BIFURCATIONS IN NONLINEAR SCHRÖDINGER EQUATIONS WITH DOUBLE WELL POTENTIALS

BY

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DISSERTATION

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ABSTRACT

In this thesis, we consider nonlinear Schrödinger equations with double well potentials with attractive and repelling nonlinearities. We discuss bifurcations along bound states, especially ground states and the first excited states, and also deal with orbital stability of the ground states. In attractive case with large separations for double wells, our results shows that the ground state must undergo the secondary symmetry breaking bifurcation, while the first excited states can be uniquely extended as long as the bifurcation of the ground state has not occurred. In repelling case with large separations for double wells, we prove that the secondary bifurcation of the ground state does not emerge, even in the strongly nonlinear regime, while the first excited state must undergo the secondary bifurcation on the first excited states.

To my parents, for their love and support.

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LIST OF ABBREVIATIONS

$\overline{z}$: the complex conjugate of $z \in \mathbb{C}$.

$\mathcal{F}f$: the Fourier transform of the function f.

$\mathcal{F}f = (2\pi)^{-n/2} \int e^{-it\xi} f(x)dx, \quad \mathcal{F}^{-1}f = (2\pi)^{-n/2} \int e^{ix\xi} f(\xi)d\xi.$

$\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x)dx$ is the scalar product in the complex Hilbert space $L^2(\mathbb{R}^n, \mathbb{C})$, while $\langle f, g \rangle_{real} = Re\langle f, g \rangle$ is the scalar product in the same set but organized now as a real Hilbert space.

$f(\epsilon) = \mathcal{O}(\epsilon) \Leftrightarrow \limsup_{\epsilon \to 0} \left| \frac{f(\epsilon)}{\epsilon} \right| < \infty$

$f(\epsilon) = o(\epsilon) \Leftrightarrow \lim_{\epsilon \to 0} \frac{f(\epsilon)}{\epsilon} = 0$

$\{\phi\}^{\perp}$ is the subspace orthogonal to ϕ in $L^2(\mathbb{R}^n, \mathbb{C})$, w.r.t the complex scalar product.

$\{\phi\}^{\perp real}$ is the subspace orthogonal to ϕ in $L^2(\mathbb{R}^n, \mathbb{C})$, w.r.t the real scalar product.

$X \lesssim Y \Longleftrightarrow \exists C > 0$ independent of X, Y such that $X \leq CY$.

$\Sigma(A)$: the spectrum of an operator A. If A is self-adjoint on a Hilbert space we will use the disjoint decomposition $\Sigma(A) = \Sigma_{ess}(A) \cup \Sigma_{disc}(A)$

$\Sigma_{ess}(A)$: the essential spectrum of the self-adjoint operator A.

$\Sigma_{disc}(A)$: the discrete spectrum of a self-adjoint operator A.

$B_\delta(x_0) = \{x : |x - x_0| < \delta\}$.

$\mathrm{ran}(A)$: the range of A.

$\ker(A)$: the kernel of A.

$\text{coker}(A)$: a cokernel of A.

$F : H^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C}),\ F(\phi, E) = (-\Delta + E + V)\phi + \sigma|\phi|^{2p}\phi.$

$L_0 = -\Delta + V(x)$

$L_+(\phi, E) = -\Delta + E + V(x) + \sigma(2p+1)|\phi(x)|^{2p}.$

$L_-(\phi, E) = -\Delta + E + V(x) + \sigma|\phi(x)|^{2p}.$

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CHAPTER 1

INTRODUCTION

The time-dependent nonlinear Schrödinger (NLS) equation :

$$iu_t = -\Delta u + V(x)u + f(|u|)u, \tag{1.1}$$

where $u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ and $V(x) : \mathbb{R}^n \to \mathbb{R}$, is an ubiquitous model in applications e.g., laser beam propagation through optical fiber [1, 2, 3], water waves [4] and Bose-Einstein Condensates (BEC's) [5]. It describes nonlinear waves propagating through different media such as the ones mentioned above.

The main results of this thesis concern the existence and bifurcation of bound states (solitary waves or solitons) i.e., solutions of the form $u(x,t) = e^{iEt}\phi(x), \phi \in H^2(\mathbb{R}^n, \mathbb{C})$, where ϕ satisfies the stationary NLS equation:

$$F(\phi, E) = -\Delta\phi(x) + V(x)\phi(x) + f(|\phi(x)|)\phi(x) + E\phi(x) = 0, \quad x \in \mathbb{R}^n, \tag{1.2}$$

Many physical applications involve these special solutions, for example: solitons in optical fibers and water waves, Bose-Einstein Condensates (BEC) in statistical physics, etc. Some of them, such as the BEC's, are bound states of lowest energy which, from now on, will be called ground states. As we shall see in the next chapter, for linear Schrödinger equations, the dynamics can be decomposed as a superposition of bound states, which is the projection onto the eigenfunctions of the discrete spectrum, and a part that scatters (disperses) to infinity, which is the projection onto the continuous spectrum. This shows that the following conjecture, which is the most important mathematical problem in scattering theory, is true for the linear case:

Asymptotic Completeness Conjecture Any initial data evolves into a superposition of bound states and a part that disperses (scatters) to infinity.

Unlike the linear case, in the nonlinear case, the conjecture has been solved only for very particular problems, for example the completely integrable cubic Schrödinger equation in 1d, or certain weakly nonlinear regimes i.e., small initial data, see [6, 7, 8, 9]. In this thesis we show how the bound states organize themselves in complicated manifolds that intersect each other, giving rise to bifurcations and changes in the stability of the underlying solutions. The bifurcations lead to very complicated dynamics and can happen, as we shall see below, even in the weakly nonlinear regime.

Albeit our techniques can be adapted to general nonlinearities, see Chapter 7 for a discussion or [10], for clarity reasons, throughout most of the thesis we will consider a power nonlinearity:

$$f(|u|) = \sigma |u|^{2p}, \quad 0 < p < \frac{2}{n-2}, \quad \sigma \in \mathbb{R} \setminus 0,$$

and the coefficient σ may be negative (attractive/focusing nonlinearity) or positive (repelling/defocusing nonlinearity).

Remark 1.1. The map F in (1.2) satisfies the gauge equivariance:

$$F(e^{i\theta}\phi, E) = e^{i\theta} F(\phi, E) \quad \text{for all} \quad 0 \leq \theta < 2\pi$$

$$F(\overline{\phi}, E) = \overline{F(\phi, E)}$$

where $\overline{\phi}$ denotes the complex conjugate of ϕ. In particular, if (ϕ, E) solves (1.2), then $(e^{i\theta}\phi, E), 0 \leq \theta < 2\pi$ and $(\overline{\phi}, E)$ also solve (1.2).

There are only few results about bifurcations along bound states of NLS equations in the strongly nonlinear regime. The non-existence of secondary bifurcation along the symmetric (even in x_1) ground states was proved in [11] when $\sigma < 0, n = 1$ and the symmetric potential $V(x)$ is C^1, monotonically increasing for $x > 0$, $V(0) < 0$ and $\lim_{x\to\infty} V(x) = 0$. In [12], the existence of secondary bifurcation along the symmetric ground state was proved and geometric analysis of an asymmetric state emerging from the ground state was considered, with cubic nonlinearity and an double well potential consisting of two Dirac delta functions.

In [13], Kirr, Kevrekidis and Pelinovsky showed the existence of secondary bifurcation along the ground state in the case of one space dimension ($n = 1$), $\sigma < 0$ and potential $V(x)$ which satisfies:

(H1') $V \in L^\infty(\mathbb{R})$,

(H2') $\lim_{|x|\to\infty} V(x) = 0$,

(H3') $V(-x) = V(x)$ for all $x \in \mathbb{R}$, (symmetric),

(H4') $-\partial_x^2 + V$ has the lowest eigenvalue $-E_0 < 0$.

Theorem 1.1 (E. Kirr, P.G. Kevrekidis, and D. Pelinovsky)**.** *Consider the stationary NLS equation* (1.2) *with $\sigma < 0$ and $V(x)$ satisfying (H1')-(H4'). The C^1 curve $E \mapsto \psi_E$ which is symmetric, real-valued solutions bifurcating from zero at E_0, undergoes another bifurcation at a finite $E_* > E_0$ provided $V(x)$ has a nondegenerate maxima at $x = 0$ and $xV'(x) \in L^\infty(\mathbb{R})$.*

The above result takes advantage of the fact that, in one space dimension ($n = 1$), all solutions of (1.2) are real-valued up to multiplication, and reduces the analysis to real valued only solutions. In this thesis we will be working in arbitrary $\mathbb{R}^n$ spaces $n \geq 1$, and consider solutions that are complex valued. However, our bifurcation results will not apply to arbitrary symmetric potentials but to double well potentials with large separation between the wells which are defined in Section 2.3. In other words we will be generalizing the results in [14], where Kirr, Kevrekidis, Shlizerman and Weinstein proved that when $\sigma < 0$, the nonlinearity is cubic ($p = 1$) and $V(x)$ is symmetric, smooth and rapidly decaying as $|x| \to \infty$, then secondary bifurcation of the ground state occurs at small L^2-norm provided a certain value depending on the distance between two lowest eigenvalue is small enough.

Theorem 1.2 (E. Kirr, P.G. Kevrekidis, E. Shlizerman, and M.I. Weinstein)**.** *Let* (E_0, ψ_0) *and* (E_1, ψ_1) *be simple eigenvalues, real eigenfunctions pairs of* $(-\Delta + V(x))\psi = -E\psi$ *with* ψ_0 *is symmetric and* ψ_1 *is anti-symmetric,* $-E_0 < -E_1 < 0$*. Assume* $\Xi[\psi_0, \psi_1] = \|\psi_0\|_{L^4}^4 - \|\psi_1\|_{L^4}^4 - 2\|\psi_0\psi_1\|_{L^2}^2 < 0$, $\|\psi_0\psi_1\|_{L^2} \neq 0$, *and* $\frac{E_0 - E_1}{\Xi[\psi_0,\psi_1]^2}$ *is sufficiently small. Then there exists a bifurcation point* $\mathcal{N}_{cr} > 0$ *such that for* $\|\psi_E\|_{L^2}^2 = \mathcal{N} > \mathcal{N}_{cr}$*, there are two branches of solutions : (a) a continuation of the symmetric branch and (b) a new asymmetric branch.*

In particular, the results in [14] and [13] can be applied to double well potentials with large separation which we will handle for the most part of this thesis, but, as opposed to us, they are restricted to the case $n = 1$ or $p = 1$.

Our main results are presented in Chapters 5 and 6. These results are obtained for double well potentials V_s where s represents the distance between the wells, whose definition and properties will be discussed in Section 2.3. Chapter 5 deals with focusing nonlinearity $\sigma < 0$ and shows that for double well potentials with large separations i.e., there exists $s_* > 0$ such that for all $s \geq s_*$, we have that the ground states bifurcating from the lowest eigenvalue $E_{0,s}$, undergo a secondary bifurcation at $E_{*,s}$ due to an eigenvalue of the linearized operator crossing zero at $(\psi_{E_{*,s}}, E_{*,s})$). Moreover, we show that the bifurcation is of pitchfork type with symmetry breaking i.e., for $E_{0,s} < E < E_{*,s}$, the ground state is unique up to rotation, orbitally stable and symmetric: $\psi_E(-x_1, x_2, \cdots, x_n) = \psi_E(x_1, x_2, \cdots, x_n)$ while for $E > E_{*,s}$ we have three ground states up to rotation: one is symmetric and orbitally unstable while two are asymmetric and orbitally stable provided $p < p_* = \frac{3+\sqrt{13}}{2}$ and orbitally unstable for $p > p_*$. Furthermore, $E_{*,s} \to w_0$ and $\psi_{E_{*,s}} \xrightarrow{H^2} 0$ as $s \to \infty$. These results generalize the one dimensional result $n = 1$ in [13, Corollary 2] and the analytic $p = 1$ result in [14]. In addition, we show that the first excited state branch bifurcating from the second eigenvalue $E_{1,s}$ can be uniquely extended (no bifurcation) beyond the weakly non-linear regime where the bifurcation along the ground state occurs. Whether bifurcations may occur in the strongly nonlinear regime remain to be studied via global, non-perturbative techniques, see [15].

In Chapter 6, we deal with defocusing (repelling) nonlinearities $\sigma > 0$. Under the same assumptions on the potentials, we prove that, in the repelling case, the first excited state branch must bifurcate at some $E_{**,s}$ and there are no bifurcations along the ground state branch, even in the strong nonlinear regime.

CHAPTER 2

LINEAR SCHRÖDINGER OPERATORS AND THEIR SPECTRAL AND DYNAMICAL PROPERTIES

In this chapter, we will review known results mostly about the spectral properties of linear Schrödinger operators with real valued potentials.

2.1 The Free Schrödinger Operator

The free Schrödinger Operator $-\Delta : H^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C})$:

$$-\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is an unbounded self-adjoint operator on L^2 with domain H^2. Its spectrum can be readily determined via Fourier transform: $\Sigma(-\Delta) = \Sigma_{ess}(-\Delta) = [0, \infty)$ with generalized eigenfunctions at $|\xi|^2 \geq 0, \xi \in \mathbb{R}^n$, given by $\frac{e^{i\xi \cdot x}}{(2\pi)^{n/2}}$. See for example [16, 17].

The Hamiltonian dynamical system generated by:

$$\begin{aligned} iu_t &= -\Delta u \text{ for } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R} \\ u(x,0) &= f(x) \text{ for } x \in \mathbb{R}^n \end{aligned} \tag{2.1}$$

can also be solved using the spectral decomposition:

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int k(\xi, t) e^{i\xi \cdot x} d\xi \tag{2.2}$$

where the coefficients $k(\xi, t)$:

$$k(\xi,t) = \frac{1}{(2\pi)^{n/2}} \int u(x,t) e^{-i\xi\cdot x} dx = \mathcal{F}(u(\cdot,t))(\xi).$$

Hence, by taking Fourier transform of (2.1) we get for each $\xi \in \mathbb{R}^n$:

$$i\frac{d}{dt}k(\xi,t) = |\xi|^2 k(\xi,t)$$

$$k(\xi,0) = \frac{1}{(2\pi)^{n/2}} \int f(x) e^{-i\xi\cdot x} dx = \mathcal{F}(f)(\xi)$$

with the solution:

$$k(\xi,t) = e^{-i|\xi|^2 t} k(\xi,0) \tag{2.3}$$

and group velocity for wave packets at frequency ξ :

$$v_g(\xi) = \nabla|\xi|^2 = 2\xi$$

showing dispersion in frequency. Its effect on general initial data can be measured in L^2 and L^∞ norms:

$$\|u(t)\|_{L^2} = \|\mathcal{F}^{-1} e^{-i|\xi|^2 t} \mathcal{F} u(0)\|_{L^2} = \|u(0)\|_{L^2} \tag{2.4}$$

$$\|u(t)\|_{L^\infty} - \|(4\pi i t)^{-n/2} \int e^{\frac{-|x-y|^2}{4it}} u(y,0) dy\|_{L^\infty} \leq |4\pi t|^{-n/2} \|u(0)\|_{L^1}. \tag{2.5}$$

i.e., the wave packets making up the initial condition disperse (scatter) to infinity, preserving the L^2 norm but their contribution in any fixed bounded domain decays zero. By interpolating between (2.4) and (2.5), we get a rate of decay for L^p norm for $2 \leq p \leq \infty$: if $f \in L^q$, $1 \leq q \leq 2$, $1/p + 1/q = 1$, then

$$\|u(t)\|_{L^p} \leq |4\pi t|^{-\frac{n(1/2-1/p)}{2}} \|u(0)\|_{L^q}.$$

See for example [17]. We will need some technical results about Sobolev spaces. We recall that Schwartz spaces:

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^n\}$$

where $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ for a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, and spaces of tempered distribution on $\mathbb{R}^n$: $\mathcal{S}'(\mathbb{R}^n)$, i.e., the dual of $\mathcal{S}(\mathbb{R}^n)$. For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, Sobolev spaces are defined as follows:

$$W^{m,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n) \text{ for } |\alpha| \leq m\}$$

and the resolvent of the free Schrödinger operator which we state here and prove the less known ones. The first result allows us to use a Fourier type norm on Sobolev spaces. We recall that the definition of the Fourier transform is extended to tempered distributions, hence, one defines as follows: For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$,

$$H^{s,p}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}u] \in L^p(\mathbb{R}^n)\}.$$

Theorem 2.1 (Mihlin multiplier theorem)**.** *If $1 < q < \infty$, and m is an integer, then*

$$W^{m,p}(\mathbb{R}^n) = H^{m,p}(\mathbb{R}^n)$$

and the norm of $W^{m,p}$ is equivalent to the norm of $H^{m,p}$:

$$\|u\|_{H^{m,p}} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{m/2}\mathcal{F}u]\|_{L^p}.$$

See [18].

Remark 2.1. We now use the Fourier type norm $\|\cdot\|_{H^{2,p}}$ on $W^{2,q}$.

Proposition 2.1. *$(-\Delta+1) : W^{2,q} \to L^q$ is unitary for all $1 < q < \infty$.*

Proof. For any $g \in W^{2,q}$, $1 < q < \infty$,

$$\|(-\Delta+1)g\|_{L^q} = \|\mathcal{F}^{-1}[(1+|\xi|^2)\mathcal{F}g]\|_{L^q} = \|g\|_{W^{2,q}}$$

□

Proposition 2.2. $\|(-\Delta+m+i)^{-1}\|_{L^q\to W^{2,q}} \le 1+\sqrt{2}$ *for all* $1<q<\infty, m\in\mathbb{R}$.

Proof. For any $g\in\mathcal{S}(\mathbb{R}^n)$ we get

$$\begin{aligned}\|(-\Delta+m+i)g\|_{L^q} &= \sup_{h\in\mathcal{S}(\mathbb{R}^n),\|h\|_{L^{q'}}\le 1} |\langle h,(-\Delta+m+i)g\rangle| \\ &= \sup_{h\in\mathcal{S}(\mathbb{R}^n),\|h\|_{L^{q'}}\le 1} |\langle \nabla h,\nabla g\rangle + \langle h,(m+i)g\rangle| \\ &\ge \sup_{h\in\mathcal{S}(\mathbb{R}^n),\|h\|_{L^{q'}}\le 1} \sqrt{m^2+1}|\langle h,g\rangle| \\ &= \sqrt{m^2+1}\|g\|_{L^q}.\end{aligned}$$

It follows that

$$\|(-\Delta+m+i)^{-1}\|_{L^q\to L^q} \le \frac{1}{\sqrt{m^2+1}}.$$

Now, since

$$(-\Delta+m+i)^{-1} = (-\Delta+1)^{-1}(\mathbb{I}+(1-m-i)(-\Delta+m+i)^{-1})$$

and

$$\begin{aligned}\|(\mathbb{I}+(1-m-i)(-\Delta+m+i)^{-1})\|_{L^q\to L^q} &\le 1+|1-m-i|\|(-\Delta+m+i)^{-1}\|_{L^q\to L^q} \\ &\le 1+\frac{\sqrt{(m-1)^2+1}}{\sqrt{m^2+1}} \\ &\le 1+\sqrt{2},\end{aligned}$$

combined with Proposition 2.1, $\|(-\Delta+m+i)^{-1}\|_{L^q\to W^{2,q}} \le 1+\sqrt{2}$. □

2.2 Schrödinger Operators with real-valued potentials

Consider the Schrödinger operator with real-valued potential $-\Delta+V : H^2(\mathbb{R}^n,\mathbb{C}) \to L^2(\mathbb{R}^n,\mathbb{C})$ where $V:\mathbb{R}^n\to\mathbb{R}$ (real-valued). We will assume the following about the potential V:

(H1) $V(x) \in L^r + L^\infty_\epsilon$ for all $1 \le r \le q$ where $q \ge \max(n/2, 2)$ for $n \ne 4, q > 2$ for $n = 4$.

Under this assumption, $-\Delta + V$ remains self-adjoint on L^2 with domain H^2. Moreover, V is a relatively compact perturbation of $-\Delta$, i.e., $V(-\Delta + i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^n)$, see [19] Chapter XIII.4. By Weyl's theorem the essential spectrum of $-\Delta + V$ is the same as $-\Delta$, i.e., $\Sigma_{ess}(-\Delta + V) = [0, \infty)$. However, the relatively compact perturbation may add a finite or countable number of isolated eigenvalues $-w_0 < -w_1 < \cdots < -w_k < \cdots < 0$ with only possible accumulation point 0, and, each of them have a finite dimensional invariant subspace, i.e., $\ker(-\Delta + V + w_k)$ is finite dimensional for all k. In what follows, we will assume that at least one such eigenvalue is supported by the potential:

(H2) The L^2 spectrum of $-\Delta + V(x)$ has the lowest negative eigenvalue $-w_0 < 0$.

This is definitely the case when V is continuous, nonzero and $V \le 0$ on $\mathbb{R}$ or $\mathbb{R}^2$, see Chapter XIII.3. Note that because of the uniform ellipticity of the second order operator $-\Delta + V$, its lowest eigenvalue is simple. i.e., $\ker(-\Delta + V + w_0)$ is one dimensional (over $\mathbb{C}$).

Our arguments will also rely on the spectral property of the family of Schrödinger operators:

$$L(E) = -\Delta + V + \tilde{\sigma}|\psi_E|^{2p}$$

where $E \mapsto \psi_E$ is a C^1 map from an interval $I \subseteq (0, \infty)$ to $H^2(\mathbb{R}^n, \mathbb{C})$, $0 < p < \dfrac{2}{n-2}$ and $\tilde{\sigma} \in \mathbb{R}$ is a constant. We will show that not only V, but also the effective potential of the operator $V_{eff}(E) = V + \tilde{\sigma}|\psi_E|^{2p}$ is a relatively compact perturbation of $-\Delta$. Note that by Sobolev embedding, $H^2(\mathbb{R}^n, \mathbb{C})$ is embedded in $L^q(\mathbb{R}^n, \mathbb{C})$ for all $2 \le q \le \dfrac{2n}{n-4}$ for $n > 4$, $2 \le q < \infty$ for $n = 4$ and $2 \le q \le \infty$ for $n < 4$. By choosing some proper $r > 0$, we can see that $\tilde{\sigma}|\psi_E|^{2p} \in L^r(\mathbb{R}^n, \mathbb{C})$. Indeed, if $n > 4$, we choose $r = \max(\frac{1}{p}, \frac{n}{2}) \Leftrightarrow 2pr = \max(2, np)$ while we choose $r = \max(\frac{1}{p}, 3) \Leftrightarrow 2pr = \max(2, 6p)$ if $n = 4$ and $r = \max(\frac{1}{p}, 2) \Leftrightarrow 2pr = \max(2, 4p)$ if $n < 4$. Then

$\||\psi_E|^{2p}\|_{L^r} = \|\psi_E\|^{2p}_{L^{2pr}} < \infty$ since $2 \leq 2pr \leq \frac{2n}{n-4}$ for $n > 4$ and $2 \leq 2pr < \infty$ for $n \leq 4$. Thus, $\tilde{\sigma}|\psi_E|^{2p} \in L^r$ for some $r \geq \max(\frac{n}{2}, 2)$ for $n \neq 4$ and $r > 2$ where $n = 4$. Hence, $\tilde{\sigma}|\psi_E|^{2p}$ is a relatively compact perturbation of $-\Delta$ and consequently, the effective potential $V_{eff}(E) = V + \tilde{\sigma}|\psi_E|^{2p}$ is a relatively compact perturbation of $-\Delta$. Therefore for all $E \in I$, $\Sigma_{ess}(L(E)) = [0, \infty)$. Moreover, the C^1 dependence of the effective potential V_{eff} on E implies C^1 dependence of simple eigenvalues of $L(E)$ on E, and, for non-simple eigenvalues, the projection operator onto their invariant subspace remains C^1 with respect to E, see [19] Chapter XII.2.

We will also need some technical results about the resolvent of the Schrödinger operator between Sobolev spaces.

Proposition 2.3. *For all* $1 < q < \infty$, $m \in \mathbb{R}$, $(-\Delta + V + m + i)^{-1} : L^q \to W^{2,q}$ *is bounded.*

Proof. Note that

$$(-\Delta + V + m + i)^{-1} = (-\Delta + m + i)^{-1}(\mathbb{I} + V(-\Delta + m + i)^{-1})^{-1}$$

$(\mathbb{I} + V(-\Delta + m + i)^{-1})^{-1} : L^q \to L^q$ is bounded because

$$\mathbb{I} + V(-\Delta + m + i)^{-1} = (-\Delta + V + m + i)(-\Delta + m + i)^{-1}$$

and the spectrum of $-\Delta, -\Delta + V$ is in $\mathbb{R}$ so the both equation in right hand side are invertible from L^q to L^q. Thus, Proposition 2.2 completes the proof. ☐

2.3 Schrödinger Operators with double-well potentials

In this section, we will discuss the Schrödinger operator with double well potentials. The main results of this thesis are based on double well potentials with large separation, which are constructed in the following manner. Consider a (single-well) potential $V_0(x)$ satisfying (H1) and (H2), then the double well potential $V = V_s$ is:

$$V \equiv V_s = T_s V_0 T_{-s} + R T_s V_0 T_{-s} R$$

where $T_{\pm s}$ and R are the translations, respectively reflection operators:

$$T_{\pm s}g(x_1, x_2, \cdots, x_n) = g(x_1 \pm s, x_2, \cdots, x_n)$$
$$Rg(x_1, x_2, \cdots, x_n) = g(-x_1, x_2, \cdots, x_n)$$

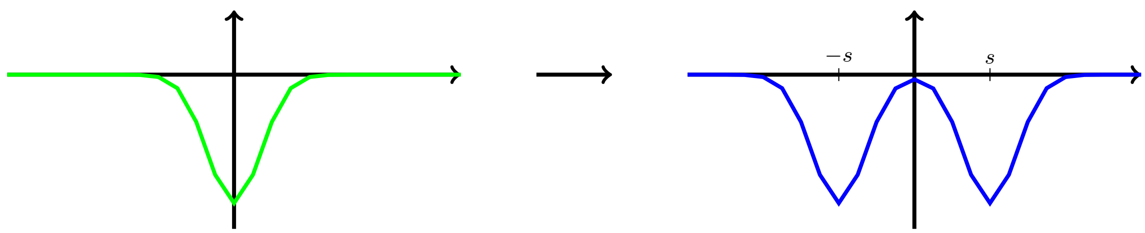

Figure 2.1: The graph in right pane is a double well potential which is constructed by translating and reflecting the single well potential in left pane.

In physical applications, large separation between the wells means that any given particle tends to “feel” only the effect of one well. Mathematically, this means that the spectrum of the operator with the double well potential has two eigenvalues very close to one eigenvalue in the spectrum of the operator with the corresponding one well potential:

Proposition 2.4 (see [14]). *Assume that $-w_0 < 0$ is a simple eigenvalue of $-\Delta + V_0$ separated from the rest of the spectrum of $-\Delta + V_0$ by a distance greater than $2d_*$. Denote by ψ_0 its corresponding eigenvector, $\|\psi_0\|_{L^2} = 1$. Then there exists $l_0 > 0$ such that for $s \geq l_0$ the following are true.*

(i) $-\Delta + V_s$ *has exactly two simple eigenvalues* $-E_{0,s}$ *and* $-E_{1,s}$ *nearer to* $-w_0$ *than* $2d_*$. *Moreover,* $\lim_{s\to\infty} E_{i,s} = w_0$, $i = 0, 1$.

(ii) *One can choose the normalized eigenvectors* $\psi_{i,s}$, $\|\psi_{i,s}\|_{L^2} = 1$, *corresponding to the eigenvalues* $-E_{i,s}$, $i = 0, 1$, *such that they satisfy*

$$\lim_{s\to\infty} \left\| \psi_{i,s} - \frac{T_s\psi_0 + (-1)^i RT_s\psi_0}{\sqrt{2}} \right\|_{H^2} = 0, \quad i = 0, 1$$

See [20] for the L^2 result and [14] Appendix for the convergence of the eigenvectors in H^2.

We need one more technical result, namely the uniform in "s" $L^2 \to H^2$ bounds for the resolvent :

Lemma 2.1. *Let $d_* > 0$. Then there exists s_0 and $M > 0$, independent of s, such that for $s \geq s_0$, $\|(-\Delta + V_s + E)^{-1}\|_{L^2 \cap \{\psi_{s,0},\psi_{s,1}\}^\perp \to H^2} \leq M$ for all $E \in \{E > 0 : dist(E, \Sigma_s \setminus \{E_{0,s}, E_{1,s}\}) \geq d_*\}$ where Σ_s is the spectrum of $-\Delta + V_s$.*

Proof. Note that

$$(-\Delta + V_s + E)^{-1} = (-\Delta + V_s + i\Omega)^{-1}[(-\Delta + V_s + E)(-\Delta + V_s + i\Omega)^{-1}]^{-1}$$

for $\Omega > 0$. The proof has two parts: first, we show that for some $\Omega > 0$, we have $\|(-\Delta + V_s + i\Omega)^{-1}\|_{L^2 \to H^2} \leq 2$. second, we prove that $\|[(-\Delta + V_s + E)(-\Delta + V_s + i\Omega)^{-1}]^{-1}\|_{L^2 \to L^2}$ is uniformly bounded.

