ESSAYS ON AUCTIONS AND MECHANISM DESIGN

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Economics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2019

Urbana, Illinois

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ABSTRACT

There are two directions in studying trading mechanisms: studying outcomes that existing mechanisms generate and designing mechanisms that satisfy desired outcomes. In this dissertation, I explore trading mechanisms in these directions.

In the first chapter, I study the first price auction with independent private valuations, wherein each bidder faces ambiguity about the probability distribution from which the other bidders' valuations for the item are drawn. Each bidder is ambiguity averse and this ambiguity is represented by a set of priors. In this informational setting, I identify a maxmin Bayesian Nash equilibrium of the auction and show that the bidders' bids and the seller's expected revenue increase with the level of the bidders' ambiguity if the bidders' valuation distribution satisfies the monotone inverse hazard rate condition. I also show that the seller's expected revenue from the first price auction is greater than that from the second price auction.

In the second chapter, I examine a trading mechanism in which traders' valuations for an item are interdependent. Trade can occur between multiple buyers and multiple sellers. The transfer rules of the trading mechanism are motivated by the second price auction. The mechanism satisfies ex-post efficiency, ex-post incentive compatibility, and ex-post individual rationality. An example in which the mechanism generates a budget deficit is provided. The result of this chapter leads to my work on an impossibility result in the next chapter.

In the third chapter, I study trading mechanisms in which traders' valuations for an indivisible item are interdependent. Trade can occur between one buyer and one seller. Under the assumption that each trader's information has a greater marginal effect on her own valuation than on the other trader's valuation, no trading mechanisms satisfying ex-post efficiency, ex-post incentive compatibility, ex-post individual rationality, and no ex-post budget deficit exist.

To my wife, Yue.

ACKNOWLEDGMENTS

I would like to express my deep gratitude to my advisor, Steve Williams, for his continuous support, advice, and encouragement. Our frequent meetings have been a great help to me to complete this thesis and become a better researcher. I was very fortunate to have Dan Bernhardt, Jorge Lemus, and Guillermo MarshallI as my committee members who have given me valuable feedback and support. I would like to thank Colleen Schultz for encouraging me to be a better teacher. I am also grateful for the kindness and help given by Carol Banks and Michelle Morfey in the department.

I am indebted to the members of the micro theory reading group for giving me advice and sharing their knowledge with me: Juan Fung, Kazuyuki Hashimoto, Chia-Ling Hsu, Blake Riley, Tom Sahajdack, Rachel Shafer, and Bei Yang. I am also thankful to my friends for their support and advice: Min Kyeong Baik, Jiyoung Chae, Heepyung Cho, Kyu Bong Cho, Henrique Fonseca, Ung Huh, Dongwoo Kim, Youngsoo Kim, Eunhye Kwak, Jae-Myun Lee, Hye Jin Park, and Seokjong Ryu.

I would like to especially thank my parents, Hangwoo Kim and Changki Back, who always love me and pray for me. Finally, I am incredibly grateful to my wife Yue who supports me, loves me, and stands by me all the time.

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Chapter 1

First-Price Auctions with Maxmin Expected Utility Bidders

1.1 Introduction

Much of the existing literature on auctions with independent private valuations makes the following assumption: Each bidder's valuation for the item is drawn from a unique prior F and this distribution is common knowledge among the bidders in the auction. This paper relaxes this assumption. There are two reasons to weaken the unique prior assumption. First, there are real-world examples in which this assumption seems strong. House auctions, merger and acquisition auctions, art auctions, and online auctions are some examples. The items sold in these auctions are unique in the sense that the bidders may not have observed similar items in the past. Thus, the bidders don't have enough information to have a unique prior. The unique prior assumption is accordingly not suitable for these type of auctions. Second, robustness of the results under the unique prior assumption can be examined by weakening the assumption. Many well-known results from the existing literature such as the revenue equivalence theorem or the equilibria of a certain auction format depend crucially on the unique prior assumption. Since Wilson (1989) emphasized the importance of studying mechanisms in a wide class of environments, robustness has been a key question in the study of mechanisms, including auctions. For these two reasons, I relax the unique prior assumption. I assume that bidders in an auction face ambiguity about the probability distribution from which each bidder's valuation is drawn and that they are ambiguity averse. My aim is to study first-price and second-price auctions under this assumption. The paper focuses on the first-price auction because the outcomes of the second-price auctions are trivial.

One of the earliest studies on ambiguity was conducted by Knight (1921). In his book, Knight differentiates ambiguity from risk. Risk refers to situations where probabilities are known; ambiguity, on the other hand, describes situations where probabilities are not known.[1] Ellsberg (1961) subsequently shows that if an agent faces ambiguity about a state of nature and is ambiguity averse, then the decision-making behavior of the agent cannot be explained by a unique belief. A decision-making rule of ambiguity averse agents is axiomatized by Gilboa and Schmeidler (1989). They introduce the maxmin expected utility model with multiple priors in which an agent has a set of multiple priors, instead of a unique prior, about a state of nature. In this model, the ambiguity of the agent is captured by the set of priors, and the agent's utility from choosing an action is its minimum expected utility across all beliefs in her prior set. She then selects the action that maximizes this minimum expected utility. I use the maxmin expected utility model developed by Gilboa and Schmeidler (1989) to explain the bidding behavior of bidders who face ambiguity about the probability distribution from which the other bidders' valuations are drawn.

Some researchers have investigated auctions with bidders with ambiguity by employing the maxmin expected utility decision rule; these researchers include Bose, Ozdenore, and Pape (2006); Bodoh-Creed (2012); and Lo (1998). The main difference between my paper and these papers is that my paper focuses on studying how the outcome of the auction changes as the level of ambiguity faced by the bidders changes. My paper quantifies the ambiguity level of each bidder from her prior set. Then, it analyzes how the bidders' ambiguity level affects their bids and the seller's expected revenue from the auction. This is an interesting question from the seller's perspective. If the seller knows how his expected revenue is impacted by the bidders' ambiguity level, then he may be able to increase his expected revenue by adjusting the bidders' information level.

The following are the main results. I identify a maxmin Bayesian Nash equilibrium of the first price auction.[2] Then, I show that the bidders' equilibrium bids and the seller's expected revenue from the auction increase with the bidders' ambiguity level if the distribution of the bidders' valuations satisfies the monotone inverse hazard rate condition. Moreover, I show that the first price auction generates a larger expected revenue for the seller than the second price auction and that the revenue gap between the two auction formats increases with the level of ambiguity faced by the bidders.

This paper defines the prior set of each bidder as follows. It is assumed that there

[1] Knight (1921) uses the term "uncertainty" instead of "ambiguity" in his book.

[2] Other papers that also study the maxmin Bayesian Nash equilibrium of first price auctions are presented in the next section.

is a probability distribution from which each bidder's valuation is independently drawn and that each bidder's prior set is a set of probability distributions in the neighborhood of this true valuation distribution. The Levy metric, an intuitive metric on the set of probability distributions that measures the maximum distance between the graphs of two cumulative distribution functions, is used to define the neighborhood. Then, the level of ambiguity that each bidder faces is represented by the size of the neighborhood. Because the ambiguity level is captured by a parameter in this prior set definition, it is possible to analyze how the bidders' ambiguity level affects the outcome of the auction. Among the probability distributions in a bidder's prior set defined by the Levy metric, we can consider two distributions: the lower bound distribution and the upper bound distribution. In the prior set, there is a distribution that first-order stochastically dominates all other distributions in the set. I use the term "the lower bound distribution" to denote this distribution because its cumulative distribution function values are lower than those of any other distributions in the set. If a bidder's belief about another bidder's valuation is the lower bound distribution, then compared to the other beliefs in her prior set, she believes that the other bidders' valuations for the item are higher. In the prior set, there is also a distribution that is first-order stocahstically dominated by all other distributions in the set. I use the term "the upper bound distribution" for this distribution. Any probability distribution whose cumulative distribution function values are between those of the lower bound and upper bound distributions are contained in the bidder's prior set.

Consider a maxmin expected utility problem faced by a bidder. Regardless of the bidder's bid, her worst belief, the expected payoff minimizing belief, in her prior set is the probability distribution that first-order stochastically dominates all the other distributions. If the bidder has this belief, then she believes that the other bidders' valuations for the item are high and there is a small chance of her winning the item. Thus, this is the bidder's worst belief. Because the worst belief does not depend on her bid, the maxmin Bayesian Nash equilibrium with multiple priors is the same as the Bayesian Nash equilibrium based on the worst belief. Thus, the maxmin Bayesian Nash equilibrium of the first-price auction is obtained by using the results on Bayesian Nash equilibrium in the first-price auction conducted by Riley and Samuelson (1981) and Monteiro (2009). Riley and Samuelson (1981) derive the equilibrium of the auction when the bidders' valuations are drawn from a continuous distribution, and Monteiro (2009) generalizes this result to the case of distributions with discontinuities.

This paper analyzes how the bidders' bidding behavior and the seller's expected revenue change as the level of ambiguity faced by the bidders changes (Proposition 1.2, 1.3). Under the assumption that the true valuation distribution satisfies the monotone inverse hazard rate condition, each bidder in the auction submits a higher bid in response to an increased level of ambiguity. If a bidder faces a higher level of ambiguity, then her worst belief is more pessimistic than the worst belief from the lower level of ambiguity. That is, if the bidder's ambiguity level increases, then she believes that the other bidders' valuations for the item are higher. Thus, to compete against the other bidders with higher valuations, she submits a higher bid. Due to the higher bids by the bidders, the seller's expected revenue is also higher. It follows that the seller's expected revenue from the auction increases with the level of bidders' ambiguity about the distribution of the other bidders' valuations.

The first price auction can be compared to the second price auction when bidders have ambiguity (Proposition 1.4). If bidders do not face ambiguity, the revenue equivalence theorem says that the first price auction and the second price auction generate the same expected revenue for the seller (Myerson, 1981). However, the first price auction generates greater expected revenue than the second price auction if there is ambiguity. Moreover, the difference in revenues between these two auction formats becomes larger as the bidders' ambiguity level rises. Consider a second price auction. If the bidders don't face ambiguity, it is a dominant strategy for them to bid their own valuations. It is still a dominant strategy even when the bidders face ambiguity because dominant strategies don't depend on the priors that agents have. Therefore, the seller's expected revenue from the second price auction does not depend on whether the bidders have ambiguity or not. As we noted, however, the seller's expected revenue from the first price auction increases with the increases in bidders' ambiguity level. That is, the sensitivity of the auction format to ambiguity is different for the first price and second price auctions, and this leads to the revenue gap between these two auction formats.

The paper is organized as follows. Section 1.2 discusses related literature. Section 1.3 introduces the model and informational assumptions and defines each bidder's prior set. Section 1.4 identifies a maxmin Bayesian Nash equilibrium of the first price auction. Section 1.5 analyzes changes in bidders' bidding behavior and the seller's expected revenue in relation to changes in the bidders' ambiguity level. Section 1.6 compares the results from the first price auction with those of the second price auction. Section 1.7 and 1.8 conclude the paper by offering future research directions.

1.2 Related Literature

Existing literature studies auctions where the bidders face ambiguity about the probability distribution from which the valuations of the other bidders are drawn and they are ambiguity averse. Lo (1998) examines first price and second price sealed-bid auctions using the maxmin expected utility model, showing that the revenue for the seller is greater from the first price auctions than the second price auctions. The main difference between my paper and Lo (1998) is that I examine how the bidders' ambiguity level affects the outcome of the auction. This is possible because each bidder's ambiguity level can be defined by a parameter determining the bidder's prior set in my paper. Bose, Ozdenore, and Pape (2006) and Bodoh-Creed (2012) characterize the revenue maximizing auction. My work focuses on one auction format, the first price auction. The first price auction is not in their set of optimal auctions. However, they mention that their optimal auctions are rarely observed in the real world unlike first price auctions.

Other papers have adopted the maxmin expected utility model to explain the decision-making behavior of agents facing ambiguity. Bergemann and Schlag (2011) investigate the monopoly pricing problem when the monopolist has ambiguity about the demand distribution. I adopt their definition of a prior set to define the bidders' prior sets. I use the Levy metric to define the neighborhood of the true distribution, which is how Bergemann and Schlag (2011) define the monopolist's prior set. Wolitzky (2016) studies properties of mechanisms using the maxmin expected utility model. He works on mechanisms in general while I focus on auctions.

1.3 Model

1.3.1. Auction

There is an indivisible item to be auctioned. There are n risk-neutral bidders and the set of the bidders is $N = \{1, 2, \ldots, n\}$. For each bidder $i \in N$, let $v_i \in [0, 1]$ denote her valuation for the item, $b_i \in [0, 1]$ denote her bid, and $b_{-i} \equiv (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ denote a profile of the bids with bidder i removed.

Consider a first price sealed-bid auction with seller's reserve price r. The bidder with the highest bid wins the item and pays her bid as long as her bid is at least r. Assume that each highest bidder wins the item with the same probability in case of a tie. Let $p_i(b_1, \ldots, b_n)$ denote the probability that bidder i wins the item and $t_i(b_1, \ldots, b_n)$ denote bidder i's expected payment to the seller when $(b_1, \ldots, b_n)$ is a profile of bids submitted by the bidders. Then, the allocation rule and the transfer rule of the auction are defined as follows: for each $i \in N$ and for each bid profile $b = (b_i, b_{-i}) \in [0, 1]^n$,

$$p_i(b_i, b_{-i}) = \begin{cases} 1 & \text{if } b_i > b_{-i}^{max} \text{ and } b_i \geq r, \\ \dfrac{1}{k} & \text{if } b_i = b_{-i}^{max} \text{ and } b_i \geq r, \\ 0 & \text{if } b_i < b_{-i}^{max} \text{ or } b_i < r, \end{cases}$$

$$t_i(b_i, b_{-i}) = \begin{cases} b_i & \text{if } b_i > b_{-i}^{max} \text{ and } b_i \geq r, \\ \dfrac{b_i}{k} & \text{if } b_i = b_{-i}^{max} \text{ and } b_i \geq r, \\ 0 & \text{if } b_i < b_{-i}^{max} \text{ or } b_i < r. \end{cases}$$

where $b_{-i}^{max} \equiv \max_{j \neq i} b_j$ and $k \equiv |\{l \in N \mid b_l = b_i\}|$. Thus, the payoff of bidder i with valuation v_i is

$$\begin{aligned} u_i(b_i, b_{-i}; v_i) &= v_i\, p_i(b_i, b_{-i}) - t_i(b_i, b_{-i}) \\ &= \begin{cases} v_i - b_i & \text{if } b_i > b_{-i}^{max} \text{ and } b_i \geq r, \\ \dfrac{v_i - b_i}{k} & \text{if } b_i = b_{-i}^{max} \text{ and } b_i \geq r, \\ 0 & \text{if } b_i < b_{-i}^{max} \text{ or } b_i < r. \end{cases} \end{aligned}$$

From this point, I assume that the auction is the first price auction with reserve price r unless stated otherwise.

1.3.2. Information

Bidder i's valuation, $v_i \in [0, 1]$, is her private information and unknown to the other bidders and the seller. Assume that each v_i is independently drawn from the continuously differentiable and strictly increasing distribution F_0 on [0,1] whose density function is f_0. Suppose that bidders don't know the distribution, that is, they face ambiguity about the distribution and that they are ambiguity averse . Each bidder knows that the valuations of other bidders are identically and independently distributed from a distribution but she doesn't know the distribution. The ambiguity of each bidder can be represented by a set of probability distributions. We assume that each bidder's set of beliefs about another bidder's valuation is the set of all probability distributions on [0,1] in the closure of ϵ-neighborhood of the distribution F_0. Following Bergemann and Schlag (2011), the Levy metric on the set of probability distributions is used to define the closure of the ϵ-neighborhood of F_0.[3] Then, bidder i's set of beliefs is given by

$$\mathcal{F}_\epsilon(F_0) = \{F \in \Delta[0,1] \,|\, F_0(v-\epsilon) - \epsilon \leq F(v) \leq F_0(v+\epsilon) + \epsilon \;\; \forall v \in [0,1]\}.$$

The Levy metric measures the maximum distance between the graphs of two cumulative distribution functions along a 45° direction. In the belief set defined by the Levy metric, there are two probability distributions on $[0,1]$, $F_0(v-\epsilon)-\epsilon$ and $F_0(v+\epsilon)+\epsilon$, that form a boundary of the set. The graph of each of these distributions is a parallel shift of the graph of the value distribution, F_0. Then, any distribution on $[0,1]$ whose graph is located between the graphs of these two distributions is in the bidder's belief set. Figure 1.1 depicts each bidder's set of beliefs when F_0 follows a uniform distribution on $[0,1]$. Any distribution on $[0,1]$ whose graph falls in the shaded area in the figure is included in the bidder's belief set. In our definition of the belief set using the Levy metric, the size of the neighborhood, ϵ, represents the level of ambiguity that the bidder has. A higher value of ϵ means a higher level of ambiguity of the bidder because the set of beliefs is larger. Because each bidder's ambiguity level is captured by a parameter, it is convenient to analyze how the bidders' ambiguity level affects the outcome of the auction. It is assumed that the auction rule, the reserve price r, and each bidder's set of beliefs $\mathcal{F}_\epsilon(F_0)$ are common knowledge.

