}$, where a subset of $\mathbb{R}^2$ is called analytic if it is the (forward) image of a Borel subset of a Polish space under a Borel mapping. Any Borel function is upper semianalytic and any upper semianalytic function is universally measurable; we refer to [20] for further background.

Proposition 2.4.8. *Let $\mu \leq_{cd} \nu$ be proper and irreducible, $f : \mathbb{R}^2 \to [0, \infty]$.*

(i) If f is upper semianalytic, then $\mathbf{S}_{\mu,\nu}(f) = \mathbf{I}^{pw}_{\mu,\nu}(f) \in [0, \infty]$.

(ii) If $\mathbf{I}^{pw}_{\mu,\nu}(f) < \infty$, there exists a dual optimizer $(\varphi, \psi, h) \in \mathcal{D}^{ci,pw}_{\mu,\nu}(f)$.

Proof. Let $f : \mathbb{R}^2 \to [0, \infty]$ be universally measurable, $P \in \mathcal{S}(\mu, \nu)$ and $(\varphi, \psi, h) \in \mathcal{D}^{ci,pw}_{\mu,\nu}(f)$. Then

$$\mu(\varphi) + \nu(\psi) \geq P[\varphi(X) + \psi(Y) + h(X)(Y - X)] \geq P(f)$$

by (2.4.1) and (2.4.2). It follows that $\mathbf{S}_{\mu,\nu}(f) \leq \mathbf{I}^{pw}_{\mu,\nu}(f)$, which is the easy inequality in (i). The proof of the converse inequality comprises of three steps. First, it is established for regular functions f by a Hahn–Banach separation argument on a suitable space of continuous functions, exploiting the closedness result of Proposition 2.4.4—alternately, a result of [41] could be applied. Second, one shows that the mappings $\mathbf{S}_{\mu,\nu}(\cdot)$ and $\mathbf{I}^{pw}_{\mu,\nu}(\cdot)$ are capacities with respect to the lattice of bounded, nonnegative upper semicontinuous functions which again uses the closedness. Finally, Choquet's capacitability theorem can be applied to extend the result from the first step to upper semianalytic f. We omit the details since these arguments are very similar to the proof of [16, Theorem 6.2].

To obtain (ii), it suffices to apply Proposition 2.4.4 with $f_n = f$ to a maximizing sequence (φ_n, ψ_n, h_n). $\square$

Global Duality

In this section, we formulate a global duality result. We shall be brief since it is little more than the combination of the preceding results for the proper irreducible case and the known martingale case; however, it requires some notation.

Let $\mu \leq_{cd} \nu$ be probability measures and let $f : \mathbb{R}^2 \to [0, \infty]$ be a Borel function.

As in the irreducible case, the primal problem is

$$\mathbf{S}_{\mu,\nu}(f) := \sup_{P \in \mathcal{S}(\mu,\nu)} P(f).$$

For the dual problem, we first recall from Proposition 2.3.4 the decompositions $\mu = \sum_{k\geq -1} \mu_k$ and $\nu = \sum_{k\geq -1} \nu_k$, where $\mu_k \leq_{cd} \nu_k$ is irreducible with domain (I_k, J_k) for $k \geq 0$ and $\mu_{-1} = \nu_{-1}$; moreover, P_{-1} is the unique element of $\mathcal{S}(\mu_{-1}, \mu_{-1})$. So far, we have focused on a proper pair (μ_0, ν_0) and its dual problem. The pairs (μ_k, ν_k) for $k \geq 1$ are in convex order (μ_k and ν_k have the same barycenter) and the corresponding martingale optimal transport has an analogous duality theory. While the arguments are different, the preceding results hold true if "convex-increasing" is replaced by "convex" and the function h is allowed to take values in $\mathbb{R}$ instead of $\mathbb{R}_+$; we refer to [16] for the proofs. The spaces corresponding to $L^{ci}(\mu,\nu)$ and $\mathcal{D}^{ci}_{\mu,\nu}(f)$ are denoted $L^c(\mu,\nu)$ and $\mathcal{D}^c_{\mu,\nu}(f)$, respectively.

Let $(\varphi, \psi, h) : \mathbb{R} \to \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \mathbb{R}$ be Borel. Since P_{-1} is concentrated on the diagonal Δ, the dual problem associated to (μ_{-1}, ν_{-1}) is trivially solved, for instance, by setting $\varphi(x) = f(x,x)$ and $\psi = h = 0$. To simplify the notation below, we set

$$L^c_{\mu_{-1},\nu_{-1}} := \{(\varphi,\psi) : \varphi + \psi \in L^1(\mu_{-1})\}$$

and $\mu_{-1}(\varphi) + \nu_{-1}(\psi) := \mu_{-1}(\varphi + \psi)$ for $(\varphi,\psi) \in L^c_{\mu_{-1},\nu_{-1}}$. Moreover, $\mathcal{D}^{c,pw}_{\mu_{-1},\nu_{-1}}(f)$ is the set of all (φ, ψ, h) with $(\varphi, \psi) \in L^c_{\mu_{-1},\nu_{-1}}$ and

$$\varphi(x) + \psi(x) \geq f(x, x), \quad x \in I_{-1}.$$

Finally, it will be convenient to define $\mathbf{S}_{\mu_{-1},\nu_{-1}}(f) := P_{-1}(f) \equiv \mu_{-1}(f(X, X))$.

We can now introduce the domain for the global dual problem which will be stated in the quasi-sure sense. A property is said to hold $\mathcal{S}(\mu, \nu)$-quasi surely, or $\mathcal{S}(\mu, \nu)$-q.s. for short, if it holds P-a.s. for all $P \in \mathcal{S}(\mu, \nu)$, or equivalently, if it holds up to a $\mathcal{S}(\mu, \nu)$-polar set.

Definition 2.4.9. Let $L(\mu, \nu)$ be the set of all Borel functions $\varphi, \psi : \mathbb{R} \to \overline{\mathbb{R}}$ such that $(\varphi, \psi) \in L^{ci}(\mu_0, \nu_0)$ and $(\varphi, \psi) \in L^c(\mu_k, \nu_k)$ for all $k \neq 0$ and

$$\sum_{k \geq -1} |\mu_k(\varphi) + \nu_k(\psi)| < \infty.$$

For $(\varphi, \psi) \in L(\mu, \nu)$, we define

$$\mu(\varphi) + \nu(\psi) := \sum_{k \geq -1} \mu_k(\varphi) + \nu_k(\psi) < \infty,$$

and $\mathcal{D}_{\mu,\nu}(f)$ is the set of all Borel functions $(\varphi, \psi, h) : \mathbb{R} \to \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \mathbb{R}$ such that $(\varphi, \psi) \in L(\mu, \nu)$, $h \geq 0$ on I_0 and

$$\varphi(X) + \psi(Y) + h(X)(Y - X) \geq f(X, Y) \quad \mathcal{S}(\mu, \nu)\text{-q.s.}$$

Finally,

$$\mathbf{I}_{\mu,\nu}(f) := \inf_{(\varphi,\psi,h)\in\mathcal{D}_{\mu,\nu}(f)} \mu(\varphi)+\nu(\psi) \ \in [0,\infty].$$

We emphasize that h is required to be nonnegative on I_0 but can take arbitrary real values outside of I_0. It is shown in Section 2.10 that nonnegativity cannot be enforced everywhere.

Before making precise the correspondence between this quasi-sure formulation and the individual components, let us recall that the intervals J_k may overlap at their endpoints, so we have to avoid counting certain things twice. Indeed, let $(\varphi_k,\psi_k,h_k)\in\mathcal{D}^{c(i),pw}_{\mu_k,\nu_k}(f)$; we claim that ψ_k can be normalized such that

$$\psi_k = 0 \quad \text{on} \quad J_k\setminus I_k. \tag{2.4.7}$$

Indeed, if J_k contains one of its endpoints, it is an atom of ν and hence ψ_k is finite on $J_k\setminus I_k$. If $k\geq 1$, we can translate ψ_k by an affine function and shift φ_k and h_k accordingly. In the supermartingale case $k=0$, we recall from Proposition 2.3.4 that $I_0=(x^*,\infty)$, so that J_0 can have at most one endpoint. As a result, we may obtain the normalization by shifting ψ_0 with a constant, which can be compensated by shifting φ_0 alone and thus respecting the requirement that $h_0\geq 0$ on I_0.

The dual domain can then be decomposed as follows.

Lemma 2.4.10. *Let $f:\mathbb{R}^2\to[0,\infty]$ be Borel, let $\mu\leq_{cd}\nu$ and let μ_k,ν_k be as in Proposition 2.3.4.*

(i) Let $(\varphi_0,\psi_0,h_0)\in\mathcal{D}^{ci,pw}_{\mu_0,\nu_0}(f)$ and $(\varphi_k,\psi_k,h_k)\in\mathcal{D}^{c,pw}_{\mu_k,\nu_k}(f)$ for $k\geq 1$, normalized

as in (2.4.7)*, and let* $\varphi_{-1}(x) = f(x,x)$ *and* $\psi_{-1} = 0$*. If* $\sum_{k\geq -1} \mu(\varphi_k) + \nu(\psi_k) < \infty$*, then*

$$\varphi := \sum_{k\geq -1} \varphi_k \mathbf{1}_{I_k}, \quad \psi := \sum_{k\geq 0} \psi_k \mathbf{1}_{J_k}, \quad h := \sum_{k\geq 0} h_k \mathbf{1}_{I_k}$$

satisfies $(\varphi, \psi, h) \in \mathcal{D}_{\mu,\nu}(f)$ *and* $\mu(\varphi) + \nu(\psi) = \sum_{k\geq -1} \mu_k(\varphi_k) + \nu_k(\psi_k)$*.*

(ii) Conversely, let $(\varphi, \psi, h) \in \mathcal{D}_{\mu,\nu}(f)$*. After changing* φ *on a* μ*-nullset and* ψ *on a* ν*-nullset, we have* $(\varphi, \psi, h) \in \mathcal{D}^{ci,pw}_{\mu_0,\nu_0}(f)$ *and* $(\varphi, \psi, h) \in \mathcal{D}^{c,pw}_{\mu_k,\nu_k}(f)$ *for* $k \neq 0$*, and*

$$\sum_{k\geq -1} \mu_k(\varphi) + \nu_k(\psi) = \mu(\varphi) + \nu(\psi) < \infty.$$

This is a direct consequence of Proposition 2.3.4; the details of the proof are analogous to [16, Lemma 7.2]. We can now state the global duality result.

Theorem 2.4.11. *Let* $f : \mathbb{R}^2 \to [0, \infty]$ *be Borel and let* $\mu \leq_{cd} \nu$*. Then*

$$\mathbf{S}_{\mu,\nu}(f) = \mathbf{I}_{\mu,\nu}(f) \in [0, \infty].$$

If $\mathbf{I}_{\mu,\nu}(f) < \infty$*, there exists an optimizer* $(\varphi, \psi, h) \in \mathcal{D}_{\mu,\nu}(f)$ *for* $\mathbf{I}_{\mu,\nu}(f)$*.*

This is a direct consequence of Proposition 2.4.8 and the corresponding result in the martingale case; the details of the proof are as in [16, Theorem 7.4].

Remark 2.4.12. The lower bound on f in Theorem 2.4.11 can easily be relaxed. Indeed, let $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be Borel and suppose that there exist $a \in L^1(\mu)$, $b \in L^1(\nu)$ such that

$$f(x, y) \geq a(x) + b(y), \quad x, y \in \mathbb{R}.$$

Then, we may apply Theorem 2.4.11 to $\bar{f} := [f(X,Y) - a(X) - b(Y)]^+$ and deduce the duality result for f as well.

2.5 Monotonicity Principle

An important consequence of the duality theorem is a monotonicity principle describing the support of optimal transports; it can be seen as a substitute for the cyclical monotonicity from classical transport theory. The following notion will be useful for our study of the canonical couplings.

Definition 2.5.1. Let π be a finite measure on $\mathbb{R}^2$ with finite first moment and let $M_0, M_1 \subseteq \mathbb{R}$ be Borel. Denote by π_1 its first marginal and by $\pi = \pi_1 \otimes \kappa$ a disintegration. A measure π' is an (M_0, M_1)-*competitor* of π if it has the same marginals and if its disintegration $\pi' = \pi_1 \otimes \kappa'$ satisfies

$$\begin{aligned}
\operatorname{bary}(\kappa'(x)) &\leq \operatorname{bary}(\kappa(x)) \quad \text{for } \pi_1\text{-a.e. } x \in M_0, \\
\operatorname{bary}(\kappa'(x)) &= \operatorname{bary}(\kappa(x)) \quad \text{for } \pi_1\text{-a.e. } x \in M_1.
\end{aligned}$$

This definition extends a concept of [13] where the barycenters are required to be equal on the whole real line. In our context, we need to distinguish three regimes for the applications in the subsequent sections: equality, inequality, and no constraint on the barycenters.

Given $\mu \leq_{cd} \nu$, we recall from Proposition 2.3.4 the sets I_k, J_k, where the labels $k \geq 1$ correspond to the martingale components, $k = 0$ is the supermartingale component, and $k = -1$ is the complement (where any transport from μ to ν is the identity). Moreover, any element of $\mathcal{S}(\mu,\nu)$ is necessarily supported by the set

$$\Sigma := \Delta \cup \bigcup_{k\geq 0} I_k \times J_k. \tag{2.5.1}$$

Theorem 2.5.2 (Monotonicity Principle). *Let $f : \mathbb{R}^2 \to [0,\infty]$ be Borel, let $\mu \leq_{cd} \nu$ be probability measures and suppose that $\mathbf{S}_{\mu,\nu}(f) < \infty$. There exist a Borel set $\Gamma \subseteq \mathbb{R}^2$ and disjoint Borel sets $M_0, M_1 \subseteq \mathbb{R}$ with the following properties, where $M := M_0 \cup M_1$.*

(i) A measure $P \in \mathcal{S}(\mu,\nu)$ is optimal for $\mathbf{S}_{\mu,\nu}(f)$ if and only if it is concentrated on Γ and $P|_{M\times\mathbb{R}}$ is a martingale.

(ii) Let $\bar\mu \leq_{cd} \bar\nu$ be probabilities on $\mathbb{R}$. If $\bar P \in \mathcal{S}(\bar\mu,\bar\nu)$ is concentrated on Γ and $\bar P|_{M\times\mathbb{R}}$ is a martingale, then $\bar P$ is optimal for $\mathbf{S}_{\bar\mu,\bar\nu}(f)$.

(iii) Let π be a finitely supported probability on $\mathbb{R}^2$ which is concentrated on Γ. Then $\pi(f) \geq \pi'(f)$ for any (M_0, M_1)-competitor π' of π that is concentrated on Σ.

If $(\varphi,\psi,h) \in \mathcal{D}_{\mu,\nu}(f)$ is a suitable[3] version of the optimizer from Theorem 2.4.11, then we can take

$$M_0 := I_0 \cap \{h > 0\},$$

$$M_1 := \cup_{k\neq 0} I_k,$$

$$\Gamma := \big\{(x,y) \in \mathbb{R}^2 :\ \varphi(x) + \psi(y) + h(x)(y-x) = f(x,y)\big\} \cap \Sigma.$$

[3] chosen as in Lemma 2.4.10 (ii)

Moreover, the assertion in (iii) remains true if π is not finitely supported, as long as $(\varphi, \psi) \in L(\pi_1, \pi_2)$, where π_1 and π_2 are the marginals of π.

Before giving the proof, let us draw a corollary stating that the supermartingale optimal transport can be decomposed as follows. On M, an optimizer $P \in \mathcal{S}(\mu, \nu)$ is also an optimizer of a martingale optimal transport problem. Thus, we think of M as the set where the supermartingale constraint is "binding," and in fact it acts like the seemingly stronger martingale constraint (thus M as in *martingale*). Whereas on $N := \mathbb{R} \setminus M$, the measure P is also an optimizer of a (Monge–Kantorovich) optimal transport problem with no constraint at all on the dynamics (N as in *no* constraint).

Corollary 2.5.3 (Extremal Decomposition)**.** *Let $f : \mathbb{R}^2 \to [0, \infty]$ be Borel and let $\mu \leq_{cd} \nu$ be probability measures such that $\mathbf{S}_{\mu,\nu}(f) < \infty$. There exists a Borel set $M \subseteq \mathbb{R}$ with the following property.*

Given an optimizer $P \in \mathcal{S}(\mu, \nu)$ for $\mathbf{S}_{\mu,\nu}(f)$, let $\mu_M = \mu|_M$ and let ν_M be the image[4] of μ_M under P. Moreover, let $\mu_N = \mu|_{\mathbb{R}\setminus M}$ and let ν_N be the image of μ_N under P. Then for the same function f as above,

(i) $P|_{M\times\mathbb{R}}$ is an optimal martingale *transport from μ_M to ν_M,*

(ii) $P|_{N\times\mathbb{R}}$ is an optimal Monge–Kantorovich *transport from μ_N to ν_N.*

A word of caution is in order: while the set M is defined without reference to P, the second marginals ν_M, ν_N in the extremal problems do depend on P. In that sense, the decomposition is non-unique—which, however, is quite natural given that the optimizer P is non-unique as well, for general f.

[4] If $P = \mu \otimes \kappa$, the image of μ_M under P is defined as the second marginal of $\mu_M \otimes \kappa$.

Remark 2.5.4. The lower bound on f in Theorem 2.5.2 and Corollary 2.5.3 can be relaxed as follows. Instead of f being nonnegative, suppose that there exist real functions $a \in L^1(\mu)$, $b \in L^1(\nu)$ such that

$$f(x, y) \geq a(x) + b(y), \quad x, y \in \mathbb{R}.$$

Then, Theorem 2.5.2 (i), (iii) as well as Corollary 2.5.3 hold as above, using Remark 2.4.12 but otherwise the same proofs. Moreover, Theorem 2.5.2 (ii) as well as the last statement in Theorem 2.5.2 hold under the condition that a, b are integrable for $\bar{\mu}, \bar{\nu}$ and π_1, π_2, respectively.

Example 2.5.5. In the context of Corollary 2.5.3, suppose that μ has no atoms and that f is smooth, of linear growth, and satisfies the Spence–Mirrlees condition $f_{xy} > 0$ and the martingale Spence–Mirrlees condition $f_{xyy} > 0$ (this is *not* one of the canonical cases studied later). Then an optimizer P exists and the corollary implies that $P|_{M \times \mathbb{R}}$ is the Left-Curtain coupling [13] between its marginals and $P|_{N \times \mathbb{R}}$ is the Hoeffding–Fréchet coupling [75, Section 3.1] between its marginals. In particular, writing $P = \mu \otimes \kappa$ and using the results of the indicated references, we immediately deduce the possible forms of the kernel: at almost every x, $\kappa(x)$ is either deterministic (the Hoeffding–Fréchet kernel) or a martingale kernel concentrated at two points (the Left–Curtain kernel). In particular, $\kappa(x)$ is never what might seem to be the typical case—a truly random process with downward drift.

In this situation, the decomposition $\mathbb{R} = M \cup N$ is in fact essentially unique (points where the transport is the identity can be assigned to either M or N). Indeed, suppose that P' is another optimizer and let κ' be the associated kernel. The above properties apply to κ' as well, and now if the decomposition were different then the optimizer $P'' = (P+P')/2$ would have a kernel $\kappa'' = (\kappa+\kappa')/2$ violating those same properties.

We mention that the coupling P is nevertheless not canonical in the sense of the Introduction: the decomposition *does* change if we replace f by a different function satisfying the same Spence–Mirrlees conditions; cf. Section 2.10 for a counterexample.

Proof of Corollary 2.5.3. Let $M = M_0 \cup M_1$ and (φ, ψ, h) be as in Theorem 2.5.2, and note that $P(f) < \infty$.

(i) We have $P_M := P|_{M\times\mathbb{R}} \in \mathcal{M}(\mu_M, \nu_M)$ by (i) of the theorem. Moreover, setting $P_N := P|_{N\times\mathbb{R}}$, any $\bar{P}_M \in \mathcal{M}(\mu_M, \nu_M)$ induces an element of $\mathcal{S}(\mu,\nu)$ via $\bar{P} := \bar{P}_M + P_N$. Thus, $\bar{P}_M(f) > P_M(f)$ would contradict the optimality of P.

(ii) This part is less direct because elements of $\Pi(\mu_N, \nu_N)$ are not supermartingales in general; we shall invoke Theorem 2.5.2(iii) with $\pi := P$. By (i) of the theorem, π is concentrated on Γ, and of course $(\varphi,\psi) \in L(\mu,\nu) = L(\pi_1,\pi_2)$. Moreover, for any $\pi'_N \in \Pi(\mu_N,\nu_N)$, the measure $\pi' = P_M + \pi'_N$ is an (M_0, M_1)-competitor of $\pi = P_M + P_N$ which is concentrated on Σ as P_N is concentrated on $I_0 \times J_0 \subseteq \Sigma$; note that $N \subseteq I_0$. Now the extension of (iii) at the end of the theorem yields $\pi(f) \geq \pi'(f)$ and hence $P_N(f) \geq \pi'_N(f)$. $\square$

Proof of Theorem 2.5.2. As $\mathbf{I}_{\mu,\nu}(f) = \mathbf{S}_{\mu,\nu}(f) < \infty$, Theorem 2.4.11 yields a dual optimizer $(\varphi,\psi,h) \in \mathcal{D}_{\mu,\nu}(f)$ and we can define Γ and M as stated.

(i) Let $P \in \mathcal{S}(\mu,\nu)$ and let $P = \mu \otimes \kappa$ be a disintegration. Recalling (2.4.1) and (2.4.2) and the analogous facts for the martingale case [16], we have

$$\begin{aligned} P(f) &\leq P[\varphi(X) + \psi(Y) + h(X)(Y - X)] \\ &= \mu(\varphi) + \nu(\psi) + \iint h(x)(y - x)\, \kappa(x, dy)\, \mu(dx) \\ &\leq \mu(\varphi) + \nu(\psi). \end{aligned}$$

Since $\mathbf{S}_{\mu,\nu}(f) = \mu(\varphi) + \nu(\psi)$, P is optimal if and only if both inequalities are equalities. As $P(f) < \infty$, the first inequality is an equality if and only if P is concentrated on Γ. Moreover, the second inequality is an equality if and only if $\int (y - x)\, \kappa(x, dy) = 0$ μ-a.e. on $\{h > 0\}$; note that the condition on κ holds automatically on the martingale components I_k, $k \geq 1$. In particular, this is equivalent to $P|_{M\times\mathbb{R}}$ being a martingale.

(ii) We choose a version of $(\varphi, \psi, h) \in \mathcal{D}_{\mu,\nu}(f)$ as in Lemma 2.4.10 (ii); moreover, we may assume that $\bar{P}(f) < \infty$. We need to show that $(\varphi, \psi, h) \in \mathcal{D}_{\bar{\mu},\bar{\nu}}(f)$; once this is established, optimality can be argued exactly as in (i) above.

(a) On the one hand, we need to show that

$$\varphi(X) + \psi(Y) + h(X)(Y - X) \geq f(X, Y) \quad \mathcal{S}(\bar{\mu}, \bar{\nu})\text{-q.s.} \tag{2.5.2}$$

For this, it suffices to prove that the domains of the irreducible components of $\bar{\mu} \leq_{cd} \bar{\nu}$ are subsets of the ones of $\mu \leq_{cd} \nu$; i.e., that $p_\mu(x) = p_\nu(x)$ implies $p_{\bar{\mu}}(x) = p_{\bar{\nu}}(x)$, for any $x \in \mathbb{R}$. Indeed, let $p_\mu(x) = p_\nu(x)$. Since $\bar{P}$ is concentrated on $\Gamma \subseteq \Sigma$, we know that $Y \leq x$ $\bar{P}$-a.s. on $\{X \leq x\}$ and $Y \geq x$ $\bar{P}$-a.s. on $\{X \geq x\}$. Writing $E[\,\cdot\,]$ for the expectation under $\bar{P}$, it follows that

$$p_{\bar{\nu}}(x) = E[(x-Y)^+] = E[(x-Y)\mathbf{1}_{X\leq x}].$$

Note that $p_\mu(x) = p_\nu(x)$ implies $x \leq x^*$, cf. Proposition 2.3.2. Recalling that $(-\infty, x^*) \subseteq M$, our assumption on $\bar{P}$ then yields that $\bar{P}|_{\{X<x\}}$ is a martingale. As a consequence,

$$E[(x-Y)\mathbf{1}_{X\leq x}] = E[(x-X)\mathbf{1}_{X\leq x}] = E[(x-X)^+] = p_{\bar{\mu}}(x)$$

and this part of the proof is complete.

(b) On the other hand, we need to show that $(\varphi, \psi) \in L(\bar{\mu}, \bar{\nu})$. By reducing to the components, we may assume without loss of generality that $(\bar{\mu}, \bar{\nu})$ is irreducible with domain (I, J). Moreover, the argument for the martingale case is contained in the proof of [16, Corollary 7.8], so we shall assume that $(\bar{\mu}, \bar{\nu})$ is proper. Let

$$\chi(y) := \inf_{x\in I} [\varphi(x) + h(x)(y-x)], \quad y \in J.$$

As $(\varphi, \psi, h) \in \mathcal{D}^{ci,pw}_{\mu,\nu}(f)$, the arguments below (2.4.2) yield that $\chi : J \to \mathbb{R}$ is concave and increasing, that $\bar{\varphi} := \varphi - \chi \geq 0$ and $\bar{\psi} := \psi + \chi \geq 0$, and that $\bar{P}[\varphi(X) + \psi(Y) + h(X)(Y-X)]$ can be computed as the $\bar{\mu}(dx)$-integral of

$$\bar{\varphi}(x) + \int \bar{\psi}(y)\,\kappa(x,dy) + \left[\chi(x) - \int \chi(y)\kappa(x,dy)\right] + h(x)(\text{bary}(\kappa(x)) - x),$$

where $\bar{P} = \bar{\mu} \otimes \kappa$. By the assumption that $\bar{P}|_{M \times \mathbb{R}}$ is a martingale and $\mathbb{R} \setminus M = \{h \leq 0\} \cap I_0 \subseteq \{h = 0\}$, either $h(x) = 0$ or $\mathrm{bary}(\kappa(x)) = x$, for $\bar{\mu}$-a.e. $x \in \mathbb{R}$. Using also that $\bar{P}$ is concentrated on Γ, we deduce that

$$\begin{aligned}
\bar{P}(f) &= \bar{P}[\varphi(X) + \psi(Y) + h(X)(Y - X)] \\
&= \int \left\{ \bar{\varphi}(x) + \int \bar{\psi}(y)\, \kappa(x, dy) + \left[\chi(x) - \int \chi(y) \kappa(x, dy) \right] \right\} \bar{\mu}(dx) \\
&= \bar{\mu}(\bar{\varphi}) + \bar{\nu}(\bar{\psi}) + (\bar{\mu} - \bar{\nu})(\chi),
\end{aligned}$$

where the last step is justified by the nonnegativity of the integrands. As $\bar{P}(f) < \infty$, we conclude that the three (nonnegative) terms on the right-hand side are finite; that is, $(\varphi, \psi) \in L(\bar{\mu}, \bar{\nu})$ with moderator χ.

(iii) Again, we may assume that $\pi(f) < \infty$. Let π' be an (M_0, M_1)-competitor of π, let $\bar{\mu}, \bar{\nu}$ be the common first and second marginals of π, π' and let $\pi = \bar{\mu} \otimes \kappa$, $\pi' = \bar{\mu} \otimes \kappa'$. If $(\varphi, \psi) \in L(\bar{\mu}, \bar{\nu})$, using $h \geq 0$ on $M_0 \subseteq I_0$ and $\mathbb{R} \setminus M \subseteq \{h = 0\}$ and the definition of the competitor yields

$$\begin{aligned}
\pi(f) &= \pi[\varphi(X) + \psi(Y) + h(X)(Y - X)] \\
&= \bar{\mu}(\varphi) + \bar{\nu}(\psi) + \int_M h(x)(\mathrm{bary}(\kappa(x)) - x)\, \bar{\mu}(dx) \\
&\geq \bar{\mu}(\varphi) + \bar{\nu}(\psi) + \int_M h(x)(\mathrm{bary}(\kappa'(x)) - x)\, \bar{\mu}(dx) \\
&= \pi'[\varphi(X) + \psi(Y) + h(X)(Y - X)] \\
&\geq \pi'(f).
\end{aligned}$$

Of course, $(\varphi, \psi) \in L(\bar{\mu}, \bar{\nu})$ holds in particular if π is finitely supported. $\square$

2.6 Shadow Construction

In this section, we introduce the Increasing and Decreasing Supermartingale Transports via an order-theoretic construction. Let $\mathfrak{M}^1(\mathbb{R})$ be the set of all finite measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which have a finite first moment, endowed with the weak convergence induced by the continuous functions of linear growth. We shall mainly use the restriction of this topology to subsets of measures of equal mass, and then it is equivalent to the Kantorovich or 1-Wasserstein distance $W(\nu, \nu') = \sup_f (\nu - \nu')(f)$, where f ranges over all 1-Lipschitz functions.

Definition 2.6.1. Let $\mu, \nu \in \mathfrak{M}^1(\mathbb{R})$. We say that μ, ν are in *positive-convex-decreasing* order, denoted $\mu \leq_{pcd} \nu$, if $\mu(\phi) \leq \nu(\phi)$ for all nonnegative, convex, decreasing functions $\phi : \mathbb{R} \to \mathbb{R}$.

We note that $\mu \leq_{pcd} \nu$ necessarily satisfy $\mu(\mathbb{R}) \leq \nu(\mathbb{R})$. In fact, the case of strict inequality is the one of interest: if $\mu(\mathbb{R}) = \nu(\mathbb{R})$, then $\mu \leq_{pcd} \nu$ is equivalent to $\mu \leq_{cd} \nu$.

Lemma 2.6.2. *Let $\mu, \nu \in \mathfrak{M}^1(\mathbb{R})$ satisfy $\mu \leq_{pcd} \nu$. Then the set*[5]

$$[\![\mu, \nu]\!] := \{\theta \in \mathfrak{M}^1(\mathbb{R}) : \mu \leq_{cd} \theta \leq \nu\}$$

[5]We think of the elements of $[\![\mu, \nu]\!]$ as lying between μ and ν, as the notation suggests. However, we caution the reader that $\mu, \nu \notin [\![\mu, \nu]\!]$ in general.

is nonempty and contains a unique least element $\mathcal{S}^\nu(\mu)$ for the convex-decreasing order:

$$\mathcal{S}^\nu(\mu) \leq_{cd} \theta \quad \textit{for all} \quad \theta \in [\![\mu, \nu]\!] \,.$$

The measure $\mathcal{S}^\nu(\mu)$ is called the shadow *of μ in ν.*

Proof. Without loss of generality, ν is a probability measure.

(i) We first show that $[\![\mu, \nu]\!]$ contains some element θ. Let λ be the Lebesgue measure on $\mathbb{R}$ and let G_ν be the quantile function of ν; that is, the left-continuous inverse of the c.d.f. of ν. We define

$$\theta := \lambda|_{[0,k]} \circ G_\nu^{-1} \quad \text{where} \quad k := \mu(\mathbb{R}) \in [0, 1].$$

This implies that $\theta \in \mathfrak{M}^1(\mathbb{R})$, that $\theta(\mathbb{R}) = k$, and that $\theta \leq \nu$. Intuitively speaking, θ is the "left-most" measure $\theta \leq \nu$ of mass k on $\mathbb{R}$; in particular, if ν admits a density f_ν, the density of θ is $f_\theta = f_\nu \mathbf{1}_{(-\infty, G_\nu(k)]}$.

Let ϕ be a convex, decreasing function; we need to show that $\mu(\phi) \leq \theta(\phi)$. To this end, we may assume that $\phi(G_\nu(k)) = 0$ by translating ϕ, and then we have

$$\mu(\phi) \leq \mu(\phi^+) \leq \nu(\phi^+) = \theta(\phi^+) = \theta(\phi)$$

since $\phi^+ = \phi$ on $G_\nu([0, k])$. As a result, $\theta \in [\![\mu, \nu]\!] \neq \emptyset$.

(ii) Next, we show that $[\![\mu, \nu]\!]$ is directed; i.e., given $\theta_i \in [\![\mu, \nu]\!]$, $i = 1, 2$ there exists $\theta \in [\![\mu, \nu]\!]$ such that $\theta \leq_{cd} \theta_i$. Indeed, let $p : \mathbb{R} \to \mathbb{R}$ be defined as the convex hull of the minimum of p_{θ_1} and p_{θ_2}. Then p is convex, and p is increasing like p_{θ_i}.

Since the asymptotic slope of the functions p_{θ_i} is given by $\theta_i(\mathbb{R}) = \mu(\mathbb{R})$, the same is true for p, and finally, $p_{\theta_i} \geq p_\mu$ yields $p \geq p_\mu$. These facts imply that p is the put function associated with a measure θ satisfying $\mu \leq_{cd} \theta \leq_{cd} \theta_i$. It remains to show that $\theta \leq \nu$, which is equivalent to $p_\nu - p$ being convex. Indeed, the fact that $p_\nu - p_{\theta_i}$ is convex for $i = 1, 2$ implies this property; cf. the end of the proof of [13, Lemma 4.6] for a detailed argument.

(iii) The set $[\![\mu, \nu]\!] \subseteq \mathfrak{M}^1(\mathbb{R})$ consists of measures with common total mass $\mu(\mathbb{R})$; we show that it is compact. Indeed, closedness is readily established. Moreover, any $\theta \in [\![\mu, \nu]\!]$ satisfies $\theta \leq \nu$. By Prokhorov's theorem, this immediately yields tightness in the weak topology induced by bounded continuous functions, and then using $\int |x| \, \nu(dx) < \infty$ yields relative compactness in $\mathfrak{M}^1(\mathbb{R})$.

(iv) It follows from (iii) that for any convex, decreasing function ϕ of linear growth, the continuous functional $\theta \mapsto \theta(\phi)$ has a nonempty compact set $\Theta_\phi \subseteq [\![\mu, \nu]\!]$ of minimizers. The directedness of $[\![\mu, \nu]\!]$ from (ii) implies that a finite intersection $\Theta_{\phi_1} \cap \cdots \cap \Theta_{\phi_n}$ is still nonempty, and then compactness shows that $\theta \mapsto \theta(\phi)$ has a common minimizer $\mathcal{S}^\nu(\mu)$ for all ϕ. The uniqueness of the minimizer follows from the fact that $\theta_1 \leq_{cd} \theta_2$ and $\theta_2 \leq_{cd} \theta_1$ imply $\theta_1 = \theta_2$. $\square$

Lemma 2.6.3. *Let $\mu, \nu \in \mathfrak{M}^1(\mathbb{R})$ satisfy $\mu \leq_{pcd} \nu$ and suppose that μ is concentrated at a single point $x \in \mathbb{R}$. Then, the shadow $\mathcal{S}^\nu(\mu)$ is of the form*

$$\mathcal{S}^\nu(\mu) = \nu|_{(a,b)} + k_a \delta_a + k_b \delta_b.$$

Among all measures $\theta \leq \nu$ with mass $\mu(\mathbb{R})$ of this form, $\mathcal{S}^\nu(\mu)$ is determined by *maximizing* $\mathrm{bary}(\theta)$ *subject to the constraint* $\mathrm{bary}(\theta) \leq x$. *Moreover,* a *and* b *can be chosen such that* $a \leq x \leq b$.

Finally, the map $\nu \mapsto \mathcal{S}^\nu(\mu)$ *is continuous when restricted to a set of measures* $\nu \in \mathfrak{M}^1(\mathbb{R})$ *of equal total mass satisfying* $\mu \leq_{pcd} \nu$.

Proof. We may assume that $\nu(\mathbb{R})=1$. Then, $\mu = k\delta_x$ for some $k \in [0,1]$, and we may focus on $k \in (0,1)$. Consider the family

$$\theta_s = \lambda|_{[s,s+k]} \circ G_\nu^{-1}, \quad s \in [0, 1-k].$$

Similarly as in the proof of Lemma 2.6.2, we have $\theta_s \leq \nu$ for all s, whereas $\mu = k\delta_x \leq_{cd} \theta_s$ if and only if $\mathrm{bary}(\theta_s) \leq x$. As $\mu \leq_{pcd} \nu$, this inequality holds true in particular for $s = 0$. The function

$$s \mapsto \mathrm{bary}(\theta_s) = \frac{1}{k}\int_0^k G_\nu(s+t)\,\lambda(dt) = \frac{1}{k}\int_0^k G_\nu(s+t+)\,\lambda(dt) \tag{2.6.1}$$

is continuous and increasing; thus, we may define s^* as the largest value in $[0, 1-k]$ for which $\mathrm{bary}(\theta_s) \leq x$, and then $\theta^* := \theta_{s^*}$ is in $[\![\mu, \nu]\!]$. We claim that θ^* is the least element in $[\![\mu, \nu]\!]$.

To show this, let $(a, b) = (G_\nu(s^*), G_\nu(s^* + k))$; then $\theta^*|_{(a,b)} = \nu|_{(a,b)}$ and θ^* is concentrated on $[a, b]$. Now let $\theta \in [\![\mu, \nu]\!]$ be arbitrary. As $\theta \leq \nu$, we see that $\theta - (\theta^* \wedge \theta)$ is concentrated on $(a, b)^c$, whereas $\theta^* - (\theta^* \wedge \theta)$ is concentrated on $[a, b]$. Moreover, we must have $\mathrm{bary}(\theta) \leq \mathrm{bary}(\theta^*)$. Indeed, this is clear if $\mathrm{bary}(\theta^*) = x$. If not, the definition of s^* implies that $\nu(b, \infty) = 0$ and then θ^* clearly has the largest barycenter among all measures $\theta \leq \nu$ with mass $\mu(\mathbb{R})$. Thus, Lemma 2.6.4 below implies that $\theta^* \leq_{cd} \theta$ and as a result, θ^* is the least element in $[\![\mu, \nu]\!]$; i.e., $\mathcal{S}^\nu(\mu) = \theta^*$.

As $\mathrm{bary}(\theta^*) \leq x$, it is clear that $a \leq x$. With the above choice of b, it may happen that $b < x$. However, by the definition of s^*, this is possible only if $\theta^*(\{b\}) = \nu(\{b\})$ and $\nu(b, \infty) = 0$. In that case, we may redefine $b := x$ without invalidating the other assertions of the lemma, and then we have $a \leq x \leq b$ as required.

It remains to verify the continuity of $\nu \mapsto \mathcal{S}^\nu(\mu)$. Let ν' be another probability measure such that $\mu \leq_{pcd} \nu'$ and let $\mathcal{S}^{\nu'}(\mu) = \theta'_{r^*}$ be the corresponding shadow constructed as above. Using the fact that the 1-Wasserstein distance satisfies

$$W(\nu, \nu') = \int_0^1 |G_\nu(t) - G_{\nu'}(t)| \, \lambda(dt) \tag{2.6.2}$$

as well as (2.6.1) and $G_{\theta_s}(r) = G_\nu(s + r)$ on $[0, k]$, we have

$$\begin{aligned} k|\,\mathrm{bary}(\theta'_t) - \mathrm{bary}(\theta_t)| &\leq W(\theta_t, \theta'_t) \leq W(\nu, \nu'), \\ k|\,\mathrm{bary}(\theta_s) - \mathrm{bary}(\theta_t)| &= W(\theta_s, \theta_t). \end{aligned}$$

These relations yield that

$$\begin{aligned} W(\mathcal{S}^\nu(\mu), \mathcal{S}^{\nu'}(\mu)) &\leq W(\theta_{s^*}, \theta_{r^*}) + W(\theta_{r^*}, \theta'_{r^*}) \\ &\leq k|\,\mathrm{bary}(\theta_{s^*}) - \mathrm{bary}(\theta_{r^*})| + W(\nu, \nu') \\ &\leq k|\,\mathrm{bary}(\theta_{s^*}) - \mathrm{bary}(\theta'_{r^*})| + 2W(\nu, \nu') \\ &= k|\,\mathrm{bary}(\mathcal{S}^\nu(\mu)) - \mathrm{bary}(\mathcal{S}^{\nu'}(\mu))| + 2W(\nu, \nu'). \end{aligned}$$

Using (2.6.1) and (2.6.2), it follows from the construction of the shadow that $\nu \mapsto \mathrm{bary}(\mathcal{S}^\nu(\mu))$ is continuous, and then the above estimate shows that $\nu \mapsto \mathcal{S}^\nu(\mu)$ is continuous. $\square$

The following property was used in the preceding proof.

Lemma 2.6.4. *Let $\mu, \nu \in \mathfrak{M}^1(\mathbb{R})$ satisfy $\mu(\mathbb{R}) = \nu(\mathbb{R})$ and $\mathrm{bary}(\mu) \geq \mathrm{bary}(\nu)$. If there exists an interval $I = (a, b)$ such that μ is concentrated on $\bar{I} := [a, b] \cap \mathbb{R}$ and ν is concentrated on I^c, then $\mu \leq_{cd} \nu$. The same is true if there exists an interval I such that $\mu - (\mu \wedge \nu)$ is concentrated on $\bar{I}$ and $\nu - (\mu \wedge \nu)$ is concentrated on I^c.*

Proof. The first claim implies the second, so we may focus on the former. We need to show that $\mu(\phi) \leq \nu(\phi)$ for any convex decreasing function ϕ. To this end, we may assume that the left endpoint a of the interval is finite and strictly smaller than the right endpoint b, as otherwise we must have $\mu = \nu = 0$; moreover, we may assume by translation that $\phi(a) = 0$. If b is finite as well, we define

$$\psi(x) := \phi(x) - \frac{\phi(b)}{b - a}(x - a), \quad x \in \mathbb{R},$$

whereas $\psi := \phi$ if $b = \infty$. Then $\psi \leq 0$ on $\bar{I}$ and $\psi \geq 0$ on I^c, which yields

$$\mu(\phi) \leq \mu(\psi^+) + \frac{\phi(b)}{b - a}[\mathrm{bary}(\mu) - a] \leq \nu(\psi^+) + \frac{\phi(b)}{b - a}[\mathrm{bary}(\nu) - a] = \nu(\phi)$$

as desired. $\square$

Since we will apply the shadow in an iterative fashion, the following additivity result is vital. The first assertion intuitively follows from the minimality of the shadow: if we transport part of a measure $\mu \leq_{pcd} \nu$ to its shadow in ν, the remaining part μ_2 of μ is still dominated by the remaining part of ν. Moreover, if we then transport μ_2 to its shadow in the remainder, the cumulative result is the same as the shadow of μ in ν.

Proposition 2.6.5. *Let $\mu_1, \mu_2, \nu \in \mathfrak{M}^1(\mathbb{R})$ satisfy $\mu_1 + \mu_2 \leq_{pcd} \nu$. Then $\mu_2 \leq_{pcd} \nu - \mathcal{S}^\nu(\mu_1)$ and*

$$\mathcal{S}^\nu(\mu_1 + \mu_2) = \mathcal{S}^\nu(\mu_1) + \mathcal{S}^{\nu - \mathcal{S}^\nu(\mu_1)}(\mu_2).$$

Proof. (i) Suppose first that $\mu_1 = k\delta_x$ is a single atom of mass $k > 0$ at $x \in \mathbb{R}$. Then by Lemma 2.6.3, there is an interval $I = (a, b)$ with $x \in \bar{I} = [a, b] \cap \mathbb{R}$ such that $\mathcal{S}^\nu(\mu)$ is concentrated on $\bar{I}$ and $\mathcal{S}^\nu(\mu)|_I = \nu|_I$. We may assume that I is a strict subset of $\mathbb{R}$, as otherwise $\mu_1 = \nu$ and $\mu_2 = 0$.

To see that $\mu_2 \leq_{pcd} \nu - \mathcal{S}^\nu(\mu_1)$, let ϕ be a nonnegative, convex, decreasing function of linear growth. Let ψ be the minimal function $\psi \geq \phi$ which is affine on I, then ψ is finite (as $I \neq \mathbb{R}$) and equal to ϕ on I^c. Then we have

$$(\nu - \mathcal{S}^\nu(\mu_1))(\phi) = (\nu - \mathcal{S}^\nu(\mu_1))(\psi) \geq (\nu - \mu_1)(\psi) \geq \mu_2(\psi) \geq \mu_2(\phi),$$

where we have used that $\nu - \mathcal{S}^\nu(\mu_1)$ is concentrated on I^c, that ψ is affine on $\bar{I}$ and $\mathcal{S}^\nu(\mu_1)$ and μ_1 are concentrated on $\bar{I}$, that $\mu_1 + \mu_2 \leq_{pcd} \nu$, and finally that $\psi \geq \phi$.

This shows that $\mu_2 \leq_{pcd} \nu - \mathcal{S}^\nu(\mu_1)$. In particular, $\mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_1)}(\mu_2)$ is well-defined. Set

$$\theta' := \mathcal{S}^\nu(\mu_1) + \mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_1)}(\mu_2).$$

Since $\mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_1)}(\mu_2) \leq \nu - \mathcal{S}^\nu(\mu_1)$, it is clear that $\theta' \leq \nu$. Using also that $\mu_1 \leq_{cd} \mathcal{S}^\nu(\mu_1)$ and $\mu_2 \leq_{cd} \mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_1)}(\mu_2)$, we have $\theta' \in [\![\mu_1 + \mu_2, \nu]\!]$.

Let θ be another element of $[\![\mu_1 + \mu_2, \nu]\!]$. Then $\theta \leq \nu$ implies $[\![\mu_1, \theta]\!] \subseteq [\![\mu_1, \nu]\!]$ and thus $\mathcal{S}^\nu(\mu_1) \leq_{cd} \mathcal{S}^\theta(\mu_1)$ by minimality. Moreover, $\theta \leq \nu$ implies $\theta - \mathcal{S}^\theta(\mu_1) \leq \nu - \mathcal{S}^\nu(\mu_1)$; to see this, we use the description of the shadow from Lemma 2.6.3 and note that the interval for θ will contain the one for ν. Now, the same minimality argument implies $\mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_1)}(\mu_2) \leq_{cd} \mathcal{S}^{\theta-\mathcal{S}^\theta(\mu_1)}(\mu_2)$. Combining these two facts, we have

$$\theta' = \mathcal{S}^\nu(\mu_1) + \mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_1)}(\mu_2) \leq_{cd} \mathcal{S}^\theta(\mu_1) + \mathcal{S}^{\theta-\mathcal{S}^\theta(\mu_1)}(\mu_2).$$

But the right-hand side equals θ; indeed, $\mathcal{S}^\theta(\mu_1) + \mathcal{S}^{\theta-\mathcal{S}^\theta(\mu_1)}(\mu_2) \leq \theta$ and both sides of this inequality have total mass $(\mu_1 + \mu_2)(\mathbb{R})$. We have shown that θ' is the minimal element in $[\![\mu_1 + \mu_2, \nu]\!]$ and thus that $\mathcal{S}^\nu(\mu_1 + \mu_2) = \theta'$.

(ii) If μ_1 is finitely supported, the claim follows by iterating the result of (i). For general μ_1, let (μ_1^n) be a sequence of finitely supported measures such that $\mu_1^n \leq_{cd} \mu_1^{n+1} \leq_{cd} \mu_1$ and $\mu_1^n \to \mu_1$ weakly; cf. [13, Lemma 2.9] for the existence of (μ_1^n), even in convex (instead of convex-decreasing) order. Under the additional condition that μ_2 is a Dirac mass, the claim follows by passing to the limit in

$$\mathcal{S}^\nu(\mu_1^n + \mu_2) = \mathcal{S}^\nu(\mu_1^n) + \mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_1^n)}(\mu_2),$$

where we use Lemma 2.6.6 below for the first two terms as well as the last assertion of Lemma 2.6.3 for the third term.

(iii) By iterating the result of (ii), we obtain the claim in the case where μ_2 is finitely supported (and μ_1 is arbitrary). To complete the proof, we approximate μ_2 by a sequence (μ_2^n) as in (ii) and pass to the limit $\mu_2^n \to \mu_2$ using Lemma 2.6.6. $\square$

The following continuity property of the shadow was used in the preceding proof.

Lemma 2.6.6. *Let $\mu_n, \mu, \nu \in \mathfrak{M}^1(\mathbb{R})$ satisfy $\mu_1 \leq_{cd} \mu_2 \leq_{cd} \cdots \leq_{cd} \mu \leq_{pcd} \nu$. Then $\mu_n \to \mu_\infty$ for some $\mu_\infty \in \mathfrak{M}^1(\mathbb{R})$, and $\mathcal{S}^\nu(\mu_n) \to \mathcal{S}^\nu(\mu_\infty)$.*

Proof. The limits are constructed by passing to the monotone limit in the associated put functions. The details are similar to the proof of [13, Proposition 4.15] and therefore omitted. $\square$

Next, we shall use the shadow mapping to construct specific supermartingale transports. Let $\mu \leq_{cd} \nu$ and suppose first that $\mu = \sum_{i=1}^n k_i \delta_{x_i}$ is finitely supported. We may transport μ to ν by first mapping $k_1\delta_{x_1}$ to its shadow in ν, continue by mapping $k_2\delta_{x_2}$ to its shadow in the "remainder" $\nu - \mathcal{S}^\nu(k_1\delta_{x_1})$ of ν, and so on. Proceeding until $i = n$, this constructs the kernel κ corresponding to a supermartingale transport $\mu \otimes \kappa \in \mathcal{S}(\mu, \nu)$. In fact, this recipe leads to a whole family of transports—the labeling of the atoms was arbitrary, and a different order in their processing will typically give rise to a different transport. There are two choices that seem canonical: left-to-right (increasing) and right-to-left (decreasing). We shall show in the subsequent sections that the corresponding transports $\vec{P}$ and $\overleftarrow{P}$ are indeed canonical in several ways.

Theorem 2.6.7. *Let $\mu \leq_{cd} \nu$.*

(i) There exists a unique measure $\vec{P}$ on $\mathbb{R} \times \mathbb{R}$ which transports $\mu|_{(-\infty,x]}$ to its shadow $\mathcal{S}^\nu(\mu|_{(-\infty,x]})$ for all $x \in \mathbb{R}$; that is, the first marginal of $\vec{P}$ equals μ and

$$\vec{P}((-\infty,x] \times A) = \mathcal{S}^\nu(\mu|_{(-\infty,x]})(A), \quad A \in \mathcal{B}(\mathbb{R}). \tag{2.6.3}$$

(ii) Similarly, there exists a unique measure $\overleftarrow{P}$ on $\mathbb{R} \times \mathbb{R}$ which transports $\mu|_{[x,\infty)}$ to its shadow $\mathcal{S}^\nu(\mu|_{[x,\infty)})$ for all $x \in \mathbb{R}$.

Moreover, those two measures are elements of $\mathcal{S}(\mu,\nu)$. We call $\vec{P}$ and $\overleftarrow{P}$ the Increasing and the Decreasing Supermartingale Transport, respectively.

Proof. The function $F(x,y) := \mathcal{S}^\nu(\mu|_{(-\infty,x]})(-\infty,y]$ is clearly increasing and right-continuous in y. Moreover, Proposition 2.6.5 implies that

$$\mathcal{S}^\nu(\mu|_{(-\infty,x_2]}) - \mathcal{S}^\nu(\mu|_{(-\infty,x_1]}) = \mathcal{S}^{\nu - \mathcal{S}^\nu(\mu|_{(-\infty,x_1]})}(\mu|_{(x_1,x_2]}) \geq 0, \quad x_1 \leq x_2$$

which yields the same properties for the variable x; note that the total mass of the right-hand side equals $\mu(x_1,x_2]$. Noting also that F has the proper normalization for a c.d.f., we conclude that F induces a unique measure $\vec{P}$ on $\mathcal{B}(\mathbb{R} \times \mathbb{R})$. It is clear that μ is the first marginal of $\vec{P}$. The second marginal is $S^\nu(\mu) \leq \nu$, and this is in fact an equality because both measures have the same mass. To conclude that $\vec{P} \in \mathcal{S}(\mu,\nu)$, it suffices to show that $\vec{P}[Y\phi(X)] \leq \vec{P}[X\phi(X)]$ for all functions ϕ of the form $\phi = \mathbf{1}_{(x_1,x_2]}$ with $x_1 < x_2$. Indeed, Proposition 2.6.5 implies that

$$\begin{aligned}\vec{P}[Y\phi(X)] &= \int y\,[\mathcal{S}^\nu(\mu|_{(-\infty,x_2]}) - \mathcal{S}^\nu(\mu|_{(-\infty,x_1]})](dy)\\ &= \text{bary}\left(\mathcal{S}^{\nu-\mathcal{S}^\nu(\mu|_{(-\infty,x_1]})}(\mu|_{(x_1,x_2]})\right)\\ &\leq \text{bary}\left(\mu|_{(x_1,x_2]}\right)\\ &= \vec{P}[X\phi(X)].\end{aligned}$$

The arguments for $\overleftarrow{P}$ are analogous. $\square$

A different construction of $\vec{P}$ and $\overleftarrow{P}$ could proceed through an approximation of the marginals by discrete measures, for which the couplings can be defined explicitly by iterating Lemma 2.6.3, and a subsequent passage to the limit. We refer to [62, Remark 2.18] for a sketch of such a construction in the martingale case.

2.7 Spence–Mirrlees Functions and Geometry of their Optimal Transports

In this section, we relate monotonicity properties of the reward function f to the geometry of the supports of the corresponding optimal supermartingale transports, where the support will be described by a pair (Γ, M) as in Theorem 2.5.2. We first introduce the relevant properties of f.

Definition 2.7.1. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *first-order Spence–Mirrlees* if

$$f(x_2, \cdot) - f(x_1, \cdot) \quad \text{is strictly increasing for all } x_1 < x_2.$$

Moreover, f is *second-order Spence–Mirrlees* if

$$f(x_2, \cdot) - f(x_1, \cdot) \quad \text{is strictly convex for all } x_1 < x_2,$$

and f is *supermartingale Spence–Mirrlees* if f is second-order Spence–Mirrlees and $-f$ is first-order Spence–Mirrlees.

We note that if f is smooth, the first and second order Spence–Mirrlees properties are equivalent to the classical cross-derivative conditions $f_{xy} > 0$ and $f_{xyy} > 0$, respectively. The latter is also called martingale Spence–Mirrlees condition in the literature on martingale optimal transport—the above terminology will be more convenient in what follows.