Let us prove the first part. For any $\Omega > 1$ and any $g \in L^2$, we have:

$$\sup_{g \in L^2, \|g\|_{L^2} = 1} \|(-\Delta + i\Omega)^{-1} g\|_{H^2} = \sup_{g \in L^2, \|g\|_{L^2} = 1} \left\| \frac{|\xi|^2 + 1}{|\xi|^2 + i\Omega} \mathcal{F}g(\xi) \right\|_{L^2} \leq 1.$$

Thus, we get

$$\|(-\Delta + i\Omega)^{-1}\|_{L^2 \to H^2} \leq 1. \tag{2.6}$$

Next, we claim that there exists $\Omega > 1$ such that $\|V_s(-\Delta + i\Omega)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{2}$. By using spectral representation and the fact that $\|V_0(-\Delta + i)^{-1}\|_{L^2 \to L^2}$ is an compact operator, we can show that

$$\lim_{\Omega \to \infty} \|V_0(-\Delta + i\Omega)^{-1}\|_{L^2 \to L^2} = 0,$$

see [19] Chapter XIII.4 Corollary 2. Hence we can choose $\Omega > 1$ such that

$$\|V_0(-\Delta + i\Omega)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{4}.$$

Now, because $T_{\pm s}$ and R are unitary and commute with $-\Delta$, we have:

$$\begin{aligned}
&\|V_s(-\Delta+i\Omega)^{-1}\|_{L^2\to L^2}\\
&\leq \|T_sV_0T_{-s}(-\Delta+i\Omega)^{-1}\|_{L^2\to L^2}+\|RT_sV_0T_{-s}R(-\Delta+i\Omega)^{-1}\|_{L^2\to L^2}\\
&= 2\|V_0(-\Delta+i\Omega)^{-1}\|_{L^2\to L^2}\leq \frac{1}{2}.
\end{aligned}$$

Consequently, $\mathbb{I}+V_s(-\Delta+i\Omega)^{-1}$ is invertible and

$$\|[\mathbb{I}+V_s(-\Delta+i\Omega)^{-1}]^{-1}\|_{L^2\to L^2}\leq 2.$$

All in all,

$$\begin{aligned}
\|(-\Delta+V_s+i\Omega)^{-1}\|_{L^2\to H^2} &= \|(-\Delta+i\Omega)^{-1}[\mathbb{I}+V_s(-\Delta+i\Omega)^{-1}]^{-1}\|_{L^2\to H^2}\\
&\leq \|(-\Delta+i\Omega)^{-1}\|_{L^2\to H^2}\|[\mathbb{I}+V_s(-\Delta+i\Omega)^{-1}]^{-1}\|_{L^2\to L^2}\\
&\leq 2.
\end{aligned}$$

Next, in order to show $\|[(-\Delta+V_s+E)(-\Delta+V_s+i\Omega)^{-1}]^{-1}\|_{L^2\cap\{\psi_{s,0},\psi_{s,1}\}^\perp\to L^2}$ is uniformly bounded, we need to prove that for any $f\in L^2\cap\{\psi_{s,0},\psi_{s,1}\}^\perp$, $\|f\|_{L^2}=1$ there exists $M>0$ such that

$$\|[(-\Delta+V_s+E)(-\Delta+V_s+i\Omega)^{-1}]f\|_{L^2}\geq M. \tag{2.7}$$

By using spectral Theorem, we have:

$$\begin{aligned}
&[(-\Delta+V_s+E)(-\Delta+V_s+i\Omega)^{-1}]f=\\
&= \Sigma_{E_{n,s}\in\Sigma_{s,disc}}\frac{E_{n,s}+E}{E_{n,s}+i\Omega}\langle\psi_{n,s},f\rangle\psi_{n,s}+\int_{\Sigma_{s,cont}}\frac{\xi+E}{\xi+i\Omega}\mu_s(\xi)fd\xi\\
&= \Sigma_{E_{n,s}\in\Sigma_{s,disc}\setminus\{E_{0,s},E_{1,s}\}}\frac{E_{n,s}+E}{E_{n,s}+i\Omega}\langle\psi_{n,s},f\rangle\psi_{n,s}+\int_{\Sigma_{s,cont}}\frac{\xi+E}{\xi+i\Omega}\mu(\xi)fd\xi
\end{aligned}$$

where $\psi_{n,s}$ are L^2-normalized eigenfunctions corresponding to the isolated eigenvalue $-E_{n,s}$ of $-\Delta+V_s$, $\Sigma_{s,cont}$ ($\Sigma_{s,disc}$) is continuous (discrete) spectrum of $-\Delta+V_s$ and

$\mu_s(\xi)$ is the spectral measure of $-\Delta + V_s$ on the continuous spectrum. The last equality is due to the fact that $f \in L^2 \cap \{\psi_{s,0}, \psi_{s,1}\}^\perp$. Therefore, we get:

$$\begin{aligned}
&\|[(-\Delta + V_s + E)(-\Delta + V_s + i\Omega)]f\|_{L^2}^2 = \\
&= \Sigma_{E_{n,s}\in\Sigma_{s,disc}} \left|\frac{E_{n,s}+E}{E_{n,s}+i\Omega}\right|^2 |\langle\psi_{n,s}, f\rangle|^2 + \int_{\Sigma_{s,cont}} \left|\frac{\xi+E}{\xi+i\Omega}\right|^2 |\mu(\xi)|^2|f|^2 d\xi \\
&\geq \min\left(\min_{E_{n,s}\in\Sigma_{s,disc}, n\neq 0,1} \left|\frac{E_{n,s}+E}{E_{n,s}+i\Omega}\right|^2, \min_{\xi\in\Sigma_{s,cont}} \left|\frac{\xi+E}{\xi+i\Omega}\right|^2\right) \|f\|_{L^2}^2 \\
&\geq \min\left(\frac{{d_*}^2}{E_{2,s}^2+\Omega^2}, \min\left(1, \frac{E^2}{\Omega^2}\right)\right).
\end{aligned}$$

The last line is due to (5.3) and $\Sigma_{s,cont} = [0,\infty)$, because $-\Delta + V_s$ is a relatively compact perturbation of $-\Delta$. Since $E_{2,s}$ approaches the second lowest eigenvalue of $-\Delta + V_0$ as $s \to \infty$, there exist some $N > 0$ such that $E_{2,s} \leq N$ for $s \geq s_0$. Thus, we obtain uniform bound:

$$||[(-\Delta + V_s + E)(-\Delta + V_s + i\Omega)^{-1}]^{-1}\|_{L^2\to L^2} \leq M = \frac{1}{\sqrt{\min\left(\frac{{d_*}^2}{N^2+\Omega^2}, \min\left(1, \frac{E^2}{\Omega^2}\right)\right)}}.$$

Combining the first and second parts, the lemma is proved. □

Remark 2.2. The proof above also shows that for $d_* > 0$, if $E \in \{E > 0 : dist(-E, \Sigma_s) \geq d_*\}$, then there exist s_0 and $M > 0$, independent of s, such that $\|(-\Delta + V_s + E)^{-1}\|_{L^2\to H^2} \leq M$ for all $s \geq s_0$.

CHAPTER 3

LOCAL BIFURCATION THEORY

In this chapter, we will review local bifurcation theory, including Implicit Function Theorem, Lyapunov-Schmidt decomposition and Morse Lemma. See for example [21].

3.1 Calculus in Banach space and Implicit Function Theorem

Let X and Y be Banach spaces and $B(X, Y)$ be the set of bounded linear maps from X to Y. Let $f : U \to Y$, be a mapping defined on an open subset of X, U.

Definition 3.1. *f has a Gâteaux derivative at $u \in U$ in the direction $x \in X$ if:*

$$\lim_{\epsilon \to 0} \frac{f(u + \epsilon x) - f(u)}{\epsilon}$$

exists and finite.

The limit is usually denoted by $df(u)[x]$ if it exists. We say that f is Gâteaux differentiable at $u \in U$ if and only if f has a Gâteaux derivative in any direction.

Definition 3.2. *f is Fréchet differentiable at $u \in U$ if there exists $A \in B(X, Y)$ such that:*

$$\lim_{|x| \to 0} \frac{\|f(u + x) - f(u) - Au\|}{\|x\|} = 0.$$

A is called a Fréchet derivative and denoted by $Df(u)$ if it exists.

Theorem 3.1. *If f is Fréchet differentiable at $u \in U$, then F is Gâteaux differentiable at u and:*

$$Df(u)[x] = df(u)[x] \text{ for all } x \in X$$

As in finite dimensions, where the Gateaux derivative corresponds to directional derivatives, the reciprocal is not true in general. However the following result holds:

Theorem 3.2. *Let V be a neighborhood of $u \in U$. Assume $df(v)[\cdot] \in B(X,Y)$ for any $v \in V$ and $df : V \to B(X,Y)$ is continuous at u. Then f is Fréchet differentiable at u.*

For example, the Gâteaux derivative of (1.2) at ϕ in the direction u is:

$$\begin{aligned} dF(\phi)[u] &= \lim_{\epsilon \to 0} \frac{F(\phi + \epsilon u) - F(\phi)}{\epsilon} \\ &= (-\Delta + V + E)u + \lim_{\epsilon \to 0} \frac{1}{\epsilon}(|\phi + \epsilon u|^{2p}(\phi + \epsilon u) - |\phi|^{2p}\phi) \\ &= (-\Delta + V + E)u + (p+1)u|\phi|^{2p} + p\bar{u}|\phi|^{2p-2}\phi^2. \end{aligned}$$

provided $\epsilon \in \mathbb{R}$, for $\epsilon \in \mathbb{C}$ the limit does not exists. Consequently we will be forced to work with the real structure over the Hilbert spaces H^2 and L^2. With this observation we have $dF : H^2(\mathbb{R}^n) \to B(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ is continuous, by the above results, the Fréchet derivative of (1.2) at each point exists and is equal to:

$$D_\phi F(\phi, E) = \begin{bmatrix} -\Delta + E & 0 \\ 0 & -\Delta + E \end{bmatrix} + \mathcal{V}(\phi) \tag{3.1}$$

where

$$\mathcal{V}(\phi) = \begin{bmatrix} V(x) + \sigma(2p+1)|\phi|^{2p} - 2\sigma p|\phi|^{2p-2}(\Im\phi)^2 & 2\sigma p|\phi|^{2p-2}\Re\phi\Im\phi \\ 2\sigma p|\phi|^{2p-2}\Re\phi\Im\phi & V(x) + \sigma|\phi|^{2p} + 2\sigma p|\phi|^{2p-2}(\Im\phi)^2 \end{bmatrix},$$

hence it is also continuous from $H^2(\mathbb{R}^n)$ to $B(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ i.e., the map F is C^1.

At a real valued ϕ, it simplifies to:

$$D_\phi F(\phi, E)[u_1 + iu_2] = L_+ u_1 + iL_- u_2 = \begin{bmatrix} L_+(\phi, E) & 0 \\ 0 & L_-(\phi, E) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{3.2}$$

where $L_\pm$ are linear, self-adjoint, Schrödinger type operators:

$$\begin{aligned} L_+(\phi, E) &= -\Delta + E + V(x) + \sigma(2p+1)|\phi|^{2p}(x) \\ L_-(\phi, E) &= -\Delta + E + V(x) + \sigma|\phi|^{2p}(x) \end{aligned} \tag{3.3}$$

The linearization of the stationary equation is important for the analysis in this thesis.

Remark 3.1. If (ϕ, E) solves (1.2) then zero is an eigenvalue of $D_\phi F(\phi, E)$ with eigenfunction $i\phi$. This can be directly seen from Remark 1.1 since $i\phi$ is the infinitesimal generator of rotation : $\phi \mapsto e^{i\theta}\phi$. In particular, if ϕ is real-valued then 0 is an eigenvalue of $L_-(\phi, E)$ with eigenfunction ϕ.

Now, we talk about Implicit Function Theorem in Banach spaces.

Theorem 3.3. *Let $f : U \to Z$ where $U \subseteq X \times Y$ is open and X, Y and Z are Banach spaces. Assume that*

1. *$f(x_0, y_0) = 0$ for $(x_0, y_0) \in U$*
2. *f is continuous at (x_0, y_0)*
3. *$D_x f(x_0, y_0) \in B(X, Z)$ is isomorphism from X to Z*
4. *$D_x f(x, y)$ exists for all $(x, y) \in U$ and $(x, y) \mapsto D_x f(x, y)$ is continuous at (x_0, y_0).*

Then

(i) *there exist $\delta > 0, r > 0$ with $\overline{B_\delta(x_0)} \times \overline{B_r(y_0)} \subseteq U$ and a unique map $u : \overline{B_r(y_0)} \to \overline{B_\delta(x_0)}$ such that the solution of $f(x, y) = 0$ in $\overline{B_\delta(x_0)} \times \overline{B_r(y_0)}$ are* *given by $(u(y), y)$. Moreover u is continuous at y_0 and if f is Fréchet differentiable at (x_0, y_0) then so is u at y_0 and*

$$D_u(y_0) = -[D_x f(x_0, y_0)]^{-1} D_y f(x_0, y_0)$$

(ii) If f is continuous on $\overline{B_\delta(x_0)} \times \overline{B_r(y_0)}$ then u is continuous on $\overline{B_r(y_0)}$. If f is Fréchet differentiable on $B_\delta(x_0) \times B_r(y_0)$ then u is differentiable on $B_r(y_0)$ and

$$D_u(y) = -[D_x f(u(y), y))]^{-1} D_y f(u(y), y)$$

(iii) If $f \in C^p$ on $B_\delta(x_0) \times B_r(y_0)$ then u is C^p on $B_r(y_0)$.

Along this trivial family of solutions $\phi \equiv 0, E \in \mathbb{R}$, we have:

$$D_\phi F(0, E) = \begin{bmatrix} -\Delta + V + E & 0 \\ 0 & -\Delta + V + E \end{bmatrix}$$

which is an isomorphism for $E \notin \Sigma(-\Delta + V) = \Sigma_{ess}(-\Delta + V) \cup \Sigma_{disc}(-\Delta + V) = [0, \infty) \cup \Sigma_{disc}(-\Delta + V)$. Therefore, the Implicit Function Theorem implies that there are no other small in H^2-norm bound states for $E > 0$, $-E \notin \Sigma_{disc}(-\Delta + V)$. It is already known that for $E < 0$ there are no other localized bound states (rapidly decaying at infinity solutions of (1.2)), see [22]. For $-E \in \Sigma_{disc}(-\Delta + V)$,

$$\Sigma_{disc}(-\Delta + V) = \{-w_0, -w_1, \cdots, -w_k, \cdots\},$$

we have at $E_k = w_k$, $\ker(-\Delta + V + E_k)$ is finite dimensional, see Section 2.2, and $\mathrm{ran}(-\Delta + V + E_k) = \ker(-\Delta + V + E_k)^\perp$ due to self-adjointness of $-\Delta + V + E_k$ on L^2. Consequently, $D_\phi F(0, E_k) : H^2(\mathbb{R}^n, \mathbb{R}) \times H^2(\mathbb{R}^n, \mathbb{R}) \to L^2(\mathbb{R}^n, \mathbb{R}) \times L^2(\mathbb{R}^n, \mathbb{R})$ will have a finite dimensional kernel, double the dimension of $\ker(-\Delta + V + E_k)$ and its range will have an orthogonal complement of finite dimension, double the dimension of $\ker(-\Delta + V + E_k)$. Hence we will be able to apply Lyapunov-Schmidt decompositon, see next section, to study bifurcation on nontrivial, nonlinear bound states for the Schrödinger equation (1.2) from $(\phi \equiv 0, E_k)$.

Note that even if $-E$ is a simple eigenvalue of $-\Delta+V$, 0 is a double eigenvalue of $D_\phi F(0,E) = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}$ where $L_+ = L_- = -\Delta + V + E$. However, the degeneracy is due to the gauge invariance, see Remark 1.1 and 3.1, and by moding out the rotations, we rigorously analyze these small nonlinear bound states, see Section 4.1 or [13, 23]. Moreover, in Section 4.2 we show that the small nonlinear bound states emerging from a simple eigenvalue of $-\Delta+V$ organize themselves in C^1 manifolds of dimension 2 $(E,\theta) \mapsto e^{i\theta}\psi_E$ which can be extended until an eigenvalue of $L_+(\psi_E, E)$ crosses zero.

3.2 Fredholm Operators and Lyapunov-Schmidt Decomposition

Definition 3.3. *Let X, Y be Banach spaces. A bounded linear operator $T : X \to Y$ is called Fredholm if:*

1. $\ker T$ *is finite dimensional and*

2. $\operatorname{ran} T$ *is closed and has a finite dimensional complement in Y, i.e., it has finite codimension.*

The index of T is defined by the number dim($\ker T$)-dim(complement of $\operatorname{ran} T$*).*

In fact, the condition that ran T is closed is redundant when its complement has finite dimension, hence it is closed:

Lemma 3.1. *Let $T : X \to Y$ be a operator so that the complement subspace for the range is closed. Then the range of T is closed.*

Now, we discuss Lyapunov-Schmidt decomposition, as explained at the end of the previous section, which is important in our local bifurcation analysis. (See for example [21].) Let X, Λ, Y be Banach spaces. Consider $f : U \to Y$ where $U \subseteq X$ is open and assume $f(x_0, \lambda_0) = 0$ for $(x_0, \lambda_0) \in U$. We are interested in the solutions of

$$f(x, \lambda) = 0 \tag{3.4}$$

near (x_0, λ_0). Assume that f is C^1 in U and $D_x f(x_0, y_0)$ is Fredholm, i.e.,

1. $\ker D_x f(x_0, y_0) = X_1$ is finite dimensional
2. $\text{ran} D_x f(x_0, y_0) = Y_1$ is closed with finite codimension.

Without loss of generality, we suppose $(x_0, \lambda_0) = (0, 0)$. We can decompose X and Y as $X = X_1 \bigoplus X_2$, i.e., for any $x \in X$ there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that $x = x_1 + x_2$, and $Y = Y_1 \bigoplus Y_2$. Let Q be the associated projection onto Y_1, which is continuous (bounded) because Y_1 is closed. Then (3.4) is equivalent to:

$$Q(f(x_1 + x_2, \lambda)) = 0 \tag{3.5}$$

$$(I - Q)(f(x_1 + x_2, \lambda)) = 0. \tag{3.6}$$

We have $D_{x_2} Q f(x_0, \lambda_0) = Q D_{x_2} f(x_0, \lambda_0) = Q D f(x_0) \mid_{X_2}$ which is an isomorphism from X_2 to Y_1.

Now, by applying Implicit Function Theorem to (3.5), there exists $\delta_1, \delta_2, \delta_3$ and a unique C^1 solution of (3.5), $x_2 : B_{\delta_1}(0) \times B_{\delta_2}(0) \to B_{\delta_3}(0)$ such that:

$$Q(f(x_1 + x_2(x_1, \lambda), \lambda)) = 0.$$

Substituting $x_2(x_1, \lambda)$ into (3.6), we finally get:

$$(I - Q)(f(x_1 + x_2(x_1, \lambda), \lambda)) = 0$$

which is a system with finitely many equations since Y_2 is finite dimensional, with finitely many unknowns since X_1 is finite dimensional.

3.3 Application to bifurcations. Morse Lemma

Consider the equation $f : X \times \Lambda \to Y$:

$$f(x, \lambda) = 0$$

where $x \in X, \lambda \in \Lambda$, X and Y are Banach spaces. We are interested in a family of solutions of the equation which depends on a parameter λ: $x(\lambda)$. We say that a bifurcation happens if the family of solution undergoes a qualitative or topological change at a certain parameter. As we have seen in the previous section, the bifurcation problem can be reduced to one finite dimensional equation as long as $D_x f$ is Fredholm.

From multivariable calculus, we know that a topological change occurs when the second derivative at a critical point is invertible. In general case we call this property non-degeneracy, i.e., the Hessian is singular at a critical point. The following theorem is Morse Lemma which can be applied to the bifurcation problems of codimension 1 at a non-degenerate critical point.

Theorem 3.4 (Morse Lemma)**.** *Consider $G : \Omega \to \mathbb{R}^n$ in $C^p(\Omega, \mathbb{R})$, $p \geq 2$, where $\Omega \subseteq \mathbb{R}^n$ is open. Suppose G satisfies $G(x_0) = 0, \nabla G(x_0) = 0$ and $D^2G(x_0)$, the Hessian matrix is invertible for some $x_0 \in \Omega$. Then, there exists change of variable near x_0, $x \mapsto \xi(x)$, such that:*

$$G(x) = \frac{1}{2}\langle D^2G(x_0)\xi(x), \xi(x)\rangle.$$

See for example [21]. Using the Morse Lemma, we can analyze the set of solutions of $G(x) = 0$ near a nondegenerate critical point x_0 by representing G as a quadratic form. When $n = 2$ and the quadratic form $\langle D^2G(x_0)\xi(x), \xi(x)\rangle$ is indefinite, then the set of solutions of $G(x) = 0$ forms two C^{p-2} distinct curves intersecting at only x_0 (transversally if $p > 2$). When $n > 2$ and $\langle D^2G(x_0)\xi(x), \xi(x)\rangle$ is indefinite, the set of solutions forms two conical hypersurfaces of dimension $n-1$ with vertex at x_0. We will use Morse Lemma in conjunction with Lyapunov-Schmidt decomposition to prove one part of one of our main results, see Theorem 5.2.

CHAPTER 4

BOUND STATES BRANCHES BIFURCATING FROM SIMPLE EIGENVALUES

In this chapter we show that the previously known small bound states bifurcating from simple eigenvalues can be uniquely continued until an eigenvalue of the linearization crosses zero. Results of this type restricted to real-valued solutions in space dimension $n = 1$ were first obtained in [13]. We generalize them to any dimension and complex-valued solutions in Section 4.2.

4.1 Small bound states bifurcating from simple eigenvalues

In this section we recall, see for example [23], the following result about local bifurcation from simple eigenvalues of the linear Schrödinger operator. We also recall that we are studying solutions of the following equation (1.2):

$$F(\phi, E) = -\Delta\phi(x) + V(x)\phi(x) + \sigma|\phi(x)|^{2p}\phi(x) + E\phi(x) = 0.$$

Proposition 4.1. *Let $-E_k$ be a simple eigenvalue of $L_0 = -\Delta + V(x)$ with L^2-normalized real-valued eigenfunction ψ_k. Then there exist $\epsilon, \delta > 0$, and real-valued function*

$$h : \{a \in \mathbb{R} : |a| < \delta\} \to H^2(\mathbb{R}^n) \cap \{\psi_k\}^{\perp real}, \quad \|h(a)\|_{H^2} = \mathcal{O}(|a|^{2p+1})$$

such that for $|E - E_k| < \epsilon$, (1.2) has a nontrivial real-valued solution $(\Psi^{real}(a), E(a))$

given by :

$$\Psi^{real}(a) = a\psi_k + h(a), \quad E(a) = E_k - \sigma|a|^{2p}\|\psi_k\|_{L^{2p+2}}^{2p+2} + \mathcal{O}(|a|^{4p}), \quad |a| < \delta, a \in \mathbb{R}.$$

Moreover, all solutions of (1.2) in the neighborhood $|E - E_k| < \epsilon, \|\Psi\|_{H^2} < \delta$ are unique up to multiplication with $e^{i\theta}$ i.e., of the form:

$$(e^{i\theta}\Psi^{real}(a), E(a)), \quad 0 \leq \theta < 2\pi,$$

or, equivalently,

$$\Psi(a) = a\psi_k + h(a)$$

where $a = \langle \psi_k, \psi(a)\rangle$, $|a| < \delta$ and $h(a)$ is extended to complex numbers via

$$h(a) = e^{i\theta}h(|a|), \quad 0 \leq \theta < 2\pi.$$

Proof. The map $F : H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R} \to L^2(\mathbb{R}^n, \mathbb{C})$ given by

$$F(\phi, E) = -\Delta\phi(x) + V(x)\phi(x) + \sigma|\phi(x)|^{2p}\phi(x) + E\phi(x)$$

is C^1 for $p > 0$ with respect to the real (not complex) Hilbert spaces L^2 and H^2. Its Fréchet derivative with respect to variable ϕ is given by (3.1), which, at $\phi \equiv 0$, becomes:

$$D_\phi F(0, E) = \begin{bmatrix} -\Delta + V + E & 0 \\ 0 & -\Delta + V + E \end{bmatrix}. \tag{4.1}$$

Since $-E_k$ is a simple eigenvalue of $L_0 = -\Delta + V$ with L^2-normalized, real-valued eigenfunction ψ_k, it follows that $D_\phi F(0, E_k)$ is a self-adjoint, Fredholm operator with

$$\ker D_\phi F(0, E_k) = \text{span}\ \{\psi_k, i\psi_k\} \quad \text{and} \quad \text{ran}\ D_\phi F(0, E_k) = [\ker D_\phi F(0, E_k)]^\perp.$$

Let

$$P_\parallel\phi = \langle\psi_k, \phi\rangle_{real}\psi_k + \langle i\psi_k, \phi\rangle_{real} i\psi_k = \langle\psi_k, \phi\rangle\psi_k, \quad P_\perp\phi = \phi - \langle\psi_k, \phi\rangle\psi_k. \tag{4.2}$$

The Lyapunov-Schmidt decomposition of (1.2) with respect to span$\{\psi_k, i\psi_k\}$ and its orthogonal complement is :

$$\phi = a\psi_k + h, \quad a = \langle \psi_k, \phi\rangle \in \mathbb{C}, \quad h = P_\perp \phi$$

$$G(E, a, h) := P_\perp F(a\psi_k + h, E) = 0 \tag{4.3}$$

$$P_\| F(a\psi_k + h, E) = 0. \tag{4.4}$$

The map $G : (0, \infty) \times \mathbb{C} \times P_\perp H^2(\mathbb{R}^n, \mathbb{C}) \to P_\perp L^2(\mathbb{R}^n, \mathbb{C})$ is C^1, and

$$G(E_k, 0, 0) = 0, \qquad D_h G(E_k, 0, 0) = L_0 + E_k.$$

The latter is an isomorphism from $P_\perp H^2$ to $P_\perp L^2$, hence, by the Implicit Function Theorem, there exists $\epsilon, \delta > 0$, and a unique C^1 map

$$h(E, a) : (E_k - \epsilon, E_k + \epsilon) \times \{a \in \mathbb{C} : |a| < \delta\} \to H^2(\mathbb{R}^n, \mathbb{C})$$

such that h solves (4.3). Moreover, (4.3) can be rewritten in the fixed point form:

$$h = -\sigma \left[P_\perp (L_0 + E) P_\perp\right]^{-1} P_\perp |a\psi_k + h|^{2p} (a\psi_k + h) \tag{4.5}$$

where the right hand side is a contraction in h, uniform in E, on a neighborhood of zero in $H^2(\mathbb{R}^n, \mathbb{C})$, provided $|E - E_k| < \epsilon,\ |a| < \delta$. Consequently we get the estimate

$$\|h(E, a)\|_{H^2} \le C|a|^{2p+1}, \tag{4.6}$$

for some constant $C > 0$ independent of both E and a, $|E - E_k| < \epsilon,\ |a| < \delta$.

Also note that our choice of ψ_k real-valued leads to the two projections, $P_\|,\ P_\perp$ having the same symmetry as F, see Remark 1.1, more precisely,

$$P_\| e^{i\theta}\phi = e^{i\theta} P_\| \phi, \quad P_\perp e^{i\theta}\phi = e^{i\theta} P_\perp \phi$$

and

$$P_\parallel \overline{\phi} = \overline{P_\parallel \phi}, \quad P_\perp \overline{\phi} = \overline{P_\perp \phi}$$

see (4.2). According to the general theory of Lyapunov-Schmidt decompositon with symmetry, see for example [24], we have:

$$\begin{aligned} h(E, e^{i\theta} a) &= e^{i\theta} h(E, a) \\ h(E, \overline{a}) &= \overline{h(E, a)} \end{aligned}$$

in particular, $h(E, a)$ is real-valued whenever $a \in \mathbb{R}$. Moreover, we can now restrict the solutions of (4.4):

$$K(E, a) := P_\parallel F(a\psi_k + h(E, a), E) = 0, \quad a \in \mathbb{C} \tag{4.7}$$

to real-valued ones i.e., $a \in \mathbb{R}$, because, for a complex valued solution $a = |a|e^{i\theta}$ we have the corresponding real solution:

$$\begin{aligned} K(E, |a|) = K(E, e^{-i\theta} a) &= P_\parallel F(e^{-i\theta} a\psi_k + e^{-i\theta} h(E, a), E) \\ &= e^{-i\theta} P_\parallel F(a\psi_k + h(E, a), E) = e^{-i\theta} K(E, a) = 0. \end{aligned}$$

For $a \neq 0$, (4.7) can be rewritten as:

$$0 = \tilde{K}(E, a) = \frac{K(E, a)}{a} = (E - E_k) + \frac{\sigma}{a} \langle \psi_k, |a\psi_k + h|^{2p} (a\psi_k + h) \rangle. \tag{4.8}$$

But, because of $p > 0$ and estimate (4.6), $\tilde{K}$ can be continuously extended at $a = 0$ by $\tilde{K}(E, 0) = E - E_k$. Now, $\tilde{K}(E, a) : (E_k - \epsilon, E_k + \epsilon) \times (-\delta, \delta) \to \mathbb{R}$, and, even though it is not C^1 for $0 < p \leq 1/2$, both $\tilde{K}$ and $\partial_E \tilde{K}$ are continuous at $(E_k, 0)$, see estimate (4.6). Moreover,

$$\tilde{K}(E_k, 0) = 0, \qquad \partial_E \tilde{K}(E_k, 0) = 1,$$

hence, by the Implicit Function Theorem, there exist δ_1 and the unique solution of

(4.8) $E(a)$ defined in $-\delta_1 < a < \delta_1$. In addition

$$|E(a) - E_k + \sigma|a|^{2p}\|\psi_0\|_{L^{2p+2}}^{2p+2}| = \mathcal{O}(|a|^{4p}).$$

Now, we can define a real-valued function $h(a)$ for $-\delta_1 < a < \delta_1$

$$h(a) = h(E(a), a).$$

Moreover, by (4.6)

$$\|h(a)\|_{H^2} = \mathcal{O}(|a|^{2p+1}).$$

In conclusion, $(e^{i\theta}\Psi^{real}(a), E(a))$, $0 \leq \theta < 2\pi$ are the only solutions of (1.2) in a small neighborhood of $(\phi = 0, E = E_k)$, where $(\Psi^{real}(a), E(a)) = (a\psi_k + h(a), E(a))$ which are real-valued.

□

Remark 4.1. For $a \neq 0$ ($\Leftrightarrow E \neq E_k$), the map $a \mapsto E(a)$ becomes C^1 and invertible, so that $\Psi(a)$ can be rewritten as $E \mapsto \Psi_E \in C^1$. Furthermore

$$|\Psi_E|^{2p} = -\frac{E - E_k}{\sigma\|\psi_k\|_{L^{2p+2}}^{2p+2}} + o(E - E_k)$$

and

$$\lim_{E\to E_k} \frac{d|\Psi_E|^{2p}}{dE} = -\frac{|\psi_k|^{2p}}{\sigma\|\psi_k\|_{L^{2p+2}}^{2p+2}}.$$

Remark 4.2. From Remark 3.1 we already know that $i\Psi_E \in \ker D_\phi F(\Psi_E, E)$ when (Ψ_E, E) are the solutions bifurcating from $(0, E_k)$ given by the previous Proposition. Hence 0 is an eigenvalue of $D_\phi F(\Psi_E, E)$, but, more importantly, it is a simple eigenvalue! Indeed, if Ψ_E is not real valued then, according to the Proposition 4.1, there is $\theta \in [0, 2\pi)$ such that $\Psi_E = e^{i\theta}\Psi_E^{real}$ where Ψ_E^{real} is a real valued solution. Consequently,

$$D_\phi F(\Psi_E, E) = e^{i\theta} D_\phi F(\Psi_E^{real}, E)e^{-i\theta},$$

which shows that their spectrum is the same, and

$$D_\phi F(\Psi_E^{real}, E) = \begin{bmatrix} L_+(\Psi_E^{real}, E) & 0 \\ 0 & L_-(\Psi_E^{real}, E) \end{bmatrix}.$$

Now, due to the continuous dependence of the spectrum of $L_\pm(\Psi_E^{real}, E)$ on E near E_k, we have that 0 remains a simple eigenvalue of L_-, but evolves into a strictly negative eigenvalue of L_+ if $\sigma < 0$ (strictly positive if $\sigma > 0$), because of comparison principle for self adjoint operators and $L_+ - L_- = \sigma 2p|\Psi_E^{real}|^{2p} < 0$ (> 0 if $\sigma > 0$.)