1.3.3. Maxmin expected utility bidders

To analyze the behavior of bidders facing ambiguity, I adopt the maxmin expected utility decision rule axiomatized by Gilboa and Schmeidler (1989). Under this decision rule, each bidder calculates the minimum expected payoff across all beliefs for each of her possible bids. Then, she chooses the bid that maximizes the minimum expected payoff. The mathematical formulation of the bidder's minimum expected payoff maximization problem is provided in the next subsection.

1.3.4. The game-theoretic auction and maxmin Bayesian Nash equilibrium

Each bidder's strategy is a bidding function $b_i : [0,1] \rightarrow [0,1]$. Let $v_{-i} \equiv (v_j)_{j\neq i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ denote a vector of bidders' valuations with

[3] See Huber and Ronchetti (2009) on robust statistics for the definition of the Levy metric.

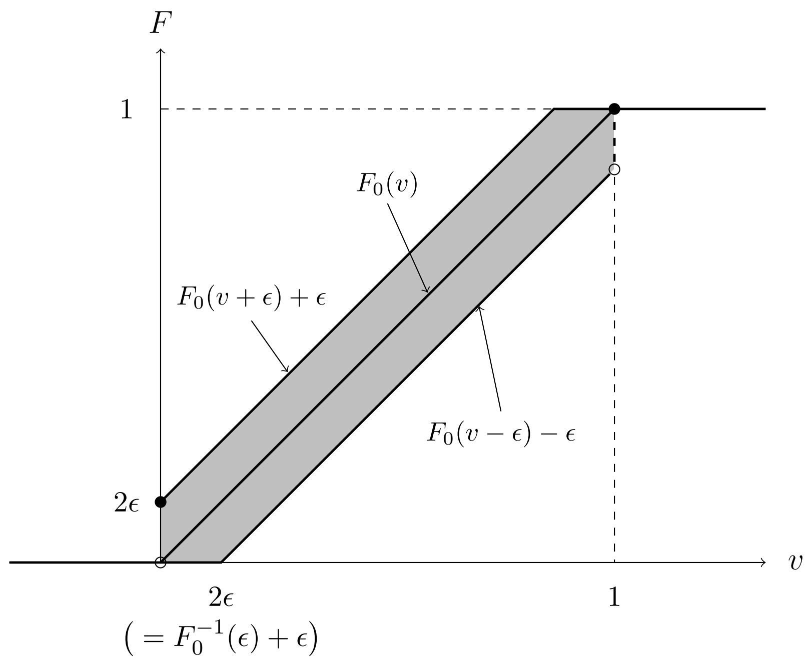

Figure 1.1: A bidder's set of beliefs when F_0 is a uniform distribution on $[0,1]$.

bidder i removed. Consider bidder i with the set of beliefs $\mathcal{F}_\epsilon(F_0)$ on another bidder's valuation. Suppose that bidder i bids b_i, her valuation for the item is v_i and $b_j(\cdot)$ is bidder j's bidding strategy for all $j \neq i$. Bidder i's minimum expected payoff from bidding b_i is defined by

$$\min_{F \in \mathcal{F}_\epsilon(F_0)} \int_{v_{-i} \in [0,1]^{n-1}} u_i\Big(b_i, \big(b_j(v_j)\big)_{j \neq i}; v_i\Big) \prod_{j \neq i} dF(v_j).$$

Bidder i's minimum expected payoff maximization problem can be defined as follows:

$$\max_{b_i \in [0,1]} \min_{F \in \mathcal{F}_\epsilon(F_0)} \int_{v_{-i} \in [0,1]^{n-1}} u_i\Big(b_i, \big(b_j(v_j)\big)_{j \neq i}; v_i\Big) \prod_{j \neq i} dF(v_j).$$

By solving this problem for each valuation $v_i \in [0,1]$, bidder i's minimum expected payoff maximizing bid, $b_i(v_i)$, when the other bidders' bidding strategies are $\big(b_j(\cdot)\big)_{j \neq i}$ can be identified. We can say that this bidding function $b_i(\cdot)$ is bidder i's best response against the other bidders' bidding functions $\big(b_j(\cdot)\big)_{j \neq i}$.

I adopt the **maxmin Bayesian Nash equilibrium** as the solution concept to investigate the behavior of the bidders in the auction. A strategy profile $\big(b_i^*(\cdot)\big)_{i=1}^n$ is a **maxmin Bayesian Nash equilibrium** if each bidder's bidding strategy is her best response against the other bidders' bidding strategies. That is, for each $i \in N$ and for each $v_i \in [0,1]$,

$$b_i^*(v_i) \in \arg\max_{b_i \in [0,1]} \min_{F \in \mathcal{F}_\epsilon(F_0)} \int_{v_{-i} \in [0,1]^{n-1}} u_i\Big(b_i, \big(b_j^*(v_j)\big)_{j \neq i}; v_i\Big) \prod_{j \neq i} dF(v_j).$$

1.4 A Maxmin Bayesian Nash Equilibrium

Assume that the seller's reserve price is at least $F_0^{-1}(\epsilon) + \epsilon$. That is,

$$r \geq F_0^{-1}(\epsilon) + \epsilon.$$

Note that $F_0^{-1}(\epsilon) + \epsilon$ is the greatest lower bound of the supports of the probability distributions in the bidders' prior set. This is assumed because otherwise the bidder with her valuation less than $F_0^{-1}(\epsilon) + \epsilon$ would update her prior set based on the valuation. For example, suppose that bidder i's valuation v_i is strictly less than $F_0^{-1}(\epsilon) + \epsilon$. Bidder i knows that value v_i is in the support of the valuation distribution. Thus, the probability distribution with a support of $[F_0^{-1}(\epsilon) + \epsilon, 1]$ cannot be included in her prior set. This problem is irrelevant in this paper because of the assumption that the reserve price is at least $F_0^{-1}(\epsilon) + \epsilon$.

Consider bidder i with valuation $v_i \in [0,1]$. Suppose that the profile of the other bidders' strategies is $\big(b_j(\cdot)\big)_{j \neq i}$ and each $b_j(\cdot)$ is a strictly increasing function. Then, bidder i's minimum expected payoff maximization problem is defined as follows:

$$\max_{b_i \in [0,1]} \min_{F \in \mathcal{F}_\epsilon(F_0)} (v_i - b_i) * Pr\,(i \text{ wins the item}).$$

That is,

$$\max_{b_i \in [0,1]} \min_{F \in \mathcal{F}_\epsilon(F_0)} (v_i - b_i) * Pr\,(b_i \geq b_j(v_j) \;\; \forall j \neq i).$$[4]

We can make two observations from this maxmin problem. First, bidder i's optimal bid b_i, the minimum expected payoff maximizing bid, is less than or equal to her valuation, v_i, because her expected payoff would be negative otherwise. Second, for each of her possible bids, $b_i \in [0,1]$, we can find the expected payoff minimizing

[4] In this formulation, I don't consider the cases of ties for convenience' sake. This does not affect the claim and the results I am going to derive in the paper.

belief, $F \in \mathcal{F}_\epsilon(F_0)$. From the objective function of the maxmin problem, the expected payoff function, we can obtain that

$$\begin{aligned}&(v_i - b_i) * Pr\left(b_i \geq b_j(v_j) \ \ \forall j \neq i\right)\\&= (v_i - b_i) * Pr\left(v_j \leq b_j^{-1}(b_i) \ \ \forall j \neq i\right)\\&= (v_i - b_i) * \prod_{j \neq i} F\big(b_j^{-1}(b_i)\big).\end{aligned}$$

Thus, for given $b_i \in [0, 1]$, the expected payoff minimizing belief minimizes bidder i's probability of winning, $\prod_{j \neq i} F\big(b_j^{-1}(b_i)\big)$. Bidder i's probability of winning is a product of cumulative probabilities, $F\big(b_j^{-1}(b_i)\big)$. Thus, a belief minimizing each of these cumulative probabilities is the expected payoff minimizing belief. Therefore, when each bidder's set of priors is given by $\mathcal{F}_\epsilon(F_0)$ defined in the previous section, the expected payoff minimizing belief is F_ϵ^* satisfying

$$F_\epsilon^*(v) = \begin{cases} 0 & \text{if } \ v < F_0^{-1}(\epsilon) + \epsilon, \\ F_0(v - \epsilon) - \epsilon & \text{if } \ F_0^{-1}(\epsilon) + \epsilon \leq v < 1, \\ 1 & \text{if } \ v \geq 1. \end{cases}$$

We can see that F_ϵ^* is continuous on $[F_0^{-1}(\epsilon) + \epsilon, 1)$ and discontinuous at 1. Notice also that F_ϵ^* is bidder i's expected payoff minimizing belief no matter which bid $b_i \in [0, 1]$ she chooses.

Example 1.1. If F_0 is a uniform distribution on $[0, 1]$, then the distribution $F_\epsilon^{U*} \in \mathcal{F}_\epsilon(F_0)$ satisfying

$$F_\epsilon^{U*}(v) = \begin{cases} 0 & \text{if } \ v < 2\epsilon, \\ v - 2\epsilon & \text{if } \ 2\epsilon \leq v < 1, \\ 1 & \text{if } \ v \geq 1. \end{cases}$$

is bidder i's expected payoff minimizing belief. In Figure 1.1, the distribution function on the bottom boundary of the shaded area corresponds to this belief.

Consider bidder i's maxmin expected payoff problem:

$$\max_{b_i \in [0,1]} \ \min_{F \in \mathcal{F}_\epsilon(F_0)} (v_i - b_i) * \prod_{j \neq i} F\big(b_j^{-1}(b_i)\big).$$

Because F_ϵ^* obtained above is the expected payoff minimizing belief for any bid b_i, this maxmin expected payoff problem is equivalent to the following problem:

$$\max_{b_i \in [0,1]} (v_i - b_i) * \prod_{j \neq i} F_\epsilon^* \big(b_j^{-1}(b_i) \big).$$

This is bidder i's expected payoff maximization problem when her belief on another bidder's valuation is F_ϵ^*. From the equivalence of these two problems, it follows that a maxmin Bayesian Nash equilibrium of the first price auction with bidders having sets of priors $\mathcal{F}_\epsilon(F_0)$ is equal to the Bayesian Nash equilibrium of the first price auction with bidders having common prior F_ϵ^*. There are many previous literature studying Bayesian Nash equilibrium of the first price auction with a common prior and thus, we can find out a maxmin Bayesian Nash equilibrium from the results of those literature. Riley and Samuelson (1981) study the unique Bayesian Nash equilibrium of the first price auction when bidders have the common belief F that is strictly increasing and differentiable. They show that the equilibrium bidding function is given by

$$b_i(v_i) = v_i - \frac{\int_{v=r}^{v_i} (F(v))^{n-1} dv}{(F(v_i))^{n-1}} \tag{1.1}$$

for $v_i \geq r$ where r is the seller's reserve price. Monteiro (2009) generalizes this result and identifies the Bayesian Nash equilibria when the common prior F has discontinuities. He shows that the equilibrium bidding strategy is equal to (1.1) at the continuous points of F and a mixed strategy at the discontinuities of F. The expected payoff minimizing belief, F_ϵ^*, in our paper has one discontinuity at $v = 1$. Thus, we can find out a maxmin Bayesian Nash equilibrium of the first price auction as follows by using the result of Monteiro (2009).

Proposition 1.1. Consider a first price sealed-bid auction with the seller's reserve price r satisfying $r \geq F_0^{-1}(\epsilon) + \epsilon$. Suppose that each bidder has a set of priors $\mathcal{F}_\epsilon(F_0)$ about another bidder's valuation for the item. Then, a profile of mixed strategies $\big(\mu_i(\cdot)\big)_{i=1}^n$ is a **maxmin Bayesian Nash equilibrium** of the auction if for each $i \in \{1, \ldots, n\}$,

$$\mu_i(v_i) = \begin{cases} \text{pure strategy, } b_i^*(v_i) = v_i - \dfrac{\int_{v=r}^{v_i} \big(F_0(v-\epsilon) - \epsilon\big)^{n-1} dv}{\big(F_0(v_i - \epsilon) - \epsilon\big)^{n-1}} & \text{if } v_i \in [r, 1), \\ \text{mixed strategy, } G & \text{if } v_i = 1, \end{cases}$$

where $G : \left[b_i^{F_\epsilon^*}(1-), b_i^{F_\epsilon^*}(1)\right] \to \left[0, 1\right]$ is a cumulative distribution function satisfying

$$G(b) = \frac{F_0(1-\epsilon)-\epsilon}{1-(F_0(1-\epsilon)-\epsilon)}\left(-1+\left(\frac{1-b_i^{F_\epsilon^*}(1-)}{1-b}\right)^{\frac{1}{n-1}}\right),$$

F_ϵ^* is the expected payoff minimizing prior, and

$$b_i^{F_\epsilon^*}(v_i) = v_i - \frac{\int_{v=r}^{v_i}\left(F_\epsilon^*(v)\right)^{n-1}dv}{\left(F_\epsilon^*(v_i)\right)^{n-1}} \qquad \text{for } v_i \in [r, 1].$$

Remark 1.1. I identify a maxmin Bayesian Nash equilibrium in the proposition. I don't show its uniqueness. There may be other equilibria.

Remark 1.2. In the maxmin Bayesian Nash equilibrium, a bidder plays the mixed strategy G only when her valuation is equal to 1. Because the true distribution F_0 is continuous, the event that the bidder's valuations for the item is equal to 1 has measure 0. Thus, we focus on the bidder's pure strategy, $v_i - \frac{\int_{v=r}^{v_i}\left(F_0(v-\epsilon)-\epsilon\right)^{n-1}dv}{\left(F_0(v_i-\epsilon)-\epsilon\right)^{n-1}}$, from this point. Let $b_i^*(v_i)$ denote this pure strategy for $v_i \in [r, 1)$.

Example 1.1. (continued.) Suppose that the true distribution F_0 is a uniform distribution on $[0, 1]$, that is, $F_0(v) = v$ for $v \in [0, 1]$. Then, bidder i's maxmin Bayesian Nash equilibrium bidding strategy for $v_i \in [r, 1)$ is given by

$$b_i^*(v_i) \;=\; v_i - \frac{\int_{v=r}^{v_i}(v-2\epsilon)^{n-1}\,dv}{(v_i-2\epsilon)^{n-1}} \;=\; \frac{n-1}{n}v_i + \frac{2\epsilon}{n} + \frac{1}{n}\frac{(r-2\epsilon)^n}{(v_i-2\epsilon)^{n-1}}.$$

If the bidders don't have any ambiguity about the other bidders' valuations ($\epsilon = 0$), then each bidder's Bayesian Nash equilibrium bidding strategy is $b_i(v_i) = \frac{n-1}{n}v_i + \frac{r^n}{nv_i^{n-1}}$.

1.5 Changes in Bidders' Ambiguity Level

From the maxmin Bayesian Nash equilibrium of the first price auction we obtained in Proposition 1.1, we can study how each bidder's equilibrium bidding behavior changes as her level of ambiguity changes. The higher value of ϵ implies the larger set of beliefs, $\mathcal{F}_\epsilon(F_0)$, and thus, the higher level of ambiguity that each bidder has. It can be shown that each bidder bids higher in response to the higher level of ambiguity if the probability distribution of the bidders' valuations, F_0, satisfies a certain condition.

Definition 1.1. The distribution F satisfies the **monotone inverse hazard rate condition** if $\dfrac{f(v)}{F(v)}$ is non-increasing in v.

Then, we can obtain the following result:

Proposition 1.2. Suppose that F_0 satisfies the monotone inverse hazard rate condition. Then, each bidder with her valuation for the item $v_i \in (r, 1)$ submits a strictly higher bid in response to an increased level of ambiguity.