Remark 2.7.2. There exist smooth, linearly growing supermartingale Spence–Mirrlees functions on $\mathbb{R}^2$.

Indeed, let φ be a smooth, bounded, strictly increasing function on $\mathbb{R}$; e.g., $\varphi(x) = \tanh(x)$. Let ψ be a smooth, linearly growing, strictly decreasing, strictly convex function on $\mathbb{R}$; e.g., $\psi(y) = (1 + y^2)^{1/2} - y$. Then,

$$g(x, y) := \varphi(x)\psi(y)$$

satisfies $g_{xy} < 0$ and $g_{xyy} > 0$, while $|g(x, y)| \leq C(1 + |y|)$ for some $C > 0$.

Next, we introduce the relevant geometric properties of the support.

Definition 2.7.3. Let $(\Gamma, M) \subseteq \mathbb{R}^2 \times \mathbb{R}$ and consider $(x_1, y_1), (x_2, y_2) \in \Gamma$ with $x_1 < x_2$. The pair (Γ, M) is

(i) *first-order left-monotone* if $y_1 \leq y_2$ whenever $x_2 \notin M$,

(ii) *first-order right-monotone* if $y_2 \leq y_1$ whenever $x_1 \notin M$.

We will also need the following properties of Γ; they are taken from [13] where they are simply called left- and right-monotonicity.

Definition 2.7.4. Let $\Gamma \subseteq \mathbb{R}^2$ and consider $(x, y_1), (x, y_2), (x', y') \in \Gamma$ with $y_1 < y_2$. Then Γ is

(i) *second-order left-monotone* if $y' \notin (y_1, y_2)$ whenever $x < x'$,

(ii) *second-order right-monotone* if $y' \notin (y_1, y_2)$ whenever $x' < x$.

For convenience, we shall use the same terminology for a pair (Γ, M) even though only Γ is relevant for the second-order properties. Yet another notion will be useful: we write

$$\Gamma^1 = \{x \in \mathbb{R} : (x, y) \in \Gamma \text{ for some } y \in \mathbb{R}\}$$

for the projection of Γ onto the first coordinate.

Definition 2.7.5. A pair $(\Gamma, M) \subseteq \mathbb{R}^2 \times \mathbb{R}$ is *nondegenerate* if

(i) for all $x \in \Gamma^1$ such that $(x, y) \in \Gamma$ for some $y > x$, there exists $y' < x$ such that $(x, y') \in \Gamma$,

(ii) for all $x \in \Gamma^1 \cap M$ such that $(x, y) \in \Gamma$ for some $y < x$, there exists $y' > x$ such that $(x, y') \in \Gamma$.

These two conditions imply that

(i') for all $x \in \Gamma^1$ there exists $y \leq x$ such that $(x, y) \in \Gamma$,

(ii') for all $x \in \Gamma^1 \cap M$ there exists $y \geq x$ such that $(x, y) \in \Gamma$.

Essentially, nondegeneracy postulates that there is a down-path at every $x \in \Gamma^1$, and also an up-path if $x \in M$. Thus, it is a natural requirement if we intend to consider supermartingales supported by Γ which are martingales on $M \times \mathbb{R}$. For later use, let us record that nondegeneracy can be assumed without loss of generality in our context.

Remark 2.7.6. Let $(\Gamma, M) \in \mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R})$, let $\mu \leq_{cd} \nu$ be probability measures and suppose there is $P \in \mathcal{S}(\mu, \nu)$ with $P(\Gamma) = 1$ such that $P|_{M \times \mathbb{R}}$ is a martingale. Then, there exists a Borel subset $\Gamma' \subseteq \Gamma$ with $P(\Gamma') = 1$ such that (Γ', M) is nondegenerate.

Proof. Let N_1' be the set of all $x \in \Gamma^1$ such that Definition 2.7.5 (i) fails. Then N_1' is universally measurable and thus we can find a Borel set $N_1 \supseteq N_1'$ such that $N_1 \backslash N_1'$ is μ-null. The fact that P is a supermartingale implies that $\Gamma_1 := \Gamma \cap \{Y > X\} \cap (N_1 \times \mathbb{R})$ is P-null. After defining similarly a set N_2 for Definition 2.7.5 (ii), the martingale property of P on $M \times \mathbb{R}$ shows that $\Gamma_2 := \Gamma \cap \{Y < X\} \cap (N_2 \times \mathbb{R})$ is P-null as well, and then we can set $\Gamma' := \Gamma \setminus (\Gamma_1 \cup \Gamma_2)$. $\square$

The first-order properties turn out to be highly asymmetric when combined with nondegeneracy. The following observation will have far-reaching consequences regarding the geometry of the coupling $\vec{P}$ and has no analogue in the left-monotone case.

Remark 2.7.7. Let (Γ, M) be first-order right-monotone and nondegenerate. Then, M is a half-line unbounded to the left within Γ^1; that is,

$$\text{if } x_1, x_2 \in \Gamma^1 \text{ satisfy } x_1 < x_2 \text{ and } x_2 \in M, \text{ then } x_1 \in M.$$

Indeed, let x_1, x_2 be as stated; then nondegeneracy yields y_1, y_2 such that $y_1 \leq x_1 < x_2 \leq y_2$ and $(x_i, y_i) \in \Gamma$. If we had $x_1 \notin M$, this would contradict first-order right-monotonicity.

With these definitions in place, we can use the monotonicity principle of Theorem 2.5.2 to infer the geometry of (Γ, M) from the properties of f. Given $\mu \leq_{cd} \nu$, we recall the corresponding intervals I_k, J_k of Proposition 2.3.4 and the corresponding set Σ of (2.5.1). In fact, the following result does not refer directly to the marginal measures; it merely uses the general shape of Σ.

Proposition 2.7.8. *Let $(\Gamma, M) \in \mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R})$ be nondegenerate, where $\Gamma \subseteq \Sigma$ and $M = M_0 \cup M_1$ with Borel sets $M_0 \subseteq I_0$ and $M_1 = \cup_{k \neq 0} I_k$, and let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose that the assertion of Theorem 2.5.2 (iii) holds; that is, if π is a finitely supported probability which is concentrated on Γ, then $\pi(f) \geq \pi'(f)$ for any (M_0, M_1)-competitor π' of π that is concentrated on Σ.*

(i) If f is first-order Spence–Mirrlees, (Γ, M) is first-order left-monotone.

(ii) If $-f$ is first-order Spence–Mirrlees, (Γ, M) is first-order right-monotone.

(iii) If f is second-order Spence–Mirrlees, Γ is second-order left-monotone.

(iv) If $-f$ is second-order Spence–Mirrlees, Γ is second-order right-monotone.

Proof. (i) Consider $(x_1, y_1), (x_2, y_2) \in \Gamma$ with $x_1 < x_2$ and suppose for contradiction that $y_2 < y_1$. The measures

$$\pi := \tfrac{1}{2}\delta_{(x_1,y_1)} + \tfrac{1}{2}\delta_{(x_2,y_2)}, \quad \pi' := \tfrac{1}{2}\delta_{(x_1,y_2)} + \tfrac{1}{2}\delta_{(x_2,y_1)}$$

have the same first marginal $\pi_1 = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2}$. Let $\pi = \pi_1 \otimes \kappa$ and $\pi' = \pi_1 \otimes \kappa'$, then

$$\mathrm{bary}(\kappa'(x_1)) < \mathrm{bary}(\kappa(x_1)), \quad \mathrm{bary}(\kappa'(x_2)) > \mathrm{bary}(\kappa(x_2)).$$

Suppose that $x_1 \notin M_1$ and $x_2 \notin M$. Then, π' is an (M_0, M_1)-competitor of π. Moreover, $x_i \notin M_1$ implies that $x_i \in I_0$ and thus $y_i \in J_0$, $i = 1, 2$ which shows that π' is supported on Σ. Thus, we must have $\pi(f) \geq \pi'(f)$. However,

$$2(\pi(f) - \pi'(f)) = (f(x_2, y_2) - f(x_1, y_2)) - (f(x_2, y_1) - f(x_1, y_1)) < 0$$

as f is first-order Spence–Mirrlees, so we have reached the desired contradiction.

Let $x_1 \in M_1$ and $x_2 \notin M$. Recalling that $M_1 = \cup_{k \neq 0} I_k = (-\infty, x^*]$, we have $y_1 \in J_k$ for some $k \neq 0$, whereas $x_2 \notin M$ implies $y_2 \in J_0$. Since J_0 is located to the right of J_k for $k \neq 0$, we must have $y_1 \leq y_2$.

(ii) Consider $(x_1, y_1), (x_2, y_2) \in \Gamma$ with $x_1 < x_2$ and suppose for contradiction that

$y_1 < y_2$. We define π, π' as in (i); then

$$\operatorname{bary}(\kappa'(x_1)) > \operatorname{bary}(\kappa(x_1)), \quad \operatorname{bary}(\kappa'(x_2)) < \operatorname{bary}(\kappa(x_2)).$$

Let $x_1 \notin M$. Then, $x_1 \in I_0 = (x^*, \infty)$ and thus $x_2 > x_1$ is in I_0 as well. In particular, $x_2 \notin M_1$ and $y_1, y_2 \in J_0$. Thus π' is an (M_0, M_1)-competitor of π that is concentrated on Σ and we reach a contradiction to $-f$ being first-order Spence–Mirrlees, similarly as in (i).

(iii) Let $(x, y_1), (x, y_2), (x', y') \in \Gamma$ satisfy $x < x'$ and assume for contradiction that $y_1 < y' < y_2$. Define $\lambda = \frac{y'-y_1}{y_2-y_1}$ and

$$\begin{aligned}
\pi &= \frac{\lambda}{2}\delta_{(x,y_1)} + \frac{1-\lambda}{2}\delta_{(x,y_2)} + \frac{1}{2}\delta_{(x',y')}, \\
\pi' &= \frac{\lambda}{2}\delta_{(x',y_1)} + \frac{1-\lambda}{2}\delta_{(x',y_2)} + \frac{1}{2}\delta_{(x,y')}.
\end{aligned}$$

Then, π and π' have the same first marginal π_1 and if $\pi = \pi_1 \otimes \kappa$ and $\pi' = \pi_1 \otimes \kappa'$, then $\kappa(x)$, $\kappa'(x)$, $\kappa(x')$, $\kappa'(x')$ all have barycenter y'. Hence, π' is an (M_0, M_1)-competitor of π, and since the shape of $\Gamma \subseteq \Sigma$ shows that π' is concentrated on Σ, we deduce that $\pi(f) \geq \pi'(f)$. However, f being second-order Spence–Mirrlees implies that $\pi(f) < \pi'(f)$.

(iv) The proof is symmetric to (iii). $\square$

2.8 Geometric Characterization of the Canonical Supermartingale Transports

In this section, we consider fixed probability measures $\mu \leq_{cd} \nu$ and show that the associated Increasing and Decreasing Supermartingale Transports $\vec{P}, \overleftarrow{P}$ (cf. Theorem 2.6.7) are characterized by geometric properties of their supports.

Theorem 2.8.1. *Let $(\Gamma, M) \in \mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R})$ be nondegenerate and let $P \in \mathcal{S}(\mu, \nu)$ be such that P is concentrated on Γ and $P|_{M\times\mathbb{R}}$ is a martingale.*

(i) If (Γ, M) is first-order right-monotone and second-order left-monotone, then P is the Increasing Supermartingale Transport $\vec{P}$.

(ii) If (Γ, M) is first-order left-monotone and second-order right-monotone, then P is the Decreasing Supermartingale Transport $\overleftarrow{P}$.

Before proving this result, we record two auxiliary lemmas.

Lemma 2.8.2. *Let $a \in \mathbb{R}$ and $\mu \leq_{cd} \nu$. If ν is concentrated on $[a, \infty)$, then so is μ, and moreover $\nu(\{a\}) \geq \mu(\{a\})$. If $\mu \leq_c \nu$, the same holds for $(-\infty, a]$.*

Proof. The first claim is easily deduced from the fact that $\mathcal{S}(\mu, \nu) \neq \emptyset$ by Proposition 2.2.1, and then the second is implied by symmetry. □

The following is [13, Lemma 5.4].

Lemma 2.8.3. *Let σ be a nontrivial signed measure on $\mathbb{R}$ with $\sigma(\mathbb{R}) = 0$ and let $\sigma = \sigma^+ - \sigma^-$ be its Hahn decomposition. There exist $a \in \operatorname{supp}(\sigma^+)$ and $b > a$ such that $\int (b - y)^+ \mathbf{1}_{[a,\infty)}(y)\, d\sigma(y) > 0$.*

We can now proceed with the proof of the theorem; we shall use the notation

$$\Gamma_x = \{y \in \mathbb{R} : (x,y) \in \Gamma\}, \quad x \in \mathbb{R}.$$

Proof of Theorem 2.8.1 (i). Given $x \in \mathbb{R}$, we set $\mu_x := \mu|_{(-\infty,x]}$ and denote by ν_x^P the second marginal of $P|_{(-\infty,x]\times\mathbb{R}}$; that is, the image of μ_x under the transport P. Since P is concentrated on Γ and has the same mass as $\vec{P}$, it suffices to show that

$$\nu_x^P = \nu_x^{\vec{P}} \tag{2.8.1}$$

for all $x \in \Gamma^1$.

In a first step we will show that (2.8.1) holds for all $x \in \Gamma^1 \cap M$. In view of Remark 2.7.7 it then follows that

$$P|_{M\times\mathbb{R}} = \vec{P}|_{M\times\mathbb{R}}. \tag{2.8.2}$$

After that we will show that (2.8.1) holds for all $x \in \Gamma^1$ if $M = \emptyset$; and this assumption will be removed in a final step.

Let us first establish an auxiliary result that will be used in steps 1 and 2. If (2.8.1) is violated for some $x \in \Gamma^1$, then the signed measure

$$\sigma := \nu_x^{\vec{P}} - \nu_x^P$$

is nontrivial and we can find $a \in \operatorname{supp}(\sigma^+)$ and $b > a$ as in Lemma 2.8.3. Note that

$\sigma^+ \leq \nu - \nu_x^P$ and that $\nu - \nu_x^P$ is the image of $\mu|_{(x,\infty)}$ under P. Hence, $a \in \operatorname{supp}(\nu - \nu_x^P)$ and as $P(\Gamma) = 1$, there exists a sequence of points

$$(x_n, a_n) \in \Gamma \quad \text{with } x < x_n \text{ and } a_n \to a. \tag{2.8.3}$$

Step 1: Equality of the martingale parts. We argue by contradiction and assume that there exists $x \in \Gamma^1 \cap M$ such that (2.8.1) is violated. We first establish that

$$\nu_x^{\vec{P}} \leq_c \nu_x^P \quad \text{and in particular} \quad \operatorname{bary}(\nu_x^{\vec{P}}) = \operatorname{bary}(\nu_x^P). \tag{2.8.4}$$

Indeed, in view of $x \in M$, Remark 2.7.7 shows that $(-\infty, x] \cap \Gamma^1 \subseteq M$ and thus $P|_{(-\infty,x]\times\mathbb{R}}$ is a martingale. Therefore, $\operatorname{bary}(\nu_x^P) = \operatorname{bary}(\mu_x)$, and moreover $\operatorname{bary}(\mu_x) \geq \operatorname{bary}(\nu_x^{\vec{P}})$ since $\vec{P}$ is a supermartingale. Thus, $\operatorname{bary}(\nu_x^P) \geq \operatorname{bary}(\nu_x^{\vec{P}})$. On the other hand, $P \in \mathcal{S}(\mu, \nu)$ implies $\nu_x^P \in [\![\mu_x, \nu]\!]$ and hence $\nu_x^{\vec{P}} \leq_{cd} \nu_x^P$ by the minimality property defining $\vec{P}$; cf. Theorem 2.6.7. In view of Proposition 2.2.1, these two facts imply (2.8.4).

Next, we show that

$$\Gamma_t \cap (a, \infty) = \emptyset, \quad t \leq a \wedge x. \tag{2.8.5}$$

Indeed, let $t \leq a \wedge x$ and suppose that $\Gamma_t \cap (a, \infty) \neq \emptyset$. Then in particular $\Gamma_t \cap (t, \infty) \neq \emptyset$ and thus nondegeneracy, more precisely Definition 2.7.5 (i), yields that $\Gamma_t \cap (-\infty, t) \neq \emptyset$ and hence $\Gamma_t \cap (-\infty, a) \neq \emptyset$. But now we obtain a contradiction to the second-order left-monotonicity of Γ by using (x_n, a_n) from (2.8.3) for (x', y') and t for x in Definition 2.7.4, for some large enough n.

Case (a): $x \in M$ and $x \leq a$. As $x \leq a$, (2.8.5) applies to all $t \leq x$ and hence $P(\Gamma) = 1$ implies that ν_x^P is concentrated on $(-\infty, a]$. In view of (2.8.4) and Lemma 2.8.2, it follows that $\nu_x^{\vec{P}}$ is concentrated on $(-\infty, a]$ as well, and $\nu_x^P(\{a\}) \geq \nu_x^{\vec{P}}(\{a\})$. Using these three facts yields

$$\begin{aligned}\int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^{\vec{P}}(dy) &= (b-a)\nu_x^{\vec{P}}(\{a\}) \\ &\leq (b-a)\nu_x^P(\{a\}) \\ &= \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^P(dy);\end{aligned}$$

that is, $\int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \sigma(dy) \leq 0$. This contradicts the choice of a and b; cf. Lemma 2.8.3.

Case (b): $x \in M$ and $a < x$. Since $a < x$, we can argue exactly as below (2.8.4) to obtain that

$$\nu_a^{\vec{P}} \leq_c \nu_a^P \quad \text{and in particular} \quad \operatorname{bary}(\nu_a^{\vec{P}}) = \operatorname{bary}(\nu_a^P). \tag{2.8.6}$$

Moreover, (2.8.5) and $P(\Gamma) = 1$ now imply that ν_a^P is concentrated on $(-\infty, a]$, and then Lemma 2.8.2 shows that

$$\nu_a^P,\ \nu_a^{\vec{P}} \text{ are concentrated on } (-\infty, a] \quad \text{and} \quad \nu_a^{\vec{P}}(\{a\}) \leq \nu_a^P(\{a\}). \tag{2.8.7}$$

Next, we establish that $\nu_x^P - \nu_a^P$ is concentrated on $[a, \infty)$. Let $a < t \le x$ be such that $\Gamma_t \neq \emptyset$. Since $x \in M$, Remark 2.7.7 yields that $t \in M$ and now nondegeneracy, cf. Definition 2.7.5 (ii'), shows that $\Gamma_t \cap [t, \infty) \neq \emptyset$. Then, using (2.8.3) and the second-order left-monotonicity of Γ yield that $\Gamma_t \cap (-\infty, a) = \emptyset$, and therefore, $\nu_x^P - \nu_a^P$ is indeed concentrated on $[a, \infty)$. We shall prove below that

$$\nu_x^{\vec{P}} - \nu_a^{\vec{P}} \le_{cd} \nu_x^P - \nu_a^P \tag{2.8.8}$$

and thus Lemma 2.8.2 shows that $\nu_x^{\vec{P}} - \nu_a^{\vec{P}}$ is concentrated on $[a, \infty)$ as well. Using these facts, (2.8.7) and that $y \mapsto (b - y)^+ \mathbf{1}_{[a,\infty)}(y)$ is convex decreasing on $[a, \infty)$ yields

$$\begin{aligned}
&\int (b - y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^{\vec{P}}(dy) \\
&\qquad = \int (b - y)^+ \mathbf{1}_{[a,\infty)}(y)\, (\nu_x^{\vec{P}} - \nu_a^{\vec{P}})(dy) + (b - a)\nu_a^{\vec{P}}(\{a\}) \\
&\qquad \le \int (b - y)^+ \mathbf{1}_{[a,\infty)}(y)\, (\nu_x^P - \nu_a^P)(dy) + (b - a)\nu_a^P(\{a\}) \\
&\qquad = \int (b - y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^P(dy).
\end{aligned}$$

This again contradicts the choice of a and b; cf. Lemma 2.8.3.

It remains to show (2.8.8). Indeed, using again that $\nu_x^P - \nu_a^P$ is concentrated on $[a, \infty)$ as well as (2.8.7), we have

$$\nu_x^P - \nu_a^P = (\nu_x^P - \nu_a^P)|_{[a,\infty)} \le (\nu - \nu_a^P)|_{[a,\infty)} \le (\nu - \nu_a^{\vec{P}})|_{[a,\infty)} \le \nu - \nu_a^{\vec{P}}.$$

On the other hand, we have $\mu|_{(a,x]} \leq_{cd} \nu_x^P - \nu_a^P$ by the supermartingale property of P, and thus

$$\nu_x^P - \nu_a^P \in \left[\!\left[\mu|_{(a,x]}, \nu - \nu_a^{\vec{P}}\right]\!\right].$$

Since $\nu_x^{\vec{P}} - \nu_a^{\vec{P}} = \mathcal{S}^{\nu-\nu_a^{\vec{P}}}(\mu|_{(a,x]})$ is the minimal element of the above set by the definition of $\vec{P}$ and the additivity of the shadow (Proposition 2.6.5), we conclude that (2.8.8) holds, and that completes the proof of Step 1.

In fact, using Remark 2.7.7 and $P(\Gamma) = 1$, this already allows us to conclude that

Step 2: $M = \emptyset$. Again, suppose there exists $x \in \Gamma^1$ such that (2.8.1) is violated. Define

$$y_x := \inf \Gamma_x.$$

If $(x', y) \in \Gamma$ and $x' < x$, first-order right-monotonicity implies that $y \geq y_x$ (since $M = \emptyset$), and the latter holds trivially for $x' = x$. Conversely, if $(x', y) \in \Gamma$ and $x < x'$, first-order right-monotonicity implies that $y \leq y_x$. As a result, P is concentrated on the set

$$(-\infty, x] \times [y_x, \infty) \quad \cup \quad (x, \infty) \times (-\infty, y_x]$$

and as $P \in \mathcal{S}(\mu, \nu)$, this implies that

$$\nu_x^P = \nu|_{(y_x,\infty)} + k\delta_{y_x}, \quad k := \mu((-\infty, x]) - \nu((y_x, \infty)).$$

This is the minimal element of $[\![\mu|_{(-\infty,x]}, \nu]\!]$ as can be seen, e.g., from Lemma 2.6.4, and thus $\nu_x^P = \nu_x^{\vec{P}}$.

Step 3: $M \neq \emptyset$. In the general case, let $\mu_M = \mu|_M$ and let ν_M^P denote the second marginal of $P|_{M\times\mathbb{R}}$. We note that $x \notin M$ yields $M \subseteq (-\infty, x]$ by Remark 2.7.7 and hence $\mu_M \leq \mu_x$.

We may apply the result proved in Step 2 to $\Gamma' = \Gamma \cap (M^c \times \mathbb{R})$, $M' = \emptyset$ and the marginals $\mu' = \mu - \mu_M$, $\nu' = \nu - \nu_M$ to deduce that $P|_{M^c\times\mathbb{R}}$ is the Increasing Supermartingale Transport from μ' to ν'. In particular,

$$\mathcal{S}^{\nu-\nu_M^P}(\mu_x - \mu_M) = \nu_x^P - \nu_M^P. \tag{2.8.9}$$

Observing that (2.8.2) implies $\nu_M^P = \nu_M^{\vec{P}} = \mathcal{S}^\nu(\mu_M)$, the additivity of the shadow (Proposition 2.6.5) shows that

$$\begin{aligned}\nu_x^{\vec{P}} = \mathcal{S}^\nu(\mu_x) &= \mathcal{S}^\nu(\mu_M) + \mathcal{S}^{\nu-\mathcal{S}^\nu(\mu_M)}(\mu_x - \mu_M) \\ &= \nu_M^P + \mathcal{S}^{\nu-\nu_M^P}(\mu_x - \mu_M)\end{aligned}$$

which equals ν_x^P by (2.8.9). As $x \notin M$ was arbitrary, this completes the proof of Theorem 2.8.1 (i). $\square$

Proof of Theorem 2.8.1 (ii). It will be convenient to reverse the notation with respect to the preceding proof: given $x \in \mathbb{R}$, we set $\mu_x := \mu|_{[x,\infty)}$ and let ν_x^P be the second marginal of $P|_{[x,\infty)\times\mathbb{R}}$. Again, we assume for contradiction that there exists $x \in \Gamma^1$ such that $\nu_x^P \neq \nu_x^{\overleftarrow{P}}$, so that the signed measure

$$\sigma := \nu_x^{\overleftarrow{P}} - \nu_x^P$$

is nontrivial and we can find $a \in \operatorname{supp}(\sigma^+)$ and $a < b$ as in Lemma 2.8.3. Similarly as in (2.8.3), there exist

$$(x_n, a_n) \in \Gamma \quad \text{with } x_n < x \text{ and } a_n \to a. \tag{2.8.10}$$

Moreover, $P \in \mathcal{S}(\mu, \nu)$ implies that $\nu_x^P \in [\![\mu_x, \nu]\!]$ and hence, by minimality,

$$\nu_x^{\overleftarrow{P}} \leq_{cd} \nu_x^P. \tag{2.8.11}$$

Case 1a: $x \in M$ and $a \leq x$. We first show that

$$\nu_x^P \quad \text{is concentrated on } [a, \infty). \tag{2.8.12}$$

Indeed, let $t \in \Gamma^1$ be such that $t > x$. Suppose that $\Gamma_t \cap (-\infty, a) \neq \emptyset$. If $t \in M$, nondegeneracy yields that $\Gamma_t \cap [t, \infty) \neq \emptyset$ and since $a \leq x < t$, (2.8.10) contradicts the second-order right-monotonicity of Γ. Hence, $t \notin M$. Since $x \in M$, nondegeneracy also yields that $\Gamma_x \cap [x, \infty) \neq \emptyset$. But now $\Gamma_t \cap (-\infty, a) \neq \emptyset$ and $a \leq x$ contradict first-order left-monotonicity as $t \notin M$. As a result, $\Gamma_t \cap (-\infty, a) = \emptyset$. To extend this to $t = x$, note that in this case we have $t \in M$. Thus, if $\Gamma_t \cap (-\infty, a) \neq \emptyset$, the nondegeneracy of Definition 2.7.5 (ii) and (2.8.10) contradict second-order right-monotonicity. We have shown that $\Gamma_t \cap (-\infty, a) = \emptyset$ for all $t \geq x$, and (2.8.12) follows. In view of (2.8.11) and Lemma 2.8.2, we conclude that

$$\nu_x^{\overleftarrow{P}} \text{ is concentrated on } [a, \infty) \quad \text{and} \quad \nu_x^{\overleftarrow{P}}(\{a\}) \leq \nu_x^P(\{a\}). \tag{2.8.13}$$

Since $(b-y)^+$ is convex and decreasing, (2.8.11), (2.8.12) and (2.8.13) then yield

$$\begin{aligned}\int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^{\overleftarrow{P}}(dy) &= \int (b-y)^+\, \nu_x^{\overleftarrow{P}}(dy) \\ &\leq \int (b-y)^+\, \nu_x^{P}(dy) \\ &= \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^{P}(dy)\end{aligned}$$

which contradicts the choice of a and b; cf. Lemma 2.8.3.

Case 1b: $x \in M$ and $x < a$. Let $t \geq a$ and suppose that $\Gamma_t \cap (-\infty, a) \neq \emptyset$. If $t \in M$, nondegeneracy yields that $\Gamma_t \cap (t,\infty) \neq \emptyset$ and since $x < a \leq t$, (2.8.10) contradicts the second-order right-monotonicity of Γ. Hence, $t \notin M$, but then $\Gamma_t \cap (-\infty, a) \neq \emptyset$ and (2.8.10) contradict first-order left-monotonicity. As a result, $\Gamma_t \cap (-\infty, a) = \emptyset$ for all $t \geq a$ and hence ν_a^P is concentrated on $[a,\infty)$. Since

$$\nu_a^{\overleftarrow{P}} \leq_{cd} \nu_a^P \tag{2.8.14}$$

can be argued as in (2.8.11), Lemma 2.8.2 then yields that

$$\nu_a^P,\ \nu_a^{\overleftarrow{P}} \text{ are concentrated on } [a,\infty) \quad \text{and} \quad \nu_a^{\overleftarrow{P}}(\{a\}) \leq \nu_a^P(\{a\}). \tag{2.8.15}$$

Next, we show that symmetrically,

$$\nu_x^P - \nu_a^P,\ \nu_x^{\overleftarrow{P}} - \nu_a^{\overleftarrow{P}} \text{ are concentrated on } (-\infty, a] \tag{2.8.16}$$

$$\text{and} \quad (\nu_x^{\overleftarrow{P}} - \nu_a^{\overleftarrow{P}})(\{a\}) \leq (\nu_x^P - \nu_x^P)(\{a\}). \tag{2.8.17}$$

Indeed, let $t \in \Gamma^1$ be such that $x \leq t < a$ and suppose that $\Gamma_t \cap (a, \infty) \neq \emptyset$. Since $\Gamma_t \cap (-\infty, t] \neq \emptyset$ by nondegeneracy, (2.8.10) contradicts second-order right-monotonicity. Thus, $\Gamma_t \cap (a, \infty) = \emptyset$ and $\nu_x^P - \nu_a^P$ is concentrated on $(-\infty, a]$. In order to conclude (2.8.16) and (2.8.17) via Lemma 2.8.2, it remains to show that $\nu_x^{\overleftarrow{P}} - \nu_a^{\overleftarrow{P}} \leq_c \nu_x^P - \nu_a^P$. Indeed, let again $t \in \Gamma^1$ be such that $x \leq t < a$. If $t \notin M$, then $\Gamma_t \cap (-\infty, t] \neq \emptyset$ and (2.8.10) contradict first-order left-monotonicity; thus $t \in M$. As a result, $P|_{[x,a)\times\mathbb{R}}$ is a martingale and $\mathrm{bary}(\nu_x^P - \nu_a^P) = \mathrm{bary}(\mu_x - \mu_a)$. Hence, we only have to show that

$$\nu_x^{\overleftarrow{P}} - \nu_a^{\overleftarrow{P}} \leq_{cd} \nu_x^P - \nu_a^P. \tag{2.8.18}$$

Using that $\nu_x^P - \nu_a^P$ is concentrated on $(-\infty, a]$ as well as (2.8.15), we have

$$\nu_x^P - \nu_a^P = (\nu_x^P - \nu_a^P)|_{(-\infty,a]} \leq (\nu - \nu_a^P)|_{(-\infty,a]} \leq (\nu - \nu_a^{\overleftarrow{P}})|_{[a,\infty)} \leq \nu - \nu_a^{\overleftarrow{P}}.$$

On the other hand, $\mu|_{[x,a)} \leq_{cd} \nu_x^P - \nu_a^P$ by the supermartingale property of P, and thus (2.8.18) follows from the minimality of $\overleftarrow{P}$ and Proposition 2.6.5. This completes the proof of (2.8.16) and (2.8.17).

Finally, we can apply (2.8.14)–(2.8.17) to find that

$$
\begin{aligned}
&\int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^{\overleftarrow{P}}(dy) \\
&\quad = \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, (\nu_x^{\overleftarrow{P}} - \nu_a^{\overleftarrow{P}})(dy) + \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_a^{\overleftarrow{P}}(dy) \\
&\quad = (b-a)(\nu_x^{\overleftarrow{P}} - \nu_a^{\overleftarrow{P}})(\{a\}) + \int (b-y)^+ \, \nu_a^{\overleftarrow{P}}(dy) \\
&\quad \leq (b-a)(\nu_x^{P} - \nu_a^{P})(\{a\}) + \int (b-y)^+ \, \nu_a^{P}(dy) \\
&\quad = \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\, \nu_x^{P}(dy)
\end{aligned}
$$

which again contradicts the choice of a and b.

Case 2: $x \notin M$. Define again $y_x = \inf \Gamma_x$; note that $y_x \leq x$ by nondegeneracy. Let $t \in \Gamma^1$ be such that $t < x$. If $\Gamma_t \cap (y_x, \infty) \neq \emptyset$, then as $x \notin M$, the definition of y_x yields a contradiction to first-order left-monotonicity. On the other hand, let $x < t$ and assume that $\Gamma_t \cap (-\infty, y_x) \neq \emptyset$. If $t \notin M$, the construction of y_x again contradicts first-order left-monotonicity; thus $t \in M$. But then nondegeneracy shows that $\Gamma_t \cap [t, \infty) \neq \emptyset$ and the definition of y_x yields a contradiction to second-order right-monotonicity. Clearly, $\Gamma_x \subseteq [y_x, \infty)$, and we have established that P must be concentrated on

$$(-\infty, x) \times (-\infty, y_x] \;\;\cup\;\; [x, \infty) \times [y_x, \infty).$$

Since $P \in \mathcal{S}(\mu, \nu)$, this implies that

$$\nu_x^P = \nu|_{(y_x,\infty)} + k\delta_{y_x}, \quad k := \mu([x,\infty)) - \nu((y_x,\infty)).$$

This is the minimal element of $[\![\mu|_{[x,\infty)}, \nu]\!]$, and thus $\nu_x^P = \nu_x^{\overleftarrow{P}}$. $\square$

2.9 Regularity of Spence–Mirrlees Functions

A supermartingale Spence–Mirrlees function f need not be (semi)continuous. For instance, if $f(x,y) = \varphi(x)\psi(y)$ for a strictly increasing function φ and a strictly convex and decreasing function ψ, then f is supermartingale Spence–Mirrlees but clearly φ need not be upper or lower semicontinuous. In general, f may have a continuum of various types of discontinuities.

However, we show in Proposition 2.9.2 below that a measurable second-order Spence–Mirrlees function is automatically continuous for a finer topology on $\mathbb{R}^2$, and this topology will be coarse enough to preserve the weak compactness of $\mathcal{S}(\mu,\nu)$. Thus, we can still deduce the existence of optimal transports (Lemma 2.9.3) for upper semicontinuous reward functions f, and in particular for supermartingale Spence–Mirrlees functions. That will allow us to apply the monotonicity principle of Theorem 2.5.2.

Before stating these results, we introduce a relaxed version of the Spence–Mirrlees conditions of Definition 2.7.1, where increase and convexity are required in the non-strict sense—we have reserved the shorter name for the object that appears more frequently.

Definition 2.9.1. We call $f : \mathbb{R}^2 \to \mathbb{R}$ *relaxed first-order Spence–Mirrlees* if

$$f(x_2,\cdot) - f(x_1,\cdot) \quad \text{is increasing for all } x_1 < x_2,$$

relaxed second-order Spence–Mirrlees if

$$f(x_2,\cdot) - f(x_1,\cdot) \quad \text{is convex for all } x_1 < x_2,$$

and *relaxed supermartingale Spence–Mirrlees* if f is relaxed second-order Spence–Mirrlees and $-f$ is relaxed first-order Spence–Mirrlees.

Proposition 2.9.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel and relaxed second-order Spence–Mirrlees. There exists a Polish topology τ on $\mathbb{R}$ such that f is $\tau \otimes \tau$-continuous. Moreover, τ refines the Euclidean topology and induces the same Borel sets.*

Proof. We begin by constructing the functions f_n; the topology will be defined in the last step.

Step 1: Regularity in y. We first suppose that f vanishes along the y-axis,

$$f(0,y) = 0, \quad y \in \mathbb{R}. \tag{2.9.1}$$

Under this hypothesis, the second-order Spence–Mirrlees condition implies that

$$\begin{cases} f(x,\cdot) \text{ is convex}, & x \geq 0, \\ f(x,\cdot) \text{ is concave}, & x \leq 0. \end{cases} \tag{2.9.2}$$

Therefore, $y \mapsto f(x,y)$ admits a finite left derivative $\partial_y f(x,0)$ at $y = 0$. We impose the further hypothesis that

$$\partial_y f(x,0) = 0, \quad x \in \mathbb{R}. \tag{2.9.3}$$

Since $y \mapsto f(x,y)$ is convex or concave, its restriction to a compact interval $K_m = [-m,m]$ is Lipschitz continuous with some optimal Lipschitz constant $\mathrm{Lip}(f(x,\cdot)|_{K_m}) < \infty$. More precisely, (2.9.2) and (2.9.3) imply that the optimal constant is the supremum of the absolute slopes of the tangents at the endpoints $y = \pm m$. The second-order Spence–Mirrlees condition implies that the absolute slopes are increasing in $|x|$; in particular,

$$\sup_{x \in K_m} \mathrm{Lip}(f(x,\cdot)|_{K_m}) = \sup_{x=\pm m} \mathrm{Lip}(f(x,\cdot)|_{K_m}) < \infty. \tag{2.9.4}$$

Step 2: Approximation. Fix $n \in \mathbb{N}$, let $y_k^n = 2^{-n}k$ for $k \in \mathbb{Z}$ and let $f_n(x,\cdot)$ be the continuous, piecewise affine approximation to $f(x,\cdot)$ along this grid; that is, for $y_k^n \le y < y_{k+1}^n$ we define

$$f_n(x,y) = \lambda f(x,y_k^n) + (1-\lambda) f(x,y_{k+1}^n), \quad \lambda := 2^n(y_{k+1}^n - y). \tag{2.9.5}$$

We then have $|f_n(x,y) - f(x,y)| \le 2^{-n}L$ for all $y \in K_m$ if L is a Lipschitz constant for $f(x,\cdot)$ on K_m. In view of (2.9.4), this shows that

$$f_n \to f \quad \text{uniformly on } K_m \times K_m, \quad \text{for all } m \in \mathbb{N}. \tag{2.9.6}$$

Step 3: Refining the Topology. Next, we introduce the topology τ. The basic idea here is that if φ is a real function with a single discontinuity at $y_0 \in \mathbb{R}$, we can change the topology on $\mathbb{R}$ by declaring y_0 an isolated point and then φ becomes continuous. More generally, [65, Theorem 13.11, Lemma 13.3] show that given a countable family of Borel functions on $\mathbb{R}$, there exists a Polish topology $\tau \subseteq \mathcal{B}(\mathbb{R})$ which renders these functions continuous and refines the Euclidean topology. In particular, we can find τ such that $f(\cdot, y_k^n)$ is τ-continuous for all n, k. As τ refines the Euclidean topology, it readily follows that the functions f_n defined in (2.9.5) are $\tau \otimes \tau$-continuous. But now (2.9.6) yields that f is continuous as well.

It remains to remove the hypotheses stated in (2.9.1) and (2.9.3). Indeed, the above shows that the claim holds for

$$\tilde{f}(x, y) := f(x, y) - f(0, y) - \partial_y [f(x, y) - f(0, y)]_{y=0} \times y.$$

It is easy to check that $\tilde{f}$ is still second-order Spence–Mirrlees if f is. We can further refine τ such that the two Borel functions subtracted on the right-hand side are τ-continuous, and then the result for f follows. $\square$

As announced, the preceding result allows us to deduce the existence of optimal transports.

Lemma 2.9.3. *Let $\mu \leq_{cd} \nu$ and let τ be a Polish topology on $\mathbb{R}$ which refines the Euclidean topology and induces the same Borel sets. Moreover, let $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be upper semicontinuous for the product topology $\tau \otimes \tau$ and suppose that f^+ is $\mathcal{S}(\mu, \nu)$-*

uniformly integrable; i.e.,

$$\lim_{N\to\infty} \sup_{P\in\mathcal{S}(\mu,\nu)} P(f^+\mathbf{1}_{f^+>N}) = 0. \tag{2.9.7}$$

Then, $\mathbf{S}_{\mu,\nu}(f) < \infty$ *and there exists an optimal* $P \in \mathcal{S}(\mu,\nu)$ *for* $\mathbf{S}_{\mu,\nu}(f)$.

Condition (2.9.7) *is satisfied in particular if* $f(x,y) \le a(x) + b(y)$ *for some functions* $a \in L^1(\mu)$ *and* $b \in L^1(\nu)$.

Proof. Standard arguments show that $\mathcal{S}(\mu,\nu)$ is compact in the usual topology of weak convergence as induced by the Euclidean metric. However, the weak topology on $\mathcal{S}(\mu,\nu)$ induced by $\tau\otimes\tau$ does not depend on the choice of the Polish topology τ as long as $\sigma(\tau) = \mathcal{B}(\mathbb{R})$; this follows from [17, Lemma 2.3]. Thus, $\mathcal{S}(\mu,\nu)$ is still weakly compact relative to $\tau\otimes\tau$.

Under the additional condition that f is bounded from above, the mapping $P \mapsto P(f)$ is upper semicontinuous by [85, Lemma 4.3]. Applying this result to $f \wedge N$ and using (2.9.7), the same extends to f as in the lemma, and the claim follows. $\square$

We remark that compactness of $\mathcal{S}(\mu,\nu)$ may fail if non-product topologies are considered on $\mathbb{R}^2$, so that the use of $\tau\otimes\tau$ is crucial.

Corollary 2.9.4. *Let* $\mu \le_{cd} \nu$ *be probability measures and let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be Borel and relaxed supermartingale Spence–Mirrlees. Suppose that there exist* $a \in L^1(\mu)$, $b \in L^1(\nu)$ *such that*

$$f(x,y) \ge a(x) + b(y), \quad x,y \in \mathbb{R}$$

and that f^+ is $\mathcal{S}(\mu,\nu)$-uniformly integrable; cf. (2.9.7). *Then,* $\mathbf{S}_{\mu,\nu}(f) < \infty$ *and* $\vec{P} \in \mathcal{S}(\mu,\nu)$ *is an optimizer. If f is supermartingale Spence–Mirrlees, the optimizer is unique.*

The analogous statement holds for $\overleftarrow{P}$ *if instead $-f$ is (relaxed) supermartingale Spence–Mirrlees.*

Proof. Let f be supermartingale Spence–Mirrlees (in the strict sense). Under the stated integrability condition, Proposition 2.9.2 and Lemma 2.9.3 show that $\mathbf{S}_{\mu,\nu}(f) < \infty$ and that an optimizer $P \in \mathcal{S}(\mu,\nu)$ exists. Now, the monotonicity principle of Theorem 2.5.2 and Remark 2.5.4 provide sets $(\Gamma, M) \in \mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R})$ such that P is concentrated on Γ, $P|_{M\times\mathbb{R}}$ is a martingale and the assertion of Theorem 2.5.2 (iii) holds. In view of Remark 2.7.6, we may assume that Γ is nondegenerate by passing to a subset of full P-measure. Proposition 2.7.8 implies that (Γ, M) is first-order right monotone and second-order left-monotone, and then Theorem 2.8.1 yields that $P = \vec{P}$.

If f is relaxed supermartingale Spence–Mirrlees, let g be as in Remark 2.7.2 and note that for each $n \in \mathbb{N}$, the function $f_n = f + (1/n)g$ is supermartingale Spence–Mirrlees in the strict sense. Since f_n satisfies the stated integrability conditions, the above shows that $\vec{P}$ is the unique optimizer for f_n. Suppose that there exists $P_* \in \mathcal{S}(\mu,\nu)$ such that $P_*(f) > \vec{P}(f)$. Then, as monotone convergence yields $P_*(f_n) \to P_*(f)$ and $\vec{P}(f_n) \to \vec{P}(f)$, it follows that $P_*(f_n) > \vec{P}(f_n)$ for n large enough, contradicting the optimality of $\vec{P}$.

The argument for $\overleftarrow{P}$ is similar. □

In the special case where $\mu \leq_c \nu$, the supermartingale transport problem specializes to martingale transport and by construction, $\vec{P}$ coincides with the Left-Curtain coupling of [13]. For completeness, we record the analogue of Corollary 2.9.4.

Corollary 2.9.5. *Let $\mu \leq_c \nu$ be probability measures and let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel and relaxed second-order Spence–Mirrlees. Suppose that there exist $a \in L^1(\mu)$, $b \in L^1(\nu)$ such that*

$$f(x, y) \geq a(x) + b(y), \quad x, y \in \mathbb{R}$$

and that f^+ is $\mathcal{S}(\mu, \nu)$-uniformly integrable; cf. (2.9.7)*. Then, $\mathbf{S}_{\mu,\nu}(f) < \infty$ and $\vec{P} \in \mathcal{S}(\mu, \nu) \equiv \mathcal{M}(\mu, \nu)$ is an optimizer, where $\vec{P}$ coincides with the Left-Curtain coupling of [13]. If f is second-order Spence–Mirrlees, the optimizer is unique.*

The analogous statement holds for $\overleftarrow{P}$ if instead $-f$ is (relaxed) supermartingale Spence–Mirrlees, and $\overleftarrow{P}$ coincides with the Right-Curtain coupling of [13].

The proof is the same as for Corollary 2.9.4—note that when $M = \mathbb{R}$, the first-order monotonicity condition is vacuous.

Finally, we also have the converse of Theorem 2.8.1 which completes the proofs for the main results as stated in the Introduction.

Corollary 2.9.6. *Let $\mu \leq_{cd} \nu$ be probability measures and let $\vec{P}$ be the associated Increasing Supermartingale Transport. There exists a nondegenerate pair $(\Gamma, M) \in \mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R})$ which is first-order right-monotone and second-order left-monotone such that $\vec{P}$ is concentrated on Γ and $\vec{P}|_{M\times\mathbb{R}}$ is a martingale.*

The analogous statement, exchanging left and right, holds for $\overleftarrow{P}$.

Proof. Let g be a supermartingale Spence–Mirrlees function as in Remark 2.7.2. We know from Corollary 2.9.4 that $\vec{P}$ is the unique optimal transport for g, and the existence of (Γ, M) follows as in the proof of Corollary 2.9.4. □

Remark 2.9.7. Corollary 2.9.6 shows, in particular, that the no-crossing properties of $\vec{P}$ and $\overleftarrow{P}$ as stated in the Introduction are true for general marginals. Together with Remark 2.7.7, it also yields that $\vec{P}$ has at most one transition from martingale kernels to proper supermartingale ones.

A martingale transport with second-order left-monotone support is the Left-Curtain coupling of its marginals and if the first marginal has no atoms, each kernel of this transport is concentrated on two points [13]. Moreover, an arbitrary transport with first-order right-monotone support is the Antitone coupling and if the first marginal has no atoms, its kernels are deterministic [75, Section 3.1]. As a result, if μ is diffuse,

(i) $\vec{P}|_{M^c \times \mathbb{R}}$ is of Monge-type,

(ii) $\vec{P}|_{M \times \mathbb{R}}$ is concentrated on the union of two graphs.

The analogue holds for $\overleftarrow{P}$, with the Right-Curtain and Hoeffding–Fréchet couplings.

2.10 Counterexamples

Duality Theory

In the Introduction and the body of the text, we have claimed that certain relaxations cannot be avoided.

In [16], we have already stated several examples related to the duality theory for the case of martingale transport. Bearing in mind that this is a special case of the supermartingale transport problem at hand, these examples still apply: If the inequality defining the dual elements is stated in the classical sense as

$$\varphi(x) + \psi(y) + h(x)(y - x) \geq f(x, y), \quad (x, y) \in \mathbb{R}^2$$

rather than the quasi-sure sense, a duality gap may occur; cf. [16, Example 8.1]. A duality gap may also occur if integrability of dual elements is required in the usual sense; i.e., $\varphi \in L^1(\mu)$, or if f has no lower bound, see [16, Examples 8.4, 8.5].

Next, let us substantiate two claims made in the body of the text. Recall that the set $\mathcal{D}^{ci,pw}_{\mu,\nu}(f)$ was defined with nonnegative functions h, whereas for $\mathcal{D}_{\mu,\nu}(f)$ nonnegativity is required only on the proper portion of the state space (Definitions 2.4.3 and 2.4.9). We shall show below that this is necessary.

(i) The requirement that the dual elements (φ, ψ, h) satisfy $h \geq 0$ would preclude existence of dual optimizers.

Second, we have claimed that the restriction to proper pairs $\mu \leq_{cd} \nu$ in Section 2.4 is necessary. While we have already seen that the proof of Proposition 2.4.4 crucially uses a nontrivial difference of the barycenters of μ and ν in order to control the slope of χ, we still owe an argument that this is indeed unavoidable.

(ii) The closedness property of $\mathcal{D}^{ci,pw}_{\mu,\nu}(f)$ asserted in Proposition 2.4.4 fails if the (irreducible) pair $\mu \leq_{cd} \nu$ is not proper, and this remains true even if, in view of (i), we were to alleviate the requirement that $h \geq 0$. Indeed, let $c_i = i^{-3}C$, $i \geq 1$, where $C > 0$ is such that $\sum c_i = 1$, and define

$$\mu := \sum_{i\geq 1} c_i \delta_i, \quad \nu := \frac{1}{3}\sum_{i\geq 1} c_i(\delta_{i-1} + \delta_i + \delta_{i+1}).$$

Moreover, let $f(x,y) = \mathbf{1}_{x\neq y}$. Following [16, Examples 8.4, 8.5] we find that $\mu \leq_{cd} \nu$ is irreducible and

$$P := \sum_{i\geq 1} c_i\, \delta_i \otimes \frac{1}{3}(\delta_{i-1} + \delta_i + \delta_{i+1}) \, \in \mathcal{S}(\mu,\nu)$$

is a primal optimizer. Clearly, $\mathrm{bary}(\mu) = \mathrm{bary}(\nu)$; i.e., the pair is not proper. Let (φ, ψ, h) be a dual optimizer. Even if we are flexible about the precise definition of the dual domain, a minimal requirement to avoid a duality gap is that $\varphi(x) + \psi(y) + h(x)(y-x) = f(x,y)$ P-a.s. and hence

$$\varphi(x) + \psi(y) + h(x)(y-x) = f(x,y), \quad (x,y) \in \mathbb{N}\times\mathbb{N}_0,\ y \in \{x-1, x, x+1\}.$$

It follows that

$$\begin{cases} \varphi(x) + \psi(x-1) - h(x) = 1, \\ \varphi(x) + \psi(x+1) + h(x) = 1, \\ \varphi(x) + \psi(x) = 0 \end{cases}$$

for $x \in \mathbb{N}$, and all solutions of this system satisfy

$$\varphi(x) = -x^2 + bx + c, \quad \psi(x) = x^2 - bx - c, \quad h(x) = -2x + b$$

for $x \in \mathbb{N}$, where $b, c \in \mathbb{R}$ are arbitrary constants. While any such triplet defines a dual optimizer in the sense of the body of the paper, we see that there is no solution satisfying $h \geq 0$, which was our claim in (i).

To argue (ii), suppose for contradiction that the closedness property of $\mathcal{D}^{ci,pw}_{\mu,\nu}(f)$ asserted in Proposition 2.4.4 were true even though $\mu \leq_{cd} \nu$ is not proper. Then, following the proofs in the body of the paper shows that the analogue of Proposition 2.4.8 would hold as well; i.e., there is no duality gap and there exists a dual optimizer in $\mathcal{D}^{ci,pw}_{\mu,\nu}(f)$. We have seen that this is not the case with the requirement that $h \geq 0$, but it fails even if this is dropped. Indeed, consider again a triplet (φ, ψ, h) satisfying the above system of equations. If $(\varphi, \psi, h) \in \mathcal{D}^{ci,pw}_{\mu,\nu}(f)$, then in particular there exists a concave and increasing moderator χ such that $\varphi - \chi \in L^1(\mu)$. Noting that μ has an infinite second moment and that $\varphi^-(x)$ has quadratic growth as $x \to \infty$ along the integers, it follows that $\chi^-(x)$ must have superlinear growth as $x \to \infty$. But then χ can certainly not be increasing, and we have reached the desired contradiction.

Two Couplings that are not Canonical

As mentioned in the Introduction, it is natural to ask if reward functions f that are first- and second-order Spence-Mirrlees are also maximized by a common supermartingale transport—i.e., $f_{xy} > 0, f_{xyy} > 0$ if f is smooth, rather than the mixed signs that were considered in the preceding sections (see also Example 2.5.5). However, it turns out that two functions f^1, f^2 satisfying these Spence–Mirrlees conditions may have different optimizers, even if the optimizer is unique for each f^i. This is shown in Example 2.10.1. The same is true when $-f^i$ are first- and second-order Spence-Mirrlees, as shown by Example 2.10.2; we confine ourselves to numerical counterexamples.

Example 2.10.1. Let μ and ν be uniformly distributed on $\{-1, 0, 1\}$ and $\{-4, -2.5, 2\}$, respectively; then $\mu \leq_{cd} \nu$. We consider the reward functions $f^1(x, y) = e^x e^y$ and $f^2(x, y) = e^x e^y + 4xy$ which satisfy $f^i_{xy} > 0$ and $f^i_{xyy} > 0$. The corresponding optimal transports can be obtained with an LP-solver; they are unique and given by

$$\pi^1 = \tfrac{5}{18}\delta_{(-1,-4)} + \tfrac{1}{18}\delta_{(-1,-2.5)} + \tfrac{5}{18}\delta_{(0,-2.5)} + \tfrac{1}{18}\delta_{(0,2)} + \tfrac{1}{18}\delta_{(1,-4)} + \tfrac{5}{18}\delta_{(1,2)},$$

$$\pi^2 = \tfrac{1}{3}\delta_{(-1,-4)} + \tfrac{7}{27}\delta_{(0,-2.5)} + \tfrac{2}{27}\delta_{(0,2)} + \tfrac{2}{27}\delta_{(1,-2.5)} + \tfrac{7}{27}\delta_{(1,2)}.$$

Their supports are shown in Figure 2.5. The transports are first- and second order left-monotone with $M = \{1\}$, but the kernels and supports are different.