In the particular case of double-well potential $V = V_s$, see Section 2.3, we will use the following stationary equation:

$$F_s(\phi, E) = -\Delta\phi(x) + V_s(x)\phi(x) + \sigma|\phi(x)|^{2p}\phi(x) + E\phi(x) = 0. \tag{4.9}$$

Recall that we assumed that $-\Delta + V_0$ has the lowest simple eigenvalue $-w_0 < 0$, and consequently, there exist two simple eigenvalues $-E_{0,s}$ and $-E_{1,s}$ of $-\Delta + V_s$ and one can choose $\psi_{0,s}$, respectively $\psi_{1,s}$, which is the real-valued L^2-normalized eigenfunction corresponding $E_{0,s}$, respectively $E_{1,s}$, see Proposition 2.4.

Proposition 4.2. *For the double well potential $V = V_s$, there exist $s_0 > 0$ such that for any $s \geq s_0$, there exist $\epsilon, \delta, r > 0$ (independent of s) such that any solution of (4.9), (Ψ_E, E), satisfying $|E - w_0| \leq \epsilon$ and $\|\Psi_E\|_{H^2} \leq r$ can be written as:*

$$\Psi_E = a\psi_{0,s} + b\psi_{1,s} + \tilde{h}(a, b, E), \quad a, b \in \mathbb{C},\ |a| \leq \delta,\ |b| \leq \delta,\ |E - w| \leq \epsilon$$

where $h : \{a, b \in \mathbb{C} : |a| \leq \delta, |b| \leq \delta\} \times \{E \in \mathbb{R} : |E - w_0| \leq \epsilon\} \to H^2 \cap \{\psi_{0,s}, \psi_{1,s}\}^\perp$ is a function which is real C^1 and the unique solution of

$$P_\perp F_s(a\psi_{0,s} + b\psi_{1,s} + \tilde{h}, E) = 0$$

in the neighborhood $|a| \leq \delta, |b| \leq \delta$ and $|E - w_0| \leq \epsilon$, where

$$P_\perp \phi = \phi - \langle \psi_{0,s}, \phi \rangle \psi_{0,s} - \langle \psi_{1,s}, \phi \rangle \psi_{1,s}.$$

Moreover, $\tilde{h}$ satisfies:

$$\tilde{h}(e^{i\theta}a, e^{i\theta}b, E) = e^{i\theta}\tilde{h}(a, b, E), \quad \theta \in [0, 2\pi),$$

$$\tilde{h}(\overline{a}, \overline{b}, E) = \overline{\tilde{h}(a, b, E)},$$

hence, for $a, b \in \mathbb{R}$, $\tilde{h}(a, b, E)$ is real-valued. In particular, the ground state ψ_E and the first excited state $\psi_1(E)$ (bifurcating at the two lowest eigenvalues $E_{0,s}$ and $E_{1,s}$ of $-\Delta + V_s$ via Proposition 4.1) are of the form:

$$\psi_E = ae^{i\theta}\psi_{0,s} + e^{i\theta}\tilde{h}(a, 0, E), \quad a \in \mathbb{R},\ |a| \leq \delta,\ \theta \in [0, 2\pi)$$

$$\psi_1(E) = be^{i\theta}\psi_{1,s} + e^{i\theta}\tilde{h}(0, b, E), \quad b \in \mathbb{R},\ |b| \leq \delta,\ \theta \in [0, 2\pi)$$

and the ground state is even in x_1 while the first excited state is odd in x_1.

Proof. From Proposition 2.4, we have

$$\lim_{s\to\infty} |E_{k,s} - w_0| = 0, \quad k = 0, 1$$

and they are separated from the rest of the spectrum i.e., there exist $l_0, d_* > 0$ such that for $s \geq l_0$ and $k = 0, 1$ we have:

$$|E_{i,s} - \tau| \geq d_*, \qquad \text{for any } \tau \in \Sigma_s \setminus \{E_{0,s}, E_{1,s}\},$$

where Σ_s is a spectrum of $-\Delta + V_s$. We first define projections onto $\psi_{0,s}$ and $\psi_{1,s}$:

$$P_{\|_0}\phi = \langle\psi_{0,s}, \phi\rangle\psi_{0,s}, \quad P_{\|_1}\phi = \langle\psi_{1,s}, \phi\rangle\psi_{1,s}$$

which implies $P_\perp\phi$ is their orthogonal complement of ϕ. Consider any solution of (4.9),(Ψ_E, E). By applying above projections we can decompose the solution as follows:

$$\Psi_E^{real} = a\psi_{0,s} + b\psi_{1,s} + \tilde{h}$$

where

$$a = \langle \psi_{0,s}, \Psi_E \rangle, \quad b = \langle \psi_{1,s}, \Psi_E \rangle, \quad \tilde{h} = P_\perp \Psi_E.$$

Also, $F_s(\Psi_E, E) = 0$ implies

$$P_\perp F_s(\Psi_E, E) = (-\Delta + V_s + E) P_\perp \Psi_E + \sigma P_\perp (|\Psi_E|^{2p} \Psi_E) = 0. \tag{4.10}$$

Then due to Lemma 2.1, there exist $s_0 > 0$, $\epsilon > 0$ and $M > 0$ such that for $s \geq s_0$, we have $\{E : |E - w_0| \leq \epsilon\} \subset \{E > 0 : dist(E, \Sigma_s \setminus \{E_{0,s}, E_{1,s}\}) \geq \frac{d_*}{2}\}$ and $\|(-\Delta + V_s + E)^{-1} P_\perp\|_{L^2 \to H^2} \leq M$ for $E \in \{E > 0 : dist(E, \Sigma_s \setminus \{E_{0,s}, E_{1,s}\}) \geq \frac{d_*}{2}\}$. Hence, (4.10) can be rewritten as:

$$\tilde{h} + \sigma(-\Delta + V_s + E)^{-1} P_\perp (|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}|^{2p} (a\psi_{0,s} + b\psi_{1,s} + \tilde{h})) = 0 \tag{4.11}$$

for $E \in \{E : |E - w_0| \leq \epsilon\}$. Hence, for $s \geq s_0$, we can define the map $G_s : \mathbb{C}^2 \times \{E : |E - w_0| \leq \epsilon\} \times H^2 \cap \{\psi_{0,s}, \psi_{1,s}\}^\perp$:

$$G_s(a, b, E, \tilde{h}) = \tilde{h} + \sigma(-\Delta + V_s + E)^{-1} P_\perp (|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}|^{2p} (a\psi_{0,s} + b\psi_{1,s} + \tilde{h}))$$

which is real C^1 for $p > 0$. Since

$$G_s(0, 0, E, 0) = 0, \quad D_{\tilde{h}} G_s(0, 0, E, 0) = \mathbb{I},$$

a direct application of the Implicit Function Theorem will give $\delta(s), r(s) > 0$ such that for $|a| \leq \delta$, $|b| \leq \delta$, $\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_H^2 < r$, (4.11) has a unique solution:

$$\tilde{h} = \tilde{h}(a, b, E) \in H^2 \cap \{\psi_{0,s}, \psi_{1,s}\}^\perp$$

which is real C^1, and we have:

$$\tilde{h}(e^{i\theta} a, e^{i\theta} b, E) = e^{i\theta} \tilde{h}(a, b, E), \quad \theta \in [0, 2\pi) \tag{4.12}$$

$$\tilde{h}(\overline{a}, \overline{b}, E) = \overline{\tilde{h}(a, b, E)} \tag{4.13}$$

by the general theory of Lyapunov-Schmidt decompositon with symmetry. In fact, δ and r are independent of s because of the contraction argument used to prove the Implicit Function Theorem. To show this, we rewrite (4.11) in the form:

$$\begin{aligned} \tilde{h} &= -\sigma(-\Delta + V_s + E)^{-1} P_\perp(|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}|^{2p}(a\psi_{0,s} + b\psi_{1,s} + \tilde{h})) \\ &:= K_s(a, b, E, \tilde{h}) \end{aligned}$$

We want to find δ, r independent of s such that

$$\|D_{\tilde{h}} K_s(a, b, E, \tilde{h})\|_{H^2 \to H^2} \leq \frac{1}{2} \tag{4.14}$$

for $|a| \leq \delta, b \leq \delta$, $\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_H^2 < r$. By Hölder's inequality, for any direction $v \in H^2$, $\|v\|_{H^2} = 1$, we have

$$\begin{aligned} &\|D_{\tilde{h}}(|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}|^{2p}(a\psi_{0,s} + b\psi_{1,s} + \tilde{h}))[v]\|_{L^2} \\ &= \|(p+1)|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}|^{2p} v + p|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}|^{2p-2}(a\psi_{0,s} + b\psi_{1,s} + \tilde{h})^2 \overline{v}\|_{L^2} \\ &\leq (2p+1)\||a\psi_{0,s} + b\psi_{1,s} + \tilde{h}|^{2p}\|_{L^q}\|v\|_{L^{q'}} \\ &\leq (2p+1)\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_{L^{4p+2}}^{2p}\|v\|_{L^{4p+2}} \\ &\leq (2p+1)C_{4p+2}\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_{L^{4p+2}}^{2p}\|v\|_{H^2} \\ &\leq (2p+1)C_{4p+2}^2\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_{H^2}^{2p} \end{aligned} \tag{4.15}$$

where $q = 2 + \frac{1}{p}$, $q' = 4p+2$. Then, combined with Lemma 2.1, (4.14) holds for $\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_H^2 < r$ with $r = \sqrt[2p]{\frac{1}{2(2p+1)|\sigma|MC_{4p+2}^2}}$ which is independent of s. Also, we need to choose $\delta, \tilde{r}$, independent of s, such that $|a| \leq \delta, |b| \leq \delta$ and $\|\tilde{h}\|_{H^2} \leq \tilde{r}$ implies

$$\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_{H^2} < r \quad \text{and} \quad \|a\psi_{0,s} + b\psi_{1,s}\|_{H^2} \leq \left(\frac{\tilde{r}}{2|\sigma|MC_{4p+2}^{2p+1}}\right)^{\frac{1}{2p+1}}.$$

If so, for $|a| \le \delta, |b| \le \delta$ and $\|\tilde{h}\|_{H^2} \le \tilde{r}$, we get

$$\begin{aligned}
&\|K_s(a,b,E,\tilde{h}) - K_s(0,0,E,0)\|_{H^2} \\
\le\ & \|K_s(a,b,E,\tilde{h}) - K_s(a,b,E,0)\|_{H^2} + \|K_s(a,b,E,0)\|_{H^2} \\
\le\ & \|D_{\tilde{h}} K_s(a,b,E,\tilde{h})\|_{H^2 \to H^2} \|\tilde{h}\|_{H^2} \\
& + \|\sigma(-\Delta + V_s + E)^{-1} P_\perp(|a\psi_{0,s} + b\psi_{1,s}|^{2p}(a\psi_{0,s} + b\psi_{1,s}))\|_{H^2} \\
\le\ & \frac{1}{2}\|\tilde{h}\|_{H^2} + |\sigma| M \|(|a\psi_{0,s} + b\psi_{1,s}|^{2p}(a\psi_{0,s} + b\psi_{1,s}))\|_{L^2} \\
\le\ & \frac{1}{2}\|\tilde{h}\|_{H^2} + |\sigma| M \||a\psi_{0,s} + b\psi_{1,s}\|_{L^{4p+2}}^{2p+1} \\
\le\ & \frac{1}{2}\|\tilde{h}\|_{H^2} + |\sigma| M C_{4p+2}^{2p+1} \|a\psi_{0,s} + b\psi_{1,s}\|_{H^2}^{2p+1} \\
\le\ & \frac{1}{2}\tilde{r} + \frac{1}{2}\tilde{r} \le \tilde{r}
\end{aligned}$$

so that the contraction argument holds independently of s. From Proposition 2.4, there exists $\tilde{l_0}$ such that for $s \ge \tilde{l_0}$, we have

$$\|\psi_{i,s}\|_{H^2} \le \left\| \frac{T_s\psi_0 + (-1)^i R T_s \psi_0}{\sqrt{2}} \right\|_{H^2} + 1 \le \frac{2}{\sqrt{2}}\|\psi_0\|_{H^2} + 1, \quad i = 0, 1$$

where ψ_0 is the L^2-normalized eigenfunction of $-\Delta + V_0$ corresponding $-w_0$ and we used the triangle inequality for the H^2 norm and the fact that translation and reflection are isometries in Sobolev spaces. By choosing $\tilde{r} < \frac{1}{2}r$ and
$\delta < \frac{1}{2(\sqrt{2}\|\psi_0\|_{H^2}+1)} \min\left(\frac{1}{2}r, \left(\frac{\tilde{r}}{2|\sigma|MC_{4p+2}^{2p+1}}\right)^{\frac{1}{2p+1}}\right)$, for $s \ge \tilde{l_0}$, we get

$$\|a\psi_{0,s} + b\psi_{1,s} + \tilde{h}\|_{H^2} \le (|a| + |b|)(\sqrt{2}\|\psi_0\|_{H^2} + 1) + \|\tilde{h}\|_{H^2} < \frac{1}{2}r + \frac{1}{2}r = r$$

and

$$\|a\psi_{0,s} + b\psi_{1,s}\|_{H^2} \le (|a| + |b|)(\sqrt{2}\|\psi_0\|_{H^2} + 1) \le \left(\frac{\tilde{r}}{2|\sigma|MC_{4p+2}^{2p+1}.}\right)^{\frac{1}{2p+1}}.$$

The first part of the theorem is proven.

Now, consider the two bound state branches given by Proposition 4.1, the ground state ψ_E (the solutions bifurcating at $E_{0,s}$) and the first excited state $\psi_1(E)$ (the solutions bifurcating at $E_{1,s}$). By the invariance of equation (4.9) under the reflection operator R,

$$F(R\phi, E) = RF(\phi, E).$$

Hence $(R\phi, E)$ is a solution whenever (ϕ, E) is. From Proposition 4.1, we have

$$\psi_E = a\psi_{0,s} + h(a)$$

with $a = \langle \psi_{0,s}, \psi_E \rangle$. Now $R\psi_E$ is also a solution of (4.9) with the representation

$$R\psi_E = a'\psi_{0,s} + h(a')$$

and since

$$a' = \langle \psi_{0,s}, R\psi_E \rangle = \langle R\psi_{0,s}, R\psi_E \rangle = \langle \psi_{0,s}, \psi_E \rangle = a$$

where we used $R\psi_{0,s} = \psi_{0,s}$, we get

$$R\psi_E = a\psi_{0,s} + h(a) = \psi_E$$

i.e., the ground state branch ψ_E is even in x_1. A similar argument, using $R\psi_{1,s} = -\psi_{1,s}$ leads to the first excited branch $\psi_1(E)$ being odd in x_1. Consequently, the ground state is always orthogonal on $\psi_{1,s}$ while the first excited state is orthogonal on $\psi_{0,s}$. Hence, combined with (4.12) and (4.13), we get

$$\psi_E = ae^{i\theta}\psi_{0,s} + e^{i\theta}\tilde{h}(a, 0, E), \quad a \in \mathbb{R},\ |a| \leq \delta,\ \theta \in [0, 2\pi)$$

$$\psi_1(E) = be^{i\theta}\psi_{1,s} + e^{i\theta}\tilde{h}(0, b, E), \quad b \in \mathbb{R},\ |b| \leq \delta,\ \theta \in [0, 2\pi).$$

□

4.2 Continuation of bound state branches

In this section we show that the branches given by Proposition 4.1 can be uniquely continued until an eigenvalue of the linearization crosses zero. We start with a local continuation result. From Remark 4.2 we already know that 0 is a simple eigenvalue of $D_\phi F(\psi_E, E)$ when (ψ_E, E) are the solutions of (1.2) constructed in the previous section. We now show that in the absence of other degeneracies i.e., if the 0 eigenvalue of the linearized operator remains simple, the C^1 manifold of solutions constructed in Proposition 4.1 can be extended.

Proposition 4.3. *Let $(\psi_{\tilde{E}}, \tilde{E}) = (e^{i\tilde{\theta}}\psi_{\tilde{E}}^{real}, \tilde{E})$ for some $\tilde{\theta} \in [0, 2\pi)$ be a solution of (1.2) obtained by rotating a real-valued solution $\psi_{\tilde{E}}^{real}$. Assume that $\ker D_\phi F(\psi_{\tilde{E}}, \tilde{E})$ is one dimensional. Then there exist $\epsilon, \delta > 0$ and a map $v : (\tilde{E}-\epsilon, \tilde{E}+\epsilon) \to H^2(\mathbb{R}^n, \mathbb{R})$ such that all solutions (ϕ, E) of (1.2) satisfying*

$$\inf_{\theta\in[0,2\pi)} \|\phi - e^{i\theta}\psi_{\tilde{E}}\|_{H^2} < \delta, \quad |E - \tilde{E}| < \epsilon$$

are of the form $(e^{i\theta}v(E), E)$ where $0 \le \theta < 2\pi$.

Proof. From Remark 3.1 and the hypothesis on the kernel, we get that

$$\ker D_\phi F(\psi_{\tilde{E}}, \tilde{E}) = \text{ span } \{i\psi_{\tilde{E}}\}.$$

Now from:

$$D_\phi F(\psi_{\tilde{E}}, \tilde{E}) = e^{i\tilde{\theta}} D_\phi F(\psi_{\tilde{E}}^{real}, \tilde{E}) e^{-i\tilde{\theta}},$$

see Remark 4.2, we get

$$\ker D_\phi F(\psi_{\tilde{E}}^{real}, \tilde{E}) = \text{ span } \{i\psi_{\tilde{E}}^{real}\}.$$

We first consider all real-valued solutions of (1.2) near $(\psi_{\tilde{E}}^{real}, \tilde{E})$. Since F transforms real functions into real functions, we can define $\tilde{F}(\phi, E) : H^2(\mathbb{R}^n, \mathbb{R}) \to L^2(\mathbb{R}^n, \mathbb{R})$, which is the restriction of the map F to real-valued functions. Furthermore, $D_\phi \tilde{F}(\psi_{\tilde{E}}^{real}, \tilde{E})$ is an isomorphism because all real-valued functions are perpendicular to $i\psi_{\tilde{E}}^{real}$ with respect to the real scalar product. Thus, by applying Implicit Function Theorem, we get the unique C^1 map $v(E)$ solving $\tilde{F}(\psi_{\tilde{E}}^{real} + v(E), E) = F(\psi_{\tilde{E}}^{real} + v(E), E) = 0$:

$$v(E) : (\tilde{E} - \epsilon_1, \tilde{E} + \epsilon_1) \to B_{\delta_1}^{real}(\psi_{\tilde{E}}^{real}) \subset H^2(\mathbb{R}^n, \mathbb{R}) \tag{4.16}$$

where $B_{\delta_1}^{real}(\psi_{\tilde{E}}^{real}) = \{\phi \in H^2(\mathbb{R}^n, \mathbb{R}) : \|\phi - \psi_{\tilde{E}}^{real}\|_{H^2} < \delta_1\}$ for some $\epsilon_1, \delta_1 > 0$.

Next consider solutions (ϕ, E) of (1.2) near $(\psi_{\tilde{E}}^{real}, \tilde{E})$ for which $\phi - \psi_{\tilde{E}}^{real}$ are perpendicular to $i\psi_{\tilde{E}}^{real}$ with respect to the real scalar product in H^2. By applying Implicit Function Theorem at $\psi_{\tilde{E}}^{real}$ to $P_\perp F(\phi, E) = 0$ where $P_\perp$ is the L^2 projection onto $\{i\psi_{\tilde{E}}^{real}\}^{\perp real}$, and the orthogonality here is with respect to the L^2 real scalar product and we restrict F to H^2 functions $\psi_{\tilde{E}}^{real} + f,\ f \perp i\psi_{\tilde{E}}^{real}$ with respect to the H^2 real scalar product, there exist $\epsilon_2, \delta_2 > 0$ and the unique C^1 map $\tilde{v}(E)$ solving $P_\perp F(\psi_{\tilde{E}}^{real} + \tilde{v}(E), E) = 0$:

$$\tilde{v}(E) : (\tilde{E} - \epsilon_2, \tilde{E} + \epsilon_2) \to B_{\delta_2}(\psi_{\tilde{E}}^{real}) \subset H^2(\mathbb{R}^n, \mathbb{C})$$

where $B_{\delta_2}(\psi_{\tilde{E}}^{real}) = \{\phi \in H^2(\mathbb{R}^n, \mathbb{C}) : \langle \phi, i\psi_{\tilde{E}}^{real}\rangle_{H^2\ real} = 0 \text{ and } \|\phi - \psi_{\tilde{E}}\|_{H^2} < \delta_2\}$.

Let $\min\{\epsilon_1, \epsilon_2\}) = \epsilon$, $\min\{\delta_1, \delta_2\} = \delta$. Since real valued functions are automatically orthogonal to $i\psi_{\tilde{E}}^{real}$ with respect to the real scalar product (in H^2) and both v and $\tilde{v}$ maps solve $P_\perp F(\psi_{\tilde{E}}^{real} + f, E) = 0$ in the H^2 ball of radius δ centered at $\psi_{\tilde{E}}^{real}$ restricted to this orthogonal subspace, by uniqueness we deduce that they coincide at least on the interval $(\tilde{E} - \epsilon, \tilde{E} + \epsilon)$.

Now let (ϕ, E) be a solution (1.2) which satisfies:

$$\inf_{\theta \in [0, 2\pi)} \|\phi - e^{i\theta}\psi_{\tilde{E}}\|_{H^2} < \delta, \quad |E - \tilde{E}| < \epsilon$$

Then there exists $\theta_* \in [0, 2\pi)$ such that $\phi - e^{i\theta_*}\psi_{\tilde{E}}^{real} \perp ie^{i\theta_*}\psi_{\tilde{E}}^{real}$ i.e., θ_* satisfies

$$\|\phi - e^{i\theta_*}\psi_{\tilde{E}}^{real}\|_{H^2} = \min_{\theta \in [0, 2\pi)} \|\phi - e^{i\theta}\psi_{\tilde{E}}\|_{H^2}.$$

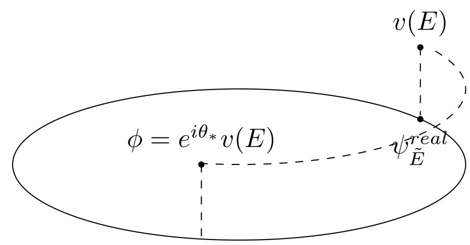

Figure 4.1

It implies that $e^{-i\theta_*}\phi$ is a solution for which $e^{-i\theta_*}\phi - \psi_{\tilde{E}}^{real}$ is perpendicular to $i\psi_{\tilde{E}}^{real}$, $e^{-i\theta_*}\phi \in B_\delta(\psi_{\tilde{E}})$. Hence, by the above argument, $(e^{-i\theta_*}\phi, E) = (v(E), E)$. $\square$

Remark 4.3. From the proof of the above Proposition, see (4.16), it is clear that the solutions in a neighborhood of $(e^{i\theta}\psi_{\tilde{E}}, \tilde{E})$ are still obtained by rotating a curve of real-valued solutions $v(E)$.

Now, by combining the two propositions with standard comparison principles for the spectrum of linear, self-adjoint operators, see Remark 4.2, we show that the branch constructed in Proposition 4.1 can be uniquely extend in parameter E, see Remark 4.1, until an eigenvalue of L_+ or L_- crosses zero or the branch reaches the boundary of the Fredholm domain i.e., $E = 0$, $E = \infty$. For clarity we will separate the attractive nonlinearity case, $\sigma < 0$, from the repelling nonlinearity case, $\sigma > 0$. The first result extends Theorem 2 in [13] to any dimensions and more general potentials.

Theorem 4.1. *Assume $\sigma < 0$, and that the hypotheses of* Proposition 4.1 *hold. Then, the (E,θ) parameterized, two dimensional manifold $(e^{i\theta}\psi_E^{real}, E)$, $0 \le \theta < 2\pi$, of solutions of* (1.2) *bifurcating from $(0, E_k)$ can be uniquely continued on a maximal interval $E \in I = (E_k, E_*)$ where:*

(i) $E_* = \infty$, *or*

(ii) $E_k < E_* < \infty$ *and there exist a sequence $\{E_n\}_{n\in\mathbb{N}} \subset I$ such that* $\lim_{n\to\infty} E_n = E_*$ *and a corresponding sequence of non-zero eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of* $L_+(\psi_{E_n}^{real}, E_n)$, *or* $L_-(\psi_{E_n}^{real}, E_n)$ *such that* $\lim_{n\to\infty}\lambda_n = 0$.

For the repelling case we have:

Theorem 4.2. *Assume $\sigma > 0$, and that the hypotheses of* Proposition 4.1 *hold. Then, the (E,θ) parameterized, two dimensional manifold $(e^{i\theta}\psi_E^{real}, E)$, $0 \le \theta < 2\pi$, of solutions of* (1.2) *bifurcating from* $(0, E_k)$ *can be uniquely continued on a maximal interval $E \in I = (E_*, E_k)$ where:*

(i) $E_* = 0$, *or*

(ii) $0 < E_* < E_k$ *and there exist a sequence* $\{E_n\}_{n\in\mathbb{N}} \subset I$ *such that* $\lim_{n\to\infty} E_n = E_*$ *and a corresponding sequence of non-zero eigenvalues* $\{\lambda_n\}_{n\in\mathbb{N}}$ *of* $L_+(\psi_{E_n}^{real}, E_n)$, *or* $L_-(\psi_{E_n}^{real}, E_n)$ *such that* $\lim_{n\to\infty} \lambda_n = 0$.

Proof. We focus first on the attractive, $\sigma < 0$ case. Let

$$E_* = \sup\{E : E > E_k, \text{ and } E \mapsto \psi_E^{real} \text{ is an extension of the map in Remark 4.1 for which 0 is not in the spectrum of } L_+ \text{ and is a simple eigenvalue of } L_-\}$$

Remark 4.2 guarantees that the set above is non-empty. Also note that $E > E_k$ in the $\sigma < 0$ case, see Proposition 4.1.

By contradiction, suppose neither of (i)-(ii) in Theorem 4.1 hold for E_*. Consequently, there exist $\tilde{E}$ and $d > 0$ such that

$$\| \left[L_+(\psi_E^{real}, E)\right]^{-1} \|_{L^2\to L^2} \le 1/d, \qquad \forall E \in \tilde{I} = [\tilde{E}, E_*). \tag{4.17}$$

This follows from the L^2 spectrum of L_+ being away from zero. Indeed, the continuous spectrum of $L_+(\psi_E^{real}, E)$ is the interval $[E, \infty)$, since both $V(x)$ and $|\psi_E|^{2p}(x)$ are relatively compact perturbations of the Laplacian, the former due to assumption (H1), while the latter is due to its exponential decay as $|x| \to \infty$, see for example [25]. Hence, the continuous spectrum is at least at distance $\tilde{E} > E_k > 0$, from 0. Moreover, the eigenvalues of L_+ depend continuously on E along the C^1 curve $E \mapsto \psi_E^{real}$ which combined with the fact that (ii) does not hold prevents the discrete spectrum from approaching zero.

By differentiating (1.2) with respect to E, we get:

$$-\left[L_+(\psi_E^{real}, E)\right]^{-1}\psi_E^{real} = \partial_E \psi_E^{real}. \tag{4.18}$$

Thus, combining (4.18) with (4.17) and with:

$$\partial_E \|\psi_E^{real}\|_{L^2}^2 = 2\langle \partial_E \psi_E^{real}, \psi_E^{real}\rangle_{L^2} \le 2\|\partial_E \psi_E^{real}\|_{L^2}\|\psi_E^{real}\|_{L^2} \le \frac{2}{d}\|\psi_E^{real}\|_{L^2}^2,$$

we get, on $E \in \tilde{I} = [\tilde{E}, E_*)$,

$$\|\psi_E^{real}\|_{L^2}^2 \le \|\psi_{\tilde{E}}^{real}\|_{L^2}^2 e^{\frac{2}{d}(E_*-\tilde{E})} = M_2^2 < \infty \tag{4.19}$$

i.e., due to E_* is finite (from negating (i)) we have that $\|\psi_E^{real}\|_{L^2}$ is uniformly bounded on $\tilde{I}$.

Next we obtain a uniform bound for $\|\psi_E^{real}\|_{L^{2p+2}}^{2p+2}$, $E \in \tilde{I}$. Consider the energy functional for $\mathcal{E} : H^1(\mathbb{R}^n, \mathbb{C}) \to \mathbb{R}$:

$$\mathcal{E}(\phi) = \int_{\mathbb{R}^n} |\nabla\phi(x)|^2 dx + \int_{\mathbb{R}^n} V(x)|\phi(x)|^2 dx + \frac{\sigma}{p+1}\int_{\mathbb{R}^n} |\phi(x)|^{2p+2} dx. \tag{4.20}$$

Note that ψ_E^{real} is a weak solution of (1.2) if and only if

$$D\mathcal{E}(\psi_E^{real})[v] = -2E\langle \psi_E^{real}, v\rangle$$

for all $v \in H^1$. If we now look at the composition, we have:

$$\frac{d\mathcal{E}}{dE} = D\mathcal{E}(\psi_E^{real})[\partial_E \psi_E^{real}] = -2E\langle \psi_E^{real}, \partial_E \psi_E^{real}\rangle$$

and by Cauchy-Schwarz inequality,

$$\left|\frac{d\mathcal{E}}{dE}\right| \le 2E\|\psi_E^{real}\|_{L^2}\|\partial_E \psi_E^{real}\|_{L^2}.$$

Thus, from (4.18) and (4.19) the derivative of the energy functional is uniformly bounded on $\tilde{I}$, which also implies $\mathcal{E}(E)$ is uniformly bounded on $\tilde{I}$:

$$\mathcal{E}(E) \leq M_{\mathcal{E}} \text{ for } E \in \tilde{I}.$$

Now, by using the weak formulation of solutions of (1.2), we get

$$\|\nabla\psi_E^{real}\|_{L^2}^2 + \int_{\mathbb{R}^n} V(x)|\psi_E^{real}(x)|^2 dx + \sigma\|\psi_E^{real}\|_{L^{2p+2}}^{2p+2} + E\|\psi_E^{real}\|_{L^2}^2 = 0. \tag{4.21}$$

Subtracting (4.21) from (4.20), we get:

$$\sigma\left(\frac{1}{p+1} - 1\right)\|\psi_E^{real}\|_{L^{2p+2}}^{2p+2} - E\|\psi_E^{real}\|_{L^2}^2 \leq M_{\mathcal{E}}$$

which is equivalent to

$$\|\psi_E^{real}\|_{L^{2p+2}}^{2p+2} \leq \frac{p+1}{|\sigma|p}(M_{\mathcal{E}} + E\|\psi_E^{real}\|_{L^2}^2) \leq \frac{p+1}{|\sigma|p}(M_{\mathcal{E}} + E_* M_2^2) = M_{2p+2}^{2p+2} \text{ for } E \in \tilde{I}. \tag{4.22}$$

i.e., we get a uniform bound for L^{2p+2} norm of ψ_E^{real} on $\tilde{I}$.