Proof. From the Proposition 1.1, it follows that bidder i's maxmin Bayesian Nash equilibrium bidding strategy is

$$b_i^*(v_i) = v_i - \frac{\int_{v=r}^{v_i} \big(F_0(v-\epsilon)-\epsilon\big)^{n-1} dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-1}}.$$

Differentiating with respect to ϵ yields

$$\begin{aligned}
&\frac{db_i^*(v_i)}{d\epsilon} = \\
&\frac{\int_{v=r}^{v_i} (n-1)\big(F_0(v-\epsilon)-\epsilon\big)^{n-2}\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-1}\big(f_0(v-\epsilon)+1\big)\,dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{2n-2}} \\
&- \frac{\int_{v=r}^{v_i} (n-1)\big(F_0(v-\epsilon)-\epsilon\big)^{n-1}\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-2}\big(f_0(v_i-\epsilon)+1\big)\,dv}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{2n-2}}
\end{aligned}$$

$$\frac{\begin{aligned} &= \int_{v=r}^{v_i} (n-1)\big(F_0(v-\epsilon)-\epsilon\big)^{n-2}\big(F_0(v_i-\epsilon)-\epsilon\big)^{n-2} \\ &\quad * \Big[\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big)\Big]dv \end{aligned}}{\big(F_0(v_i-\epsilon)-\epsilon\big)^{2n-2}}. \tag{1.2}$$

Consider the following term in the brackets in (1.2):

$$\Big[\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big)\Big].$$

It is given that $v \leq v_i$. It can be shown that the value of this term is strictly positive if $v < v_i$. There are two possible cases: $f_0(v-\epsilon) \geq f_0(v_i-\epsilon)$ and $f_0(v-\epsilon) < f_0(v_i-\epsilon)$. Consider $f_0(v-\epsilon) \geq f_0(v_i-\epsilon)$. Because F_0 is strictly increasing, it follows that $F_0(v_i-\epsilon)-\epsilon > F_0(v-\epsilon)-\epsilon$. Thus, $\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big)-\big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big) > 0$. Now suppose that $f_0(v-\epsilon) < f_0(v_i-\epsilon)$. Because F_0 satisfies the monotone inverse hazard rate condition, it follows that $\frac{f_0(v-\epsilon)}{F_0(v-\epsilon)} \geq \frac{f_0(v_i-\epsilon)}{F_0(v_i-\epsilon)}$, that is, $F_0(v_i-\epsilon)f_0(v-\epsilon) - F_0(v-\epsilon)f_0(v_i-\epsilon) \geq 0$. Thus, we obtain that

$$
\begin{aligned}
&F_0(v_i-\epsilon)f_0(v-\epsilon) - F_0(v-\epsilon)f_0(v_i-\epsilon) \geq 0 \\
&\Rightarrow \quad \big(F_0(v_i-\epsilon)-\epsilon\big)f_0(v-\epsilon) - \big(F_0(v-\epsilon)-\epsilon\big)f_0(v_i-\epsilon) > 0 \\
&\Rightarrow \quad \big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big) - \big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big) > 0.
\end{aligned}
$$

The second inequality is obtained from the supposition that $f_0(v-\epsilon) < f_0(v_i-\epsilon)$. The third inequality is obtained from strictly increasing F_0. By investigating the two possible cases, we have that $\big(F_0(v_i-\epsilon)-\epsilon\big)\big(f_0(v-\epsilon)+1\big) - \big(F_0(v-\epsilon)-\epsilon\big)\big(f_0(v_i-\epsilon)+1\big)$ is strictly positive if $v < v_i$. Therefore, the value of the integral of (1.2) is strictly positive.■

Example 1.1. (continued.) If F_0 is a uniform distribution on $[0,1]$, then its inverse hazard rate function, $\frac{1}{v}$, is decreasing in v. Therefore, we can apply Proposition 1.2 and say that each bidder responds to increased ambiguity with a higher bid when F_0 is a uniform distribution.

Because the bidders increase their bids in response to an increased level of ambiguity, the seller's expected revenue from the auction also increases.

Proposition 1.3. Suppose that F_0 satisfies the monotone inverse hazard rate condition. Then, the seller's expected revenue from the auction when bidders face ambiguity is greater than the one when bidders don't face ambiguity. Moreover, the seller's expected revenue increases as the level of ambiguity faced by the bidders increases.

Proof. Suppose that each bidder in the auction has a prior set $\mathcal{F}_\epsilon(F_0)$ about the other bidders' valuations. Let $b^{\epsilon *}(\cdot)$ denote the maxmin Bayesian Nash equilibrium bidding strategy of each bidder we obtained in Proposition 1.1. Then, the seller's expected revenue is defined by

$$R(\epsilon) = \int_{v=r}^{1} b^{\epsilon *}(v)\, n\big(F_0(v)\big)^{n-1} f_0(v)\, dv.$$

Consider two numbers, $\epsilon_1 \in [0,1]$ and $\epsilon_2 \in [0,1]$, satisfying $\epsilon_1 < \epsilon_2$. Note that a bidder having prior set $\mathcal{F}_{\epsilon_2}(F_0)$ faces the higher level of ambiguity than a bidder having prior set $\mathcal{F}_{\epsilon_1}(F_0)$. Note also that if $\epsilon_1 = 0$, then the prior set $\mathcal{F}_{\epsilon_1}(F_0)$ is a singleton and it implies that the bidder does not have ambiguity. Because $\epsilon_1 < \epsilon_2$, it follows that $b^{\epsilon_1 *}(v) < b^{\epsilon_2 *}(v)$ for all $v \in (r, 1)$ from the result of Proposition 1.2. Therefore, $R(\epsilon_1) < R(\epsilon_2)$.■

1.6 A Comparison with the Second Price Auction

The first price auction is compared to another popular auction format, the second price auction, when bidders have ambiguity about the probability distribution from which the other bidders' valuations for the item are drawn.

Bidders in the second price auction have an incentive to bid their own valuations even when they face ambiguity. Consider the second price auction with bidders facing no ambiguity. In this case, there is a well-known result that truthful bidding from each bidder forms a dominant strategy equilibrium. Since it is a dominant strategy, each bidder's incentive for truthful bidding does not depend on her belief about the other bidders' valuations. Thus, even when the bidders face ambiguity and have multiple beliefs, truthful bidding forms a dominant strategy equilibrium in the second price auction.

The bidders in the second price auction always bid their true valuations, no matter whether they have ambiguity or not about the distribution of the others' valuations. Thus, the seller's expected revenue from the second price auction does not depend on whether the bidders face ambiguity. However, we know from proposition 1.3 that the seller's expected revenue from the first price auction increases as the bidders' ambiguity level in the auction increases. When bidders don't face ambiguity, there is a well-known revenue equivalence result between first price and second price auctions. From these results on the seller's expected revenue from two auction formats, it follows that:

Proposition 1.4. Consider a first price auction and a second price auction with the seller's reserve price r. Suppose that the true distribution, F_0, from which each bidder's valuation is drawn satisfies the monotone inverse hazard rate condition, the bidders face ambiguity about the distribution and they are ambiguity averse, and each bidder's set of priors is $\mathcal{F}_\epsilon(F_0)$. Then the seller can obtain the higher expected revenue from the first price auction than the second price auction. Moreover, the difference in expected revenues from two auction formats increases as the level of ambiguity faced by the bidders increases.

1.7 Conclusion

This paper analyzes the first price auction where each bidder faces ambiguity about the probability distribution from which the other bidders' valuations are independently drawn and is ambiguity averse. The maxmin expected utility model with multiple priors axiomatized by Gilboa and Schmeidler (1989) is used to solve the bidders' optimal bidding problems. The bidders' equilibrium bidding functions and the seller's expected revenue from the auction are identified. Moreover, it is shown that the bidders' equilibrium bids and the seller's expected revenue increase as the bidders' ambiguity level increases. It is also determined that the seller's expected revenue from the first price auction is greater than that of the second price auction when the bidders face ambiguity.

1.8 Future Research Directions

Asymmetry in bidders' ambiguity levels. I assumed that the level of ambiguity is the same for all bidders. As a next step, I plan to relax this assumption and assign different ambiguity levels to bidders.

Endogenous participation of bidders. I assumed that the number of bidders is exogenously given and showed that the seller's expected revenue from the first price auction increases with the level of ambiguity faced by bidders. Suppose that bidders can choose to participate in the auction or not. In this case, increased level of ambiguity may discourage bidders from participating in the auction and thus, it may reduce the seller's expected revenue from the auction. Endogenous participation of bidders can lead to a different result from this paper.

Chapter 2

Multilateral Trading Mechanism with Interdependent Values

2.1 Introduction

In real life, there are many trading situations in which multiple buyers and sellers report his or her own information, and an institution allocates resources based on these reported information. Examples include stock exchanges such as the New York Stock Exchange where the prices of stocks are determined based on the reported demand and supply from traders. A possible issue in this trading situation is building an institution, that is, a trading rule that has desired properties. [1]

In this paper, I study one trading mechanism in a market with multiple buyers and multiple sellers, that is, a multilateral trading mechanism. The setting is as follows. There are many buyers and sellers. Each trader can buy or sell at most one indivisible item. Each trader is given her own real-valued type that is private information for the trader. Therefore, this market can be analyzed as a game of incomplete information. Traders in the market have interdependent values. That is, each trader's valuation for one unit of the item may depend on types of all traders, not only on her own type. Interdependent values are observed in many practical situations. For example, in a stock exchange, a trader's assessment of the value of a stock may depend on the other traders' information concerning the stock. Each trader has quasi-linear utility that consists of a valuation for the item and a monetary transfer.

This paper introduces a direct revelation mechanism under which each trader in the market reports her own type. Allocation of the items and transfers are determined based on these reported types. The mechanism that I consider satisfies ex-post incentive compatibility, ex-post individual rationality, and ex-post efficiency. Because it is ex-post incentive compatible, no trader has an incentive to

[1] Desirable properties include incentive compatibility, individual rationality, budget balance, and economic efficiency, which are defined formally in a later section.

misreport her type even after the other traders' true types are revealed. And voluntary participation is guaranteed in this mechanism because it is ex-post individually rational. Finally, the mechanism is ex-post efficient therefore the items are owned by the traders who value the item the most after the mechanism operates.

A brief explanation of how the mechanism satisfies ex-post incentive compatibility and ex-post efficiency is as follows. First, the mechanism allocates the good efficiently based on reported types, and the transfer rules are defined in a way that induces each trader to report her type honestly. How the winner's payment is determined in the second price auction motivates the transfer rules of the mechanism. The winning bidder's payment is her lowest bid that would still make her the winner in the second price auction. In the mechanism, transfers are defined by using the similar logic in consideration of the market with multiple buyers and multiple sellers with interdependent valuations. Then, each trader under the mechanism has an incentive to report her true type. From these honestly reported types and the allocation rule, the mechanism allocates the items efficiently.

In sum, my goal is to propose a multilateral trading mechanism satisfying desired properties in an interdependent values setting. The paper is organized as follows. Section 2.2 discusses related literature. Section 2.3 introduces models and related assumptions. Section 2.4 defines the trading mechanism. Section 2.5 studies desired properties of the mechanism. Section 2.6 presents an example. Section 2.7 and 2.8 conclude with future research directions.

2.2 Previous Literature

Previous researches have studied trading mechanisms in markets with buyers and sellers with incomplete information. Myerson and Satterthwaite (1983) consider the impossibility result of the trading between one buyer and one seller with private valuations for an item.[2] They show that a trading mechanism satisfying Bayesian incentive compatibility, interim individual rationality, and ex-ante budget balance cannot be ex-post efficient.

Dasgupta and Maskin (2000) study an efficient auction when bidders' valuations for items are interdependent. They show that a generalized VCG auction

[2]Private valuations imply that each trader's valuation for an item depends solely on her own type.

for interdependent values can achieve ex-post efficiency when traders' types are single-dimensional.[3] The main difference between my paper and Dasgupta and Maskin (2000) is that I analyze a two-sided problem in which all buyers and sellers can affect the price and allocation of the items, in contrast to an auction in which only buyers can. Also, each trader under my mechanism reports her own type that is a single value, whereas each buyer in the auction by Dasgupta and Maskin (2000) reports a bidding function that is a function of other buyers' valuations.

Jehiel and Moldovanu (2001) obtain an impossibility result about mechanisms with interdependent valuations. They show that if agents have multi-dimensional types, then a mechanism satisfying both efficiency and Bayes-Nash incentive compatibility does not exist. In my paper, I assume that each trader's type is one-dimensional. As a result, my mechanism can satisfy both ex-post efficiency and ex-post incentive compatibility.

Kojima and Yamashita (2014) propose a trading mechanism in the market where there are multiple buyers and sellers with interdependent valuations. They develop a mechanism called the groupwise-price mechanism under which a market is divided into many submarkets. They show that the groupwise-price mechanism satisfies ex-post incentive compatibility, ex-post individual rationality, ex-post budget balance, and asymptotic efficiency. The mechanism in my paper is efficient with any finite number of traders. However, it may not satisfy ex-post budget balance.

2.3 Model

Consider a market with m buyers and n sellers. Let $B = \{b_1, b_2, \cdots, b_m\}$ be a set of buyers and $S = \{s_1, s_2, \cdots, s_n\}$ be a set of sellers. Each trader buys or sells at most one unit of an indivisible item. Each buyer $b_i \in B$ observes her own type $t_{b_i} \in [0, 1]$ and each seller $s_j \in S$ observes her own type $t_{s_j} \in [0, 1]$. The type t_i is trader i's private information. Let $t = (t_{b_1}, t_{b_2}, \cdots, t_{b_m}, t_{s_1}, t_{s_2}, \cdots, t_{s_n})$ be a profile of types of all traders, $t_{-b_i} = (t_{b_1}, \cdots, t_{b_{i-1}}, t_{b_{i+1}}, \cdots, t_{b_m}, t_{s_1}, \cdots, t_{s_n})$ be a profile of types with buyer b_i removed, and $t_{-s_j} = (t_{b_1}, \cdots, t_{b_m}, t_{s_1}, \cdots, t_{s_{j-1}}, t_{s_{j+1}}, \cdots, t_{s_n})$ be a profile of types with seller s_j removed. For each trader $i \in B \cup S$, $v_i(t)$ denotes i's valuation for the item. The valuation $v_i(t)$ depends on i's

[3] See Clarke (1971), Groves (1973), and Vickrey (1961) for VCG auction.

own type t_i as well as other traders' types t_{-i}. Thus, traders' valuations for the item are interdependent. We assume the following about the valuation function $v_i(\cdot)$.

A1. $v_i(\cdot)$ is continuously differentiable for all $i \in B \cup S$.

A2. $v_i(\cdot)$ is non-decreasing in t_j for all $i, j \in B \cup S$:

$$\frac{\partial v_i}{\partial t_j} \geq 0.$$

A3. $v_i(\cdot)$ is strictly increasing in t_i for all $i \in B \cup S$:

$$\frac{\partial v_i}{\partial t_i} > 0.$$

A4. A trader i's type t_i has a greater effect on her own valuation $v_i(\cdot)$ than on the other trader j's valuation $v_j(\cdot)$ for all i and $j \neq i$:

$$\frac{\partial v_i}{\partial t_i} > \frac{\partial v_j}{\partial t_i}.$$

Traders have quasi-linear utility functions. Buyer b_i's utility is $v_{b_i}(t) - p$ if the buyer purchases the item and pays price p, and it is 0 if the buyer does not trade and pays nothing. Seller s_j's utility is $p - v_{s_j}(t)$ if she sells her item and receives payment p, and it is 0 if she does not participate in the trade and receives nothing. We assume that the valuation functions, $v_{b_1}(\cdot), \cdots, v_{b_m}(\cdot), v_{s_1}(\cdot), \cdots, v_{s_n}(\cdot)$, are common knowledge among the buyers and sellers, as well as the auctioneer.

A direct revelation mechanism is a pair of functions (π, τ) defined on the set of type profiles, $[0,1]^{m+n}$. The functions π and τ represent the allocation rule and the transfer rule of the mechanism respectively. Specifically, for trader $i \in B \cup S$, the function $\pi_i(t)$ denotes the probability that the trade occurs for i when the reported type profile is t. The function $\tau_i(t)$ is the expected transfer for trader i when the reported type profile is t: $\tau_b(t)$ is the expected transfer buyer b pays for $b \in B$, and $\tau_s(t)$ is the expected transfer seller s receives for $s \in S$. Thus, buyer b's ex-post utility is $v_b(t)\pi_b(t') - \tau_b(t')$ and seller s's ex-post utility is $\tau_s(t') - v_s(t)\pi_s(t')$ when the reported type profile is $t' = (t'_{b_1}, \cdots, t'_{b_m}, t'_{s_1}, \cdots, t'_{s_n})$ and the true type profile is $t = (t_{b_1}, \cdots, t_{b_m}, t_{s_1}, \cdots, t_{s_n})$.

A mechanism (π, τ) is *ex-post incentive compatible* if

$$v_b(t_b, t_{-b})\pi_b(t_b, t_{-b}) - \tau_b(t_b, t_{-b}) \geq v_b(t_b, t_{-b})\pi_b(t'_b, t_{-b}) - \tau_b(t'_b, t_{-b})$$

for each t_b, t'_b, t_{-b}, and b, and

$$\tau_s(t_s, t_{-s}) - v_s(t_s, t_{-s})\pi_s(t_s, t_{-s}) \geq \tau_s(t'_s, t_{-s}) - v_s(t_s, t_{-s})\pi_s(t'_s, t_{-s})$$

for each t_s, t'_s, t_{-s}, and s.

If a mechanism is ex-post incentive compatible, then it is the best response ex-post for each trader to report her true type given that the other traders report their own true types. Thus, reporting true types forms an ex-post Nash equilibrium.

A mechanism (π, τ) is *ex-post individually rational* if

$$v_b(t)\pi_b(t) - \tau_b(t) \geq 0 \quad \text{for each } t \text{ and } b, \text{ and}$$
$$\tau_s(t) - v_s(t)\pi_s(t) \geq 0 \quad \text{for each } t \text{ and } s.$$

In an ex-post individually rational mechanism, each trader obtains non-negative utility after all true types of the traders are revealed. Thus, each trader has an incentive to participate in the trading process.

A mechanism is *ex-post budget balanced* if

$$\sum_{i=1}^{m} \tau_{b_i}(t) = \sum_{j=1}^{n} \tau_{s_j}(t) \quad \text{for each } t.$$

That is, the total transfers paid by buyers equal the total transfers received by sellers.