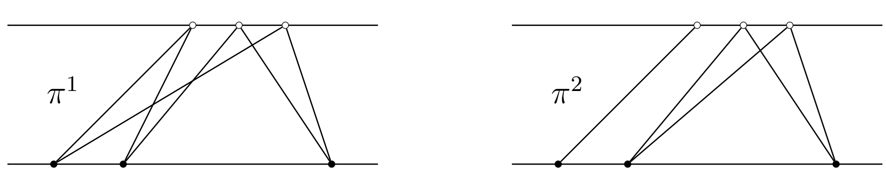

Figure 2.5: The optimal transports from Example 2.10.1

Example 2.10.2. Let μ and ν be uniformly distributed on $\{-1, 0, 1\}$ and $\{-4, -2.5, 0.5\}$, respectively; then again $\mu \leq_{cd} \nu$. We consider the reward functions $f^1(x, y) = -e^x e^y$ and $f^2(x, y) = -e^x e^y - 4xy$ which are the negatives of the functions in Example 2.10.1; they satisfy $f^i_{xy} < 0$ and $f^i_{xyy} < 0$. The corresponding (unique) optimal transports are given by

$$\pi^1 = \tfrac{1}{9}\delta_{(-1,-4)} + \tfrac{2}{9}\delta_{(-1,0.5)} + \tfrac{2}{9}\delta_{(0,-2.5)} + \tfrac{1}{9}\delta_{(0,0.5)} + \tfrac{2}{9}\delta_{(1,-4)} + \tfrac{1}{9}\delta_{(1,-2.5)},$$

$$\pi^2 = \tfrac{1}{6}\delta_{(-1,-2.5)} + \tfrac{1}{6}\delta_{(-1,0.5)} + \tfrac{1}{6}\delta_{(0,-2.5)} + \tfrac{1}{6}\delta_{(0,0.5)} + \tfrac{1}{3}\delta_{(1,-4)}.$$

Their supports are shown in Figure 2.6. The transports are first- and second order right-monotone with $M = \{-1\}$, but the kernels and supports are different.

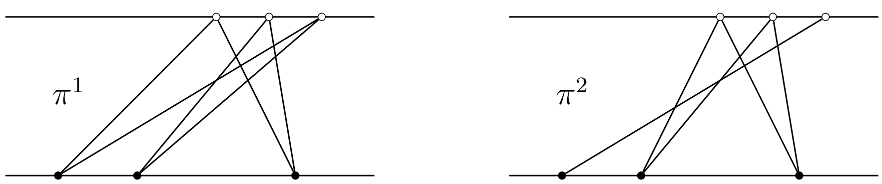

Figure 2.6: The optimal transports from Example 2.10.2

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— 3 —

Multiperiod Martingale Transport

This chapter is based on the article [72] of the same title, authored by Marcel Nutz, Florian Stebegg and Xiaowei Tan. It is forthcoming in Stochastic Processes and its Applications.

3.1 Introduction

Let $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ be a vector of probability measures μ_t on the real line. A measure P on $\mathbb{R}^{n+1}$ whose marginals are given by $\boldsymbol{\mu}$ is called a coupling (or transport) of $\boldsymbol{\mu}$, and the set of all such measures is denoted by $\Pi(\boldsymbol{\mu})$. We shall be interested in couplings P that are martingales; that is, the identity $X = (X_0, \dots, X_n)$ on $\mathbb{R}^{n+1}$ is a martingale under P. Hence, we will assume that all marginals have a finite first moment and denote by $\mathcal{M}(\boldsymbol{\mu})$ the set of martingale couplings. A classical result of Strassen [82] shows that $\mathcal{M}(\boldsymbol{\mu})$ is nonempty if and only if the marginals are in convex order, denoted by $\mu_{t-1} \leq_c \mu_t$ and defined by the requirement that $\mu_{t-1}(\phi) \leq \mu_t(\phi)$ for any convex function ϕ, where $\mu(\phi) := \int \phi \, d\mu$.

The first goal of this paper is to introduce and study a family of "canonical" couplings $P \in \mathcal{M}(\mu)$ that we call left-monotone. These couplings specialize to the Left-Curtain coupling of [13] in the one-step case $n = 1$ and share, broadly speaking, several properties reminiscent of the Hoeffding–Fréchet coupling of classical optimal transport. Indeed, left-monotone couplings will be characterized by order-theoretic minimality properties, as simultaneous optimal transports for certain classes of reward (or cost) functions, and through no-crossing conditions on their supports.

The second goal is to develop a strong duality theory for multiperiod martingale optimal transport, along the lines of [16] for the one-period martingale case and [66] for the classical optimal transport problem. That is, we introduce a suitable dual optimization problem and show the absence of a duality gap as well as the existence of dual optimizers for general transport reward (or cost) functions. The duality result is a crucial tool for the study of the left-monotone couplings.

We also develop similar results for a variant of our problem where the intermediate marginals $\mu_1, \dots, \mu_{n-1}$ are not prescribed (Section 3.9), but we shall focus on the full marginal case for the purpose of the Introduction.

Left-Monotone Transports

For the sake of orientation, let us first state the main result and then explain the terminology contained therein. The following is a streamlined version—the results in the body of the paper are stronger in some technical aspects.

Theorem 3.1.1. *Let $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ be in convex order and $P \in \mathcal{M}(\boldsymbol{\mu})$ a martin-*

gale transport between these marginals. The following are equivalent:

(i) P *is a simultaneous optimal transport for* $f(X_0, X_t)$, $1 \le t \le n$ *whenever* $f : \mathbb{R}^2 \to \mathbb{R}$ *is a smooth second-order Spence–Mirrlees function.*

(ii) P *is concentrated on a left-monotone set* $\Gamma \subseteq \mathbb{R}^{n+1}$.

(iii) P *transports* $\mu_0|_{(-\infty,a]}$ *to the obstructed shadow* $\mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0|_{(-\infty,a]})$ *in step* t, *for all* $1 \le t \le n$ *and* $a \in \mathbb{R}$.

There exists $P \in \mathcal{M}(\boldsymbol{\mu})$ *satisfying (i)–(iii), and any such* P *is called a left-monotone transport. If* μ_0 *is atomless, then* P *is unique.*

Let us now discuss the items in the theorem.

(i) Optimal Transport. This property characterizes P as a simultaneous optimal transport. Given a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, we may consider the martingale optimal transport problem with reward f (or cost $-f$),

$$\mathbf{S}_{\boldsymbol{\mu}}(f) = \sup_{P \in \mathcal{M}(\boldsymbol{\mu})} P(f); \tag{3.1.1}$$

recall that $P(f) = \mathbb{E}^P[f(X_0, \dots, X_n)]$. A Lipschitz function $f \in C^{1,2}(\mathbb{R}^2; \mathbb{R})$ is called a *smooth second-order Spence–Mirrlees* function if it satisfies the cross-derivative condition $f_{xyy} > 0$; this has also been called the martingale Spence–Mirrlees condition in analogy to the classical Spence–Mirrlees condition $f_{xy} > 0$. Given such a function of two variables and $1 \le t \le n$, we may consider the n-step martingale optimal transport problem with reward $f(X_0, X_t)$. Characterization (i) states that a left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ is an optimizer simultaneously for the n transport problems $f(X_0, X_t)$, $1 \le t \le n$, for some (and then all) smooth second-order Spence–Mirrlees functions f.

In the one-step case, a corresponding result holds for the Left-Curtain coupling [13]; here the simultaneous optimization becomes a single one. In view of the characterization in (i), an immediate consequence is that if there exists $P \in \mathcal{M}(\boldsymbol{\mu})$ such that all bivariate projections $P_{0t} = P \circ (X_0, X_t)^{-1} \in \mathcal{M}(\mu_0, \mu_t)$ are of Left-Curtain type, then P is left-monotone. However, such a transport does not exist unless the marginals satisfy a very specific condition (see Proposition 3.6.9), and in general the bivariate projections of a left-monotone transport are *not* of Left-Curtain type.

(ii) Geometry. The second item characterizes P through a geometric property of its support. A set $\Gamma \subseteq \mathbb{R}^{n+1}$ will be called *left-monotone* if it has the following no-crossing property for all $1 \le t \le n$: Let $\boldsymbol{x} = (x_0, \dots, x_{t-1})$, $\boldsymbol{x}' = (x'_0, \dots, x'_{t-1}) \in \mathbb{R}^t$ and

$$y^-, y^+, y' \in \mathbb{R} \quad \text{with} \quad y^- < y^+$$

be such that $(\boldsymbol{x}, y^+), (\boldsymbol{x}, y^-), (\boldsymbol{x}', y')$ are in the projection of Γ to the first $t+1$ coordinates. Then,

$$y' \notin (y^-, y^+) \quad \text{whenever} \quad x_0 < x'_0.$$

That is, if we consider two paths in Γ starting at x_0 and coinciding up to $t-1$, and a third path starting at x'_0 to the right of x_0, then at time t the third path cannot step in-between the first two—this is illustrated in Figure 3.1. Item (ii) states that a left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ can be characterized by the fact that it is

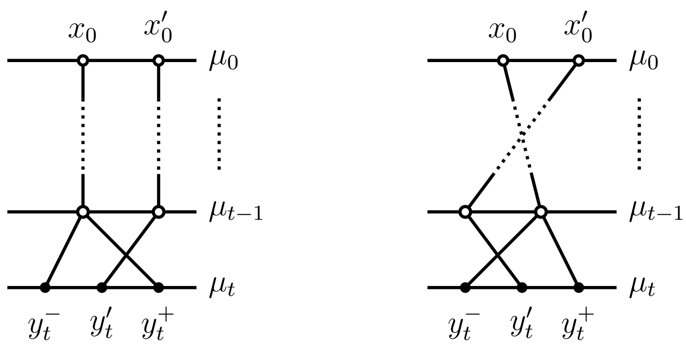

Figure 3.1: Two examples of forbidden configurations in left-monotone sets.

concentrated on a left-monotone set Γ. (In Theorem 3.7.16 we shall state a stronger result: we can find a left-monotone set that carries all left-monotone transports at once.)

In the one-step case $n = 1$, left-monotonicity coincides with the Left-Curtain property of [13]. However, we emphasize that for $t > 1$, our no-crossing condition differs from the Left-Curtain property of the bivariate projection $(X_0, X_t)(\Gamma)$ as the latter would not contain the restriction that the first two paths have to coincide up to $t - 1$ (see also Example 3.6.10). This corresponds to the mentioned fact that the bivariate marginal P_{0t} need not be of Left-Curtain type. On the other hand, the geometry of the projection $(X_{t-1}, X_t)(\Gamma)$ is also quite different from the Left-Curtain one, as our condition may rule out third paths crossing from the right *and* left at $t - 1$, depending on the starting point x'_0 rather than the location of x'_{t-1}.

(iii) Convex Ordering. This property characterizes left-monotone transports in an order-theoretic way and will be used in the existence proof. To explain the idea, suppose that μ_0 consists of finitely many atoms at $x_1, \dots, x_N \in \mathbb{R}$. Then, for any fixed t, a coupling of μ_0 and μ_t can be defined by specifying a "destination" measure for each atom. We consider all chains[1] $\mu_0|_{x_i} \leq_c \theta_1 \leq_c \cdots \leq_c \theta_t$ of measures θ_s in convex order that satisfy the marginals constraints $\theta_s \leq \mu_s$ for $s \leq t$. Of these chains, keep only the terminal measures θ_t and compare them according to the convex order. The *obstructed shadow of* $\mu_0|_{x_1}$ *in* μ_t *through* $\mu_1, \ldots, \mu_{t-1}$, denoted $\mathcal{S}^{\mu_1,\ldots,\mu_t}(\mu_0|_{x_i})$, is defined as the unique least element[2] among the θ_t. A particular coupling of μ_0 and μ_t is the one that successively maps the atoms $\mu_0|_{x_i}$ to their obstructed shadows in the remainder of μ_t, starting with the left-most atom x_i and continuing from left to right. In the case of general measures, we consider the restrictions $\mu_0|_{(-\infty,a]}$ instead of successively mapping the atoms. Characterization (iii) then states that a left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ maps $\mu_0|_{(-\infty,a]}$ to its obstructed shadow at date t for all $1 \leq t \leq n$ and $a \in \mathbb{R}$. This shows in particular that the bivariate projections $P_{0t} = P \circ (X_0, X_t)^{-1}$ of a left-monotone coupling are uniquely determined. In the body of the text, we shall also give an alternative definition of the obstructed shadow by iterating unobstructed shadows through the marginals up to date t; see Section 3.6.

The above specializes to the construction of [13] for the one-step case, which corresponds to the situation of $t = 1$ where there are no intermediate marginals obstructing the shadow. When $t > 1$, the obstruction by the intermediate marginals once again entails that P_{0t} need not be of Left-Curtain type. More precisely, Characterization (iii) gives rise to a sharp criterion (Proposition 3.6.9) on the marginals $\boldsymbol{\mu}$, describing exactly when this coincidence arises.

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[1] Here $\mu_0|_{x_i}$ denotes a Dirac measure of mass $\mu_0(\{x_i\})$ at x_i.

[2] See Definition 3.6.6 and Lemma 3.6.7 for details on this construction.

(Non-)Uniqueness. We have seen above that for a left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ the bivariate projections P_{0t}, $1 \leq t \leq n$ are uniquely determined. In particular, for $n = 1$, we recover the result of [13] that the left-monotone coupling is unique. For $n > 1$, the situation turns out to be quite different depending on the nature of the first marginal. On the one extreme, we shall see that when μ_0 is atomless, there is a unique left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$. Moreover, P has a degenerate structure reminiscent of Brenier's theorem: it can be disintegrated as $P = \mu_0 \otimes \kappa_1 \otimes \cdots \otimes \kappa_n$ where each one-step transport kernel κ_t is concentrated on the graphs of two functions. On the other extreme, if μ_0 is a Dirac mass, the typical case is that there are infinitely many left-monotone couplings—see Section 3.8 for a detailed discussion. We shall also show that left-monotone transports are not Markovian in general, even if uniqueness holds (Example 3.7.17).

Duality

The analysis of left-monotone transports is based on a duality result that we develop for general reward functions $f : \mathbb{R}^{n+1} \to (-\infty, \infty]$ with an integrable lower bound. Formally, the dual problem (in the sense of linear programming) for the transport problem $\mathbf{S}_{\boldsymbol{\mu}}(f) = \sup_{P \in \mathcal{M}(\boldsymbol{\mu})} P(f)$ is the minimization

$$\mathbf{I}_{\boldsymbol{\mu}}(f) := \inf_{(\boldsymbol{\phi}, H)} \sum_{t=0}^{n} \mu_t(\phi_t)$$

where the infimum is taken over vectors $\boldsymbol{\phi} = (\phi_0, \ldots, \phi_n)$ of real functions and predictable processes $H = (H_1, \ldots, H_n)$ such that

$$\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n \geq f; \tag{3.1.2}$$

here $(H \cdot X)_n := \sum_{t=1}^{n} H_t\,(X_t - X_{t-1})$ is the discrete-time integral. The desired result (Theorem 3.5.2) states that there is no duality gap, i.e. $\mathbf{I}_{\boldsymbol{\mu}}(f) = \mathbf{S}_{\boldsymbol{\mu}}(f)$, and that the dual problem is attained whenever it is finite. From the analysis for the one-step case in [16] we know that this assertion fails for the above naive formulation of the dual, and requires several relaxations regarding the integrability of the functions ϕ_t and the domain $\mathcal{V} \subseteq \mathbb{R}^{n+1}$ where the inequality (3.1.2) is required. Specifically, the inequality needs to be relaxed on sets that are $\mathcal{M}(\boldsymbol{\mu})$-polar; i.e. not charged by any transport $P \in \mathcal{M}(\boldsymbol{\mu})$. These sets are characterized in Theorem 3.3.1 where we show that the $\mathcal{M}(\boldsymbol{\mu})$-polar sets are precisely the (unions of) sets which project to a two-dimensional polar set of $\mathcal{M}(\mu_{t-1}, \mu_t)$ for some $1 \leq t \leq n$.

The duality theorem gives rise to a monotonicity principle (Theorem 3.5.4) that underpins the analysis of the left-monotone couplings. Similarly to the cyclical monotonicity condition in classical transport, it allows one to study the geometry of the support of optimal transports for a given function f.

Background and Related Literature

The martingale optimal transport problem (3.1.1) was introduced in [9] with the dual problem as a motivation. Indeed, in financial mathematics the function f is understood as the payoff of a derivative written on the underlying X and (3.1.2) corresponds to superhedging f by statically trading in European options $\phi_t(X_t)$ and dynamically trading in the underlying according to the strategy H. The value $\mathbf{I}_{\boldsymbol{\mu}}(f)$ then corresponds to the lowest price of f for which the seller can enter a model-free hedge $(\boldsymbol{\phi}, H)$ if the marginals $X_t \sim \mu_t$ are known from option market data. In [9], it was shown (with the above, "naive" formulation of the dual problem) that there is no duality gap if f is sufficiently regular, whereas dual existence was shown to fail even in regular cases. The idea of model-free hedging as well as the connection to Skorokhod embeddings goes back to [56]; we refer to [21, 22, 28, 57, 73, 84] for further references. A specific multiperiod martingale optimal transport problem also arises in the study of the maximum maximum of a martingale given n marginals [52].

The one-step case $n = 1$ has been studied in great detail. In particular, [13] introduced the Left-Curtain coupling and pioneered numerous ideas underlying Theorem 3.1.1, [51] provided an explicit construction of that coupling, and [62] established the stability with respect to the marginals. Our duality results specialize to the ones of [16] when $n = 1$. Unsurprisingly, we shall exploit many arguments and results from these papers wherever possible. As indicated above, and as will be seen in the proofs below, the multistep case allows for a richer structure and necessitates novel ideas; for instance, the analysis of the polar sets (Theorem 3.3.1) is surprisingly involved. Other works in the one-step martingale case have studied reward functions f such as forward start straddles [58, 59] or Asian payoffs [81]. We also refer to [44, 67] for recent developments with multidimensional marginals.

One-step martingale optimal transport problems can alternately be studied as optimal Skorokhod embedding problems with marginal constraints; cf. [7, 19, 10, 12]. A multi-marginal extension [8] of [7] is in preparation at the time of writing and the authors have brought to our attention that it will offer a version of Theorem 3.1.1 in the Skorokhod picture, at least in the case where μ_0 is atomless and some further conditions are satisfied. The Skorokhod embedding problem with multi-marginal constraint was also studied in [48].

A multi-step coupling quite different from ours can be obtained by composing in a Markovian fashion the Left-Curtain transport kernels from μ_{t-1} to μ_t, $1 \leq t \leq n$, as discussed in [51]. In [61] the continuous-time limits of such couplings for $n \to \infty$ are studied to find solutions of the so-called Peacock problem [54] where the marginals for a continuous-time martingale are prescribed; see also [50] and [63] for other continuous-time results with full marginal constraint. Early contributions related to the continuous-time martingale transport problem include [35, 36, 43, 68, 80, 83].

The remainder of the paper is organized as follows. Section 3.2 fixes basic terminology and recalls the necessary results from the one-step case. In Section 3.3, we characterize the polar structure of $\mathcal{M}(\boldsymbol{\mu})$. Section 3.4 introduces and analyzes the space that is the domain of the dual problem in Section 3.5, where we state the duality theorem and the monotonicity principle. Section 3.6 introduces left-monotone transports by the shadow construction and Section 3.7 develops the equivalent characterizations in terms of support and optimality properties. The (non-)uniqueness of left-monotone transports is discussed in Section 3.8. We conclude with the analysis of the problem with unconstrained intermediate marginals in Section 3.9.

3.2 Preliminaries

Throughout this paper, μ_t, μ, ν denote finite measures on $\mathbb{R}$ with finite first moment, the total mass not necessarily being normalized. Generalizing the notation from the Introduction to a vector $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ of such measures, we will write $\Pi(\boldsymbol{\mu})$ for the set of couplings; that is, measures P on $\mathbb{R}^{n+1}$ such that $P \circ X_t^{-1} = \mu_t$ for $0 \leq t \leq n$ where $X = (X_0, \ldots, X_n) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the identity. Moreover, $\mathcal{M}(\boldsymbol{\mu})$ is the subset of all $P \in \Pi(\boldsymbol{\mu})$ that are martingales, meaning that

$$\int X_s \mathbf{1}_A(X_0, \ldots, X_s) dP = \int X_t \mathbf{1}_A(X_0, \ldots, X_s) dP$$

for all $s \leq t$ and Borel sets $A \in \mathfrak{B}(\mathbb{R}^{s+1})$.

We denote by $\mathbb{F} = \{\mathfrak{F}_t\}_{0 \leq t \leq n}$ the canonical filtration $\mathfrak{F}_t := \sigma(X_0, \ldots, X_t)$. As usual, an $\mathbb{F}$-predictable process $H = \{H_t\}_{1 \leq t \leq n}$ is a sequence of real functions on $\mathbb{R}^{n+1}$ such that H_t is $\mathfrak{F}_{t-1}$-measurable; i.e. $H_t = h_t(X_0, \ldots, X_{t-1})$ for some Borel-measurable $h_t : \mathbb{R}^t \to \mathbb{R}$. Given an $\mathbb{F}$-predictable process H, the discrete stochastic integral $\{(H \cdot X)_t\}_{0 \leq t \leq n}$ is defined by

$$(H \cdot X)_t := \sum_{s=1}^{t} H_s \cdot (X_s - X_{s-1}).$$

If X is a martingale under some measure P, then $H \cdot X$ is a generalized (not necessarily integrable) martingale in the sense of generalized conditional expectations; cf. [60, Proposition 1.64].

We say that $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ is in convex order if $\mu_{t-1} \leq_c \mu_t$ for all $1 \leq t \leq n$;

that is, $\mu_{t-1}(\phi) \leq \mu_t(\phi)$ for any convex function $\phi : \mathbb{R} \to \mathbb{R}$. This implies that μ_{t-1} and μ_t have the same total mass. The order can also be characterized by the potential functions

$$u_{\mu_t} : \mathbb{R} \to \mathbb{R}, \quad u_{\mu_t}(x) := \int |x-y| \, \mu_t(dy).$$

The following properties are elementary:

(i) u_{μ_t} is nonnegative and convex,

(ii) $\partial^+ u_{\mu_t}(x) - \partial^- u_{\mu_t}(x) = 2\mu_t(\{x\})$,

(iii) $\lim_{|x|\to\infty} u_{\mu_t}(x) = \infty \mathbf{1}_{\mu_t \neq 0}$,

(iv) $\lim_{|x|\to\infty} u_{\mu_t}(x) - \mu_t(\mathbb{R})|x - \mathrm{bary}(\mu_t)| = 0$,

where ∂^+ and ∂^- denote the right and left derivatives, respectively, and $\mathrm{bary}(\mu_t) = (\int x d\mu_t)/\mu_t(\mathbb{R})$ is the barycenter. We can therefore extend u_{μ_t} continuously to $\bar{\mathbb{R}} = [-\infty, \infty]$. The following result of Strassen is classical (cf. [82]; the last statement is obtained as e.g. in [42, Corollary 2.95]).

Proposition 3.2.1. *Let $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ be finite measures on $\mathbb{R}$ with finite first moments and equal total mass. The following are equivalent:*

(i) $\mu_0 \leq_c \cdots \leq_c \mu_n$,

(ii) $u_{\mu_0} \leq \cdots \leq u_{\mu_n}$,

(iii) $\mathcal{M}(\boldsymbol{\mu}) \neq \emptyset$,

(iv) there exist stochastic kernels $\kappa_t(x_0, \ldots, x_{t-1}, dx_t)$ such that

$$\int |x_t| \, \kappa_t(x_0, \ldots, x_{t-1}, dx_t) < \infty \textit{ and } \int x_t \, \kappa_t(x_0, \ldots, x_{t-1}, dx_t) = x_{t-1}$$

for all $(x_0, \dots, x_t) \in \mathbb{R}^t$ *and* $1 \le t \le n$*, and*

$$\mu_t = (\mu_0 \otimes \kappa_1 \otimes \cdots \otimes \kappa_n) \circ (X_t)^{-1} \quad \textit{for all } 0 \le t \le n.$$

All kernels will be stochastic (i.e. normalized) in what follows. A kernel κ_t with the first property in (iv) is called *martingale kernel.*

The One-Step Case

For the convenience of the reader, we summarize some results from [13] and [16] for the one-step problem ($n = 1$) which will be used later on. In this section we write (μ, ν) instead of (μ_0, μ_1) for the given marginals in convex order.

Definition 3.2.2. The pair $\mu \le_c \nu$ is *irreducible* if the set $I = \{u_\mu < u_\nu\}$ is connected and $\mu(I) = \mu(\mathbb{R})$. In this situation, let J be the union of I and any endpoints of I that are atoms of ν; then (I, J) is the *domain* of $\mathcal{M}(\mu, \nu)$.

The first result is a decomposition of the transport problem into irreducible parts; cf. [13, Theorem 8.4].

Proposition 3.2.3. *Let* $\mu \le_c \nu$ *and let* $(I_k)_{1 \le k \le N}$ *be the (open) components of* $\{u_\mu < u_\nu\}$*, where* $N \in \{0, 1, \dots, \infty\}$*. Set* $I_0 = \mathbb{R} \setminus \cup_{k \ge 1} I_k$ *and* $\mu_k = \mu|_{I_k}$ *for* $k \ge 0$*, so that* $\mu = \sum_{k \ge 0} \mu_k$*. Then, there exists a unique decomposition* $\nu = \sum_{k \ge 0} \nu_k$ *such that*

$$\mu_0 = \nu_0 \quad \textit{and} \quad \mu_k \le_c \nu_k \quad \textit{for all} \quad k \ge 1,$$

and this decomposition satisfies $I_k = \{u_{\mu_k} < u_{\nu_k}\}$ *for all* $k \geq 1$. *Moreover, any* $P \in \mathcal{M}(\mu, \nu)$ *admits a unique decomposition* $P = \sum_{k \geq 0} P_k$ *such that* $P_k \in \mathcal{M}(\mu_k, \nu_k)$ *for all* $k \geq 0$.

We observe that the measure P_0 in Proposition 3.2.3 transports μ_0 to itself and is concentrated on $\Delta_0 := \Delta \cap I_0^2$ where $\Delta = \{(x, x) : x \in \mathbb{R}\}$ is the diagonal. Thus, the transport problem with index $k = 0$ is not actually an irreducible one, but we shall nevertheless refer to (I_0, I_0) as the domain of this problem. When we want to emphasize the distinction, we call (I_0, I_0) the *diagonal domain* and $(I_k, J_k)_{k \geq 1}$ the *irreducible domains* of $\mathcal{M}(\mu, \nu)$. Similarly, the sets $V_k := I_k \times J_k$, $k \geq 1$ will be called the *irreducible components* and $V_0 := \Delta_0$ will be called the *diagonal component* of $\mathcal{M}(\mu, \nu)$. This terminology refers to the following result of [16, Theorem 3.2] which essentially states that the components are the only sets that can be charged by a martingale transport. We call a set $B \subseteq \mathbb{R}^2$ $\mathcal{M}(\mu, \nu)$-polar if it is P-null for all $P \in \mathcal{M}(\mu, \nu)$, where a nullset is, as usual, any set contained in a Borel set of zero measure.

Proposition 3.2.4. *Let* $\mu \leq_c \nu$ *and let* $B \subseteq \mathbb{R}^2$ *be a Borel set. Then* B *is* $\mathcal{M}(\mu, \nu)$-*polar if and only if there exist a* μ-*nullset* N_μ *and a* ν-*nullset* N_ν *such that*

$$B \subseteq (N_\mu \times \mathbb{R}) \cup (\mathbb{R} \times N_\nu) \cup \left(\bigcup_{k \geq 0} V_k \right)^c .$$

The following result of [16, Lemma 3.3] will also be useful; it is the main ingredient in the proof of the preceding proposition.

Lemma 3.2.5. *Let $\mu \leq_c \nu$ be irreducible and let π be a finite measure on $\mathbb{R}^2$ whose marginals π_1, π_2 satisfy*[3] *$\pi_1 \leq \mu$ and $\pi_2 \leq \nu$. Then, there exists $P \in \mathcal{M}(\mu, \nu)$ such that P dominates π in the sense of absolute continuity.*

3.3 The Polar Structure

The goal of this section is to identify all obstructions to martingale transports imposed by the marginals $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$, and thus, conversely, the sets that can indeed be charged. We recall that a subset B of $\mathbb{R}^{n+1}$ is called $\mathcal{M}(\boldsymbol{\mu})$-polar if it is a P-nullset for all $P \in \mathcal{M}(\boldsymbol{\mu})$. The result for the one-step case in Proposition 3.2.4 already exhibits an obvious type of polar set $B \subseteq \mathbb{R}^{n+1}$: if for some t there is an $\mathcal{M}(\mu_{t-1}, \mu_t)$-polar set $B' \subseteq \mathbb{R}^2$ such that $B \subseteq \mathbb{R}^{t-1} \times B' \times \mathbb{R}^{n-t}$, then B must be $\mathcal{M}(\boldsymbol{\mu})$-polar. The following shows that unions of such sets are in fact the only polar sets of $\mathcal{M}(\boldsymbol{\mu})$.

Theorem 3.3.1 (Polar Structure)**.** *Let $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ be in convex order. Then a Borel set $B \subseteq \mathbb{R}^{n+1}$ is $\mathcal{M}(\boldsymbol{\mu})$-polar if and only if there exist μ_t-nullsets N_t such that*

$$B \;\subseteq\; \bigcup_{t=0}^{n} (X_t)^{-1}(N_t) \;\cup\; \bigcup_{t=1}^{n} (X_{t-1}, X_t)^{-1} \left(\bigcup_{k \geq 0} V_k^t \right)^c \tag{3.3.1}$$

where $(V_k^t)_{k \geq 1}$ are the irreducible components of $\mathcal{M}(\mu_{t-1}, \mu_t)$ and V_0^t is the corresponding diagonal component.

Before stating the proof, we introduce some additional terminology. The second

[3] By $\pi_1 \leq \mu$ we mean that $\pi_1(A) \leq \mu(A)$ for every Borel set $A \subseteq \mathbb{R}$.

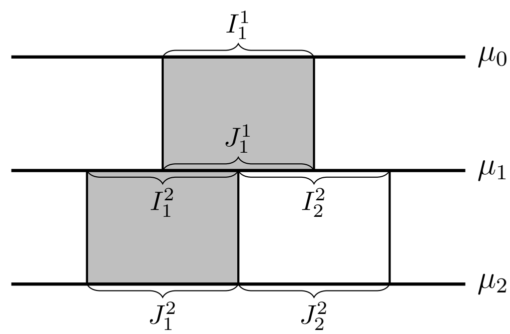

Figure 3.2: The shaded area represents $V_{\boldsymbol{k}}$ for $\boldsymbol{k} = (1, 1)$.

part of (3.3.1) can be expressed as

$$\begin{aligned}\bigcup_{t=1}^{n} (X_{t-1}, X_t)^{-1} \left(\bigcup_{k \geq 0} V_k^t \right)^c &= \left(\bigcap_{t=1}^{n} \bigcup_{k \geq 0} (X_{t-1}, X_t)^{-1} (V_k^t) \right)^c \\ &= \left(\bigcup_{k_1, \ldots, k_n \geq 0} \bigcap_{t=1}^{n} (X_{t-1}, X_t)^{-1} (V_{k_t}^t) \right)^c . \end{aligned} \tag{3.3.2}$$

For every $\boldsymbol{k} = (k_1, \ldots, k_n)$, the set

$$V_{\boldsymbol{k}} = \bigcap_{t=1}^{n} (X_{t-1}, X_t)^{-1} (V_{k_t}^t) \subseteq \mathbb{R}^{n+1}$$

as occurring in the last expression of (3.3.2) will be referred to as an *irreducible component* of $\mathcal{M}(\boldsymbol{\mu})$; these sets are disjoint since $V_k^t \cap V_{k'}^t = \emptyset$ for $k \neq k'$. Moreover, we call their union

$$\mathcal{V} = \cup_{\boldsymbol{k}} V_{\boldsymbol{k}}$$

the *effective domain* of $\mathcal{M}(\boldsymbol{\mu})$.

Roughly speaking, an irreducible component $V_{\boldsymbol{k}}$ is a chain of irreducible components from the individual steps $(t-1, t)$. In the one-step case considered in [13, 16], it was possible and useful to decompose the transport problem into its irreducible components and study those separately to a large extent; cf. Proposition 3.2.3. This is impossible in the multistep case, as illustrated by the following example.

Example 3.3.2. Consider the two-step martingale transport problem with marginals $\mu_0 = \delta_0$, $\mu_1 = \frac{1}{2}(\delta_{-1}+\delta_1)$ and $\mu_2 = \frac{1}{4}(\delta_{-2}+2\delta_0+\delta_2)$. Then the irreducible components are given by

$$\begin{aligned}
V_{00} &= \{(x,x,x) : x \notin (-2,2)\} \\
V_{01} &= \{(x,x) : x \in (-2,-1]\} \times [-2,0] \\
V_{02} &= \{(x,x) : x \in [1,2)\} \times [0,2] \\
V_{10} &= (-1,1) \times \{0\} \times \{0\} \\
V_{11} &= (-1,1) \times [-1,0) \times [-2,0] \\
V_{12} &= (-1,1) \times (0,1] \times [0,2].
\end{aligned}$$

There is only one martingale transport $P \in \mathcal{M}(\boldsymbol{\mu})$, given by

$$P = \frac{1}{4}(\delta_{(0,-1,-2)} + \delta_{(0,-1,0)} + \delta_{(0,1,0)} + \delta_{(0,1,2)}).$$

While P is supported on $V_{11} \cup V_{12}$, it cannot be decomposed into two martingale parts that are supported on V_{11} and V_{12}, respectively: V_{11} and V_{12} are disjoint, but $P|_{V_{11}} = \frac{1}{4}(\delta_{(0,-1,-2)} + \delta_{(0,-1,0)})$ is not a martingale.

The main step in the proof of Theorem 3.3.1 will be the following lemma.

Lemma 3.3.3. *Let $V_{\boldsymbol{k}}$ be an irreducible component of $\mathcal{M}(\boldsymbol{\mu})$ and consider a measure π concentrated on $V_{\boldsymbol{k}}$ such that $\pi_t \leq \mu_t$ for $t = 0, \ldots, n$. Then there exists a transport $P \in \mathcal{M}(\boldsymbol{\mu})$ which dominates π in the sense of absolute continuity.*

Deferring the proof, we first show how this implies the theorem.

Proof of Theorem 3.3.1. Clearly $(X_t)^{-1}(N_t)$ is $\mathcal{M}(\boldsymbol{\mu})$-polar for $t = 0, \ldots, n$ and $(X_{t-1}, X_t)^{-1}\left(\bigcup_{k\geq 0} V_k^t\right)^c$ is $\mathcal{M}(\boldsymbol{\mu})$-polar for $t = 1, \ldots, n$. This shows that (3.3.1) is sufficient for $B \subseteq \mathbb{R}^{n+1}$ to be $\mathcal{M}(\boldsymbol{\mu})$-polar.

Conversely, suppose that (3.3.1) does not hold; we show that B is not $\mathcal{M}(\boldsymbol{\mu})$-polar. In view of (3.3.2), by passing to a subset of B if necessary, we may assume that

$$B \subseteq \mathcal{V} = \bigcup_{\boldsymbol{k}} V_{\boldsymbol{k}} = \bigcup_{\boldsymbol{k}} \bigcap_{t=1}^{n} (X_{t-1}, X_t)^{-1}(V_{k_t}^t).$$

We may also assume that there are no μ_t-nullsets N_t such that $B \subseteq \cup_{t=0}^n (X_t)^{-1}(N_t)$. By a result of classical optimal transport [18, Proposition 2.1], this entails that B is not $\Pi(\boldsymbol{\mu})$-polar; i.e. we can find a measure $\rho \in \Pi(\boldsymbol{\mu})$ such that $\rho(B) > 0$.

We now write $B = \bigcup_{\boldsymbol{k}} B \cap V_{\boldsymbol{k}}$. As $\rho(B) = \sum_{\boldsymbol{k}} \rho(B \cap V_{\boldsymbol{k}}) > 0$, we can find some $\boldsymbol{k}$ such that $\rho(B \cap V_{\boldsymbol{k}}) > 0$. But then $\pi := \rho|_{V_{\boldsymbol{k}}}$ satisfies the assumptions of Lemma 3.3.3 which yields $P \in \mathcal{M}(\boldsymbol{\mu})$ such that $P \gg \pi$. In particular, $P(B) > 0$ and B is not $\mathcal{M}(\boldsymbol{\mu})$-polar. $\square$

Proof of Lemma 3.3.3

The reasoning for Lemma 3.3.3 follows an induction on the number n of time steps; its rigorous formulation requires a certain amount of control over subsequent steps of the transport problem. Thus, we first state a more quantitative version of (the core part of) the lemma that is tailored to the inductive argument.

Definition 3.3.4. Let $\boldsymbol{\mu}$ be in convex order and $\mathcal{V}$ the effective domain of $\mathcal{M}(\boldsymbol{\mu})$. We say that a finite measure π has a *compact support family* if there are disjoint compact product sets[4] $K_1, \dots, K_m \subseteq \mathcal{V}$ with $\pi(\cup_i K_i) = \pi(\mathbb{R}^{n+1})$ such that $K_i \subseteq V_{\boldsymbol{k}_i}$ for some irreducible component $V_{\boldsymbol{k}_i}$ for all $i = 1, \dots, m$.

Definition 3.3.5. Let $\boldsymbol{\mu}$ be in convex order, $t \leq n$ and $\sigma \leq \mu_t$ a finite measure on $\mathbb{R}$. If $t = n$, we say that σ is *diagonally compatible* (with $\boldsymbol{\mu}$) if there is a finite family of compact sets $L_1, \dots, L_m \subseteq \mathbb{R}$ with $\sigma(\cup_i L_i) = \sigma(\mathbb{R})$. Whereas if $t < n$, we require in addition that for every i, either (a) $L_i \subseteq I_k$ for some irreducible component (I_k, J_k) of $\mathcal{M}(\mu_t, \mu_{t+1})$ or (b) $L_i \subseteq I_0$ and there is $t+1 \leq t' \leq n$ such that $L_i \subseteq I_0^s$ for the diagonal components of $\mathcal{M}(\mu_s, \mu_{s+1})$ for all $t \leq s < t'$ and $L_i \subseteq I_k^{t'}$ for some (non-diagonal) irreducible component $(I_k^{t'}, J_k^{t'})$ of $\mathcal{M}(\mu_{t'}, \mu_{t'+1})$, where we set $I_k^n = J_k^n = \mathbb{R}$ for notational convenience.

Lemma 3.3.6. *Let $t < n$ and let $L \subseteq I_0$ be a compact interval contained in the diagonal component of $\mathcal{M}(\mu_t, \mu_{t+1})$ such that $\mu_t(L) > 0$. There exist a compact interval $L' \subseteq L$ with $\mu_t(L') > 0$ and $t+1 \leq t' \leq n$ such that $L' \subseteq I_0^s$ for the diagonal component of $\mathcal{M}(\mu_s, \mu_{s+1})$ for all $t \leq s < t'$ and $L' \subseteq I_k^{t'}$ for some (non-diagonal)*

[4] By a compact product set we mean a set $K = A_0 \times \cdots \times A_n$ where each $A_t \subseteq \mathbb{R}$ is compact.

irreducible component $(I_k^{t'}, J_k^{t'})$ *of* $\mathcal{M}(\mu_{t'}, \mu_{t'+1})$*, where we again set* $I_k^n = J_k^n = \mathbb{R}$ *for notational convenience.*

Proof. The statement is trivially satisfied for $t = n-1$ as we can just take $L' = L$. For $t < n-1$, consider the family of irreducible components (I_k^{t+1}, J_k^{t+1}) of $\mathcal{M}(\mu_{t+1}, \mu_{t+2})$. We distinguish three cases.

(i) First, consider the case where $L \cap I_k^{t+1} = \emptyset$ for all $k \geq 1$, then L is contained in the diagonal component of $\mathcal{M}(\mu_{t+1}, \mu_{t+2})$.

(ia) If $L = \{x\}$ consists of a single point with positive mass, then we can conclude by induction from the result for $t + 1$.

(ib) If no endpoint of L is on the boundary of some component I_k^t, then observe that $\mu_t|_L = \mu_{t+1}|_L$. We can find $L' \subseteq L$ from the statement of the lemma for $t + 1$. Then L' gives the result as $\mu_t(L') = \mu_{t+1}(L') > 0$.

(ic) If L contains more than one point, and also the endpoint of some component I_k^t. When this endpoint x has positive point mass, we can set $L' = \{x\}$ and conclude as in (ia). If the endpoint has zero mass, we can find $\bar{L} \subseteq L$ compact with $\mu_t(\bar{L}) > 0$ that does not contain this endpoint and argue as in (ib). (Observe that there might be at most two endpoints.)

(ii) Next, let $k \geq 1$ be such that $\mu_{t+1}(L \cap I_k^{t+1}) > 0$ (and in particular $L \cap I_k^{t+1} \neq \emptyset$). Then we can find a compact interval $L' \subseteq L \cap I_k^{t+1}$ such that $\mu_t(L') > 0$ and we directly see that L' satisfies the statement of the lemma.

(iii) Finally, suppose that there is $k \geq 1$ with $L \cap I_k^{t+1} \neq \emptyset$ but $\mu_t(L \cap I_k^{t+1}) = 0$. In particular this means that $L \not\subseteq I_k^{t+1}$. It furthermore means that $I_k^{t+1} \not\subseteq L$, as otherwise $\mu_{t+1}(I_k^{t+1}) = \mu_t(I_k^{t+1}) = \mu_t(L \cap I_k^{t+1}) = 0$ which contradicts the definition of I_k^{t+1}. As L is a compact interval and I_k^{t+1} is an open interval, we have that $L \backslash I_k^{t+1}$ is a compact interval and $\mu_t(L \backslash I_k^{t+1}) = \mu_t(L) > 0$. Notice that there can be at most two such components I_k^{t+1} for fixed L and we will be in case (i) after removing both of them if necessary. □

Lemma 3.3.7. *Let $t \leq n$ and let $J \subseteq \mathbb{R}$ be an interval such that $\mu_t(J) > 0$. Then we can find a compact interval $K \subseteq J$ with $\mu_t(K) > 0$ such that $\mu_t|_K$ is diagonally compatible.*

Proof. The case $t = n$ is trivial. Thus, let $t < n$. We consider the family $\{I_k\}_{k \geq 1}$ of open sets corresponding to the irreducible components of $\mathcal{M}(\mu_t, \mu_{t+1})$ and distinguish two cases.

(i) There is some $k \geq 1$ such that $\mu_t(I_k \cap J) > 0$. In this case, we can choose a compact interval $K \subseteq I_k \cap J$ such that $\mu_t(K) > 0$.

(ii) Now suppose that $\mu_t(I_k \cap J) = 0$ for all $k \geq 1$. Then we first notice that there are at most two components I_{k_1},I_{k_2} so that $I_{k_i} \cap J \neq \emptyset$ and $J \backslash (I_{k_1} \cup I_{k_2})$ is still a nonempty interval with positive μ_t-mass, since I_k cannot be contained in J. We can therefore assume without loss of generality that $J \subseteq I_0$ and is compact. Now we can apply Lemma 3.3.6 to find a subinterval $K \subseteq J$ such that $\mu_t|_K$ is diagonally compatible. □

Lemma 3.3.8. *Let $t \leq n$ and let π be a measure on $\mathbb{R}^{t+1}$ that has a compact support family with respect to $\mu_0, \ldots, \mu_t$ and satisfies $\pi_s \leq \mu_s$ for $s \leq t$. In addition, suppose that π_t is diagonally compatible.*

Then there is a martingale measure Q on $\mathbb{R}^{t+1}$ that dominates π in the sense of absolute continuity and has a compact support family with respect to $\mu_0, \ldots, \mu_t$ and satisfies $Q_s \leq \mu_s$ for $s \leq t$. In addition, Q_t can be chosen to be diagonally compatible. Finally, Q can be chosen such that $dQ = g d\pi + d\sigma$ where the density g is bounded and the measure σ is singular with respect to π.

Proof. We proceed by induction on t. For $t = 0$ there is nothing to prove; we can set $Q = \pi$.

Consider $t \geq 1$ and assume that the lemma has already been shown for $(t-1)$-step measures. We disintegrate

$$\pi = \pi' \otimes \kappa(x_0, \ldots, x_{t-1}, dx_t) \tag{3.3.3}$$

and observe that π' satisfies the conditions of the lemma. In particular, π'_{t-1} must be diagonally compatible: the compact sets that it is supported on are either contained in irreducible components of $\mathcal{M}(\mu_{t-1}, \mu_t)$ or in the diagonal component. Any such compact subset of the diagonal component of $\mathcal{M}(\mu_{t-1}, \mu_t)$ must correspond to one of the finitely many compact sets in the support of π_t so that they inherit the compatibility property from these sets.

By the induction assumption, we then find a martingale measure $Q' \gg \pi'$ on $\mathbb{R}^t$ with the stated properties. In particular, the marginal Q'_{t-1} is diagonally compatible with $\boldsymbol{\mu}$.

Again, let $\{I_k\}_{k\geq 1}$ be the open intervals from the irreducible domains (I_k, J_k) of $\mathcal{M}(\mu_{t-1}, \mu_t)$ and let I_0 denote the corresponding diagonal domain. We shall construct a martingale kernel $\hat{\kappa}$ by suitably manipulating κ. Let us observe that since π is concentrated on $\mathcal{V}$ and has a compact support family with respect to $\mu_0, \dots, \mu_t$, the following hold for π'-a.e. $\boldsymbol{x} = (x_0, \dots, x_{t-1}) \in \mathbb{R}^t$ and a finite family of compact sets L_i with properties (a) or (b) from Definition 3.3.5:

- $\kappa(\boldsymbol{x}, \cdot) = \delta_{x_{t-1}}$ whenever $x_{t-1} \in I_0$,
- $\kappa(\boldsymbol{x}, \cdot)$ is concentrated on some L_i with $L_i \subseteq J_k$ for $x_{t-1} \in I_k$ with $k \geq 1$ and $Q'_{t-1}(I_k) > 0$.

By changing κ on a π'-nullset, we may assume that these two properties hold for all $\boldsymbol{x} \in \mathbb{R}^t$.

Step 1. Next, we argue that we may change Q' and κ such that the marginal $(Q' \otimes \kappa)_t = (Q' \otimes \kappa) \circ X_t^{-1}$ satisfies

$$(Q' \otimes \kappa)_t \leq \mu_t. \tag{3.3.4}$$

Indeed, recall that $dQ' = dQ'_{abs} + d\sigma' = g' d\pi' + d\sigma'$ where the density g' is bounded and σ' is singular with respect to π'. Using the Lebesgue decomposition theorem, we find a Borel set $A \subseteq \mathbb{R}^t$ such that $\sigma'(A) = \sigma'(\mathbb{R}^t)$ and $\pi'(A) = 0$. By scaling Q' with a constant we may assume that $g' \leq 1/2$. As $\pi_t \leq \mu_t$, the marginal $(Q'_{abs} \otimes \kappa)_t$ is then bounded by $\frac{1}{2}\mu_t$, and it remains to bound $(\sigma' \otimes \kappa)_t$ in the same way.

Note that $Q'_{t-1} \leq \mu_{t-1}$ implies $\sigma'_{t-1} \leq \mu_{t-1}$. We may change κ arbitrarily on the set A without invalidating (3.3.3). Indeed, for each irreducible component (I_k, J_k) of $\mathcal{M}(\mu_{t-1}, \mu_t)$ we choose and fix a compact interval $K_k \subseteq J_k$ with $\mu_t(K_k) > 0$

such that $\mu_t|_{K_k}$ is diagonally compatible; this is possible by Lemma 3.3.7. For $\boldsymbol{x} = (x_0, \dots, x_{t-1}) \in A$ such that $x_{t-1} \in I_k$ we then define

$$\kappa(\boldsymbol{x}, \cdot) := \frac{1}{\mu_t(K_k)} \mu_t|_{K_k}.$$

Set $\epsilon_k = \mu_t(K_k)/\mu_{t-1}(I_k)$. Then

$$\epsilon := \inf_{k:\, Q'_{t-1}(I_k)>0} \epsilon_k \wedge 1$$

is strictly positive because there are only finitely many k with $Q'_{t-1}(I_k) > 0$ (this is the purpose of the induction assumption that Q'_{t-1} is diagonally compatible). As $\sigma'_{t-1} \leq \mu_{t-1}$, we may scale Q' once again to obtain $\sigma'_{t-1} \leq \frac{\epsilon}{6}\mu_{t-1}$. We now have

$$(\sigma'|_{\mathbb{R}^{t-1}\times I_k} \otimes \kappa)_t = \sigma'_{t-1}(I_k) \frac{1}{\mu_t(K_k)} \mu_t|_{K_k} \leq \frac{\epsilon}{6} \frac{\mu_{t-1}(I_k)}{\mu_t(K_k)} \mu_t|_{K_k} \leq \frac{1}{6} \mu_t|_{K_k}.$$

For the diagonal domain I_0 the corresponding inequality holds because we have $\kappa(\boldsymbol{x}, \cdot) = \delta_{x_{t-1}}$ for $x_{t-1} \in I_0$ and $\sigma'_{t-1}|_{I_0} \leq \frac{1}{6}\mu_{t-1}|_{I_0} \leq \frac{1}{6}\mu_t|_{I_0}$. As a consequence, we have $(\sigma' \otimes \kappa)_t \leq \frac{1}{2}\mu_t$ as desired, so that we may assume (3.3.4) in what follows.

Step 2. We now construct a martingale kernel $\hat{\kappa}$ such that $Q = Q' \otimes \hat{\kappa}$ has the required properties. For a fixed irreducible component (I_k, J_k) we have that $Q'_{t-1}|_{I_k} = Q'_{t-1}|_K$ for some compact $K \subseteq I_k$. We can find compact intervals $B^-, B^+ \subseteq J_k$ with $\mu_t(B^-) > 0$ and $\mu_t(B^+) > 0$ such that B^- is to the left of K and B^+ is to the right of K, in the sense that $x < y < z$ for $x \in B^-$, $y \in K$ and $z \in B^+$. By Lemma 3.3.7, we can further assume that we have $B^+ \subseteq I^t_k$ and $B^- \subseteq I^t_{k'}$ for some $k, k' \geq 0$, where $(I_l^t)_{l\geq 0}$ belong to the components of $\mathcal{M}(\mu_t, \mu_{t+1})$, and that $\mu_t|_{B^\pm}$ is diagonally compatible.

Next, we define two nonnegative functions $\boldsymbol{x} \mapsto \varepsilon^-(\boldsymbol{x}), \varepsilon^+(\boldsymbol{x})$ for $\boldsymbol{x} = (x_0, \ldots, x_{t-1}) \in \mathbb{R}^{t-1} \times K$ as follows:

- for $\boldsymbol{x}$ such that $\mathrm{bary}(\kappa(\boldsymbol{x}, \cdot)) < x_{t-1}$, let ε^+ be the unique number such that $\kappa(\boldsymbol{x}, \cdot) + \varepsilon^+(\boldsymbol{x}) \cdot \mu_t|_{B^+}$ has barycenter x_{t-1},
- for $\boldsymbol{x}$ such that $\mathrm{bary}(\kappa(\boldsymbol{x}, \cdot)) > x_{t-1}$, let ε^- be the unique number such that $\kappa(\boldsymbol{x}, \cdot) + \varepsilon^-(\boldsymbol{x}) \cdot \mu_t|_{B^-}$ has barycenter x_{t-1},
- $\varepsilon^\pm(\boldsymbol{x}) = 0$ otherwise.

Observe that these numbers always exist because B^- and B^+ have positive mass and positive distance from the points $x_{t-1} \in K$. We now define the martingale kernel $\hat{\kappa}$ by

$$\hat{\kappa}(\boldsymbol{x}) := c(\varepsilon^- \cdot \mu_t|_{B^-} + \kappa + \varepsilon^+ \cdot \mu_t|_{B^+})$$

where $0 < c \leq 1$ is a normalizing constant such that $\hat{\kappa}$ is again a stochastic kernel. We also define $\hat{\kappa}(\boldsymbol{x}) = \kappa(\boldsymbol{x})$ for $\boldsymbol{x}$ on the diagonal domain.

For each $k \geq 1$, let $B_k^\pm$ denote the sets associated with I_k as above. Once again, the number

$$C := \frac{1}{3} \inf_{k\colon Q'_{t-1}(I_k)>0} [\mu_t(B_k^-) \wedge \mu_t(B_k^+)]$$

is strictly positive because there are only finitely many k with $Q'_{t-1}(I_k) > 0$. We can now define

$$Q := C \cdot (Q' \otimes \hat{\kappa}).$$

Then Q is a martingale transport whose marginals satisfy $Q_s \leq Q'_s \leq \mu_s$ for $0 \leq s \leq t-1$ whereas $Q_t \leq \mu_t$ by (3.3.4), the construction of $\hat{\kappa}$ and the choice of C; indeed, for every $x_{t-1} \in I_k^t$ we have

$$
\begin{aligned}
3C\hat{\kappa}(\boldsymbol{x}) &\leq 3C\varepsilon^- \cdot \mu_t|_{B^-} + 3C\kappa + 3C\varepsilon^+ \cdot \mu_t|_{B^+} \\
&\leq \mu_t|_{B^-} + \kappa + \mu_t|_{B^+} \leq 2\mu_t + \kappa.
\end{aligned}
$$

To see that Q_t is diagonally compatible, observe that Q_t is supported by a finite family of compact sets consisting of the following:

- a finite family of compact sets $\bar{L}_i \subseteq I_0$ such that $Q'_{t-1}|_{\bar{L}_i}$ is diagonally compatible (from the induction hypothesis that Q'_{t-1} is diagonally compatible),
- a finite family of compact sets $L_i \subseteq J_k$ for some $k \geq 1$ with $Q'_{t-1}(I_k) > 0$ such that $Q_t|_{L_i} \leq \mu_t|_{L_i}$ is diagonally compatible, and
- the sets $B_k^{\pm}$ for the finitely many k such that $Q'_{t-1}(I_k) > 0$, where $Q_t|_{B_k^{\pm}} \leq \mu_t|_{B_k^{\pm}}$ is diagonally compatible.

It remains to check that Q has the required decomposition with respect to π. Indeed, $\hat{\kappa}$ can be decomposed as

$$
\hat{\kappa} = c\kappa + (1-c)\kappa^{\perp}
$$

where $\kappa^{\perp}$ is singular to κ. Recalling the decomposition $Q' = Q'_{abs} + \sigma'$, we then have

$$
Q' \otimes \hat{\kappa} = cQ'_{abs} \otimes \kappa \; + \; (1-c)Q'_{abs} \otimes \kappa^{\perp} \; + \; \sigma' \otimes \hat{\kappa}.
$$

The last two terms are singular with respect to $\pi = \pi' \otimes \kappa$, and the first term is absolutely continuous with bounded density. $\square$

Proof of Lemma 3.3.3. Let π be a measure with marginals $\pi_t \leq \mu_t$ for all t which is concentrated on some irreducible component $V = V_{\boldsymbol{k}}$ and thus, in particular, on the effective domain $\mathcal{V}$.