Finally, we use a standard regularity argument, see for example [26, Theorem 8.1.1], to obtain H^2 uniform bounds for $\psi_E^{real},\ E \in \tilde{I}$ from the uniform bounds in L^2 and L^{2p+2}. We start by rewriting ψ_E^{real} as:

$$\psi_E^{real} = (-\Delta + V + i)^{-1}[(i - E_k)\psi_E^{real} + \sigma(2p+1)|\psi_E^{real}|^{2p}\psi_E^{real}] \tag{4.23}$$

It is known that for any $1 < q < \infty$ there exists $l_q > 0$ such that:

$$\|(-\Delta + V + i)^{-1}\|_{L^q \to W^{2,q}} \leq l_q \tag{4.24}$$

because

$$(-\Delta + V + i)^{-1} = (-\Delta + i)^{-1}(\mathbb{I} + V(-\Delta + i)^{-1})^{-1}$$

and both $(-\Delta + i)^{-1} : L^q \to W^{2,q}$ and $(\mathbb{I} + V(-\Delta + i)^{-1})^{-1} : L^q \to L^q$ are bounded. See Section 2.1 and 2.2.

Define the sequence q_j by

$$q_0 = 2p+2, \quad \frac{1}{q_j} = \frac{2p+1}{q^{j-1}} - \frac{2}{n} = (2p+1)^j \left(\frac{1}{2p+2} - \frac{1}{np} + \frac{1}{np(2p+1)^j} \right) \text{ for } j \geq 1$$

where n is the space dimension. Since $p < \frac{2}{n-2}$, ($p < \infty$ when $n = 1, 2$), we have:

$$\frac{1}{q^{j+1}} - \frac{1}{q^j} = \frac{2p+1}{q^j} - \frac{2}{n} - \frac{1}{q^j} = (2p+1)^j \left(\frac{2p}{2p+2} - \frac{2}{n} \right) < 0 \tag{4.25}$$

i.e., $\frac{1}{q_j}$ is decreasing and not Cauchy hence:

$$\lim_{j \to \infty} \frac{1}{q^j} = -\infty. \tag{4.26}$$

From (4.22), since $q_0 = 2p + 2$, we have $\|\psi_E^{real}\|_{L^{q_0}} \leq M_{q_0}$, for all $E \in \tilde{I}$. It implies that $\||\psi_E^{real}|^{2p}|\psi_E^{real}\|_{L^{\frac{q_0}{2p+1}}} \leq (M_{q_0})^{2p+1}$, and by combining with (4.24),

$$\|(-\Delta + V + i)^{-1}|\psi_E^{real}|^{2p}|\psi_E^{real}\|_{W^{2,\frac{q_0}{2p+1}}} \leq l_{\frac{q_0}{2p+1}}(M_{q_0})^{2p+1}.$$

Meanwhile, we have:

$$\|(-\Delta + V + i)^{-1}(i - E)\psi_E^{real}\|_{W^{2,2}} \leq l_2 \sqrt{E_*^2 + 1} M_2.$$

Note that

$$W^{2,\frac{q_0}{2p+1}} \text{ is embedded in } L^q \text{ for all } q \geq \frac{q_0}{2p+1} \text{ such that } \frac{1}{q} \geq \frac{2p+1}{q_0} - \frac{2}{n} = \frac{1}{q_1}.$$

For clarity, we split the analysis in two cases depending on the space dimension, $n \leq 4$ and $n > 4$. In the first case, $n \leq 4$, we know that

$$W^{2,2} \text{ is embedded in } L^q \text{ for all } q \geq 2.$$

Since both $W^{2,\frac{q_0}{2p+1}}$ and $W^{2,2}$ are embedded in L^{q_1}, by taking C_{q_1} to be the maximum of the two embedding constants, we get:

$$\|\psi_E^{real}\|_{L^{q_1}} \leq C_{q_1}(l_{\frac{q_0}{2p+1}}(M_{q_0})^{2p+1} + l_2\sqrt{E_*^2+1}M_2) = M_{q_1} \text{ for all } E \in \tilde{I}.$$

We can repeat this process and obtain uniform bounds for ψ_E^{real}, $E \in \tilde{I}$, in L^{q_j}, $\frac{1}{q_j} = \frac{2p+1}{q^{j-1}} - \frac{2}{n}$, $j \geq 1$ until q_{j+1} becomes negative, see (4.26), i.e.,

$$\frac{2p+1}{q_l} \geq \frac{2}{n} \text{ for } 0 \leq l \leq j-1 \text{ and } \frac{2p+1}{q_j} \leq \frac{2}{n}.$$

But $\frac{2p+1}{q_j} \leq \frac{2}{n}$ implies that

$$W^{2,\frac{q_j}{2p+1}} \text{ is embedded in } L^q \text{ for all } q \geq \frac{q_j}{2p+1}.$$

In particular we can fix a $q > 4p+2$ such that

$$\|\psi_E^{real}\|_{L^q} \leq M_q \qquad \text{for all } E \in \tilde{I}. \tag{4.27}$$

For the second case, $n > 4$,

$$W^{2,2} \text{ is embedded in } L^q \text{ for all } q \geq 2 \text{ such that } \frac{1}{q} \geq \frac{1}{2} - \frac{2}{n} = \frac{n-4}{2n}.$$

We can repeat the above process and obtain uniform bounds for ψ_E^{real}, $E \in \tilde{I}$, in L^{q_j}, $\frac{1}{q_j} = \frac{2p+1}{q^{j-1}} - \frac{2}{n}$, $j \geq 1$ until $W^{2,2}$ no longer embeds in $L^{q_{j+1}}$ i.e.,

$$\frac{2p+1}{q_l} \geq \frac{1}{2} \text{ for } 0 \leq l \leq j-1 \text{ and } \frac{2p+1}{q_j} < \frac{1}{2}.$$

Note that such j exists due to (4.26) and $q_0 = 2p+2$. Moreover, in this $n > 4$ case we now have $q_j > 0$ and $\frac{2p+1}{q_j} < \frac{1}{2}$ implies $q_j > 4p+2$. Hence we get (4.27) with $q = q_j$.

We now finish the H^2 bounds. By applying Riesz-Thorin interpolation for the L^{4p+2} norm of ψ_E^{real}, $E \in \tilde{I}$, using (4.22) and (4.27) we get that there exists $M_{4p+2} > 0$

such that $\||\psi_E^{real}|^{2p}|\psi_E^{real}\|_{L^2} \le (M_{4p+2})^{2p+1}$, for all $E \in \tilde{I}$, which combined with (4.19) and the identity (4.23)gives uniform bounds in H^2 :

$$\|\psi_E^{real}\|_{H^2} \le \tilde{M} \qquad \text{for all } E \in \tilde{I}. \tag{4.28}$$

We now show that $E \mapsto \psi_E^{real}$ has limit in H^2 at E_*. We again start with the L^2 limit. Now, for any sequence $\{E_n\}_{n\in\mathbb{N}} \subset \tilde{I}$ such that $\lim_{n\to\infty} E_n = E_*$,

$$\|\psi_{E_m}^{real} - \psi_{E_l}^{real}\|_{L^2} \le \|\partial_E \psi_E^{real}\|_{L^2}|E_m - E_l| \le \frac{M}{d}|E_m - E_l| \to 0 \quad \text{ as } m, l \to \infty.$$

Therefore, by completeness of $L^2(\mathbb{R}^n, \mathbb{R})$, there exists $\psi_{E_*}^{real} \in L^2(\mathbb{R}^n, \mathbb{R})$ such that

$$\lim_{n\to\infty} \|\psi_{E_n}^{real} - \psi_{E_*}^{real}\|_{L^2} = 0.$$

In fact, $\psi_{E_*}^{real}$ is independent on sequences $\{E_n\}$ and consequently,

$$\lim_{E\to E_*} \|\psi_E^{real} - \psi_{E_*}^{real}\|_{L^2} = 0.$$

Also, since $\psi_{E_n}^{real}$ is bounded in $H^2(\mathbb{R}^n, \mathbb{R})$, there exists a subsequence E_{n_k} and $\tilde{\psi} \in H^2(\mathbb{R}^n, \mathbb{R})$ such that $\psi_{E_{n_k}}^{real} \rightharpoonup \tilde{\psi}$ in H^2 as $k \to \infty$. It implies that $\psi_{E_{n_k}}^{real} \rightharpoonup \tilde{\psi}$ in L^2 as $k \to \infty$, hence $\psi_{E_*} = \tilde{\psi} \in H^2(\mathbb{R}^n, \mathbb{R})$. Moreover, $\|\psi_{E_*}\|_{H^2} \le \liminf_{k\to\infty} \|\psi_{E_{n_k}}\|_{H^2} \le \tilde{M}$. To simplify the notation, let us redenote $E_{n_k} = E_k, \psi_{E_{n_k}}^{real} = \psi_{E_k}^{real}$. As we did in (4.23), we can rewrite $\psi_{E_k}^{real}$ as:

$$\psi_{E_k}^{real} = (-\Delta + V + i)^{-1}[(i - E_k)\psi_{E_k}^{real} + \sigma(2p+1)|\psi_{E_k}^{real}|^{2p}\psi_{E_k}^{real}]. \tag{4.29}$$

Clearly, $(-\Delta + V + i)^{-1}(i - E_k)\psi_{E_k}^{real} \to (-\Delta + V + i)^{-1}(i - E_*)\psi_{E_*}^{real}$ in H^2. Now, we show that the remaining part also converges in H^2. Let $f : L^{4p+2} \to L^2$ as

$f(\phi) = |\phi|^{2p}\phi$. Then by mean value Theorem for the Fréchet derivative, we get:

$$\begin{aligned} & \| |\psi_{E_k}^{real}|^{2p}\psi_{E_k}^{real} - |\psi_{E_*}^{real}|^{2p}\psi_{E_*}^{real}\|_{L^2} \\ \leq & \ \|Df_\phi[(1-\alpha_k)|\psi_{E_k}^{real}| + \alpha_k|\psi_{E_*}^{real}|]|\psi_{E_k}^{real} - \psi_{E_*}^{real}|\|_{L^2} \\ \leq & \ \|(2p+1)[(1-\alpha_k)|\psi_{E_k}^{real}| + \alpha_k|\psi_{E_*}^{real}|]^{2p}|\psi_{E_k}^{real} - \psi_{E_*}^{real}|\|_{L^2} \\ \leq & \ (2p+1)\|[(1-\alpha_k)|\psi_{E_k}^{real}| + \alpha_k|\psi_{E_*}^{real}|]^{2p}\|_{L^{4p+2}\to L^2}\|\psi_{E_k}^{real} - \psi_{E_*}^{real}\|_{L^{4p+2}} \end{aligned} \tag{4.30}$$

for some $0 < \alpha_k < 1$. By the Hölder's inequality, for any $v \in L^{4p+2}$, $\|v\|_{L^{4p+2}} = 1$, we have:

$$\begin{aligned} \|[(1-\alpha_k)|\psi_{E_k}^{real}| + \alpha_k|\psi_{E_*}^{real}|]^{2p}v\|_{L^2} &\leq \|[(1-\alpha_k)|\psi_{E_k}^{real}| + \alpha_k|\psi_{E_*}^{real}|]^{2p}\|_{L^q}\|v\|_{L^{4p+2}} \\ &= \|[(1-\alpha_k)|\psi_{E_k}^{real}| + \alpha_k|\psi_{E_*}^{real}|]\|_{L^{2pq}}^{2p}\|v\|_{L^{4p+2}} \\ &\leq ((1-\alpha_k)\|\psi_{E_k}^{real}\|_{L^{2pq}} + \alpha_k\|\psi_{E_*}^{real}\|_{L^{2pq}})^{2p}\|v\|_{L^{4p+2}} \\ &\leq (\sup_{E\in[\tilde{E},E_*]}\|\psi_E^{real}\|_{L^{2pq}})^{2p}\|v\|_{L^{4p+2}} \\ &\lesssim \tilde{M}^{2p} \end{aligned} \tag{4.31}$$

where $q = 2 + \frac{1}{p}$. The Minkowski inequality can be applied to the third line because $2pq = 4p + 2 > 2$. The last inequality is due to the fact that H^2 is embedded in L^{2pq} because of Sobolev embedding Theorem and $0 < p < \frac{2}{n-2}$. Thus, $\|[(1-\alpha_k)|\psi_{E_k}^{real}| + \alpha_k|\psi_{E_*}^{real}|]^{2p}\|_{L^{4p+2}\to L^2}$ is uniformly bounded. Moreover, by using Riesz Thorin interpolation theorem we get:

$$\begin{aligned} \|\psi_{E_k}^{real} - \psi_{E_*}^{real}\|_{L^{4p+2}} &\leq \|\psi_{E_k}^{real} - \psi_{E_*}^{real}\|_{L^2}^a\|\psi_{E_k}^{real} - \psi_{E_*}^{real}\|_{L^{\frac{2n}{n-4}}}^{1-a} \\ &\lesssim \|\psi_{E_k}^{real} - \psi_{E_*}^{real}\|_{L^2}^a(2\tilde{M})^{1-a} \end{aligned} \tag{4.32}$$

where a satisfies $0 < a < 1$, $\frac{1}{4p+2} = \frac{a}{2} + \frac{(1-a)(n-4)}{2n}$. (This a exists because $2 < 4p+2 < \frac{2n+4}{n-2} < \frac{2n}{n-4}$.) Since the right hand side of (4.32) converges to 0 as $E_k \to E_*$, it follows that, combining with (4.31), the right hand side of (4.30) converges to 0 as $E_k \to E_*$.

Therefore, the right hand side of (4.29) converges to $(-\Delta + V + i)^{-1}[(i - E_*)\psi_{E_*}^{real} + \sigma(2p+1)|\psi_{E_*}^{real}|^{2p}\psi_{E_*}^{real}]$ in H^2, while the left hand side converges to $\psi_{E_*}^{real}$ in H^2 as $E_k \to E_*$. It follows that $(\psi_{E_*}^{real}, E_*)$ is a solution of (1.2).

Now Proposition 4.3 can be applied to $(\psi_{E_*}^{real}, E_*)$ because its spectral assumpptions follows from negating (ii). Consequently $E \mapsto \psi_E^{real}$ can be C^1 extended past E_* which contradicts the choice of E_*.

For the repelling, $\sigma > 0$, case we define

$$\begin{aligned} E_* \ = \ & \inf \{\tilde{E} \ : \ \tilde{E} < E_k, \text{ and } E \mapsto \psi_E^{real} \text{ is an extension of the map in Remark 4.1 for} \\ & \text{which 0 is not in the spectrum of } L_+ \text{ and is a simple eigenvalue of } L_-\}. \end{aligned}$$

As before, Remark 4.2 guarantees that the set above is non-empty but in this case $E < E_k$, see Proposition 4.1. As before, we prove Theorem 4.2 by contradiction. Obvious adaptation of the above argument e.g., the continuous spectrum of the linearized operator is now at distance at least E_* from zero, where $E_* > 0$ by negating (i) in Theorem 4.2, leads to the existence of a unique C^1 extension of the $E \mapsto \psi_E^{real}$ map below E_* contradicting the choice of E_*. $\square$

CHAPTER 5

DOUBLE WELL POTENTIALS WITH ATTRACTIVE NONLINEARITY

In this chapter we show that the attractive nonlinearity causes a symmetry breaking bifurcation along the ground state branch in problems with double well potentials with large separation. The bifurcation is due to an eigenvalue of the linearized operator crossing zero. The eigenvalue corresponds to an antisymmetric eigenfunction (i.e., odd in x_1) while up to the bifurcation point the ground states were symmetric (i.e., even in x_1). Therefore, the bifurcation is of pitchfork type with the symmetric ground state branch continuing past the bifurcation point but becoming unstable, while the emerging asymmetric branch is stable for low power nonlinearities, $p \leq \dfrac{3+\sqrt{13}}{2}$, but unstable for higher power nonlinearities. These results extend the one dimensional ones in [13, Corollary 2] and the cubic nonlinearity results in [14]. As for the first excited branch we show that it can be uniquely continued much further compared to the ground state branch, including parts of the strongly nonlinear regime.

5.1 Bifurcations of ground states

In this section we analyze in detail the branch of symmetric ground states bifurcating from zero at the lowest eigenvalue of a Schrödinger operator with double well potential, see Proposition 4.2. We first show that for large enough separation of wells, case (ii) holds in our previous continuation Theorem 4.1 i.e., an eigenvalue of the linearized operator approaches zero, see Theorem 5.1 where we also approximate the end point E_*.. Then, we identify the eigenvalue as corresponding to an antisymmetric eigenfunction and, by first restricting our analysis to symmetric solutions, we infer that a limit point where the eigenvalue is zero does exist and the symmetric branch can be continued past it. Moreover, a pitchfork bifurcation occurs at the limit point and an asymetric branch of ground states emerges from it, see Theorem 5.2.

Theorem 5.1. *Let $\sigma < 0$, $p > 0$ and potential $V = V_s$ be a double-well potential. Consider the branch of solutions $(\psi_E, E) = (e^{i\theta}\psi_E^{real}, E)$, ($\psi_E^{real}$ is a real-valued, $\theta \in [0, 2\pi]$) of* (1.2) *which bifurcates from the lowest eigenvalue $-E_{0,s}$ of $L_0 = -\Delta + V_s(x)$, see* Proposition 4.1. *Then there exists $s_* > 0$ such that for all $s \geq s_*$, there exists $E_{*,s}$, $E_{0,s} < E_{*,s} < \infty$ such that this branch can be uniquely continued on $(E_{0,s}, E_{*,s})$. Moreover, as the parameter E approaches the endpoint $E_{*,s}$, the second eigenvalue of $L_+(\psi_E^{real}, E)$, denoted by $\lambda(E)$, which is simple, and only the second eigenvalue approaches 0 i.e.,*

$$\lim_{E \nearrow E_{*,s}} \lambda(E) = 0.$$

Proof. Suppose (i) in Theorem 4.1 holds. Let us rewrite (1.2):

$$F(\phi, E) = -\Delta\phi + V_s\phi + E\phi + \sigma|\phi|^{2p}\phi = 0 \tag{5.1}$$

for a double-well potential V_s. By 2.4, the two lowest eigenvalues $-E_{0,s} < -E_{1,s}$ of $-\Delta + V_s$ are simple and satisfy

$$\lim_{s\to\infty} |E_{k,s} - w_0| = 0, \quad k = 0, 1 \tag{5.2}$$

where $-w_0$ is the lowest eigenvalue of $-\Delta + V_0$. Moreover, they are separated from the rest of the spectrum i.e., there exist $s_0, d_* > 0$ such that for $s > s_0$ and $k = 0, 1$ we have:

$$|E_{i,s} - \tau| \geq d_*, \qquad \forall \tau \in \Sigma \setminus \{E_{0,s}, E_{1,s}\}, \tag{5.3}$$

where Σ is a spectrum of $-\Delta + V_s$. Furthermore, the L^2-normalized real-valued eigenfunctions $\psi_{0,s}, \psi_{1,s}$ of $-\Delta + V_s$ corresponding to the eigenvalues $E_{0,s}, E_{1,s}$ satisfy

$$\lim_{s\to\infty} \left\| \psi_{i,s} - \frac{T_s\psi_0 + (-1)^i R T_s \psi_0}{\sqrt{2}} \right\|_{H^2} = 0, \quad i = 0, 1 \tag{5.4}$$

where ψ_0 is the L^2-normalized eigenfunction of $-\Delta + V_0$ corresponding to the lowest eigenvalue $-w_0$.

From Theorem 4.1 we have a unique two dimensional manifold of solutions of (5.1), $(E, \theta) \mapsto e^{i\theta}\psi_E^{real}$, $\theta \in \mathbb{R}$, $[E_{0,s}, E_{*,s})$ where $E_{*,s} = \infty$ since we are assuming that (i) holds. Note that the map is C^1 on $(E_{0,s}, \infty)$, and $\psi_{E_{0,s}} = 0$, where according to Proposition 4.1, we can use the parametrization given by the projection onto the eigenfunction of $-\Delta + V_s$ corresponding to eigenvalue $E_{0,s}$.

$$(\psi_E, E) = (e^{i\theta}\psi^{real}(a), E(a)) \quad \text{for} \quad a = \langle \psi_{0,s}, \psi^{real}(a) \rangle,\ |a| < \delta, \quad \theta \in [0, 2\pi),$$

for some $\delta > 0$. Moreover, the following estimates hold:

$$\psi_E = e^{i\theta}\psi^{real}(a) = a e^{i\theta}\psi_{0,s} + \mathcal{O}(|a|^{2p+1}) \text{ i.e., } \exists C_1 > 0 : \ \|\psi_E - a e^{i\theta}\psi_{0,s}\|_{H^2} \le C_1 |a|^{2p+1} \tag{5.5}$$

and there exists $C_2 > 0$ such that

$$\left| E - \left(E_{0,s} - \sigma \|\psi_{0,s}\|_{L^{2p+2}}^{2p+2} |a|^{2p} \right) \right| \le C_2 |a|^{4p}. \tag{5.6}$$

First, we note that the estimates (5.5) and (5.6) are uniform in the parameter measuring the distance between wells i.e., there is $s_0 > 0$ such that the constants C_1, C_2 can be chosen independent of $s \ge s_0$. This follows from the fact that, for $s \ge s_0$ given by (5.3), the estimates (5.5)-(5.6) rely on contraction mapping theorem for h, see (4.5):

$$h = -\sigma \left[P_{\perp,s}(-\Delta + V_s + E) P_{\perp,s} \right]^{-1} |a\psi_{0,s} + h|^{2p} (a\psi_{0,s} + h)$$

where $P_{\perp,s}\phi = \phi - \langle \psi_{0,s}, \phi \rangle \psi_{0,s}$. Since $P_{\perp,s}(-\Delta + V_s + E)P_{\perp,s}$ restricted to even functions is invertible with uniformly bounded inverse on $\{E > 0 : \mathrm{dist}(-E, \Sigma \setminus \{E_{0,s}, E_{1,s}\}) \geq d_*\}$, we can choose the Lipschitz constant for the map on the right hand side, and, consequently, the C_1, C_2 above independent of s, see Lemma 2.1 for a complete argument.

Now, since the dependence in $E > E_{0,s}$ of the ground state branch is at least C^1, see Remark 4.1, so is the dependence of the second eigenvalue of $L_+(\psi_E^{real}, E),\ \lambda(E, s)$ for as long as it remains simple. The continuous dependence of the discrete spectrum of $L_+(\psi_E^{real}, E)$ on E together with the non-discrete (continuous) spectrum being given by the interval $[E, \infty)$ and the lowest two e-values of $L_+(0, E_{0,s})$ being separated from rest of the spectrum, see (5.3), also gives us $\delta > 0$, which, as above, can be chosen independent of $s > s_0$ such that, for all $E \in [E_{0,s}, E_{0,s} + \delta]$:

$$|\lambda(E, s) - \tau| \geq \frac{d_*}{2}, \qquad \forall \tau \in \Sigma_E \setminus \{\lambda_0(E, s), \lambda(E, s)\}, \tag{5.7}$$

where Σ_E is the spectrum of $L_+(\psi_E^{real}, E)$ and $\lambda_0(E, s)$ is its lowest eigenvalue. Moreover, a collision between the first and second eigenvalue of $L_+(\psi_E^{real}, E)$ will make its lowest eigenvalue non-simple in contradiction with the uniform ellipticity of this operator. Consequently $\lambda(E, s),\ E_{0,s} \leq E \leq E_{0,s} + \delta$ remains simple, and together with its L^2-normalized real-valued eigenfunction η_E depends C^1 on E and satisfies:

$$L_+(\psi_E^{real}, E)\eta_E = \lambda \eta_E.$$

By differentiating the above with respect to E, we have:

$$L_+(\psi_E^{real}, E)\frac{d\eta_E}{dE} + \eta_E + \sigma(2p+1)\eta_E \frac{d}{dE}|\psi_E^{real}|^{2p} = \frac{d\lambda}{dE}\eta_E + \lambda \frac{d\eta_E}{dE}.$$

Taking the scalar product with η_E, we get:

$$1 + (2p+1)\sigma \int_{\mathbb{R}^n} \eta_E^2 \frac{d}{dE}|\psi_E^{real}|^{2p} dx = \frac{d\lambda}{dE}. \tag{5.8}$$

Thus, using Remark 4.1, see also (5.5), (5.6), we have for $a \to 0 : \lim_{a\to 0} E(a) = E_{0,s}$

and

$$\lim_{a\to 0}\frac{d}{dE}|\psi_E^{real}|^{2p}=\lim_{E\to E_{0,s}}\frac{d}{dE}|\psi_E^{real}|^{2p}=\frac{\psi_{0,s}^{2p}}{-\sigma\|\psi_{0,s}\|_{L^{2p+2}}^{2p+2}}.$$

Then by the continuous dependence with respect to a of L_+, we obtain:

$$\frac{d\lambda}{dE}(a=0,s)=\lim_{a\to 0}\left[1+(2p+1)\sigma\int_{\mathbb{R}^n}\eta_E^2\frac{d}{dE}|\psi_E^{real}|^{2p}dx\right]=1-\frac{2p+1}{\|\psi_{0,s}\|_{L^{2p+2}}^{2p+2}}\int_{\mathbb{R}^n}\psi_{1,s}^2\psi_{0,s}^{2p}dx.$$

Furthermore, using (5.4) we get:

$$\lim_{s\to\infty}\int_{\mathbb{R}^n}\psi_{1,s}^2\psi_{0,s}^{2p}dx=2^{-p}\|\psi_0\|_{L^{2p+2}}^{2p+2} \tag{5.9}$$

$$\lim_{s\to\infty}\|\psi_{0,s}\|_{L^{2p+2}}^{2p+2}=2^{-p}\|\psi_0\|_{L^{2p+2}}^{2p+2}. \tag{5.10}$$

Therefore, we have

$$\lim_{s\to\infty}\lim_{a\to 0}\frac{d\lambda}{dE}(a,s)=-2p<0.$$

We will show that the estimate of η_E in the formula above are uniform in s which combined with the uniform estimate of $\frac{d}{dE}|\psi_E^{real}|^{2p}$ given by (5.5) and (5.6), see also the paragraph below them, gives

$$\lim_{\substack{s\to\infty\\ a\to 0}}\frac{d\lambda}{dE}(a,s)=\lim_{s\to\infty}\lim_{a\to 0}\frac{d\lambda}{dE}(a,s)=-2p<0. \tag{5.11}$$

Indeed, the eigenvalue-eigenfunction problem for $L_+(\psi_E^{real},E)$ can be rewritten as $H(\eta,\lambda,E): H^2(\mathbb{R}^n,\mathbb{R})_{odd}\times\mathbb{R}\times\mathbb{R}\to L^2(\mathbb{R}^n,\mathbb{R})\times\mathbb{R}$:

$$H(\eta,\lambda,E)=\begin{bmatrix}L_+(\psi_E^{real},E)\eta-\lambda\eta\\ \langle\eta,\eta\rangle-1\end{bmatrix}$$

where $H^2(\mathbb{R}^n,\mathbb{R})_{odd}$ is $H^2(\mathbb{R}^n,\mathbb{R})$ *restricted to odd functions.* Now this problem satisfies the hypothesis of Implicit Function Theorem at $(\eta,\lambda,E)=(\psi_{1,s},E_{0,s}-E_{1,s},E_0,s)$ because $H(\psi_{1,s},E_{0,s}-E_{1,s},E_0,s)=\begin{bmatrix}0\\0\end{bmatrix}$, the Frechét derivative of H with respect to (η, λ)

$$D_{(\eta,\lambda)} H(\eta, \lambda, E) = \begin{bmatrix} L_+(\psi_E^{real}, E) - \lambda & -\eta \\ 2\langle \eta, \cdot \rangle & 0 \end{bmatrix}$$

is obviously continuous in (η, λ) uniformly in s, and

$$D_{(\eta,\lambda)} H(\psi_{1,s}, E_{0,s} - E_{1,s}, E_0) = \begin{bmatrix} -\Delta + V_s + E_{1,s} & -\psi_{1,s} \\ 2\langle \psi_{1,s}, \cdot \rangle & 0 \end{bmatrix}$$

is invertible with inverse that is uniformly bounded in s when restricted to odd η. Indeed, combined with (5.4) and Lemma 2.1,

$$[D_{(\eta,\lambda)} H(\psi_{1,s}, E_{0,s} - E_{1,s}, E_0)]^{-1} = \begin{bmatrix} (-\Delta + V_s + E_{1,s})^{-1} - \langle \psi_{1,s}, \cdot \rangle \psi_{1,s} & \frac{1}{2}\psi_{1,s} \\ -\langle \psi_{1,s}, \cdot \rangle & 0 \end{bmatrix}$$

is uniformly bounded in s. To get the uniform estimate of η_E, we will use the fixed point argument in the proof of Implicit Function Theorem. Let $A_s := D_{(\eta,\lambda)} H(\psi_{1,s}, E_{0,s} - E_{1,s}, E_0)$. Note that $H(\eta, \lambda, E) = \vec{0}$ is equivalent to $K(\eta, \lambda, E) = \begin{bmatrix} \eta \\ \lambda \end{bmatrix} - \begin{bmatrix} \psi_{1,s} \\ E_{0,s} - E_{1,s} \end{bmatrix}$ where

$$K(\eta, \lambda, E) = \begin{bmatrix} \eta \\ \lambda \end{bmatrix} - \begin{bmatrix} \psi_{1,s} \\ E_{0,s} - E_{1,s} \end{bmatrix} - A_s^{-1} H(\eta, \lambda, E)$$

The Frechét derivative of $K(\eta, \lambda, E)$ with respect to (η, λ) is:

$$D_{(\eta,\lambda)} K(\eta, \lambda, E) = \mathbb{I} - A_s^{-1} D_{(\eta,\lambda)} H(\eta, \lambda, E)$$

and at $(\psi_{1,s}, E_{0,s} - E_{1,s}, E_0)$, we get

$$D_{(\eta,\lambda)} K(\psi_{1,s}, E_{0,s} - E_{1,s}, E_0) = 0.$$

Since $D_{(\eta,\lambda)} H(\eta, \lambda, E)$ is continuous with respect to (η, λ) uniformly in s and A_s^{-1} is uniformly bounded in s, there exist $r, t > 0$, independent of s, such that

$$\|D_{(\eta,\lambda)} K(\eta, \lambda, E)\| \leq \frac{1}{2}, \quad \text{for all } (\eta, \lambda, E) \in \overline{B_r(\psi_{1,s}, E_{0,s} - E_{1,s})} \times \overline{B_t(E_0)}.$$

By Implicit Function theorem, there exists the solution of $H(\eta, \lambda, E) = 0$, (η_E, λ_E) and by the fixed point argument, we get:

$$\begin{aligned} & \|\eta_E - \psi_{1,s}\|_{H^2} + |\lambda_E - (E_{0,s} - E_{1,s})| && (5.12) \\ \leq\ & \frac{1}{2}\|A_s^{-1}\|_{L^2\times\mathbb{R}\to H^2\times\mathbb{R}}\|L_+(\psi_E^{real}, E)\psi_{1,s} - (E_{0,s} - E_{1,s})\psi_{1,s}\|_{L^2} + \frac{1}{2}r && (5.13) \\ \leq\ & \frac{1}{2}\|A_s^{-1}\|_{L^2\times\mathbb{R}\to H^2\times\mathbb{R}}\|E\psi_{1,s} + \sigma|\psi_E|^{2p}\psi_{1,s}\|_{L^2} + \frac{1}{2}r && (5.14) \end{aligned}$$

and last bound converges to 0 uniformly in s due to (5.4) and (5.5). As a result, we get the uniformly (in s) estimate of η_E. In conclusion, (5.11) holds.