A mechanism (π, τ) is *ex-post efficient* if it allocates the items to the traders with n highest valuations. That is, assume that the traders' valuations based on reported types profile t are listed in ascending order as follows:

$$v_{(1)}(t) \leq v_{(2)}(t) \leq \cdots \leq v_{(m+n)}(t).$$

If an ex-post efficient mechanism is applied, then the items are possessed by n traders whose valuations correspond to $v_{(m+1)}(t), v_{(m+2)}(t), \cdots$ and $v_{(m+n)}(t)$ respectively.[4]

2.4 Mechanism

In my mechanism, the allocations and payments are defined as follows:
Consider buyers $b_1, b_2, \cdots, b_m$ with true types $t_{b_1}, t_{b_2}, \cdots, t_{b_m}$ respectively, and sellers $s_1, s_2, \cdots, s_n$ with true types $t_{s_1}, t_{s_2}, \cdots, t_{s_n}$ respectively. Each trader i in $B \cup S$ reports a type $t_i^{'}$ simultaneously. After types are reported, traders' valuations for the item, $v_{b_i}(t^{'})$ for $b_i \in B$ and $v_{s_j}(t^{'})$ for $s_j \in S$, can be calculated based on their reported types, $t^{'}$. Then the items are allocated efficiently based on these

[4] In the case of tie at $v_{(m+1)}(t)$, that is, $|\{i \in B \cup S | v_i(t) = v_{(m+1)}(t)\}| \geq 2$, it does not matter who acquires the item among them.

valuations from the reported types. More formally, the procedure of allocation is defined as follows: arrange these valuations in ascending order:

$$v_{(1)}(t^{'}) \leq v_{(2)}(t^{'}) \leq \cdots \leq v_{(m+n)}(t^{'}).$$

Then, consider the following sets of traders:

$$B_\alpha = \{b_i \in B | v_{b_i}(t^{'}) \geq v_{(m+1)}(t^{'})\},$$
$$S_\alpha = \{s_j \in S | v_{s_j}(t^{'}) \leq v_{(m)}(t^{'})\}.$$

B_α is the set of buyers whose valuation at $t^{'}$ is at least $v_{(m+1)}(t^{'})$, and S_α is the set of sellers whose valuation is no more than $v_{(m)}(t^{'})$. Depending on the relationship between $v_{(m)}(t^{'})$ and $v_{(m+1)}(t^{'})$, there are two cases to be considered to define the allocation rule.[5] First, if $v_{(m+1)}(t^{'}) > v_{(m)}(t^{'})$, that is, if $v_{(m+1)}(t^{'})$ is strictly greater than $v_{(m)}(t^{'})$, then $|B_\alpha| = |S_\alpha|$. In this case, the buyers in B_α and the sellers in S_α participate in trade, and buyers in $B \backslash B_\alpha$ and sellers in $S \backslash S_\alpha$ do not trade. In the second case that $v_{(m)}(t^{'}) = v_{(m+1)}(t^{'})$, $|B_\alpha|$ may not be equal to $|S_\alpha|$. In this case, k buyers in B_α and k sellers in S_α participate in trade where

$$k := min\{|B_\alpha|, |S_\alpha|\}.$$

The k traders in each set are determined by the following procedure. Consider the following subsets of B_α and S_α.

$$B_s = \{b_i \in B | v_{b_i}(t^{'}) > v_{(m+1)}(t^{'})\},$$
$$S_s = \{s_j \in S | v_{s_j}(t^{'}) < v_{(m)}(t^{'})\}.$$

Then,

$$B_\alpha \backslash B_s = \{b_i \in B | v_{b_i}(t^{'}) = v_{(m+1)}(t^{'})\},$$
$$S_\alpha \backslash S_s = \{s_j \in S | v_{s_j}(t^{'}) = v_{(m)}(t^{'})\}.$$

All buyers in B_s and $k - |B_s|$ randomly chosen buyers in $B_\alpha \backslash B_s$ participate in trade. Similarly, all sellers in S_s and $k - |S_s|$ randomly chosen sellers in $S_\alpha \backslash S_s$ participate in trade.[6] Thus, k traders from each side participate in trade.

From the allocation rule of this mechanism, we can note that the k buyers with the highest valuations and the k sellers with the lowest valuations participate in trade and that all buyers who trade have weakly higher valuations for the item than

[5]The allocation rule of this mechanism is the same as the k-double auction studied by Rustichini, Satterthwaite, and Williams (1994).

[6]See Appendix A for well-definedness of the allocation rule.

all sellers who trade. Also, all buyers who don't participate in trade have lower valuations for the item than all sellers who don't participate in trade. Thus, under this allocation rule, the items are assigned efficiently based on the reported types of traders.

Formally, the allocation rule of the mechanism is defined as follows:

For $b_i \in B$,

$$\pi_{b_i}(t^{'}) = \begin{cases} 1 & \text{if} \quad v_{b_i}(t^{'}) > v_{(m+1)}(t^{'}), \\ \dfrac{k - |B_s|}{|B_\alpha| - |B_s|} & \text{if} \quad v_{b_i}(t^{'}) = v_{(m+1)}(t^{'}), \\ 0 & \text{if} \quad v_{b_i}(t^{'}) < v_{(m+1)}(t^{'}). \end{cases}$$

and for $s_j \in S$,

$$\pi_{s_j}(t^{'}) = \begin{cases} 1 & \text{if} \quad v_{s_j}(t^{'}) < v_{(m)}(t^{'}), \\ \dfrac{k - |S_s|}{|S_\alpha| - |S_s|} & \text{if} \quad v_{s_j}(t^{'}) = v_{(m)}(t^{'}), \\ 0 & \text{if} \quad v_{s_j}(t^{'}) > v_{(m)}(t^{'}). \end{cases}$$

I now turn to the transfers of the mechanism. Consider buyer b_i. If trade does not occur for her, no transfer is made. If trade occurs, her lowest reported type, $\hat{t}_{b_i} \in [0, 1]$, at which trade still occurs for her with positive probability can be identified. Then, the buyer's payment is her valuation at that type. To define this formally, we define the following set.

$$T_{b_i}(t^{'}_{-b_i}) = \{t_{b_i} \in [0, 1] : v_{b_i}(t_{b_i}, t^{'}_{-b_i}) \geq v_{(m+1)}(t_{b_i}, t^{'}_{-b_i})\}.$$

$T_{b_i}(t^{'}_{-b_i})$ is a set of reported types of b_i at which trade happens with strictly positive probability when other traders report $t^{'}_{-b_i}$. Then,

$$\hat{t}_{b_i} = \inf T_{b_i}(t^{'}_{-b_i}),$$

and the buyer's payment is,

$$\hat{v}_{b_i} = v_{b_i}(\hat{t}_{b_i}, t^{'}_{-b_i}).$$

Remark 2.1. From this transfer rule, we can see the relationship between this mechanism and a second price auction. In a second price auction, the winner's payment is her minimum bid such that she would still be the winner. Note that the price for buyer b_i when trade happens, $v_{b_i}(\hat{t}_{b_i}, t^{'}_{-b_i})$, does not depend on her reported type $t^{'}_{b_i}$.

Remark 2.2. It is shown in the next section that $T_{b_i}(t^{'}_{-bi})$ is a connected and closed set that takes the form of $[\hat{t}_{b_i}, 1]$ when it is not empty. This means that there is a cutoff type $\hat{t}_{b_i}$ so that trade does not occur if b_i reports a type less than $\hat{t}_{b_i}$, and trade occurs with positive probability if she reports a type greater than or equal to $\hat{t}_{b_i}$. Note also that in the case that b_i participates in trade no matter what she reports, that is, $T_{b_i}(t^{'}_{-b_i}) = [0, 1]$, the transfer of b_i is $\hat{v}_{b_i} = v_{b_i}(\hat{t}_{b_i} = 0, t^{'}_{-b_i})$.

Consider seller s_j. If s_j does not participate in trade, the transfer is 0. If she participates in trade, then her highest reported type, $\hat{t}_{s_j} \in [0, 1]$, which makes trade still occur for her with positive probability can be identified. Then, the seller receives the amount $\hat{v}_{s_j}$ equal to the valuation for the good at that highest type. It can also be defined formally.

$$T_{s_j}(t^{'}_{-s_j}) = \{t_{s_j} \in [0, 1] : v_{s_j}(t_{s_j}, t^{'}_{-s_j}) \leq v_{(m)}(t_{s_j}, t^{'}_{-s_j})\}.$$

$T_{s_j}(t^{'}_{-s_j})$ is a set of reported types of s_j at which trade occurs with positive probability given that others report $t^{'}_{-s_j}$. Then,

$$\hat{t}_{s_j} = \sup T_{s_j}(t^{'}_{-s_j}),$$

and the seller receives

$$\hat{v}_{s_j} = v_{s_j}(\hat{t}_{s_j}, t^{'}_{-s_j}).$$

It is shown in the next section that $T_{s_j}(t^{'}_{-s_j})$ takes the form of $[0, \hat{t}_{s_j}]$ when it is not empty. Thus, the seller participates in trade with positive probability if and only if she reports a type less than or equal to $\hat{t}_{s_j}$.

Based on the description above, the transfer rules are defined as follows:

For $b_i \in B$,

$$\tau_{b_i}(t^{'}) = \begin{cases} v_{b_i}(\hat{t}_{b_i}, t^{'}_{-b_i}) & \text{if} \quad v_{b_i}(t^{'}) > v_{(m+1)}(t^{'}), \\ \dfrac{k - |B_s|}{|B_\alpha| - |B_s|} v_{b_i}(\hat{t}_{b_i}, t^{'}_{-b_i}) & \text{if} \quad v_{b_i}(t^{'}) = v_{(m+1)}(t^{'}), \\ 0 & \text{if} \quad v_{b_i}(t^{'}) < v_{(m+1)}(t^{'}). \end{cases}$$

and for $s_j \in S$,

$$\tau_{s_j}(t^{'}) = \begin{cases} v_{s_j}(\hat{t}_{s_j}, t^{'}_{-s_j}) & \text{if} \quad v_{s_j}(t^{'}) < v_{(m)}(t^{'}), \\ \dfrac{k - |S_s|}{|S_\alpha| - |S_s|} v_{s_j}(\hat{t}_{s_j}, t^{'}_{-s_j}) & \text{if} \quad v_{s_j}(t^{'}) = v_{(m)}(t^{'}), \\ 0 & \text{if} \quad v_{s_j}(t^{'}) > v_{(m)}(t^{'}). \end{cases}$$

2.5 Results

In this section, we study the desirable properties of the mechanism. First, we investigate how a trader's reported type affects her own allocation in the following two lemmas.

Lemma 2.1. Suppose that (A1) - (A4) hold. For $b_i \in B$, if $T_{b_i}(t'_{-b_i}) \neq \varnothing$, then $T_{b_i}(t'_{-b_i}) = [\hat{t}_{b_i}, 1]$ where $\hat{t}_{b_i} = \inf T_{b_i}(t'_{-b_i})$. Also, for $s_j \in S$, if $T_{s_j}(t'_{-s_j}) \neq \varnothing$, then $T_{s_j}(t'_{-s_j}) = [0, \hat{t}_{s_j}]$ where $\hat{t}_{s_j} = \sup T_{s_j}(t'_{-s_j})$.

Proof. Consider buyer b_i and assume that $T_{b_i}(t'_{-b_i}) \neq \varnothing$. $T_{b_i}(t'_{-b_i}) = \{t_{b_i} \in [0,1] : v_{b_i}(t_{b_i}, t'_{-b_i}) \geq v_{(m+1)}(t_{b_i}, t'_{-b_i})\}$ by the definition. Consider the statement $v_{b_i}(t) \geq v_{(m+1)}(t)$. $v_{b_i}(t) \geq v_{(m+1)}(t)$ holds if and only if there are at least m traders other than b_i whose valuations at t are less than or equal to b_i's valuation evaluated at t. Let's represent this formally. Let Λ be a set of all subsets with cardinality m of $B \cup S \setminus \{b_i\}$. That is,

$$\Lambda = \{I_1, I_2, \cdots, I_{\binom{m+n-1}{m}}\}$$

where each element I_k is a set of m traders in $B \cup S \setminus \{b_i\}$. Also, for $I_k \in \Lambda$,

$$I_k = \{I_k^1, I_k^2, \cdots, I_k^m\}$$

where each element I_k^l for $l = 1, 2, \cdots, m$ is a trader. Then,

$$\begin{aligned} v_{b_i}(t) \geq v_{(m+1)}(t) \quad \text{iff} \quad & (v_{b_i}(t) \geq v_{I_1^1}(t), v_{b_i}(t) \geq v_{I_1^2}(t), \cdots, \text{ and } v_{b_i}(t) \geq v_{I_1^m}(t)), \\ & \text{or} \\ & (v_{b_i}(t) \geq v_{I_2^1}(t), v_{b_i}(t) \geq v_{I_2^2}(t), \cdots, \text{ and } v_{b_i}(t) \geq v_{I_2^m}(t)), \\ & \text{or} \\ & \vdots \\ & \text{or} \\ & (v_{b_i}(t) \geq v_{I^1_{\binom{m+n-1}{m}}}(t), v_{b_i}(t) \geq v_{I^2_{\binom{m+n-1}{m}}}(t), \cdots, \text{ and } v_{b_i}(t) \geq v_{I^m_{\binom{m+n-1}{m}}}(t)). \end{aligned}$$

By using the above equivalence, we can obtain another expression for $T_{b_i}(t'_{-b_i})$ as follows:

$$T_{b_i}(t'_{-b_i}) = \{t_{b_i} \in [0,1] : v_{b_i}(t_{b_i}, t'_{-b_i}) \geq v_{(m+1)}(t_{b_i}, t'_{-b_i})\}$$
$$= \bigcup_{k=1}^{\binom{m+n-1}{m}} \bigcap_{l=1}^{m} \{t_{b_i} \in [0,1] : v_{b_i}(t_{b_i}, t'_{-b_i}) \geq v_{I_k^l}(t_{b_i}, t'_{-b_i})\}$$

Each set $\{t_{b_i} \in [0,1] : v_{b_i}(t_{b_i}, t'_{-b_i}) \geq v_{I_k^l}(t_{b_i}, t'_{-b_i})\}$ is closed because of (A1), the continuity of $v_i(\cdot)$. Then, $T_{b_i}(t'_{-b_i})$ is also closed because it is finite union of finite intersection of these closed sets. Thus, $T_{b_i}(t'_{-b_i})$ is compact and contains its own infimum. Therefore, $\hat{t}_{b_i} \in T_{b_i}(t'_{-b_i})$.

We can show that $T_{b_i}(t'_{-b_i}) = [\hat{t}_{b_i}, 1]$. If $t'_{b_i} < \hat{t}_{b_i}$, then $t'_{b_i} \notin T_{b_i}(t'_{-b_i})$ by the definition of infimum. Consider $\hat{t}_{b_i} \in T_{b_i}(t'_{-b_i})$. This implies that there exists $I_k \in \Lambda$ such that $v_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) \geq v_{I_k^l}(\hat{t}_{b_i}, t'_{-b_i})$ for all $l = 1, 2, \cdots, m$. Thus, if $t'_{b_i} > \hat{t}_{b_i}$, then $v_{b_i}(t'_{b_i}, t'_{-b_i}) > v_{I_k^l}(t'_{b_i}, t'_{-b_i})$ for all $l = 1, 2, \cdots, m$ by (A4), $\dfrac{\partial v_i}{\partial t_i} > \dfrac{\partial v_j}{\partial t_i}$. It follows that $v_{b_i}(t'_{b_i}, t'_{-b_i}) \geq v_{(m+1)}(t'_{b_i}, t'_{-b_i})$ and thus $t'_{b_i} \in T_{b_i}(t'_{-b_i})$. Therefore, $T_{b_i}(t'_{-b_i}) = [\hat{t}_{b_i}, 1]$.

We can show that $T_{s_j}(t'_{-s_j}) = [0, \hat{t}_{s_j}]$ for seller s_j by using the same argument as above for the buyer. For the seller, we can use the following equivalence: $v_{s_j}(t) \leq v_{(m)}(t)$ holds if and only if there are at least n traders other than s_j whose valuations at t are at least as high as the valuation of s_j at t. ■

From the lemma, we note that each buyer has at most one cutoff reported type, $\hat{t}_{b_i}$, depending on other traders' reported types. The buyer participates in trade with positive probability if and only if she reports a type greater than or equal to $\hat{t}_{b_i}$. Each seller also has at most one cutoff type, $\hat{t}_{s_j}$, so that the trade occurs for her with positive probability if and only if she reports a type less than or equal to the cutoff reported type.

Although we know that b_i (s_j) participates in trade with strictly positive probability if she reports a type greater (less) than or equal to her cutoff, we don't know the exact probability of trade. Thus, in the next lemma, we obtain the exact probability of trade for each reported type value.

Lemma 2.2 (Buyer). Consider buyer $b_i \in B$. Suppose that t'_{-b_i} is a profile of reported types of traders other than b_i and $T_{b_i}(t'_{-b_i}) = [\hat{t}_{b_i}, 1] \neq \varnothing$ for some $\hat{t}_{b_i} > 0$. If b_i reports t'_{b_i} less than $\hat{t}_{b_i}$, then she does not trade. If she reports t'_{b_i} equal to $\hat{t}_{b_i}$, then she trades with probability p where $0 < p \leq 1$. And if she reports t'_{b_i} greater than $\hat{t}_{b_i}$, then she participates in trade with probability 1. That is,

$$\pi_{b_i}(t'_{b_i}, t'_{-b_i}) = \begin{cases} 1 & \text{if} \quad t'_{b_i} > \hat{t}_{b_i}, \\ p & \text{if} \quad t'_{b_i} = \hat{t}_{b_i}, \\ 0 & \text{if} \quad t'_{b_i} < \hat{t}_{b_i}. \end{cases}$$

where $0 < p = \dfrac{k - |B_s|}{|B_\alpha| - |B_s|} \leq 1$.