Step 1. We first decompose $\pi = \sum_{m=1}^{\infty} \pi^m$ such that each π^m satisfies the requirements of Lemma 3.3.8 with $t = n$.

Indeed, let $V = \cap_{t=1}^{n}(X_{t-1}, X_t)^{-1}(V_{k_t}^t)$ and suppose first that $k_t \neq 0$ for $1 \leq t \leq n$. Then, we can write V as a product of nonempty intervals: $V = A_0 \times \cdots \times A_n$ where $A_0 = I_{k_1}^1$, $A_n = J_{k_n}^n$ and $A_t = J_{k_t}^t \cap I_{k_{t+1}}^{t+1}$ for $1 < t < n$. Thus, we can choose increasing families of compact intervals K_t^m such that $A_t = \cup_{m \geq 1} K_t^m$ for all t. Setting $\pi^1 := \pi|_{\prod_{t=0}^{n} K_t^1}$ and $\pi^m := \pi|_{\prod_{t=0}^{n} K_t^m \setminus \prod_{t=0}^{n} K_t^{m-1}}$ for $m > 1$ yields the required decomposition.

If $k_t = 0$ for one or more $1 \leq t \leq n$, we have $V \subseteq A_0 \times \cdots \times A_n$, where A_t is defined as above when $k_t \neq 0 \neq k_{t+1}$ but we use $\mathbb{R}$ instead of $J_{k_t}^t$ when $k_t = 0$ and $\mathbb{R}$ instead of $I_{k_{t+1}}^{t+1}$ when $k_{t+1} = 0$. After these modifications, π^m can be defined as above; recall that diagonal components are always closed.

Step 2. For each of the measures π^m, Lemma 3.3.8 yields a martingale measure $Q^m \gg \pi^m$ with the properties stated in the lemma. In particular, each Q^m has a compact support family. We show below that there exist $P^m \in \mathcal{M}(\boldsymbol{\mu})$ such that $P^m \gg Q^m$, and then $P := \sum 2^{-m} P^m$ satisfies $P \in \mathcal{M}(\boldsymbol{\mu})$ and $P \gg \pi$ as desired.

To complete the proof, it remains to show that for fixed $m \geq 1$ there exist $0 <$

$\epsilon < 1$ and $\bar{Q}^m \in \mathcal{M}(\boldsymbol{\mu} - \epsilon(Q_0^m, \ldots, Q_n^m))$, as we may then conclude by setting $P^m := \epsilon Q^m + \bar{Q}^m \in \mathcal{M}(\boldsymbol{\mu})$. By Proposition 3.2.1, the set $\mathcal{M}(\boldsymbol{\mu} - \epsilon(Q_0^m, \ldots, Q_n^m))$ is nonempty if the marginals are in convex order, or equivalently if the potential functions satisfy

$$u_{\mu_{t-1}} - \epsilon u_{Q_{t-1}^m} \leq u_{\mu_t} - \epsilon u_{Q_t^m} \tag{3.3.5}$$

for $t = 1, \ldots, n$. Thus, it suffices to find $\epsilon > 0$ with this property for fixed t, and we have reduced to a question about a one-step martingale transport problem. Indeed, we have $u_{\mu_{t-1}} \leq u_{\mu_t}$ on $\mathbb{R}$. Since Q^m has a compact support family and in particular is supported by $\mathcal{V}$, there is a finite collection of compact sets $K_j \subseteq \mathbb{R}$ such that each K_j is contained in one of the intervals $I_{k_j}^{t-1}$ from the decomposition of (μ_{t-1}, μ_t) into irreducible components, Q^m transports mass from K_j to itself for each j, and Q^m is the identical Monge transport on the complement $(\cup_j K_j)^c$. On each K_j, Steps (a) and (b) in the proof of [16, Lemma 3.3] yield $\epsilon > 0$ such that (3.3.5) holds on K_j, and we can choose $\epsilon > 0$ independently of j since there are finitely many j. On the other hand, (3.3.5) trivially holds on $(\cup_j K_j)^c$ since $u_{Q_{t-1}^m} = u_{Q_t^m}$ on that set. This completes the proof. □

3.4 The Dual Space

In this section we introduce the domain of the dual optimization problem and show that it has a certain closedness property. The latter will be crucial for the duality theorem in the subsequent section.

We shall need a generalized notion of integrability for the elements of the dual space. To this end, we first recall the integral for concave functions as detailed in [16, Section 4.1].

Definition 3.4.1. Let $\mu \leq_c \nu$ be irreducible with domain (I, J) and let $\chi : J \to \mathbb{R}$ be a concave function. We define

$$(\mu - \nu)(\chi) := \frac{1}{2} \int_I (u_\mu - u_\nu) d\chi'' + \int_{J \setminus I} |\Delta \chi| d\nu \in [0, \infty]$$

where $-\chi''$ is the (locally finite) second derivative measure of $-\chi$ on I and $|\Delta\chi|$ is the absolute magnitude of the jumps of χ at the boundary points $J \setminus I$.

Remark 3.4.2. As shown in [16, Lemma 4.1], this integral is well-defined and satisfies

$$(\mu - \nu)(\chi) = \int_I \left[\chi(x) - \int_J \chi(y)\, \kappa(x, dy) \right] \mu(dx)$$

for any $P = \mu \otimes \kappa \in \mathcal{M}(\mu, \nu)$. Moreover, it coincides with the difference $\mu(\chi) - \nu(\chi)$ of the usual integrals when $\chi \in L^1(\mu) \cap L^1(\nu)$.

For later reference, we record two more properties of the integral.

Lemma 3.4.3. *Let $\mu \leq_c \nu$ be irreducible with domain (I, J) and let $\chi : J \to \mathbb{R}$ be concave.*

(i) Assume that I has a finite right endpoint r and $\chi(a) = \chi'(a) = 0$ for some

$a \in I$. Then $\chi \le 0$ and $\chi \mathbf{1}_{[a,\infty)}$ is concave. If ν has an atom at r, then

$$\chi(r) \ge -\frac{C}{\nu(\{r\})}(\mu - \nu)(\chi \mathbf{1}_{[a,\infty)})$$

for a constant $C \ge 0$ depending only on μ, ν.

(ii) For $a, b \in \mathbb{R}$, the concave function $\bar{\chi}(x) := \chi(x) + ax + b$ satisfies

$$(\mu - \nu)(\bar{\chi}) = (\mu - \nu)(\chi).$$

Proof. The first part is [16, Remark 4.6] and the second part follows directly from $\bar{\chi}'' = \chi''$ and $\Delta\bar{\chi} = \Delta\chi$. $\square$

Let us now return to the multistep case with a vector $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ of measures in convex order and introduce $\boldsymbol{\mu}(\boldsymbol{\phi}) := \sum_{t=0}^{n} \mu_t(\phi_t)$ in cases where we do not necessarily have $\phi_t \in L^1(\mu_t)$. As mentioned previously, in contrast to [16], the multistep transport problem does not decompose into irreducible components, forcing us to directly give a global definition of the integral.

Definition 3.4.4. Let $\boldsymbol{\phi} = (\phi_0, \dots, \phi_n)$ be a vector of Borel functions $\phi_t : \mathbb{R} \to \bar{\mathbb{R}}$. A vector $\boldsymbol{\chi} = (\chi_1, \dots, \chi_n)$ of Borel functions $\chi_t : \mathbb{R} \to \mathbb{R}$ is called a *concave moderator* for $\boldsymbol{\phi}$ if for $1 \le t \le n$,

(i) $\chi_t|_J$ is concave for every domain (I, J) of an irreducible component of $\mathcal{M}(\mu_{t-1}, \mu_t)$,

(ii) $\chi_t|_{I_0} \equiv 0$ for the diagonal domain I_0 of $\mathcal{M}(\mu_{t-1}, \mu_t)$,

(iii) $\phi_t - \chi_{t+1} + \chi_t \in L^1(\mu_t)$, where $\chi_{n+1} \equiv 0$. We also convene that $\chi_0 \equiv 0$. The *moderated integral* of $\boldsymbol{\phi}$ is then defined by

$$\boldsymbol{\mu}(\boldsymbol{\phi}) := \sum_{t=0}^{n} \mu_t(\phi_t - \chi_{t+1} + \chi_t) + \sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k(\chi_t) \in (-\infty, \infty], \qquad (3.4.1)$$

where $(\mu_{t-1} - \mu_t)^k(\chi_t)$ denotes the integral of Definition 3.4.1 on the k-th irreducible component of $\mathcal{M}(\mu_{t-1}, \mu_t)$.

Remark 3.4.5. The moderated integral is independent of the choice of the moderator $\boldsymbol{\chi}$. To see this, consider a second moderator $\tilde{\boldsymbol{\chi}}$ for $\boldsymbol{\phi}$; then we have $(\tilde{\chi}_{t+1} - \chi_{t+1}) - (\tilde{\chi}_t - \chi_t) \in L^1(\mu_t)$. We may assume that (3.4.1) is finite for at least one of the moderators. Using Remark 3.4.2 with arbitrary κ_t such that $\mu_{t-1} \otimes \kappa_t \in \mathcal{M}(\mu_{t-1}, \mu_t)$ for $1 \leq t \leq n$, as well as Fubini's theorem for kernels,

$$\begin{aligned}
&\sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k(\chi_t) - (\mu_{t-1} - \mu_t)^k(\tilde{\chi}_t) \\
&\quad = \int \cdots \int \sum_{t=1}^{n} \chi_t(x_{t-1}) - \chi_t(x_t)\, \kappa_n(x_{n-1}, dx_n) \cdots \kappa_1(x_0, dx_1)\mu_0(dx_0) \\
&\qquad - \int \cdots \int \sum_{t=1}^{n} \tilde{\chi}_t(x_{t-1}) - \tilde{\chi}_t(x_t)\, \kappa_n(x_{n-1}, dx_n) \cdots \kappa_1(x_0, dx_1)\mu_0(dx_0) \\
&\quad = \sum_{t=0}^{n} \mu_t((\chi_{t+1} - \tilde{\chi}_{t+1}) - (\chi_t - \tilde{\chi}_t)).
\end{aligned}$$

It now follows that (3.4.1) yields the same value for both moderators.

For later reference, we also record the following property.

Remark 3.4.6. If $\boldsymbol{\chi}$ is a concave moderator, Definition 3.4.4 (ii) implies that

$$\chi_t = \sum_{k\geq 1} \chi_t|_{I_k^t} = \sum_{k\geq 1} \chi_t|_{J_k^t}$$

where (I_k^t, J_k^t) is the k-th irreducible domain of $\mathcal{M}(\mu_{t-1}, \mu_t)$.

Next, we introduce the space of functions which have a finite integral in the moderated sense.

Definition 3.4.7. We denote by $L^c(\boldsymbol{\mu})$ the space of all vectors $\boldsymbol{\phi}$ admitting a concave moderator $\boldsymbol{\chi}$ with $\sum_{t=1}^n \sum_{k\geq 1} (\mu_{t-1} - \mu_t)^k(\chi_t) < \infty$.

It follows that $\boldsymbol{\mu}(\boldsymbol{\phi})$ is finite for $\boldsymbol{\phi} \in L^c(\boldsymbol{\mu})$, and we have $\boldsymbol{\mu}(\phi) = \sum_t \mu_t(\phi_t)$ for $\boldsymbol{\phi} \in \Pi_{t=0}^n L^1(\mu_t)$. The definition is also consistent with the expectation under martingale transports, in the following sense.

Lemma 3.4.8. *Let $\boldsymbol{\phi} \in L^c(\boldsymbol{\mu})$ and let $H = (H_1, \dots, H_n)$ be $\mathbb{F}$-predictable. If*

$$\sum_{t=0}^n \phi_t(X_t) + (H \cdot X)_n$$

is bounded from below on the effective domain $\mathcal{V}$ of $\mathcal{M}(\boldsymbol{\mu})$, then

$$\boldsymbol{\mu}(\boldsymbol{\phi}) = P\left[\sum_{t=0}^n \phi_t(X_t) + (H \cdot X)_n\right], \quad P \in \mathcal{M}(\boldsymbol{\mu}).$$

Proof. Let $P \in \mathcal{M}(\boldsymbol{\mu})$, let $\boldsymbol{\chi}$ be a concave moderator for $\boldsymbol{\phi}$, and assume without loss of generality that 0 is the lower bound. Using Remark 3.4.6, we have that

$\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n$ equals

$$\sum_{t=0}^{n} (\phi_t - \chi_{t+1} + \chi_t)(X_t) + \sum_{t=1}^{n} \sum_{k \geq 1} (\chi_t|_{I_k^t}(X_{t-1}) - \chi_t|_{J_k^t}(X_t)) + (H \cdot X)_n \geq 0.$$

By assumption, the functions $(\phi_t - \chi_{t+1} + \chi_t)(X_t)$ are P-integrable. Therefore, the negative part of the remaining expression must also be P-integrable. Writing $P_t := P \circ (X_0, \ldots, X_t)^{-1}$ and using that $(\chi_t|_{J_k^t})^+$ has linear growth, we see that for any disintegration $P = P_{n-1} \otimes \kappa_n$,

$$\begin{aligned} \int \left[\sum_{t=1}^{n} \sum_{k \geq 1} (\chi_t|_{I_k^t}(X_{t-1}) - \chi_t|_{J_k^t}(X_t)) + (H \cdot X)_n \right] \kappa_n(X_0, \ldots, X_{n-1}, dX_n) \\ = \sum_{t=1}^{n-1} \sum_{k \geq 1} (\chi_t|_{I_k^t}(X_{t-1}) - \chi_t|_{J_k^t}(X_t)) + (H \cdot X)_{n-1} \\ + \sum_{k \geq 1} \int \left[\chi_n|_{I_k^n}(X_{n-1}) - \chi_n|_{J_k^n}(X_n) \right] \kappa_n(X_0, \ldots, X_{n-1}, dX_n). \end{aligned}$$

Iteratively integrating with kernels such that $P_t = P_{t-1} \otimes \kappa_t$ and observing that we can apply Fubini's theorem to $\sum_{t=1}^{n} \sum_{k \geq 1} (\chi_t|_{I_k^t}(X_{t-1}) - \chi_t|_{J_k^t}(X_t)) + (H \cdot X)_n$ as its negative part is P-integrable, we obtain

$$P \left[\sum_{t=1}^{n} \sum_{k \geq 1} (\chi_t|_{I_k^t}(X_{t-1}) - \chi_t|_{J_k^t}(X_t)) + (H \cdot X)_n \right] = \sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k (\chi_t)$$

and the result follows. □

We can now define our dual space. It will be convenient to work with nonnegative reward functions f for the moment—we shall relax this constraint later on; cf. Remark 3.5.3.

Definition 3.4.9. Let $f : \mathbb{R}^{n+1} \to [0, \infty]$. We denote by $\mathcal{D}_{\boldsymbol{\mu}}(f)$ the set of all pairs $(\boldsymbol{\phi}, H)$ where $\boldsymbol{\phi} \in L^c(\boldsymbol{\mu})$ and $H = (H_1, \ldots, H_n)$ is an $\mathbb{F}$-predictable process such that

$$\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n \geq f \quad \text{on} \quad \mathcal{V}.$$

By Lemma 3.4.8, the expectation of the left hand side under any $P \in \mathcal{M}(\boldsymbol{\mu})$ is given by the moderated integral $\boldsymbol{\mu}(\boldsymbol{\phi})$; this will be seen as the dual cost of $(\boldsymbol{\phi}, H)$ when we consider the dual problem $\inf_{(\boldsymbol{\phi},H)\in\mathcal{D}_{\boldsymbol{\mu}}(f)} \boldsymbol{\mu}(\boldsymbol{\phi})$ in Section 3.5 below.

The following closedness property is the key result about the dual space.

Proposition 3.4.10. *Let $f^m : \mathbb{R}^{n+1} \to [0, \infty]$, $m \geq 1$ be a sequence of functions such that*

$$f^m \to f \qquad \textit{pointwise}$$

and let $(\boldsymbol{\phi}^m, H^m) \in \mathcal{D}_{\boldsymbol{\mu}}(f^m)$ be such that $\sup_m \boldsymbol{\mu}(\boldsymbol{\phi}^m) < \infty$. Then there exist $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)$ with

$$\boldsymbol{\mu}(\boldsymbol{\phi}) \leq \liminf_{m\to\infty} \boldsymbol{\mu}(\boldsymbol{\phi}^m).$$

Proof of Proposition 3.4.10

An attempt to prove Proposition 3.4.10 directly along the lines of [16] runs into a technical issue in controlling the concave moderators. Roughly speaking, they do not allow sufficiently many normalizations; this is related to the aforementioned fact that the multistep problem cannot be decomposed into its components. We shall introduce a generalized dual space with families of functions indexed by the components, and prove a "lifted" version of Proposition 3.4.10 in this larger space. Once that is achieved, we can infer the closedness result in the original space as well. (The reader willing to admit Proposition 3.4.10 may skip this subsection without much loss of continuity.)

Definition 3.4.11. Let $\boldsymbol{\phi} = \{\phi_t^k : 0 \le t \le n,\, k \ge 0\}$ be a family of Borel functions, consisting of one function $\phi_t^k : J_k^t \to \bar{\mathbb{R}}$ for each irreducible component (I_k^t, J_k^t) of $\mathcal{M}(\mu_{t-1}, \mu_t)$ as indexed by $k \ge 1$ and $1 \le t \le n$, functions $\phi_t^0 : I_0^t \to \bar{\mathbb{R}}$ for the diagonal components I_0^t indexed by $1 \le t \le n$, and a single function $\phi_0^0 : \mathbb{R} \to \bar{\mathbb{R}}$ for $t = 0$. Similarly, let $\boldsymbol{\chi} = \{\chi_t^k : 1 \le t \le n,\, k \ge 0\}$ be a family of functions, consisting of one concave function $\chi_t^k : J_k^t \to \mathbb{R}$ for each irreducible component (I_k^t, J_k^t) and Borel functions $\chi_t^0 : I_0^t \to \mathbb{R}$ for the diagonal components. We also convene that $\chi_0^0 \equiv 0$ and define the functions[5] $\chi_t := \sum_{k\ge 0} \chi_t^k|_{I_k^t}$ for $t = 1, \dots, n$, as well as $\chi_{n+1} \equiv 0$.

We call $\boldsymbol{\chi}$ a *concave moderator* for $\boldsymbol{\phi}$ if for all $t = 0, \dots, n$ and $k \ge 0$,

$$\phi_t^k + \chi_t^k - \chi_{t+1} \in L^1(\mu_t^k)$$

and the sum $\sum_{k\ge 0} \mu_t^k(\phi_t^k + \chi_t^k - \chi_{t+1})$ converges in $(-\infty, \infty]$, where μ_t^k is the second marginal of the k-th irreducible component in the decomposition of $\mathcal{M}(\mu_{t-1}, \mu_t)$ as in Proposition 3.2.3 and $\mu_0^0 \equiv \mu_0$. The generalized[6] moderated integral is then defined

[5]The restriction to I_k^t is important to avoid "double counting" in the sums. Note that the intervals J may overlap at their endpoints.

[6]This integral is not related to the notion of a generalized martingale.

by

$$\boldsymbol{\mu}(\boldsymbol{\phi}) := \sum_{t=0}^{n} \sum_{k \geq 0} \mu_t^k(\phi_t^k + \chi_t^k - \chi_{t+1}) + \sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k(\chi_t^k).$$

We denote by $L^{c,g}(\boldsymbol{\mu})$ the set of all families $\boldsymbol{\phi}$ which admit a concave moderator $\boldsymbol{\chi}$ such that

$$\sum_{t=0}^{n} \sum_{k \geq 0} |\mu_t^k(\phi_t^k + \chi_t^k - \chi_{t+1})| + \sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k(\chi_t^k) < \infty.$$

For $\boldsymbol{\phi} \in L^{c,g}(\boldsymbol{\mu})$, the value of $\boldsymbol{\mu}(\boldsymbol{\phi})$ is independent of the choice of the moderator $\boldsymbol{\chi}$. This is shown similarly as in Remark 3.4.5. We can now introduce the generalized dual space.

Definition 3.4.12. Let $f : \mathbb{R}^{n+1} \to [0, \infty]$. We denote by $\mathcal{D}_{\boldsymbol{\mu}}^g(f)$ the set of all pairs $(\boldsymbol{\phi}, H)$ where $\boldsymbol{\phi} \in L^{c,g}(\boldsymbol{\mu})$, $H = (H_1, \ldots, H_n)$ is $\mathbb{F}$-predictable, and

$$\sum_{t=0}^{n} \phi_t^{k_t}(x_t) + (H \cdot \boldsymbol{x})_n \geq f(\boldsymbol{x})$$

for all $\boldsymbol{x} = (x_0, \ldots, x_n)$ and $\boldsymbol{k} = (k_0, \ldots, k_n)$ such that $(x_{t-1}, x_t) \in (I_{k_t}^t, J_{k_t}^t)$ for some (irreducible or diagonal) component[7] and $t = 1, \ldots, n$.

We observe that for any $\boldsymbol{x} \in \mathcal{V}$ the corresponding $\boldsymbol{k} = (k_0, \ldots, k_n)$ is uniquely defined, where the index $k_0 \equiv 0$ exists purely for notational convenience.

For later reference, the following lemma elaborates on certain degrees of freedom in choosing elements of $\mathcal{D}_{\boldsymbol{\mu}}^g(f)$.

[7] Given an irreducible component (I, J), the notation $(x, y) \in (I, J)$ means that $x \in I, y \in J$, whereas for a diagonal component (I_0, I_0) it is to be understood as $x = y \in I_0$.

Lemma 3.4.13. *Let $(\boldsymbol{\phi}, H) \in \mathcal{D}^g_{\boldsymbol{\mu}}(f)$ and let $\boldsymbol{\chi}$ be a corresponding concave moderator. Let $1 \le t \le n$, let (I^t_k, J^t_k) be the domain of an irreducible component of $\mathcal{M}(\mu_{t-1}, \mu_t)$ and $c_1, c_2 \in \mathbb{R}$. Introduce new families $(\tilde{\boldsymbol{\phi}}, \tilde{H})$ and $\tilde{\boldsymbol{\chi}}$ by either (i) or (ii):*

(i) Define

$$\tilde{\phi}^k_t(y) = \phi^k_t(y) - (c_1 y - c_2), \qquad \tilde{\chi}^k_t(y) = \chi^k_t(y) + (c_1 y - c_2),$$

$$\tilde{\phi}^{k'}_{t-1}(x) = \phi^{k'}_{t-1}(x) + (c_1 x - c_2)|_{I^t_k}, \qquad \tilde{\chi}^{k'}_{t-1} = \chi^{k'}_{t-1},$$

$$\tilde{\phi}^{k'}_s = \phi^{k'}_s, \qquad \tilde{\chi}^{k'}_s = \chi^{k'}_s \quad \textit{for } s \notin \{t-1, t\},$$

$$\tilde{H}_t = H_t + c_1|_{X^{-1}_{t-1}(I^t_k)}, \qquad \tilde{H}_s = H_s \quad \textit{for } s \neq t$$

where k' runs over all components of the corresponding step in the subscript.

(ii) Define

$$\tilde{\phi}^0_t = \phi^0_t + \chi^0_t - \chi^0_{t+1}|_{I^t_0}, \qquad \tilde{\chi}^0_t = 0, \qquad \textit{and}$$

$$\tilde{\phi}^k_t = \phi^k_t - \chi^0_{t+1}, \qquad \tilde{\chi}^k_t = \chi^k_t \qquad \textit{for } k \ge 1,\ t = 0, \dots, n.$$

Then $(\tilde{\boldsymbol{\phi}}, \tilde{H}) \in \mathcal{D}^g_{\boldsymbol{\mu}}(f)$ and $\tilde{\boldsymbol{\chi}}$ is a corresponding concave moderator. Moreover, we have

$$\sum_{t=0}^n \phi^{k_t}_t(x_t) + (H \cdot \boldsymbol{x})_n = \sum_{t=0}^n \tilde{\phi}^{k_t}_t(x_t) + (\tilde{H} \cdot \boldsymbol{x})_n \quad \textit{and}$$

$$\phi^k_t + \chi^k_t - \chi_{t+1} = \tilde{\phi}^k_t + \tilde{\chi}^k_t - \tilde{\chi}_{t+1} \qquad \textit{for all } k \ge 1,\ t = 0, \dots, n,$$

as well as $\boldsymbol{\mu}(\boldsymbol{\phi}) = \boldsymbol{\mu}(\tilde{\boldsymbol{\phi}})$.

Proof. (i) If $\boldsymbol{x}$ is such that $(x_{t-1}, x_t) \notin I_k^t \times J_k^t$, then $\tilde{\phi}_t^{k_t}(x_t) = \phi_t^{k_t}(x_t)$ for $t = 0, \ldots, n$ and $\tilde{H}(\boldsymbol{x}) = H(\boldsymbol{x})$. Otherwise,

$$\begin{aligned}\tilde{\phi}_t^{k_t}(x_t) + \tilde{\phi}_{t-1}^{k_{t-1}}(x_{t-1}) + \tilde{H}_t(x_t - x_{t-1}) = \phi_t^{k_t}(x_t) + \phi_{t-1}^{k_{t-1}}(x_{t-1})& \\ + H_t(x_t - x_{t-1}),& \\ \tilde{\phi}_t^k + \tilde{\chi}_t^k - \tilde{\chi}_{t+1} = \phi_t^k + \chi_t^k - \chi_{t+1}, \text{ and}& \\ \tilde{\phi}_{t-1}^{k'} + \tilde{\chi}_{t-1}^{k'} - \tilde{\chi}_t = \phi_{t-1}^{k'} + \chi_{t-1}^{k'} - \chi_t.&\end{aligned}$$

Along with the fact that $(\mu_t - \mu_{t-1})^k(\chi_t^k) = (\mu_t - \mu_{t-1})^k(\tilde{\chi}_t^k)$, these identities imply the assertions.

(ii) Similarly as in (i), the terms in question coincide by construction. $\square$

Remark 3.4.14. The modification of Lemma 3.4.13 (i) can be applied simultaneously for infinitely many k's without difficulties. In this case we set

$$\tilde{\phi}_{t-1}^{k'}(x) := \phi_{t-1}^{k'}(x) + \sum_{k \geq 1} (c_1^k x - c_2^k)|_{I_k^t},$$

as well as $\tilde{\phi}_t^k(y) = \phi_t^k(y) - (c_1^k y - c_2^k)$ and $\tilde{\chi}_t^k(y) = \chi_t^k(y) + (c_1^k y - c_2^k)$ for the components $k \geq 1$ in step t. The pointwise equalities still hold as above and in particular, the moderated integral does not change.

Remark 3.4.15. Any $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)$ induces an element $(\boldsymbol{\phi}^g, H) \in \mathcal{D}_{\boldsymbol{\mu}}^g(f)$ with $\boldsymbol{\mu}(\boldsymbol{\phi}^g) = \boldsymbol{\mu}(\boldsymbol{\phi})$ by choosing some concave moderator $\boldsymbol{\chi}$ for $\boldsymbol{\phi}$ and setting

$$\phi_t^k := \phi_t|_{J_k^t}, \qquad \chi_t^k := \chi_t|_{J_k^t}.$$

We now show the analogue to Lemma 3.4.8 for the generalized dual space.

Lemma 3.4.16. *Let $\boldsymbol{\phi} \in L^{c,g}(\boldsymbol{\mu})$ and let $H = (H_1, \ldots, H_n)$ be $\mathbb{F}$-predictable. If*

$$\sum_{t=0}^{n} \phi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t) + (H \cdot \boldsymbol{x})_n$$

is bounded from below on the effective domain $\mathcal{V}$ of $\mathcal{M}(\boldsymbol{\mu})$, then

$$\boldsymbol{\mu}(\phi) = P\left[\sum_{t=0}^{n} \phi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t) + (H \cdot \boldsymbol{x})_n\right], \quad P \in \mathcal{M}(\boldsymbol{\mu}).$$

Proof. Let $P \in \mathcal{M}(\boldsymbol{\mu})$, let $\boldsymbol{\chi}$ be a concave moderator for $\boldsymbol{\phi}$ such that $\chi_t^0 \equiv 0$ and assume that 0 is the lower bound. It is easy to see that $\sum_{t=0}^{n} \phi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t) + (H \cdot \boldsymbol{x})_n$ equals

$$\sum_{t=0}^{n}(\phi_t^{\boldsymbol{k}_t(\boldsymbol{x})} - \chi_{t+1} + \chi_t^{\boldsymbol{k}_t(\boldsymbol{x})})(x_t) + \sum_{t=1}^{n}(\chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_{t-1}) - \chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t)) + (H \cdot \boldsymbol{x})_n \geq 0.$$

By assumption $\sum_{t=0}^{n}(\phi_t^{\boldsymbol{k}_t(\boldsymbol{x})} - \chi_{t+1} + \chi_t^{\boldsymbol{k}_t(\boldsymbol{x})})(x_t)$ is P-integrable. Therefore, the negative part of the remaining expression must also be P-integrable. Writing $P_t := P \circ (X_0, \ldots, X_t)^{-1}$ and using that $(\chi_t^k)^+$ has linear growth, we see that for any disintegration $P = P_{n-1} \otimes \kappa_n$,

$$\int \left[\sum_{t=1}^{n} (\chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_{t-1}) - \chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t)) + (H \cdot \boldsymbol{x})_n \right] \kappa_n(x_0, \ldots, x_{n-1}, dx_n)$$
$$= \sum_{t=1}^{n-1} (\chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_{t-1}) - \chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t)) + (H \cdot \boldsymbol{x})_{n-1}$$
$$+ \int \left[(\chi_n^{\boldsymbol{k}_n(\boldsymbol{x})}(x_{n-1}) - \chi_n^{\boldsymbol{k}_n(\boldsymbol{x})}(x_n)) \right] \kappa_n(x_0, \ldots, x_{n-1}, dx_n).$$

Iteratively integrating with kernels such that $P_t = P_{t-1} \otimes \kappa_t$ and observing that we can apply Fubini's theorem to $\sum_{t=1}^{n} (\chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_{t-1}) - \chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t)) + (H \cdot \boldsymbol{x})_n$ as its negative part is P-integrable, we obtain

$$P \left[\sum_{t=1}^{n} (\chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_{t-1}) - \chi_t^{\boldsymbol{k}_t(\boldsymbol{x})}(x_t)) + (H \cdot \boldsymbol{x})_n \right] = \sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k (\chi_t^k)$$

and the result follows. □

Next, we establish that lifting from $\mathcal{D}_{\boldsymbol{\mu}}(f)$ to $\mathcal{D}_{\boldsymbol{\mu}}^g(f)$ does not change the range of dual costs.

Proposition 3.4.17. *Let $f : \mathbb{R}^{n+1} \to [0, \infty]$. We have*

$$\{\boldsymbol{\mu}(\boldsymbol{\phi}^g) : (\boldsymbol{\phi}^g, H) \in \mathcal{D}_{\boldsymbol{\mu}}^g(f)\} = \{\boldsymbol{\mu}(\boldsymbol{\phi}) : (\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)\}.$$

Proof. Remark 3.4.15 shows the inclusion "$\supseteq$." To show the reverse, we may apply Lemma 3.4.13 (i) together with Remark 3.4.14 to modify a given pair $(\boldsymbol{\phi}^g, H) \in \mathcal{D}_{\boldsymbol{\mu}}^g(f)$ such that $\phi_t^k(x) = 0$ for $x \in J_k^t \backslash I_k^t$, for all irreducible domains (I_k^t, J_k^t) of

$\mathcal{M}(\mu_{t-1},\mu_t)$ and $1 \le t \le n$. Here we have used that $x \in J_k^t \backslash I_k^t$ implies $\mu_k^t(\{x\}) > 0$, cf. Definition 3.2.2, and therefore $\phi^g \in L^{c,g}(\boldsymbol{\mu})$ implies $\phi_t^k(x) \in \mathbb{R}$; that is, such endpoints can indeed be shifted to 0 by adding affine functions to ϕ_t^k.

Let $\boldsymbol{\chi}^g$ be a concave moderator for $\boldsymbol{\phi}^g$. Using Lemma 3.4.3 (ii) and again Lemma 3.4.13 as above, we can modify χ_t^k to satisfy $\chi_t^k(x) = 0$ for $x \in J_k^t \backslash I_k^t$, for all irreducible domains (I_k^t, J_k^t) of $\mathcal{M}(\mu_{t-1},\mu_t)$ and $1 \le t \le n$. Here, the finiteness of χ_t^k at the endpoints follows from Lemma 3.4.3 (i) and $(\mu_{t-1}-\mu_t)^k(\chi_t^k) < \infty$.

Still denoting the modified dual element by $(\boldsymbol{\phi}^g, H)$, we define $\boldsymbol{\phi} \in L^c(\boldsymbol{\mu})$ and a corresponding concave moderator $\boldsymbol{\chi}$ by

$$\phi_t(x) := \phi_t^k(x), \quad \chi_t(x) := \chi_t^k(x), \qquad \text{for } x \in J_k^t;$$

they are well-defined since ϕ_t^k and χ_t^k vanish at points that belong to more than one set J_k^t. We have $\boldsymbol{\mu}(\boldsymbol{\phi}) = \boldsymbol{\mu}(\boldsymbol{\phi}^g)$ by construction and the result follows. □

Definition 3.4.18. Let $1 \le t \le n$ and $x_t \in \mathbb{R}$. A sequence $\boldsymbol{x} = (x_0, \dots, x_t)$ is a *predecessor path* of x_t if there are indices $(k_0, \dots, k_t)$ such that $(x_{s-1}, x_s) \in (I_{k_s}^s, J_{k_s}^s)$ for some component (irreducible or diagonal) of $\mathcal{M}(\mu_{s-1},\mu_s)$, for all $1 \le s \le t$. We write $\mathbb{k}(\boldsymbol{x})$ for the (unique) associated sequence $(k_0, \dots, k_t)$ followed by the path $\boldsymbol{x}$ in the above sense, and $\Psi_t^k(x_t)$ for the set of all predecessor paths with $k_t = k$.

These notions will be useful in the next step towards the closedness result, which is to "regularize" the concave moderators. For concreteness in some of the expressions below, we convene that $\infty - \infty := \infty$.

Lemma 3.4.19. *Let $(\boldsymbol{\phi}, H) \in \mathcal{D}^g_{\boldsymbol{\mu}}(0)$. There is a concave moderator $\boldsymbol{\chi}$ of $\boldsymbol{\phi}$ such that*

$$\phi^k_t + \chi^k_t - \chi_{t+1} \geq 0 \quad \text{on } J^t_k \quad \text{for all} \quad t = 0, \dots, n, \quad k \geq 1, \text{ and} \tag{3.4.2}$$

$$\phi^0_t + \chi^0_t - \chi_{t+1} \geq 0 \quad \mu_t\text{-a.s. on } I^t_0 \quad \text{for all} \quad t = 1, \dots, n. \tag{3.4.3}$$

As a consequence,

$$\sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k (\chi^k_t) \leq \boldsymbol{\mu}(\boldsymbol{\phi}).$$

Proof. Fix $1 \leq t \leq n$ and let (I^t_k, J^t_k) be the domain of some component of $\mathcal{M}(\mu_{t-1}, \mu_t)$. We define $\boldsymbol{\chi} = (\chi^k_t)$ by $\chi^0_0 = 0$ and

$$\chi^k_t(x_t) = \inf_{\boldsymbol{x} \in \Psi^k_t(x_t)} \left\{ \sum_{s=0}^{t-1} \phi^{\mathbb{k}_s(\boldsymbol{x})}_s(x_s) + (H \cdot \boldsymbol{x})_t \right\};$$

then χ^k_t is concave on J^t_k for $k \geq 1$ as an infimum of affine functions.

We first show that

$$\{\chi^k_t = +\infty\} \subseteq \left\{ \phi^{k'}_{t-1} = +\infty \right\} \cup \{\chi^{k'}_{t-1} = +\infty\}.$$

In particular, such points only exist after a chain of diagonal components from a point where $\phi^k_t(x_t) = \infty$. Suppose $\chi^k_t(x_t) = +\infty$ and $k \geq 1$, then the predecessor paths of x_t agree with the predecessor paths of all of J^k_t up to $t-1$, but $\{\sum_{s=0}^{t-1} \phi^{\mathbb{k}_s(\boldsymbol{x})}_s(x_s) < \infty\}$ must hold $\mathcal{M}(\boldsymbol{\mu})$-q.s. as $\boldsymbol{\phi} \in L^{c,g}(\boldsymbol{\mu})$. We must therefore have $x_t \in I^0_t$. Then, by definition, $\chi^0_t(x_t) = \chi^k_{t-1}(x_t) + \phi^k_{t-1}(x_t)$ and the claim follows.

Next, we verify that $\boldsymbol{\chi}$ satisfies (3.4.2) and (3.4.3). For notational convenience we for now set $\chi_{n+1} \equiv \inf_{\boldsymbol{x}\in\mathcal{V}} \left\{\sum_{s=0}^{n} \phi_s^{\mathbb{k}_s(\boldsymbol{x})}(x_s) + (H\cdot\boldsymbol{x})_n\right\} \geq 0$. Restricting the infimum in the definition of $\boldsymbol{\chi}$ to the set of paths $\boldsymbol{x}$ with $x_{t+1} = x_t \in I_{k'}^{t+1} \cap J_k^t$ yields

$$
\begin{aligned}
\chi_{t+1}(x_t) = \chi_{t+1}^{k'}(x_t) &= \inf_{\boldsymbol{x}\in\Psi_{t+1}^{k'}(x_t)} \left\{\sum_{s=0}^{t} \phi_s^{\mathbb{k}_s(\boldsymbol{x})}(x_s) + (H\cdot\boldsymbol{x})_{t+1}\right\} \\
&\leq \inf_{\boldsymbol{x}\in\Psi_t^{k}(x_t)} \left\{\sum_{s=0}^{t-1} \phi_s^{\mathbb{k}_s(\boldsymbol{x})}(x_s) + (H\cdot\boldsymbol{x})_{t}\right\} + \phi_t^k(x_t) \\
&= \chi_t^k(x_t) + \phi_t^k(x_t).
\end{aligned}
$$

Since $\cup_{k'\geq 0} I_{k'}^{t+1} = \mathbb{R}$, this will imply (3.4.2) after we check that $\chi_t^k > -\infty$ for $k \geq 1$ and $\chi_t^0 > -\infty$ holds μ_t^0-a.s., which also implies that $\chi_t > -\infty$ holds μ_{t-1}-almost surely. We show this inductively for $t \geq 1$.

Clearly $\chi_{n+1} \geq 0 > -\infty$. Now, for $t \leq n$ the induction hypothesis is that $\chi_{t+1} > -\infty$ holds almost surely μ_t.

From $\boldsymbol{\phi} \in L^{c,g}$ and $\chi_{t+1} > -\infty$ μ_t-a.s. we have that

$$
\phi_t^k < \infty, \quad \chi_{t+1} > -\infty \quad \text{hold } \mu_t^k\text{-a.s.}
$$

As χ_t^k is concave and J_t^k is the convex hull of the topological support of μ_t^k we then get $\chi_t^k > -\infty$ on all of J_t^k from the previous inequality.

For $k = 0$, the inequality yields $\{\chi_t^0 = -\infty\} \subseteq \{\chi_{t+1} = -\infty\} \cup \{\phi_t^0(x_t) = \infty\}$ and both of these sets are μ_t nullsets. Finally $\mu_{t-1}(\{\chi_t = -\infty\}) = 0$ as this is a subset of the diagonal component where μ_{t-1} is dominated by μ_t.

Set $\bar\phi_t^k := \phi_t^k + \chi_t^k - \chi_{t+1}|_{J_k^t}$ for $0 \leq t \leq n$; then $\bar\phi_t^k \geq 0$. Moreover, choose an arbitrary $P \in \mathcal{M}(\boldsymbol{\mu})$ with disintegration $P = \mu_0 \otimes \kappa_1 \otimes \cdots \otimes \kappa_n$ for some stochastic kernels $\kappa_t(x_0, \ldots, x_{t-1}, dx_t)$. From Lemma 3.4.16 we know that

$$\boldsymbol{\mu}(\boldsymbol{\phi}) = P\left[\sum_{t=0}^{n} \phi_t^{\Bbbk_t(X)}(X_t) + (H \cdot X)_n\right] < \infty.$$

We can therefore apply Fubini's theorem for kernels as in the proof of Lemma 3.4.16 to the expression

$$\begin{aligned} 0 &\leq \sum_{t=0}^{n} \phi_t^{\Bbbk_t(\boldsymbol{x})}(x_t) + (H \cdot \boldsymbol{x})_n \\ &= \sum_{t=0}^{n} \bar\phi_t^{\Bbbk_t(\boldsymbol{x})}(x_t) + \sum_{t=1}^{n} \Big(\chi_t(x_{t-1}) - \chi_t^{\Bbbk_t(\boldsymbol{x})}(x_t)\Big) + (H \cdot \boldsymbol{x})_n \end{aligned}$$

and obtain

$$P\left[\sum_{t=0}^{n} \phi_t^{\Bbbk_t(X)}(X_t) + (H \cdot X)_n\right] = \sum_{t=0}^{n} \sum_{k \geq 0} \mu_t^k(\bar\phi_t^k) + \sum_{t=1}^{n} \sum_{k \geq 1} (\mu_{t-1} - \mu_t)^k(\chi_t^k)$$

which shows that the right hand side is finite, and therefore $\boldsymbol{\chi}$ is a concave moderator for $\boldsymbol{\phi}$. Finally, the second claim follows from $\mu_t^k(\bar\phi_t^k) \geq 0$. $\square$

The last tool for our closedness result is a compactness property for concave functions in the one-step case; cf. [16, Proposition 5.5].

Proposition 3.4.20. *Let $\mu \leq_c \nu$ be irreducible with domain (I, J) and let $a \in I$ be* *the common barycenter of μ and ν. Let $\chi_m : J \to \mathbb{R}$ be concave functions such that*[8]

$$\chi_m(a) = \chi_m'(a) = 0 \qquad \textit{and} \qquad \sup_{m\geq 1}(\mu-\nu)(\chi_m) < \infty.$$

There exists a subsequence χ_{m_k} which converges pointwise on J to a concave function $\chi : J \to \mathbb{R}$, and $(\mu-\nu)(\chi) \leq \liminf_k (\mu-\nu)(\chi_{m_k})$.

We are now ready to state and prove the analogue of Proposition 3.4.10 in the generalized dual.

Proposition 3.4.21. *Let $f^m : \mathbb{R}^{n+1} \to [0,\infty]$, $m \geq 1$ be a sequence of functions such that*

$$f^m \to f \qquad \textit{pointwise}$$

and let $(\boldsymbol{\phi}^m, H^m) \in \mathcal{D}^g_{\boldsymbol{\mu}}(f^m)$ be such that $\sup_m \boldsymbol{\mu}(\boldsymbol{\phi}^m) < \infty$. Then there exist $(\boldsymbol{\phi}, H) \in \mathcal{D}^g_{\boldsymbol{\mu}}(f)$ with

$$\boldsymbol{\mu}(\boldsymbol{\phi}) \leq \liminf_{m\to\infty} \boldsymbol{\mu}(\boldsymbol{\phi}^m).$$

Proof. Since $(\boldsymbol{\phi}^m, H^m) \in \mathcal{D}^g_{\boldsymbol{\mu}}(f^m)$ and $f^m \geq 0$, we can introduce a sequence of concave moderators $\boldsymbol{\chi}_m$ as in Lemma 3.4.19. A normalization of $(\boldsymbol{\phi}^m, H^m)$ as in Lemma 3.4.13 (i) and (ii), in the general form of Remark 3.4.14, allows us to assume without loss of generality that $\chi^0_{t,m} \equiv 0$ and $\chi^k_{t,m}(a^k_t) = (\chi^k_{t,m})'(a^k_t) = 0$, where a^k_t is the barycenter of μ^k_t—this modification is the main merit of lifting to the generalized dual space. While the generalized dual gives enough degrees of freedom to choose this normalization, the dual without the generalization does not. This is related to

[8]To be specific, let us convene that χ_m' is the left derivative—this is not important here.

the possible overlap of the intervals I, J at the different times t; see also Figure 3.2 and the paragraph preceding Example 3.3.2.

By passing to a subsequence as in Proposition 3.4.20 for each component and using a diagonal argument, we obtain pointwise limits $\chi_t^k : J_k^t \to \mathbb{R}$ for $\chi_{t,m}^k$ after passing to another subsequence.

Since $\phi_{t,m}^k + \chi_{t,m}^k - \chi_{t+1,m} \geq 0$ on J_k^t[9] and $\chi_{t,m}^k \to \chi_t^k$ as well as $\chi_{t+1,m} \to \chi_{t+1}$, we can apply Komlos' lemma (in the form of [33, Lemma A1.1] and its remark) to find convex combinations $\tilde{\phi}_{t,m}^k \in \operatorname{conv}\{\phi_{t,m}^k, \phi_{t,m+1}^k, \dots\}$ which converge μ_t^k-a.s. for $0 \leq t \leq n$. We may assume without loss of generality that $\tilde{\phi}_{t,m}^k = \phi_{t,m}^k$. Thus, we can set

$$\begin{aligned} &\phi_t^k := \limsup \phi_{t,m}^k \quad \text{ on } J_k^t \quad \text{for } t = 1, \dots, n, \\ &\phi_0 := \liminf \phi_{0,m} \end{aligned}$$

to obtain

$$\phi_{t,m}^k \to \phi_t^k \quad \mu_t^k\text{-a.s.} \quad \text{ and } \quad \phi_t^k + \chi_t^k - \chi_{t+1} \geq 0 \text{ on } J_k^t.$$

[9] Observe that this inequality will still hold after modifying $\boldsymbol{\phi}$ and $\boldsymbol{\chi}$ as in Lemma 3.4.13.

We can now apply Fatou's lemma and Proposition 3.4.20 to deduce that

$$\begin{aligned}
\boldsymbol{\mu}(\boldsymbol{\phi}) &= \sum_{t=0}^{n}\sum_{k\geq 0}\mu_t^k(\phi_t^k + \chi_t^k - \chi_{t+1}) + \sum_{t=1}^{n}\sum_{k\geq 1}(\mu_{t-1}-\mu_t)^k(\chi_t^k) \\
&\leq \sum_{t=0}^{n}\sum_{k\geq 0}\liminf \mu_t^k(\phi_{t,m}^k + \chi_{t,m}^k - \chi_{t+1,m}) \\
&\quad + \sum_{t=1}^{n}\sum_{k\geq 1}\liminf(\mu_{t-1}-\mu_t)^k(\chi_{t,m}^k) \\
&\leq \liminf \left[\sum_{t=0}^{n}\sum_{k\geq 0}\mu_t^k(\phi_{t,m}^k + \chi_{t,m}^k - \chi_{t+1,m}) + \sum_{t=1}^{n}\sum_{k\geq 1}(\mu_{t-1}-\mu_t)^k(\chi_{t,m}^k)\right] \\
&= \liminf \boldsymbol{\mu}(\boldsymbol{\phi}^m) < \infty.
\end{aligned}$$

In particular, we see that $\boldsymbol{\phi} \in L^{c,g}(\boldsymbol{\mu})$ with concave moderator $\boldsymbol{\chi}$.

It remains to construct the predictable process $H = (H_1, \dots, H_n)$. With a mild abuse of notation, we shall identify $H_t(x_0, \dots, x_n)$ with the corresponding function of $(x_0, \dots, x_{t-1})$ in this proof.

We first define for each $\boldsymbol{k} = (k_0, \dots, k_t)$ and $\boldsymbol{x} = (x_0, \dots, x_t)$ such that $\boldsymbol{k} = \Bbbk(\boldsymbol{x})$, the functions $G_{t,m}^{\boldsymbol{k}}$ and $G_t^{\boldsymbol{k}}$ by

$$\begin{aligned}
G_{t,m}^{\boldsymbol{k}}(\boldsymbol{x}) &:= \sum_{s=0}^{t}\phi_{s,m}^{k_s}(x_s) + \sum_{s=1}^{t}H_{s,m}(x_0, \dots, x_{s-1})\cdot(x_s - x_{s-1}), \\
G_t^{\boldsymbol{k}}(\boldsymbol{x}) &:= \liminf G_{t,m}^{\boldsymbol{k}}(\boldsymbol{x}).
\end{aligned}$$

Given $\boldsymbol{k} = (k_0, \dots, k_t)$, we write $\boldsymbol{k'} = (k_0, \dots, k_{t-1})$. We claim that there exists an

$\mathbb{F}$-predictable process H such that for all $1 \leq t \leq n$,

$$G^{\boldsymbol{k}'}_{t-1}(x_0,\dots,x_{t-1}) + \phi^{k_t}_t(x_t) + H_t(x_0,\dots,x_{t-1}) \cdot (x_t - x_{t-1}) \geq G^{\boldsymbol{k}}_t(x_0,\dots,x_t). \quad (3.4.4)$$

Once this is established, the proposition follows by induction since $G^{(0)}_0(x_0) = \phi_0(x_0)$ and $G^{\boldsymbol{k}}_n(x_0,\dots,x_n) \geq f(x_0,\dots,x_n)$.

To prove the claim, write g^{conc} for the concave hull of a function g and observe that

$$\begin{aligned}
\liminf[G^{\boldsymbol{k}'}_{t-1,m}&(x_0,\dots,x_{t-1}) + H_{t,m}(x_0,\dots,x_{t-1}) \cdot (x_t - x_{t-1})] \\
&\geq \liminf[(G^{\boldsymbol{k}}_{t,m}(x_0,\dots,x_{t-1},\cdot) - \phi^{k_t}_{t,m}(\cdot))^{\mathrm{conc}}(x_t)] \\
&\geq [\liminf(G^{\boldsymbol{k}}_{t,m}(x_0,\dots,x_{t-1},\cdot) - \phi^{k_t}_{t,m}(\cdot)]^{\mathrm{conc}}(x_t) \\
&\geq [G^{\boldsymbol{k}}_t(x_0,\dots,x_{t-1},\cdot) - \phi^{k_t}_t(\cdot)]^{\mathrm{conc}}(x_t) \\
&=: \hat{\phi}^{\boldsymbol{k}}_t(x_0,\dots,x_{t-1},x_t).
\end{aligned}$$

By construction, $\hat{\phi}^{\boldsymbol{k}}_t$ is concave in the last variable and satisfies

$$G^{\boldsymbol{k}'}_{t-1}(x_0,\dots,x_{t-1}) \geq \hat{\phi}^{\boldsymbol{k}}_t(x_0,\dots,x_{t-1},x_{t-1}).$$

Let $\partial_t\hat{\phi}^{\boldsymbol{k}}_t$ denote the left partial derivative in the last variable and set

$$H^{\boldsymbol{k}}_t(x_0,\dots,x_{t-1}) := \partial_t\hat{\phi}^{\boldsymbol{k}}_t(x_0,\dots,x_{t-1},x_{t-1})$$

for $\boldsymbol{k}_t \geq 1$ and $H_t^{\boldsymbol{k}}(x_0, \ldots, x_{t-1}) = 0$ for $\boldsymbol{k}_t = 0$; then we have

$$
\begin{aligned}
&G_{t-1}^{\boldsymbol{k}'}(x_0, \ldots, x_{t-1}) + H_t^{\boldsymbol{k}}(x_0, \ldots, x_{t-1}) \cdot (x_t - x_{t-1}) \\
&\qquad \geq \hat{\phi}_t^{\boldsymbol{k}}(x_0, \ldots, x_{t-1}, x_{t-1}) + H_t^{\boldsymbol{k}}(x_0, \ldots, x_{t-1}) \cdot (x_t - x_{t-1}) \\
&\qquad \geq \hat{\phi}_t^{\boldsymbol{k}}(x_0, \ldots, x_{t-1}, x_t) \\
&\qquad \geq G_t^{\boldsymbol{k}}(x_0, \ldots, x_t) - \phi_t^{k_t}(x_t).
\end{aligned}
$$

Finally, for any $(x_0, \ldots, x_{t-1}) \in \mathbb{R}^t$, we define $H_t(x_0, \ldots, x_{t-1})$ as

$$
\begin{cases}
H_t^{\boldsymbol{k}}(x_0, \ldots, x_{t-1}), & \text{if } \boldsymbol{k} = \Bbbk(x_0, \ldots, x_{t-1}, x_t) \text{ for some } x_t \in \mathbb{R} \\
0, & \text{otherwise;}
\end{cases}
$$

this is well-defined since $\Bbbk(x_0, \ldots, x_t)$ depends only on $(x_0, \ldots, x_{t-1})$. The predictable process H satisfies (3.4.4) and thus the proof is complete. □

Proof of Proposition 3.4.10. In view of Remark 3.4.15 and Proposition 3.4.17, the result follows from Proposition 3.4.21. □

3.5 Duality Theorem and Monotonicity Principle

The first goal of this section is a duality result for the multistep martingale transport problem; it establishes the absence of a duality gap and the existence of optimizers in the dual problem. (As is well known, an optimizer for the primal problem only exists under additional conditions, such as continuity of f.) The second goal is a monotonicity principle describing the geometry of optimal transports; it will be a consequence of the duality result.

As above, we consider a fixed vector $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ of marginals in convex order. The primal and dual problems as defined follows.

Definition 3.5.1. Let $f : \mathbb{R}^{n+1} \to [0, \infty]$. The *primal problem* is

$$\mathbf{S}_{\boldsymbol{\mu}}(f) := \sup_{P \in \mathcal{M}(\boldsymbol{\mu})} P(f) \in [0, \infty],$$

where $P(f)$ refers to the outer integral if f is not measurable. The *dual problem* is

$$\mathbf{I}_{\boldsymbol{\mu}}(f) := \inf_{(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)} \boldsymbol{\mu}(\boldsymbol{\phi}) \in [0, \infty].$$

We recall that a function $f : \mathbb{R}^{n+1} \to [0, \infty]$ is called *upper semianalytic* if the sets $\{f \geq c\}$ are analytic for all $c \in \mathbb{R}$, where a subset of $\mathbb{R}^{n+1}$ is called analytic if it is the image of a Borel subset of a Polish space under a Borel mapping. Any Borel function is upper semianalytic and any upper semianalytic function is universally measurable; we refer to [20, Chapter 7] for background. The following is the announced duality result.