Therefore, there exists s_1 and ϵ_1 such that:

$$\frac{d\lambda}{dE}(a, s) < -p \quad \text{for all } s > s_1, |a| < \epsilon_1$$

and, by Remark 4.1, see also (5.6), there exist $\epsilon \leq \delta$ such that:

$$\frac{d\lambda}{dE}(E, s) < -p \quad \text{for all } s > s_1, |E - E_{0,s}| < \epsilon. \tag{5.15}$$

Also, from (5.2), there exists s_2 such that for $a = 0 : E(a = 0) = E_{0,s}$, and

$$0 < \lambda(E_{0,s}, s) = E_{0,s} - E_{1,s} < \epsilon p \quad \text{for all } s > s_2. \tag{5.16}$$

Now let $s_* = \max\{\tilde{s}_0, s_1, s_2\}$. From (5.15) and (5.16), we conclude that, for any $s > s_*$, the graph of $E \mapsto \lambda(E, s)$ is below the graph of $y = -p(x - E_{0,s}) + \epsilon p$ which becomes negative at $E = E_{0,s} + \epsilon$, see Figure 5.1. Consequently $\lambda(E, s)$ must cross zero at some $E = E_{*,s}$, $E_{0,s} < E_{*,s} < E_{0,s} + \epsilon$, which contradicts our assumption that $E_{*,s} = \infty$, i.e. that (i) in Theorem 4.1 holds. Consequently (ii) in the same theorem holds with the second eigenvalue of L_+ approaching zero at the end of the maximal interval of unique continuation and remaining simple on this interval. All the other spectrum except the lowest e-value is at distance at least $d_*/2$ from zero while the lowest e-value $\lambda_0(E)$ cannot approach zero as following argument. Assume that there exists a sequence $E_n \nearrow E_{*,s}$ such that $\lambda_0(E_n) \to 0$ as $n \to \infty$. By min-max principle,

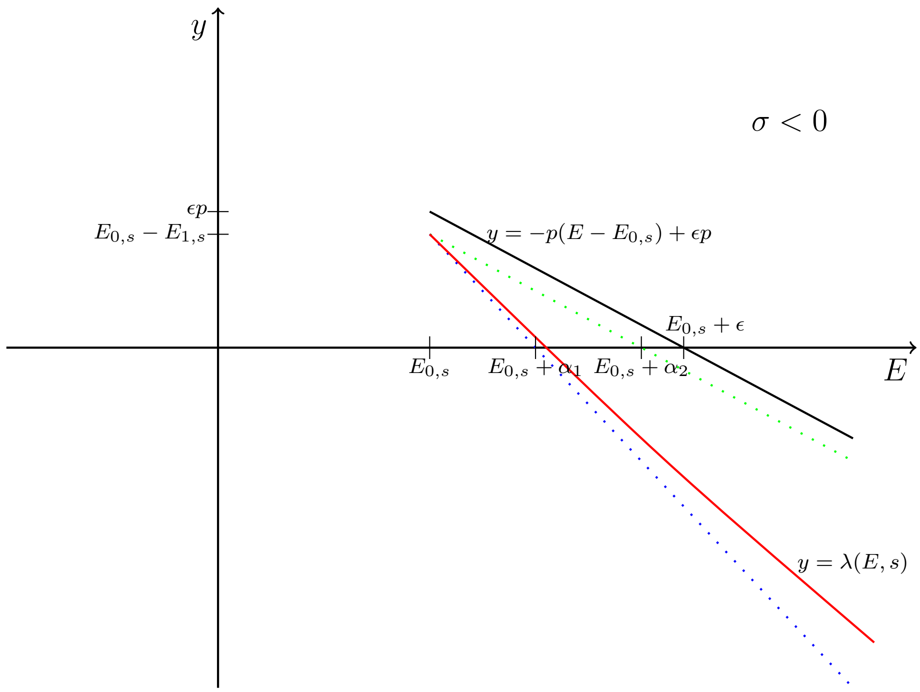

Figure 5.1: The above figure shows the branch of second eigenvalue of L_+, $\lambda(E,s)$ (the red line), crosses zero at some finite E since $y = -p(E - E_{0,s}) + \epsilon p$ (the black line) crosses zero. The blue and green dotted lines denote the lines with slope $-2p$, $-p$ whose x-intercept is $E_{0,s} + \alpha_1 = E_{0,s} + \frac{E_{0,s} - E_{1,s}}{2p}$, $E_{0,s} + \alpha_2 = E_{0,s} + \frac{E_{0,s} - E_{1,s}}{p}$, respectively. They give an approximation of x-intercept of $\lambda(E,s)$, $E = E_{*,s}$, corresponding to $a_*(s)$.

we have:

$$\begin{aligned}
\lambda_0(E_n) &\leq \frac{1}{\|\psi_{E_n}^{real}\|_{L^2}^2} \langle L_+(\psi_{E_n}^{real}, E_n)\psi_{E_n}^{real}, \psi_{E_n}^{real}\rangle \\
&= \frac{1}{\|\psi_{E_n}^{real}\|_{L^2}^2} \left[\langle L_- \psi_{E_n}^{real}, \psi_{E_n}^{real}\rangle + 2p\sigma \langle |\psi_{E_n}^{real}|^{2p} \psi_{E_n}^{real}, \psi_{E_n}^{real}\rangle\right] = 2p\sigma \frac{\|\psi_{E_n}^{real}\|_{L^{2p+2}}^{2p+2}}{\|\psi_{E_n}^{real}\|_{L^2}^2}
\end{aligned}$$

Since $\lambda_0(E_n) \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \|\psi_{E_n}^{real}\|_{L^{2p+2}}^{2p+2} = 0$. However, by plugging $\psi_{E_n}^{real}$

into (5.1) and taking L^2 scalar product with $\frac{\psi_{E_n}^{real}}{\|\psi_{E_n}^{real}\|_{L^2}^2}$, we have:

$$\begin{aligned}
-E_{0,s} &\leq \langle(-\Delta + V_s)\psi_{E_n}^{real}, \frac{\psi_{E_n}^{real}}{\|\psi_{E_n}^{real}\|_{L^2}^2}\rangle = \langle(-\sigma|\psi_{E_n}^{real}|^{2p}\psi_{E_n}^{real} - E_n\psi_{E_n}^{real}), \frac{\psi_{E_n}^{real}}{\|\psi_{E_n}^{real}\|_{L^2}^2}\rangle \\
&= -\sigma\frac{\|\psi_{E_n}^{real}\|_{L^{2p+2}}^{2p+2}}{\|\psi_{E_n}^{real}\|_{L^2}^2} - E_n.
\end{aligned}$$

Thus, by taking limit to $n \to \infty$, this inequality shows that $-E_{0,s} \leq -E_{*,s}$, which contradicts that $E_{*,s} > E_{0,s}$. Therefore, the lowest eigenvalue cannot approach zero and only the second lowest eigenvalue $\lambda(E, s)$ crosses zero.

Now, the theorem is completely proven and we have, for double well potentials with large separation $s > s_*$,

$$E_{0,s} + \frac{E_{0,s} - E_{1,s}}{2p} \lesssim E_{*,s} < E_{0,s} + \frac{E_{0,s} - E_{1,s}}{p}. \tag{5.17}$$

Note that, by Remark 4.1, this corresponds to a unique $|a| = a_*(s)$:

$$a_*(s) \approx \left(\frac{E_{0,s} - E_{1,s}}{2p}\right)^{1/2p}.$$

□

Remark 5.1. Note that the estimate, given in (5.17), shows that the maximal interval of unique continuation for this ground state branch is actually small, since $\lim_{s\to\infty} |E_{0,s} - E_{1,s}| = 0$, in fact it is exponetially small in parameter s, see [20]. This also implies that the bifurcation that follows happens in small amplitude regimes, at the corresponding amplitude:

$$a_*(s) \approx \left(\frac{E_{0,s} - E_{1,s}}{2p}\right)^{1/2p}$$

see Remark 4.1.

Now, we show that the second eigenvalue of the linearization L_+ crossing zero at

$E = E_{*,s}$ leads to a symmetry breaking bifurcation along the ground state branch. This is the main result of this section:

Theorem 5.2. *Let $\sigma < 0, p > \frac{1}{2}$ and $V = V_s$ be a double-well potential and consider the branch of solutions $(\psi_E, E) = (e^{i\theta}\psi_E^{real}, E), (\psi_E^{real}$ is a real-valued, $\theta \in [0, 2\pi])$ of (1.2) which bifurcates from the lowest eigenvalue $-E_0 = -E_{0,s}$ of $L_0 = -\Delta + V_s(x)$. Then for all $s \geq s_*$ (s_* is the number from Theorem 5.1), there exists $E_* < \infty$ such that (E_0, E_*) is the maximal interval on which this branch can be uniquely continued. Moreover, the set of solutions of (1.2) past E_* in a $H^2 \times \mathbb{R}$ neighborhood of (ψ_{E_*}, E_*) consists of exactly two surfaces of class at least $C^{[2p]-1}$ intersecting along the circle $e^{i\theta}\psi_{E_*}^{real}, 0 \leq \theta < 2\pi$. Each of these surfaces is obtained by rotating (multiplicity by $e^{i\theta}$) a curve of real valued solutions of (1.2).*

Proof. We will let $E_{0,s} = E_0$ and $E_{*,s} = E_*$ for fixed s. Theorem 5.1 already guarantees the existence of a finite E_*, such that the branch of solutions (ψ_E, E) can be uniquely continued on the interval (E_0, E_*), and the second eigenvalue of $L_+(\psi_E^{real}, E)$, $\lambda(E)$, and only it approaches zero as $E \nearrow E_*$, while remaining simple on this interval. Its corresponding L^2 normalized eigenfunction must be odd in x_1 because $L_+(\psi_E^{real}, E)$ commutes with the reflection operator R and the eigenfunction is odd at the $(0, E_0)$ end point. By restricting our analysis to the Banach subspace of even functions in H^2, (note that this branch if formed by even functions, see Proposition 4.2), we deduce that $L_+(\psi_E^{real}, E)$ *restricted to even functions* has no eigenvalue approaching zero as $E \nearrow E_*$. Indeed, Theorem 5.1 guarantees that only an eigenvalue corresponding to an odd eigenvector $\lambda(E)$ approaches zero and this eigenvalue is removed by the restriction to even functions. Applying Theorem 4.1 in this restricted Banach spaces we deduce that the branch can be extended past E_*. In particular there exists a unique $(\psi_{E_*}^{real}, E_*)$ on this branch.

Let ϕ_* be the L^2-normalized real-valued eigenfunction of $L_+(\psi_{E_*}^{real}, E_*)$ corresponding to its zero eigenvalue. Since $RL_+(\psi_{E_*}^{real}, E_*) = L_+(\psi_{E_*}^{real}, E_*)R$, where R is the reflection operator and 0 is a simple eigenvalue, ϕ_* is anti-symmetric in x_1. Furthermore, for fixed $\theta \in [0, 2\pi)$, $(e^{i\theta}\psi_E^{real}, E)$ is uniquely continued on the same interval. In order to prove the existence of the bifurcation at E_*, we use Lyapunov-Schmidt decomposition and Morse Lemma. The map $F(\phi, E) : H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R} \to L^2(\mathbb{R}^n, \mathbb{C})$:

$$F(\phi, E) = -\Delta\phi(x) + V(x)\phi(x) + \sigma|\phi(x)|^{2p}\phi(x) + E\phi(x)$$

has the Fréchet derivative at (ϕ, E) where ϕ is real-valued:

$$D_\phi F(\phi, E) = \begin{bmatrix} L_+(\phi, E) & 0 \\ 0 & L_-(\phi, E) \end{bmatrix}.$$

Since $i\psi_{E_*}^{real}$ is the L^2-normalized eigenfunction of $L_-(\psi_{E_*}^{real}, E_*)$ corresponding to its zero eigenvalue, $D_\phi F(\psi_{E_*}^{real}, E_*)$ is a Fredholm operator with

$$\ker D_\phi F(\psi_{E_*}^{real}, E_*) = \text{span}\{\phi_*, i\psi_{E_*}^{real}\},\ \text{ran } D_\phi F(\psi_{E_*}^{real}, E_*) = [\ker D_{\phi_*} F(\psi_{E_*}^{real}, E_*)]^\perp.$$

Let $\ker D_\phi F(\psi_{E_*}^{real}, E_*)$ =span$\{\phi_*, i\psi_{E_*}^{real}\} = X_1$ and $\text{Ran} D_\phi F(\psi_{E_*}^{real}, E_*) = X_2$. Then,

$$P_{\|\phi_*}\phi = \langle\phi_*, \phi\rangle_{real}\phi_*, \quad P_{\|i\psi_{E_*}^{real}}\phi = \langle i\psi_{E_*}^{real}, \phi\rangle_{real} i\psi_{E_*}^{real}, \quad P_\perp\phi = \phi - P_{\|\phi_*}\phi - P_{\|i\psi_{E_*}^{real}}\phi$$

are three orthogonal projections on L^2 =span$\{\phi_*, i\psi_{E_*}^{real}\} \oplus X^2$. By applying the Lyapunov-Schmidt decomposition at $(\psi_{E_*}^{real}, E_*)$, the equation (1.2) is equivalent to three following equations:

$$P_\perp F(\psi_{E_*}^{real} + a_1\phi_* + a_2 i\psi_{E_*}^{real} + k(a_1, a_2, E), E) = 0 \tag{5.18}$$

$$P_{\|\phi_*} F(\psi_{E_*}^{real} + a_1\phi_* + a_2 i\psi_{E_*}^{real} + k(a_1, a_2, E), E) = 0 \tag{5.19}$$

$$P_{\|i\psi_{E_*}^{real}} F(\psi_{E_*}^{real} + a_1\phi_* + a_2 i\psi_{E_*}^{real} + k(a_1, a_2, E), E) = 0 \tag{5.20}$$

where $a_1\phi_* = P_{\|\phi_*}(\phi - \psi_{E_*}^{real})$, $a_2 i\psi_{E_*}^{real} = P_{\|i\psi_{E_*}^{real}}(\phi - \psi_{E_*}^{real})$, $k = P_\perp(\phi - \psi_{E_*}^{real})$. Therefore, by Implicit Function Theorem, we get :

Lemma 5.1. *There is an unique $C^{[2p]+1}$ map $k : U \to L^2 \cap \{\phi_*, i\psi_{E_*}\}^\perp$ in some neighborhood $W \subset H^2 \times \mathbb{R}$ of (ψ_{E_*}, E_*), $U \subset \mathbb{R}^3$ of $(0, 0, E_*)$ such that for any* *solution* (ϕ, E) *of* (1.2),

$$\exists! a_1, a_2 \textit{ such that} (a_1, a_2, E) \in U, \quad \phi = \psi_{E_*} + a_1\phi_* + a_2 i\psi_{E_*}^{real} + k(a_1, a_2, E)$$

where $a_1 = \langle \phi - \psi_{E_*}^{real}, \phi_* \rangle_{real}$, $a_2 = \langle \phi - \psi_{E_*}^{real}, i\psi_{E_*} \rangle_{real}$ *and*

$$\langle \phi_*, F(\psi_{E_*}^{real} + a_1\phi_* + a_2 i\psi_{E_*}^{real} + k(a_1, a_2, E), E) \rangle_{real} = 0 \tag{5.21}$$

$$\langle i\psi_{E_*}^{real}, F(\psi_{E_*}^{real} + a_1\phi_* + a_2 i\psi_{E_*}^{real} + k(a_1, a_2, E), E) \rangle_{real} = 0. \tag{5.22}$$

Our strategy is to make the LHS of (6.6) identically vanish so that Morse Lemma can be applied, see Nirenberg [21] and [13]. To vanish the LHS of (6.6), we use a similar argument in Proposition 2.

First, assume $\phi - \psi_{E_*}^{real} \perp i\psi_{E_*}^{real}$ with respect to real scalar product. Then $a_2 = 0$ and $\phi = \psi_{E_*}^{real} + a_1\phi_* + k(a_1, 0, E)$. We claim that in this case $k(a_1, 0, E)$ must be real valued, hence ϕ is also real valued. To show this, we solve again (1.2) under restriction $a_2 = 0$ and $P_\perp(\phi - \psi_{E_*})$ is real valued. Define $F_\perp : \mathbb{R} \times H^2(\mathbb{R}^n, \mathbb{R}) \cap X_2 \times \mathbb{R} \to X_2 \cap L^2(\mathbb{R}^n, \mathbb{R})$:

$$F_\perp(a_1, k(a_1, 0, E), E)) = P_\perp F(\psi_{E_*}^{real} + a_1\phi_* + k(a_1, 0, E)).$$

This is well-defined because $P_\perp F(\phi, E)$ maps from $H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R}$ to $X_2 \cap L^2(\mathbb{R}^n, \mathbb{C})$ and for real-valued k, we have real-valued $\psi_{E_*}^{real} + a_1\phi_* + k(a_1, 0, E)$ and $F(\phi, E)$ and $P_\perp$ maps from real-valued to real-valued functions. Now

$$D_k F_\perp(0, 0, E_*) = L_+(\psi_{E_*}^{real}, E_*)$$

is an isomorphism from $H^2(\mathbb{R}^n, \mathbb{R}) \cap X_2 \times \mathbb{R} \to X_2 \cap L^2(\mathbb{R}^n, \mathbb{R})$. By Implicit Function Theorem, $\exists \delta_1, \delta_2$ and a unique $C^{[2p]+1}$ function $\tilde{k} : (-\delta_1, \delta_1) \times (E_* - \delta_2, E_* + \delta_2) \to X_2 \cap L^2(\mathbb{R}^n, \mathbb{R})$ such that $(a_1, \tilde{k}(a_1, E), E)$ is the unique solution of $F_\perp(a_1, k(a_1, 0, E), E))$ in a neighborhood $\tilde{W} \subset (-\delta_1, \delta_1) \times H^2(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R}$ of $(0, 0, E_*)$. It gives another unique real-valued solution

$$\phi = \psi_{E_*}^{real} + a_1\phi_* + \tilde{k}(a_1, E)$$

of (5.18) when $a_2 = 0$. By uniqueness, $\tilde{k}(a_1, E) = k(a_1, 0, E)$. It follows that (6.6) is zero since $F(\psi_{E_*}^{real} + a_1\phi_* + k(a_1, 0, E))$ is real valued.

Next, consider the case $\phi - \psi_{E_*}^{real} \not\perp i\psi_{E_*}^{real}$. In this case, we can use the argument in Proposition 4.3, i.e.,there exists θ_* such that

$$\|e^{i\theta_*}\psi_{E_*}^{real} - \phi\|_{H^2} = \inf_{\theta\in[0,2\pi)} \|e^{i\theta}\psi_{E_*}^{real} - \phi\|_{H^2}.$$

Then, as we have seen in Propostion 4.3, $\phi - e^{i\theta_*}\psi_{E_*}^{real} \perp ie^{i\theta_*}\psi_{E_*}^{real}$, which is equivalent to $e^{-i\theta_*}\phi - \psi_{E_*}^{real} \perp i\psi_{E_*}^{real}$. By apply the Lyapunov-Schmidt decomposition to $e^{-i\theta_*}\phi$, we get

$$e^{-i\theta_*}\phi = \psi_{E_*}^{real} + \langle e^{-i\theta_*}\phi - \psi_{E_*}^{real}, \phi_*\rangle\phi_* + k(a, 0, E).$$

Since the right-hand-side of the above equation is real-valued, $\langle i\psi_{E_*}^{real}, F(e^{-i\theta_*}\phi, E)\rangle = 0$. To finish the proof, use the same argument in [13]. □

5.2 Stability analysis

In this section we show that the pitchfork bifurcation given by Theorem 5.2 leads to a change in the orbital stability of the ground states. We start by defining orbital stability and by recalling a well known result we will subsequently use.

Definition 5.1 (see [27]). *The family of bound states $\{\Psi_E e^{-iE\theta} : \theta \in [0, 2\pi)\}$ is orbitally stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that if the initial data $u(x, 0) = u_0$ satisfies*

$$\inf_{\theta\in[0,2\pi)} \|u_0(\cdot) - \Psi_E(\cdot)e^{i\theta}\|_{H^2} < \delta$$

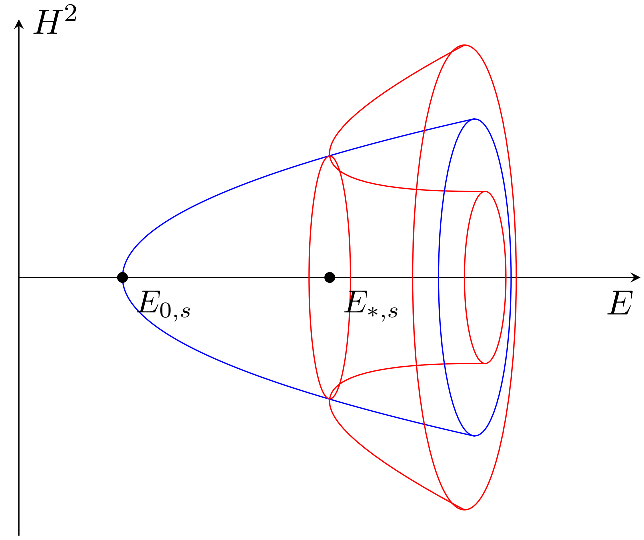

Figure 5.2: Bifurcation diagram for ground states of NLS equation with a double well potential and attractive nonlinearity. The blue line represents the symmetric state bifurcating from a trivial solution at $E_{0,s}$ and being continuing even past $E_{*,s}$. The red line represents the asymmetric state bifurcating from the ground states at $E_{*,s}$. The set of solutions is two surfaces of curves intersecting along the circle at $E_{*,s}$ and there are three branches up to rotation for $E > E_{*,s}$: one is symmetric and the other two is asymmetric.

then the solution $u(x,t)$ *of* (1.1) *satisfies*

$$\inf_{\theta\in[0,2\pi)} \|u(\cdot,t) - \Psi_E(\cdot)e^{i\theta}\|_{H^2} < \epsilon.$$

Theorem 5.3 (see [28, 29, 27, 30, 31])**.** *Let* $n_-(L_+)$ = *the number of negative eigenvalues of* L_+ *along the branch of solutions.*

(i) Suppose $n_-(L_+) = 1$ *and* L_- *is nonnegative. If*

$$\frac{d}{dE}\|\Psi_E\|^2_{L^2} > 0,$$

then Ψ_E *is orbitally stable.*

(ii) Suppose L_- *is nonnegative. If* $n_-(L_+) \geq 2$, *or,* $n_-(L_+) = 1$ *and* $\frac{d}{dE}\|\Psi_E\|^2_{L^2} < 0$, *then* Ψ_E *is orbitally unstable.*

The following theorem guarantees that the manifolds of ground states near the bifurcation point are of class C^2 for all $p \geq 1/2$ which allows us to calculate the quantities in Theorem 5.3 along each branch emerging from the bifurcation point, hence determine its orbital stability. The theorem generalizes the one dimensional result in [13, Corollary 2]

Theorem 5.4. *Let $\sigma < 0, p \geq \frac{1}{2}$, and $V = V_s$ be a double-well potential and $s \geq s_*$ where s_* satisfying Theorem 5.2. The two $C^{[2p]-1}$ surfaces in a $H^2 \times \mathbb{R}$ neighborhood of (ψ_{E_*}, E_*) which we obtained in Theorem 5.2 are in fact C^2, and :*

(a) the first surface, (ψ_E, E), is a continuation of the symmetric ground states past $E = E_*$. It is also even in x_1 and orbitally unstable past E_*.

(b) the second surface, $(\phi(a), E(a))$ is the new asymmetric states past $E = E_*$ such that

$$\phi(a) = e^{i\theta}\psi_{E_*}^{real} + ae^{i\theta}\phi_* + e^{i\theta}k(a, E) \quad \textit{for some } \theta \in [0, 2\pi)$$

$$E(a) = E_* + \frac{Q}{2}a^2 + o(a^2)$$

where $a = \langle e^{-i\theta}\phi - \psi_{E_*}^{real}, \phi_*\rangle \in \mathbb{R}$, $k(a, E)$ is real-valued and

$$Q = -\frac{1}{\lambda'(E_*)}\left[\frac{1}{3}(2p+1)2p(2p-1)\sigma\langle\phi_*^2, (\psi_{E_*}^{real})^{2p-2}\phi_*^2\rangle \right. \tag{5.23}$$
$$\left. -(2p+1)^2(2p)^2\sigma^2\langle(\psi_{E_*}^{real})^{2p-1}\phi_*, L_+(0, E_*)^{-1}(\psi_{E_*}^{real})^{2p-1}\phi_*^2\rangle\right].$$

ϕ is orbitally stable if $Q > 0$ and $R > 0$ and orbitally unstable if $Q < 0$, or $Q > 0$ and $R < 0$ where

$$R = \lim_{E \to E_*} \frac{d\|\phi\|_{L^2}^2}{dE} = 2\frac{\lambda'(E_*)}{Q} + N'(E_*), \quad N(E) = \|\psi_E\|_{L^2}^2. \tag{5.24}$$

Proof. For part (a), consider the Fréchet derivative of $F(\phi, E)$ at $(\psi_{E_*}^{real}, E_*)$:

$$D_\phi F(\psi_{E_*}^{real}, E_*) = \begin{bmatrix} L_+(\psi_{E_*}^{real}, E_*) & 0 \\ 0 & L_-(\psi_{E_*}^{real}, E_*) \end{bmatrix}.$$

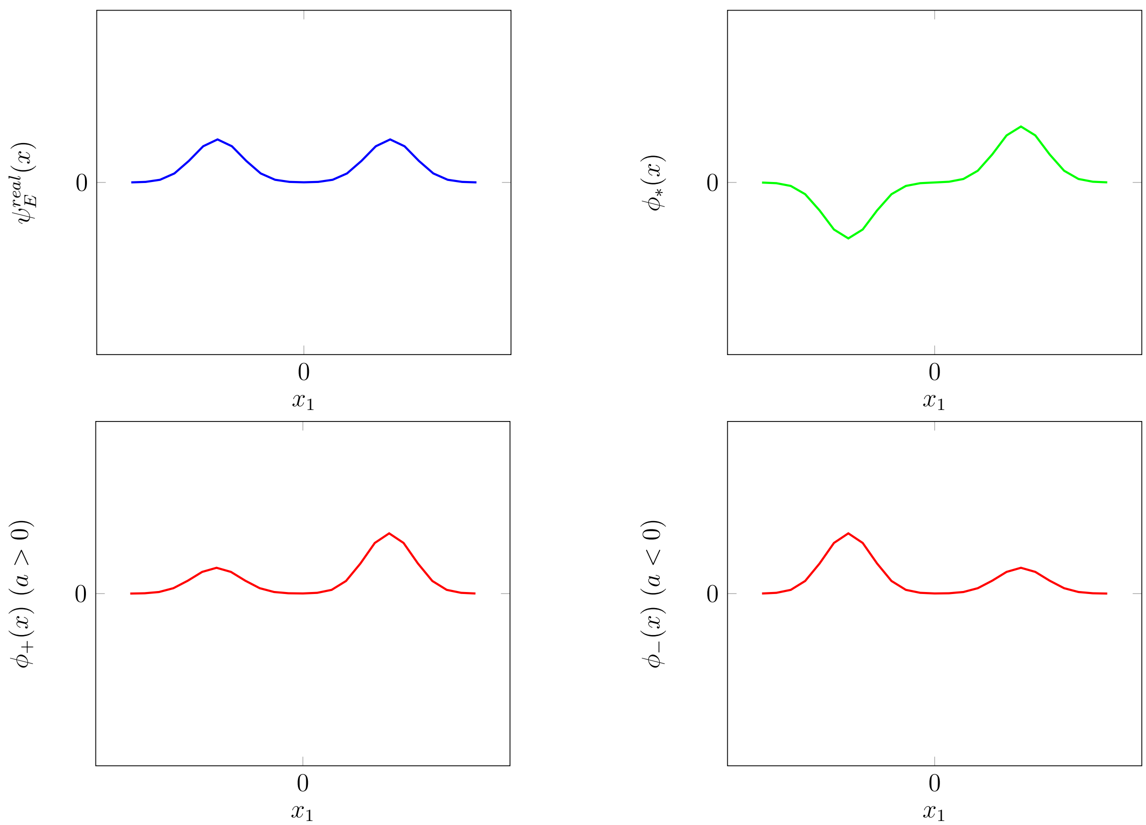

Figure 5.3: The figure shows what three (real-valued) branches look like for $E > E_{*,s}$. The top left (blue) subplot represents the real-valued symmetric state ψ_E^{real} which is even in x_1. The top right (green) subplot represents the L^2-normalized real-valued eigenfunction of $L_+(\psi_{E_*}^{real},, E_*)$ corresponding to zero eigenvalue, ϕ_*, which is odd in x_1. The bottom left and right (red) subplot represent the two real-valued asymmetric states. Since the asymmetric state is written by $\phi(a) = e^{i\theta}\psi_{E_*}^{real} + ae^{i\theta}\phi_* + e^{i\theta}k(a,E)$, there are two real-valued asymmetric states for $a > 0$ or $a < 0$. Let ϕ_+ and ϕ_- be the real-valued asymmetric states: $\psi_{E_*}^{real} + a\phi_* + k(a,E)$ for $a > 0$ or $a < 0$ resp. They are the sum of one odd function $a\phi_*$ and two even functions ψ_E^{real} and $k(a,E)$.