Proof. If b_i reports t'_{b_i} less than $\hat{t}_{b_i}$, then $v_{b_i}(t'_{b_i}, t'_{-b_i}) < v_{(m+1)}(t'_{b_i}, t'_{-b_i})$ because $\hat{t}_{b_i}$ is a infimum of $T_{b_i}(t'_{-b_i})$. Thus, $\pi_{b_i}(t'_{b_i}, t'_{-b_i}) = 0$ and the trade does not occur by the allocation rule of the mechanism.

Suppose that b_i reports $t'_{b_i} = \hat{t}_{b_i}$. Because $\hat{t}_{b_i} \in T_{b_i}(t'_{-b_i})$, it follows that $v_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) \geq v_{(m+1)}(\hat{t}_{b_i}, t'_{-b_i}) \geq v_{(m)}(\hat{t}_{b_i}, t'_{-b_i})$. It can be shown that $v_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) = v_{(m+1)}(\hat{t}_{b_i}, t'_{-b_i}) = v_{(m)}(\hat{t}_{b_i}, t'_{-b_i})$. Suppose to the contrary that $v_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) > v_{(m)}(\hat{t}_{b_i}, t'_{-b_i})$. This implies that there are m traders in $B \cup S \setminus \{b_i\}$ such that their valuations for the item are strictly less than b_i's valuation for the item at $(\hat{t}_{b_i}, t'_{-b_i})$. Then, by (A1), the continuity of $v_i(\cdot)$, there exists $\epsilon > 0$ such that these m traders' valuations are strictly less than b_i's valuation at $(\hat{t}_{b_i} - \epsilon, t'_{-b_i})$. Therefore, $v_{b_i}(\hat{t}_{b_i} - \epsilon, t'_{-b_i}) \geq v_{(m+1)}(\hat{t}_{b_i} - \epsilon, t'_{-b_i})$. Thus, it follows that $\hat{t}_{b_i} - \epsilon \in T_{b_i}(t'_{-b_i})$. This statement, however, contradicts the definition that $\hat{t}_{b_i}$ is an infimum of $T_{b_i}(t'_{-b_i})$. Therefore, $v_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) = v_{(m+1)}(\hat{t}_{b_i}, t'_{-b_i}) = v_{(m)}(\hat{t}_{b_i}, t'_{-b_i})$. Then, by the allocation rule of the mechanism, $\pi_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) = \dfrac{k - |B_s|}{|B_\alpha| - |B_s|}$. We can show that this probability is always strictly greater than 0. Note first that $|B_\alpha| > |B_s|$ at $(\hat{t}_{b_i}, t'_{-b_i})$ because $v_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) = v_{(m+1)}(\hat{t}_{b_i}, t'_{-b_i})$. Also, $|S_\alpha| > |B_s|$.[7] Thus, $\pi_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) = \dfrac{k - |B_s|}{|B_\alpha| - |B_s|} > 0$ where $k := min\{|B_\alpha|, |S_\alpha|\}$.

Suppose that b_i reports t'_{b_i} strictly greater than $\hat{t}_{b_i}$. We obtain that $v_{b_i}(\hat{t}_{b_i}, t'_{-b_i}) = v_{(m+1)}(\hat{t}_{b_i}, t'_{-b_i}) = v_{(m)}(\hat{t}_{b_i}, t'_{-b_i})$ in the previous paragraph. That is, at $(\hat{t}_{b_i}, t'_{-b_i})$, b_i's valuation is equal to the $(m+1)$st smallest valuation among all traders. Now, consider the traders' valuations at (t'_{b_i}, t'_{-b_i}) where $t'_{b_i} > \hat{t}_{b_i}$. Because of (A4), $\dfrac{\partial v_i}{\partial t_i} > \dfrac{\partial v_j}{\partial t_i}$, the relative ranking of b_i's valuation does not fall in moving from $(\hat{t}_{b_i}, t'_{-b_i})$ to (t'_{b_i}, t'_{-b_i}). That is, $v_{b_i}(t'_{b_i}, t'_{-b_i}) \geq v_{(m+1)}(t'_{b_i}, t'_{-b_i})$. Thus, there are two possible cases: $v_{b_i}(t'_{b_i}, t'_{-b_i}) > v_{(m+1)}(t'_{b_i}, t'_{-b_i})$ or $v_{b_i}(t'_{b_i}, t'_{-b_i}) = v_{(m+1)}(t'_{b_i}, t'_{-b_i})$. If $v_{b_i}(t'_{b_i}, t'_{-b_i}) > v_{(m+1)}(t'_{b_i}, t'_{-b_i})$, then $\pi_{b_i}(t'_{b_i}, t'_{-b_i}) = 1$ by the allocation rule of the mechanism. Consider the case that $v_{b_i}(t'_{b_i}, t'_{-b_i}) = v_{(m+1)}(t'_{b_i}, t'_{-b_i})$. In this

[7]See Appendix A.

case, $v_{(m+1)}(t^{'}_{b_i}, t^{'}_{-b_i}) > v_{(m)}(t^{'}_{b_i}, t^{'}_{-b_i})$ because of (A4), $\dfrac{\partial v_i}{\partial t_i} > \dfrac{\partial v_j}{\partial t_i}$. Thus, buyer b_i participates in trade with probability 1 by the allocation rule. ■

By the same argument, we can obtain the following lemma for the sellers' side.

Lemma 2.2 (Seller). Consider seller $s_j \in S$. Suppose that $t^{'}_{-s_j}$ is a profile of reported types of the traders other than s_j and $T_{s_j}(t^{'}_{-s_j}) = [0, \hat{t}_{s_j}] \neq \varnothing$ for some $\hat{t}_{s_j} < 1$. If s_j reports $t^{'}_{s_j}$ greater than $\hat{t}_{s_j}$, then she does not trade. If she reports $t^{'}_{s_j}$ equal to $\hat{t}_{s_j}$, then she participates in trade with probability p where $0 < p \leq 1$. And if she reports $t^{'}_{s_j}$ less than $\hat{t}_{s_j}$, then she trades with probability 1. That is,

$$\pi_{s_j}(t^{'}_{s_j}, t^{'}_{-s_j}) = \begin{cases} 1 & \text{if} \quad t^{'}_{s_j} < \hat{t}_{s_j}, \\ p & \text{if} \quad t^{'}_{s_j} = \hat{t}_{s_j}, \\ 0 & \text{if} \quad t^{'}_{s_j} > \hat{t}_{s_j}. \end{cases}$$

where $0 < p = \dfrac{k - |S_s|}{|S_\alpha| - |S_s|} \leq 1$.

By using the lemmas, we can study the properties of the mechanism.

Theorem 2.1. Consider multilateral trading with the mechanism defined above. Suppose that (A1) - (A4) hold. Then, the mechanism is

(1) ex-post incentive compatible,

(2) ex-post individually rational,

(3) and ex-post efficient.

Proof. Suppose that m buyers, $b_1, b_2, \cdots, b_m$, and n sellers, $s_1, s_2, \cdots, s_n$, have their true types $t_{b_1}, t_{b_2}, \cdots, t_{b_m}$, and $t_{s_1}, t_{s_2}, \cdots, t_{s_n}$, respectively.

Ex-post incentive compatibility. Consider buyers first. Consider buyer $b_i \in B$. Suppose that the other traders report their true types, t_{-b_i}. We need to show that buyer b_i's best response is reporting her true type, t_{b_i}. By using lemma 2.1, we can identify following three possible cases depending on the values of t_{-b_i}.

Case 1. Buyer b_i does not participates in trade no matter what she reports. That is, $T_{b_i}(t_{-b_i}) = \varnothing$.

Case 2. Buyer b_i trades with probability 1 no matter which type she reports. That is, $T_{b_i}(t_{-b_i}) = [0, 1]$ and $\Big(v_{b_i}(0, t_{-b_i}) > v_{(m+1)}(0, t_{-b_i})$ or $v_{b_i}(0, t_{-b_i}) = v_{(m+1)}(0, t_{-b_i}) > v_{(m)}(0, t_{-b_i})\Big)$.

Case 3. There is a cutoff reported type, $\hat{t}_{b_i}$, such that b_i participates in trade with strictly positive probability if and only if she reports a type greater than or equal to $\hat{t}_{b_i}$. That is, $T_{b_i}(t_{-b_i}) = [\hat{t}_{b_i}, 1]$.

Let's show that reporting true type t_{b_i} is the best response for buyer b_i in each case.

In case 1, it is trivial that reporting her true type is the best response.

In case 2, no matter what b_i reports, trade occurs with probability 1 and she pays $v_{b_i}(0, t_{-b_i})$. Thus, her utility is $v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(0, t_{-b_i})$ and does not depend on her reported type. Therefore, reporting true type is her best response.

In case 3, $T_{b_i}(t_{-b_i}) = [\hat{t}_{b_i}, 1]$. From lemma 2.2, it follows that there are three possible utilities for b_i depending on her reported type $t^{'}_{b_i}$. If b_i reports a type $t^{'}_{b_i} < \hat{t}_{b_i}$, then trade does not occur for her and her utility is 0. If she reports a type $t^{'}_{b_i} = \hat{t}_{b_i}$, then she participates in trade with probability $\frac{k-|B_s|}{|B_\alpha|-|B_s|}$ and her utility is $\frac{k-|B_s|}{|B_\alpha|-|B_s|}\big(v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i})\big)$.[8] And if she reports a type $t^{'}_{b_i} > \hat{t}_{b_i}$, then she participates in trade with probability 1 and obtains utility $v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i})$. Now, for each possible value of true type, t_{b_i}, let's show that reporting true type gives b_i the highest utility among these three possible utilities. Suppose that $t_{b_i} < \hat{t}_{b_i}$. If b_i reports this true type, then she receives the utility is 0. This is greater than the other two possible utilities, $\frac{k-|B_s|}{|B_\alpha|-|B_s|}\big(v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i})\big)$ and $v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i})$, because $t_{b_i} < \hat{t}_{b_i}$ and $\frac{\partial v_i}{\partial t_i} > 0$ from (A3). Suppose that b_i's true type $t_{b_i} = \hat{t}_{b_i}$. If b_i reports this true type, then she obtains the utility $\frac{k-|B_s|}{|B_\alpha|-|B_s|}\big(v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i})\big)$, which is 0. This is equal to the other two possible utilities, $v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i}) (= 0)$ and 0. Suppose that $t_{b_i} > \hat{t}_{b_i}$. By reporting this true type, b_i has the utility $v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i})$. This is greater than or equal to the other two possible utilities, $\frac{k-|B_s|}{|B_\alpha|-|B_s|}\big(v_{b_i}(t_{b_i}, t_{-b_i}) - v_{b_i}(\hat{t}_{b_i}, t_{-b_i})\big)$ and 0, because $t_{b_i} > \hat{t}_{b_i}$ and $\frac{\partial v_i}{\partial t_i} > 0$ from (A3).

By investigating these three possible cases, we showed that buyer b_i's best response is reporting her true type t_{b_i} when the other traders report their true types. By using the same argument, we can also show that each seller s_j has an incentive to report her true type. Therefore, reporting true types forms an ex-post Nash equilibrium.

Ex-post individual rationality. In the proof of ex-post incentive compatibility above, we can find out that each trader always obtains non-negative ex-post utility in the truth-telling ex-post Nash equilibrium.

Ex-post efficiency. Consider buyers and sellers who participate in trade under the mechanism. k buyers with the highest valuations for the item and k sellers with the lowest valuations for the item participate in trade. Each buyer b_i who

[8] $0 < \frac{k-|B_s|}{|B_\alpha|-|B_s|} \leq 1$ from lemma 2.2.

purchases an item has valuation $v_{b_i}(t) \geq v_{(m+1)}(t)$, and each seller s_j who sells an item has valuation $v_{s_j}(t) \leq v_{(m)}(t)$. Thus, each of these buyers has weakly higher valuation for the item than each of these sellers. Consider buyers and sellers who do not participate in trade under the mechanism. Each of these buyers has strictly lower valuation for the item than each of these sellers. Therefore, the mechanism allocates the items efficiently. ■

An example in the next section provides a market where the mechanism generates a budget deficit.

2.6 Example

Consider a market with two buyers and two sellers. That is, $B = \{b_1, b_2\}$ and $S = \{s_1, s_2\}$. Suppose that a profile of their types is $t = (t_{b_1}, t_{b_2}, t_{s_1}, t_{s_2}) = (\frac{3}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4})$ and valuation functions for one indivisible item are defined as follows:

$$\begin{aligned}
v_{b_1}(t_{b_1}, t_{b_2}, t_{s_1}, t_{s_2}) &= \tfrac{1}{2}t_{b_1} + \tfrac{1}{4}t_{b_2} + \tfrac{1}{8}t_{s_1} + \tfrac{1}{8}t_{s_2},\\
v_{b_2}(t_{b_1}, t_{b_2}, t_{s_1}, t_{s_2}) &= \tfrac{1}{16}t_{b_1} + \tfrac{3}{4}t_{b_2} + \tfrac{5}{32}t_{s_1} + \tfrac{1}{32}t_{s_2},\\
v_{s_1}(t_{b_1}, t_{b_2}, t_{s_1}, t_{s_2}) &= \tfrac{1}{12}t_{b_1} + \tfrac{1}{6}t_{b_2} + \tfrac{2}{3}t_{s_1} + \tfrac{1}{12}t_{s_2},\\
v_{s_2}(t_{b_1}, t_{b_2}, t_{s_1}, t_{s_2}) &= \tfrac{1}{6}t_{b_1} + \tfrac{1}{6}t_{b_2} + \tfrac{1}{6}t_{s_1} + \tfrac{1}{2}t_{s_2}.
\end{aligned}$$

Note that these valuation functions satisfy (A1) - (A4).

Under the multilateral trading mechanism, each trader reports her true type in an equilibrium. That is, buyer b_1 reports $t_{b_1} = \frac{3}{4}$, buyer b_2 reports $t_{b_2} = \frac{1}{3}$, seller s_1 reports $t_{s_1} = \frac{1}{2}$, and seller s_2 reports $t_{s_2} = \frac{1}{4}$. Then, we can obtain that $v_{(1)}(t) = v_{b_2}(t) = 0.383 < v_{(2)}(t) = v_{s_2}(t) = 0.389 < v_{(3)}(t) = v_{s_1}(t) = 0.472 < v_{(4)}(t) = v_{b_1}(t) = 0.552$. Thus, b_1 and s_2 trade an item and b_2 and s_1 do not trade by the allocation rule of the mechanism.

How much transfer b_1 pays and s_2 receives for an item can be obtained from Figures. In Figure 2.1, valuations of traders based on their reported types are presented when the traders other than b_1 report their true types. The figure shows that b_1's lowest reported type that makes her still participate in trade with positive probability, $\hat{t}_{b_1}$, is determined by the intersection of the lines for b_1 and b_2. Thus, solving $v_{b_1}(\hat{t}_{b_1}, t_{b_2} = \frac{1}{3}, t_{s_1} = \frac{1}{2}, t_{s_2} = \frac{1}{4}) = v_{b_2}(\hat{t}_{b_1}, t_{b_2} = \frac{1}{3}, t_{s_1} = \frac{1}{2}, t_{s_2} = \frac{1}{4})$ yields $\hat{t}_{b_1} = 0.363$. Therefore, b_1 pays $\hat{v}_{b_1} = v_{b_1}(\hat{t}_{b_1}, t_{-b_1}) = 0.359$ by the transfer rules of the mechanism. We can use the same argument to find out how much s_2

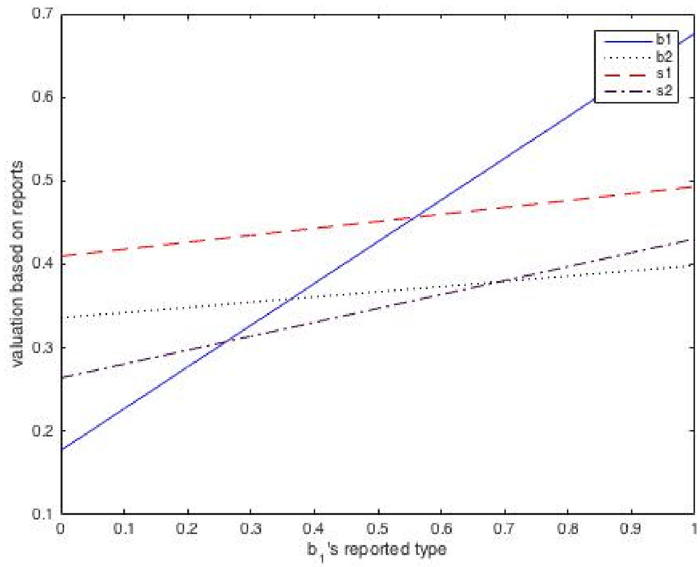

Figure 2.1: Valuations of traders depending on b_1's report when b_2 reports $t_{b_2} = \frac{1}{3}$, s_1 reports $t_{s_1} = \frac{1}{2}$, and s_2 reports $t_{s_2} = \frac{1}{4}$.