Theorem 3.5.2 (Duality)**.** *Let* $f : \mathbb{R}^{n+1} \to [0, \infty]$.

(i) If f *is upper semianalytic, then* $\mathbf{S}_{\boldsymbol{\mu}}(f) = \mathbf{I}_{\boldsymbol{\mu}}(f) \in [0, \infty]$.

(ii) If $\mathbf{I}_{\boldsymbol{\mu}}(f) < \infty$*, there exists a dual optimizer* $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)$.

Proof. Given our preceding results, much of the proof follows the lines of the corresponding result for the one-step case in [16, Theorem 6.2]; therefore, we shall be brief. We mention that the present theorem is slightly more general than the cited one in terms of the measurability condition (f is upper semianalytic instead of Borel); this is due to the global proof given here.

Step 1. Using Lemma 3.4.8 we see that $\mathbf{S}_{\boldsymbol{\mu}}(f) \leq \mathbf{I}_{\boldsymbol{\mu}}(f)$ holds for all upper semicontinuous $f : \mathbb{R}^{n+1} \to [0, \infty]$.

Step 2. Using the de la Vallée–Poussin theorem and our assumption that the marginals have a finite first moment, there exist increasing, superlinearly growing functions $\zeta_{\mu_t} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x \mapsto \zeta_{\mu_t}(|x|)$ is μ_t-integrable for all $0 \leq t \leq n$. Define

$$\zeta(x_0, \dots, x_n) := 1 + \sum_{t=0}^{n} \zeta_{\mu_t}(|x_t|)$$

and let C_ζ be the vector space of all continuous functions f such that f/ζ vanishes at infinity. Then, a Hahn–Banach separation argument can be used to show that $\mathbf{S}_{\boldsymbol{\mu}}(f) \geq \mathbf{I}_{\boldsymbol{\mu}}(f)$ holds for all $f \in C_\zeta$; the details of the argument are the same as in the proof of [16, Lemma 6.4].

Step 3. Let f be bounded and upper semicontinuous; then there exists a sequence of bounded continuous functions $f^m \in C_b(\mathbb{R}^{n+1})$ which decrease to f pointwise. As $C_b(\mathbb{R}^{n+1}) \subseteq C_\zeta$, we have $\mathbf{S}_{\boldsymbol{\mu}}(f^m) = \mathbf{I}_{\boldsymbol{\mu}}(f^m)$ for all m by the first two steps.

Let $\mathcal{U}$ be the set of all bounded, nonnegative, upper semicontinuous functions on $\mathbb{R}^{n+1}$. We recall that a map $\mathbf{C} : [0, \infty]^{\mathbb{R}^{n+1}} \to [0, \infty]$ is called a $\mathcal{U}$-capacity if it is monotone, sequentially continuous upwards on $[0, \infty]^{\mathbb{R}^{n+1}}$ and sequentially continuous downwards on $\mathcal{U}$. The functional $f \mapsto \mathbf{S}_{\boldsymbol{\mu}}(f)$ is a $\mathcal{U}$-capacity; this follows from the weak compactness of $\mathcal{M}(\boldsymbol{\mu})$ and the arguments in [66, Propositions 1.21, 1.26].

It follows that $\mathbf{S}_{\boldsymbol{\mu}}(f^m) \to \mathbf{S}_{\boldsymbol{\mu}}(f)$. By the monotonicity of $f \mapsto \mathbf{I}_{\boldsymbol{\mu}}(f)$ and Step 1 we obtain

$$\mathbf{I}_{\boldsymbol{\mu}}(f) \leq \lim \mathbf{I}_{\boldsymbol{\mu}}(f^m) = \lim \mathbf{S}_{\boldsymbol{\mu}}(f^m) = \mathbf{S}_{\boldsymbol{\mu}}(f) \leq \mathbf{I}_{\boldsymbol{\mu}}(f).$$

Step 4. Since $\mathbf{S}_{\boldsymbol{\mu}} = \mathbf{I}_{\boldsymbol{\mu}}$ on $\mathcal{U}$ by Step 3, $\mathbf{I}_{\boldsymbol{\mu}}$ is sequentially downward continuous on $\mathcal{U}$ like $\mathbf{S}_{\boldsymbol{\mu}}$. On the other hand, Proposition 3.4.10 implies that it is sequentially upwards continuous on $[0,\infty]^{\mathbb{R}^{n+1}}$. As a result, $\mathbf{I}_{\boldsymbol{\mu}}$ is a $\mathcal{U}$-capacity.

Step 5. Let $f : \mathbb{R}^{n+1} \to [0,\infty]$ be upper semianalytic. For any $\mathcal{U}$-capacity $\mathbf{C}$, Choquet's capacitability theorem shows that

$$\mathbf{C}(f) = \sup\{\mathbf{C}(g) : g \in \mathcal{U}, g \leq f\}.$$

As $\mathbf{S}_{\boldsymbol{\mu}}$ and $\mathbf{I}_{\boldsymbol{\mu}}$ are $\mathcal{U}$-capacities that coincide on $\mathcal{U}$, it follows that $\mathbf{S}_{\boldsymbol{\mu}}(f) = \mathbf{I}_{\boldsymbol{\mu}}(f)$. This completes the proof of (i).

Step 6. To see that the infimum $\mathbf{I}_{\boldsymbol{\mu}}(f)$ is attained if it is finite, we merely need to apply Proposition 3.4.10 with the constant sequence $f^m = f$. □

We can easily relax the lower bound on f.

Remark 3.5.3. Let $f : \mathbb{R}^{n+1} \to (-\infty,\infty]$ and suppose there exist $\boldsymbol{\phi} \in \prod_{t=0}^n L^1(\mu_t)$ and a predictable process H such that

$$f \geq \sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n \quad \text{on} \quad \mathcal{V}.$$

Then we can apply Theorem 3.5.2 to $[f - \sum_{t=0}^{n} \phi_t(X_t) - (H \cdot X)_n]^+$ and obtain the analogue of its assertion for f.

The duality result gives rise to a monotonicity principle describing the support of optimal martingale transports, in the spirit of the cyclical monotonicity condition from classical transport theory. The following generalizes the results of [13, Lemma 1.11] and [16, Corollary 7.8] for the one-step martingale transport problem.

Theorem 3.5.4 (Monotonicity Principle)**.** *Let $f : \mathbb{R}^{n+1} \to [0, \infty]$ be Borel and suppose that $\mathbf{S}_{\boldsymbol{\mu}}(f) < \infty$. There exists a Borel set $\Gamma \subseteq \mathbb{R}^{n+1}$ with the following properties.*

(i) A measure $P \in \mathcal{M}(\boldsymbol{\mu})$ is concentrated on Γ if and only if it is optimal for $\mathbf{S}_{\boldsymbol{\mu}}(f)$.

(ii) Let $\bar{\boldsymbol{\mu}} = (\bar{\mu}_0, \dots, \bar{\mu}_n)$ be another vector of marginals in convex order. If $\bar{P} \in \mathcal{M}(\bar{\boldsymbol{\mu}})$ is concentrated on Γ, then $\bar{P}$ is optimal for $\mathbf{S}_{\bar{\boldsymbol{\mu}}}(f)$.

Indeed, if $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)$ is an optimizer for $\mathbf{I}_{\boldsymbol{\mu}}(f)$, then we can take

$$\Gamma := \left\{ \sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n = f \right\} \cap \mathcal{V}.$$

Proof. As $\mathbf{S}_{\boldsymbol{\mu}}(f) < \infty$, Theorem 3.5.2 shows that $\mathbf{I}_{\boldsymbol{\mu}}(f) = \mathbf{S}_{\boldsymbol{\mu}}(f) < \infty$ and that there exists a dual optimizer $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)$. In particular, we can define Γ as above.

(i) As $0 \leq f$ and $P(f) \leq \mathbf{S}_{\boldsymbol{\mu}}(f) < \infty$ for all $P \in \mathcal{M}(\boldsymbol{\mu})$, we see that f is P-integrable for all $P \in \mathcal{M}(\boldsymbol{\mu})$. Since $\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n \geq 0$ on the effective domain $\mathcal{V}$, and $P\left[\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n\right] = \boldsymbol{\mu}(\boldsymbol{\phi}) = \mathbf{I}_{\boldsymbol{\mu}}(f) < \infty$ by Lemma 3.4.8, we also obtain the P-integrability of $\sum_{t=0}^n \phi_t(X_t) + (H \cdot X)_n$. In particular,

$$0 \le P\left[\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n - f\right] = \boldsymbol{\mu}(\boldsymbol{\phi}) - P(f) = \mathbf{S}_{\boldsymbol{\mu}}(f) - P(f)$$

and equality holds if and only if P is concentrated on Γ.

(ii) We may assume that $\bar{P}$ is a probability measure with $\bar{P}(f) < \infty$. As a first step, we show that the effective domain $\bar{\mathcal{V}}$ of $\mathcal{M}(\bar{\boldsymbol{\mu}})$ is a subset of the effective domain $\mathcal{V}$ of $\mathcal{M}(\boldsymbol{\mu})$. To that end, it is sufficient to show that if $1 \le t \le n$ and $x \in \mathbb{R}$ are such that $u_{\mu_{t-1}}(x) = u_{\mu_t}(x)$, then $u_{\bar{\mu}_{t-1}}(x) = u_{\bar{\mu}_t}(x)$, and if moreover $\partial^+ u_{\mu_{t-1}}(x) = \partial^+ u_{\mu_t}(x)$, then $\partial^+ u_{\bar{\mu}_{t-1}}(x) = \partial^+ u_{\bar{\mu}_t}(x)$, and similarly for the left derivative ∂^- (cf. Proposition 3.2.3). Indeed, for t and x such that $u_{\mu_{t-1}}(x) = u_{\mu_t}(x)$, our assumption that $\Gamma \subseteq \mathcal{V}$ implies

$$\Gamma \subseteq (X_{t-1}, X_t)^{-1}\big((-\infty, x]^2 \cup [x, \infty)^2\big).$$

Using also that $\mathbb{E}^{\bar{P}}[X_t|\mathfrak{F}_{t-1}] = X_{t-1}$ and that $\bar{P}$ is concentrated on Γ,

$$\begin{aligned}
u_{\bar{\mu}_{t-1}}(x) &= \mathbb{E}^{\bar{P}}[|X_{t-1} - x|] \\
&= \mathbb{E}^{\bar{P}}[(X_{t-1} - x)\mathbf{1}_{X_{t-1} \ge x}] + \mathbb{E}^{\bar{P}}[(x - X_{t-1})\mathbf{1}_{X_{t-1} \le x}] \\
&= \mathbb{E}^{\bar{P}}[(X_t - x)\mathbf{1}_{X_{t-1} \ge x}] + \mathbb{E}^{\bar{P}}[(x - X_t)\mathbf{1}_{X_{t-1} \le x}] \\
&= \mathbb{E}^{\bar{P}}[|X_t - x|] = u_{\bar{\mu}_t}(x)
\end{aligned}$$

as desired. If in addition $\partial^+ u_{\mu_{t-1}}(x) = \partial^+ u_{\mu_t}(x)$, then $\Gamma \subseteq \mathcal{V}$ implies

$$\Gamma \subseteq (X_{t-1}, X_t)^{-1}\big((-\infty, x]^2 \cup (x, \infty)^2\big).$$

As $\bar{P}$ is concentrated on Γ, it follows that

$$\begin{aligned}\partial^+ u_{\bar{\mu}_{t-1}}(x) &= \bar{P}[X_{t-1} \leq x] - \bar{P}[X_{t-1} > x] \\ &= \bar{P}[X_t \leq x] - \bar{P}[X_t > x] = \partial^+ u_{\bar{\mu}_t}(x)\end{aligned}$$

as desired. The same argument can be used for the left derivative and we have shown that $\bar{\mathcal{V}} \subseteq \mathcal{V}$.

In view of that inclusion, the inequality $\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n \geq f$ holds on $\bar{\mathcal{V}}$. Since $\bar{P}$ is concentrated on Γ,

$$\bar{P}\left[\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n\right] = \bar{P}(f) < \infty.$$

We may follow the arguments in the proof of Lemma 3.4.19 to construct a moderator $\boldsymbol{\chi}$ and establish that $(\boldsymbol{\phi}, H) \in \mathcal{D}^g_{\bar{\boldsymbol{\mu}}}(f)$, where we are implicitly using the embedding detailed in Remark 3.4.15. (Note that the proof of Lemma 3.4.19 uses the condition $(\boldsymbol{\phi}, H) \in \mathcal{D}^g_{\bar{\boldsymbol{\mu}}}(0)$ only to establish $\bar{P}\left[\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n\right] < \infty$. In the present situation the latter is known a priori and the condition is not needed.) Then, we can modify $\boldsymbol{\chi}$ as in the proof of Proposition 3.4.17 to see that $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\bar{\boldsymbol{\mu}}}(f)$. As a result, we may apply Lemma 3.4.8 to obtain that

$$\bar{P}(f) = \bar{P}\left[\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n\right] = \bar{\boldsymbol{\mu}}(\boldsymbol{\phi}),$$

whereas for any other $P' \in \mathcal{M}(\bar{\boldsymbol{\mu}})$ we have

$$P'(f) \leq P'\left[\sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n\right] = \bar{\boldsymbol{\mu}}(\boldsymbol{\phi}) = \bar{P}(f).$$

This shows that $\bar{P} \in \mathcal{M}(\bar{\boldsymbol{\mu}})$ is optimal. $\square$

3.6 Left-Monotone Transports

In this section we define left-monotone transports through a shadow property and prove their existence.

Preliminaries

Before moving on to the n-step case, we recall the essential definitions and results regarding the one-step version of the left-monotone transport (also called the Left-Curtain coupling). The first notion is the so-called shadow, and it will be useful to define it for measures $\mu \leq_{pc} \nu$ in *positive convex order*, meaning that $\mu(\phi) \leq \nu(\phi)$ for any nonnegative convex function ϕ. Clearly, this order is weaker than the convex order $\mu \leq_c \nu$, and it is worth noting that μ may have a smaller mass than ν. The following is the result of [13, Lemma 4.6].

Lemma 3.6.1. *Let $\mu \leq_{pc} \nu$. Then the set*

$$\llbracket \mu, \nu \rrbracket := \{\theta : \mu \leq_c \theta \leq \nu\}$$

is non-empty and contains a unique least element $\mathcal{S}^\nu(\mu)$ for the convex order:

$$\mathcal{S}^\nu(\mu) \leq_c \theta \text{ for all } \theta \in \llbracket \mu, \nu \rrbracket .$$

The measure $\mathcal{S}^\nu(\mu)$ is called the shadow *of μ in ν.*

It will be useful to have the following picture in mind: if μ is a Dirac measure, its shadow in ν is a measure θ of equal mass and barycenter, chosen such as to have minimal variance subject to the constraint $\theta \leq \nu$.

The second notion is a class of reward functions.

Definition 3.6.2. A Borel function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *second-order Spence–Mirrlees* if $y \mapsto f(x', y) - f(x, y)$ is strictly convex for any $x < x'$.

We note that if f is sufficiently differentiable, this can be expressed as the cross-derivative condition $f_{xyy} > 0$ which has also been called the martingale Spence–Mirrlees condition, in analogy to the classical Spence–Mirrlees condition $f_{xy} > 0$.

In the one-step case, the left-monotone transport is unique and can be characterized as follows; cf. [13, Theorems 4.18, 4.21, 6.1] where this transport is called the Left-Curtain coupling, as well as [71, Theorem 1.2] for the third equivalence in the stated generality.

Proposition 3.6.3. *Let $\mu \leq_c \nu$ and $P \in \mathcal{M}(\mu,\nu)$. The following are equivalent:*

(i) For all $x \in \mathbb{R}$ and $A \in \mathfrak{B}(\mathbb{R})$,

$$P[(-\infty, x] \times A] = \mathcal{S}^\nu(\mu|_{(-\infty,x]})(A).$$

(ii) P is concentrated on a Borel set $\Gamma \subseteq \mathbb{R}^2$ satisfying

$$(x, y^-), (x, y^+), (x', y') \in \Gamma, \ \ x < x' \quad \Rightarrow \quad y' \notin (y^-, y^+).$$

(iii) P is an optimizer of $\mathbf{S}_{\mu,\nu}(f)$ for some (and then all) $f : \mathbb{R}^2 \to \mathbb{R}$ second-order Spence–Mirrlees such that there exist functions $a \in L^1(\mu)$, $b \in L^1(\nu)$ with $|f(x,y)| \leq a(x) + b(y)$.

There exists a unique measure $\bar{P} \in \mathcal{M}(\mu,\nu)$ satisfying (i)–(iii), and $\bar{P}$ is called the (one-step) left-monotone transport.

If μ is a discrete measure, the characterization in (i) can be understood as follows: the left-monotone transport $\bar{P}$ processes the atoms of μ from left to right, mapping each one of them to its shadow in the remaining target measure.

Next, we record two more results about shadows that will be used below. The first one, cited from [14, Theorem 3.1], generalizes the above idea in the sense that the atoms are still mapped to their shadows but can be processed in any given order; in the general (non-discrete) case, such an order is defined by a coupling π from the uniform measure to μ.

Proposition 3.6.4. *Let $\mu \leq_c \nu$ and $\pi \in \Pi(\lambda, \mu)$ where λ denotes the Lebesgue measure on $[0,1]$. Then there exists a unique measure $Q \in \Pi(\lambda, \mu, \nu)$ on $\mathbb{R}^3$ such that $Q \circ (X_0, X_1)^{-1} = \pi$ and*

$$Q|_{[0,s]\times\mathbb{R}\times\mathbb{R}} \circ (X_1, X_2)^{-1} \in \mathcal{M}(\pi_s, \mathcal{S}^\nu(\pi_s)), \quad s \in \mathbb{R},$$

where $\pi_s := \pi|_{[0,s]\times\mathbb{R}} \circ (X_1)^{-1}$.

We shall also need the following facts about shadows.

Lemma 3.6.5. *(i) Let μ_1, μ_2, ν be finite measures satisfying $\mu_1 + \mu_2 \leq_{pc} \nu$. Then $\mu_2 \leq_{pc} \nu - \mathcal{S}^\nu(\mu_1)$ and $\mathcal{S}^\nu(\mu_1 + \mu_2) = \mathcal{S}^\nu(\mu_1) + \mathcal{S}^{\nu - \mathcal{S}^\nu(\mu_1)}(\mu_2)$.*

(ii) Let μ, ν_1, ν_2 be finite measures such that $\mu \leq_{pc} \nu_1 \leq_c \nu_2$. Then, it follows that $\mathcal{S}^{\nu_1}(\mu) \leq_{pc} \nu_2$. Moreover, $\mathcal{S}^{\nu_2}(\mathcal{S}^{\nu_1}(\mu)) = \mathcal{S}^{\nu_2}(\mu)$ if and only if $\mathcal{S}^{\nu_1}(\mu) \leq_c \mathcal{S}^{\nu_2}(\mu)$.

Proof. Part (i) is [13, Theorem 4.8]. To obtain the first statement in (ii), we observe that $\mathcal{S}^{\nu_1}(\mu) \leq \nu_1 \leq_c \nu_2$ and hence

$$\mathcal{S}^{\nu_1}(\mu)(\phi) \leq \nu_1(\phi) \leq \nu_2(\phi)$$

for any nonnegative convex function ϕ. Turning to the second statement, the "only if" implication follows directly from the definition of the shadow in Lemma 3.6.1. To show the reverse implication, suppose that $\mathcal{S}^{\nu_1}(\mu) \leq_c \mathcal{S}^{\nu_2}(\mu)$. Then, we have

$$\mu \leq_c \mathcal{S}^{\nu_1}(\mu) \leq_c \mathcal{S}^{\nu_2}(\mathcal{S}^{\nu_1}(\mu)) \leq \nu_2 \quad \text{and} \quad \mathcal{S}^{\nu_1}(\mu) \leq_c \mathcal{S}^{\nu_2}(\mu) \leq \nu_2.$$

These inequalities imply that

$$\mathcal{S}^{\nu_2}(\mathcal{S}^{\nu_1}(\mu)) \in [\![\mu, \nu_2]\!] \quad \text{and} \quad \mathcal{S}^{\nu_2}(\mu) \in [\![\mathcal{S}^{\nu_1}(\mu), \nu_2]\!]\,,$$

and now the minimality property of the shadow shows that

$$\mathcal{S}^{\nu_2}(\mu) \leq_c \mathcal{S}^{\nu_2}(\mathcal{S}^{\nu_1}(\mu)) \quad \text{and} \quad \mathcal{S}^{\nu_2}(\mathcal{S}^{\nu_1}(\mu)) \leq_c \mathcal{S}^{\nu_2}(\mathcal{S}^{\nu_2}(\mu)) = \mathcal{S}^{\nu_2}(\mu)$$

as desired. □

Construction of a Multistep Left-Monotone Transport

Our next goal is to define and construct a multistep left-monotone transport. The following concept will be crucial.

Definition 3.6.6. Let $\mu_0 \leq_{pc} \mu_1 \leq_c \cdots \leq_c \mu_n$. For $1 \leq t \leq n$, the *obstructed shadow of μ_0 in μ_t through $\mu_1, \ldots, \mu_{t-1}$* is iteratively defined by

$$\mathcal{S}^{\mu_1,\ldots,\mu_t}(\mu_0) := \mathcal{S}^{\mu_t}(\mathcal{S}^{\mu_1,\ldots,\mu_{t-1}}(\mu_0)).$$

The obstructed shadow is well-defined due to Lemma 3.6.5 (ii). An alternative definition is provided by the following characterization.

Lemma 3.6.7. *Let $\mu_0 \leq_{pc} \mu_1 \leq_c \cdots \leq_c \mu_n$ and $1 \leq t \leq n$. Then $\mathcal{S}^{\mu_1,\ldots,\mu_t}(\mu_0)$ is the* *unique least element of the set*

$$[\![\mu_0, \mu_t]\!]^{\mu_1,\dots,\mu_{t-1}} := \{\theta_t \le \mu_t : \exists \theta_s \le \mu_s,\ 1 \le s \le t-1,\ \mu_0 \le_c \theta_1 \le_c \cdots \le_c \theta_t\}$$

for the convex order; that is, $\mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0) \le_c \theta$ *for all elements* θ.

Proof. For $t = 1$ this holds by the definition of the shadow in Lemma 3.6.1. For $t > 1$, we inductively assume that $\mathcal{S}^{\mu_1,\dots,\mu_{t-1}}(\mu_0)$ is the least element of $[\![\mu_0, \mu_{t-1}]\!]^{\mu_1,\dots,\mu_{t-2}}$. Consider an arbitrary element $\theta_t \in [\![\mu_0, \mu_t]\!]^{\mu_1,\dots,\mu_{t-1}}$ and fix some

$$\mu_0 \le_c \theta_1 \le_c \cdots \le_c \theta_{t-1} \le_c \theta_t \quad \text{with} \quad \theta_s \le \mu_s, \quad 1 \le s \le t-1.$$

Then, $\theta_{t-1} \in [\![\mu_0, \mu_{t-1}]\!]^{\mu_1,\dots,\mu_{t-2}}$ and in particular $\mathcal{S}^{\mu_1,\dots,\mu_{t-1}}(\mu_0) \le_c \theta_{t-1}$. Recall that $\mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0)$ is defined as the least element for $\le_c$ of

$$\begin{aligned} [\![\mathcal{S}^{\mu_1,\dots,\mu_{t-1}}(\mu_0), \mu_t]\!] &= \{\theta \le \mu_t : \mathcal{S}^{\mu_1,\dots,\mu_{t-1}}(\mu_0) \le_c \theta\} \\ &\supseteq \{\theta \le \mu_t : \theta_{t-1} \le_c \theta\} \ni \theta_t. \end{aligned}$$

Hence, $\mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0) \le_c \theta_t$, and as $\theta_t \in [\![\mu_0, \mu_t]\!]^{\mu_1,\dots,\mu_{t-1}}$ was arbitrary, this shows that $\mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0)$ is a least element of $[\![\mu_0, \mu_t]\!]^{\mu_1,\dots,\mu_{t-1}}$. The uniqueness of the least element follows from the general fact that $\theta_t^1 \le_c \theta_t^2$ and $\theta_t^2 \le_c \theta_t^1$ imply $\theta_t^1 = \theta_t^2$. $\square$

We can now state the main result of this section.

Theorem 3.6.8. *Let* $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ *be in convex order. Then there exists* $P \in$

$\mathcal{M}(\boldsymbol{\mu})$ *such that the bivariate projections* $P_{0t} := P \circ (X_0, X_t)^{-1}$ *satisfy*

$$P_{0t}[(-\infty, x] \times A] = \mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0|_{(-\infty,x]})(A) \quad \textit{for} \quad x \in \mathbb{R},\ A \in \mathfrak{B}(\mathbb{R}),$$

for all $1 \le t \le n$. *Any such* $P \in \mathcal{M}(\boldsymbol{\mu})$ *is called a* left-monotone transport.

We observe that an n-step left-monotone transport is defined purely in terms of its bivariate projections $P \circ (X_0, X_t)^{-1}$. In the one-step case, this completely determines the transport. For $n > 1$, we shall see that there can be multiple (and then infinitely many) left-monotone transports; in fact, they form a convex compact set. This will be discussed in more detail in Section 3.8, where it will also be shown that uniqueness does hold if μ_0 is atomless.

Proof of Theorem 3.6.8. Step 1. We first construct measures $\pi_t \in \Pi(\lambda, \mu_t)$, $0 \le t \le n$ such that

$$\pi_t|_{[0,\mu_0((-\infty,x])]\times\mathbb{R}} \circ X_1^{-1} = \mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0|_{(-\infty,x]})$$

for all $x \in \mathbb{R}$, as well as measures $Q_t \in \Pi(\lambda, \mu_{t-1}, \mu_t)$, $1 \le t \le n$ such that

$$\begin{aligned} &Q_t|_{[0,\mu_0((-\infty,x])]\times\mathbb{R}\times\mathbb{R}} \circ (X_1, X_2)^{-1} \in \\ &\qquad\qquad \mathcal{M}\big(\mathcal{S}^{\mu_1,\dots,\mu_{t-1}}(\mu_0|_{(-\infty,x]}), \mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0|_{(-\infty,x]})\big) \end{aligned} \tag{3.6.1}$$

for all $x \in \mathbb{R}$. Indeed, for $t = 0$, we take $\pi_0 \in \Pi(\lambda, \mu_0)$ to be the quantile[10] coupling. Then, applying Proposition 3.6.4 to π_0 yields the measure Q_1, and we can define

[10]The quantile coupling (or Fréchet–Hoeffding coupling) is given by the law of $(F_\lambda^{-1}, F_{\mu_0}^{-1})$ under λ, where $F_{\mu_0}^{-1}$ is the inverse c.d.f. of μ_0.

$\pi_1 := Q_1 \circ (X_0, X_2)^{-1}$. Proceeding inductively, applying Proposition 3.6.4 to π_{t-1} yields Q_t which in turn allows us to define $\pi_t := Q_t \circ (X_0, X_2)^{-1}$.

Step 2. For $1 \leq t \leq n$, consider a disintegration $Q_t = \pi_{t-1} \otimes \kappa_t$ of Q_t. By (3.6.1), we may choose $\kappa_t(s, x_{t-1}, dx_t)$ to be a martingale kernel; that is,

$$\int x_t \, \kappa_t(s, x_{t-1}, dx_t) = x_{t-1}$$

holds for all $(s, x_{t-1}) \in \mathbb{R}^2$. We now define a measure $\pi \in \Pi(\lambda, \mu_0, \ldots, \mu_n)$ on $\mathbb{R}^{n+2}$ via

$$\pi = \pi_0 \otimes \kappa_1 \otimes \cdots \otimes \kappa_n.$$

Then, π satisfies

$$\pi \circ (X_0, X_t)^{-1} = \pi_{t-1} \quad \text{and} \quad \pi \circ (X_0, X_t, X_{t+1})^{-1} = Q_t$$

for $1 \leq t \leq n$, and setting $P = \pi \circ (X_1, \ldots, X_{n+1})^{-1}$ yields the theorem. $\square$

The following result studies the bivariate projections P_{0t} of a left-monotone transport and shows in particular that P_{0t} may differ from the Left-Curtain coupling [13] in $\mathcal{M}(\mu_0, \mu_t)$.

Proposition 3.6.9. *Let $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ be in convex order and let $P \in \mathcal{M}(\boldsymbol{\mu})$ be a left-monotone transport. The following are equivalent:*

(i) The bivariate projection $P_{0t} = P \circ (X_0, X_t)^{-1} \in \mathcal{M}(\mu_0, \mu_t)$ is left-monotone for all $1 \leq t \leq n$.

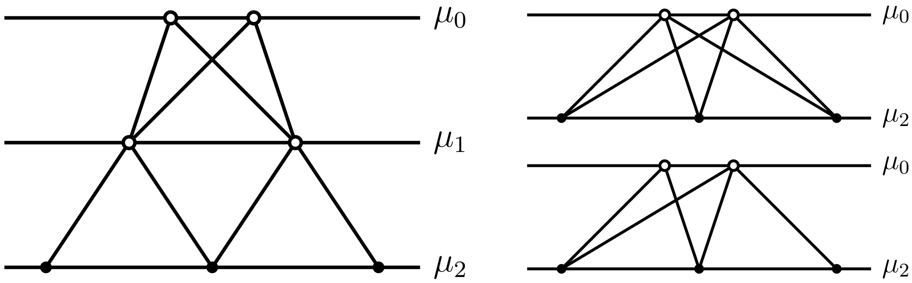

Figure 3.3: The left panel shows the support of the left-monotone transport P from Example 3.6.10. The right panel shows the support of P_{02} (top) and the support of the left-monotone transport in $\mathcal{M}(\mu_0, \mu_2)$ (bottom). The elements of the support are represented by the diagonal lines.

(ii) The marginals $\boldsymbol{\mu}$ *satisfy*

$$\mathcal{S}^{\mu_1}(\mu_0|_{(-\infty,x]}) \leq_c \cdots \leq_c \mathcal{S}^{\mu_n}(\mu_0|_{(-\infty,x]}) \quad \textit{for all} \quad x \in \mathbb{R}. \tag{3.6.2}$$

Proof. Given $\mu \leq \mu_0$, an iterative application of Lemma 3.6.5 (ii) shows that the obstructed shadows coincide with the ordinary shadows, i.e. $\mathcal{S}^{\mu_1,\dots,\mu_t}(\mu) = \mathcal{S}^{\mu_t}(\mu)$ for $1 \leq t \leq n$, if and only if $\mathcal{S}^{\mu_1}(\mu) \leq_c \cdots \leq_c \mathcal{S}^{\mu_n}(\mu)$. The proposition follows by applying this observation to $\mu = \mu_0|_{(-\infty,x]}$. $\square$

The following example illustrates the proposition and shows that (3.6.2) may indeed fail.

Example 3.6.10. Consider the marginals

$$\mu_0 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1, \quad \mu_1 = \frac{1}{2}\delta_{-2} + \frac{1}{2}\delta_2, \quad \mu_2 = \frac{1}{4}\delta_{-4} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_4.$$

Then the set $\mathcal{M}(\boldsymbol{\mu})$ consists of a single transport P; cf. the left panel of Figure 3.3.

Thus, P is necessarily left-monotone. Similarly, $P_{01} = P \circ (X_0, X_1)^{-1}$ is the unique element of $\mathcal{M}(\mu_0, \mu_1)$. However, $P_{02} = P \circ (X_0, X_2)^{-1}$ is given by

$$\frac{3}{16}\delta_{(-1,-4)} + \frac{1}{4}\delta_{(-1,0)} + \frac{1}{16}\delta_{(-1,4)} + \frac{1}{16}\delta_{(1,-4)} + \frac{1}{4}\delta_{(1,0)} + \frac{3}{16}\delta_{(1,4)}$$

whereas the unique left-monotone transport in $\mathcal{M}(\mu_0, \mu_2)$ can be found to be

$$\frac{1}{8}\delta_{(-1,-4)} + \frac{3}{8}\delta_{(-1,0)} + \frac{1}{8}\delta_{(1,-4)} + \frac{1}{8}\delta_{(1,0)} + \frac{1}{4}\delta_{(1,4)}.$$

Therefore, there exists no transport $P \in \mathcal{M}(\boldsymbol{\mu})$ such that both P_{01} and P_{02} are left-monotone, and Proposition 3.6.9 shows that (3.6.2) fails.

Remark 3.6.11. Of course, all our results on left-monotone transports have "right-monotone" analogues, obtained by reversing the orientation on the real line (i.e. replacing $x \mapsto -x$ everywhere).

3.7 Geometry and Optimality Properties

In this section we introduce the optimality properties for transports and the geometric properties of their supports that were announced in the Introduction, and prove that they equivalently characterize left-monotone transports.

Geometry of Optimal Transports for Reward Functions of Spence–Mirrlees Type

The first goal is to show that optimal transports for specific reward functions are concentrated on sets $\Gamma \subseteq \mathbb{R}^{n+1}$ satisfying certain no-crossing conditions that we introduce next. Given $1 \le t \le n$, we write

$$\Gamma^t = \{(x_0, \ldots, x_t) \in \mathbb{R}^{t+1} : (x_0, \ldots, x_n) \in \Gamma \text{ for some } (x_{t+1}, \ldots, x_n) \in \mathbb{R}^{n-t}\}$$

for the projection of Γ onto the first $t+1$ coordinates.

Definition 3.7.1. Let $\Gamma \subseteq \mathbb{R}^{n+1}$ and $1 \le t \le n$. Consider $\boldsymbol{x} = (x_0, \ldots, x_{t-1})$, $\boldsymbol{x}' = (x'_0, \ldots, x'_{t-1}) \in \mathbb{R}^t$ and $y^+, y^-, y' \in \mathbb{R}$ with $y^- < y^+$ such that $(\boldsymbol{x}, y^+), (\boldsymbol{x}, y^-), (\boldsymbol{x}', y') \in \Gamma^t$. Then, the projection

$$\Gamma^t \text{ is } \textit{left-monotone} \quad \text{if} \quad y' \notin (y^-, y^+) \text{ whenever } x_0 < x'_0.$$

The set Γ is left-monotone[11]—it will be clear from the context what is meant. if Γ^t is left-monotone for all $1 \le t \le n$.

We also need the following notion.

Definition 3.7.2. Let $\Gamma \subseteq \mathbb{R}^{n+1}$ and $1 \le t \le n$. The projection Γ^t is *nondegenerate* if for all $\boldsymbol{x} = (x_0, \ldots, x_{t-1}) \in \mathbb{R}^t$ and $y \in \mathbb{R}$ such that $(\boldsymbol{x}, y) \in \Gamma^t$, the following hold:

(i) if $y > x_{t-1}$, there exists $y' < x_{t-1}$ such that $(\boldsymbol{x}, y') \in \Gamma^t$;

(ii) if $y < x_{t-1}$, there exists $y' > x_{t-1}$ such that $(\boldsymbol{x}, y') \in \Gamma^t$.

[11] This terminology for Γ is abusive since $\Gamma = \Gamma^n$ is in fact a projection itself

The set Γ is called nondegenerate[12] if Γ^t is nondegenerate for all $1 \leq t \leq n$.

Broadly speaking, this definition says that for any path to the right in Γ there exists a path to the left, and vice versa. For a set supporting a martingale, nondegeneracy is not a restriction, in the following sense.

Remark 3.7.3. Let $\boldsymbol{\mu}$ be in convex order, $\mathcal{V}$ its effective domain and $\Gamma \subseteq \mathcal{V}$.

(i) There exists a nondegenerate, universally measurable set $\Gamma' \subseteq \Gamma$ such that $P(\Gamma') = 1$ for all $P \in \mathcal{M}(\boldsymbol{\mu})$ with $P(\Gamma) = 1$.

(ii) Fix $P \in \mathcal{M}(\boldsymbol{\mu})$ with $P(\Gamma) = 1$. There exists a nondegenerate, Borel-measurable set $\Gamma'_P \subseteq \Gamma$ such that $P(\Gamma'_P) = 1$.

Proof. Let N_t be the set of all $\boldsymbol{x} \in \Gamma^t$ such that (i) or (ii) of Definition 3.7.2 fail. If P is a martingale with $P(\Gamma) = 1$, we see that $N_t \times \mathbb{R}^{n-t+1}$ is P-null. Moreover, N_t is universally measurable (as the projection of a Borel set) and we can set

$$\Gamma' := \Gamma \setminus \bigcup_{t=1}^{n} (N_t \times \mathbb{R}^{n-t+1})$$

to prove (i). Turning to (ii), universal measurability implies that there exists a Borel set $N'_t \supseteq N_t$ such that $N'_t \setminus N_t$ is P_{t-1}-null, where $P_{t-1} = P \circ (X_0, \ldots, X_{t-1})^{-1}$. We can then set $\Gamma'_P := \Gamma \setminus \cup_{t=1}^{n} (N'_t \times \mathbb{R}^{n-t+1})$. $\square$

Next, we introduce a notion of competitors along the lines of [13, Definition 1.10].

Definition 3.7.4. Let π be a finite measure on $\mathbb{R}^{t+1}$ whose marginals have finite first moments and consider a disintegration $\pi = \pi_t \otimes \kappa$, where π_t is the projection of π

[12] Footnote 11 applies here as well.

onto the first t coordinates. A measure $\pi' = \pi_t \otimes \kappa'$ is a *t-competitor* of π if it has the same last marginal and

$$\mathrm{bary}(\kappa(\boldsymbol{x}, \cdot)) = \mathrm{bary}(\kappa'(\boldsymbol{x}, \cdot)) \quad \text{for } \pi_t\text{-a.e.} \quad \boldsymbol{x} = (x_0, \dots, x_{t-1}).$$

Using these definitions, we now formulate a variant of the monotonicity principle stated in Theorem 3.5.4 (i) that will be convenient to infer the geometry of Γ.

Lemma 3.7.5. *Let $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ be in convex order, $1 \le t \le n$ and let $\bar{f} : \mathbb{R}^{t+1} \to [0, \infty)$ be Borel. Consider $f(X_0, \dots, X_n) := \bar{f}(X_0, \dots, X_t)$ and suppose that $\mathbf{I}_{\boldsymbol{\mu}}(f) < \infty$. Let $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f)$ be an optimizer for $\mathbf{I}_{\boldsymbol{\mu}}(f)$ with the property that $\phi_s \equiv H_s \equiv 0$ for $s = t+1, \dots, n$ and define the set*

$$\Gamma := \left\{ \sum_{t=0}^{n} \phi_t(X_t) + (H \cdot X)_n = f \right\} \cap \mathcal{V}.$$

Let π be a finitely supported probability on $\mathbb{R}^{t+1}$ which is concentrated on Γ^t. Then $\pi(\bar{f}) \ge \pi'(\bar{f})$ for any t-competitor π' of π that is concentrated on $\mathcal{V}^t$.

Proof. Recall that the projections π_t and π'_t onto the first t coordinates coincide. Thus,

$$\begin{aligned}
\pi[H_t \cdot (X_t - X_{t-1})] &= \int H_t \cdot (\mathrm{bary}(\kappa(X_0, \dots, X_{t-1}, \cdot) - X_{t-1}) d\pi_t \\
&= \int H_t \cdot (\mathrm{bary}(\kappa'(X_0, \dots, X_{t-1}, \cdot) - X_{t-1}) d\pi'_t \\
&= \pi'[H_t \cdot (X_t - X_{t-1})].
\end{aligned}$$

Using also that the last marginals coincide, we deduce that

$$\pi[\bar{f}] = \pi\left[\sum_{s=0}^{t}\phi_s(X_s) + (H\cdot X)_t\right] = \pi'\left[\sum_{s=0}^{t}\phi_s(X_s) + (H\cdot X)_t\right] \geq \pi'[\bar{f}].$$

□

Next, we formulate an intermediate result relating optimality for Spence–Mirrlees reward functions to left-monotonicity of the support.

Lemma 3.7.6. *Let $1 \leq t \leq n$ and let $\Gamma \subseteq \mathcal{V}$ be a subset such that Γ^t is nondegenerate. Moreover, let $f : \mathbb{R}^{t+1} \to \mathbb{R}$ be of the form $f(X_0, \dots, X_t) = \bar{f}(X_0, X_t)$ for a second-order Spence–Mirrlees function $\bar{f}$. Assume that for any finitely supported probability π that is concentrated on Γ^t and any t-competitor π' of π that is concentrated on $\mathcal{V}^t$, we have $\pi(f) \geq \pi'(f)$. Then, the projection Γ^t is left-monotone.*

Proof. Consider $(\boldsymbol{x}, y_1), (\boldsymbol{x}, y_2), (\boldsymbol{x}', y') \in \Gamma^t$ satisfying $x_0 < x_0'$ and suppose for contradiction that $y_1 < y' < y_2$. We define $\lambda = \frac{y_2 - y'}{y_2 - y_1}$ and

$$\pi = \frac{\lambda}{2}\delta_{(\boldsymbol{x},y_1)} + \frac{1-\lambda}{2}\delta_{(\boldsymbol{x},y_2)} + \frac{1}{2}\delta_{(\boldsymbol{x}',y')}$$
$$\pi' = \frac{\lambda}{2}\delta_{(\boldsymbol{x}',y_1)} + \frac{1-\lambda}{2}\delta_{(\boldsymbol{x}',y_2)} + \frac{1}{2}\delta_{(\boldsymbol{x},y')}.$$

Then π and π' have the same projection $\pi_t = \pi_t'$ on the first t marginals and their last marginals also coincide. Moreover, disintegrating $\pi = \pi_t \otimes \kappa$ and $\pi' = \pi_t \otimes \kappa'$, the measures $\kappa(\boldsymbol{x}), \kappa(\boldsymbol{x}'), \kappa'(\boldsymbol{x}), \kappa(\boldsymbol{x}')$ all have barycenter y'. Therefore, π and π' are t-competitors. We must also have that π' is concentrated on $\mathcal{V}^t$, by the shape of $\mathcal{V}$.

Now our assumption implies that $\pi(f) \geq \pi'(f)$, but the second-order Spence–Mirrlees property of $\bar{f}$ implies that $\pi(f) < \pi'(f)$. $\square$

Geometry of Left-Monotone Transports

Next, we establish that transports with left-monotone support are indeed left-monotone in the sense of Theorem 3.6.8.

Theorem 3.7.7. *Let $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ be in convex order and let $P \in \mathcal{M}(\boldsymbol{\mu})$ be concentrated on a nondegenerate, left-monotone set $\Gamma \subseteq \mathbb{R}^{n+1}$. Then P is left-monotone.*

Before stating the proof of the theorem, we record two auxiliary results about measures on the real line. The first one is a direct consequence of Proposition 3.2.1.

Lemma 3.7.8. *Let $a < b$ and $\mu \leq_c \nu$. If ν is concentrated on $(-\infty, a]$, then so is μ, and moreover $\nu(\{a\}) \geq \mu(\{a\})$. The analogue holds for $[b, \infty)$.*

The second result is [13, Lemma 5.2].

Lemma 3.7.9. *Let σ be a nontrivial signed measure on $\mathbb{R}$ with $\sigma(\mathbb{R}) = 0$ and let $\sigma = \sigma^+ - \sigma^-$ be its Hahn decomposition. There exist $a \in \operatorname{supp}(\sigma^+)$ and $b > a$ such that $\int (b-y)^+ \mathbf{1}_{[a,\infty)}\, d\sigma(y) > 0$.*

We can now give the proof of the theorem; it is inspired by [13, Theorem 5.3] which corresponds to the case $n = 1$.

Proof of Theorem 3.7.7. Since the case $n = 1$ is covered by Proposition 3.6.3, we may assume that the theorem has been proved for transports with $n - 1$ steps and focus on the induction argument.

For every $x \in \mathbb{R}$ we denote by μ_x^t the marginal $(P|_{(-\infty,x]\times\mathbb{R}^n}) \circ X_t^{-1}$. In particular, we then have $\mu_x^0 = \mu_0|_{(-\infty,x]}$ and μ_x^t is the image of μ_x^0 under P after t steps. For the sake of brevity, we also set $\nu_x^t := \mathcal{S}^{\mu_1,\ldots,\mu_t}(\mu_x^0)$. By definition, P is left-monotone if $\mu_x^t = \nu_x^t$ for all $x \in \mathbb{R}$ and $t \le n$, and by the induction hypothesis, we may assume that this holds for $t \le n-1$.

We argue by contradiction and assume that there exists $x \in \mathbb{R}$ such that $\mu_x^n \neq \nu_x^n$. Then, the signed measure

$$\sigma := \nu_x^n - \mu_x^n$$

is nontrivial and we can find $a < b$ with $a \in \operatorname{supp}(\sigma^+)$ as in Lemma 3.7.9. Observe that $\sigma^+ \le \mu_n - \mu_x^n$ where $\mu_n - \mu_x^n$ is the image of $\mu_n|_{(x,\infty)}$ under P. Hence, $a \in \operatorname{supp}(\mu_n - \mu_x^n)$ and as P is concentrated on Γ, we conclude that there exists a sequence of points

$$\boldsymbol{x}^m = (x_0^m, \ldots, x_n^m) \in \Gamma \quad \text{with } x < x_0^m \text{ and } x_n^m \to a. \tag{3.7.1}$$

Moreover, by the characterization of the obstructed shadow in Lemma 3.6.7, we must have

$$\nu_x^n \le_c \mu_x^n$$

as $\mu_x^n \in [\![\mu_x^0, \mu_n]\!]^{\mu_1,\ldots,\mu_{n-1}}$ due to the fact that μ_x^n is the image of μ_x^0 under a martingale transport.

Step 1. We claim that for all $\boldsymbol{x} = (x_0, \ldots, x_{n-1})$ with $x_0 \le x$ and $x_{n-1} \le a$, it holds that

$$\Gamma_{\boldsymbol{x}} \cap (a, \infty) = \emptyset,$$

where $\Gamma_{\boldsymbol{x}} = \{y \in \mathbb{R} : (\boldsymbol{x}, y) \in \Gamma\}$ is the section of Γ at $\boldsymbol{x}$. By way of contradiction, assume that for some $\boldsymbol{x}$ with $x_0 \leq x$ and $x_{n-1} \leq a$ we have $\Gamma_{\boldsymbol{x}} \cap (a, \infty) \neq \emptyset$, then in particular $\Gamma_{\boldsymbol{x}} \cap (x_{n-1}, \infty) \neq \emptyset$. In view of the nondegeneracy of Γ, we conclude that $\Gamma_{\boldsymbol{x}} \cap (-\infty, x_{n-1}) \neq \emptyset$ and hence that $\Gamma_{\boldsymbol{x}} \cap (-\infty, a) \neq \emptyset$. This yields a contradiction to the left-monotonicity of Γ by using $\boldsymbol{x}^m$ from (3.7.1) for $\boldsymbol{x}'$ in Definition 3.7.1 for large enough m, and the proof of the claim is complete.

Step 2. Similarly, we can show that for all $\boldsymbol{x} = (x_0, \dots, x_{n-1})$ with $x_0 \leq x$ and $x_{n-1} \geq a$,

$$\Gamma_{\boldsymbol{x}} \cap (-\infty, a) = \emptyset.$$

Step 3. Next, we consider the marginals

$$\mu^t_{x,a} := \left(P|_{(-\infty,x]\times\mathbb{R}^{n-2}\times(-\infty,a]\times\mathbb{R}}\right) \circ X_t^{-1}.$$

Then, in particular, $\mu^{n-1}_{x,a} = \mu^{n-1}_x|_{(-\infty,a]}$ and $\mu^n_{x,a}$ is the image of $\mu^{n-1}_{x,a}$ under the last step of P. Step 1 of the proof thus implies that $\mu^n_{x,a}$ is concentrated on $(-\infty, a]$. We also write

$$\nu^n_{x,a} := \mathcal{S}^{\mu_n}(\mu^{n-1}_x|_{(-\infty,a]}).$$

We have $\mu^{n-1}_{x,a} \leq_c \mu^n_{x,a}$ as $\mathcal{M}(\mu^{n-1}_{x,a}, \mu^n_{x,a}) \neq \emptyset$, and $\mu^n_{x,a} \leq \mu^n_x \leq \mu_n$. Therefore,

$$\nu^n_{x,a} \leq_c \mu^n_{x,a} \tag{3.7.2}$$

by the minimality of the shadow. Next, we show that

$$\nu_x^n - \nu_{x,a}^n \leq_c \mu_x^n - \mu_{x,a}^n. \tag{3.7.3}$$

Observe that $\mu_x^n - \mu_{x,a}^n$ is the image of $\mu_x^{n-1}|_{(a,\infty)}$ under P and therefore concentrated on $[a,\infty)$ by Step 2. Using this observation, that $\mu_{x,a}^n$ is concentrated on $(-\infty,a]$ as mentioned above, and the fact that $\nu_{x,a}^n(\{a\}) \leq \mu_{x,a}^n(\{a\})$ as a consequence of (3.7.2) and Lemma 3.7.8, we have

$$\mu_x^n - \mu_{x,a}^n = (\mu_x^n - \mu_{x,a}^n)|_{[a,\infty)} \leq (\mu_n - \mu_{x,a}^n)|_{[a,\infty)} \leq (\mu_n - \nu_{x,a}^n)|_{[a,\infty)} \leq \mu_n - \nu_{x,a}^n.$$

We also have $\mu_x^{n-1}|_{(a,\infty)} \leq_c \mu_x^n - \mu_{x,a}^n$ since the latter measure is the image of the former under P. Together with the preceding display, we have established that

$$\mu_x^n - \mu_{x,a}^n \in \left[\!\left[\mu_x^{n-1}|_{(a,\infty)}, \mu_n - \nu_{x,a}^n \right]\!\right].$$

On the other hand,

$$\nu_x^n - \nu_{x,a}^n = \mathcal{S}^{\mu_n - \nu_{x,a}^n}(\mu_x^{n-1}|_{(a,\infty)})$$

from the additivity property of the shadow in Lemma 3.6.5 (i), and therefore (3.7.3) follows by the minimality of the shadow.

Step 4. Recall from Step 3 that $\mu_{x,a}^n$ is concentrated on $(-\infty,a]$ and that $\mu_x^n - \mu_{x,a}^n$ is concentrated on $[a,\infty)$. Therefore, $\nu_{x,a}^n$ is concentrated on $(-\infty,a]$ and $\nu_x^n - \nu_{x,a}^n$ is concentrated on $[a,\infty)$, by Lemma 3.7.8. Moreover, we have $\nu_{x,a}^n(\{a\}) \leq \mu_{x,a}^n(\{a\})$ by the same lemma, and finally, the function $y \mapsto (b-y)^+ \mathbf{1}_{[a,\infty)}(y)$ is convex on $[a,\infty)$ as $a<b$. Using these facts and (3.7.3),

$$
\begin{aligned}
&\int (b-y)^+ \mathbf{1}_{[a,\infty)}(y) \nu_x^n(dy) \\
&\qquad = \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)(\nu_x^n - \nu_{x,a}^n)(dy) + (b-a)\nu_{x,a}^n(\{a\}) \\
&\qquad \le \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)(\mu_x^n - \mu_{x,a}^n)(dy) + (b-a)\mu_{x,a}^n(\{a\}) \\
&\qquad = \int (b-y)^+ \mathbf{1}_{[a,\infty)}(y)\mu_x^n(dy).
\end{aligned}
$$

This contradicts the choice of a and b, cf. Lemma 3.7.9, and thus completes the proof. □

Optimality Properties

In this section we relate left-monotone transports and left-monotone sets to the optimal transport problem for Spence–Mirrlees functions.

Theorem 3.7.10. *For $1 \le t \le n$, let $f_t : \mathbb{R}^2 \to \mathbb{R}$ be second-order Spence–Mirrlees functions such that $|f_t(x,y)| \le a_0(x) + a_t(y)$ for some $a_0 \in L^1(\mu_0)$ and $a_t \in L^1(\mu_t)$. There exists a universally measurable, nondegenerate, left-monotone set $\Gamma' \subseteq \mathbb{R}^{n+1}$ such that any simultaneous optimizer $P \in \mathcal{M}(\boldsymbol{\mu})$ for $\mathbf{S}_{\boldsymbol{\mu}}(f_t(X_0,X_t))$, $1 \le t \le n$ is concentrated on Γ'. In particular, any such P is left-monotone.*

Proof. The last assertion follows by an application of Theorem 3.7.7, so we may focus on finding Γ'. For each $1 \le t \le n$, we use Theorem 3.5.2 and Remark 3.5.3 to find a dual optimizer $(\boldsymbol{\phi}, H) \in \mathcal{D}_{\boldsymbol{\mu}}(f_t)$ for $\mathbf{I}_{\boldsymbol{\mu}}(f_t(X_0, X_t))$ and define the Borel set

$$\Gamma(t) := \left\{ \sum_{s=0}^{n} \phi_s(X_s) + (H \cdot X)_n = f_t \right\} \cap \mathcal{V}.$$

Here, we may choose a dual optimizer such that $\phi_s \equiv H_s \equiv 0$ for $s = t+1, \dots, n$. (This can be seen by applying Theorem 3.5.2 to the transport problem involving only the marginals $(\mu_0, \dots, \mu_t)$ and taking the corresponding dual optimizer.) Theorem 3.5.4 shows that any simultaneous optimizer $P \in \mathcal{M}(\boldsymbol{\mu})$ is concentrated on $\Gamma(t)$ for all t, and hence also on the Borel set

$$\Gamma := \bigcap_{t=1}^{n} \Gamma(t).$$

Using Remark 3.7.3 (i), we find a universally measurable, nondegenerate subset $\Gamma' \subseteq \Gamma$ with the same property. Since the projection $(\Gamma')^t$ is contained in the projection $(\Gamma(t))^t$, Lemma 3.7.5 and Lemma 3.7.6 yield that $(\Gamma')^t$ is left-monotone for all t; that is, Γ' is left-monotone. $\square$

Remark 3.7.11. In Theorem 3.7.10, if we only wish to find a nondegenerate, left-monotone set $\Gamma'_P \subseteq \mathbb{R}^{n+1}$ such that a given simultaneous optimizer $P \in \mathcal{M}(\boldsymbol{\mu})$ is concentrated on Γ'_P, then we may choose Γ'_P to be Borel instead of universally measurable. This follows by replacing the application of Remark 3.7.3 (i) by Remark 3.7.3 (ii) in the proof.