Note that $\psi_{E_*}^{real} = \lim_{E \nearrow E_*} \psi_E^{real}$, see Theorem 3 in [13]. It implies that $\psi_{E_*}^{real}$ is even in x_1. Thus, $D_\phi F(\psi_{E_*}^{real}, E_*)$ transforms even functions into even functions (in x_1). Moreover, we note that $\mathrm{Ker} D_\phi F(\psi_{E_*}^{real}, E_*)$ =span$\{\phi_*, i\psi_{E_*}^{real}\}$ and ϕ_*, which is the eigenfunction corresponding the second eigenvalue of $L_+(\psi_{E_*}^{real}, E_*)$, is odd. Therefore, the functional $D_* : H^2(\mathbb{R}^n, \mathbb{R})_{even} \to L^2(\mathbb{R}^n, \mathbb{R})_{even}$, which is the functional $D_\phi F(\psi_{E_*}^{real}, E_*)$ restricted in real-valued and even functions, is an isomorphism. By using Implicit Function Theorem, a set of real-valued, even solutions is uniquely extended in a neighborhood of $(\psi_{E_*}^{real}, E_*)$. Next, consider even solutions which are perpendicular to $i\psi_{E_*}^{real}$. Then we can apply Implicit Function Theorem again to the functional $P_\perp D_\phi F(\psi_{E_*}^{real}, E_*)_{even} : H^2(\mathbb{R}^n, \mathbb{C})_{even} \cap [\text{span}\{i\psi_{E_*}^{real}\}]^\perp \to L^2(\mathbb{R}^n, \mathbb{C})_{even} \cap [\text{span}\{i\psi_{E_*}^{real}\}]^\perp$ where $P_\perp$ is the projection onto $[\text{span}\{i\psi_{E_*}^{real}\}]^\perp$. Moreover, the unique continuation from above coincides with the unique real-valued even continuation due to the similar argument in Proposition 4.3. Now, we can extend the symmetric ground states ψ_E in the all even function space near E_*. If there is a even solution near (ψ_{E_*}, E_*) which is not orthogonal to $i\psi_{E_*}$, then there is a $\theta_* \in [0, 2\pi)$ such that the solution is orthogonal to $ie^{i\theta_*}\psi_{E_*}$. Using the similar argument in Proposition 4.3, we conclude that the ground states is continued past E_* in the even function space.

Moreover, $E \mapsto \psi_E$ is C^2. Since ψ_E is the eigenfunction corresponding the lowest eigenvalue of $L_-(\psi_E, E)$ we can choose the real-valued strictly positive function ψ_E^{real} for any E near E_*. Therefore, $F(\psi_E^{real}, E) = (-\Delta + V + E)\psi_E^{real} + |\psi_E^{real}|^{2p}\psi_E^{real}$ is C^2 in E for $p \geq 1/2$ and hence, $L_+ = D_\phi F$ is C^1. Differentiating $F(\psi_E^{real}, E) \equiv 0$, we get

$$\frac{d\psi_E^{real}}{dE} = -(L_+)^{-1}\psi_E^{real}.$$

This follows that ψ_E^{real} is C^2 in E. Also, since all even solutions near (ψ_{E_*}, E_*) is of the form $(\psi_E, E) = (e^{i\theta}\psi_E^{real}, E)$, $\theta \in [0, 2\pi)$, ψ_E is C^2 in E. From Theorem 5.1, the second eigenvalue of $L_+(\psi_E, E), \lambda(E)$ is negative for $E > E_*$ while $L_-(\psi_E, E)$ does not have strictly negative eigenvalues. If follows that $e^{iEt}\psi_E$ is orbitally unstable for $E > E_*$ by Theorem 5.2 in [14].

Now, we will prove part (b). From Theorem 5.2, the solution of (1.2) past E_* can be decomposed as

$$\phi = e^{i\theta}\psi_{E_*}^{real} + ae^{i\theta}\phi_* + e^{i\theta}k(a,E) \tag{5.25}$$

for some $\theta \in [0, 2\pi)$ where $a = \langle e^{-i\theta}\phi - \psi_{E_*}^{real}, \phi_* \rangle \in \mathbb{R}$, $k(a, E)$ is real-valued. Consider the continuation of symmetric ground states (ψ_E, E). Then $a = \langle e^{-i\theta}\psi_E, \phi_* \rangle = 0$, hence

$$\psi_E = e^{i\theta}\psi_{E_*}^{real} + e^{i\theta}k(0,E) \quad \text{for some}\, \theta \in [0, 2\pi) \tag{5.26}$$

$$\langle \phi_*, F(\psi_{E_*}^{real} + k(0,E), E) \rangle \equiv 0. \tag{5.27}$$

Let $F_{\|_{\phi_*}} = \langle \phi_*, F(\psi_{E_*}^{real} + a\phi_* + k(a,E), E) \rangle$. Then in the right hand side of equation, a can be factored out. Define a function $g(a, E) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$g(a,E) = \begin{cases} \frac{F_{\|_{\phi_*}}(a,E) - F_{\|_{\phi_*}}(0,E)}{a}, \text{ if}\, a \neq 0 \\ \frac{\partial F_{\|_{\phi_*}}}{\partial a}(0,E), \quad \text{if}\, a = 0 \end{cases} \tag{5.28}$$

Then the second surface of solutions of (1.2) when $a \neq 0$ must satisfy

$$g(a, E) = 0.$$

We will show that

(a) $g(0, E_*) = 0$

(b) $g \in C^1$ in a neighborhood of $(a = 0, E = E_*)$

(c) $\frac{\partial g}{\partial E_*}(0, E_*) \neq 0, \frac{\partial g}{\partial a}(0, E_*) = 0.$

If (a)-(c) hold, Implicit Function Theorem implies that there exists a unique C^1 curve $a \mapsto E(a)$ for $|a| < \epsilon$ for some ϵ with $E(0) = E_*$. Moreover from (c) we get

$$\frac{dE}{da}(0) = -\frac{\frac{\partial g}{\partial a}(0, E_*)}{\frac{\partial g}{\partial E_*}(0, E_*)} = 0. \tag{5.29}$$

We also will show $a \mapsto E(a)$ is C^2 and

$$\frac{d^2E}{da^2}(0) = Q \tag{5.30}$$

where Q is (5.23). Then

$$E(a) = E_* + \frac{Q}{2}a^2 + o(a^2). \tag{5.31}$$

Since k is C^1, we get

$$\frac{\partial k}{\partial a} = -(P_\perp L_+)^{-1} P_\perp L_+ \phi_*, \quad \frac{\partial k}{\partial E} = -(P_\perp L_+)^{-1} P_\perp [\psi^{real}_{E_*} + k] \tag{5.32}$$

and $a \mapsto \phi(a) = e^{i\theta}(\psi^{real}_{E_*} + a\phi_* + k(a, E(a)))$ is C^2 since

$$\frac{\partial \phi}{\partial a} = e^{i\theta}(\phi_* + \frac{\partial k}{\partial a} + \frac{\partial k}{\partial E}E'(a)) = e^{i\theta}(\phi_* - (P_\perp L_+)^{-1} P_\perp [L_+\phi_* + E'(a)(\psi^{real}_{E*} + k)]) \in C^1.$$

Moreover, we show

$$\lambda'(E_*) = 1 + (2p+1)2p\sigma \int_{\mathbb{R}^n} (\psi^{real}_{E_*})^{2p-1} \phi_*^2 \frac{\partial k}{\partial E}(0, E_*) dx. \tag{5.33}$$

This is because of following facts. Using continuous dependence of spectral decomposition of $L_+(\psi_E, E)$, with respect to E,

$$\lim_{E \to E_*} \|\eta_E - \phi_*\|_{H^2} = 0$$

From the fact $\frac{\partial \psi_E}{\partial E} = -L_+^{-1}\psi_E$ and (5.32), we get

$$\lim_{E \to E_*} \| \frac{\partial \psi^{real}_E}{\partial E} - \frac{\partial k}{\partial E}(0, E_*) \|_{H^2}.$$

Therefore, by (5.8), (5.33) is proved.

Now, using (5.25),(5.32), (5.33) and the properties of the partial derivatives of

$k(a,E)$, we obtain

$$\lambda_1(a) = -\lambda'(E_*)Qa^2 + o(a^2) \tag{5.34}$$

$$\|\phi(a)\|_{L^2}^2 = N(E_*) + 1/2(2\lambda'(E_*) + QN'(E_*))a^2 + o(a^2) = N(E_*) + 1/2QRa^2 + o(a^2) \tag{5.35}$$

where $\lambda_1(a)$ is the second eigenvalue of $L_+(\phi(a), E(a))$. By theorem 5.3, when $Q < 0$ or $Q > 0, R < 0$, ϕ is unstable and when $Q > 0$ and $R > 0$, ϕ is stable by (5.34) and (5.35).

Now we need to prove (a)-(c) and (5.30). By (5.28), $g(0, E_*) = \dfrac{\partial F_{\|\phi_*}}{\partial a}(0, E_*) = 0$ since $\dfrac{\partial F_{\|\phi_*}}{\partial a}(a, E) = \langle \phi_*, L_+[\phi_* + \frac{\partial k}{\partial a}]\rangle$ and $L_+(\psi_{E_*}^{real}, E_*)\phi_* = 0$. Therefore, (a) is proved. For (b), we consider two cases. First, when $a \neq 0$, (b) is clear since for $p \geq 1/2$, F is C^1 over real functions. To prove (b) for $a = 0$, we need to prove

$$\lim_{a\to 0, E\to E_*} \frac{\partial g}{\partial a}(a, E) \quad \text{exists, and} \lim_{a\to 0, E\to E_*} \frac{\partial g}{\partial E} = \frac{\partial^2 F_{\|\phi_*}}{\partial E \partial a}(0, E_*) \tag{5.36}$$

since $\dfrac{\partial^2 F_{\|\phi_*}}{\partial E \partial a}(0, E_*)$ is continuous in E. Consider the first limit $\lim_{a\to 0, E\to E_*} \dfrac{\partial g}{\partial a}(a, E)$. For $a \neq 0$, we have

$$\frac{\partial g}{\partial a}(a, E) = -\frac{F_{\|\phi_*}(a, E) - F_{\|\phi_*}(0, E)}{a^2} + \frac{\langle \phi_*, L_+(a, E)[\phi_* + \frac{\partial k}{\partial a}(a, E)]\rangle}{a}$$

where $L_+(a, E) = L_+(\psi_{E_*}^{real} + a\phi_* + k(a, E), E)$. Adding and subtract $\frac{1}{a}\langle \phi_*, L_+(a, E)[\phi_* + \frac{\partial k}{\partial a}(0, E)]\rangle$, we get

$$\begin{aligned}
&\lim_{a\to 0, E\to E_*} \frac{\partial g}{\partial a}(a, E) = \\
= &\lim_{a\to 0, E\to E_*} -\frac{F_{\|\phi_*}(a, E) - F_{\|\phi_*}(0, E) - a\langle \phi_*, L_+(a, E)[\phi_* + \frac{\partial k}{\partial a}(0, E)]\rangle}{a^2} \\
+ &\lim_{a\to 0, E\to E_*} \frac{\langle \phi_*, L_+(a, E)[\phi_* + \frac{\partial k}{\partial a}(a, E)]\rangle - \langle \phi_*, L_+(a, E)[\phi_* + \frac{\partial k}{\partial a}(0, E)]\rangle}{a} \\
= &I_1 + I_2.
\end{aligned}$$

Note that $I_1 = -\frac{1}{2} I_2$ provided I_2 exists. For $p > 1/2$, we have

$$I_2 = \langle \phi_*, \partial_a L_+(0, E_*)[\phi_* + \frac{\partial k}{\partial a}(0, E_*)] + L_+(0, E_*)\frac{\partial^2 k}{\partial a^2}(0, E_*)\rangle = 0$$

because k is C^2 and L_+ is C^1. For $p = 1/2$, we get

$$\begin{aligned} I_2 = & \lim_{a\to 0, E\to E_*} \frac{\langle \phi_*, (L_+(a,E) - L_+(0,E))[\phi_* + \frac{\partial k}{\partial a}(a,E)]\rangle}{a} \\ & + \lim_{a\to 0, E\to E_*} \frac{\langle \phi_*, L_+(0,E)[\frac{\partial k}{\partial a}(a,E) - \frac{\partial k}{\partial a}(0,E)]\rangle}{a} = I_3 + I_4. \end{aligned}$$

For I_3, we note that

$$\begin{aligned} & (L_+(a,E) - L_+(0,E))[\phi_* + \frac{\partial k}{\partial a}(a,E)] = \\ = \; & 2\sigma(|\psi_{E_*}^{real} + a\phi_* + k(a,E)| - |\psi_{E_*}^{real} + k(0,E))|[\phi_* + \frac{\partial k}{\partial a}(a,E)] \\ \le \; & 2\sigma|a\phi_* + k(a,E) - k(0,E)| \left|\phi_* + \frac{\partial k}{\partial a}(a,E)\right| \\ \le \; & 2\sigma|a| \left(|\phi_*| + \left|\frac{\partial k}{\partial a}(a',E)\right|\right) \left|\phi_* + \frac{\partial k}{\partial a}(a,E)\right| \quad \text{for some} \quad |a'| < |a|. \end{aligned}$$

Therefore, the integrand is bounded by an integrable function:

$$\begin{aligned} & \left|\frac{\phi_*(L_+(a,E) - L_+(0,E))[\phi_* + \frac{\partial k}{\partial a}(a,E)](x)}{a}\right| \\ & \qquad \le 2\sigma \left[|\phi_*|^2(x) + |\phi_*|(x)\left|\frac{\partial k}{\partial a}(a',E)\right|(x)\right] \left|\phi_* + \frac{\partial k}{\partial a}(a,E)\right| \end{aligned}$$

since ϕ_*, $\frac{\partial k}{\partial a} \in L^2(\mathbb{R}^n) \cap L^{3=2p+2}(\mathbb{R}^n)$. Moreover, since $k \in H^2$ is continuous in (a, E),

$$\lim_{a\to 0, E\to E_*} \|(\psi_{E_*}^{real} + a\phi_* + k(a,E)) - (\psi_{E_*}^{real} + k(0,E_*))\|_{H^2} = 0.$$

It implies that for any sequence $\{a_n\}_{n\in\mathbb{Z}}, \{E_n\}_{n\in\mathbb{Z}}$ such that $(a_n, E_n) \to (0, E_*)$, there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{Z}}, \{E_{n_k}\}_{k\in\mathbb{Z}}$ such that

$$\lim_{k\to\infty} \psi^{real}_{E_*} + a_{n_k}\phi_* + k(a_{n_k}, E_{n_k})(x) = \psi^{real}_{E_*} + k(0, E_*)(x)$$

for almost everywhere x. Therefore there exists N_x such that for $k > N_x$, $\psi^{real}_{E_*} + a_{n_k}\phi_* + k(a_{n_k}, E_{n_k})(x) > 0$ a.e. x because $\psi^{real}_{E_*} + k(0, E_*)(x) > 0$ and we have pointwise convergence a.e x :

$$\lim_{k\to\infty} \frac{|\psi^{real}_{E_*} + a_{n_k}\phi_* + k(a_{n_k}, E_{n_k})| - |\psi^{real}_{E_*} + k(0, E_*)|(x)}{a_{n_k}}$$
$$= \lim_{k\to\infty} \frac{a_{n_k}\phi_*(x) + k(a_{n_k}, E_{n_k}) - k(0, E_*)(x)}{a_{n_k}} = \phi_*(x) + \frac{\partial k}{\partial a}(0, E_*)(x).$$

Thus, by Lebesque Dominated Convergence Theorem we have

$$I_3 = \langle \phi_*, 2\sigma(\phi_* + \frac{\partial k}{\partial a}(0, E_*))^2 \rangle = 0.$$

Similarly, the integrand of I_4 is in L^1 and

$$\lim_{a\to 0, E\to E_*} \frac{\frac{\partial k}{\partial a}(a, E) - \frac{\partial k}{\partial a}(0, E_*)}{a} = -(P_\perp L_+)^{-1} P_\perp \left(\frac{\partial L_+}{\partial a}(0.E_*) \right) \left[\phi_* + \frac{\partial k}{\partial a}(0, E_*) \right].$$

Therefore, we get

$$I_4 = \langle \phi_*, -L_+(0, E_*)(P_\perp L_+)^{-1} P_\perp 2\sigma \left[\phi_* + \frac{\partial k}{\partial a}(0, E_*) \right]^2 \rangle = 0.$$

Thus, the existence of $\lim_{a\to 0, E\to E_*} \frac{\partial g}{\partial a}(a, E)$ is proved. Similarly, we can show

$$\lim_{a\to 0, E\to E_*} \frac{\partial g}{\partial E} = \frac{\partial^2 F_{\|\phi_*}}{\partial E \partial a}(0, E_*).$$

For $a \neq 0$,

$$\frac{\partial g}{\partial E} = \frac{1}{a}\left(\frac{\partial F_{\|_{\phi_*}}}{\partial E}(a,E) - \frac{\partial F_{\|_{\phi_*}}}{\partial E}(0,E)\right)$$
$$= \frac{1}{a}\left(a + \langle \phi_*, L_+(a,E)\frac{\partial k}{\partial E}(a,E)\rangle - \langle \phi_*, L_+(0,E)\frac{\partial k}{\partial E}(0,E)\rangle\right).$$

By adding and subtract $\frac{1}{a}L_+(0,E)\frac{\partial k}{\partial E}(a,E)$ we get

$$\lim_{a\to 0, E\to E_*}\frac{\partial g}{\partial E} = 1 + \lim_{a\to 0, E\to E_*}\frac{\langle \phi_*, (L_+(a,E) - L_+(0,E))\frac{\partial k}{\partial E}(a,E)\rangle}{a}$$
$$+ \lim_{a\to 0, E\to E_*}\frac{\langle \phi_*, L_+(0,E)\left(\frac{\partial k}{\partial E}(a,E) - \frac{\partial k}{\partial E}(0,E)\right)\rangle}{a} = 1 + \tilde{I}_1 + \tilde{I}_2.$$

The same argument of I_1 and I_2 in limit of $\frac{\partial g}{\partial a}$ gives

$$\tilde{I}_1 = (2p+1)2p\sigma\langle \phi_*, (\psi_{E_*}^{real} + k(0,E_*))^{2p-1}(\phi_* + \frac{\partial k}{\partial a}(0,E_*))\frac{\partial k}{\partial E}(0,E_*)\rangle$$

$$\tilde{I}_2 = \langle \phi_*, L_+(0,E_*)\frac{\partial^2 k}{\partial E\partial a}(0,E_*)\rangle$$

which implies

$$\lim_{a\to 0, E\to E_*}\frac{\partial g}{\partial E} = \frac{\partial^2 F_{\|_{\phi_*}}}{\partial E\partial a}(0,E_*) = \lambda'(E_*) \neq 0.$$

Thus, (b) and (c) are proved and there exists C^1 curve $a \mapsto E(a)$ in some neighborhood $(0,E_*)$ with

$$E(0) = E_*, \quad \frac{dE}{da}(0) = -\frac{\frac{\partial g}{\partial a}(0,E_*)}{\frac{\partial g}{\partial E}(0,E_*)} = 0.$$

Furthermore, we can show that $E(a)$ is C^2, which is clear for $a \neq 0, p > 1/2$ since g is C^2. For $a \neq 0, p = 1/2$, we can use the fact that $\psi_E(a) = \psi_{E_*}^{real} + a\phi_* + k(a, E(a))$ is a solution of the elliptic equation $L_-\psi_E(a) = 0$ which implies $\psi_{E_*}^{real} + a\phi_* + k(a, E(a))$

is strictly positive for all x. Therefore, we can prove that $\frac{\partial g}{\partial a}$ and $\frac{\partial g}{\partial E}$ are C^1 in (a, E) by using the similar argument for (5.36), which includes pointwise convergence. For $a = 0$,

$$\begin{aligned} E''(0) &= \lim_{a\to 0} \frac{\frac{dE}{da}(a) - \frac{dE}{da}(0)}{a-0} = -\frac{1}{\lambda'(E_*)} \lim_{a\to 0} \frac{1}{a}\frac{\partial g}{\partial a}(a, E(a)) \\ &= -\frac{1}{\lambda'(E_*)}\Big[\frac{1}{3}(2p+1)2p(2p-1)\sigma\langle \phi_*^2, (\psi_{E_*}^{real})^{2p-2}\phi_*^2\rangle \\ &\quad - (2p+1)^2(2p)^2\sigma^2\langle (\psi_{E_*}^{real})^{2p-1}\phi_*, -L_+(0,E_*)^{-1}(\psi_{E_*}^{real})^{2p-1}\phi_*^2\rangle\Big]. \end{aligned}$$

The limit of $\frac{1}{a}\frac{\partial g}{\partial a}(a, E(a))$ can be obtained by the same argument as the limit of $\frac{\partial g}{\partial a}(a, E(a))$, only except that for $1/2 < p < 1$, in order to apply Lebesque Dominated Convergence Theorem we need:

$$|\psi_{E_*}^{real} + a\phi_* + k(a, E(a))|^{2p-2}|\phi_*| \in L^2 \cap L^\infty. \tag{5.37}$$

Since ϕ_* and $\psi_E(a) = \psi_{E_*}^{real} + a\phi_* + k(a, E(a)) > 0$ are solutions of the uniform elliptic equations, by Theorem A.3 in [25], we have:

$$|\phi_*(x)| \le C(\delta)e^{(-\sqrt{E_*-\delta}|x|)}, \quad \psi_E(a) \ge C(\epsilon)e^{-(\sqrt{E_*+\epsilon}|x|)}$$

for $\delta > 0$ and $\epsilon > E(a) - E_*$. Thus, for $E(a) - E_* < \tilde{\epsilon} < \dfrac{E_*}{(2-2p)^2} - E_*$, we can choose $0 < \delta < E_* - (2-2p)^2(E_* + \tilde{\epsilon})$ in which case :

$$|\psi(a)|^{2p-2}|\phi_*| < Ce^{-\eta|x|}$$

where $\eta = (E_* - \delta)^{\frac{1}{2}} - (2-2p)(E_* + \tilde{\epsilon})^{\frac{1}{2}} > 0$. It implies (5.37) and finishes the proof. □

Corollary 5.1. *Under the assumptions of Theorem 5.4, the asymmetric states in Theorem 5.4-(b) is orbitally stable if* $p < p_* = \dfrac{3+\sqrt{13}}{2}$ *and orbitally unstable if* $p > p_*$.

Proof. We will figure out the sign of $Q = Q(s), R = R(s)$ with

$$\lim_{s\to\infty} a_*^{2-2p}(s)Q(s) = -\sigma\frac{2^{2-p}}{3}(2p+1)(p+1)\|\psi_0\|_{L^{2p+2}}^{2p+2} \tag{5.38}$$

$$\lim_{s\to\infty} a_*^{2-2p}(s)R(s) = \frac{2^p(-p^2+3p+1)}{-\sigma(2p+1)(p+1)p\|\psi_0\|_{L^{2p+2}}^{2p+2}}. \tag{5.39}$$

By above the limits and choosing some $\tilde{s_*}$ which is larger than s_*, $Q(s) > 0$ for all $s \geq \tilde{s_*}$ and $R(s) > 0$ if $p < p_* = \dfrac{3+\sqrt{13}}{2}$ while $R(s) < 0$ if $p > p_*$ for all $s \geq \tilde{s_*}$.

It remains to compute (5.38) and (5.39). For (5.38), we already showed that

$$\lim_{s\to\infty} \lambda'(E_*) = -2p.$$

Moreover, by (5.4) and the uniform estimate in s, (5.5), we get

$$\lim_{\substack{s\to\infty\\ a\to 0}} \int_{\mathbb{R}^n} a^{-q}(\psi_E^{real})^q \eta_E^{2k} dx = 2^{1-q/2-k}\|\psi_0\|_{L^{q+2k}}^{q+2k}, \qquad \text{for all } q \geq 0,\ k = 1, 2, ... \tag{5.40}$$

Recall that η_E is the L^2-normalized real-valued eigenfunction corresponding to the second eigenvalue of $L_+(\psi_E^{real}, E)$. Thus, we obtain

$$\lim_{s\to\infty} a_*^{2-2p}\langle \phi_*^2, (\psi_{E_*}^{real})^{2p-2}\phi_*^2\rangle = 2^{-p}\|\psi_0\|_{L^{2p+2}}^{2p+2}.$$

The only remaining part for Q is calculating $a^{2-2p}\langle(\psi_{E_*}^{real})^{2p-1}\phi_*, L_+(0, E_*)^{-1}(\psi_{E_*}^{real})^{2p-1}\phi_*^2\rangle$. Since $L_+(0, E_*)\psi_{E_*}^{real} = L_+(\psi_{E_*}^{real}, E_*)\psi_{E_*}^{real} = \sigma 2p(\psi_{E_*}^{real})^{2p+1}$, for even functions, it is equivalent to

$$L_+(0, E_*)^{-1}(\psi_{E_*}^{real})^{2p+1} = \frac{1}{2p\sigma}\psi_{E_*}^{real}.$$

Also, since

$$(\psi_{E_*}^{real})^{2p-1}\phi_*^2 = (\psi_{E_*}^{real})^{2p-1}\left[\frac{(\psi_{E_*}^{real})^2}{\|\psi_{E_*}^{real}\|_{L^2}^2} + \left(\phi_*^2 - \frac{(\psi_{E_*}^{real})^2}{\|\psi_{E_*}^{real}\|_{L^2}^2}\right)\right],$$

we have

$$\langle(\psi_{E_*}^{real})^{2p-1}\phi_*, L_+(0,E_*)^{-1}(\psi_{E_*}^{real})^{2p-1}\phi_*^2\rangle = I_1 + I_2 + I_3$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\|\psi_{E_*}^{real}\|_{L^2}^4}\langle(\psi_{E_*}^{real})^{2p+1}\phi_*, L_+(0,E_*)^{-1}(\psi_{E_*}^{real})^{2p+1}\rangle = \frac{1}{2p\sigma\|\psi_{E_*}^{real}\|_{L^2}^4}\int_{\mathbb{R}^n}(\psi_{E_*}^{real})^{2p+2}dx \\
I_2 &= \frac{2}{\|\psi_{E_*}^{real}\|_{L^2}^2}\langle L_+(0,E_*)^{-1}(\psi_{E_*}^{real})^{2p+1}, (\psi_{E_*}^{real})^{2p-1}\left(\phi_*^2 - \frac{(\psi_{E_*}^{real})^2}{\|\psi_{E_*}^{real}\|_{L^2}^2}\right)\rangle \\
&= \frac{1}{p\sigma\|\psi_{E_*}^{real}\|_{L^2}^2}\int_{\mathbb{R}^n}(\psi_{E_*}^{real})^{2p}\left(\phi_*^2 - \frac{(\psi_{E_*}^{real})^2}{\|\psi_{E_*}^{real}\|_{L^2}^2}\right)dx \\
I_3 &= \langle(\psi_{E_*}^{real})^{2p-1}\left(\phi_*^2 - \frac{(\psi_{E_*}^{real})^2}{\|\psi_{E_*}^{real}\|_{L^2}^2}\right), L_+(0,E_*)^{-1}(\psi_{E_*}^{real})^{2p-1}\left(\phi_*^2 - \frac{(\psi_{E_*}^{real})^2}{\|\psi_{E_*}^{real}\|_{L^2}^2}\right)\rangle.
\end{aligned}$$

Using (5.40), $a_*^{2-2p}I_1$ and $a_*^{2-2p}I_2$ converges to zero as $s \to \infty$. Also,we can expand I_3 by using spectral Theorem. Due to the facts that the only first two eigenvalues of $L_+(0,E_*)$ approaches zero and I_3 contains only even functions, we get

$$\lim_{s\to\infty} a_*^{2-2p}I_3 = \lim_{s\to\infty} a_*^{2-2p}\frac{|\langle(\psi_{E_*}^{real})^{2p-1}\left(\phi_*^2 - \frac{(\psi_{E_*}^{real})^2}{\|\psi_{E_*}^{real}\|_{L^2}^2}\right), \eta_{0,E_*}\rangle|^2}{\lambda_0(E_*)}$$

where $\lambda_0(E_*)$ is the lowest eigenvalue of $L_+(0,E_*)$ and η_{0,E_*} is the corresponding eigenfunction. By L'Hospital's rule, the latter becomes zero because the derivative of the numerator is zero, while the derivative of denominator is:

$$\lim_{s\to\infty}\lambda_0'(E_*) = -2p < 0$$

by the similar argument of (5.11). Combining all computations, we get (5.38).

Finally, in order to show (5.39), we compute $N'(E_*)$ first. Using (5.5) and (5.6), we have

$$N'(E_*) = \lim_{a\to a_*} 2\langle\frac{d\psi_E^{real}}{da}\left(\frac{dE}{da}\right)^{-1}, \psi_E^{real}\rangle = \lim_{a\to a_*}\frac{2a + O(|a|^{2p+1})}{-2p\sigma\|\psi_{0,s}\|_{L^{2p+2}}^{2p+2}a^{2p-1} + O(|a|^{4p-1})}.$$

Taking the limit in s, we get

$$\lim_{s\to\infty} a_*^{2p-2} N'(E_*) = \frac{2^p}{-\sigma p \|\psi_0\|_{L^{2p+2}}^{2p+2}}.$$

Combining this with (5.11), (5.38) and (5.24), the definition of R, we attain (5.39).

□

5.3 Unique continuation of the first excited states

In this section we show that the secondary bifurcation occurring along the ground state branch and analyzed above does not occur along the first excited branch. More precisely, we can prove that the branch $\psi_1(E)$, which bifurcates from 0 at the second lowest eigenvalue E_1, can be uniquely continued to large values of E and $\|\psi_1(E)\|_{L^2}$.