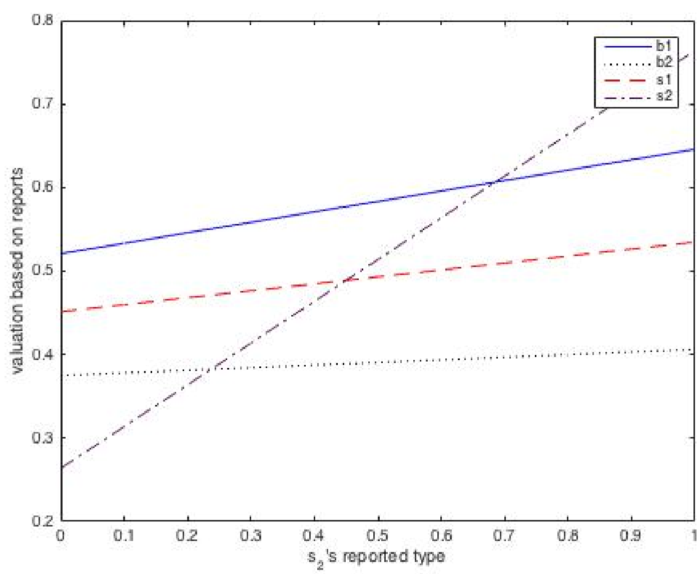

Figure 2.2: Valuations of traders depending on s_2's report when b_1 reports $t_{b_1} = \frac{3}{4}$, b_2 reports $t_{b_2} = \frac{1}{3}$, and s_1 reports $t_{s_1} = \frac{1}{2}$.

receives for an item. In Figure 2.2, s_2's highest possible reported type at which she participates in trade with positive probability, $\hat{t}_{s_2}$, is a value where the lines for s_1 and s_2 intersect. Thus, $\hat{t}_{s_2} = 0.45$ and s_2 receives $\hat{v}_{s_2} = v_{s_2}(\hat{t}_{s_2}, t_{-s_2}) = 0.489$ by the transfer rules of the mechanism.

Buyer b_2 and seller s_1 don't trade an item and hence make no transfers. Thus, the total transfers paid by buyers is $0.359(= 0.359 + 0)$, and the total transfers received by sellers is $0.489(= 0 + 0.489)$. Because $0.359 < 0.489$, ex-post budget deficits occur in this example.

In sum, in the truth-telling ex-post Nash equilibrium under the multilateral trading mechanism, trade occurs between buyer b_1 and seller s_2, the ex-post utility of buyer b_1 is

$$v_{b_1}(t) - \hat{v}_{b_1} = 0.552 - 0.359 = 0.193,$$

the ex-post utility of seller s_2 is

$$\hat{v}_{s_2} - v_{s_2}(t) = 0.489 - 0.389 = 0.1,$$

and the ex-post utilities of b_2 and s_1 are 0.

2.7 Conclusion

In this paper, a trading mechanism is introduced for a market that has multiple buyers and multiple sellers with interdependent valuations for an item. The mechanism is ex-post efficient, ex-post incentive compatible, and ex-post individually rational. The paper provides an example in which a budget deficit occurs. The allocation and transfer rules are inspired by the second-price auction, therefore, the mechanism allocates the items efficiently based on the traders' honest reports.

2.8 Future Research Directions

Myerson and Satterthwaite (1983) show that a bilateral trading mechanism satisfying Bayesian incentive compatibility, interim individual rationality, ex-post efficiency, and ex-ante budget balance does not exist when traders have private valuations for an item. My mechanism features all of these desired properties except ex-post budget balance. An example that the mechanism generates a budget deficit was provided in a section. This means that there is a potential for showing the impossibility result for interdependent values setting; establishing a counterpart of the impossibility theorem by Myerson and Satterthwaite (1983) for the case of interdependent values is a future research topic.

Chapter 3

An Impossibility Result on Bilateral Trading Mechanisms with Interdependent Values

3.1 Introduction

In a stock market, there are multiple buyers and sellers and they trade stock shares. Each trader has information about the stock. If a trader somehow discovers the other traders' information, it will affect the trader's valuation for the stock. Therefore, each trader's valuation depends on her information as well as information held by the other traders. Traders in a stock market have interdependent values in this sense.

This paper studies trading mechanisms in a market where a buyer and a seller have interdependent values for an item. Myerson and Satterthwaite (1983) study bilateral trading mechanisms in a market where traders have private values, that is, each trader's valuation for an item solely depends on her information. Their impossibility theorem shows that there is no bilateral trading mechanism satisfying ex-post efficiency, Bayesian incentive compatibility, interim individual rationality, and ex-ante budget balance. My paper examines whether there exists a bilateral trading mechanism satisfying these desired properties when traders have interdependent values.

The paper assumes that each bidder's information has a greater marginal effect on her valuation than on the other trader's valuation. Art markets and stock markets are examples that fit this assumption. In an art market, for instance, traders' valuations for an art work strongly depend on personal preferences. Thus, each trader's type has a greater effect on her valuation than on others' valuations. The paper also assumes that the interior of the buyer's possible valuation set and the interior of the seller's possible valuation set have nonempty intersection. This means that it is unknown ex-ante whether trade should occur or not for efficient allocation. Under these assumptions, the paper shows that no bilateral trading mechanism with interdependent values can satisfy ex-post efficiency, ex-post incentive compatibility, ex-post individual rationality and no ex-post budget deficit.

Jehiel and Moldovanu (2001) study trading mechanisms with interdependent values. They show that there exists a trading mechanism satisfying Bayesian incentive compatibility and ex-post efficiency if each agent's information is one-dimensional. My paper adds two more properties, ex-post individual rationality and no ex-post budget deficit, and shows the nonexistence of the mechanism featuring these four properties. I discuss details in the later section.

The paper is organized as follows. Section 3.2 specifies the assumptions, especially those on the traders' valuation functions, and defines the market, mechanisms, and the desired properties of the mechanisms. Section 3.3 consists of three propositions: Proposition 3.1 derives necessary conditions for ex-post incentive compatibility of the bilateral trading mechanisms. Proposition 3.2 obtains necessary conditions for ex-post efficiency, ex-post individual rationality, and no ex-post budget deficit. Proposition 3.3 shows the nonexistence of the mechanism satisfying all these properties by contradiction. Section 3.4 discusses differences between this paper and Jehiel and Moldovanu (2001). Section 3.5 concludes.

3.2 Model

There is one buyer, b, and one seller, s. The seller has one unit of an indivisible item and the buyer can purchase this item. Each trader has her own real-valued type, $t_b \in [\underline{t}_b, \bar{t}_b]$ and $t_s \in [\underline{t}_s, \bar{t}_s]$, that is her private information. The traders have interdependent valuations for the item. A trader's estimate for her own value for the item may depend on the information held by the other.[1] Thus, in our setting, each trader's valuation for the item depends on both her own type and the other trader's type. That is, if (t_b, t_s) is a profile of trader types, then $v_b(t_b, t_s)$ and $v_s(t_b, t_s)$ are valuations of the buyer and the seller for the item respectively. We assume the following on the valuation functions.

A1. The item is a good that provides utility to traders:

$$v_i(t_b, t_s) \geq 0 \quad \text{for all } t_b, t_s, \text{ and } i \in \{b, s\}.$$

A2. $v_i(\cdot)$ is continuously differentiable for $i \in \{b, s\}$.

A3. $v_i(\cdot)$ is non-decreasing in t_j for all $i, j \in \{b, s\}$:

[1]The interdependent value model includes the private value model as a special case. Results also hold for the private value model.

$$\frac{\partial v_i}{\partial t_j} \geq 0.$$

A4. $v_i(\cdot)$ is strictly increasing in t_i for $i \in \{b, s\}$:

$$\frac{\partial v_i}{\partial t_i} > 0.$$

A5. A trader i's type t_i has a greater marginal effect on her own valuation $v_i(\cdot)$ than on the other's valuation $v_j(\cdot)$:

$$\frac{\partial v_i}{\partial t_i} > \frac{\partial v_j}{\partial t_i}.$$

A6. There are type profiles, $(t_b^1, t_s^1), (t_b^2, t_s^2) \in (\underline{t}_b, \bar{t}_b) \times (\underline{t}_s, \bar{t}_s)$, at which $v_b(t_b^1, t_s^1) > v_s(t_b^1, t_s^1)$ and $v_b(t_b^2, t_s^2) < v_s(t_b^2, t_s^2)$ respectively. Thus, it is unknown ex-ante whether trade should happen or not for efficient allocation.

Each trader has a quasi-linear utility consisting of the valuation and transfer. If the buyer and seller trade the item at price p, then the buyer's utility is $v_b(t_b, t_s) - p$, and the seller's utility is $p - v_s(t_b, t_s)$. Their utility is 0 if they don't trade and no transfer are made. The valuation functions, $v_b(\cdot)$ and $v_s(\cdot)$, and the utility functions are common knowledge.

We focus on direct revelation mechanisms in which the traders report their types then the allocation of the item and the transfers for the traders are determined based on these reported types. A direct revelation mechanism is a profile of functions (π, τ_b, τ_s) defined on the set of type profiles, $[\underline{t}_b, \bar{t}_b] \times [\underline{t}_s, \bar{t}_s]$. When (t_b, t_s) is a reported type profile, $\pi(t_b, t_s) \in [0, 1]$ is the probability that trade occurs, $\tau_b(t_b, t_s)$ is the expected monetary transfer the buyer pays, and $\tau_s(t_b, t_s)$ is the expected monetary transfer the seller receives. Thus, if (t_b, t_s) is the true type profile and $(t_b^{'}, t_s^{'})$ is reported type profile, then the buyer's ex-post utility is $v_b(t_b, t_s)\pi(t_b^{'}, t_s^{'}) - \tau_b(t_b^{'}, t_s^{'})$ and the seller's ex-post utility is $\tau_s(t_b^{'}, t_s^{'}) - v_s(t_b, t_s)\pi(t_b^{'}, t_s^{'})$.

A mechanism (π, τ_b, τ_s) is *ex-post incentive compatible* if

$$v_b(t_b, t_s)\pi(t_b, t_s) - \tau_b(t_b, t_s) \geq v_b(t_b, t_s)\pi(t_b^{'}, t_s) - \tau_b(t_b^{'}, t_s) \quad \text{for each } t_b, t_s, \text{ and } t_b^{'},$$

$$\text{and } \tau_s(t_b, t_s) - v_s(t_b, t_s)\pi(t_b, t_s) \geq \tau_s(t_b, t_s^{'}) - v_s(t_b, t_s)\pi(t_b, t_s^{'}) \quad \text{for each } t_b, t_s, \text{ and } t_s^{'}.$$

In an ex-post incentive compatible mechanism, each trader has an incentive to report her true type and honest reporting forms an ex-post Nash equilibrium.

We restrict attention to ex-post incentive compatible direct revelation mechanisms in the paper because the revelation principle holds.[2]

A mechanism (π, τ_b, τ_s) is *ex-post individually rational* if

$$\begin{aligned} v_b(t_b, t_s)\pi(t_b, t_s) - \tau_b(t_b, t_s) \geq 0 \quad &\text{for each } t_b \text{ and } t_s, \\ \text{and } \tau_s(t_b, t_s) - v_s(t_b, t_s)\pi(t_b, t_s) \geq 0 \quad &\text{for each } t_b \text{ and } t_s. \end{aligned}$$

Each trader volunteers to participate in an ex-post individually rational mechanism because she obtains non-negative utility after all types are revealed.

A mechanism (π, τ_b, τ_s) satisfies *no ex-post budget deficit* if

$$\tau_b(t_b, t_s) \geq \tau_s(t_b, t_s) \quad \text{for each } t_b \text{ and } t_s.$$

A mechanism (π, τ_b, τ_s) is *ex-post efficient* if

$$\pi(t_b, t_s) = \begin{cases} 1 & \text{if } v_b(t_b, t_s) \geq v_s(t_b, t_s), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, trade occurs if and only if the buyer's valuation weakly exceeds the seller's valuation based on the reported types.

3.3 Results

Let's first discuss necessary conditions for certain properties.

Proposition 3.1. *Suppose that (A1) - (A5) hold. If a mechanism (π, τ_b, τ_s) is ex-post incentive compatible, then*
(i) $\pi(t_b, t_s), \tau_b(t_b, t_s)$, and the buyer's ex-post payoff $v_b(t_b, t_s)\pi(t_b, t_s) - \tau_b(t_b, t_s)$ are weakly increasing in t_b when t_s is fixed, and
(ii) $\pi(t_b, t_s), \tau_s(t_b, t_s)$, and the seller's ex-post payoff $\tau_s(t_b, t_s) - v_s(t_b, t_s)\pi(t_b, t_s)$ are weakly decreasing in t_s when t_b is fixed.

Proof. Only *(i)* is proven here as the proof of *(ii)* is similar. Consider $t_b, t_b^{'} \in [\underline{t}_b, \overline{t}_b]$, and $t_s \in [\underline{t}_s, \overline{t}_s]$. Ex-post incentive compatibility implies that

$$\begin{aligned} v_b(t_b, t_s)\pi(t_b, t_s) - \tau_b(t_b, t_s) &\geq v_b(t_b, t_s)\pi(t_b^{'}, t_s) - \tau_b(t_b^{'}, t_s), \\ \text{and } v_b(t_b^{'}, t_s)\pi(t_b^{'}, t_s) - \tau_b(t_b^{'}, t_s) &\geq v_b(t_b^{'}, t_s)\pi(t_b, t_s) - \tau_b(t_b, t_s). \end{aligned} \tag{3.1}$$

[2] See Appendix B for the proof of the revelation principle.

It follows that[3]

$$\begin{aligned}
&\big(v_b(t_b,t_s) - v_b(t_b^{'},t_s)\big)\pi(t_b^{'},t_s) \\
&\leq \Big(v_b(t_b,t_s)\pi(t_b,t_s) - \tau_b(t_b,t_s)\Big) - \Big(v_b(t_b^{'},t_s)\pi(t_b^{'},t_s) - \tau_b(t_b^{'},t_s)\Big) \\
&\leq \big(v_b(t_b,t_s) - v_b(t_b^{'},t_s)\big)\pi(t_b,t_s).
\end{aligned}$$

If $t_b > t_b^{'}$, then $\pi(t_b,t_s) \geq \pi(t_b^{'},t_s)$ by (A4) and this inequality. Thus, $\pi(t_b,t_s)$ is weakly increasing in t_b. Also, if $t_b > t_b^{'}$, then $v_b(t_b,t_s)\pi(t_b,t_s) - \tau_b(t_b,t_s) \geq v_b(t_b^{'},t_s)\pi(t_b^{'},t_s) - \tau_b(t_b^{'},t_s)$ because $\big(v_b(t_b,t_s) - v_b(t_b^{'},t_s)\big)\pi(t_b^{'},t_s) \geq 0$. Therefore, the buyer's ex-post payoff, $v_b(t_b,t_s)\pi(t_b,t_s) - \tau_b(t_b,t_s)$, is weakly increasing in t_b.

Inequality (3.1) from ex-post incentive compatibility implies that

$$\tau_b(t_b^{'},t_s) - \tau_b(t_b,t_s) \geq v_b(t_b,t_s)\big(\pi(t_b^{'},t_s) - \pi(t_b,t_s)\big).$$

If $t_b^{'} > t_b$, then the right-hand side of this inequality is non-negative because $\pi(t_b,t_s)$ is increasing in t_b and $v_b(t_b,t_s)$ is always positive from (A1). Thus, $\tau_b(t_b^{'},t_s) \geq \tau_b(t_b,t_s)$ and therefore, $\tau_b(t_b,t_s)$ is weakly increasing in t_b. ■

Proposition 3.2 states the necessary conditions for ex-post efficiency, ex-post individual rationality, and no ex-post budget deficit.

Proposition 3.2. *Suppose that (A1) - (A6) hold and a mechanism* (π, τ_b, τ_s) *satisfies ex-post efficiency, ex-post individual rationality, and no ex-post budget deficit. Then:*

i) There exists a closed interval $I(t_s^*) = [t_s^* - \epsilon, t_s^* + \epsilon]$ *for some* $t_s^* \in (\underline{t}_s, \overline{t}_s)$ *and* $\epsilon > 0$, *and a strictly increasing differentiable function* $g : I(t_s^*) \to [\underline{t}_b, \overline{t}_b]$ *such that*

$$v_b\big(g(t_s), t_s\big) = v_s(g(t_s), t_s) \text{ for all } t_s \in I(t_s^*).$$

ii) For each $(t_b, t_s) \in g(I(t_s^*)) \times I(t_s^*)$, *the following hold:*

$$\begin{aligned}
&\text{If } t_b > g(t_s), \text{ then } v_b(t_b,t_s) > v_s(t_b,t_s), \pi(t_b,t_s) = 1, \\
&\qquad\qquad \text{and } v_s(t_b,t_s) \leq \tau_s(t_b,t_s) \leq \tau_b(t_b,t_s) \leq v_b(t_b,t_s). \\
&\text{If } t_b = g(t_s), \text{ then } v_b(t_b,t_s) = v_s(t_b,t_s) = \tau_b(t_b,t_s) = \tau_s(t_b,t_s), \text{ and } \pi(t_b,t_s) = 1. \\
&\text{If } t_b < g(t_s), \text{ then } v_b(t_b,t_s) < v_s(t_b,t_s), \pi(t_b,t_s) = 0, \text{ and } \tau_b(t_b,t_s) = \tau_s(t_b,t_s) = 0.
\end{aligned}$$

We can refer to Figure 3.1 for a better understanding of this proposition. In Figure 3.1, we can find out the positive slope curve on which the buyer's valuation

[3] This part of the proof follows Theorem 1 of Myerson and Satterthwaite (1983).