The following is a converse to Theorem 3.7.10.

Theorem 3.7.12. *Given $1 \leq t \leq n$, let $f \in C^{1,2}(\mathbb{R}^2)$ be such that $f_{xyy} \geq 0$ and suppose that the following integrability condition holds:*

$$\begin{cases} f(X_0, X_t), \quad f(0, X_t), \quad f(X_0, 0), \quad \bar{h}(X_0)X_0, \quad \bar{h}(X_0)X_t \\ \\ \text{are } P\text{-integrable for all } P \in \mathcal{M}(\boldsymbol{\mu}), \end{cases} \tag{3.7.4}$$

where $\bar{h}(x) := \partial_y|_{y=0}[f(x,y) - f(0,y)]$. Then every left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ is an optimizer for $\mathbf{S}_{\boldsymbol{\mu}}(f)$.

The integrability condition clearly holds when f is Lipschitz continuous; in particular, a smooth second-order Spence–Mirrlees function (as defined in the Introduction) satisfies the assumptions of the theorem for any $\boldsymbol{\mu}$.

The proof will be given by an approximation based on the following building blocks for Spence–Mirrlees functions; the construction is novel and may be of independent interest.

Lemma 3.7.13. *Let $1 \leq t \leq n$ and let $f(X_0, \dots, X_n) := \mathbf{1}_{(-\infty,a]}(X_0)\varphi(X_t)$ for a concave function φ and $a \in \mathbb{R}$. Then every left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ is an optimizer for $\mathbf{S}_{\boldsymbol{\mu}}(f)$.*

Proof. In view of Lemma 3.6.7, this follows directly by applying the defining shadow property from Theorem 3.6.8 with $x = a$. □

The integrability condition (3.7.4) implies that setting

$$g(x, y) := f(x, 0) + f(0, y) - f(0, 0) + \bar{h}(x)y,$$

the three terms constituting

$$g(X_0, X_t) = [f(X_0, 0) + \bar{h}(X_0)X_0] + [f(0, X_t) - f(0, 0)] + [\bar{h}(X_0)(X_t - X_0)]$$

are P-integrable and $P[g(X_0, X_t)]$ is constant over $P \in \mathcal{M}(\boldsymbol{\mu})$. By replacing f with $f - g$, we may thus assume without loss of generality that

$$f(x, 0) = f(0, y) = f_y(x, 0) = 0 \quad \text{for all} \quad (x, y) \in \mathbb{R}^2. \tag{3.7.5}$$

After this normalization, integration by parts yields the representation

$$f(x, y) = \int_0^y \int_0^x (y - t) f_{xyy}(s, t)\, ds\, dt. \tag{3.7.6}$$

Lemma 3.7.14. *Theorem 3.7.12 holds under the following additional condition: there exists a constant $c > 0$ such that*

$$x \mapsto f(x, y) \textit{ is constant on } \{x > c\} \textit{ and on } \{x < -c\},$$

$$y \mapsto f(x, y) \textit{ is affine on } \{y > c\} \textit{ and on } \{y < -c\}.$$

Proof. Integration by parts implies that for all $(x,y) \in \mathbb{R}^2$, we have the representation

$$\begin{aligned} f(x,y) = &- \int_{-c}^{c} \int_{-c}^{c} \mathbf{1}_{(-\infty,s]}(x)(y-t)^+ f_{xyy}(s,t)\, ds\, dt \\ &+ [f(x,-c) - (-c) f_y(x,-c)] \\ &+ [f(c,y) - f(c,-c) - f_y(c,-c)(y-(-c))] \\ &+ f_y(x,-c)y. \end{aligned}$$

The last three terms are of the form $g(x,y) = \tilde{\phi}(x) + \tilde{\psi}(y) + \tilde{h}(x)y$ and of linear growth due to the additional condition. Hence, as above, $P'[g(X_0, X_t)] = C$ is constant for $P' \in \mathcal{M}(\boldsymbol{\mu})$. If $P \in \mathcal{M}(\boldsymbol{\mu})$ is left-monotone and $P' \in \mathcal{M}(\boldsymbol{\mu})$ is arbitrary, Fubini's theorem and Lemma 3.7.13 yield that

$$\begin{aligned} P[f] &= - \int_{-c}^{c} \int_{-c}^{c} P[\mathbf{1}_{(-\infty,s]}(x)(y-t)^+] f_{xyy}(s,t)\, ds\, dt + C \\ &\geq - \int_{-c}^{c} \int_{-c}^{c} P'[\mathbf{1}_{(-\infty,s]}(x)(y-t)^+] f_{xyy}(s,t)\, ds\, dt + C \\ &= P'[f], \end{aligned}$$

where P, P' are understood to integrate with respect to (x,y) and the application of Fubini's theorem is justified by the nonnegativity of the integrand. □

Proof of Theorem 3.7.12. Let f be as in the theorem. We shall construct functions f^m, $m \geq 1$ satisfying the assumption of Lemma 3.7.14 as well as $P[f^m] \to P[f]$ for all $P \in \mathcal{M}(\boldsymbol{\mu})$. Once this is achieved, the theorem follows from the lemma.

Indeed, we may assume that f is normalized as in (3.7.5). Let $m \geq 1$ and let

$\rho_m : \mathbb{R} \to [0,1]$ be a smooth function such that $\rho_m = 1$ on $[-m, m]$ and $\rho_m = 0$ on $[-m-1, m+1]^c$. In view of (3.7.6), we define f^m by

$$f^m(x,y) = \int_0^y \int_0^x (y-t) f_{xyy}(s,t) \rho_m(s) \rho_m(t) \, ds \, dt.$$

It then follows that f^m satisfies the assumptions of Lemma 3.7.14 with the constant $c = m+1$. Moreover, we have

$$0 \le f^m(x,y) \le f^{m+1}(x,y) \le f(x,y) \quad \text{for} \quad x \ge 0$$

and the opposite inequalities for $x \le 0$, as well as $f^m(x,y) \to f(x,y)$ for all (x,y).

Let $P \in \mathcal{M}(\boldsymbol{\mu})$. Since f is P-integrable, applying monotone convergence separately on $\{x \ge 0\}$ and $\{x \le 0\}$ yields that $P[f^m] \to P[f]$, and the proof is complete. $\square$

Remark 3.7.15. The function

$$\bar{f}(x,y) := \tanh(x)\sqrt{1+y^2}$$

satisfies the conditions of Theorem 3.7.12 for all marginals $\boldsymbol{\mu}$ in convex order, since the latter are assumed to have a finite first moment.

We can now collect the preceding results to obtain, in particular, the equivalences stated in Theorem 3.1.1.

Theorem 3.7.16. *Let $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ be in convex order. There exists a left-monotone, nondegenerate, universally measurable set $\Gamma \subseteq \mathbb{R}^{n+1}$ such that for any $P \in \mathcal{M}(\boldsymbol{\mu})$, the following are equivalent:*

(i) P is an optimizer for $\mathbf{S}_{\boldsymbol{\mu}}(f(X_0, X_t))$ whenever f is a smooth second-order Spence–Mirrlees function and $1 \le t \le n$,

(ii) P is concentrated on Γ,

(ii') P is concentrated on a left-monotone set,

(iii) P is left-monotone; i.e. P_{0t} transports $\mu_0|_{(-\infty,a]}$ to $\mathcal{S}^{\mu_1,\dots,\mu_t}(\mu_0|_{(-\infty,a]})$ for all $1 \le t \le n$ and $a \in \mathbb{R}$.

Moreover, there exists $P \in \mathcal{M}(\boldsymbol{\mu})$ satisfying (i)–(iii).

Proof. Let Γ be the set provided by Theorem 3.7.10 for the function $f_t = \bar{f}$ of Remark 3.7.15. Given $P \in \mathcal{M}(\boldsymbol{\mu})$, Theorem 3.7.10 shows that (i) implies (ii) which trivially implies (ii'). Theorem 3.7.7 and Remark 3.7.3 show that (ii') implies (iii), and Theorem 3.7.12 shows that (iii) implies (i). Finally, the existence of a left-monotone transport was stated in Theorem 3.6.8. □

We conclude this section with an example showing that left-monotone transports are not Markovian in general, even if they are unique and (3.6.2) holds for $\boldsymbol{\mu}$.

Example 3.7.17. Consider the marginals

$$\mu_0 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1, \quad \mu_1 = \frac{3}{4}\delta_0 + \frac{1}{4}\delta_2, \quad \mu_2 = \frac{1}{8}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{8}\delta_1 + \frac{1}{4}\delta_2.$$

The transport $P \in \mathcal{M}(\boldsymbol{\mu})$ given by

$$P = \frac{1}{2}\delta_{(0,0,0)} + \frac{1}{8}\delta_{(1,0,-1)} + \frac{1}{8}\delta_{(1,0,1)} + \frac{1}{4}\delta_{(1,2,2)}$$

is left-monotone because its support is left-monotone (Figure 3.4), and it is clearly not Markovian. On the other hand, it is not hard to see that this is the only way to build a left-monotone transport in $\mathcal{M}(\boldsymbol{\mu})$.

3.8 Uniqueness of Left-Monotone Transports

In this section we consider the (non-)uniqueness of left-monotone transports. It turns out the presence of atoms in μ_0 is important in this respect—let us start with the following simple observation.

Remark 3.8.1. Let $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ be in convex order. If μ_0 is a Dirac mass, then every $P \in \mathcal{M}(\boldsymbol{\mu})$ is left-monotone. Indeed, $\mathcal{M}(\mu_0, \mu_t)$ is a singleton for every $1 \leq t \leq n$, hence P_{0t} must be the (one-step) left-monotone transport.

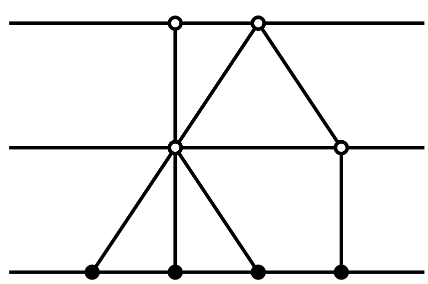

Figure 3.4: Support of the non-Markovian transport in Example 3.7.17.

Exploiting this observation, the following shows that left-monotone transports need not be unique when $n \geq 2$.

Example 3.8.2. Let $\mu_0 = \delta_0,\ \mu_1 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,\ \mu_2 = \frac{3}{8}\delta_{-2} + \frac{1}{4}\delta_0 + \frac{3}{8}\delta_2$. By the remark, any element in $\mathcal{M}(\boldsymbol{\mu})$ is left-monotone. Moreover, $\mathcal{M}(\boldsymbol{\mu})$ is a continuum since $\mathcal{M}(\mu_1, \mu_2)$ contains the convex hull of the two measures

$$P_l = \frac{1}{4}\delta_{(-1,-2)} + \frac{1}{4}\delta_{(-1,0)} + \frac{1}{8}\delta_{(1,-2)} + \frac{3}{8}\delta_{(1,2)},$$
$$P_r = \frac{3}{8}\delta_{(-1,-2)} + \frac{1}{8}\delta_{(-1,2)} + \frac{1}{4}\delta_{(1,0)} + \frac{1}{4}\delta_{(1,2)}.$$

The corresponding supports are depicted in Figure 3.5.

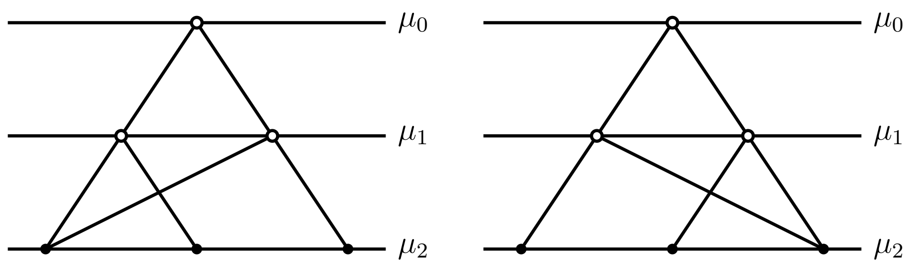

Figure 3.5: Supports of two left-monotone transports for the same marginals.

The example illustrates that non-uniqueness can typically be expected when μ_0 has atoms. On the other hand, we have the following uniqueness result.

Theorem 3.8.3. *Let $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ be in convex order. If μ_0 is atomless, there exists a unique left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$.*

The remainder of this section is devoted to the proof. Let us call a kernel $\kappa(x, dy)$ *binomial* if for all $x \in \mathbb{R}$, the measure $\kappa(x, dy)$ consists of (at most) two point masses.

A martingale transport will be called binomial if it can be disintegrated using only binomial kernels. We shall show that when μ_0 is atomless, any left-monotone transport is a binomial martingale, and then conclude the uniqueness via a convexity argument.

The first step is the following set-theoretic result.

Lemma 3.8.4. *Let $k \geq 1$ be an integer and $\Gamma \subseteq \mathbb{R}^{t+1}$. For $\boldsymbol{x} \in \mathbb{R}^t$, we denote by $\Gamma_{\boldsymbol{x}} := \{y \in \mathbb{R} : (\boldsymbol{x}, y) \in \Gamma\}$ the section at $\boldsymbol{x}$. If the set*

$$\{\boldsymbol{x} \in \mathbb{R}^t : |\Gamma_{\boldsymbol{x}}| \geq k\}$$

is uncountable, then it has an accumulation point. More precisely, there are $\boldsymbol{x} = (x_0, \ldots, x_t) \in \mathbb{R}^t$ and $y_1 < \cdots < y_k$ in $\Gamma_{\boldsymbol{x}}$ such that for all $\epsilon > 0$ there exist $\boldsymbol{x}' = (x_0', \ldots, x_t') \in \mathbb{R}^t$ and $y_1' < \cdots < y_k'$ in $\Gamma_{\boldsymbol{x}'}$ satisfying

(i) $\|\boldsymbol{x} - \boldsymbol{x}'\| < \epsilon$,

(ii) $x_0 < x_0'$,

(iii) $\max_{i=1,\ldots,k} |y_i - y_i'| < \epsilon$.

Proof. The proof is similar to the one of [13, Lemma 3.2] and therefore omitted. □

The following statement on the binomial structure generalizes a result of [13] for the one-step case and is of independent interest.

Proposition 3.8.5. *Let $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ be in convex order and let μ_0 be atomless. There exists a universally measurable set $\Gamma \subseteq \mathbb{R}^{n+1}$ such that every left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ is concentrated on Γ and such that for all $1 \leq t \leq n$ and*

$\boldsymbol{x} \in \mathbb{R}^t$,

$$|\{y \in \mathbb{R} : (X_0, \ldots, X_t)^{-1}(\boldsymbol{x}, y) \cap \Gamma \neq \emptyset\}| \leq 2. \tag{3.8.1}$$

In particular, every left-monotone transport $P \in \mathcal{M}(\boldsymbol{\mu})$ is a binomial martingale.

Proof. Let Γ be as in Theorem 3.7.16; then every left-monotone $P \in \mathcal{M}(\boldsymbol{\mu})$ is concentrated on Γ. Let A_t be the set of all $\boldsymbol{x} \in \mathbb{R}^t$ such that (3.8.1) fails. Suppose that A_t is uncountable; then Lemma 3.8.4 yields points $\boldsymbol{x}, \boldsymbol{x}'$ such that for some $y_1, y_2 \in \Gamma^t_{\boldsymbol{x}}$ and $y \in \Gamma^t_{\boldsymbol{x}'}$ we have $y_1 < y < y_2$. This contradicts the left-monotonicity of Γ (Definition 3.7.1), thus A_t must be countable. Hence, $(X_0, \ldots, X_{t-1})^{-1}(A_t)$ is Borel and P-null for all $P \in \mathcal{M}(\boldsymbol{\mu})$, as μ_0 is atomless. The set $\Gamma' = \Gamma \setminus \cup_{t=1}^n (X_0, \ldots, X_{t-1})^{-1}(A_t)$ then has the required properties. $\square$

Proof of Theorem 3.8.3. We will prove this result using induction on n. For $n = 1$ the result holds by Proposition 3.6.3, with or without atoms. To show the induction step, let P' be the unique left-monotone transport in $\mathcal{M}(\mu_0, \ldots, \mu_{n-1})$ and let $P_1 = P' \otimes \kappa_1$ and $P_2 = P' \otimes \kappa_2$ be disintegrations of two n-step left-monotone transports. Then,

$$\frac{P_1 + P_2}{2} = P' \otimes \frac{\kappa_1 + \kappa_2}{2}$$

is again left-monotone, and Proposition 3.8.5 yields that $(\kappa_1 + \kappa_2)/2$ must be a binomial kernel P'-a.s. Using also the martingale property of κ_1 and κ_2, this can only be true if $\kappa_1 = \kappa_2$ holds P'-a.s., and therefore $P_1 = P_2$. $\square$

3.9 Free Intermediate Marginals

In this section we discuss a variant of our transport problem where the intermediate marginal constraints $\mu_1, \dots, \mu_{n-1}$ are omitted; that is, only the first and last marginals μ_0, μ_n are prescribed. (One could similarly adapt the results to a case where some, but not all of the intermediate marginals are given.)

The primal space will be denoted by $\mathcal{M}^n(\mu_0, \mu_n)$ and consists of all martingale measures P on $\mathbb{R}^{n+1}$ such that $\mu_0 = P \circ (X_0)^{-1}$ and $\mu_n = P \circ (X_n)^{-1}$. To make the connection with the previous sections, we note that

$$\mathcal{M}^n(\mu_0, \mu_n) = \bigcup \mathcal{M}(\boldsymbol{\mu})$$

where the union is taken over all vectors $\boldsymbol{\mu} = (\mu_0, \mu_1, \dots, \mu_{n-1}, \mu_n)$ in convex order.

Polar Structure

We first characterize the polar sets of $\mathcal{M}^n(\mu_0, \mu_n)$. To that end, we introduce an analogue of the irreducible components.

Definition 3.9.1. Let $\mu_0 \leq_c \mu_n$ and let $(I_k, J_k) \subseteq \mathbb{R}^2$ be the corresponding irreducible domains in the sense of Proposition 3.2.3. The *n-step components* of $\mathcal{M}^n(\mu_0, \mu_n)$ are the sets[13]

(i) $I_k^n \times J_k$, where $k \geq 1$,

(ii) $I_0^{n+1} \cap \Delta_n$,

[13] A superscript m indicates the m-fold Cartesian product; Δ_n is the diagonal in $\mathbb{R}^{n+1}$.

(iii) $I_k^t \times \{p\}^{n-t+1}$, where $p \in J_k \setminus I_k$ and $1 \le t \le n$, $k \ge 1$.

The characterization then takes the following form.

Theorem 3.9.2 (Polar Structure)**.** *Let $\mu_0 \le_c \mu_n$. A Borel set $B \subseteq \mathbb{R}^{n+1}$ is $\mathcal{M}^n(\mu_0, \mu_n)$-polar if and only if there exist a μ_0-nullset N_0 and a μ_n-nullset N_n such that*

$$B \;\subseteq\; (N_0 \times \mathbb{R}^n) \,\cup\, (\mathbb{R}^n \times N_n) \,\cup\, \Big(\bigcup V_j\Big)^c$$

where the union runs over all n-step components V_j of $\mathcal{M}^n(\mu_0, \mu_n)$.

It turns out that our previous results can be put to work to prove the theorem, by means of the following lemma which may be of independent interest.

Lemma 3.9.3. *Let $\mu \le_c \nu$ be irreducible with domain (I, J) and let ρ be a probability concentrated on J. Then, there exists a probability $\mu \le_c \theta \le_c \nu$ satisfying $\theta \gg \rho$ such that $\mu \le_c \theta$ and $\theta|_I \le_c (\nu - \theta|_{J\setminus I})$ are both irreducible.*

Proof. Step 1. We first assume that $\rho = \delta_x$ for some $x \in J$ and show that there exists θ satisfying

$$\mu \le_c \theta \le_c \nu \quad \text{and} \quad \theta \gg \delta_x.$$

If ν has an atom at x, we can choose $\theta = \nu$. Thus, we may assume that $\nu(\{x\}) = 0$ and in particular that $x \in I$. Let a be the common barycenter of μ and ν and suppose that $x < a$. For all $b \in \mathbb{R}$ and $0 \le c \le \nu(\{b\})$, the measure

$$\nu_{b,c} := \nu|_{(-\infty,b)} + c\delta_b$$

satisfies $\nu_{b,c} \leq \nu$, and as $x < a$ there are unique b, c such that $\mathrm{bary}(\nu_{b,c}) = x$. Setting $\alpha = \nu_{b,c}$ and $\epsilon_0 = \alpha(\mathbb{R})$, we then have $\epsilon_0\delta_x \leq_c \alpha \leq \nu$, and a similar construction yields this result for $x \geq a$. The existence of such α implies that

$$\epsilon\delta_x \leq_{pc} \nu, \quad 0 \leq \epsilon \leq \epsilon_0$$

and thus the shadow $\mathcal{S}^\nu(\epsilon\delta_x)$ is well-defined. This measure is given by the restriction of ν to an interval (possibly including fractions of atoms at the endpoints); cf. [13, Example 4.7]. Moreover, the interval is bounded after possibly reducing the mass ϵ_0. Thus, for all $\epsilon < \epsilon_0$, the difference of potential functions

$$u_{\mathcal{S}^\nu(\epsilon\delta_x)} - u_{\epsilon\delta_x} \geq 0$$

vanishes outside a compact interval, and it converges uniformly to zero as $\epsilon \to 0$.

On the other hand, as $\mu \leq_c \nu$ is irreducible, the difference $u_\nu - u_\mu \geq 0$ is uniformly bounded away from zero on compact subsets of I and has nonzero derivative on $J \setminus I$. Together, it follows that

$$u_\nu - u_{\mathcal{S}^\nu(\epsilon\delta_x)} + u_{\epsilon\delta_x} \geq u_\mu \tag{3.9.1}$$

for small enough $\epsilon > 0$, so that

$$\theta := \nu - \mathcal{S}^\nu(\epsilon\delta_x) + \epsilon\delta_x$$

satisfies $\mu \leq_c \theta \leq_c \nu$; moreover, $\theta \gg \delta_x$ as $\nu(\{x\}) = 0$.

Step 2. We turn to the case of a general probability measure ρ on J. By Step 1, we can find a measure θ_x for each $x \in J$ such that

$$\mu \leq_c \theta_x \leq_c \nu \quad \text{and} \quad \theta_x \gg \delta_x.$$

The map $x \mapsto \theta_x$ can easily be chosen to be measurable (by choosing the ϵ for (3.9.1) in a measurable way). We can then define the probability measure

$$\theta'(A) := \int_J \theta_x(A)\rho(dx), \quad A \in \mathfrak{B}(\mathbb{R})$$

which satisfies $\mu \leq_c \theta' \leq_c \nu$. Moreover, we have $\theta' \gg \rho$; indeed, if $A \in \mathfrak{B}(\mathbb{R})$ is a θ'-nullset, then $\theta_x(A) = 0$ for ρ-a.e. x and thus $\rho(A) = 0$ as $\theta_x \gg \delta_x$.

Finally, $\theta := (\mu + \theta' + \nu)/3$ shares these properties. As $u_\mu < u_\nu$ on I due to irreducibility, we have $u_\mu < u_\theta < u_\nu$ on I and it follows that $\mu \leq_c \theta$ and $\theta|_I \leq_c (\nu - \theta|_{J\setminus I})$ are irreducible. $\square$

Lemma 3.9.4. *Let $\mu_0 \leq_c \mu_n$ and let π be a measure on $\mathbb{R}^{n+1}$ which is concentrated on an n-step component V of $\mathcal{M}^n(\mu_0, \mu_n)$ and whose first and last marginals satisfy*

$$\pi_0 \leq \mu_0, \quad \pi_n \leq \mu_n.$$

Then there exists $P \in \mathcal{M}^n(\mu_0, \mu_n)$ such that $P \gg \pi$.

Proof. If $V = I_0^{n+1} \cap \Delta_n$, then π must be an identical transport and we can take P to be any element of $\mathcal{M}(\mu_0, \mu_0, \ldots, \mu_0, \mu_n)$. Thus, we may assume that V is of type (i) or (iii) in Definition 3.9.1, and then, by fixing $k \geq 1$, that $\mu_0 \leq_c \mu_n$ is irreducible with domain (I, J).

Using Lemma 3.9.3, we can find intermediate marginals μ_t with

$$\mu_0 \leq_c \mu_1 \leq_c \cdots \leq_c \mu_{n-1} \leq_c \mu_n$$

such that $\mu_t \gg \pi_t$ for all $1 \leq t \leq n-1$, and each of the steps $\mu_{t-1} \leq_c \mu_t$, $1 \leq t \leq n$ has a single irreducible domain given by (I, J) as well as (possibly) a diagonal component on $J \setminus I$. We note that V is an irreducible component of $\mathcal{M}(\mu_0, \mu_1, \ldots, \mu_n)$ as introduced after Theorem 3.3.1.

Let $f_t = d\pi_t / d\mu_t$ be the Radon–Nikodym derivative of the marginal at date t. For $m \geq 1$, we define the measure $\pi^m \ll \pi$ by

$$\pi^m(dx_0, \ldots, dx_n) = 2^{-m} \left(\prod_{t=1}^{n-1} \mathbf{1}_{f_t(x_t) \leq 2^m} \right) \pi(dx_0, \ldots, dx_n).$$

Then, the marginals π_t^m satisfy the stronger condition $\pi_t^m \leq \mu_t$ for $0 \leq t \leq n$. Thus, we can apply Lemma 3.3.3 to $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n)$ and the irreducible component V, to find $P^m \in \mathcal{M}(\boldsymbol{\mu}) \subseteq \mathcal{M}^n(\mu_0, \mu_n)$ such that $P^m \gg \pi^m$. Noting that $\sum_{m \geq 1} 2^{-m} \pi^m \gg \pi$, we see that $P := \sum_{m \geq 1} 2^{-m} P^m \gg \pi$ satisfies the requirements of the lemma. □

Proof of Theorem 3.9.2. The result is deduced from Lemma 3.9.4 by following the argument in the proof of Theorem 3.3.1. □

Duality

In this section we formulate a duality theorem for the transport problem with free intermediate marginals.

Definition 3.9.5. Let $f : \mathbb{R}^{n+1} \to [0, \infty]$. The *primal problem* is

$$\mathbf{S}^n_{\mu_0,\mu_n}(f) := \sup_{P \in \mathcal{M}^n(\mu_0,\mu_n)} P(f) \in [0, \infty]$$

and the dual problem is

$$\mathbf{I}^n_{\mu_0,\mu_n}(f) := \inf_{(\phi,\psi,H) \in \mathcal{D}^n_{\mu_0,\mu_n}(f)} \mu_0(\phi) + \mu_n(\psi) \in [0, \infty],$$

where $\mathcal{D}^n_{\mu_0,\mu_n}(f)$ consists of all triplets (ϕ, ψ, H) such that $(\phi, \psi) \in L^c(\mu_0, \mu_n)$ and $H = (H_1, \ldots, H_n)$ is $\mathbb{F}$-predictable with

$$\phi(X_0) + \psi(X_n) + (H \cdot X)_n \geq f \quad \mathcal{M}^n(\mu_0, \mu_n)\text{-q.s.}$$

i.e. the inequality holds P-a.s. for all $P \in \mathcal{M}^n(\mu_0, \mu_n)$.

The analogue of Theorem 3.5.2 reads as follows.

Theorem 3.9.6 (Duality)**.** *Let* $f : \mathbb{R}^{n+1} \to [0, \infty]$.

(i) If f is upper semianalytic, then $\mathbf{S}^n_{\mu_0,\mu_n}(f) = \mathbf{I}^n_{\mu_0,\mu_n}(f) \in [0, \infty]$.

(ii) If $\mathbf{I}^n_{\mu_0,\mu_n}(f) < \infty$, *there exists a dual optimizer* $(\phi, \psi, H) \in \mathcal{D}^n_{\mu_0,\mu_n}(f)$.

The main step for the proof is again a closedness result. We shall only discuss the case where $\mu_0 \leq_c \mu_n$ is irreducible; the extension to the general case can be obtained along the lines of Section 3.4.

Proposition 3.9.7. *Let $\mu_0 \leq_c \mu_n$ be irreducible and let $f^m : \mathbb{R}^{n+1} \to [0,\infty]$ be a sequence of functions such that $f^m \to f$ pointwise. Moreover, let $(\phi^m, \psi^m, H^m) \in \mathcal{D}^n_{\mu_0,\mu_n}(f^m)$ be such that $\sup_m \mu_0(\phi^m) + \mu_n(\psi^m) < \infty$. Then there exist $(\phi, \psi, H) \in \mathcal{D}^n_{\mu_0,\mu_n}(f)$ such that*

$$\mu_0(\phi) + \mu_n(\psi) \leq \liminf_{m\to\infty} \mu_0(\phi^m) + \mu_n(\psi^m).$$

Proof. Let μ_t, $1 \leq t \leq n-1$ be such that $\boldsymbol{\mu} = (\mu_0, \dots, \mu_n)$ is in convex order and $\mu_{t-1} \leq_c \mu_t$ is irreducible for all $1 \leq t \leq n$; such μ_t are easily constructed by prescribing their potential functions. Setting $\boldsymbol{\phi}^m = (\phi^m, 0, \dots, 0, \psi^m)$ we have $(\boldsymbol{\phi}^m, H^m) \in \mathcal{D}^g_{\boldsymbol{\mu}}(f^m)$ and can thus apply Proposition 3.4.21 to obtain $(\boldsymbol{\phi}, H) \in \mathcal{D}^g_{\boldsymbol{\mu}}(f)$. The construction in the proof of that proposition yields $\phi_t \equiv 0$ for $1 \leq t \leq n-1$. Therefore, $(\phi_0, \phi_n, H) \in \mathcal{D}^n_{\mu_0,\mu_n}(f)$ and

$$\mu_0(\phi_0) + \mu_n(\phi_n) = \boldsymbol{\mu}(\boldsymbol{\phi}) \leq \liminf_{m\to\infty} \boldsymbol{\mu}(\boldsymbol{\phi}^m) = \liminf_{m\to\infty} \mu_0(\phi^m) + \mu_n(\psi^m).$$

□

Proof of Theorem 3.9.6. On the strength of Proposition 3.9.7, the proof is analogous to the one of Theorem 3.5.2. □

Monotone Transport

The analogue of our result on left-monotone transports is somewhat degenerate: with unconstrained intermediate marginals, the corresponding coupling is the identical transport in the first $n-1$ steps and the (one-step) left-monotone transport in the last step. The full result runs as follows.

Theorem 3.9.8. *Let $P \in \mathcal{M}^n(\mu_0, \mu_n)$. The following are equivalent:*

(i) P is a simultaneous optimizer for $\mathbf{S}^n_{\mu_0,\mu_n}(f(X_0, X_t))$ for all smooth second-order Spence–Mirrlees functions f and $1 \leq t \leq n$.

(ii) P is concentrated on a left-monotone set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\Gamma^{n-1} = \{(x, \dots, x) : x \in \Gamma^0\}.$$

(iii) For $0 \leq t \leq n-1$, we have $P \circ (X_t)^{-1} = \mu_0$ and $P \circ (X_t, X_n)^{-1}$ is the (one-step) left-monotone transport in $\mathcal{M}(\mu_0, \mu_n)$.

There exists a unique $P \in \mathcal{M}^n(\mu_0, \mu_n)$ satisfying (i)–(iii).

Proof. A transport P as in (iii) exists and is unique, because the identical transport between equal marginals and the left-monotone transport in $\mathcal{M}(\mu_0, \mu_n)$ exist and are unique; cf. Proposition 3.6.3. The equivalence of (ii) and (iii) follows from the same proposition and the fact that the only martingale transport from μ_0 to μ_0 is the identity.

Let $P \in \mathcal{M}^n(\mu_0, \mu_n)$ satisfy (i). In particular, P is then an optimizer for $\mathbf{S}^n_{\mu_0,\mu_n}(f(X_0, X_n))$, which by Proposition 3.6.3 implies that $P_{0n} = P \circ (X_0, X_n)^{-1}$ is the (one-step) left-monotone transport in $\mathcal{M}(\mu_0, \mu_n)$. For $t = 1, \dots, n-1$, P is an optimizer for $\mathbf{S}^n_{\mu_0,\mu_n}(-\mathbf{1}_{\{X_0 \le a\}}|X_t - b|)$, for all $a, b \in \mathbb{R}$. This implies that P_{0t} transports $\mu_0|_{(-\infty,a]}$ to the minimal element of $\{\theta : \mu_0|_{(-\infty,a]} \le_c \theta \le_{pc} \mu_n\}$ in the sense of the convex order, which is $\theta = \mu_0|_{(-\infty,a]}$. Therefore, P_{0t} must be the identical transport for $t = 1, \dots, n-1$ and all but the last marginal are equal to μ_0.

Conversely, let $P \in \mathcal{M}^n(\mu_0, \mu_n)$ have the properties from (iii). Then, P is optimal for $\mathbf{S}^n_{\mu_0,\mu_n}(-\mathbf{1}_{\{X_0 \le a\}}(X_t - b)^+)$ for all $1 \le t \le n$ and this can be extended to the optimality (i) for smooth second-order Spence–Mirrlees functions as in the proof of Theorem 3.7.12. $\square$

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— 4 —

Robust Pricing and Hedging around the Globe

This chapter is based on the article [53] of the same title, authored by Sebastian Herrmann and Florian Stebegg. It is forthcoming in Annals of Applied Probability.

4.1 Introduction

This paper studies the robust pricing and superhedging of derivative securities with a payoff of the form

$$F(X, A) = f\Big(\int_{[0,T]} X_t \,\mathrm{d}\, A_t \Big). \tag{4.1.1}$$

Here, f is a nonnegative Borel function, X is a càdlàg price process (realized on the Skorokhod space), and A is chosen by the buyer from a given set $\mathcal{A}$ of exercise rights. More precisely, $\mathcal{A}$ is a set of so-called averaging processes, i.e., nonnegative and nondecreasing adapted càdlàg processes A with $A_T \equiv 1$. Setting $\mathcal{A} = \{\mathbf{1}_{[\![\tau,T]\!]} : \tau \text{ a } [0,T]\text{-valued stopping time}\}$ or $\mathcal{A} = \{t \mapsto t/T\}$ reduces (4.1.1) to the relevant special cases of American- or Asian-style derivatives, respectively:

$$f(X_\tau) \quad \text{or} \quad f\Big(\frac{1}{T}\int_0^T X_t \,\mathrm{d}\, t\Big). \tag{4.1.2}$$

Other relevant examples are Bermudan options and European options with intermediate maturity (cf. Examples 4.3.3–4.3.4).

Robust pricing problem. Let μ and ν be probability measures on $\mathbb{R}$. We denote by $\mathcal{M}(\mu,\nu)$ the set of (continuous-time) martingale couplings between μ and ν, i.e., probability measures P under which X is a martingale with marginal distributions $X_0 \overset{P}{\sim} \mu$ and $X_T \overset{P}{\sim} \nu$. The value of the *primal problem*

$$\mathbf{S} := \sup_{P\in\mathcal{M}(\mu,\nu)} \sup_{A\in\mathcal{A}} E^P\left[F(X,A)\right] \tag{4.1.3}$$

can be interpreted as the maximal model-based price for F over all models which are consistent with the given marginals.

If $\mathcal{A}$ is a singleton, then (4.1.3) is a so-called (continuous-time) martingale optimal transport problem. This problem was introduced (for general payoffs) by Beiglböck, Henry-Labordère, and Penkner [9] in a discrete-time setting and by Galichon, Henry-Labordère, and Touzi [43] in continuous time; cf. the survey [84].

Semi-static superhedging problem. The formal dual problem to (4.1.3) has a natural interpretation as a superhedging problem.[1] Loosely speaking, a semi-static

[1] The primal problem (4.1.3) can be viewed as an optimization over finite measures P with three constraints: two marginal constraints and the martingale constraint. Its formal dual problem is

superhedge is a triplet (φ, ψ, H) consisting of functions φ, ψ and a suitable process H such that for every $A \in \mathcal{A}$, the superhedging inequality holds:

$$\varphi(X_0) + \psi(X_T) + \int_0^T H_{t-}^A \, \mathrm{d}\, X_t \geq F(X, A) \quad \text{pathwise.} \tag{4.1.4}$$

Here, the strategy $H = H^A$ may depend in an adapted way on A (cf. Section 3.2 for a precise formulation). For the example of an American-style payoff, this means that at the chosen exercise time τ, the buyer communicates her decision to exercise to the seller, who can then adjust the dynamic part of his hedging strategy (cf. [6, Section 3]). The left-hand side in (4.1.4) is the payoff of a static position in two European-style derivatives on X plus the final value of a self-financing dynamic trading strategy in X. The inequality (4.1.4) says that the final value of this semi-static portfolio dominates the payoff F for every choice of exercise right and "all" price paths. The initial cost to set up a semi-static superhedge (φ, ψ, H) equals the price $\mu(\varphi) + \nu(\psi)$ of the static part.[2] The formal *dual problem* to (4.1.3),

$$\mathbf{I} := \inf\{\mu(\varphi) + \nu(\psi) : (\varphi, \psi, H) \text{ is a semi-static superhedge}\}, \tag{4.1.5}$$

amounts to finding the cheapest semi-static superhedge (if it exists) and its initial cost, the so-called robust superhedging price.

the Lagrange dual problem where suitable functions φ and ψ and a suitable process H are used as Lagrange multipliers for the marginal and martingale constraints, respectively.

[2]We use the common notation $\mu(\varphi)$ for the integral of φ against μ.

Main objectives and relaxation of the dual problem. We are interested in *strong duality*, i.e., $\mathbf{S} = \mathbf{I}$, and *dual attainment*, i.e., the existence of a dual minimizer.

Dual attainment requires a suitable relaxation of the dual problem. Indeed, for the discrete-time martingale optimal transport problem, Beiglböck, Henry-Labordère, and Penkner [9] show strong duality for upper semicontinuous payoffs but provide a counterexample that shows that dual attainment can fail even if the payoff function is bounded and continuous. Beiglböck, Nutz, and Touzi [16] achieve strong duality and dual attainment for general payoffs and marginals in the one-step case by relaxing the dual problem in two aspects. First, they only require the superhedging inequality to hold in the quasi-sure sense, i.e., outside a set which is a nullset under every one-step martingale coupling between μ and ν. The reason is that the marginal constraints may introduce barriers on the real line which (almost surely) cannot be crossed by *any* martingale with these marginals; this was first observed by Hobson [30] (see also Cox [26] and Beiglböck and Juillet [14]). These barriers partition the real line into intervals and the marginal laws into so-called irreducible components. Then strong duality and dual attainment can be reduced to proving the same results for each irreducible component [30, 16]. Second, Beiglböck, Nutz, and Touzi [16] extend the meaning of the expression $\mu(\varphi) + \nu(\psi)$ to certain situations where both individual integrals are infinite. For example, it can happen that $\mu(\varphi) = -\infty$ and $\nu(\psi) = \infty$, but the price $E^P[\varphi(X_0) + \psi(X_T)]$ of the combined static part is well-defined, finite, and invariant under the choice of $P \in \mathcal{M}(\mu, \nu)$. In this situation, this price is still denoted by $\mu(\varphi) + \nu(\psi)$. We employ both relaxations for the precise definition of the dual problem in Section 4.3.

In continuous time, Dolinsky and Soner [35, 36] show strong duality for uniformly continuous payoffs satisfying a certain growth condition. They use the integration by parts formula to define the stochastic integral $\int_0^T H_{t-} \, \mathrm{d}\, X_t$ pathwise for finite variation integrands H. However, dual attainment cannot be expected in this class in general. For our payoffs (4.1.1), we need to allow integrands that are of finite variation whenever they are bounded but can become arbitrarily large or small on certain price paths. As the integrands are not of finite variation on these paths, the meaning of the pathwise integral needs to be extended appropriately.

For the purpose of the introduction, we discuss our results and methodology in a non-rigorous fashion, ignoring all aspects relating to the relaxation of the dual problem.

Main results. We prove strong duality and dual attainment for payoffs of the form (4.1.1) under mild conditions on f and $\mathcal{A}$ for irreducible marginals (Theorem 4.3.9); all results can be extended to general marginals along the lines of [16, Section 7]. The key idea is the reduction of the primal and dual problems to simpler auxiliary problems, which do not depend on the set $\mathcal{A}$ of exercising rights. In particular, our results cover American-style derivatives $f(X_\tau)$ for Borel-measurable f and Asian-style derivatives $f(\frac{1}{T}\int_0^T X_t \, \mathrm{d}\, t)$ for lower semicontinuous f and show that both derivatives have (perhaps surprisingly) the same robust superhedging prices and structurally similar semi-static superhedges.

Methodology. Our methodology relies on two crucial observations which allow us to bound the primal problem from below and the dual problem from above by simpler auxiliary primal and dual problems, respectively. To obtain a primal lower bound, we show that for any law θ which is in convex order between μ and ν, there is a sequence $(P_n)_{n\geq 1} \subset \mathcal{M}(\mu, \nu)$ such that the law of $\int_{[0,T]} X_t \,\mathrm{d}\, A_t$ under P_n converges weakly to θ if A is a suitable averaging process. This allows us to bound $\mathbf{S}$ from below by the value of the auxiliary primal problem

$$\widetilde{\mathbf{S}} := \sup_{\mu\leq_c\theta\leq_c\nu} \theta(f).$$

(The converse inequality also holds (cf. Lemma 4.4.1), so that in fact $\mathbf{S} = \widetilde{\mathbf{S}}$.)

Regarding the dual upper bound, we prove (modulo technicalities) that if φ is concave and ψ is convex such that $\varphi + \psi \geq f$, then (φ, ψ, H) is a semi-static superhedge, where the dynamic part H is given explicitly in terms of φ and ψ by

$$H_t := \varphi'(X_0) - \int_{[0,t]} \{\varphi'(X_0) + \psi'(X_s)\} \,\mathrm{d}\, A_s. \tag{4.1.6}$$

This allows us to bound $\mathbf{I}$ from above by the value of the auxiliary dual problem

$$\widetilde{\mathbf{I}} := \inf\{\mu(\varphi) + \nu(\psi) : \varphi \text{ concave, } \psi \text{ convex, } \varphi + \psi \geq f\}.$$

As a consequence, strong duality and dual attainment for $\mathbf{S}$ and $\mathbf{I}$ follow from the same assertions for the simpler problems $\widetilde{\mathbf{S}}$ and $\widetilde{\mathbf{I}}$, which we prove by adapting the techniques of [16]. Moreover, our reduction of the dual problem implies that if (φ, ψ)

is optimal for $\widetilde{\mathbf{I}}$, then it is also the static part of an optimal semi-static superhedge and the dynamic part H can be computed ex-post via (4.1.6). This dramatically decreases the complexity of the superhedging problem: the optimization over two functions *and a process* satisfying an inequality constraint *on the Skorokhod space* is reduced to an optimization over two functions satisfying an inequality constraint on $\mathbb{R}$.

Our methodology reveals that many derivatives have the same robust superhedging prices and semi-static superhedges. Indeed, $\widetilde{\mathbf{I}}$ and $\widetilde{\mathbf{S}}$ do not depend on the set $\mathcal{A}$ of exercise rights granted to the buyer, and this independence transfers to $\mathbf{S}$ and $\mathbf{I}$ under mild conditions on f and $\mathcal{A}$. For example, if f is lower semicontinuous, then the Asian-style derivative $f(\frac{1}{T}\int_0^T X_t \,\mathrm{d}\,t)$, the American-style derivative $f(X_\tau)$, and the European-style derivative $f(X_{T'})$ (for a fixed maturity $T' \in (0,T)$) all have the same robust superhedging price (Remark 4.3.10). This invariance breaks down when more than two marginals are given.

Related literature. Much of the extant literature on robust superhedging in semi-static settings is concerned with strong duality and dual attainment. The results vary in their generality and explicitness as well as their precise formulation. The semi-static setting, where call options are available as additional hedging instruments, dates back to Hobson's seminal paper [56] on the lookback option.[3] Many other specific exotic derivatives (mostly without special exercise rights) have been analyzed in this framework in the past two decades; cf. the survey [57].

[3]We note that the superhedging strategies described in [56] are actually dynamic in the call options.

Securities with special exercise rights have been studied in the context of American-style derivatives in discrete-time settings. Bayraktar, Huang, and Zhou [6] obtain a duality result for a somewhat different primal problem (cf. [6, Theorem 3.1]) and show that duality may fail in their setting if they formulate their primal problem in analogy to the present paper; see also [39] for related results with portfolio constraints. Hobson and Neuberger [31] (based on Neuberger's earlier manuscript [2]) resolve this issue by adopting a weak formulation for the primal problem: instead of optimizing only over martingale measures on a fixed filtered path space, the optimization there runs over filtered probability spaces supporting a martingale and thereby allows richer information structures and hence more stopping times. We also refer to [1, 40, 32] for recent developments in this regard. We note that all these papers permit significant restrictions on the set of possible price paths (e.g., binomial trees) while we allow all càdlàg paths. This difference may be the reason why strong duality holds in our setting without any relaxation of the primal problem.

The case of an Asian-style payoff as in (4.1.2) has been studied in the case of a Dirac initial law μ. For convex or concave f, Stebegg [81] shows strong duality and dual attainment. For nonnegative Lipschitz f, Cox and Källblad [5] obtain a PDE characterization of the maximal model-based price for finitely supported ν. Bayraktar, Cox, and Stoev [38] provide a similar, but not identical PDE for the corresponding pricing problem for American-style payoffs as in (4.1.2). A consequence of our main duality result is that the Asian and American pricing problems are the same, so that both these PDEs have the same (viscosity) solution.

Organization of the paper. The remainder of the article is organized as follows. In Section 2, we recall basic results on the convex order and potential functions, introduce the generalized integral of [16] and its relevant properties, and present the extension of the pathwise definition of the stochastic integral for finite variation integrands. Section 4.3 introduces the robust pricing and semi-static superhedging problems and presents our duality result. The duality between the auxiliary problems, the structure of their optimizers, and their relation to the robust pricing and superhedging problem are treated in Section 4.4. In Section 4.5, we provide simple geometric constructions of primal and dual optimizers for risk reversals and butterfly spreads. Finally, some counterexamples are collected in Section 4.6.

4.2 Preliminaries

Fix a time horizon T and let $\Omega = D([0,T];\mathbb{R})$ be the space of real-valued càdlàg paths on $[0,T]$. We endow Ω with the Skorokhod topology and denote by $\mathcal{F}$ the corresponding Borel σ-algebra, by $X = (X_t)_{t\in[0,T]}$ the canonical process on Ω, and by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the (raw) filtration generated by X. Unless otherwise stated, all probabilistic notions requiring a filtration pertain to $\mathbb{F}$.

For any process $Y = (Y_t)_{t\in[0,T]}$ on Ω, we set $Y_{0-} = 0$, so that the jump of Y at time 0 is $\Delta Y_0 = Y_0$.

Martingale measures and convex order

Let μ and ν be finite[4] measures on $\mathbb{R}$ with finite first moment. We denote by $\Pi(\mu,\nu)$ the set of (continuous-time) couplings of μ and ν, i.e., finite measures P on Ω such that $P \circ X_0^{-1} = \mu$ and $P \circ X_T^{-1} = \nu$. If, in addition, the canonical process X is a martingale under P (defined in the natural way if P is not a probability measure), then we write $P \in \mathcal{M}(\mu,\nu)$ and say that P is a (continuous-time) martingale coupling between μ and ν.

We also consider discrete-time versions of these notions. To wit, we denote by $\Pi^d(\mu,\nu)$ the set of finite measures Q on $\mathbb{R}^2$ with marginal distributions μ and ν and by $\mathcal{M}^d(\mu,\nu)$ the subset of measures Q under which the canonical process on $\mathbb{R}^2$ is a martingale (in its own filtration). The sets $\Pi^d(\mu,\theta,\nu)$ and $\mathcal{M}(\mu,\theta,\nu)$ of finite measures on $\mathbb{R}^3$ with prescribed marginal distributions are defined analogously.

We write $\mu \leq_c \nu$ if μ and ν are in *convex order* in the sense that $\mu(\varphi) \leq \nu(\varphi)$ holds for any convex function $\varphi : \mathbb{R} \to \mathbb{R}$. In this case, μ and ν have the same mass and the same barycenter $\operatorname{bary}(\mu) := \frac{1}{\mu(\mathbb{R})} \int x \, \mu(\mathrm{d}\, x)$.

The potential function $u_\mu : \mathbb{R} \to [0,\infty]$ of μ is defined as

$$u_\mu(x) := \int |x-y| \, \mu(\mathrm{d}\, y). \tag{4.2.1}$$

We refer to [14, Section 4.1] for basic properties of potential functions. In particular, the following relationship between the convex order, potential functions, and martingale measures is well known.

[4]As in [16], using finite measures as opposed to probability measures turns out to be useful.

Proposition 4.2.1. *Let μ and ν be finite measures with finite first moments and $\mu(\mathbb{R}) = \nu(\mathbb{R})$. Then the following are equivalent: (i) $\mu \leq_c \nu$, (ii) $u_\mu \leq u_\nu$, (iii) $\mathcal{M}^d(\mu, \nu) \neq \emptyset$, and (iv) $\mathcal{M}(\mu, \nu) \neq \emptyset$.*

An analogous result holds for three marginals μ, θ, ν, the corresponding potential functions, and the set $\mathcal{M}^d(\mu, \theta, \nu)$.

We recall the following definition from [16, Definition 2.2] (see also [14, Definition A.3]).

Definition 4.2.2. A pair of finite measures $\mu \leq_c \nu$ is called *irreducible* if the set $I = \{u_\mu < u_\nu\}$ is connected and $\mu(I) = \mu(\mathbb{R})$. In this situation, let J be the union of I and any endpoints of I that are atoms of ν; then (I, J) is the *domain* of $\mu \leq_c \nu$.

We work with irreducible $\mu \leq_c \nu$ for the remainder of this article.

Generalized integral

Let $\mu \leq_c \nu$ be irreducible with domain (I, J). Beiglböck and Juillet [14, Section A.3] and Beiglböck, Nutz, and Touzi [16, Section 4] appropriately extend the meaning of the expression $\mu(\varphi) + \nu(\psi)$ to the case where the individual integrals are not necessarily finite. We present here a slight extension of their work in order to deal with intermediate laws $\mu \leq_c \theta \leq_c \nu$ for which the pairs $\mu \leq_c \theta$ and $\theta \leq_c \nu$ may not be irreducible.

For the rest of this article, whenever we write $\mu \leq_c \nu$ for any two measures μ and ν, we implicitly assume that both measures are finite and have a finite first moment. Throughout this section, we fix $\mu \leq_c \theta_1 \leq_c \theta_2 \leq_c \nu$.

Definition 4.2.3. Let $\chi : J \to \mathbb{R}$ be concave. Denote by $-\chi''$ the second derivative measure of the convex function $-\chi$ and by $\Delta\chi$ the possible jumps of χ at the endpoints of I. We set

$$(\theta_1 - \theta_2)(\chi) := \frac{1}{2}\int_I (u_{\theta_1} - u_{\theta_2}) \,\mathrm{d}\,\chi'' + \int_{J\setminus I} |\Delta\chi| \,\mathrm{d}(\theta_2 - \theta_1) \in [0, \infty]. \tag{4.2.2}$$

The right-hand side is well defined in $[0, \infty]$ because $u_{\theta_1} \leq u_{\theta_2}$ on I and $\theta_1(\{b\}) \leq \theta_2(\{b\})$ for $b \in J \setminus I$.

If $\theta_1 = \mu$ and $\theta_2 = \nu$, then (4.2.2) coincides with Equation (4.2) in [16] because μ is concentrated on I. As in [16], there is an alternative representation of $(\theta_1 - \theta_2)(\chi)$ in terms of an iterated integral with respect to a disintegration of a (one-step) martingale coupling on $\mathbb{R}^2$:

Lemma 4.2.4. *Let $\chi : J \to \mathbb{R}$ be concave and let $Q \in \mathcal{M}^d(\theta_1, \theta_2)$. For any disintegration $Q = \theta_1 \otimes \kappa$, we have*

$$(\theta_1 - \theta_2)(\chi) = \int_J \left[\chi(x_1) - \int_J \chi(x_2)\kappa(x_1, \mathrm{d}\, x_2)\right] \theta_1(\mathrm{d}\, x_1).$$

Proof. The proof of [16, Lemma 4.1] does not use that $\mu \leq_c \nu$ is irreducible. Moreover, for $\bar{\chi} : J \to \mathbb{R}$ concave and continuous, the same arguments as in the proof of [16, Lemma 4.1] yield

$$\frac{1}{2}\int_I (u_{\theta_1} - u_{\theta_2}) \,\mathrm{d}\,\bar{\chi}'' = \int_J \left[\bar{\chi}(x_1) - \int_J \bar{\chi}(x_2)\,\kappa(x_1, \mathrm{d}\, x_2)\right] \theta_1(\mathrm{d}\, x_1). \tag{4.2.3}$$

(Note that $\int_J \bar{\chi}(x_2)\,\kappa(x_1, \mathrm{d}\,x_2) = \bar{\chi}(x_1)$ for boundary points $x_1 \in J \setminus I$ because κ is a martingale kernel concentrated on J.)