Theorem 5.5. *Let $\sigma < 0, V = V_s$ be a double well potential. Let $\psi_1(E)$ be the first excited state bifurcating from 0 at $E_{1,s}$. Then there exist $\tilde{s}$ and $c > 0$ such that for all $s \geq \tilde{s}$, the first excited branch can be extended at least on $(E_{1,s}, E_{1,s} + c)$.*

Proof. Let $\lambda_{i,s}^-(E)$ respectively $\lambda_{i,s}^+(E)$, be the i-th eigenvalue of $L_-(\psi_1(E), E)$ respectively $L_+(\psi_1(E), E)$, for $V = V_s$ and let $\mathcal{N} = \|\psi_1(E)\|_{L^2}$. If (i) in Theorem 4.1 holds, the proof is done because $\psi_1(E)$ can be uniquely continued on $(E_{1,s}, \infty)$. Thus, suppose (ii) in Theorem 4.1 holds: $\psi_1(E)$ can be uniquely extend on $I = (E_{1,s}, E_{*,s})$ where $E_{*,s}$ is finite and there exists a sequence $\{E_n\}_{n\in\mathbb{N}} \subset I$ such that $\lim_{n\to\infty} E_n = E_{*,s}$, and a corresponding sequence of nonzero eigenvalues of $L_+(\psi_1(E_n), E_n)$ or $L_-(\psi_1(E_n), E_n)$, $\{\lambda_n\}_{m\in\mathbb{N}}$, such that $\lim_{n\to\infty} \lambda_n = 0$. We will show that for some $c > 0$, $E_{*,s}$ must be greater than $E_{1,s} + c$ for large enough s.
Let us focus on the first and second eigenvalues of linearizations. Since $L_-(\psi_1(E_{1,s}), E_{1,s}) = L_0 + E_{1,s}$, $\lambda_{1,s}^-(E_{1,s})$ is $E_{1,s} - E_{0,s} < 0$. Using continuity of discrete eigenvalues, we have $\lambda_{2,s}^-(E) \equiv 0$ for all $E_{1,s} \leq E < E_{*,s}$ because 0 is an eigenvalue of $L_-(\psi_1(E), E)$ for all $E_{1,s} \leq E < E_{*,s}$ and $\lambda_{2,s}^-(E_{1,s}) = 0$, see Remark 4.2. We note that eigenfunctions corresponding to $\lambda_{1,s}^-(E)$ are even in x_1, see Proposition 4.2. By Theorem

4.1, $\psi_1(E)$ can be smoothly extended past $E_{*,s}$ in the Banach space *restricted to odd functions.* In particular, there exists a solution $(\psi_1(E_{*,s}), E_{*,s})$ unique up to rotation *restricted to odd functions.*. Assume that $\lambda^-_{1,s}(E_n) = \lambda_n$. By continuity of discrete eigenvalues, $\lambda^-_{1,s}(E_{*,s}) = 0 = \lambda^-_{2,s}(E_{*,s})$, which is a contradiction with the simplicity of the lowest eigenvalue of the second order elliptic operator $L_-(\psi_{E_{*,s}}, E_{*,s})$.

Similarly to L_-, $\lambda^+_{1,s}(E_{1,s})$ is $E_1 - E_0 < 0$ at $\mathcal{N} = 0$ and $\lambda^+_{2,s}(E_{1,s})$ is 0. We can show that λ_n cannot be $\lambda^+_{2,s}(E_n)$ because there exists $\delta > 0, d > 0$ such that

$$\lambda^+_{2,s}(E) \leq -d < 0 \quad \text{for all } E \in [E_{*,s} - \delta, E_{*,s}).$$

Indeed, by the min-max principle, we have for any $n \in \mathbb{N}$:

$$\begin{aligned}
\lambda^+_{2,s}(E_n) &= \inf_{\phi \in H^2, \|\phi\|_{L^2}=1, \phi \perp \psi^+_{0,s}(E_n)} \langle \phi, L_+(\psi_1(E_n), E_n)\phi \rangle \\
&\leq \frac{1}{\|\psi_1(E_n)\|^2_{L^2}} \langle \psi_1(E_n), L_+(\psi_1(E_n), E_n)\psi_1(E_n) \rangle \\
&= \frac{1}{\|\psi_1(E_n)\|^2_{L^2}} \langle \psi_1(E_n), L_-(\psi_1(E_n), E_n)\psi_1(E_n) + 2p\sigma |\psi_1(E_n)|^{2p} \psi_1(E_n) \rangle \\
&= 2p\sigma \frac{\|\psi_1(E_n)\|^{2p+2}_{L^{2p+2}}}{\|\psi_1(E_n)\|^2_{L^2}} < 0
\end{aligned}$$

where let $\psi^+_{0,s}(E)$ be the eigenfunction corresponding $\lambda^+_{1,s}(E)$ and we used that $\psi^+_{0,s}(E_n)$ is even, $\psi_1(E_n)$ is odd hence $\psi_1(E_n) \perp \psi^+_{0,s}(E_n)$. Let us assume that $\lambda^+_{2,s}(E_n) \to 0$ as $E_n \nearrow E_{*,s}$. Then from the above inequality, we get:

$$\lim_{n\to\infty} \frac{\|\psi_1(E_n)\|^{2p+2}_{L^{2p+2}}}{\|\psi_1(E_n)\|^2_{L^2}} = 0.$$

However, since $L_-(\psi_1(E_n), E_n)\psi_1(E_n) = 0$, by taking the inner product with $\frac{\psi_1(E_n)}{\|\psi_1(E_n)\|^2_{L^2}}$, we obtain:

$$-E_n - \sigma \frac{\|\psi_1(E_n)\|^{2p+2}_{L^{2p+2}}}{\|\psi_1(E_n)\|^2_{L^2}} = \langle \frac{\psi_1(E_n)}{\|\psi_1(E_n)\|_{L^2}}, (-\Delta + V) \frac{\psi_1(E_n)}{\|\psi_1(E_n)\|_{L^2}} \rangle \geq -E_{1,s}$$

The last inequality comes from the min-max principle, for $-\Delta + V$:

$$-E_{1,s} = \inf_{\phi \in H^2, \|\phi\|_{L^2}=1, \phi \perp \psi_{0,s}} \langle \phi, (-\Delta + V)\phi \rangle$$

by plugging in $\phi = \frac{\psi_1(E_n)}{\|\psi_1(E_n)\|_{L^2}}$ which is odd hence orthogonal to $\psi_{0,s}$ which is even. By passing to the limit as $n \to \infty$, we get $E_{1,s} \geq E_{*,s}$, which is a contradiction. Therefore, λ_n cannot be $\lambda^+_{2,s}(E_n)$ and it follows that λ_n also cannot be $\lambda^+_{1,s}(E_n)$ which is less than $\lambda^+_{2,s}(E_n)$. In conclusion, the sequence of eigenvalues from the assumption, λ_n, cannot be the first or the second eigenvalues of linearizations L_- and L_+.

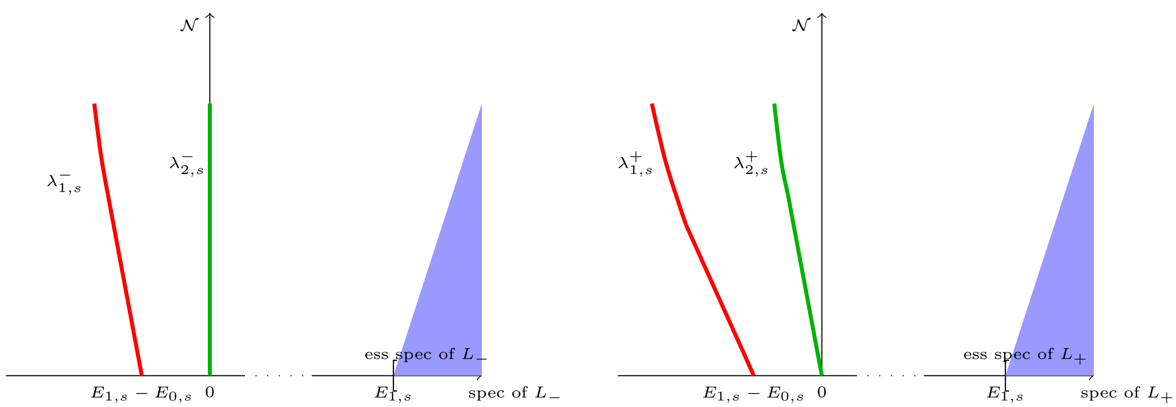

Figure 5.4: $\lambda^-_{i,s}$ and $\lambda^+_{i,s}$ denote the i-th eigenvalue of L_- and L_+ respectively, and $\mathcal{N} = \|\psi_1(E)\|^2_{L^2}$. The left graph shows that $\lambda^-_{1,s}$ cannot cross zero because of the simplicity of the lowest eigenvalue of elliptic second order operator. The right graph shows that $\lambda^+_{1,s}$ and $\lambda^+_{2,s}$ must be less than $\lambda^-_{1,s}$ and $\lambda^-_{2,s}$ resp., since $L_+ - L_- = \sigma 2p|\psi_1|^{2p} < 0$, which eliminates the possibility that $\lambda^+_{1,s}$ and $\lambda^+_{2,s}$ cross zero.

Therefore, if (ii) in Theorem 4.1 holds, λ_n must be the third eigenvalue of $L_+(\psi_1(E_n), E_n)$ or $L_-(\psi_1(E_n), E_n)$. If the branch of third eigenvalue of linearizations crosses zero, the third eigenvalue of $L_+(\psi_1(E), E)$ would cross zero first because $L_+(\psi_1(E), E) < L_-(\psi_1(E), E)$ implies $\lambda^+_{3,s}(E_n) \leq \lambda^-_{3,s}(E_n)$. Hence, we can consider only the case that $\lambda^+_{3,s}(E_n) = \lambda_n$. Let d_* be the distance between the lowest eigenvalue and the rest of the spectrum of the operator with the single well potential. Namely,

$$d_* = dist(-w_0, \Sigma(-\Delta + V_0) \setminus \{-w_0\}) \tag{5.41}$$

where $-w_0$ is the lowest eigenvalue of $-\Delta+V_0$. By Proposition 4.1, we get that there exists $\delta_1>0$ such that for $|a|<\delta_1$:

$$\|\psi_1(E)-a\psi_{1,s}\|_{H^2}\le C_{1,s}|a|^{2p+1} \tag{5.42}$$

$$|E-E_{1,s}-\sigma\|\psi_{1,s}\|^{2p+2}_{L^{2p+2}}|a|^{2p}|\le C_{2,s}|a|^{4p} \tag{5.43}$$

where $a=\langle\psi_{1,s},\psi_1(E)\rangle$. By the same argument of (5.5) and (5.6), there exist s_1, C_1 and C_2 independent of $s\ge s_1$ such that $C_{1,s}\le C_1$, $C_{2,s}\le C_2$, and C_1, respectively C_2, satisfies (5.42), respectively (5.43) for any $s\ge s_1$. Therefore, we obtain:

$$\begin{aligned}
&\|L_+(\psi_1(E),E)-(-\Delta+V_s+E_{1,s})\|_{H^2\to L^2}\\
&= \|(-\Delta+V_s+E)+\sigma(2p+1)|\psi_1(E)|^{2p}-(-\Delta+V_s+E_{1,s})\|_{H^2\to L^2}\\
&= \|E-E_{1,s}+\sigma(2p+1)|\psi_1(E)|^{2p}\|_{H^2\to L^2}\\
&\le |\sigma\|\psi_{1,s}\|^{2p+2}_{L^{2p+2}}|a|^{2p}|+|\sigma|(2p+1)\|\psi_{1,s}\|^{2p}_{L^{4p}}|a|^{2p}+C_2|a|^{4p}+C_1|a|^{4p^2+2p}
\end{aligned} \tag{5.44}$$

Let $-w_1$ be the second lowest eigenvalue of $-\Delta+V_0$, if there exists. Otherwise, let $w_1=0$. Then we note that

$$d_*=w_0-w_1.$$

Moreover, there exists s_2 such that for $s\ge s_2$,

$$|E_{1,s}-w_0|<\frac{d_*}{4},$$

$$|E_{2,s}-w_1|<\frac{d_*}{4}.$$

To emphasis that $L_+(\psi_1(E),E)=-\Delta+V_s+E+\sigma(2p+1)|\psi_1(E)|^{2p}$ depends on s, let us use the notation $L_+^s(\psi_1(E),E)=L_+(\psi_1(E),E)$. Now, we claim that for $\lambda=E_{1,s}-w_0+\frac{d_*}{2}$, there exists $\tilde{s}\ge s_2$, $\delta>0$ such that $(L_+^s(\psi_1(E),E))-\lambda)^{-1}:L^2\to H^2$ is uniformly bounded for $s\ge\tilde{s}$ and $|a|<\delta$. It follows that there is no eigenvalue of $L_+^s(\psi_1(E),E)$ crosses $E=E_{1,s}-w_0+\frac{d_*}{2}$, which implies that no eigenvalue of $L_+^s(\psi_1(E),E)$ crosses 0. Due to the fact that $E_{1,s}-w_0+\frac{d_*}{2}$ is greater than zero and

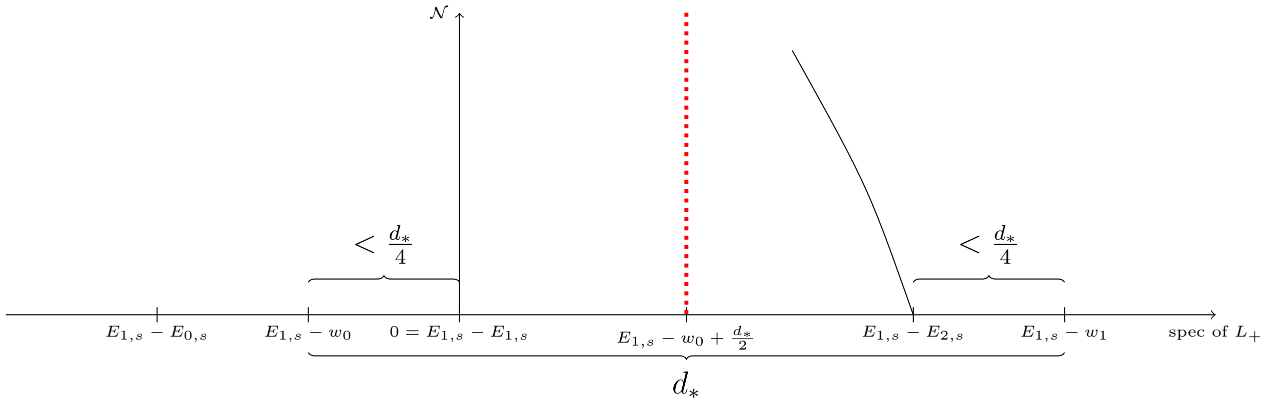

Figure 5.5

less than $E_{1,s} - E_{2,s}$ for $s \geq s_2$, and eigenvalues of $L_+^s(\psi_1(E), E)$ are continuous with respect to a, we can conclude that $\lambda_{3,s}^+(E)$ cannot cross zero. To show this, let us rewrite $(L_+^s(\psi_1(E), E)) - \lambda)^{-1}$ as:

$$(L_+^s(\psi_1(E), E)) - \lambda)^{-1} = (-\Delta + V_s + E_{1,s} - \lambda)^{-1}[\mathbb{I} + (B - A)(-\Delta + V_s + E_{1,s} - \lambda)^{-1}]^{-1}.$$

where $B - A = L_+^s(\psi_1(E), E) - (-\Delta + V_s + E_{1,s})$. Since $E_{1,s} - \lambda = w_0 - \frac{d_*}{2}$, we have

$$dist(-(E_{1,s} - \lambda), \Sigma(-\Delta + V_s)\}) > \frac{d_*}{4}.$$

By Remark 2.2, there exists $s_M \geq s_2, M > 0$ such that for all $s \geq s_M$, we have:

$$\|(-\Delta + V_s + E_{1,s} - \lambda)^{-1}\|_{L^2 \to H^2} \leq M. \tag{5.45}$$

Next, by (5.44), there exists $\tilde{s}_M$ and δ_M such that for $s \geq \tilde{s}_M$, $|E - E_{1,s}| < \delta_M$, we have:

$$\|(B - A)\|_{H^2 \to L^2} \leq \frac{1}{2M}.$$

It follows that

$$\begin{aligned}
&\|(B-A)(-\Delta+V_s+E_{1,s}-\lambda)^{-1}\|_{L^2\to L^2}\\
&\le \|(B-A)\|_{H^2\to L^2}\|(-\Delta+V_s+E_{1,s}-\lambda)^{-1}\|_{L^2\to H^2}\\
&\le \frac{1}{2M}\cdot M\\
&\le \frac{1}{2}.
\end{aligned}$$

Consequently, for $s\ge \max\{s_M,\tilde{s_M}\}$ and $|E-E_{1,s}|<\delta_M$, $\mathbb{I}+(B-A)(-\Delta+V_s+E_{1,s}-\lambda)^{-1}$ is invertible and

$$\|[\mathbb{I}+(B-A)(-\Delta+V_s+E_{1,s}-\lambda)^{-1}]^{-1}\|_{L^2\to L^2}\le 2.$$

Combined with (5.45), we obtain and for $s\ge \max\{s_M,\tilde{s_M}\}$ and $|E-E_{1,s}|<\delta_M$:

$$\|(L_+^s(\psi_1(E),E))-\lambda)^{-1}\|_{L^2\to H^2}\le 2M.$$

By (5.43), there exist s_1 and δ which is independent of s such that $|a|<\delta$ implies $|E-E_{1,s}|<\delta_M$. Hence, for $s\ge\tilde{s}$, where $\tilde{s}=\max\{s_M,\tilde{s_M},s_1\}$, and for $|a|<\delta$:

$$\|(L_+^s(\psi_1(E),E))-\lambda)^{-1}\|_{L^2\to H^2}\le 2M$$

which complete the claim. By the above argument, we obtain that for $s\ge\tilde{s}$:

$$|\lambda_{3,s}^+(E)-0|\ge E_{1,s}-w_0+\frac{d_*}{2}>\frac{d_*}{4}\quad \text{for } |a|<\delta. \tag{5.46}$$

Now, we show that there exists $c>0$ such that for all $s\ge\tilde{s}$, $\psi_1(E)$ can be extended on $(E_{1,s},E_{1,s}+c)$. From Proposition 4.1, there exists $\tilde{\delta}$ such that E is defined as a increasing function of a for $|a|<\tilde{\delta}$:

$$E(a)=E(|a|)=E_{1,s}-\sigma|a|^{2p}\|\psi_{1,s}\|_{L^{2p+2}}^{2p+2}+\mathcal{O}(|a|^{4p}).$$

Let $c=E(\min\{\delta,\tilde{\delta}\})-E_{1,s}$. Then for any $E_n\in(E_{1,s},E_{1,s}+c)$, where E_n is the sequence of eigenvalues from the assumption, $|\lambda^+_{3,s}(E_n) - 0| > \frac{d_*}{4}$. It completes the theorem. $\square$

Corollary 5.2. *Let $\sigma < 0$, $V = V_s$ be a double well potential. Let $\psi_1(E)$ be the first excited state bifurcating from 0 at $E_{1,s}$. Then there exists $\tilde{s}_1$ such that for all $s \geq \tilde{s}_1$, $\psi_1(E)$ can be parametrized by $\mathcal{N} = \|\psi_1(E)\|_{L^2}$ and it can be extended at least on the interval $(0, \|\psi_{E_{*,s}}\|_{L^2})$ where $\|\psi_{E_{*,s}}\|_{L^2}$ is a L^2 norm of the ground state bifurcation point.*

Proof. From Proposition 4.1, there exists δ such that $\mathcal{N} = \|\psi_1(E)\|_{L^2}$ is an increasing function of E on $(E_{1,s}, E_{1,s} + \delta)$. By Theorem 5.5, there exists $\tilde{s}$ such that for any $s \geq \tilde{s}$, $\psi_1(E)$ can be uniquely (up to rotation) extended at least on the interval $(0, \|\psi_1(\tilde{E}_s)\|_{L^2})$ where $\tilde{E}_s = E_{1,s} + \min\{\delta, c\}$. Since $a \mapsto E(a)$ is invertible near $a = 0$, see Remark 1.1, we can replace the interval of the previous statement $(0, \|\psi_1(\tilde{E}_s)\|_{L^2})$ with $(0, \|\psi_1(a = \epsilon_*)\|_{L^2})$ for some $\epsilon_* > 0$.

By Theorem 5.1, there exists s_* such that for any $s \geq s_*$, the ground state ψ_E bifurcates at

$$|a_*(s)| \approx \left(\frac{E_{0,s} - E_{1,s}}{-2p}\right)^{1/2p}.$$

In fact, $0 < |a_*(s)| \leq \left(\frac{E_{0,s} - E_{1,s}}{-p}\right)^{1/2p}$. By (5.2),

$$\lim_{s\to\infty} a_*(s) = 0. \tag{5.47}$$

From Proposition 4.1, we have

$$\|\psi_E\|^2_{L^2} \leq |a|^2 \|\psi_{0,s}\|^2_{L^2} + \|h(a)\|^2_{L^2}$$
$$|a|^2 \|\psi_{1,s}\|^2_{L^2} \leq \|\psi_1(E)\|^2_{L^2} \leq |a|^2 \|\psi_{1,s}\|^2_{L^2} + \|h_1(a)\|^2_{L^2}$$

where $h(a)$ respectively $h_1(a)$ are orthogonal complements of ψ_E respectively $\psi_1(E)$ to $\psi_{0,s}$ respectively $\psi_{1,s}$. Also, $\|h(a)\|^2_{L^2}, \|h_1(a)\|^2_{L^2} = \mathcal{O}(|a|^{4p+2})$. Therefore, combining with (5.47), we can choose some s_1 such that:

$$|a_*(s)| < \frac{\epsilon_*}{\sqrt{2}}$$

$$\|h(a_*(s))\|_{L^2}^2 = \mathcal{O}(|a_*(s)|^{4p+2}) < \frac{\epsilon_*^2}{8}\|\psi_0\|_{L^2}^2$$

for any $s \geq s_1$. Moreover, by (5.4), there exists s_2 such that for any $s \geq s_2$:

$$|\|\psi_{i,s}\|_{L^2}^2 - \|\psi_0\|_{L^2}^2| < \frac{\|\psi_0\|_{L^2}^2}{4} \quad i = 1, 2.$$

Let $\max\{\tilde{s}, s_1, s_2\} = \tilde{s_1}$. Then, for any $s \geq \tilde{s_1}$,

$$\|\psi_{E_*}\|_{L^2}^2 \leq |a_*(s)|^2\|\psi_{0,s}\|_{L^2}^2 + \|h(a_*(s))\|_{L^2}^2 < \frac{\epsilon_*^2}{2}\left(\|\psi_0\|_{L^2}^2 + \frac{\|\psi_0\|_{L^2}^2}{4}\right) + \frac{\epsilon_*^2}{8}\|\psi_0\|_{L^2}^2$$
$$= \frac{3\epsilon_*^2}{4}\|\psi_0\|_{L^2}^2 \leq \epsilon_*^2\|\psi_{1,s}\|_{L^2}^2 \leq \|\psi_1(a = \epsilon_*)\|_{L^2}^2.$$

Thus, for $s \geq \tilde{s_1}$, $\psi_1(E)$ can be extended at least on the interval $(0, \|\psi_{E_*}\|_{L^2}) \subseteq (0, \psi_1(a = \epsilon_*)\|_{L^2}^2)$.

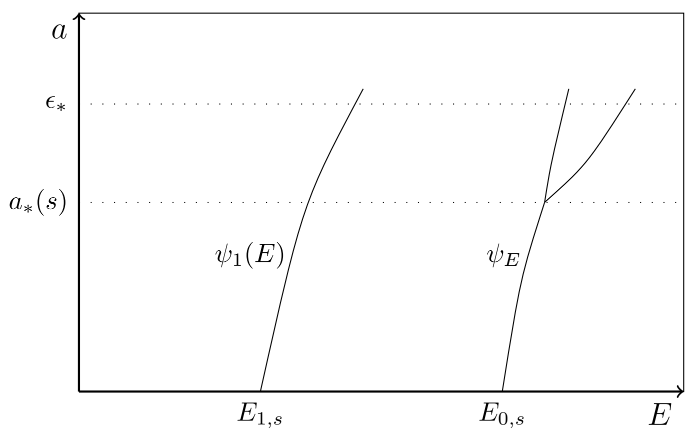

Figure 5.6

□

Remark 5.2. We just proved that bifurcations along the first excited branch in attractive case cannot happen at small amplitude (perturbation regime). The existence of such bifurcations at large amplitude is left to a forthcoming paper which uses global techniques, see [15].

CHAPTER 6

DOUBLE WELL POTENTIALS WITH REPELLING NONLINEARITY

In this chapter we show, for the first time to our knowledge, that the repelling nonlinearity causes a symmetry breaking bifurcation along the first excited state branch in problems with double well potentials with large separation. Our techniques also show that the ground state branch can be uniquely continued past the low amplitude regime. Note that the results in [32] show that a connected component of this branch reaches $E = 0$. We claim and show that there are no bifurcation along this branch until it reaches $E = 0$.

6.1 Bifurcations of first excited states

In this section we analyze in detail the branch of antisymmetric ground states bifurcating from zero at the second lowest eigenvalue of a Schrödinger operator with double well potential, see Proposition 4.2. We first show that for large enough separation of wells, case (ii) holds in our previous continuation Theorem 4.2 i.e., an eigenvalue of the linearized operator approaches zero. Then, we identify the eigenvalue as corresponding to a symmetric eigenfunction and, by first restricting our analysis to antisymmetric solutions, we infer that a limit point where the eigenvalue is zero does exist and the antisymmetric branch can be continued past it. Moreover, a pitchfork bifurcation occurs at the limit point and an asymetric branch of first excited states emerges from it, see Theorem 6.2.

Theorem 6.1. *Let $\sigma > 0$ and potential $V = V_s$ be a double-well potential. Consider the branch of solutions $(\psi_1(E), E) = (e^{i\theta}\psi_1(E)^{real}, E), (\psi_1(E)^{real}$ is a real-valued, $\theta \in [0, 2\pi])$ of (1.2) which bifurcates from the second lowest eigenvalue $-E_{1,s}$ of $L_0 = -\Delta + V_s(x)$. Then there exists $s_{**} > 0$ such that for all $s \geq s_{**}$, the lowest eigenvalue of $L_+(\psi_1(E), E)$, denote it by $\mu(E)$, and only the lowest eigenvalue approaches 0 as $E \searrow E_{**,s}$ for some $E_{**,s}, 0 < E_{**,s} < E_{1,s}$.*

Proof. Assume that (i) in Theorem 4.2 holds. Let us use the same notation in proof of Theorem 5.1. From Proposition 1, there exist $\delta > 0$ and solutions of (5.1) :

$$(\psi_1(E), E) = (e^{i\theta}\psi_1(a), E(a)) \quad \text{for} \quad |a| < \delta, \quad \theta \in [0, 2\pi)$$

bifurcating from $(0, E_{1,s})$ such that

$$\psi_1(E) = e^{i\theta}\psi_1(a) = ae^{i\theta}\psi_{1,s} + \mathcal{O}(|a|^{2p+1}), \quad \text{i.e. } \|\psi_1(E) - ae^{i\theta}\psi_{1,s}\|_{H^2} = \mathcal{O}(|a|^{2p+1}) \tag{6.1}$$

$$E = E_{1,s} - \sigma\|\psi_{1,s}\|^{2p+2}_{L^{2p+2}}|a|^{2p} + \mathcal{O}(|a|^{4p}) \tag{6.2}$$

where a parameter $a = \langle\psi_{1,s}, \psi_1^{real}(E)\rangle$. As shown in Theorem 5.1, for some s_0 we can find the uniform estimates for (6.1) and (6.2) when $s \geq s_0$, by using the contraction argument for h:

$$h = -\sigma\left[P_{\perp,s}(-\Delta + V_s + E)P_{\perp,s}\right]^{-1}|a\psi_{1,s} + h|^{2p}(a\psi_{1,s} + h)$$

where $P_{\perp,s}\phi = \phi - \langle\psi_{1,s}, \phi\rangle\psi_{1,s}$.

Let μ, ζ_E be the lowest eigenvalue and the corresponding eigenfunctions of $L_+(\psi_1(E), E)$. Then we obtain:

$$\frac{d\mu}{dE}(a, s) = 1 + (2p+1)\int_{\mathbb{R}^n}\zeta_E^2\frac{d}{dE}|\psi_1(E)|^{2p}dx.$$

Using (6.1), (6.2) and continuous dependence with recpect to a of L_+, we get:

$$\frac{d\mu}{dE}(0,s) = \lim_{a\to 0}\left[1+(2p+1)\sigma\int_{\mathbb{R}^n}\zeta_E^2\frac{d}{dE}|\psi_1(E)|^{2p}dx\right] = 1-\frac{2p+1}{\|\psi_{1,s}\|_{L^{2p+2}}^{2p+2}}\int_{\mathbb{R}^n}\psi_{0,s}^2|\psi_{1,s}|^{2p}dx.$$

From (5.4) we get:

$$\lim_{s\to\infty}\int_{\mathbb{R}^n}\psi_{0,s}^2|\psi_{1,s}|^{2p}dx = 2^{-p}\|\psi_0\|_{L^{2p+2}}^{2p+2},$$

$$\lim_{s\to\infty}\|\psi_{1,s}\|_{L^{2p+2}}^{2p+2} = 2^{-p}\|\psi_0\|_{L^{2p+2}}^{2p+2}.$$

By the same argument in Theorem 1, there exists s_1, s_2 and ϵ such that:

$$\frac{d\mu}{dE}(a,s) < -p < 0 \quad \text{for all } s > s_1,\ |E_{1,s} - E| < \epsilon$$

$$-\epsilon p < \mu(E_{1,s}, s) = E_{1,s} - E_{0,s} < 0 \quad \text{for all } s > s_2.$$

Unlike the focusing case, E decreases as a increases. Let s_{**} =max$\{s_0, s_1, s_2\}$. Then for any $s > s_{**}$, $\mu(E,s)$ must cross zero once at some $E = E_{**,s}, 0 < E_{**,s} < E_{1,s}$. This contradicts (i) in Theorem 4.2. Therefore, (ii) must hold with the lowest eigenvalue of L_+ approaching zero at the end of the maximal interval of unique continuation. By the similar argument in Theorem 5.1, approximations of $E_{**,s}$ and $|a| = a_{**}(s)$, with large separation $s > s_{**}$, is given by:

$$E_{1,s} + \frac{E_{1,s} - E_{0,s}}{p} \lesssim E_{**,s} < E_{1,s} + \frac{E_{1,s} - E_{0,s}}{2p} \tag{6.3}$$

and

$$a_{**}(s) \approx \left(\frac{E_{0,s} - E_{1,s}}{2p}\right)^{1/2p}. \tag{6.4}$$

□

Now, we show that the second eigenvalue of the linearization L_+ crossing zero at $E = E_{**,s}$ leads to a pitchfork bifurcation along the first excited state branch.