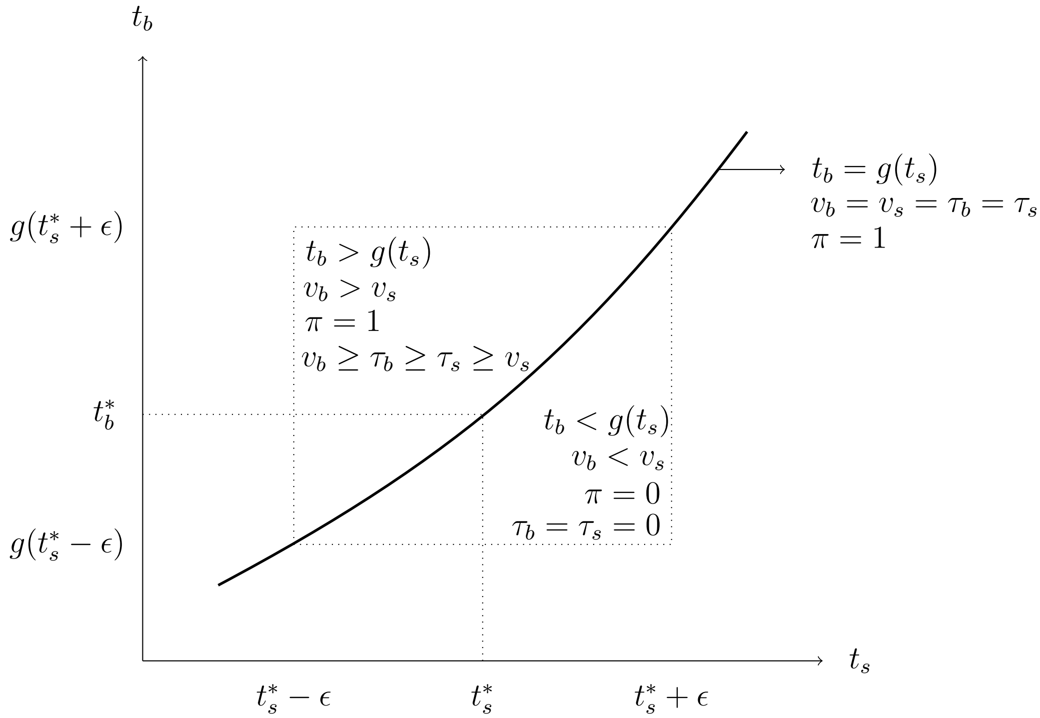

Figure 3.1: The necessary conditions for ex-post efficiency, ex-post individual rationality, and no ex-post budget deficit.

is equal to the seller's valuation for the item. In the upper-left area of this curve, the buyer's valuation for the item is greater than the seller's valuation. Thus, if the traders report their types honestly, then trade happens because the mechanism is ex-post efficient. Also, the transfers for the traders are between the buyer's valuation and the seller's valuation by individual rationality and no budget deficit. In the lower-right area of the curve, the seller's valuation for the item is greater than the buyer's valuation. Therefore, if the traders report their true types, trade does not occur due to ex-post efficiency, and no transfers are made because of ex-post individual rationality and no ex-post budget deficit of the mechanism.

Proof of Proposition 3.2. It follows from (A2) and (A6) that there exists $(t_b^*, t_s^*) \in (\underline{t}_b, \bar{t}_b) \times (\underline{t}_s, \bar{t}_s)$ such that $v_b(t_b^*, t_s^*) = v_s(t_b^*, t_s^*)$. Define a function $H(t_b, t_s) := v_b(t_b, t_s) - v_s(t_b, t_s)$. Then, $H(t_b^*, t_s^*) = 0$ and we can apply the implicit function theorem to show *i)*.[4]

I now turn to the proof of *ii)* of the proposition. Consider the closed rectangular region $g(I(t_s^*)) \times I(t_s^*)$. We can partition this set into three subsets depending on the relationship between t_b and t_s: $t_b = g(t_s)$, $t_b > g(t_s)$, and $t_b < g(t_s)$.

[4]See Appendix B for the detailed proof.

If (t_b, t_s) in $g(I(t_s^*)) \times I(t_s^*)$ satisfies $t_b = g(t_s)$, then it follows that $v_b(t_b, t_s) = v_s(t_b, t_s)$ from the property of $g(\cdot)$ in result *i)*. Thus, $\pi(t_b, t_s) = 1$ because the mechanism is ex-post efficient. If we plug this into the ex-post individual rationality and no ex-post budget deficit constraints, $v_b(t_b, t_s)\pi(t_b, t_s) - \tau_b(t_b, t_s) \geq 0$, $\tau_s(t_b, t_s) - v_s(t_b, t_s)\pi(t_b, t_s) \geq 0$, and $\tau_b(t_b, t_s) \geq \tau_s(t_b, t_s)$, then we can have that $v_b(t_b, t_s) \geq \tau_b(t_b, t_s) \geq \tau_s(t_b, t_s) \geq v_s(t_b, t_s)$. It leads to $v_b(t_b, t_s) = \tau_b(t_b, t_s) = \tau_s(t_b, t_s) = v_s(t_b, t_s)$ because $v_b(t_b, t_s) = v_s(t_b, t_s)$.

Consider $(t_b, t_s) \in g(I(t_s^*)) \times I(t_s^*)$ satisfying $t_b > g(t_s)$. Then, $v_b(t_b, t_s) > v_s(t_b, t_s)$ because $v_b(g(t_s), t_s) = v_s(g(t_s), t_s)$ and $\partial v_b/\partial t_b > \partial v_s/\partial t_b$. Because the mechanism is ex-post efficient, it follows that $\pi(t_b, t_s) = 1$. We can also get that $v_b(t_b, t_s) \geq \tau_b(t_b, t_s) \geq \tau_s(t_b, t_s) \geq v_s(t_b, t_s)$ from the ex-post individual rationality and no ex-post budget deficit constraints.

Consider the last case where $t_b < g(t_s)$. By using the same reasoning as used in the previous cases, we can obtain that $v_b(t_b, t_s) < v_s(t_b, t_s), \pi(t_b, t_s) = 0$, and $\tau_b(t_b, t_s) = \tau_s(t_b, t_s) = 0$. ■

We study the necessary conditions for the properties of the mechanism in Proposition 3.1 and Proposition 3.2. In the next proposition, it is shown that there is no mechanism satisfying all these properties by deriving a contradiction from these necessary conditions.

Proposition 3.3. *Suppose that (A1) - (A6) hold. Then, there is no bilateral trading mechanism satisfying ex-post efficiency, ex-post incentive compatibility, ex-post individual rationality, and no ex-post budget deficit.*

Proof. Suppose there is a mechanism (π, τ_b, τ_s) satisfying ex-post efficiency, ex-post incentive compatibility, ex-post individual rationality, and no ex-post budget deficit. From Proposition 3.2, we can obtain the positive slope curve, $g : I(t_s^*) \rightarrow [\underline{t}_b, \bar{t}_b]$, on which the buyer's valuation and the seller's valuation for the item are equal and the rectangular region, $g\big(I(t_s^*)\big) \times I(t_s^*)$, as in Figure 1.

Consider a set of type profiles, $\textbf{int}\Big(g\big(I(t_s^*)\big) \times I(t_s^*)\Big) \cap \{(t_b, t_s) : t_b > g(t_s)\}$, that is, the upper-left area of the curve $t_b = g(t_s)$ in an open rectangular region $\textbf{int}\Big(g\big(I(t_s^*)\big) \times I(t_s^*)\Big)$ in Figure 3.1. $v_b(t_b, t_s)$ is weakly increasing in t_s from (A3). There are two cases depending on how the value of $v_b(t_b, t_s)$ changes as the value of t_s changes in this region:

Case 1. There exists a type pair, $(t_b^0, t_s^0) \in \textbf{int}\Big(g\big(I(t_s^*)\big) \times I(t_s^*)\Big) \cap \{(t_b, t_s) : t_b > g(t_s)\}$, and $\delta > 0$ such that $v_b(t_b^0, t_s)$ is strictly increasing in t_s on an interval $(t_s^0 - \delta, t_s^0 + \delta)$. That is, $v_b(t_b, t_s)$ is strictly increasing in t_s on some intervals in the region.

Case 2. For all $(t_b, t_s) \in \textbf{int}\Big(g\big(I(t_s^*)\big) \times I(t_s^*)\Big) \cap \{(t_b, t_s) : t_b > g(t_s)\}$, the value of $v_b(t_b, t_s)$ does not depend on the value of t_s.

We derive a contradiction in each case.

The first case is depicted in Figure 3.2. Consider a pair of types $(t_b^0, g^{-1}(t_b^0))$. Then, it follows from Proposition 3.2 that $v_b\big(t_b^0, g^{-1}(t_b^0)\big) = v_s\big(t_b^0, g^{-1}(t_b^0)\big) = \tau_b\big(t_b^0, g^{-1}(t_b^0)\big) = \tau_s\big(t_b^0, g^{-1}(t_b^0)\big)$. Also, we have that $v_b(t_b^0, t_s) > v_s(t_b^0, t_s)$ and $v_b(t_b^0, t_s) \geq \tau_b(t_b^0, t_s) \geq \tau_s(t_b^0, t_s) \geq v_s(t_b^0, t_s)$ for $t_s < g^{-1}(t_b^0)$ from Proposition 3.2. Because $v_b(t_b^0, t_s)$ is strictly increasing in t_s on some intervals including t_s^0 and $t_s^0 < g^{-1}(t_b^0)$, it follows that $v_b\big(t_b^0, t_s^0\big) < v_b\big(t_b^0, g^{-1}(t_b^0)\big)$. Therefore,

$$\tau_s\big(t_b^0, t_s^0\big) \leq v_b\big(t_b^0, t_s^0\big) < v_b\big(t_b^0, g^{-1}(t_b^0)\big) = \tau_s\big(t_b^0, g^{-1}(t_b^0)\big).$$

However, because the mechanism is ex-post incentive compatible, $\tau_s(t_b^0, t_s)$ is weakly decreasing in t_s from Proposition 3.1, thus, $\tau_s\big(t_b^0, t_s^0\big) \geq \tau_s\big(t_b^0, g^{-1}(t_b^0)\big)$.

We now turn to the second case, which is depicted in Figure 3.3. Consider a type profile $(t_b^{'}, t_s^{'}) \in \textbf{int}\Big(g\big(I(t_s^*)\big) \times I(t_s^*)\Big) \cap \{(t_b, t_s) : t_b > g(t_s)\}$. We can see that $t_s^{'} < g^{-1}(t_b^{'})$. Because $v_b(t_b^{'}, t_s)$ does not depend on t_s for $t_s < g^{-1}(t_b^{'})$ and $v_b(t_b^{'}, t_s)$ is continuous, it follows that $v_b\big(t_b^{'}, t_s^{'}\big) = v_b\big(t_b^{'}, g^{-1}(t_b^{'})\big)$. Thus, if we apply Proposition 3.2 and repeat the same procedure as we did in the first case, we have

$$\tau_s\big(t_b^{'}, t_s^{'}\big) \leq \tau_b\big(t_b^{'}, t_s^{'}\big) \leq v_b\big(t_b^{'}, t_s^{'}\big) = v_b\big(t_b^{'}, g^{-1}(t_b^{'})\big) = \tau_s\big(t_b^{'}, g^{-1}(t_b^{'})\big). \quad (3.2)$$

By ex-post incentive compatibility of the mechanism, $\tau_s(t_b^{'}, t_s)$ is weakly decreasing in t_s from Proposition 3.1. Therefore,

$$\tau_s\big(t_b^{'}, t_s^{'}\big) \geq \tau_s\big(t_b^{'}, g^{-1}(t_b^{'})\big). \quad (3.3)$$

It follows from (3.2) and (3.3) that

$$\tau_s(t_b^{'}, t_s^{'}) = \tau_b(t_b^{'}, t_s^{'}) = v_b(t_b^{'}, t_s^{'}).$$

Thus, we obtain that $\tau_b(t_b, t_s) = v_b(t_b, t_s)$ for all $(t_b, t_s) \in \textbf{int}\Big(g\big(I(t_s^*)\big) \times I(t_s^*)\Big) \cap \{(t_b, t_s) : t_b > g(t_s)\}$.

However, this result contradicts ex-post incentive compatibility of the mechanism. Consider a type pair $(\hat{t_b}, \hat{t_s}) \in \textbf{int}\Big(g\big(I(t_s^*)\big) \times I(t_s^*)\Big) \cap \{(t_b, t_s) : t_b > g(t_s)\}$. Suppose that the seller reports her true type, $\hat{t_s}$. If the buyer reports her true type,

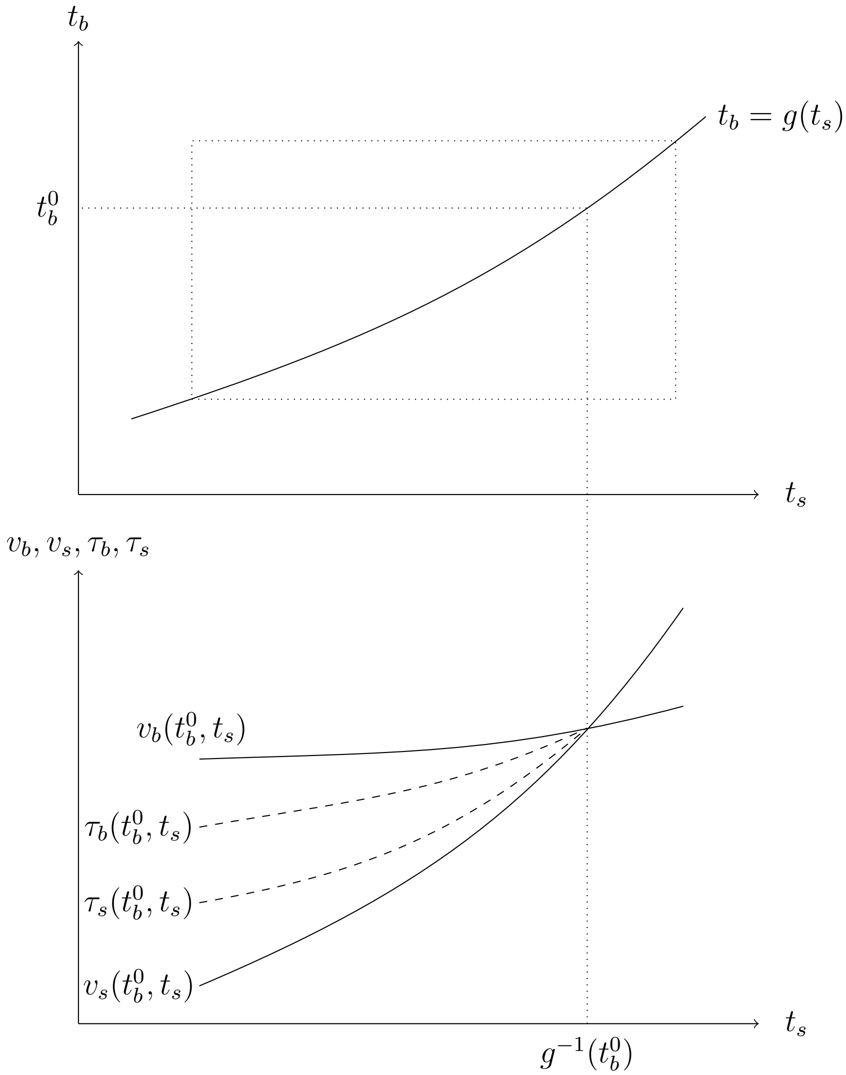

Figure 3.2: Case 1. $v_b(t_b, t_s)$ is strictly increasing in t_s for some intervals.