For a general concave $\chi : J \to \mathbb{R}$, we write $\chi = \bar{\chi} - |\Delta\chi|\mathbf{1}_{J\setminus I}$ with $\bar{\chi}$ continuous. Then we can replace $\bar{\chi}$ with χ on the left-hand side of (4.2.3) and the integrand on the right-hand side reads as

$$\chi + |\Delta\chi|\mathbf{1}_{J\setminus I} - \int_J \chi(x_2)\,\kappa(\cdot, \mathrm{d}\,x_2) - \int_{J\setminus I} |\Delta\chi(x_2)|\,\kappa(\cdot, \mathrm{d}\,x_2).$$

Integrating this against θ_1 and using Fubini's theorem yields

$$\int_J \left[\chi(x_1) - \int_J \chi(x_2)\,\kappa(x_1, \mathrm{d}\,x_2)\right] \theta(\mathrm{d}\,x_1) + \int_{J\setminus I} |\Delta\chi|\ \mathrm{d}\,\theta_1 - \int_{J\setminus I} |\Delta\chi|\ \mathrm{d}\,\theta_2.$$

Together with (4.2.3), this proves the claim. □

It can be shown as in [16] that $(\theta_1 - \theta_2)(\chi) = \theta_1(\chi) - \theta_2(\chi)$ if at least one of the individual integrals is finite.

We can now define the integral $\theta_1(\varphi) + \theta_2(\psi)$ as in [16, Definition 4.7].

Definition 4.2.5. Let $\varphi : J \to \overline{\mathbb{R}}$ and $\psi : J \to \overline{\mathbb{R}}$ be Borel functions. If there exists a concave function $\chi : J \to \mathbb{R}$ such that $\varphi - \chi \in L^1(\theta_1)$ and $\psi + \chi \in L^1(\theta_2)$, we say that χ is a *concave moderator* for (φ, ψ) with respect to $\theta_1 \leq_c \theta_2$ and set

$$\theta_1(\varphi) + \theta_2(\psi) := \theta_1(\varphi - \chi) + \theta_2(\psi + \chi) + (\theta_1 - \theta_2)(\chi) \in (-\infty, \infty]. \tag{4.2.4}$$

As in [16], the expression $\theta_1(\varphi) + \theta_2(\psi)$ defined in (4.2.4) does not depend on the choice of the concave moderator.

Definition 4.2.6. We write $L^c(\theta_1, \theta_2)$ for the space of pairs of Borel functions $\varphi, \psi : J \to \overline{\mathbb{R}}$ which admit a concave moderator χ with respect to $\theta_1 \leq_c \theta_2$ such that $(\theta_1 - \theta_2)(\chi) < \infty$.

We next present additional properties of the notions introduced above.

Lemma 4.2.7. *Let $(\varphi, \psi) \in L^c(\theta_1, \theta_2)$.*

(i) φ is finite on atoms of θ_1. If φ is concave, then $\varphi < \infty$ on J and $\varphi > -\infty$ on the interior of the convex hull of the support of θ_1.

(ii) ψ is finite on atoms of θ_2. If ψ is convex, then $\psi > -\infty$ on J and $\psi < \infty$ on the interior of the convex hull of the support of θ_2.

(iii) If $a, b : \mathbb{R} \to \mathbb{R}$ are affine, then $(\varphi + a, \psi + b) \in L^c(\theta_1, \theta_2)$ and

$$\theta_1(\varphi + a) + \theta_2(\psi + b) = \{\theta_1(\varphi) + \theta_2(\psi)\} + \theta_1(a) + \theta_2(b).$$

Proof. We only prove (iii). Let χ be a concave moderator for (φ, ψ) with respect to $\theta_1 \leq_c \theta_2$. Then $\varphi - \chi \in L^1(\theta_1)$, $\psi + \chi \in L^1(\theta_2)$, and $(\theta_1 - \theta_2)(\chi) < \infty$. Being affine, a and b are θ_1- and θ_2-integrable. It follows that χ is also a concave moderator for $(\varphi + a, \psi + b)$ with respect to $\theta_1 \leq_c \theta_2$ and that $(\varphi + a, \psi + b) \in L^c(\theta_1, \theta_2)$. The last assertion is a direct computation. □

Remark 4.2.8. Recall that I is the interior of the convex hull of the support of ν and that J is the union of I and any endpoints of I that are atoms of ν. Hence, Lemma 4.2.7 (ii) shows in particular, that if $(\varphi, \psi) \in L^c(\theta_1, \nu)$ with ψ convex, then ψ is finite on J.

We conclude this section with a number of calculation rules for the integrals defined above when φ is concave and ψ is convex.

Lemma 4.2.9. *Let $\mu \leq_c \theta_1 \leq_c \theta_2 \leq_c \theta_3 \leq_c \nu$ (where the pair $\mu \leq_c \nu$ is irreducible) and let $(\varphi, \psi) \in L^c(\theta_1, \theta_3)$ be such that φ is concave and finite, ψ is convex and finite, and $\varphi + \psi$ is bounded from below by a concave θ_3-integrable function.*

(i) φ and $-\psi$ are concave moderators for (φ, ψ) with respect to $\theta_1 \leq_c \theta_3$.

(ii) $(\varphi, \psi) \in L^c(\theta_1, \theta_2) \cap L^c(\theta_2, \theta_3)$.

(iii) $\theta_1(\varphi) + \theta_2(\psi) \leq \theta_1(\varphi) + \theta_3(\psi)$.

(iv) $\theta_2(\varphi) + \theta_3(\psi) \leq \theta_1(\varphi) + \theta_3(\psi)$.

Proof. Denote by ξ a concave θ_3-integrable lower bound for $\varphi + \psi$. By the concavity of ξ, we have $\theta_1(\xi) \geq \theta_2(\xi) \geq \theta_3(\xi) > -\infty$, so that ξ is also θ_1- and θ_2-integrable.

(i): Regarding the concave moderator property of φ, it suffices to show that $\varphi + \psi$ is θ_3-integrable. Denote by φ' the left-derivative of the concave function φ on I. Then for $(x_1, x_3) \in I \times J$,

$$\xi(x_3) \leq \varphi(x_3) + \psi(x_3) \leq \varphi(x_1) + \psi(x_3) + \varphi'(x_1)(x_3 - x_1). \tag{4.2.5}$$

Fix any $Q \in \mathcal{M}^d(\theta_1, \theta_3)$. Then (4.2.5) also holds Q-a.e. on $J \times J$ (setting $\varphi' = 0$ on $J \setminus I$ for example); this uses that any mass in a point of $J \setminus I$ stays put under a martingale transport plan. Since ξ is θ_3-integrable, the negative part of the right-hand side in (4.2.5) is Q-integrable. Then it can be argued as in [16, Remark 4.10] that the right-hand side in (4.2.5) is Q-integrable. It follows that $\varphi + \psi$ is θ_3-integrable.

Regarding the assertion about $-\psi$, it suffices to show that $\varphi + \psi$ is θ_1-integrable. We have

$$\begin{aligned}\xi(x_1) &\le \varphi(x_1) + \psi(x_1)\\ &= [\varphi(x_1) + \psi(x_3) + \varphi'(x_1)(x_3 - x_1)]\\ &\quad + [\psi(x_1) - \psi(x_3) - \varphi'(x_1)(x_3 - x_1)] \quad Q\text{-a.e. on } J \times J.\end{aligned} \tag{4.2.6}$$

By the above, the first term on the right-hand side is Q-integrable. Thus, the negative part of the second term is also Q-integrable. Hence, we may integrate the second term iteratively using Fubini's theorem as in [16, Remark 4.10]. The Q-integral equals $-(\theta_1 - \theta_3)(-\psi) \le 0$. In particular, the right-hand side in (4.2.6) is Q-integrable. It follows that $\varphi + \psi$ is θ_1-integrable.

(ii)–(iv): We know from the above that $\varphi+\psi$ is θ_3-integrable. It follows that φ is a concave moderator for (φ, ψ) with respect to $\theta_2 \le_c \theta_3$. Because $\theta_1 \le_c \theta_2$, we have that $u_{\theta_1} \le u_{\theta_2}$ and $\theta_1(\{b\}) \le \theta_2(\{b\})$ for $b \in J \setminus I$. Thus, $(\theta_2 - \theta_3)(\varphi) \le (\theta_1 - \theta_3)(\varphi) < \infty$ (cf. Definition 4.2.3). Hence, $(\varphi, \psi) \in L^c(\theta_2, \theta_3)$ and

$$\begin{aligned}\theta_2(\varphi) + \theta_3(\psi) &= \theta_2(\varphi - \varphi) + \theta_3(\varphi + \psi) + (\theta_2 - \theta_3)(\varphi)\\ &\le \theta_1(\varphi - \varphi) + \theta_3(\varphi + \psi) + (\theta_1 - \theta_3)(\varphi)\\ &= \theta_1(\varphi) + \theta_3(\psi).\end{aligned}$$

One can show similarly that $(\varphi, \psi) \in L^c(\theta_1, \theta_2)$ and that $\theta_1(\varphi) + \theta_2(\psi) \leq \theta_1(\varphi) + \theta_3(\psi)$. □

Pathwise stochastic integration

For any $\mathbb{F}$-adapted càdlàg process H of finite variation, the integral $H_- \bullet X_T = \int_{(0,T]} H_{t-} \, \mathrm{d}\, X_t$ can be defined *pathwise*, i.e., for each $\omega \in \Omega$ individually, via integration by parts as follows:

$$H_- \bullet X_T := X_T H_T - X_0 H_0 - \int_{(0,T]} X_t \, \mathrm{d}\, H_t, \tag{4.2.7}$$

where the integral on the right-hand side is the pathwise Lebesgue–Stieltjes integral. Setting $H_{0-} = 0$, so that $\Delta H_0 = H_0$, we can recast (4.2.7) as

$$H_- \bullet X_T = (X_T - X_0) H_0 + \int_{(0,T]} (X_T - X_t) \, \mathrm{d}\, H_t. \tag{4.2.8}$$

For any martingale measure P, if the (standard) stochastic integral of H_- with respect to X exists, then it is P-indistinguishable from the pathwise stochastic integral.

We need to give a sensible meaning to the integral $H_- \bullet X_T$ for certain integrands H which are not necessarily of finite variation, but may diverge in finite time.

Example 4.2.10. The following example motivates our extension of the pathwise stochastic integral for finite variation integrands. Let $\mu = \delta_0$ and $\nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. Then $\mu \leq_c \nu$ are irreducible with domain $(I, J) = ((-1, 1), [-1, 1])$. Consider a payoff function f which is convex on $[-1, 1]$ and has infinite (one-sided) derivatives at -1

and 1, e.g., $f(x) = 1-\sqrt{1-x^2}\mathbf{1}_{[-1,1]}(x)$, A semi-static superhedge for the Asian-style derivative $f(\frac{1}{T}\int_0^T X_t \,\mathrm{d}\,t)$ can be derived as follows. By Jensen's inequality and the convexity of f, for every path of X that evolves in $[-1,1]$,

$$\begin{aligned} f\Big(\frac{1}{T}\int_0^T X_t \,\mathrm{d}\,t\Big) &\leq \int_0^T f(X_t)\,\frac{\mathrm{d}\,t}{T} \leq \int_0^T \big(f(X_T) - f'(X_t)(X_T - X_t)\big)\,\frac{\mathrm{d}\,t}{T} \\ &= f(X_T) - \int_0^T (X_T - X_t)f'(X_t)\,\frac{\mathrm{d}\,t}{T}. \end{aligned}$$

Comparing this with (4.2.8), a semi-static superhedge for the Asian-style derivative is obtained from a European-style derivative with payoff $f(X_T)$ maturing at T and a dynamic trading strategy H with $H_0 = 0$ and dynamics $\mathrm{d}\,H_t = -f'(X_t)\frac{\mathrm{d}\,t}{T}$. Then H is of finite variation whenever X stays away from the boundaries of $(-1,1)$. But, as X approaches -1 or 1, the derivative $f'(X_t)$ becomes arbitrarily large (in absolute value), and H may fail to be of finite variation. It turns out, however, that the integral $\int_0^T (X_T - X_t)f'(X_t)\,\frac{\mathrm{d}\,t}{T}$ is still well defined on these paths. The reason is that when paths of X come arbitrarily close to 1, say, then for any martingale coupling $P \in \mathcal{M}(\mu,\nu)$, $X_T = 1$ P-a.s. on these paths (because $J = [-1,1]$), so that $X_T - X_t$ becomes small and counteracts the growth of $f'(X_t)$.

We shall define a pathwise stochastic integral for $\mathbb{F}$-adapted càdlàg integrators X and integrands $\hat{H}_-$ of the form

$$\hat{H}_t = h_0 + \int_{(0,t]} h_s \,\mathrm{d}\,Y_s \tag{4.2.9}$$

for an $\mathbb{F}$-adapted càdlàg process $Y = (Y_t)_{t\in[0,T]}$ of finite variation and an $\mathbb{F}$-adapted process $h = (h_t)_{t\in[0,T]}$—even in certain situations where the right-hand side of (4.2.9) is not finite. The idea is to formally substitute (4.2.9) into (4.2.8), formally use the associativity of Lebesgue–Stieltjes integrals, and then employ the resulting expression as a definition for a pathwise stochastic integral. In particular, this expression is P-indistinguishable from the stochastic integral of $\hat{H}_-$ with respect to X for any martingale measure P and can still be interpreted as the gains from self-financing trading in X according to the trading strategy $\hat{H}_-$.

We first introduce a set of integrands for this integral.

Definition 4.2.11. Let $\Omega' \subset D([0,T];\mathbb{R})$. We denote by $L(\Omega')$ the set of pairs (h, Y) consisting of an $\mathbb{F}$-adapted process h and an $\mathbb{F}$-adapted càdlàg process Y of finite variation such that the process $((X_T - X_t)h_t)_{t\in[0,T]}$ is $\mathrm{d}\,Y$-integrable on $(0,T]$ for each path in Ω'.

If Y is an $\mathbb{F}$-adapted càdlàg process of finite variation, then $(1, Y) \in L(\Omega')$ for any $\Omega' \subset D([0,T];\mathbb{R})$ (because any càdlàg function is bounded on compact intervals).

We fix a set $\Omega' \subset D([0,T];\mathbb{R})$ for the rest of this section.

Definition 4.2.12. For $H = (h, Y) \in L(\Omega')$, we set

$$H \diamond X_T := (X_T - X_0)h_0 + \int_{(0,T]} (X_T - X_t)h_t \,\mathrm{d}\,Y_t \quad \text{on} \quad \Omega'. \tag{4.2.10}$$

We note that the Lebesgue–Stieltjes integral on the right-hand side of (4.2.10) is well defined and finite by the definition of $L(\Omega')$. The following result shows that for pathwise bounded h, $H \diamond X_T$ coincides with $\hat{H}_- \bullet X_T$ for $\hat{H}$ as in (4.2.9).

Proposition 4.2.13. *Let $H = (h, Y) \in L(\Omega')$ and $\omega \in \Omega'$. If the function $t \mapsto h_t(\omega)$ is bounded on $[0, T]$, then*

$$(H \diamond X_T)(\omega) = (\hat{H}_- \bullet X_T)(\omega),$$

where $\hat{H} = h_0 + \int_{(0,\cdot]} h \, \mathrm{d}\, Y$.

If we set $h_0 = Y_0$ and $h_t = 1$ for $t \in (0, T]$ for an $\mathbb{F}$-adapted càdlàg process Y of finite variation, then $H = (h, Y) \in L(\Omega)$ and by Proposition 4.2.13,

$$H \diamond X_T = Y_- \bullet X_T \quad \text{on} \quad \Omega.$$

So the integral $H \diamond X_T$ embeds all pathwise stochastic integrals $Y_- \bullet X_T$.

Proof of Proposition 4.2.13. Since $h(\omega)$ is bounded on $[0, T]$, $\hat{H}_t(\omega) = h_0(\omega) + \int_{(0,t]} h_s(\omega) \, \mathrm{d}\, Y_s(\omega)$ is a well-defined càdlàg finite variation function on $[0, T]$. Thus, by (4.2.8),

$$\hat{H}_- \bullet X_T = (X_T - X_0)h_0 + \int_{(0,T]} (X_T - X_s) h_s \, \mathrm{d}\, Y_s = H \diamond X_T. \qquad \square$$

4.3 Robust pricing and superhedging problems

Throughout this section, we fix an irreducible pair $\mu \leq_c \nu$ with domain (I, J) and a Borel function $f : \mathbb{R} \to \overline{\mathbb{R}}$ which is bounded from below by a ν-integrable concave function.

Pricing problem

Our pricing and hedging duality applies to a wide range of exotic derivatives including American options, fixed strike Asian options, Bermudan options, and European options with an intermediate maturity. We now describe this class of derivative securities.

Definition 4.3.1. A nonnegative $\mathbb{F}$-adapted nondecreasing càdlàg process $A = (A_t)_{t\in[0,T]}$ is called an *averaging process* if $A_T(\omega) = 1$ for every $\omega \in \Omega$. If in addition $A_0(\omega) = 0$ and $\Delta A_T(\omega) = 0$ for each $\omega \in \Omega$, then A is called an *interior averaging process.* If in addition there is $t \in (0,T)$ such that $A_t(\omega) = 0$ for each $\omega \in \Omega$, then A is called a *strictly interior averaging process.*

Recall that we set $A_{0-} = 0$ and note that for each $\omega \in \Omega$, $A(\omega)$ can be identified with a Borel probability measure on $[0,T]$. If A is an interior averaging process, then this probability measure is supported on $(0,T)$, and if A is a strictly interior averaging process then its support is (uniformly in ω) contained in $[t,T)$ for some $t \in (0,T)$.

Given a nonempty set $\mathcal{A}$ of averaging processes, we consider a derivative security whose payoff at time T is

$$f\Big(\int_{[0,T]} X_t \,\mathrm{d}\, A_t\Big), \tag{4.3.1}$$

where $A \in \mathcal{A}$ is chosen by the buyer and the seller observes $(A_s)_{s\in[0,t]}$ at time t. Then the *robust model-based price* is defined as

$$\mathbf{S}_{\mu,\nu}(f,\mathcal{A}) = \sup_{P\in\mathcal{M}(\mu,\nu)} \sup_{A\in\mathcal{A}} E^P\left[f\Big(\int_{[0,T]} X_t \,\mathrm{d}\, A_t\Big)\right]. \tag{4.3.2}$$

In other words, $\mathbf{S}_{\mu,\nu}(f,\mathcal{A})$ is the highest model-based price of the derivative security (4.3.1) among all martingale models which are consistent with the given marginal distributions.

Remark 4.3.2. One can show that for each $P \in \mathcal{M}(\mu,\nu)$ and each averaging process A, the law of $\int_{[0,T]} X_t \,\mathrm{d}\, A_t$ under P is in convex order between μ and ν; cf. Lemma 4.4.1. Because f is by assumption bounded from below by a ν-integrable concave function, the expectations in (4.3.2) are well defined.

Important special cases are obtained for specific choices of $\mathcal{A}$.

Example 4.3.3 (No special exercise rights)**.** Setting $\mathcal{A} = \{A\}$ deprives the buyer of any special exercise rights and reduces (4.3.2) to the more familiar robust pricing problem

$$\sup_{P\in\mathcal{M}(\mu,\nu)} E^P[F]$$

for the derivative security $F = f(\int_{[0,T]} X_t \,\mathrm{d}\, A_t)$.

(i) *Asian options.* Setting $A_t = t/T$ recovers the Asian-style derivative $f(\frac{1}{T}\int_0^T X_t \,\mathrm{d}\, t)$; this includes fixed strike Asian puts and calls, but not floating strike Asian options. This robust pricing problem is analyzed in [5].

(ii) *European options.* Setting $A_t = \mathbf{1}_{[T',T]}(t)$ yields a European-style payoff $f(X_{T'})$ with an intermediate maturity $T' \in (0,T)$.

Example 4.3.4 (Special exercise rights)**.** Fix a nonempty set $\mathcal{T}$ of $[0,T]$-valued $\mathbb{F}$-stopping times, and consider $\mathcal{A} = \{\mathbf{1}_{[\![\tau,T]\!]} : \tau \in \mathcal{T}\}$. Then (4.3.2) reduces to

$$\sup_{P\in\mathcal{M}(\mu,\nu)} \sup_{\tau\in\mathcal{T}} E^P\left[f(X_\tau)\right]. \tag{4.3.3}$$

(i) *American options.* If $\mathcal{T}$ consists of all $[0,T]$-valued $\mathbb{F}$-stopping times, then (4.3.3) is the robust American option pricing problem analyzed in [38].

(ii) *Bermudan options.* Bermudan options with exercise dates $0 \le T_1 < \cdots < T_n \le T$ are covered by choosing $\mathcal{T}$ to be the set of $\{T_1,\dots,T_n\}$-valued $\mathbb{F}$-stopping times.

Superhedging problem

In the case of robust semi-static superhedging of American options, it is well known that a pricing-hedging duality can in general only hold if the seller of the option can adjust the dynamic part of his trading strategy after the option has been exercised; cf. [6, Section 3]. In other words, the buyer has to communicate her decision of exercising to the seller at the time of exercising. The analog in our setting is that the seller observes A_t at time t. That is, his trading strategy can be "adapted" to the averaging process chosen by the buyer.

To make this precise, let $\hat{\Omega}$ be the cartesian product of Ω and the set of non-negative, nondecreasing, càdlàg functions $a : [0,T] \to [0,1]$ with $a(T) = 1$. As $\hat{\Omega}$

is a subspace of the Skorokhod space $D([0,T];\mathbb{R}\times[0,1])$, we can equip it with the subspace Skorokhod topology and denote by $\hat{\mathcal{F}}$ the corresponding Borel σ-algebra. We write $\hat{\mathbb{F}}=(\hat{\mathcal{F}}_t)_{t\in[0,T]}$ for the (raw) filtration generated by the canonical process on $\hat{\Omega}$. For any process Z on $\hat{\Omega}$ and any averaging process A (on Ω), we define the process Z^A on Ω by

$$Z_t^A(\omega)=Z_t(\omega,A(\omega)),\quad \omega\in\Omega.$$

Note that if Z is $\hat{\mathbb{F}}$-adapted, then Z^A is $\mathbb{F}$-adapted, and if Z is càdlàg or of finite variation, then so is Z^A.

Next, we define a suitable set of paths for the hedging problem. Let $\Omega_{\mu,\nu}\subset\Omega$ denote the subset of paths which start in I, evolve in J, and are "captured" if they approach the boundary ∂J:

$$\begin{aligned}\Omega_{\mu,\nu}:=\{\omega\in\Omega:\ &\omega_0\in I,\ \omega_t\in J \text{ for all } t\in(0,T],\\ &\text{if } \omega_{t-}\in\partial J,\text{ then } \omega_u=\omega_{t-} \text{ for all } u\in[t,T],\text{ and}\\ &\text{if } \omega_t\in\partial J,\text{ then } \omega_u=\omega_t \text{ for all } u\in[t,T]\}.\end{aligned}\tag{4.3.4}$$

One can show that every martingale coupling between μ and ν is concentrated on $\Omega_{\mu,\nu}$:[5]

Lemma 4.3.5. $\Omega_{\mu,\nu}\in\mathcal{F}$ *and* $P[\Omega_{\mu,\nu}]=P[\Omega]$ *for every* $P\in\mathcal{M}(\mu,\nu)$.

[5]The fact that μ and ν are concentrated on I and J, respectively, together with the martingale property implies that P-a.e. path has the first two properties in (4.3.4). The other two properties can be shown similarly to the fact that nonnegative supermartingales are almost surely captured in zero (cf. [64, Lemma 7.31]).

We are now ready to define the trading strategies for the robust superhedging problem.

Definition 4.3.6. A *semi-static trading strategy* is a triplet (φ, ψ, H) consisting of a pair of functions $(\varphi, \psi) \in L^c(\mu, \nu)$ and a pair $H = (h_t, Y_t)_{t \in [0,T]}$ of $\hat{\mathbb{F}}$-adapted processes on $\hat{\Omega}$ such that

$$H^A := (h^A, Y^A) \in L(\Omega_{\mu,\nu}) \quad \text{for every averaging process } A. \tag{4.3.5}$$

The portfolio value at time T of a semi-static trading strategy is given by the sum of the static part with payoffs $\varphi(X_0)$ and $\psi(X_T)$ and the gains $H^A \diamond X_T$ of the dynamic part:

$$\varphi(X_0) + \psi(X_T) + H^A \diamond X_T. \tag{4.3.6}$$

The initial cost to set up this position is equal to the initial price of the static part:

$$\mu(\varphi) + \nu(\psi). \tag{4.3.7}$$

We now turn our attention to semi-static trading strategies which dominate the payoff (4.3.1) of our derivative security for each path in $\Omega_{\mu,\nu}$ and every averaging process in $\mathcal{A}$.

Definition 4.3.7. A semi-static trading strategy (φ, ψ, H) is called a *semi-static* *superhedge (for f and $\mathcal{A}$)* if for every $A \in \mathcal{A}$,

$$f\Big(\int_{[0,T]} X_t \,\mathrm{d}\, A_t\Big) \leq \varphi(X_0) + \psi(X_T) + H^A \diamond X_T \quad \text{on} \quad \Omega_{\mu,\nu} \tag{4.3.8}$$

and

$$E^P\left[\varphi(X_0) + \psi(X_T) + H^A \diamond X_T\right] \leq \mu(\varphi) + \nu(\psi), \quad P \in \mathcal{M}(\mu,\nu). \tag{4.3.9}$$

The set of semi-static superhedges for f and $\mathcal{A}$ is denoted by $\mathcal{D}_{\mu,\nu}(f,\mathcal{A})$.

The requirement (4.3.9) is an admissibility condition. It demands that for every $P \in \mathcal{M}(\mu,\nu)$, the portfolio value, consisting of both the static and the dynamic part, is a one-step P-supermartingale between the time at which the static part is set up and time T. In other words, the expectation of the terminal portfolio value (4.3.6) is less than or equal to the initial portfolio value (4.3.7).

We define the *robust superhedging price (for f and $\mathcal{A}$)* as the "minimal" initial capital required to set up a semi-static superhedge for f and $\mathcal{A}$:[6]

$$\mathbf{I}_{\mu,\nu}(f,\mathcal{A}) = \inf_{(\varphi,\psi,H)\in\mathcal{D}_{\mu,\nu}(f,\mathcal{A})} \{\mu(\varphi) + \nu(\psi)\}. \tag{4.3.10}$$

Weak and strong duality

Weak duality between the robust pricing and hedging problems is an immediate consequence of their definitions:

[6]We use the convention $\inf \emptyset = \infty$.

Lemma 4.3.8 (Weak duality)**.** *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be Borel and bounded from below by a ν-integrable concave function and let $\mathcal{A}$ be a nonempty set of averaging processes. Then*

$$\mathbf{S}_{\mu,\nu}(f, \mathcal{A}) \leq \mathbf{I}_{\mu,\nu}(f, \mathcal{A}).$$

Proof. Let $P \in \mathcal{M}(\mu, \nu)$, $A \in \mathcal{A}$, and $(\varphi, \psi, H) \in \mathcal{D}_{\mu,\nu}(f, \mathcal{A})$ (there is nothing to show if this set is empty). Taking P-expectations in (4.3.8) and using (4.3.9) shows that $E^P\left[f(\int_{[0,T]} X_t \,\mathrm{d}\, A_t)\right] \leq \mu(\varphi) + \nu(\psi)$.

This proves the claim as P, A, and (φ, ψ, H) were arbitrary. □

With an additional mild assumption on either $\mathcal{A}$ or f, we obtain strong duality and the existence of dual minimizers:

Theorem 4.3.9. *Let $\mu \leq_c \nu$ be irreducible, let $f : \mathbb{R} \to [0, \infty]$ be Borel, and let $\mathcal{A}$ be a set of averaging processes. Suppose that one of the following two conditions holds:*

- *f is lower semicontinuous and $\mathcal{A}$ contains an interior averaging process;*
- *$\mathcal{A}$ contains a strictly interior averaging process.*

Then

$$\mathbf{S}_{\mu,\nu}(f, \mathcal{A}) = \mathbf{I}_{\mu,\nu}(f, \mathcal{A}) \in [0, \infty]$$

and this value is independent of $\mathcal{A}$ as long as one of the two conditions above holds. Moreover, if $\mathbf{I}_{\mu,\nu}(f, \mathcal{A}) < \infty$, then there exists an optimizer $(\varphi, \psi, H) \in \mathcal{D}_{\mu,\nu}(f, \mathcal{A})$ for $\mathbf{I}_{\mu,\nu}(f, \mathcal{A})$.

Remark 4.3.10.

(i) For fixed f, the robust model-based price $\mathbf{S}_{\mu,\nu}(f, \mathcal{A})$ is invariant under the choice of the set $\mathcal{A}$ (as long as the assumptions of Theorem 4.3.9 hold). In particular, American, Bermudan, and European options with intermediate maturity (cf. Examples 4.3.3–4.3.4) all have the same robust model-based price (because the corresponding sets $\mathcal{A}$ all contain a strictly interior averaging process). If f is lower semicontinuous, this extends to the Asian-style option of Example 4.3.3 (i). If more than two marginals are given, then the robust model-based prices of these derivatives typically differ; see Example 4.6.3.

(ii) Derivatives of the form (4.3.1) that depend distinctly on X_0 and/or X_T such as $f(\frac{1}{2}(X_0 + X_T))$ are not covered by Theorem 4.3.9 ($\mathcal{A}$ does not contain an *interior* averaging process). In these cases, the robust model-based price is still bounded above by the corresponding robust model-based price of, say, the European-style derivative $f(X_{T/2})$. However, the inequality is typically strict; see Example 4.6.4.

Remark 4.3.11.

(i) Theorem 4.3.9 can be extended to non-irreducible marginals along the lines of [16, Section 7].

(ii) Strong duality continues to hold if we restrict ourselves to finite variation strategies; cf. Remark 4.4.15 for an outline of the argument. It is an open question whether there is (in general) a dual minimizer (φ, ψ, H) with a dynamic part H of finite variation.

We defer the proof of Theorem 4.3.9 to the end of Section 4.4. The idea is as follows. We bound the pricing problem from below and the hedging problem from above by auxiliary maximization and minimization problems, respectively, and show that strong duality holds between those two auxiliary problems. Then all four problems have equal value and in particular strong duality for the pricing and hedging problems holds. Moreover, we show that the auxiliary dual problem admits a minimizer and that every element in the dual space of the auxiliary problem gives rise to a semi-static superhedge with the same cost. Then, in particular, the minimizer of the auxiliary dual problem yields an optimal semi-static superhedge for f and $\mathcal{A}$ (which is independent of $\mathcal{A}$).

4.4 Auxiliary problems

Throughout this section, we fix an irreducible pair $\mu \leq_c \nu$ with domain (I, J) and a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ which is bounded from below by a ν-integrable concave function.

The auxiliary primal and dual problems are formally derived in Section 4.4. They are rigorously introduced in Sections 4.4–4.4 and proved to be lower and upper bounds of the robust model-based price and the robust superhedging price, respectively. Their strong duality is proved in Section 4.4. Finally, structural properties of primal and dual optimizers of the auxiliary problems are studied in Section 4.4.

Motivation

The key property of payoffs of the form (4.3.1) is that the law of $\int_{[0,T]} X_t \,\mathrm{d}\, A_t$ under $P \in \mathcal{M}(\mu, \nu)$ is in convex order between μ and ν. In this section, we explain this observation and how it can be used to estimate the robust pricing problem from below and the robust superhedging problem from above.

Let $P \in \mathcal{M}(\mu, \nu)$ and let τ be a $[0, T]$-valued $\mathbb{F}$-stopping time. An application of the optional stopping theorem and Jensen's inequality shows that for any convex function ψ,

$$\mu(\psi) = E^P\left[\psi(X_0)\right] = E^P\left[\psi(E^P\left[X_\tau \mid \mathcal{F}_0\right])\right] \leq E^P\left[\psi(X_\tau)\right] \qquad \text{and}$$

$$\nu(\psi) = E^P\left[\psi(X_T)\right] \geq E^P\left[\psi(E^P\left[X_T \mid \mathcal{F}_\tau\right])\right] = E^P\left[\psi(X_\tau)\right],$$

so that the law of X_τ under P is in convex order between μ and ν.

Using a time change argument and again Jensen's inequality and the optional stopping theorem, it can be shown that this property generalizes to the random variable $\int_{[0,T]} X_t \,\mathrm{d}\, A_t$ for an averaging process A.

Lemma 4.4.1. *Let $P \in \mathcal{M}(\mu, \nu)$ and let A be an averaging process. Then the law of $\int_{[0,T]} X_t \,\mathrm{d}\, A_t$ under P is in convex order between μ and ν.*

In the sequel, we write $\mathbf{S} = \mathbf{S}_{\mu,\nu}(f, \mathcal{A})$ and $\mathbf{I} = \mathbf{I}_{\mu,\nu}(f, \mathcal{A})$ for brevity. Lemma 4.4.1 implies that

$$\mathbf{S} \leq \sup_{\mu \leq_c \theta \leq_c \nu} \theta(f) =: \widetilde{\mathbf{S}}.$$

We show in Section 4.4 that also the converse inequality holds under mild assumptions on f and $\mathcal{A}$. Thus, $\mathbf{S} = \widetilde{\mathbf{S}}$ and one is led to expect that $\mathbf{I} = \widetilde{\mathbf{I}}$ for a suitable dual problem $\widetilde{\mathbf{I}}$ to $\widetilde{\mathbf{S}}$.

Let us thus formally derive the Lagrange dual problem for $\widetilde{\mathbf{S}}$. Dualizing the constraint $\mu \leq_c \theta \leq_c \nu$ suggests to consider the Lagrangian

$$L(\theta, \psi_1, \psi_2) := \theta(f) + (\theta(\psi_1) - \mu(\psi_1)) + (\nu(\psi_2) - \theta(\psi_2)), \tag{4.4.1}$$

where *convex* functions ψ_1, ψ_2 are taken as Lagrange multipliers.[7] Then the Lagrange dual problem is

$$\widetilde{\mathbf{I}} = \inf_{\psi_1,\psi_2} \sup_{\theta} L(\theta, \psi_1, \psi_2) = \inf_{\psi_1,\psi_2} \sup_{\theta} \{\theta(f + \psi_1 - \psi_2) - \mu(\psi_1) + \nu(\psi_2)\}$$

where the infima are taken over convex functions and the suprema are taken over finite measures. Viewing the finite measure θ as a Lagrange multiplier for the constraint $f \leq -\psi_1 + \psi_2$ and relabeling $\varphi = -\psi_1$ and $\psi = \psi_2$, we obtain

$$\widetilde{\mathbf{I}} = \inf\{\mu(\varphi) + \nu(\psi) : \varphi \text{ concave, } \psi \text{ convex, and } \varphi + \psi \geq f\}. \tag{4.4.2}$$

In the precise definition of $\widetilde{\mathbf{I}}$ in Section 4.4, $\mu(\varphi)+\nu(\psi)$ is understood in the generalized sense of Definition 4.2.5 and the inequality $\varphi+\psi \geq f$ is required to hold on J. We then show that each feasible element (φ, ψ) for $\widetilde{\mathbf{I}}$ entails an element $(\varphi, \psi, H) \in \mathcal{D}_{\mu,\nu}(f, \mathcal{A})$

[7] Note that the last two terms in (4.4.1) are nonnegative for all convex ψ_1, ψ_2 if and only if the primal constraint $\mu \leq_c \theta \leq_c \nu$ holds.

(Proposition 4.4.7), which implies that $\mathbf{I} \leq \widetilde{\mathbf{I}}$.

Combining the above with the weak duality inequality (Lemma 4.3.8) yields

$$\widetilde{\mathbf{S}} = \mathbf{S} \leq \mathbf{I} \leq \widetilde{\mathbf{I}}.$$

Hence, strong duality and dual attainment for the robust pricing and superhedging problems reduce to the same assertions for the simpler auxiliary problems, which are proved in Section 4.4.

Auxiliary primal problem

Consider the *auxiliary primal problem*

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \sup_{\mu \leq_c \theta \leq_c \nu} \theta(f), \tag{4.4.3}$$

where $\theta(f)$ is understood as the outer integral if f is not Borel-measurable. Under suitable conditions on f and $\mathcal{A}$, the primal value $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$ is a lower bound for the robust model-based price (4.3.2):

Proposition 4.4.2. *Let $\mathcal{A}$ be a set of averaging processes. Suppose that one of the following two sets of conditions holds:*

(i) $\mathcal{A}$ contains an interior averaging process and f is lower semicontinuous and bounded from below by a ν-integrable concave function $\varphi : J \to \overline{\mathbb{R}}$;

(ii) $\mathcal{A}$ contains a strictly interior averaging process and f is Borel.

Then

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f) \leq \mathbf{S}_{\mu,\nu}(f, \mathcal{A}).$$

The proof of Proposition 4.4.2 is given at the end of this section. It is based on the following construction of measures in $\mathcal{M}(\mu,\nu)$ under which the law of $\int_{[0,T]} X_t \,\mathrm{d}\, A_t$ equals (approximately or exactly) a given θ. This construction also highlights the importance of $\mathcal{A}$ containing an *interior* averaging process, which does not put any mass on the times 0 and T at which the marginal distributions of X are given; see Example 4.6.4 for a counterexample.

Lemma 4.4.3. *Let $\mu \leq_c \theta \leq_c \nu$.*

(i) There is a sequence $(P_n)_{n\geq 1} \subset \mathcal{M}(\mu,\nu)$ such that

$$\mathcal{L}^{P_n}\Big(\int_{[0,T]} X_t \,\mathrm{d}\, A_t\Big) \xrightarrow{n\to\infty} \theta \quad \textit{weakly}$$

for every interior averaging process A.

(ii) If A is a strictly interior averaging process, then there is $P \in \mathcal{M}(\mu,\nu)$ (depending on A) such that $\mathcal{L}^{P}\big(\int_{[0,T]} X_t \,\mathrm{d}\, A_t\big) = \theta$.

Proof. (i): By the two-step adaptation of Proposition 4.2.1, there exists a measure $Q \in \mathcal{M}^d(\mu,\theta,\nu)$. For all n large enough, let $\iota^n : \mathbb{R}^3 \to \Omega$ be the embedding of $\mathbb{R}^3$ in Ω which maps (y_1,y_2,y_3) to the piecewise constant path

$$[0,T] \ni t \mapsto y_1 \mathbf{1}_{[0,\frac{1}{n})}(t) + y_2 \mathbf{1}_{[\frac{1}{n},T)}(t) + y_3 \mathbf{1}_{\{T\}}(t) \tag{4.4.4}$$

(which jumps (at most) at times $\frac{1}{n}$ and T), and denote by $P_n := Q \circ (\iota^n)^{-1}$ the associated pushforward measure. Then $P_n \in \mathcal{M}(\mu, \nu)$ by the corresponding properties of Q. Moreover, denoting the canonical process on $\mathbb{R}^3$ by (Y_1, Y_2, Y_3) and setting $A^n = A \circ \iota^n$ for an interior averaging process A, we have

$$\begin{aligned}
\int_{[0,T]} (\iota^n)_t \,\mathrm{d}\, A^n_t - Y_2 &= Y_1 A^n_{\frac{1}{n}-} + Y_2(A^n_{T-} - A^n_{\frac{1}{n}-}) + Y_3 \Delta A^n_T - Y_2 A_T \\
&= (Y_1 - Y_2) A^n_{\frac{1}{n}-} + (Y_3 - Y_2)\Delta A^n_T \qquad (4.4.5) \\
&= (Y_1 - Y_2) A^n_{\frac{1}{n}-} \quad \text{on} \quad \mathbb{R}^3,
\end{aligned}$$

where we use the properties $A_T = 1$ and $\Delta A_T = 0$ of an interior averaging process.

By construction, the law of $\int_{[0,T]} (\iota^n)_t \,\mathrm{d}\, A^n_t$ under Q coincides with the law of $\int_{[0,T]} X_t \,\mathrm{d}\, A_t$ under P_n and the law of Y_2 under Q is θ. It thus suffices to prove that the right-hand side in (4.4.5) converges to zero in $L^1(Q)$ as $n \to \infty$. To this end, note that $|Y_1 - Y_2| \leq |Y_1| + |Y_2|$ is Q-integrable because μ and θ have finite first moments. Thus, by dominated convergence, it is enough to show that $A^n_{\frac{1}{n}-} \to 0$ pointwise as $n \to 0$. So fix $(y_1, y_2, y_3) \in \mathbb{R}^3$. Since A is $\mathbb{F}$-adapted, $A_{\frac{1}{n}-}(\omega)$ only depends on the values of the path ω on the interval $[0, \frac{1}{n})$. In view of the embedding (4.4.4), this means that

$$A^n_{\frac{1}{n}-}(y_1, y_2, y_3) = A_{\frac{1}{n}-}(\iota^n(y_1, y_2, y_3)) = A_{\frac{1}{n}-}(y_1 \mathbf{1}_{[0,T]}),$$

where $y_1 \mathbf{1}_{[0,T]}$ denotes the constant path at y_1. Hence, the asserted pointwise convergence follows from the fact that $A_0 = 0$ and A is right-continuous.

(ii): If A is a strictly interior averaging process, then the last expression in (4.4.5) is identically zero for n large enough and setting $P = P_n$ gives the desired result. $\square$

Remark 4.4.4. Part (i) of Lemma 4.4.3 remains true if we restrict ourselves to martingale measures with almost surely continuous paths. The analog of part (ii) for continuous martingales requires the additional assumption that there exists $t < T$ such that $A_t \equiv 1$.

The main ingredient for this assertion is [78, Theorem 11]: for every discrete time-martingale $\{Y_n\}_{n\geq 0}$, there is a continuous-time martingale $\{Z_t\}_{t\geq 0}$ with continuous sample paths such that the processes $\{Y_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ have the same (joint) distribution.

Proof of Proposition 4.4.2. Let $\mu \leq_c \theta \leq_c \nu$. Assume first that condition (ii) holds and let A be a strictly interior averaging process. Then by Lemma 4.4.3 (ii), there is $P \in \mathcal{M}(\mu, \nu)$ such that $\mathcal{L}^P(\int_{[0,T]} X_t \, \mathrm{d}\, A_t) = \theta$. Hence,

$$\theta(f) = E^P\left[f\Big(\int_{[0,T]} X \, \mathrm{d}\, A\Big)\right] \leq \mathbf{S}_{\mu,\nu}(f, \mathcal{A}).$$

As θ was arbitrary, the claim follows.

Next, assume instead that condition (i) holds and let A be an interior averaging process and φ as in condition (i). By Lemma 4.4.3 (i), there is a sequence $(P_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\mu, \nu)$ such that $\theta_n := \mathcal{L}^{P_n}(\int_{[0,T]} X_t \, \mathrm{d}\, A_t) \to \theta$ weakly. Define $f_k = f \vee (-k)$, $k \geq 1$. Then f_k is bounded from below and lower semicontinuous, so $\liminf_{n\to\infty} \theta_n(f_k) \geq \theta(f_k)$ by the Portmanteau theorem.

Fix $\varepsilon > 0$. Choose first k large enough such that $\nu((\varphi+k)^-) \le \frac{\varepsilon}{2}$ and then N large enough such that $\theta_n(f_k) - \theta(f_k) \ge -\frac{\varepsilon}{2}$ for all $n \ge N$. Using that $0 \le f_k - f \le (\varphi+k)^-$ and that $(\varphi + k)^-$ is convex, we obtain for $n \ge N$,

$$\begin{aligned}\theta_n(f) - \theta(f) &= \theta_n(f - f_k) + (\theta_n(f_k) - \theta(f_k)) + \theta(f_k - f)\\ &\ge -\theta_n((\varphi + k)^-) - \frac{\varepsilon}{2} \ge -\nu((\varphi + k)^-) - \frac{\varepsilon}{2} \ge -\varepsilon.\end{aligned}$$

Thus, $\liminf_{n\to\infty} \theta_n(f) \ge \theta(f)$. Now the claim follows from

$$\theta(f) \le \liminf_{n\to\infty} \theta_n(f) = \liminf_{n\to\infty} E^{P_n}\left[f\Big(\int_{[0,T]} X_t \,\mathrm{d}\, A_t\Big)\right] \le \mathbf{S}_{\mu,\nu}(f, \mathcal{A}). \qquad \square$$

Auxiliary dual problem

Consider the *auxiliary dual problem*

$$\widetilde{\mathbf{I}}_{\mu,\nu}(f) = \inf_{(\varphi,\psi)\in\widetilde{\mathcal{D}}_{\mu,\nu}(f)} \{\mu(\varphi) + \nu(\psi)\}, \tag{4.4.6}$$

where $\widetilde{\mathcal{D}}_{\mu,\nu}(f)$ denotes the set of $(\varphi, \psi) \in L^c(\mu, \nu)$ with concave $\varphi : J \to \overline{\mathbb{R}}$ and convex $\psi : J \to \overline{\mathbb{R}}$ such that $\varphi + \psi \ge f$ on J.

The dual value $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$ is an upper bound for the robust superhedging price (4.3.10):

Proposition 4.4.5. *Let $f : \mathbb{R} \to [0, \infty]$ be Borel. Then $\mathbf{I}_{\mu,\nu}(f, \mathcal{A}) \le \widetilde{\mathbf{I}}_{\mu,\nu}(f)$.*

Proposition 4.4.5 follows immediately from the next result (Proposition 4.4.7) which shows that every $(\varphi, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ gives rise to a semi-static superhedge for f and $\mathcal{A}$. More precisely, the semi-static superhedge is of the form (φ, ψ, H) and the dynamic part H can be explicitly written in terms of the "derivatives" of φ and ψ.

Given a convex function $\psi : J \to \mathbb{R}$, a Borel function $\psi' : I \to \mathbb{R}$ is called a *subderivative* of ψ if for every $x_0 \in I$, $\psi'(x_0)$ belongs to the subdifferential of ψ at x_0, i.e.,

$$\psi(x) - \psi(x_0) \geq \psi'(x_0)(x - x_0), \quad x \in J.$$

Symmetrically, for a concave function $\varphi : J \to \mathbb{R}$, a Borel function $\varphi' : I \to \mathbb{R}$ is called a *superderivative* of φ if $-\varphi'$ is a subderivative of $-\varphi$.

Remark 4.4.6. If $(\varphi, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ and $f > -\infty$ on J, then φ and ψ are both finite (so that sub- and superderivatives are well defined). Indeed, we already know from Remark 4.2.8 that ψ is finite on J. Moreover, $\varphi < \infty$ on J by Lemma 4.2.7 (i) and if $f > -\infty$ on J, then $\varphi \geq f - \psi > -\infty$, so that also φ is finite on J.

Proposition 4.4.7. *Let $f : \mathbb{R} \to [0, \infty]$ be Borel and let $(\varphi, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$. Denoting the canonical process on $\hat{\Omega}$ by (X, A), define the $\hat{\mathbb{F}}$-adapted process $h = (h_t)_{t \in [0,T]}$ (on $\hat{\Omega}$) by*

$$\begin{aligned} h_0 &= \varphi'(X_0)(1 - A_0) - \psi'(X_0)A_0, \\ h_t &= -\varphi'(X_0) - \psi'(X_t), \quad t \in (0, T], \end{aligned} \tag{4.4.7}$$

where φ' is any superderivative of φ, ψ' is any subderivative of ψ, and we set $\varphi' = \psi' = 0$ on $\mathbb{R} \setminus I$. Set $H = (h, A)$. Then $(\varphi, \psi, H) \in \mathcal{D}_{\mu,\nu}(f, \mathcal{A})$ for any nonempty set

$\mathcal{A}$ *of averaging processes.*

The proof of Proposition 4.4.7, given at the end of this section, relies on the following two technical lemmas. The definition of $\Omega_{\mu,\nu}$ in (4.3.4) is crucial for the first one. We recall that (real-valued) càdlàg functions are bounded on compact intervals.

Lemma 4.4.8. *Let ψ and ψ' be as in Proposition 4.4.7. For each $\omega \in \Omega_{\mu,\nu}$, the function $[0,T] \ni t \mapsto (\omega_T - \omega_t)\psi'(\omega_t)$ is bounded.*

Proof. Fix $\omega \in \Omega_{\mu,\nu}$ and write $I = (l, r)$ with $l, r \in \overline{\mathbb{R}}$. We consider three cases: (i) $J = I$, (ii) $J = [l, r)$, and (iii) $J = [l, r]$. The case $(l, r]$ is symmetric to (ii).

(i): Suppose that $J = I = (l, r)$. We claim that ω evolves in a compact (and hence strict) subset of I. Suppose for the sake of contradiction hat $\inf_{t\in[0,T]} \omega_t = l \in [-\infty, \infty)$. Then there is a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, T]$ such that $\lim_{n\to\infty} \omega_{t_n} = l$. Passing to a subsequence if necessary, this sequence may be chosen to be either (strictly) increasing or nonincreasing to a limit $t^\star \in [0, T]$. Then, as ω is càdlàg, $\omega_{t^\star-} = l$ or $\omega_{t^\star} = l$. But then $\omega_{t^\star} = l$ in any case by the definition of $\Omega_{\mu,\nu}$, a contradiction to $\omega_{t^\star} \in J = I$. Thus, $\inf_{t\in[0,T]} \omega_t > l$ and symmetrically $\sup_{t\in[0,T]} \omega_t < r$. This proves the claim. It follows that $(\omega_T - \omega_t)\psi'(\omega_t)$ is bounded over $t \in [0, T]$ because the subderivative ψ' is bounded on compact subsets of I.

(ii): Suppose that $J = [l, r)$, i.e., ν has an atom in $l > -\infty$. If ω evolves in I, then we can argue as in (i). We may thus assume that $t^\star := \inf\{t \in [0, T] : \omega_t = l\} \in (0, T]$. Then, as ω is càdlàg and by the definition of $\Omega_{\mu,\nu}$, we have $\omega_u = l$ for all $u \in [t^\star, T])$.

In particular, $\omega_T = l$ and it is enough to show that $[0, t^\star) \ni t \mapsto (\omega_T - \omega_t)\psi'(\omega_t)$ is bounded.

We can argue similarly as in (i) that $r' := \sup_{t\in[0,T]} \omega_t < r$, so that the path ω evolves in the compact interval $[l, r']$. Because ψ is convex and finite on (l, r), ψ' is bounded from above on $[l, r']$. It follows that $t \mapsto (\omega_T - \omega_t)\psi'(\omega_t)$ is bounded from below on $[0, t^\star)$. To show that this function is also bounded from above, we observe that by the convexity of ψ,

$$(\omega_T - \omega_t)\psi'(\omega_t) \leq \psi(\omega_T) - \psi(\omega_t) = \psi(l) - \psi(\omega_t). \tag{4.4.8}$$

Now $\psi(l)$ is finite because ν has an atom at l, and ψ is bounded from below on $[l, r']$ because it is finite and convex on the compact interval $[l, r']$. Using this in (4.4.8) shows the assertion.

(iii): Suppose that $J = [l, r]$, i.e., ν has atoms at $l > -\infty$ and $r < \infty$. As in (ii), we may assume that ω hits one of the endpoints of J before T. By symmetry, we may assume that ω hits l. By definition of $\Omega_{\mu,\nu}$, the path ω is then bounded away from the right endpoint r (otherwise it would be captured in r), i.e., $\sup_{t\in[0,T]} \omega_t < r$. Now the same argument as in (ii) proves the assertion. $\square$

The second technical lemma is an adaptation of [16, Remark 4.10] to our setting. It is used to show the admissibility condition (4.3.9) of the semi-static trading strategy in Proposition 4.4.7.

Lemma 4.4.9. *Let $(\varphi, \psi) \in L^c(\mu, \nu)$ and let $g_0, g_1 : J \to \mathbb{R}$ be Borel. Let τ be a*

$[0,T]$*-valued* $\mathbb{F}$*-stopping time such that*

$$\varphi(X_0) + \psi(X_T) + g_0(X_0)(X_\tau - X_0) + g_1(X_\tau)(X_T - X_\tau) \tag{4.4.9}$$

is bounded from below on $\Omega_{\mu,\nu}$*. Then for all* $P \in \mathcal{M}(\mu,\nu)$*,*

$$E^P\left[\varphi(X_0) + \psi(X_T) + g_0(X_0)(X_\tau - X_0) + g_1(X_\tau)(X_T - X_\tau)\right] = \mu(\varphi) + \nu(\psi).$$

Proof. Let χ be a concave moderator for (φ,ψ) with respect to $\mu \leq_c \nu$ and let θ be the law of X_τ. By optional stopping, $\mu \leq_c \theta \leq_c \nu$. We expand (4.4.9) to

$$\begin{aligned}(\varphi - \chi)(X_0) + (\psi + \chi)(X_T) + [\chi(X_0) - \chi(X_T) \\ + g_0(X_0)(X_\tau - X_0) + g_1(X_\tau)(X_T - X_\tau)],\end{aligned} \tag{4.4.10}$$

and observe that the first two terms are P-integrable. Then the assumed lower bound yields that the last term has a P-integrable negative part. We can therefore apply Fubini's theorem and evaluate its integral iteratively. To this end, let Q be the law of (X_0, X_τ, X_T) on the canonical space $\mathbb{R}^3$ with a disintegration

$$\mathrm{d}\,Q = \mu(\mathrm{d}\,x_0) \otimes \kappa_0(x_0, \mathrm{d}\,x_1) \otimes \kappa_1(x_0, x_1, \mathrm{d}\,x_2)$$

for martingale kernels κ_0 and κ_1. In view of the definition of $\mu(\varphi) + \nu(\psi)$ in (4.2.4), we have to show that the P-expectation of the last term in (4.4.10) is $(\mu - \nu)(\chi)$.

To this end, we observe that for $\mu \otimes \kappa_0$-a.e. $(x_0, x_1) \in J^2$,

$$\begin{aligned}\int_J &[\chi(x_0) - \chi(x_2) + g_0(x_0)(x_1 - x_0) + g_1(x_1)(x_2 - x_1)] \; \kappa_1(x_0, x_1, \mathrm{d}\, x_2) \\ &= \int_J [\chi(x_0) - \chi(x_2) + g_0(x_0)(x_1 - x_0)] \; \kappa_1(x_0, x_1, \mathrm{d}\, x_2) \qquad (4.4.11) \\ &= \chi(x_0) - \int_J \chi(x_2)\kappa_1(x_0, x_1, \mathrm{d}\, x_2) + g_0(x_0)(x_1 - x_0).\end{aligned}$$

Integrating the left-hand side of (4.4.11) against $\mu \otimes \kappa_0$ gives the P-expectation of the last term in (4.4.10). It thus remains to show that the corresponding integral of the right-hand side equals $(\mu - \nu)(\chi)$. Integrating the right-hand side of (4.4.11) first against $\kappa_0(x_0, \mathrm{d}\, x_1)$ yields for μ-a.e. $x_0 \in J$,

$$\chi(x_0) - \int_J \chi(x_2)\, \kappa(x_0, \mathrm{d}\, x_2), \qquad (4.4.12)$$

where $\kappa(x_0, \cdot) = \int_J \kappa_1(x_1, \cdot)\, \kappa_0(x_0, \mathrm{d}\, x_1)$ is again a martingale kernel. Finally, the integral of (4.4.12) against μ is

$$\int_J \left[\chi(x_0) - \int_J \chi(x_2)\, \kappa(x_0, \mathrm{d}\, x_2)\right] \mu(\mathrm{d}\, x_0).$$

Noting that $\mu \otimes \kappa$ is a disintegration of a one-step martingale measure on $\mathbb{R}^2$ with marginals μ and ν, the last term equals $(\mu - \nu)(\chi)$ by Lemma 4.2.4. $\square$

Proof of Proposition 4.4.7. First, we show that (φ, ψ, H) is a semi-static trading strategy. As h and A are clearly $\hat{\mathbb{F}}$-adapted and $(\varphi, \psi) \in L^c(\mu, \nu)$ by assumption, it remains to check condition (4.3.5) (with Y^A replaced by A). So fix an averaging process A and note that $H^A = (h^A, A)$. The only nontrivial part in proving $H^A \in L(\Omega_{\mu,\nu})$ is to show that $(X_T - X_t)h_t^A$ is $\mathrm{d}\,A$-integrable on $(0,T]$ for each path in $\Omega_{\mu,\nu}$. To this end, note that $\varphi'(X_0)$ and $\psi'(X_0)$ are finite because $X_0 \in I$. It thus suffices to show that $(X_T - X_t)\psi'(X_t)$ is bounded on $[0,T]$ for each path in $\Omega_{\mu,\nu}$; this is the content of Lemma 4.4.8.

Second, we show the superhedging property (4.3.8). Fix an averaging process A and a path in $\Omega_{\mu,\nu}$. To ease the notation, we write h instead of h^A in the following. Note, however, that h^A has the same formal expression as h in (4.4.7), but with A being the fixed averaging process (and not the second component of the canonical process on $\hat{\Omega}$).

Using the definitions of $H \diamond X_T$ and h as well as the fact that $A_0 = \Delta A_0$, we obtain

$$
\begin{aligned}
H \diamond X_T &= (X_T - X_0)h_0 + \int_{(0,T]} (X_T - X_t)h_t \,\mathrm{d}\,A_t \\
&= (X_T - X_0)\varphi'(X_0) - \int_{[0,T]} (X_T - X_t)(\varphi'(X_0) + \psi'(X_t)) \,\mathrm{d}\,A_t.
\end{aligned}
$$

Then, using that $\mathrm{d}\,A$ is a probability measure on $[0,T]$, the concavity of φ and the convexity of ψ, and Jensen's inequality, we can estimate

$$\begin{aligned} H \diamond X_T &= \int_{[0,T]} \varphi'(X_0)(X_t - X_0) \,\mathrm{d}\, A_t - \int_{[0,T]} \psi'(X_t)(X_T - X_t) \,\mathrm{d}\, A_t \\ &\geq \varphi'(X_0) \left(\int_{[0,T]} X_t \,\mathrm{d}\, A_t - X_0 \right) - \int_{[0,T]} (\psi(X_T) - \psi(X_t)) \,\mathrm{d}\, A_t \\ &\geq \varphi \left(\int_{[0,T]} X_t \,\mathrm{d}\, A_t \right) - \varphi(X_0) - \psi(X_T) + \int_{[0,T]} \psi(X_t) \,\mathrm{d}\, A_t \\ &\geq \varphi \left(\int_{[0,T]} X_t \,\mathrm{d}\, A_t \right) - \varphi(X_0) - \psi(X_T) + \psi \left(\int_{[0,T]} X_t \,\mathrm{d}\, A_t \right). \end{aligned} \tag{4.4.13}$$

Rearranging terms and using that $\varphi + \psi \geq f$ on J, we find

$$\varphi(X_0) + \psi(X_T) + H \diamond X_T \geq f \left(\int_{[0,T]} X_t \,\mathrm{d}\, A_t \right).$$

Third, we show the admissibility condition (4.3.9). Fix an averaging process A and $P \in \mathcal{M}(\mu, \nu)$. Define the family of $\mathbb{F}$-stopping times C_s, $s \in (0,1)$, by

$$C_s = \inf\{t \in [0,T] : A_t > s\}$$

and note that $0 \leq C_s \leq T$ for $s \in (0,1)$ because $A_T = 1$. Then using the family C_s as a time change (cf. [77, Proposition 0.4.9]) for the integral in (4.4.13) yields

$$\begin{aligned} &\varphi(X_0) + \psi(X_T) + H \diamond X_T \\ &= \int_0^1 \{\varphi(X_0) + \psi(X_T) + \varphi'(X_0)(X_{C_s} - X_0) - \psi'(X_{C_s})(X_T - X_{C_s})\} \,\mathrm{d}\, s. \end{aligned} \tag{4.4.14}$$

Now, suppose that the integrand in (4.4.14) is bounded from below, uniformly over

$s \in (0,1)$ and $\omega \in \Omega_{\mu,\nu}$. Then by Lemma 4.4.9, the P-expectation of the integrand equals $\mu(\varphi)+\nu(\psi)$ for each $s \in (0,1)$. Using this together with Tonelli's theorem and (4.4.14) gives

$$E^P\left[\varphi(X_0) + \psi(X_T) + H \diamond X_T\right] = \mu(\varphi) + \nu(\psi),$$

so that (4.3.9) holds.

It remains to show that the integrand in (4.4.14) is uniformly bounded from below. This follows from concavity of φ and convexity of ψ together with the fact that $\varphi + \psi \geq f \geq 0$ on J:

$$\begin{aligned}
&\varphi(X_0) + \psi(X_T) + \varphi'(X_0)(X_t - X_0) - \psi'(X_t)(X_T - X_t) \\
&\quad \geq \varphi(X_t) + \psi(X_t) \geq f(X_t) \geq 0, \quad t \in [0,T].
\end{aligned}$$

This completes the proof. □

Duality

We now turn to the duality between the auxiliary problems $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$ and $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$.

Theorem 4.4.10. *Let $\mu \leq_c \nu$ be irreducible with domain (I,J) and let $f : \mathbb{R} \to [0,\infty]$.*

(i) If f is upper semianalytic, then $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \widetilde{\mathbf{I}}_{\mu,\nu}(f) \in [0,\infty]$.

(ii) If $\widetilde{\mathbf{I}}_{\mu,\nu}(f) < \infty$, then there exists a dual minimizer $(\varphi,\psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$.

A couple of remarks are in order.

Remark 4.4.11. We only state the duality for one irreducible component. One can formulate and prove the full duality for arbitrary marginals $\mu \leq_c \nu$ in analogy to [16, Section 7]. We omit the details in the interest of brevity.

Remark 4.4.12. The lower bound on f in Theorem 4.4.10 can be relaxed. Indeed, suppose that $f : \mathbb{R} \to \overline{\mathbb{R}}$ is upper semianalytic and bounded from below by an affine function g.

We first consider the primal problem. Because g is affine and any $\mu \leq_c \theta \leq_c \nu$ has the same mass and barycenter as μ,

$$\theta(f - g) = \theta(f) - \theta(g) = \theta(f) - \mu(g).$$

Thus,

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f - g) = \widetilde{\mathbf{S}}_{\mu,\nu}(f) - \mu(g). \tag{4.4.15}$$

Regarding the dual problem, we note that $(\varphi, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f - g)$ if and only if $(\varphi + g, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ and that by Lemma 4.2.7 (iii),

$$\mu(\varphi) + \nu(\psi) = \{\mu(\varphi + g) + \nu(\psi)\} - \mu(g).$$

Hence,

$$\widetilde{\mathbf{I}}_{\mu,\nu}(f-g) = \widetilde{\mathbf{I}}_{\mu,\nu}(f) - \mu(g). \tag{4.4.16}$$

Because $f-g$ is nonnegative, the left-hand sides of (4.4.15)–(4.4.16) coincide by Theorem 4.4.10 (i). Therefore, $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \widetilde{\mathbf{I}}_{\mu,\nu}(f) \in (-\infty, \infty]$.

Moreover, if $\widetilde{\mathbf{I}}_{\mu,\nu}(f) < \infty$, then also $\widetilde{\mathbf{I}}_{\mu,\nu}(f-g) < \infty$ and a dual minimizer $(\varphi,\psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f-g)$ for $\widetilde{\mathbf{I}}_{\mu,\nu}(f-g)$ exists by Theorem 4.4.10 (ii). Now the above shows that $(\varphi+g,\psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ is a dual minimizer for $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$.

The proof of Theorem 4.4.10 is based on several preparatory results. We start with the crucial closedness property of the dual space in the spirit of [16, Proposition 5.2].

Proposition 4.4.13. *Let $\mu \leq_c \nu$ be irreducible with domain (I,J), let $f, f_n : J \to [0,\infty]$ be such that $f_n \to f$ pointwise, and let $(\varphi_n,\psi_n) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f_n)$ with $\sup_n\{\mu(\varphi_n) + \nu(\psi_n)\} < \infty$. Then there is $(\varphi,\psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ such that $\mu(\varphi)+\nu(\psi) \leq \liminf_{n\to\infty}\{\mu(\varphi_n)+\nu(\psi_n)\}$.*

Proof. Let $h_n = \varphi_n' : I \to \mathbb{R}$ be a superderivative of the concave function φ_n. As

$$\varphi_n(x) + \psi_n(y) + h_n(x)(y-x) \geq \varphi_n(y) + \psi_n(y) \geq f_n(y) \geq 0, \quad (x,y) \in I \times J,$$

(φ_n, ψ_n, h_n) is in the dual space $\mathcal{D}^c_{\mu,\nu}(0)$ of [16]. Hence, following the line of reasoning in the proof of [16, Proposition 5.2] (which is based on Komlos' lemma; we recall that convex combinations of convex (concave) functions are again convex (concave)), we may assume without loss of generality that

$$\varphi_n \to \bar{\varphi} \quad \mu\text{-a.e.} \quad \text{and} \quad \psi_n \to \bar{\psi} \quad \nu\text{-a.e.}$$

for some $(\bar{\varphi}, \bar{\psi}) \in L^c(\mu, \nu)$. Moreover, the arguments in [16] also show that $\mu(\bar{\varphi}) + \nu(\bar{\psi}) \leq \liminf_{n\to\infty}\{\mu(\varphi_n) + \nu(\psi_n)\}$.

Now, define the functions $\varphi, \psi : J \to \overline{\mathbb{R}}$ by $\varphi := \liminf_{n\to\infty} \varphi_n$ and $\psi := \limsup_{n\to\infty} \psi_n$. Then φ is convex, ψ is concave, $\varphi = \bar{\varphi}$ μ-a.e., and $\psi = \bar{\psi}$ ν-a.e. In particular, $(\varphi, \psi) \in L^c(\mu, \nu)$ and $\mu(\varphi) + \nu(\psi) \leq \liminf_{n\to\infty}\{\mu(\varphi_n) + \nu(\psi_n)\}$. Furthermore, as $\varphi_k + \psi_k \geq f_k$ on J, we have for each n that

$$\inf_{k\geq n} \varphi_k + \sup_{k\geq n} \psi_k \geq \inf_{k\geq n} (\varphi_k + \psi_k) \geq \inf_{k\geq n} f_k \quad \text{on } J.$$

Sending $n \to \infty$ gives $\varphi + \psi \geq f$. In summary, $(\varphi, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$. $\square$

We proceed to show strong duality for bounded upper semicontinuous functions.

Lemma 4.4.14. *Let $f : \mathbb{R} \to [0, \infty]$ be bounded and upper semicontinuous. Then $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \widetilde{\mathbf{I}}_{\mu,\nu}(f)$.*

The proof is based on a Hahn–Banach separation argument similar to [16, Lemma 6.4].

Proof. We first show the weak duality inequality. Let $\mu \leq_c \theta \leq_c \nu$ and $(\varphi, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$. In particular, $\varphi + \psi$ is bounded from below. Then by Lemma 4.2.9 (iii)–(iv),

$$\theta(f) \leq \theta(\varphi + \psi) = \theta(\varphi) + \theta(\psi) \leq \theta(\varphi) + \nu(\psi) \leq \mu(\varphi) + \nu(\psi), \tag{4.4.17}$$

and the inequality $\widetilde{\mathbf{S}}_{\mu,\nu}(f) \leq \widetilde{\mathbf{I}}_{\mu,\nu}(f)$ follows.

The converse inequality is based on a Hahn–Banach argument, so let us introduce a suitable space. By the de la Vallée–Poussin theorem, there is an increasing convex function $\zeta_\nu : \mathbb{R}_+ \to \mathbb{R}_+$ of superlinear growth such that $x \mapsto \zeta_\nu(|x|)$ is ν-integrable. Now, set $\zeta(x) = 1 + \zeta_\nu(|x|)$, $x \in \mathbb{R}$, and denote by C_ζ the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that f/ζ vanishes at infinity. We endow C_ζ with the norm $\|f\|_\zeta := \|f/\zeta\|_\infty$. With this notation, the same arguments as in the proof of [16, Lemma 6.4] show that the dual space C_ζ^* of continuous linear functionals on C_ζ can be represented by finite signed measures.

Fix $f \in C_\zeta$. Then

$$-\zeta(x)\|f\|_\zeta \leq f(x) \leq \zeta(x)\|f\|_\zeta, \quad x \in J. \tag{4.4.18}$$

Because ζ_ν is convex and $x \mapsto \zeta_\nu(|x|)$ is ν-integrable, we have $\theta(\zeta) \leq \nu(\zeta) < \infty$ for all $\mu \leq_c \theta \leq_c \nu$. This together with (4.4.18) shows that $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$ is finite. Thus, adding a suitable constant to f, we may assume that $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = 0$. For the following Hahn–Banach argument, we consider the convex cone

$$K := \{g \in C_\zeta : \widetilde{\mathbf{I}}_{\mu,\nu}(g) \leq 0\}.$$

Proposition 4.4.13 implies that K is closed.

Suppose for the sake of contradiction that $\widetilde{\mathbf{I}}_{\mu,\nu}(f) > 0$. Then, by the Hahn–Banach theorem, K and f can be strictly separated by a continuous linear functional on C_ζ.

That is, there is a finite signed measure ρ such that $\rho(f) > 0$ and $\rho(g) \leq 0$ for all $g \in K$. For any compactly supported nonnegative continuous function $g \in C_\zeta$, we have $\widetilde{\mathbf{I}}_{\mu,\nu}(-g) \leq 0$. That is, $-g \in K$ and hence $\rho(-g) \leq 0$. This shows that ρ is a (nonnegative) finite measure. Multiplying ρ by a positive constant if necessary, we may assume that ρ has the same mass as μ and ν. Next, let ψ be convex and of linear growth. Then $\psi\nu(\mathbb{R}) - \nu(\psi) \in K$ and $-\psi\mu(\mathbb{R}) + \mu(\psi) \in K$. Using that $\rho \leq 0$ for these two functions yields $\mu(\psi) \leq \rho(\psi) \leq \nu(\psi)$. We conclude that $\mu \leq_c \rho \leq_c \nu$. But now $\rho(f) > 0$ contradicts $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = 0$. Thus, $\widetilde{\mathbf{I}}_{\mu,\nu}(f) \leq \widetilde{\mathbf{S}}_{\mu,\nu}(f)$.

Finally, let f be bounded and upper semicontinuous and choose $f_n \in C_b(\mathbb{R}) \subseteq C_\zeta$ such that $f_n \searrow f$. By the above, we have $\widetilde{\mathbf{S}}_{\mu,\nu}(f_n) = \widetilde{\mathbf{I}}_{\mu,\nu}(f_n)$ for all n. We show below that $\lim_{n\to\infty} \widetilde{\mathbf{S}}_{\mu,\nu}(f_n) = \widetilde{\mathbf{S}}_{\mu,\nu}(f)$. Using this and the monotonicity of $\widetilde{\mathbf{I}}_{\mu,\nu}$, we obtain

$$\widetilde{\mathbf{I}}_{\mu,\nu}(f) \leq \lim_{n\to\infty} \widetilde{\mathbf{I}}_{\mu,\nu}(f_n) = \lim_{n\to\infty} \widetilde{\mathbf{S}}_{\mu,\nu}(f_n) = \widetilde{\mathbf{S}}_{\mu,\nu}(f) \leq \widetilde{\mathbf{I}}_{\mu,\nu}(f).$$

So strong duality holds for bounded upper semicontinuous functions.

It remains to argue that $\lim_{n\to\infty} \widetilde{\mathbf{S}}_{\mu,\nu}(f_n) = \widetilde{\mathbf{S}}_{\mu,\nu}(f)$. We show more generally that $\widetilde{\mathbf{S}}_{\mu,\nu}$ is continuous along decreasing sequences of bounded upper semicontinuous functions. So let $f_n \searrow f$ be a convergent sequence of bounded upper semicontinuous functions. Fix $\varepsilon > 0$ and set $\ell := \lim_{n\to\infty} \widetilde{\mathbf{S}}_{\mu,\nu}(f_n)$. Then for each n, $\ell \leq \mathbf{S}_{\mu,\nu}(f_n) < \infty$ and thus the set

$$A_n := \{\mu \leq_c \theta \leq_c \nu : \theta(f_n) \geq \ell - \varepsilon\}$$

is nonempty. Moreover, each A_n is a closed subset of the weakly compact set $\{\theta : \mu \leq_c \theta \leq_c \nu\}$ and $A_{n+1} \subseteq A_n$. Therefore, there exists a θ' in the intersection $\cap_{n\geq 1} A_n$. We then obtain by monotone convergence that

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f) \geq \theta'(f) = \lim_{n\to\infty} \theta'(f_n) \geq \ell - \varepsilon.$$

This implies that $\widetilde{\mathbf{S}}_{\mu,\nu}(f) \geq \ell$ as ε was arbitrary. The converse inequality follows from the monotonicity of $\widetilde{\mathbf{S}}_{\mu,\nu}$. This completes the proof. □

Proof of Theorem 4.4.10. (i): This is a consequence of Lemma 4.4.14 and a capacitability argument that is almost verbatim to [16, Section 6]. The same arguments can be found in [66]. We therefore omit these elaborations.

(ii): Applying Proposition 4.4.13 to the constant sequence $f_n = f$ and a minimizing sequence $(\varphi_n, \psi_n) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ of $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$ yields a dual minimizer. □

We are now in a position to prove the duality between the robust pricing and superhedging problems.

Proof of Theorem 4.3.9. By Proposition 4.4.2, Lemma 4.3.8, and Proposition 4.4.5,

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f) \leq \mathbf{S}_{\mu,\nu}(f, \mathcal{A}) \leq \mathbf{I}_{\mu,\nu}(f, \mathcal{A}) \leq \widetilde{\mathbf{I}}_{\mu,\nu}(f),$$

and Theorem 4.4.10 shows that $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \widetilde{\mathbf{I}}_{\mu,\nu}(f)$. Hence,

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \mathbf{S}_{\mu,\nu}(f, \mathcal{A}) = \mathbf{I}_{\mu,\nu}(f, \mathcal{A}) = \widetilde{\mathbf{I}}_{\mu,\nu}(f). \tag{4.4.19}$$

In particular, the quantities in (4.4.19) are all independent of the choice of $\mathcal{A}$ (as long as one of the two conditions of Theorem 4.3.9 holds).

If $\mathbf{I}_{\mu,\nu}(f,\mathcal{A}) < \infty$, then $\widetilde{\mathbf{I}}_{\mu,\nu}(f) < \infty$ and hence there is an optimizer $(\varphi,\psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ for $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$. Then Proposition 4.4.7 provides an $H = (h, A)$ such that $(\varphi,\psi,H) \in \mathcal{D}_{\mu,\nu}(f,\mathcal{A})$. By (4.4.19) and the definition of $\mathbf{I}_{\mu,\nu}(f,\mathcal{A})$, (φ,ψ,H) is an optimizer for $\mathbf{I}_{\mu,\nu}(f,\mathcal{A})$. □

Remark 4.4.15. Strong duality (without dual attainment) for the robust pricing and superhedging problems continues to hold if we restrict ourselves to trading strategies whose dynamic part is of finite variation.

First, observe that the process $\hat{H}$ defined by (4.2.9) is of finite variation when h is bounded. Recalling the definition (4.4.7) of h in Proposition 4.4.7, we see that h is bounded on $\{\omega : \omega_T \in J^\circ\}$ as these paths are bounded in a compact subset of J, on which ψ' is bounded. This will more generally hold for almost all paths if ψ' is uniformly bounded on J. Therefore, strong duality (and dual attainment in strategies of finite variation) holds if J is open.

Second, consider the case $J = [a,b)$ for some $-\infty < a < b \leq \infty$. Suppose that the assumptions of Theorem 4.3.9 hold and that $\mathbf{I}_{\mu,\nu}(f,\mathcal{A}) < \infty$, and let $(\varphi,\psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ be a dual auxiliary optimizer. Then $\psi(a) < \infty$ as ν has an atom at a (cf. Lemma 4.2.7). If $\psi'(a) > -\infty$, then the same argument as above shows that the dynamic trading strategy constructed in Proposition 4.4.7 is of finite variation. If

$\psi'(a) = -\infty$, then we construct a sequence of functions

$$\psi_k(x) := \begin{cases} \psi(x) & \text{for } x \geq a + \frac{1}{k}, \\ \psi(a) + k(x-a)(\psi(a+\frac{1}{k}) - \psi(a)) & \text{for } x < a + \frac{1}{k}. \end{cases}$$

that approximates ψ by linear interpolation on the interval $[a, a+\frac{1}{k}]$. We then have $\psi_k \searrow \psi$ and $\mu(\varphi) + \nu(\psi_k) \searrow \mu(\varphi) + \nu(\psi) = \widetilde{\mathbf{I}}_{\mu,\nu}(f)$ as $k \to \infty$. Since $\psi_k'(a) > -\infty$, the associated process $H^{(k)}$ is of finite variation almost surely. The cases $J = (a, b]$ and $J = [a, b]$ are analogous.

Structure of primal and dual optimizers

If a primal optimizer to the auxiliary problem exists, we can derive some necessary properties for the dual optimizer.

Proposition 4.4.16. *Let $\mu \leq_c \nu$ be irreducible with domain (I, J) and let $f : \mathbb{R} \to [0, \infty]$ be Borel. Suppose that $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \widetilde{\mathbf{I}}_{\mu,\nu}(f)$, that $\mu \leq_c \theta \leq_c \nu$ is an optimizer for $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$, and that $(\varphi, \psi) \in \widetilde{\mathcal{D}}_{\mu,\nu}(f)$ is an optimizer for $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$. Then*

(i) $\varphi + \psi = f$ *θ-a.e.,*

(ii) φ *is affine on the connected components of* $\{u_\mu < u_\theta\}$*,*

(iii) ψ *is affine on the connected components of* $\{u_\theta < u_\nu\}$*,*

(iv) φ *does not have a jump at a finite endpoint b of J if $\theta(\{b\}) > 0$, and*

(v) ψ *does not have a jump at a finite endpoint b of J if $\theta(\{b\}) < \nu(\{b\})$.*

Proof. As in the proof of Lemma 4.4.14, we obtain (cf. (4.4.17)) that

$$\theta(f) \leq \theta(\varphi + \psi) \leq \mu(\varphi) + \nu(\psi).$$

By the absence of a duality gap as well as the optimality of θ and (φ, ψ), all inequalities are equalities:

$$\theta(f) = \theta(\varphi + \psi) = \mu(\varphi) + \nu(\psi). \tag{4.4.20}$$

Now (i) follows from the first equality in (4.4.20) and the fact that $\varphi + \psi \geq f$ on J. Rearranging the second equality, we can write

$$\begin{aligned} 0 &= \{\mu(\varphi) + \nu(\psi)\} - \theta(\varphi + \psi) \\ &= \{\mu(\varphi) + \nu(\psi)\} - \{\theta(\varphi) + \nu(\psi)\} + \{\theta(\varphi) + \nu(\psi)\} - \theta(\varphi + \psi). \end{aligned}$$

Using the definition (4.2.4) of the first three expressions (using φ as a concave moderator for the first two terms and $-\psi$ for the third; cf. Lemma 4.2.9 (i)), we obtain

$$\begin{aligned} 0 &= (\mu - \nu)(\varphi) - (\theta - \nu)(\varphi) + (\theta - \nu)(-\psi) \\ &= (\mu - \theta)(\varphi) + (\theta - \nu)(-\psi), \end{aligned} \tag{4.4.21}$$

where the last equality is a direct consequence of the definitions of $(\mu - \nu)(\varphi)$ and $(\theta - \nu)(\varphi)$ (cf. (4.2.2)). Both terms on the right-hand side of (4.4.21) are nonnegative by definition and hence must vanish:

$$0 = (\mu - \theta)(\varphi) = \int_I (u_\mu - u_\theta)\, \mathrm{d}\,\varphi'' + \int_{J\setminus I} |\Delta\varphi|\, \mathrm{d}\,\theta$$

and similarly for $(\theta - \nu)(-\psi)$. This implies that $\varphi'' = 0$ on $\{u_\mu < u_\theta\}$ (which is assertion (ii)) and that $|\Delta\varphi| = 0$ for every endpoint of J on which θ has an atom (which is assertion (iv)). The proofs of (iii) and (v) are similar. □

The next result shows that for upper semicontinuous f, there is a maximizer for $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$ which is maximal with respect to the convex order. We omit the proof in the interest of brevity.

Proposition 4.4.17. *Let $\mu \leq_c \nu$ be irreducible and let $f : \mathbb{R} \to [0, \infty]$ be upper semicontinuous and bounded from above by a convex, continuous, and ν-integrable function. Furthermore, fix a strictly convex function $g : \mathbb{R} \to \mathbb{R}$ with linear growth, and consider the "secondary" optimization problem*

$$\sup_{\theta \in \Theta(f)} \theta(g), \tag{4.4.22}$$

where $\Theta(f) := \{\theta : \mu \leq_c \theta \leq_c \nu \text{ and } \theta(f) \geq \widetilde{\mathbf{S}}_{\mu,\nu}(f)\}$ is the set of optimizers of the auxiliary primal problem.

(i) $\Theta(f)$ is non-empty, convex, and weakly compact and (4.4.22) admits an optimizer.

(ii) Any optimizer θ of (4.4.22) has the following properties:

- *θ is maximal in $\Theta(f)$ with respect to the convex order.*
- *If O is an open interval such that $O \subseteq \{u_\theta < u_\nu\}$ and $f|_O$ is convex, then $\theta(O) = 0$.*
- *If K is an interval such that $K^\circ \subseteq \{u_\mu < u_\theta\}$, $f|_K$ is strictly concave, and $\theta(K) > 0$, then $\theta|_K$ is concentrated in a single atom.*

The following example shows that the set optimizers for $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$ can have multiple maximal or minimal elements with respect to the convex order; there is in general no greatest or least element for this partially ordered set.

Example 4.4.18. Let $\mu = \delta_0$ and $\nu = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$ and let f be piecewise linear with $f(-1) = f(1) = 3$, $f(-1/2) = f(1/2) = 2$, and $f(0) = 0$. We claim that there is no greatest or least primal optimizer.

We construct candidate primal and dual optimizers as follows. On the primal side, set $\theta_1 = \frac{2}{3}\delta_{-\frac{1}{2}} + \frac{1}{3}\delta_1$ and $\theta_2 = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_{\frac{1}{2}}$. On the dual side, set $\varphi \equiv 0$ and let ψ be the convex function that interpolates linearly between $\psi(-1) = \psi(1) = 3$ and $\psi(0) = 1$. Direct computations yield $\theta_1(f) = \theta_2(f) = \frac{7}{3} = \nu(\psi)$ which shows that θ_1 and θ_2 are primal optimizers and that (φ, ψ) is a dual optimizer.

First, we show that there is no primal optimizer which dominates both θ_1 and θ_2 in convex order. Indeed, one can check that $u_\nu = \max(u_{\theta_1}, u_{\theta_2})$, so that ν is the only feasible primal element which dominates both θ_1 and θ_2 in convex order. But $\nu(f) = 2 < \frac{7}{3}$ and therefore ν is not optimal.

Second, we show that there is no primal optimizer which is dominated by both θ_1 and θ_2. Indeed, one can check that $\{u_\mu < \min(u_{\theta_1}, u_{\theta_2})\} = (-\frac{1}{2}, \frac{1}{2})$, so that every feasible primal element that is dominated by both θ_1 and θ_2 must be concentrated on $[-\frac{1}{2}, \frac{1}{2}]$. But $f \le 2$ on $[-\frac{1}{2}, \frac{1}{2}]$, so that no primal optimizer can be concentrated on this interval.

We conclude this section with an example that shows that primal attainment does not hold in general if f is not upper semicontinuous.

Example 4.4.19. Let $\mu = \delta_0$, $\nu = \frac{1}{2}(\delta_{-1} + \delta_1)$, and set $f(x) := |x|\mathbf{1}_{(-1,1)}(x)$. Then $\mu \le_c \nu$ is irreducible with domain $((-1,1), [-1,1])$. Considering the sequence $\theta_n := \frac{1}{2}(\delta_{-1+\frac{1}{n}} + \delta_{1-\frac{1}{n}})$, one can see that $\widetilde{\mathbf{S}}_{\mu,\nu}(f) \ge 1$. But there is no $\mu \le_c \theta \le_c \nu$ such that $\theta(f) \ge 1$ because $f < 1$ on $[-1,1]$.

4.5 Examples

Two common payoff functions are risk reversals and butterfly spreads. In this section, we provide solutions to the auxiliary primal and dual problems for these payoffs. Throughout this section, we fix irreducible marginals $\mu \le_c \nu$ and denote their common total mass and first moment by m_0 and m_1, respectively.

Risk reversals

The payoff function of a risk reversal is of the form

$$f(x) = -(a - x)_+ + (x - b)_+,$$

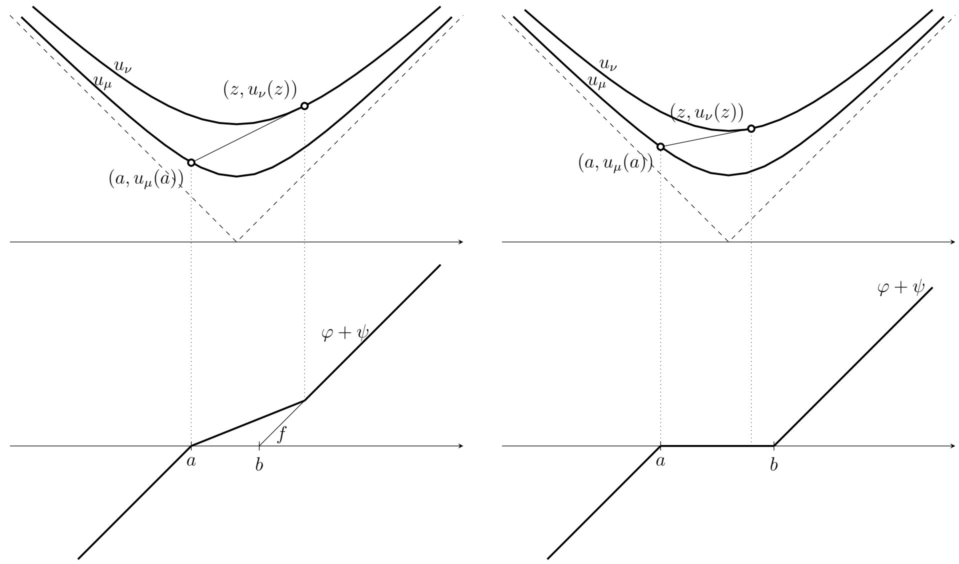

Figure 4.1: Construction of the potential functions of the optimal intermediate laws θ (top) and the dual optimizers $\varphi + \psi$ (bottom) for a risk reversal as described in Proposition 4.5.1; $z > b$ in the left panel and $z < b$ in the right panel.

for fixed $a < b$. The following result provides a simple geometric construction of the primal and dual optimizers in terms of the potential functions of μ and ν.[8] We recall that any convex function u lying between the potential functions u_μ and u_ν is the potential function of a measure θ which is in convex order between μ and ν (cf., e.g., [79]).

Proposition 4.5.1. *Consider the line through the point $(a, u_\mu(a))$ of maximal slope lying below (or on) the graph of u_ν; cf. Figure 4.1. This line is either (i) a tangent line to the graph of u_ν with a tangent point $(z, u_\nu(z))$ for some $z \in (a, \infty)$ or (ii) the asymptote line for the graph of u_ν near $+\infty$.*[9]

[8] The authors thank David Hobson for the idea of this construction.

[9] Note that case (ii) can only happen when $(a, \mu(a))$ lies on the increasing part of the dashed potential function in Figure 4.1. In particular, in this case, μ is concentrated on the left of a.

In case (i), define the concave function φ and the convex function ψ by

$$\varphi(x) = -\alpha(x-a)_+,$$

$$\psi(x) = x - a + \alpha(x - (z \vee b))_+,$$

where $\alpha = (b-a)/((z \vee b) - a)$. Moreover, let u be the unique convex function that coincides with u_μ on $(-\infty, a]$ and with u_ν on $[z, \infty)$ and is affine on $[a, z]$ (i.e., u coincides on $[a, z]$ with the tangent line considered above). Denote by θ the unique measure with potential function $u_\theta = u$. In case (ii), set $\varphi(x) = 0$, $\psi(x) = x - a$, and $\theta = \mu$.

Then, θ is an optimizer for the auxiliary primal problem $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$, (φ, ψ) is an optimizer for the auxiliary dual problem $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$, and the common optimal optimal value is given in terms of the potential functions of μ and ν by

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \widetilde{\mathbf{I}}_{\mu,\nu}(f) = \begin{cases} m_1 - \frac{a+b}{2} m_0 + \frac{b-a}{2} \frac{u_\nu(z \vee b) - u_\mu(a)}{(z \vee b) - a} & \textit{in case (i)}, \\ m_1 - a m_0 & \textit{in case (ii)}. \end{cases}$$

Proof. We first note that θ and (φ, ψ) are admissible elements for the auxiliary primal and dual problems, respectively. Indeed, by construction, u_θ is convex and lies between u_μ and u_θ. Thus, the associated measure θ satisfies $\mu \leq_c \theta \leq_c \nu$. Moreover, a straightforward computation shows that $\varphi + \psi \geq f$, and $(\varphi, \psi) \in L^c(\mu, \nu)$ by Lemma 4.2.9.

By the weak duality inequality (4.4.17) (this also holds if f is bounded from below by an affine function; cf. Remark 4.4.12), $\theta(f) \leq \mu(\varphi)+\nu(\psi)$ holds for any admissible primal and dual elements. It thus suffices to show that $\theta(f) = \mu(\varphi) + \nu(\psi)$ for our particular choices for θ and (φ, ψ).

Case (i): Using the identity $(t-s)_+ = \frac{1}{2}(|t-s| + t - s)$, the integrals $\theta(f), \mu(\varphi)$, and $\nu(\psi)$ can be expressed in terms of the potential functions of μ, θ, and ν as follows:

$$\begin{aligned}
\theta(f) &= \frac{1}{2}(u_\theta(b) - u_\theta(a)) + m_1 - \frac{a+b}{2} m_0, \\
\mu(\varphi) &= -\frac{\alpha}{2}(u_\mu(a) + m_1 - a m_0), \\
\nu(\psi) &= m_1 - a m_0 + \frac{\alpha}{2}(u_\nu(z \vee b) + m_1 - (z \vee b) m_0).
\end{aligned}$$

Substituting $\alpha = (b-a)/((z \vee b) - a)$ and simplifying gives

$$\begin{aligned}
\mu(\varphi) + \nu(\psi) &= m_1 + \frac{1}{2}((a - (z \vee b))\alpha - 2a) m_0 - \frac{\alpha}{2}(u_\mu(a) - u_\nu(z \vee b)) \\
&= m_1 - \frac{a+b}{2} m_0 + \frac{b-a}{2} \frac{u_\nu(z \vee b) - u_\mu(a)}{(z \vee b) - a}.
\end{aligned}$$

Hence,

$$\theta(f) - (\mu(\varphi) + \nu(\psi)) = \frac{b-a}{2} \left(\frac{u_\theta(b) - u_\theta(a)}{b-a} - \frac{u_\nu(z \vee b) - u_\mu(a)}{(z \vee b) - a} \right),$$

and it suffices to show that the two quotients inside the brackets are equal. To this end, we distinguish two cases. On the one hand, if $z \leq b$, it is enough to observe that $u_\theta(a) = u_\mu(a)$ and $u_\theta(b) = u_\nu(b)$ by the construction of $u = u_\theta$. On the other hand, if $z \geq b$, then the two quotients are the same because $u = u_\theta$ is affine on $[a, z] \supset [a, b]$

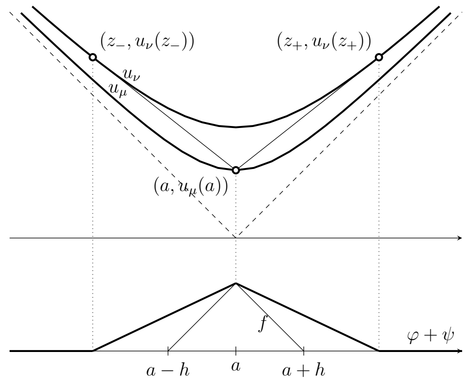

Figure 4.2: Construction of the potential function of the optimal intermediate law θ (top) and the dual optimizer $\varphi + \psi$ (bottom) for a butterfly spread as described in Proposition 4.5.2.

and coincides with u_μ at a and with u_ν at z.

Case (ii): On the one hand, since $\theta = \mu$ and μ is concentrated on the left of a, we have $\theta(f) = \mu(f) = m_1 - am_0$. On the other hand, $\mu(\varphi) + \nu(\psi) = \int (x-a)\, \nu(\mathrm{d}\, x) = m_1 - am_0$. $\square$

Butterfly spreads

The payoff function of a butterfly spread is of the form

$$
\begin{aligned}
f(x) &= (x-(a-h))_+ - 2(x-a)_+ + (x-(a+h))_+ \\
&= \tfrac{1}{2}|x-a+h| - |x-a| + \tfrac{1}{2}|x-a-h|,
\end{aligned}
$$

for fixed a and $h > 0$. We have the following analog to Proposition 4.5.1; we omit the proof.

Proposition 4.5.2. *Consider the two lines l_+, l_- through the point $(a, u_\mu(a))$ of max-*

imal and minimal slope, respectively, lying below (or on) the graph of u_ν. *We distinguish the cases (i+)* l_+ *is a tangent line with tangent point* $(z_+, u_\nu(z_+))$, *(ii+)* l_+ *is an asymptote, (i−)* l_- *is a tangent line with tangent point* $(z_-, u_\nu(z_-))$, *(ii−)* l_- *is an asymptote. In case (ii±), we set* $z_\pm = \pm\infty$.

Let u *be the convex function that coincides with* u_ν *on* $(-\infty, z_-] \cup [z_+, \infty)$ *and is affine on* $[z_-, a]$ *and on* $[a, z_+]$, *and define the concave function* φ *and the convex function* ψ *by*

$$\varphi(x) = -(\alpha+\beta)(x-a)_+,$$
$$\psi(x) = \alpha(x-(z_- \wedge (a-h)))_+ + \beta(x-(z_+ \vee (a+h)))_+,$$

where $\alpha = \frac{h}{a-(z_- \wedge (a-h))}$ *and* $\beta = \frac{h}{(z_+ \vee (a+h))-a}$. *Here, in the asymptote cases (ii±),* φ, ψ *need to be interpreted as the limiting functions that arise as* $z_\pm \to \pm\infty$.[10]

Then, the intermediate law θ *with potential function* $u_\theta = u$ *is an optimizer for the auxiliary primal problem* $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$, (φ, ψ) *is an optimizer for the auxiliary dual problem* $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$, *and the common optimal optimal value is given in terms of the potential functions of* μ *and* ν *by*

$$\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \widetilde{\mathbf{I}}_{\mu,\nu}(f) = \tfrac{h}{2}(s_+ + s_-),$$

[10] For instance, if $z_- = -\infty$ and $z_+ < \infty$, then $\varphi(x) = -\beta(x-a)_+$ and $\psi(x) = h + \beta(x-(z_+ \vee (a+h)))_+$.

where

$$s_+ = \begin{cases} \frac{u_\nu(z_+ \vee (a+h)) - u_\mu(a)}{(z_+ \vee (a+h)) - a} & \textit{in case (i+)}, \\ 1 & \textit{in case (ii+)}, \end{cases}$$

$$s_- = \begin{cases} \frac{u_\nu(z_- \wedge (a-h)) - u_\mu(a)}{a - (z_- \wedge (a-h))} & \textit{in case (i-)}, \\ -1 & \textit{in case (ii-)}. \end{cases}$$

4.6 Counterexamples

In this section, we give four counterexamples. Example 4.6.1 shows that strong duality for the auxiliary problems may fail for general (not necessarily irreducible) marginals if the dual elements φ, ψ are required to be *globally* concave and convex, respectively. Example 4.6.2 shows that strong duality may fail if the dual elements φ and ψ are required to be μ- and ν-integrable, respectively. Example 4.6.3 shows that the robust model-based prices of Asian- and American-style derivatives are typically not equivalent when more than two marginals are given. Example 4.6.4 shows that the equality $\mathbf{S}_{\mu,\nu}(f, \mathcal{A}) = \widetilde{\mathbf{S}}_{\mu,\nu}(f)$ may fail when the assumptions of Proposition 4.4.2 are violated.

Example 4.6.1 (Duality gap with globally convex/concave dual elements)**.** Let $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, let ν be the uniform distribution on $(-2, 2)$, and set $f(x) := |x|^{-\frac{1}{2}}$, $x \in \mathbb{R}$ (with $f(0) = \infty$).

First, we show that $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$ is finite. Fix any $\mu \leq_c \theta \leq_c \nu$. Computing the potential functions u_μ and u_ν shows that $\mu \leq_c \nu$ and that $\{u_\mu < u_\nu\} = I_1 \cup I_2$

with $I_1 = (-2, 0)$ and $I_2 = (0, 2)$. Because ν does not have an atom at the common boundary 0 of I_1 and I_2, also θ cannot have an atom at 0. Thus, we can write $\theta = \theta_1 + \theta_2$ with

$$\frac{1}{2}\delta_{-1} \leq_c \theta_1 \leq_c \nu|_{I_1} \quad \text{and} \quad \frac{1}{2}\delta_1 \leq_c \theta_2 \leq_c \nu|_{I_2}.$$

Since f is convex when restricted to I_1 or I_2, we have

$$\theta(f) = \theta_1(f) + \theta_2(f) \leq \nu|_{I_1}(f) + \nu|_{I_2}(f) = \nu(f) < \infty.$$

It follows $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \nu(f) < \infty$.

Second, let φ be concave and ψ be convex such that $\varphi + \psi \geq f$. We show that then necessarily $\mu(\varphi) + \nu(\psi) = \infty$. To this end, we may assume that $\varphi < \infty$ on $\mathrm{supp}(\mu) = \{-1, 1\}$. Then $\varphi < \infty$ everywhere by concavity. Thus, evaluating $\varphi + \psi \geq f$ at 0 implies that $\psi(0) = \infty$. Therefore, $\psi = \infty$ on $(-\infty, 0]$ or on $[0, \infty)$ by the convexity of ψ. In both cases, we have $\mu(\varphi) + \nu(\psi) = \infty$.

Example 4.6.2 (Duality gap with individually integrable dual elements)**.** We consider the marginals

$$\mu := C \sum_{n\geq 1} n^{-3}\mu_n \quad \text{and} \quad \nu := C \sum_{n\geq 1} n^{-3}\nu_n,$$

where $C := (\sum_{n\geq 1} n^{-3})^{-1}$, $\mu_n := \delta_n$ and $\nu_n := \frac{1}{3}(\delta_{n-1} + \delta_n + \delta_{n+1})$ for $n \geq 1$. These are the same marginals as in [16, Example 8.5] where it is shown that $\mu \leq_c \nu$ is

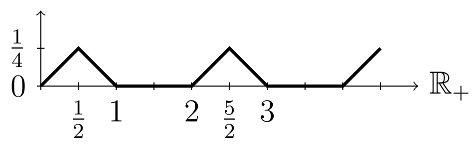

Figure 4.3: The function f in Example 4.6.2.

irreducible with domain $((0,\infty),[0,\infty))$. We now let $f:\mathbb{R}_+\to[0,1]$ be the piecewise linear function through the points given by $f(n)=0$ and $f(2n+\frac{1}{2})=\frac{1}{4}$ for $n\geq 0$; cf. Figure 4.3.

We proceed to construct candidates for optimizers for $\widetilde{\mathbf{S}}_{\mu,\nu}(f)$ and $\widetilde{\mathbf{I}}_{\mu,\nu}(f)$. For the primal problem, define the sequence $(\bar{\theta}_n)_{n\geq 1}$ by

$$\bar{\theta}_n = \begin{cases} \frac{1}{3}(\delta_{n-1}+2\delta_{n+\frac{1}{2}}) & \text{for } n \text{ even,} \\ \frac{1}{3}(2\delta_{n-\frac{1}{2}}+\delta_{n+1}) & \text{for } n \text{ odd,} \end{cases}$$

and set $\bar{\theta}:=C\sum_{n\geq 1}n^{-3}\bar{\theta}_n$. One can check that $\mu_n\leq_c\bar{\theta}_n\leq_c\nu_n$ and compute $\bar{\theta}_n(f)=\frac{1}{6}$. Hence, $\mu\leq_c\bar{\theta}\leq_c\nu$ (by linearity of potential functions in the measure) and $\bar{\theta}(f)=\frac{1}{6}$.

We now turn to the dual problem. Let $\bar{\varphi}$ and $\bar{\psi}$ be the unique concave and convex functions, respectively, with second derivative measures

$$-\bar{\varphi}''=\sum_{n\geq 0}\delta_{2n+\frac{1}{2}} \quad \text{and} \quad \bar{\psi}''=\frac{1}{2}\sum_{n\geq 1}\delta_n$$

and $\bar{\varphi}(0)=\bar{\psi}(0)=0$, $\bar{\varphi}'(0)=f'(0)=\frac{1}{2}$, and $\bar{\psi}'(0)=0$. The "initial conditions" are chosen such that $f(0)=\bar{\varphi}(0)+\bar{\psi}(0)$ and $f'(0)=\bar{\varphi}'(0)+\bar{\psi}'(0)$ and the choice of the second derivative measures ensures that φ and ψ pick up the negative and positive curvature of f, respectively. Thus, $\bar{\varphi} + \bar{\psi} = f$ on $\mathbb{R}_+$ by construction. We proceed to compute $\mu(\bar{\varphi}) + \nu(\bar{\psi})$ in the sense of Definition 4.2.3. (The individual integrals are infinite because $\bar{\varphi}$ and $\bar{\psi}$ have quadratic growth while μ and ν have no second moments.) To this end, we note that $\bar{\varphi} + \bar{\psi} = f$ vanishes on the support of ν. This implies that $\bar{\varphi}$ is a concave moderator for $(\bar{\varphi}, \bar{\psi})$ with respect to $\mu \leq_c \nu$. We can then compute

$$\begin{aligned}\mu(\bar{\varphi}) + \nu(\bar{\psi}) &= \mu(\bar{\varphi} - \bar{\varphi}) + \nu(\bar{\psi} + \bar{\varphi}) + (\mu - \nu)(\bar{\varphi}) = (\mu - \nu)(\bar{\varphi}) \\ &= C \sum_{n \geq 1} n^{-3} (\mu_n - \nu_n)(\bar{\varphi}).\end{aligned}$$

Fix $n \geq 1$. Because $\bar{\varphi}$ is continuous, we have

$$(\mu_n - \nu_n)(\bar{\varphi}) = \frac{1}{2} \int_I (u_{\mu_n} - u_{\nu_n}) \, \mathrm{d}\, \bar{\varphi}''. \tag{4.6.1}$$

The difference $u_{\mu_n} - u_{\nu_n}$ vanishes outside $(n-1, n+1)$ and on this interval, $\bar{\varphi}''$ is concentrated on either $n - \frac{1}{2}$ (if n is odd) or on $n + \frac{1}{2}$ (if n is even) with mass 1. Therefore, the right-hand side of (4.6.1) collapses to $\frac{1}{2}(u_{\mu_n} - u_{\nu_n})(n \pm \frac{1}{2}) = \frac{1}{6}$. It follows that $\mu(\bar{\varphi}) + \nu(\bar{\psi}) = \frac{1}{6} = \bar{\theta}(f)$. Hence, by (weak) duality, $\bar{\theta}$ and $(\bar{\varphi}, \bar{\psi})$ are primal and dual optimizers, respectively.

We are now in a position to argue that no dual optimizer lies in $L^1(\mu) \times L^1(\nu)$. Suppose for the sake of contradiction that $(\varphi, \psi) \in L^1(\mu) \times L^1(\nu)$ is a dual optimizer and note that $\mathrm{supp}(\bar{\theta}) = \{0.5, 1, 2, 2.5, 3, \ldots\}$. We have $\varphi + \psi = f$ $\bar{\theta}$-a.e. by Proposition 4.4.16 (i). One can show that the following modifications of (φ, ψ) do not affect its optimality nor the individual integrability of φ and ψ; we omit the tedious details. First, ψ is replaced by its piecewise linear interpolation at the atoms of ν. Second, φ is replaced by its piecewise linear interpolation at the kinks of f. Third, a suitable convex function is added to φ and subtracted from ψ (preserving their concavity and convexity, respectively) such that the second derivative measures $-\varphi''$ and ψ'' become singular.

Because $\varphi + \psi = f$ on $\operatorname{supp}(\bar{\theta})$ and both sides are piecewise linear, we conclude that $\varphi + \psi = f$ holds on $[\frac{1}{2}, \infty)$. As $-\varphi''$ and ψ'' are singular, φ and ψ must then account for the negative and positive curvature of f, respectively. It follows that both φ and ψ have quadratic growth. Since μ and ν do not have a second moment, we conclude that $\mu(\varphi) = -\infty$ and $\nu(\psi) = \infty$, a contradiction.

Example 4.6.3 (Different robust model-based prices for Asian- and American-style derivatives for multiple marginals)**.** For $n \geq 2$ given marginals $\mu_0 \leq_c \mu_1 \leq_c \cdots \leq_c \mu_n$ corresponding to the time points $0, 1, \ldots, n$ (say), the robust model-based price $\mathbf{S}_{\mu_0,\ldots,\mu_n}(f, \mathcal{A})$ is defined analogously. But this robust model-based price now depends non-trivially on $\mathcal{A}$, as the following example shows. Fix a strictly convex function f. On the one hand, if $\mathcal{A}$ corresponds to American-style derivatives, then one can check that $\mathbf{S}_{\mu_0,\ldots,\mu_n}(f, \mathcal{A}) = \mu_n(f)$. On the other hand, for Asian-style derivatives, i.e., $\mathcal{A}' = \{t \mapsto t/n\}$, Jensen's inequality yields

$$f\left(\frac{1}{n}\int_0^n X_t \,\mathrm{d}\,t\right) \leq \frac{1}{n}\sum_{i=0}^{n-1} f\left(\int_i^{i+1} X_t \,\mathrm{d}\,t\right),$$

so that

$$\mathbf{S}_{\mu_0,\ldots,\mu_n}(f,\mathcal{A}') \le \frac{1}{n}\sum_{i=1}^{n}\mu_i(f) \le \mu_n(f).$$

For a generic choice of marginals, both inequalities are strict. Hence, the robust model-based price of an Asian-style derivative with a strictly convex payoff function is typically smaller than that of the corresponding American-style derivative.

Example 4.6.4 (Necessity of the assumptions of Proposition 4.4.2)**.**

(i) We show that $\mathbf{S}_{\mu,\nu}(f,\mathcal{A}) = \widetilde{\mathbf{S}}_{\mu,\nu}(f)$ may fail if $\mathcal{A}$ does not contain an interior averaging process. Set $\mathcal{A} = \{A\} = \{t \mapsto \frac{1}{2} + \frac{1}{2}\mathbf{1}_{\{t=T\}}\}$, so that $\int_0^T X_t dA_t = (X_0 + X_T)/2$, and consider $f(x) = x^2$. Then, using the martingale property of X under any $P \in \mathcal{M}(\mu,\nu)$, one can check that $\mathbf{S}_{\mu,\nu}(f,\mathcal{A}) = (3\mu(f) + \nu(f))/4$, whereas $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = \nu(f)$ since f is convex. Now, choose μ and ν such that $\mu(f) < \nu(f)$ (f is strictly convex). Then, $\mathbf{S}_{\mu,\nu}(f,\mathcal{A}) < \widetilde{\mathbf{S}}_{\mu,\nu}(f)$.

(ii) We show that $\mathbf{S}_{\mu,\nu}(f,\mathcal{A}) = \widetilde{\mathbf{S}}_{\mu,\nu}(f)$ may fail if $\mathcal{A}$ contains an interior averaging process but f is not lower semicontinuous. Set $\mathcal{A} = \{t \mapsto t/T\}$ and $f(x) = \mathbf{1}_{\{|x|\ge 1\}}$, and choose $\mu = \delta_0$ and $\nu = (\delta_1 + \delta_{-1})/2$. On the one hand, since $\nu(f) = 1$ and $f \le 1$, we have $\widetilde{\mathbf{S}}_{\mu,\nu}(f) = 1$. On the other hand, we claim that $\mathbf{S}_{\mu,\nu}(f,\mathcal{A}) = 0$. To this end, fix $P \in \mathcal{M}(\mu,\nu)$. Since P-a.e. path of X starts in 0, evolves in $[-1,1]$, and is right-continuous, $\left|\frac{1}{T}\int_0^T X_t \,\mathrm{d}\,t\right| < 1$ P-a.s. Thus, $E^P\big[f(\frac{1}{T}\int_0^T X_t \,\mathrm{d}\,t)\big] = 0$. Since $P \in \mathcal{M}(\mu,\nu)$ was arbitrary, $\mathbf{S}_{\mu,\nu}(f,\mathcal{A}) = 0$.

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