$\hat{t_b}$, then her ex-post payoff is 0 because

$$v_b(\hat{t_b}, \hat{t_s})\pi(\hat{t_b}, \hat{t_s}) - \tau_b(\hat{t_b}, \hat{t_s}) = v_b(\hat{t_b}, \hat{t_s}) * 1 - v_b(\hat{t_b}, \hat{t_s}) = 0.$$

If the buyer reports a type $\hat{t_b} - \eta$ where $\eta > 0$ is sufficiently small so that $\hat{t_b} - \eta > g(\hat{t_s})$ and $\pi(\hat{t_b} - \eta, \hat{t_s}) = 1$, then her ex-post payoff is strictly positive because

$$v_b(\hat{t_b}, \hat{t_s})\pi(\hat{t_b} - \eta, \hat{t_s}) - \tau_b(\hat{t_b} - \eta, \hat{t_s}) = v_b(\hat{t_b}, \hat{t_s}) - v_b(\hat{t_b} - \eta, \hat{t_s}) > 0.$$

That is, the buyer can be strictly better off by reporting $\hat{t_b} - \eta$ instead of her true type, $\hat{t_b}$. ■

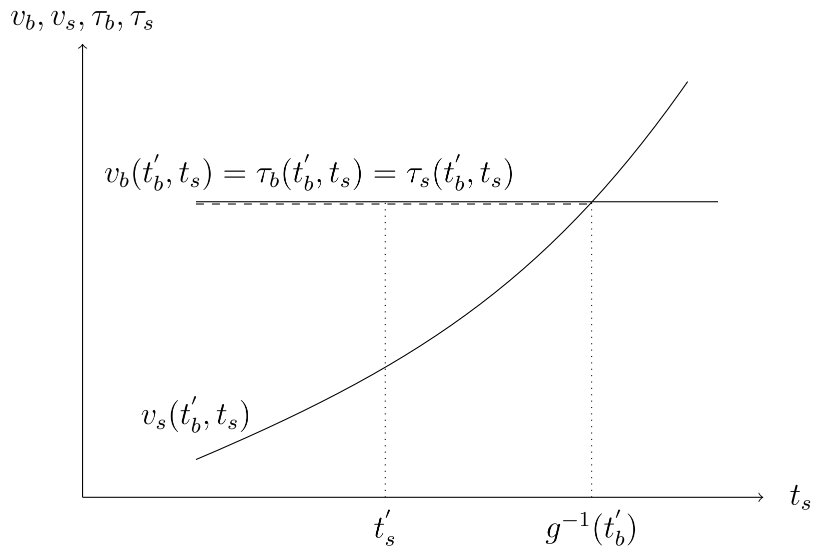

Figure 3.3: Case 2. $v_b(t_b, t_s)$ does not depend on t_s.

3.4 A Comparison with Jehiel and Moldovanu (2001)

In this section, I explain how my work differs from Jehiel and Moldovanu (2001).

Both Jehiel and Moldovanu (2001) and my research study interdependent values models: an agent's estimate for her own value for an item may depend on the signal held by the others. Jehiel and Moldovanu (2001) show that if each agent's signal is one-dimensional, then it is possible to have direct revelation mechanisms satisfying efficiency and Bayesian incentive compatibility. My research shows that it is impossible to have mechanisms satisfying efficiency, ex-post incentive compatibility, ex-post individual rationality, and no ex-post budget deficit.

Jehiel and Moldovanu (2001) identify conditions under which there exist direct revelation mechanisms satisfying efficiency and Bayesian incentive compatibility. The condition cannot be satisfied if each agent's signal is multi-dimensional. This is their impossibility result. However, the condition can be satisfied if the signal of each agent is one-dimensional. I study the one-dimensional signals and the model of my research satisfies this condition.[5] Thus, according to the result by Jehiel and Moldovanu (2001), some of the mechanisms I study in my research satisfy both

[5] See Appendix B for detailed explanation.

efficiency and Bayesian incentive compatibility.[6] I show, however, that if we add two more properties, individual rationality and no budget deficit, then there is no mechanisms satisfying all of these four properties. This difference between Jehiel and Moldovanu (2001) and my research is summarized in Table 3.1.

	Agents	Value Function
Jehiel & Moldovanu (2001)	n agents	Linear in signals $\left(V^i(s^i, s^{-i}) = \sum_j a_i^j s_i^j\right)$
My research	1 buyer and 1 seller	General $V^i(s^i, s^{-i})$

	Signals Dimension	Results
Jehiel & Moldovanu (2001)	Multi-dimensional	EF, BIC impossible
	One-dimensional	EF, BIC possible
My research	One-dimensional	EF, EPIC, EPIR, no EPBD impossible

Table 3.1: The difference between Jehiel and Moldovanu (2001) and my research.

3.5 Conclusion

Under the assumption that each trader's information has a greater effect on her own valuation than on the other trader's valuation, no bilateral trading mechanisms with interdependent values satisfy efficiency, ex-post incentive compatibility, ex-post individual rationality, and no ex-post budget deficit.

[6]The mechanism I introduce in the second chapter of the dissertation is an example because it satisfies efficiency and ex-post incentive compatibility.

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Appendix A

Proofs for Chapter 2

A.1 Well-Definedness of the Allocation Rule

In the second case of the allocation rule of the mechanism, $v_{(m)}(t^{'}) = v_{(m+1)}(t^{'})$. We need to show that $0 \leq k - |B_s| \leq |B_\alpha \backslash B_s|$, and $0 \leq k - |S_s| \leq |S_\alpha \backslash S_s|$ for the well-defined allocation rule. The number of traders for each range of valuations is summarized in Table A.1. We know that there are less than n traders

	Buyers	Sellers
$v > v_{(m+1)}(= v_{(m)})$	$\lvert B_s\rvert$	$n - \lvert S_\alpha\rvert$
$v = v_{(m+1)}(= v_{(m)})$	$\lvert B_\alpha\rvert - \lvert B_s\rvert$	$\lvert S_\alpha\rvert - \lvert S_s\rvert$
$v < v_{(m+1)}(= v_{(m)})$	$m - \lvert B_\alpha\rvert$	$\lvert S_s\rvert$

Table A.1: The number of valuations in each range

whose valuation is greater than $v_{(m+1)}(t^{'})$. Thus, from the first row of the table, $|B_s| + (n - |S_\alpha|) < n$. That is, $|B_s| - |S_\alpha| < 0$. Similarly, from the third row of the table, we can obtain that $(m - |B_\alpha|) + |S_s| < m$. Thus, $|S_s| - |B_\alpha| < 0$.

There are two cases depending on the values of $|B_\alpha|$ and $|S_\alpha|$. First, suppose that $|B_\alpha| \leq |S_\alpha|$, that is, $k = |B_\alpha|$. Because $B_s \subseteq B_\alpha$, $|B_\alpha \backslash B_s| = |B_\alpha| - |B_s| \geq 0$. Thus, $0 \leq k - |B_s| \leq |B_\alpha \backslash B_s|$. Also, from $|S_s| - |B_\alpha| < 0$, $|S_\alpha \backslash S_s| = |S_\alpha| - |S_s|$, and $|B_\alpha| \leq |S_\alpha|$, we can obtain that $0 < k - |S_s| \leq |S_\alpha \backslash S_s|$. Consider the second case in which $|B_\alpha| > |S_\alpha|$, that is, $k = |S_\alpha|$. From $|B_s| - |S_\alpha| < 0$, and $|B_\alpha \backslash B_s| = |B_\alpha| - |B_s|$, it follows that $0 < k - |B_s| \leq |B_\alpha \backslash B_s|$. Also, because $S_s \subseteq S_\alpha$, $|S_\alpha \backslash S_s| = |S_\alpha| - |S_s| \geq 0$. Thus, $0 \leq k - |S_s| \leq |S_\alpha \backslash S_s|$. ■

Appendix B

Proofs for Chapter 3

B.1 The Revelation Principle for the Ex-Post Nash Equilibrium.

Let Γ_i denote the set of actions for trader $i \in \{b, s\}$ and $\sigma_i : [\underline{t}_i, \bar{t}_i] \to \Gamma_i$ denote trader i's strategy. $\sigma = (\sigma_b, \sigma_s)$ is a profile of the strategies. Suppose that $\pi : \Gamma_b \times \Gamma_s \to [0, 1]$ is the probability that the trade happens between the traders, $\tau_b : \Gamma_b \times \Gamma_s \to \mathbb{R}$ is the expected transfer that the buyer pays, and $\tau_s : \Gamma_b \times \Gamma_s \to \mathbb{R}$ is the expected transfer that the seller receives. Each trader has a quasi-linear utility consisting of the valuation and transfer. A game is defined as $(\Gamma_b, \Gamma_s, \pi, \tau_b, \tau_s)$. If $\Gamma_b = [\underline{t}_b, \bar{t}_b]$ and $\Gamma_s = [\underline{t}_s, \bar{t}_s]$, then the game is a direct revelation game and (π, τ_b, τ_s) is a direct revelation mechanism.

The Revelation Principle. If $\sigma = (\sigma_b, \sigma_s)$ is a Nash equilibrium in the game $(\Gamma_b, \Gamma_s, \pi,$
$\tau_b, \tau_s)$, then honest reporting by each trader forms an ex-post Nash equilibrium in the game $([\underline{t}_b, \bar{t}_b], [\underline{t}_s, \bar{t}_s], \pi \circ \sigma, \tau_b \circ \sigma, \tau_s \circ \sigma)$.

Proof. Suppose that $\sigma = (\sigma_b, \sigma_s)$ is a Nash equilibrium in the game $(\Gamma_b, \Gamma_s, \pi, \tau_b, \tau_s)$. Consider the direct revelation game $([\underline{t}_b, \bar{t}_b], [\underline{t}_s, \bar{t}_s], \pi \circ \sigma, \tau_b \circ \sigma, \tau_s \circ \sigma)$. Let's first show that the buyer's best response in the revelation game is reporting her type honestly when the seller reports her type honestly. The proof for the seller's best response is similar so it is omitted. Consider the types $t_b, t_b^{'} \in [\underline{t}_b, \bar{t}_b]$, and $t_s \in [\underline{t}_s, \bar{t}_s]$. We need to show that

$$\begin{aligned} & v_b(t_b, t_s) * (\pi \circ \sigma)(t_b, t_s) - (\tau_b \circ \sigma)(t_b, t_s) \\ & \geq v_b(t_b, t_s) * (\pi \circ \sigma)(t_b^{'}, t_s) - (\tau_b \circ \sigma)(t_b^{'}, t_s). \end{aligned}$$

Because $\sigma = (\sigma_b, \sigma_s)$ is a Nash equilibrium in the game $(\Gamma_b, \Gamma_s, \pi, \tau_b, \tau_s)$, it follows that

$$\begin{aligned} &v_b(t_b, t_s)\pi(\sigma_b(t_b), \sigma_s(t_s)) - \tau_b(\sigma_b(t_b), \sigma_s(t_s)) \\ &\geq v_b(t_b, t_s)\pi(\sigma_b^{'}(t_b), \sigma_s(t_s)) - \tau_b(\sigma_b^{'}(t_b), \sigma_s(t_s)) \end{aligned}$$

for any strategy $\sigma_b^{'}$.

Define the buyer's strategy $\sigma_b^{'}$ as follows:

$$\sigma_b^{'}(t) = \sigma_b(t_b^{'}) \textit{ for all } t \in [\underline{t}_b, \bar{t}_b].$$

Then, we can obtain that

$$\begin{aligned} &v_b(t_b, t_s)\pi(\sigma_b(t_b), \sigma_s(t_s)) - \tau_b(\sigma_b(t_b), \sigma_s(t_s)) \\ &\geq v_b(t_b, t_s)\pi(\sigma_b(t_b^{'}), \sigma_s(t_s)) - \tau_b(\sigma_b(t_b^{'}), \sigma_s(t_s)). \end{aligned}$$

Thus, it leads to

$$\begin{aligned} &v_b(t_b, t_s) * (\pi \circ \sigma)(t_b, t_s) - (\tau_b \circ \sigma)(t_b, t_s) \\ &\geq v_b(t_b, t_s) * (\pi \circ \sigma)(t_b^{'}, t_s) - (\tau_b \circ \sigma)(t_b^{'}, t_s). \blacksquare \end{aligned}$$

B.2 Proof of *i)* of Proposition 3.2.

Define $H(t_b, t_s) := v_b(t_b, t_s) - v_s(t_b, t_s)$ for $(t_b, t_s) \in (\underline{t}_b, \bar{t}_b) \times (\underline{t}_s, \bar{t}_s)$. Then, $H(t_b, t_s)$ is continuously differentiable by (A2) and $H(t_b^*, t_s^*) = 0$. Also, from (A5), we can obtain that

$$\begin{aligned} \frac{\partial H(t_b, t_s)}{\partial t_b} &= \frac{\partial v_b(t_b, t_s)}{\partial t_b} - \frac{\partial v_s(t_b, t_s)}{\partial t_b} > 0, \\ \text{and } \frac{\partial H(t_b, t_s)}{\partial t_s} &= \frac{\partial v_b(t_b, t_s)}{\partial t_s} - \frac{\partial v_s(t_b, t_s)}{\partial t_s} < 0. \end{aligned}$$

Thus, we can apply the implicit function theorem and have the following results:

There exists a closed interval $I(t_s^*) = [t_s^* - \epsilon, t_s^* + \epsilon]$ for some $\epsilon > 0$ and a continuously differentiable function $g : I(t_s^*) \to [\underline{t}_b, \bar{t}_b]$ such that

1) $t_b^* = g(t_s^*)$,

2) $H(g(t_s), t_s) = 0$, that is, $v_b(g(t_s), t_s) = v_s(g(t_s), t_s)$ for all $t_s \in I(t_s^*)$, and

3) $\frac{dg(t_s)}{dt_s} = -\frac{\frac{\partial H(g(t_s), t_s)}{\partial t_s}}{\frac{\partial H(g(t_s), t_s)}{\partial t_b}} > 0$ for all $t_s \in I(t_s^*)$. ■

B.3 A Comparison with Jehiel and Moldovanu (2001)

In this section, I introduce the model of Jehiel and Moldovanu (2001). Then, I show how the model of my research satisfies the condition derived by Jehiel and Moldovanu (2001). By doing so, we can see the similarity and difference between the research by Jehiel and Moldovanu (2001) and my research. Also, we can see the implication of their research in my work and how my research is different from theirs.

B.3.1. The model of Jehiel and Moldovanu (2001)

There are N agents and K alternatives. $s^i \in S^i \subseteq \mathbb{R}^{K \times N}$ is a signal of agent i that is her private information where $s^i = (s^i_{kj})_{1 \leq k \leq K, 1 \leq j \leq N}$ and s^i_{kj} affects the agent j's utility in alternative k. Agent i's valuation function for alternative k is given by $V^i_k(s^1_{ki}, s^2_{ki}, \cdots, s^N_{ki})$ where $V^i_k(s^1_{ki}, s^2_{ki}, \cdots, s^N_{ki}) = \sum_{j=1}^{N} a^j_{ki} s^j_{ki}$ with the assumption that $a^i_{ki} \geq 0 \;\; \forall k \;\; \forall i$. If agent i obtains transfer x_i in alternative k, then her utility is $\sum_{j=1}^{N} a^j_{ki} s^j_{ki} + x_i$.

A direct revelation mechanism is a pair of functions (p, x) defined on the set of profiles of signals. $p : \prod_{i=1}^{N} S^i \to \mathbb{R}^K$ is a probability assignment function where $p_k(s^1, \cdots, s^N)$ is the probability that alternative k is chosen when reported signals are $(s^1, s^2, \cdots, s^N)$, and $x : \prod_{i=1}^{N} S^i \to \mathbb{R}^N$ is a transfer rule where $x_i(s^1, \cdots, s^N)$ is the payment received by agent i.

If $s^i \in S^i \subseteq \mathbb{R}$, then the signal of each agent is one-dimensional. In this case, the valuation function of agent i for alternative k is $V^i_k(s^1, s^2, \cdots, s^N) = \sum_{j=1}^{N} a^j_{ki} s^j$. Jehiel and Moldovanu (2001) derive the following sufficient condition for the existence of efficient and Bayesian incentive compatible direct revelation mechanism when the signal of each agent is one-dimensional.

$$\forall i \ \forall k \ \forall k' \qquad a^i_{ki} > a^i_{k'i} \Rightarrow \sum_{j=1}^{N} a^i_{kj} > \sum_{j=1}^{N} a^i_{k'j}. \tag{B.1}$$

B.3.2. My research and Jehiel and Moldovanu (2001)

First, I show that the model of my research satisfies condition (B.1), the sufficient condition for the existence of efficient and Bayesian incentive compatible mechanisms, derived by Jehiel and Moldovanu (2001) if I assume that the valuation functions in my model are linear in types of the agents.

The model of my research can be written again based on the notation by Jehiel and Moldovanu (2001) as follows:

There are two alternatives $(K = 2)$. Assume that alternative 1 $(k = 1)$ is that the buyer purchases the item from the seller, and alternative 2 $(k = 2)$ is that the buyer does not purchase the item. Assume also that the valuation function of each agent is linear in types of the agents. Then, the valuation function for each agent for each alternative, $V^i_k(t^b, t^s)$ for $k \in \{1, 2\}$ and $i \in \{b, s\}$ is given by,

$$V^b_1(t^b, t^s) = a^b_{1b} t^b + a^s_{1b} t^s, \qquad V^s_1(t^b, t^s) = 0 \cdot t^b + 0 \cdot t^s,$$
$$V^b_2(t^b, t^s) = 0 \cdot t^b + 0 \cdot t^s, \qquad V^s_2(t^b, t^s) = a^b_{2s} t^b + a^s_{2s} t^s$$

where $a^b_{1b} > a^b_{2s} \geq 0$, and $a^s_{2s} > a^s_{1b} \geq 0$ from the assumptions of my research, $\dfrac{\partial v_i}{\partial t^j} \geq 0$, $\dfrac{\partial v_i}{\partial t^i} > 0$, and $\dfrac{\partial v_i}{\partial t^i} > \dfrac{\partial v_j}{\partial t^i}$ $\forall i \in \{b, s\}, j \neq i$. We can check that these coefficients, $\{a^j_{ki}\}_{k \in \{1,2\}, i,j \in \{b,s\}}$, satisfy condition (4) derived by Jehiel and Moldovanu (2001). This implies that some of the mechanisms I study in my research satisfy efficiency and Bayesian incentive compatibility. However, I show that if we add two more properties, ex-post individual rationality and no ex-post budget deficit, then no mechanisms satisfying all of these four properties exist.