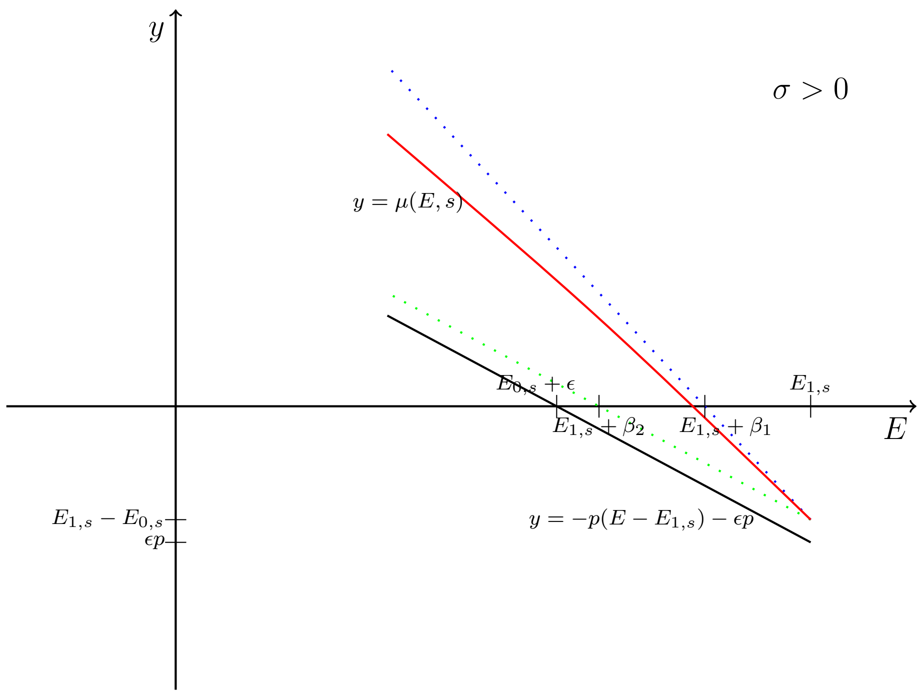

Figure 6.1: The above figure shows the branch of the lowest eigenvalue of L_+, $\mu(E, s)$ (the red line), crosses zero at some finite E since $y = -p(E - E_{1,s}) - \epsilon p$ (the black line) crosses zero. The blue and green dotted lines denote the lines with slope $-2p$, $-p$ whose x-intercept is $E_{1,s} + \beta_1 = E_{1,s} + \frac{E_{1,s} - E_{0,s}}{2p}$, $E_{1,s} + \beta_2 = E_{1,s} + \frac{E_{1,s} - E_{0,s}}{p}$, respectively. They give an approximation of x-intercept of $\mu(E)$, $E = E_{**,s}$, corresponding to $a_{**}(s)$.

Theorem 6.2. *Let $\sigma > 0, p > \frac{1}{2}$ and $V = V_s$ be a double-well potential and consider the branch of solutions $(\psi_1(E), E) = (e^{i\theta}\psi_1^{real}(E), E), (\psi_1^{real}(E)$ is a real-valued, $\theta \in [0, 2\pi])$ of (1.2) which bifurcates from the second lowest eigenvalue $-E_1 = -E_{1,s}$ of $L_0 = -\Delta + V_s(x)$. Then for all $s \geq s_{**}$, there exists $E_{**} < \infty$ such that (E_1, E_{**}) is the maximal interval on which this branch can be uniquely continued. Moreover, the set of solutions of (1.2) past E_{**} in a $H^2 \times \mathbb{R}$ neighborhood of $(\psi_1(E_{**}), E_{**})$ consists of exactly two surfaces of class at least $C^{[2p]-1}$ intersecting along the circle $e^{i\theta}\psi_1(E_{**})^{real}, 0 \leq \theta < 2\pi$. Each of these surfaces is obtained by rotating (multiplicity by $e^{i\theta}$) a curve of real valued solutions of (1.2).*

Proof. We will let $E_{1,s} = E_1$ and $E_{**,s} = E_{**}$ for fixed s. Theorem 6.1 already guarantees the existence of a finite E_{**}, such that the branch of solutions $(\psi_1(E), E)$ can be uniquely continued on the interval (E_1, E_{**}), and the lowest eigenvalue of $L_+(\psi_1^{real}(E), E)$, $\mu(E)$, and only it approaches zero as $E \nearrow E_{**}$, while remaining simple on this interval. Its corresponding eigenfunction must be even in x_1, since $L_+(\psi_1(E), E)$ commutes with the reflection operator and the eigenfunction is even at $(0, E_1)$. Therefore, $L_+(\psi_1^{real}(E), E)$ *restricted to odd functions* has no eigenvalue approaching zero as $E \nearrow E_{**}$, which implies that $\psi_1(E)$ can be extended past E_{**} in the Banach space restricted to odd functions. In particular, there exists a (real-valued) unique continuation $\psi_1(E_{**})$ at E_{**} on the branch. Let τ_{**} be the L^2-normalized real-valued eigenfunction of $L_+(\psi_1(E_{**})^{real}, E_{**})$ corresponding to its zero eigenvalue. τ_{**} is even in x_1, as mentioned above. Also, for fixed $\theta \in [0, 2\pi)$, $(e^{i\theta}\psi_1^{real}(E), E)$ is uniquely continued on (E_1, E_{**}). The map $F(\phi, E) : H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R} \to L^2(\mathbb{R}^n, \mathbb{C})$:

$$F(\phi, E) = -\Delta\phi(x) + V(x)\phi(x) + \sigma|\phi(x)|^{2p}\phi(x) + E\phi(x)$$

has the Fréchet derivative at (ϕ, E) where ϕ is real-valued:

$$D_\phi F(\phi, E) = \begin{bmatrix} L_+(\phi, E) & 0 \\ 0 & L_-(\phi, E) \end{bmatrix}.$$

Since $i\psi_1(E_{**})^{real}$ is the L^2-normalized eigenfunction of $L_-(\psi_1(E_{**})^{real}, E_{**})$ corresponding to its zero eigenvalue, $D_\phi F(\psi_1(E_{**})^{real}, E_{**})$ is a Fredholm operator with

$$\ker D_\phi F(\psi_1(E_{**})^{real}, E_{**}) = \text{span}\{\tau_{**}, i\psi_1(E_{**})^{real}\},$$

$$\text{ran } D_\phi F(\psi_1(E_{**})^{real}, E_{**}) = [\ker D_\phi F(\psi_1(E_{**})^{real}, E_{**})]^\perp.$$

Let $\ker D_\phi F(\psi_1(E_{**})^{real}, E_{**})$ =span$\{\tau_{**}, i\psi_1(E_{**})^{real}\} = X_1$ and $\text{Ran} D_\phi F(\psi_1(E_{**})^{real}, E_{**}) = X_2$. Then,

$$P_{\|\tau_{**}}\phi = \langle \tau_{**}, \phi\rangle_{real}\tau_{**}, \quad P_{\|i\psi_1(E_{**})^{real}}\phi = \langle i\psi_1(E_{**})^{real}, \phi\rangle_{real} i\psi_1(E_{**})^{real},$$

$$P_\perp \phi = \phi - P_{\|\tau_{**}}\phi - P_{\|i\psi_1(E_{**})^{real}}\phi$$

are three orthogonal projections on L^2 =span$\{\tau_{**}, i\psi_1(E_{**})^{real}\} \oplus X^2$. By applying the Lyapunov-Schmidt decomposition at $(\psi_1(E_{**})^{real}, E_*)$, the equation (1.2) is equivalent to three following equations:

$$P_\perp F(\psi_1(E_{**})^{real} + a_1\tau_{**} + a_2 i\psi_1(E_{**})^{real} + k(a_1, a_2, E), E) = 0$$

$$P_{\|\tau_{**}} F(\psi_1(E_{**})^{real} + a_1\tau_{**} + a_2 i\psi_1(E_{**})^{real} + k(a_1, a_2, E), E) = 0$$

$$P_{\|i\psi_1(E_{**})^{real}} F(\psi_1(E_{**})^{real} + a_1\tau_{**} + a_2 i\psi_1(E_{**})^{real} + k(a_1, a_2, E), E) = 0$$

where $a_1\tau_{**} = P_{\|\tau_{**}}(\phi - \psi_1(E_{**})^{real})$, $a_2 i\psi_1(E_{**})^{real} = P_{\|i\psi_1(E_{**})^{real}}(\phi - \psi_1(E_{**})^{real})$, $k = P_\perp(\phi - \psi_1(E_{**})^{real})$. Therefore, by Implicit Function Theorem, we get :

Lemma 6.1. *There is an unique $C^{[2p]+1}$ map $k : U \to L^2 \cap \{\tau_{**}, i\psi_1(E_{**})\}^\perp$ in some neighborhood $W \subset H^2 \times \mathbb{R}$ of $(\psi_1(E_{**}), E_{**})$, $U \subset \mathbb{R}^3$ of $(0, 0, E_{**})$ such that for any solution (ϕ, E) of (1.2),*

$$\exists! a_1, a_2 \text{ such that} (a_1, a_2, E) \in U, \quad \phi = \psi_1(E_{**}) + a_1\tau_{**} + a_2 i\psi_1(E_{**})^{real} + k(a_1, a_2, E)$$

where $a_1 = \langle \phi - \psi_1(E_{**})^{real}, \tau_{**}\rangle_{real}$, $a_2 = \langle \phi - \psi_1(E_{**})^{real}, i\psi_1(E_{**})\rangle_{real}$ and

$$\langle \tau_{**}, F(\psi_1(E_{**})^{real} + a_1\tau_{**} + a_2 i\psi_1(E_{**})^{real} + k(a_1, a_2, E), E)\rangle_{real} = 0 \tag{6.5}$$

$$\langle i\psi_1(E_{**})^{real}, F(\psi_1(E_{**})^{real} + a_1\tau_{**} + a_2 i\psi_1(E_{**})^{real} + k(a_1, a_2, E), E)\rangle_{real} = 0. \quad (6.6)$$

Our strategy is to make the LHS of (6.6) identically vanish so that Morse Lemma can be applied, see Nirenberg [21] and [13]. To vanish the LHS of (6.6), we use a similar argument in Proposition 2.

First, assume $\phi - \psi_1(E_{**})^{real} \perp i\psi_1(E_{**})^{real}$ with respect to real scalar product. Then $a_2 = 0$ and $\phi = \psi_1(E_{**})^{real} + a_1\tau_{**} + k(a_1, 0, E)$. We claim that in this case $k(a_1, 0, E)$ must be real valued, hence ϕ is also real valued. To show this, we solve again (1.2) under restriction $a_2 = 0$ and $P_\perp(\phi - \psi_1(E_{**}))$ is real valued. Define $F_\perp : \mathbb{R} \times H^2(\mathbb{R}^n, \mathbb{R}) \cap X_2 \times \mathbb{R} \to X_2 \cap L^2(\mathbb{R}^n, \mathbb{R})$:

$$F_\perp(a_1, k(a_1, 0, E), E)) = P_\perp F(\psi_1(E_{**})^{real} + a_1\tau_{**} + k(a_1, 0, E)).$$

This is well-defined because $P_\perp F(\phi, E)$ maps from $H^2(\mathbb{R}^n, \mathbb{C}) \times \mathbb{R}$ to $X_2 \cap L^2(\mathbb{R}^n, \mathbb{C})$ and for real-valued k, we have real-valued $\psi_1(E_{**})^{real} + a_1\tau_{**} + k(a_1, 0, E)$ and $F(\phi, E)$ and $P_\perp$ maps from real-valued to real-valued functions. Now

$$D_k F_\perp(0, 0, E_{**}) = L_+(\psi_1(E_{**})^{real}, E_{**})$$

is an isomorphism from $H^2(\mathbb{R}^n, \mathbb{R}) \cap X_2 \times \mathbb{R} \to X_2 \cap L^2(\mathbb{R}^n, \mathbb{R})$. By Implicit Function Theorem, $\exists \delta_1, \delta_2$ and a unique $C^{[2p]+1}$ function $\tilde{k} : (-\delta_1, \delta_1) \times (E_* - \delta_2, E_* + \delta_2) \to X_2 \cap L^2(\mathbb{R}^n, \mathbb{R})$ such that $(a_1, \tilde{k}(a_1, E), E)$ is the unique solution of $F_\perp(a_1, k(a_1, 0, E), E))$ in a neighborhood $\tilde{W} \subset (-\delta_1, \delta_1) \times H^2(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R}$ of $(0, 0, E_*)$. It gives another unique real-valued solution

$$\phi = \psi_1(E_{**})^{real} + a_1\tau_{**} + \tilde{k}(a_1, E)$$

of (5.18) when $a_2 = 0$. By uniqueness, $\tilde{k}(a_1, E) = k(a_1, 0, E)$. It follows that (6.6) is zero since $F(\psi_1(E_{**})^{real} + a_1\tau_{**} + k(a_1, 0, E))$ is real valued.

Next, consider the case $\phi - \psi_1(E_{**})^{real} \not\perp i\psi_1(E_{**})^{real}$. In this case, we can use the argument in Proposition 4.3, i.e.,there exists θ_* such that

$$\|e^{i\theta_*}\psi_1(E_{**})^{real} - \phi\|_{H^2} = \inf_{\theta\in[0,2\pi)} \|e^{i\theta}\psi_1(E_{**})^{real} - \phi\|_{H^2}.$$

Then, as we have seen in Propostion 4.3, $\phi - e^{i\theta_*}\psi_1(E_{**})^{real} \perp ie^{i\theta_*}\psi_1(E_{**})^{real}$, which is equivalent to $e^{-i\theta_*}\phi - \psi_1(E_{**})^{real} \perp i\psi_1(E_{**})^{real}$. By apply the Lyapunov-Schmidt decomposition to $e^{-i\theta_*}\phi$, we get

$$e^{-i\theta_*}\phi = \psi_1(E_{**})^{real} + \langle e^{-i\theta_*}\phi - \psi_1(E_{**})^{real}, \tau_{**}\rangle\tau_{**} + k(a, 0, E).$$

Since the right-hand-side of the above equation is real-valued, $\langle i\psi_{E_*}^{real}, F(e^{-i\theta_*}\phi, E)\rangle = 0$. To finish the proof, use the same argument in [13]. □

6.2 Unique continuation of the ground states

In this section we improve the global bifurcation result for ground states in [32]. We not only show by a different technique that the ground state branch bifurcating from zero at the lowest eigenvalue of the Schrödinger operator can be continued until $E = 0$ but also that the continuation is unique i.e., there are no bifurcations along this branch. We essentially use a comparison principle for the linearized operators combine with our continuation Theorem 4.2. While our nonlinearity is a particular example of the ones considered in [32], see also [11], we expect that the technique we use can be extended to more general nonlinearities.

Theorem 6.3. *Let $\sigma > 0$ and the potential V which satisfies (H1), (H2) and*

$$\liminf_{R\to\infty |x|\geq R} V(x) = 0, \tag{6.7}$$

$$\inf_{x\in\mathbb{R}^n} V(x) > -\infty. \tag{6.8}$$

Then the ground state branch ψ_E bifurcating from 0 at E_0 can be uniquely extended (to the left) to the maximal interval $(0, E_0)$.

Proof. Let $\lambda_i^-(E)$ respectively $\lambda_i^+(E)$, be the i-th eigenvalue of $L_-(\psi_E, E)$ respectively $L_+(\psi_E, E)$, and let $\mathcal{N} = \|\psi_E\|_{L^2}$. If (i) in Theorem 4.2 holds, we are done. Suppose (ii) in Theorem 4.2 holds i.e., ψ_E can be uniquely extended to (E_*, E_0) where $0 < E_* < E_0$ and there exists a sequence $\{E_n\}_{n\in\mathbb{N}} \subset I$ such that $\lim_{n\to\infty} E_n = E_*$ and a corresponding non-zero eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $L_+(\psi_E, E)$ or $L_-(\psi_E, E)$ such that $\lim_{n\to\infty} \lambda_n = 0$. We will show that (ii) cannot hold by showing that no eigenvalues of linearizations can accumulate to zero.

We note that $\lambda_1^+(E_0) = 0$ and $\lambda_2^+(E_0) = E_0 - E_1 > 0$ at $\mathcal{N} = 0$. We will first show that $\lim_{n\to\infty} \lambda_1^+(E_n)$ is strictly positive, so λ_n cannot be $\lambda_1^+(E_n)$ as well as $\lambda_i^+(E_n)$ for $i = 2, 3, \cdots$, by using the results in [32]. Let $S = \{(\psi_E^{real}, E) \in H^2(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R} | (\psi_E' E)$ be the branch of real-positive solution of (1.2) bifurcating at $E_0\}$ and $S_{E_*} = \{(\psi_E^{real}, E) \in S | E \in [E_*, E_0)\}$. Since $E_* > 0$, we can choose $\epsilon > 0$ such that $E_* - \epsilon > 0$. Due to the hypothesis of potential V, (6.7) and (6.8), there exists $R_\epsilon > 0$ such that

$$V(x) + \sigma|s|^{2p} \geq -\epsilon \quad \text{ for all } |x|^2 + s^2 \geq R_\epsilon^2.$$

Set $\Omega = \{x \in \mathbb{R}^n | \psi_E^{real} > R_\epsilon,\ (\psi_E, E) \in S_{E*}\}$ and suppose that $\Omega \neq \emptyset$. Then we have

$$\Delta\psi_E^{real}(x) = (V(x) + \sigma|\psi_E^{real}(x)|^{2p} + E)\psi_E^{real}(x) \geq (E_* - \epsilon)\psi_E^{real}(x) > 0 \quad \text{ on } \Omega,$$

and $\psi_E^{real}(x) = R_\epsilon$ on $\partial\Omega$. By the weak maximum principle, we get

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u = R_\epsilon$$

which implies $\Omega = \emptyset$, so that $\psi_E^{real}(x) \leq R_\epsilon$ for $E_* \leq E < E_0$ and for all $x \in \mathbb{R}^n$.

To find the standard upper bound for ψ_E^{real}, let $\eta(x) = R_\epsilon e^{-\sqrt{E_*-\epsilon}(|x|-R_\epsilon)}$. Then η is positive, continuous, $\eta \to 0$ as $|x| \to \infty$ and

$$\Delta\eta(x) = (E_* - \epsilon)\eta(x) - \frac{2\sqrt{E_* - \epsilon}}{|x|}\eta(x) \leq (E_* - \epsilon)\eta(x), \quad \eta(x) = R_\epsilon \quad \text{for } |x| = R_\epsilon,$$

Moreover, we have for $|x| \geq R_\epsilon$,

$$\Delta\psi_E^{real}(x) = (V(x) + \sigma|\psi_E^{real}(x)|^{2p} + E)\psi_E^{real}(x) \geq (E_* - \epsilon)\psi_E^{real}(x).$$

By [33, Theorem 2.1], $\psi_E^{real} \leq \eta$ for $|x| \geq R_\epsilon$. It follows that, combined with $\psi_E^{real} \leq R_\epsilon$ for $|x| \leq R_\epsilon$,

$$\psi_E^{real}(x) \leq R_\epsilon e^{-\sqrt{E_*-\epsilon}(|x|-R_\epsilon)} = \eta(x) \quad \text{for all } x \in \mathbb{R}^n \tag{6.9}$$

for $(\psi_E^{real}, E) \in S_{E_*}$.

Now, we will show that (6.9) implies the existence of the limit point of ψ_E at E_*. Since we are under the assumption that (ii) in Theorem 4.2 holds, there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $E_n \to E_*$ as $n \to \infty$ and a corresponding non-zero eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $L_+(\psi_E, E)$ or $L_-(\psi_E, E)$ with $\lim_{n\to\infty} \lambda_n = 0$. $\psi_E^{real}(x)$ is bounded in H^2 for $E \in [E_*, E_0)$ because ψ_E^{real} can be rewritten as

$$\psi_E^{real} = (-\Delta + V + E_* + i)^{-1}[(i + E_* - E)\psi_E^{real} - \sigma|\psi_E^{real}|^{2p}\psi_E^{real}],$$

and $(-\Delta+V+E_*+i)^{-1} : L^2 \to H^2$ is bounded, $(i+E_*-E)\|\psi_E^{real}\|_{L^2}$ is bounded by $\sqrt{1+(E_*-E_0)^2}\|\eta\|_{L^2}$ and $\||\psi_E^{real}|^{2p}\psi_E^{real}\|_{L^2}$ is bounded by $\|\eta\|_{L^{4p+2}}^{2p+1}$ where η is given in (6.9). Hence, there exists a subsequence $\psi_{E_{n_k}}^{real}$ and $\tilde{\psi} \in H^2$ such that $\psi_{E_{n_k}}^{real} \rightharpoonup \tilde{\psi}$ as $k \to \infty$ in H^2. In fact, $\psi_{E_{n_k}}^{real}$ converges to $\tilde{\psi}$ in L^q for $2 < q < \frac{2n}{n-4}$ ($2 < q \leq \infty$ if $n \leq 4$) by the following argument, see also [26, Lemma 1.7.2]. Fix $\epsilon > 0$ and $R > 0$ to be chosen later. Then for all $2 < q < \frac{2n}{n-4}$ ($2 < q \leq \infty$ if $n \leq 4$), we have

$$\begin{aligned} \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^q} &= \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^q(\{|x|<R\})} + \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^q(\{|x|\geq R\})} \\ &\leq \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^q(\{|x|<R\})} + \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^2}. \end{aligned} \tag{6.10}$$

The last inequality comes from Riesz Thorin interpolation theorem. By (6.9), $\psi_{E_{n_k}}^{real}(x) \to 0$ as $|x| \to \infty$ uniformly for all $n \geq 1$. Hence, we can choose R large enough such that

$$\|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^\infty(\{|x|\geq R\})}^{\frac{p-2}{p}} \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^2} \leq \frac{\epsilon}{2}.$$

Next, since $\psi_{E_{n_k}}^{real}|_{\{|x|<R\}}$ is bounded in $H^2(\{|x|<R\})$, by Rellich's compactness theorem there exists a subsequence, denoted by E_{n_k} for simplicity, such that $\psi_{E_{n_k}}^{real}|_{\{|x|<R\}} \to \tilde{\psi}|_{\{|x|<R\}}$ as $k \to \infty$ in $L^q(\{|x|<R\})$ for $2 < q < \frac{2n}{n-4}$ ($2 < q \leq \infty$ if $n \leq 4$). Thus, for large enough k, we have

$$\|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^q(\{|x|<R\})} \leq \frac{\epsilon}{2}.$$

Therefore, by (6.10), $\psi_{E_{n_k}}^{real} \to \tilde{\psi}$ as $k \to \infty$ in L^q. Furthermore, $\psi_{E_{n_k}}^{real}$ can be rewritten as:

$$\psi_{E_{n_k}}^{real} = (-\Delta + E_*)^{-1}[(E_* - E_{n_k})\psi_{E_{n_k}}^{real} - V\psi_{E_{n_k}}^{real} - \sigma|\psi_{E_{n_k}}^{real}|^{2p}\psi_{E_{n_k}}^{real}] \tag{6.11}$$

and the right hand side, $(-\Delta + E_*)^{-1}[(E_* - E_{n_k})\psi_{E_{n_k}}^{real} - V\psi_{E_{n_k}}^{real} - \sigma|\psi_{E_{n_k}}^{real}|^{2p}\psi_{E_{n_k}}^{real}]$, converges to $(-\Delta + E_*)^{-1}[V\tilde{\psi}^{real} - \sigma|\tilde{\psi}|^{2p}\tilde{\psi}]$ as $k \to \infty$ so that $\tilde{\psi}$ is in H^2 and a solution of (1.2). Indeed, we note that $(-\Delta + E_*)^{-1} : L^2 \to H^2$ and

$$\|(E_* - E_{n_k})[\psi_{E_{n_k}}^{real} - \tilde{\psi}]\|_{L^2} \leq |E_* - E_{n_k}| \cdot 2 \sup_{E \in [E_*, E_0)} \|\psi_E^{real}\|_{L^\infty} \to 0 \quad \text{as } k \to \infty$$

since $\|\psi_E^{real}\|_{L^\infty} \leq R_\epsilon$ on $[E_*, E_0)$. Also, we have

$$\||\psi_{E_{n_k}}^{real} - \tilde{\psi}|^{2p}(\psi_{E_{n_k}}^{real} - \tilde{\psi})\|_{L^2} = \|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^{4p+2}}^{2p+1} \to 0 \quad \text{as } k \to \infty$$

since $4p + 2 < \dfrac{2n}{n-4}$. It remains to show that $V(\psi_{E_{n_k}}^{real} - \tilde{\psi})$ converges to 0 in L^2. For $n \leq 4$, $[\psi_{E_{n_k}}^{real}$ converges $\tilde{\psi}$ in L^∞, so

$$\|V(\psi_{E_{n_k}}^{real} - \tilde{\psi})\|^{L^2} \leq \|V\|_{L^2}\|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^\infty} \to 0 \quad \text{as } k \to \infty.$$

For $n > 4$, the hypothesis of V, (H1), implies that there exists $r = 2 + \delta$ for some $\delta > 0$ such that $V \in L^r + L^\infty_\epsilon$. It follows that for all $\epsilon > 0$, there exist V_1 and V_2 such that $V = V_1 + V_2$, $\|V_1\|_{L^r} < \infty$, $\|V_2\|_{L^\infty} < \epsilon$. Thus, we obtain

$$\|V(\psi_{E_{n_k}}^{real} - \tilde{\psi})\|_{L^2} \leq \|V_1\|_{L^r}\|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^{r'}} + \|V_2\|_{L^\infty}\|\psi_{E_{n_k}}^{real} - \tilde{\psi}\|_{L^2} \tag{6.12}$$

where r' satisfies $\frac{1}{2} = \frac{1}{r} + \frac{1}{r'}$. By Riesz Thorin interpolation, for any $q > \frac{2n-1}{n-4}$, we have

$$\begin{aligned} \|\psi^{real}_{E_{n_k}} - \tilde{\psi}\|_{L^q} &\leq \|\psi^{real}_{E_{n_k}} - \tilde{\psi}^{real}\|^{\frac{2}{q}}_{L^{\frac{2n-1}{n-4}}} \|\psi^{real}_{E_{n_k}} - \tilde{\psi}\|^{\frac{q-2}{q}}_{L^\infty} \\ &\leq \|\psi^{real}_{E_{n_k}} - \tilde{\psi}\|^{\frac{2}{q}}_{L^{\frac{2n-1}{n-4}}} \cdot (2R_\epsilon)^{\frac{q-2}{q}} \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

It follows that $\|V_1\|_{L^r} \|\psi^{real}_{E_{n_k}} - \tilde{\psi}\|_{L^{r'}} \to 0$ as $k \to \infty$. Also, since ψ^{real}_E is bounded in L^2 on $[E_*, E_0)$ and $\|V_2\| < \epsilon$ for any $\epsilon > 0$, the left hand side of (6.12) converges 0 as $k \to \infty$, which complete the argument that $\tilde{\psi}$ is in H^2 and a solution of (1.2). Thus, we have a limit point of ψ^{real}_E at E_*, say $\psi^{real}_{E_*} \in H^2$.

Now, we assume, for the sake of contradiction, $\lim_{n\to\infty} \lambda^+_1(E_n) = \lambda^+_1(E_*) = 0$. Let ϕ_* be the L^2-normalized eigenfunction of $L_+(\psi^{real}_{E_*}, E_*)$ corresponding to $\lambda^+_1(E_*) = 0$. By the min-max principle, for any $\phi \in H^2$, $\|\phi\|_{L^2} = 1$, we have

$$\langle \phi, L_-(\psi_E, E)\phi \rangle \geq 0 = \lambda^-_1(E). \tag{6.13}$$

Therefore, we have

$$\begin{aligned} 0 &\leq \langle \phi_*, L_-(\psi^{real}_{E_*}, E_*) \rangle \\ &\leq \langle \phi_*, L_+(\psi^{real}_{E_*}, E_*) \rangle - 2p\sigma \langle \phi_*, |\psi^{real}_{E_*}|^{2p} \phi_* \rangle \\ &= -2p\sigma \langle \phi_*, |\psi^{real}_{E_*}|^{2p} \phi_* \rangle \leq 0. \end{aligned}$$

Thus, $\psi_{E_*} \equiv 0$ because if not, ψ_{E_*} is strictly positive so that $\phi_* = 0$, a.e., which contradicts the fact that ϕ_* is the L^2-normalized eigenfunction. However, $L_-(\psi_{E_*} = 0, E_*) = -\Delta + V_s + E_*$ has a negative eigenvalue $E_* - E_0 < 0$ which contradicts the continuity of eigenvalues of $L_-(\psi_E, E)$ with respect to E and the simplicity of the lowest eigenvalue of $L_-(\psi_E, E)$.

It remains to show that nonzero eigenvalues of $L_-(\psi_{E_n}, E_n)$ cannot be λ_n. Since $L_-(\psi_{E_0}, E_0) = L_0 + E_0$, $\lambda^-_1(E_0)$ is 0 at $\mathcal{N} = 0$ so that $\lambda^-_1(E) \equiv 0$ for all $0 < E < E_0$ by continuity of discrete eigenvalues, see Remark 4.2. Also, $\lambda^-_2(E_0) = E_0 - E_1 > 0$ at $\mathcal{N} = 0$. We note that the eigenfunction corresponding to $\lambda^-_2(E)$ is odd in x_1, see Proposition 4.2. By Theorem 4.2, ψ_E can be smoothly extended past E_* in the Banach space *restricted to even functions.* In particular, there exists (ψ_{E_*}, E_*) unique up to rotation *restricted to even functions.* Suppose $\lambda_2^-(E_n) = \lambda_n$. By continuity of discrete eigenvalues, $\lambda_2^-(E_*) = 0 = \lambda_1^-(E_*)$ which lead a contradiction for the simplicity of the lowest eigenvalue of the second order elliptic operator $L_-(\psi_{E_*}, E_*)$, so that λ_n cannot be the larger eigenvalues $\lambda_2^-(E_n)$. This implies that λ_n also cannot be the larger eigenvalues $\lambda_i^-(E_n)$ for all $i = 3, 4, \cdots$.

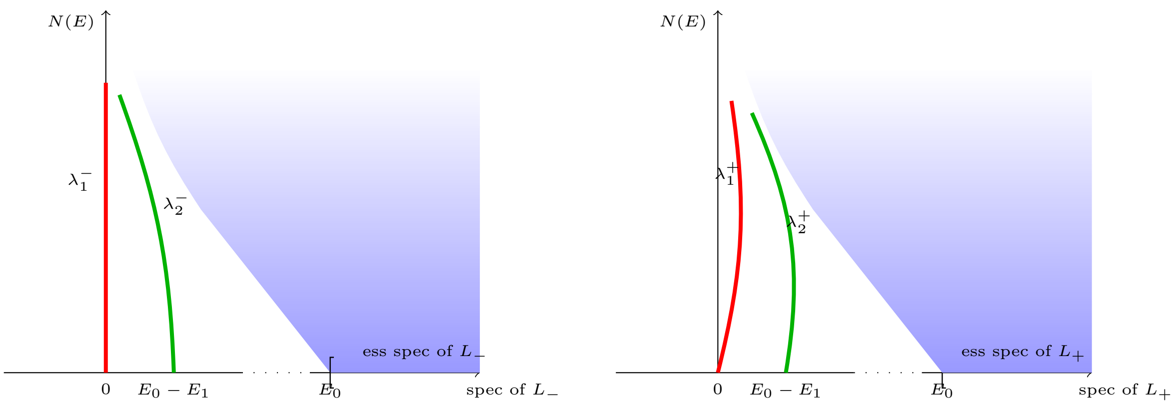

Figure 6.2: As in Figure 5.4, λ_i^- and λ_i^+ denote the i-th eigenvalue of linearization and $N(E) = \|\psi_E\|_{L^2}^2$. Similarly to Theorem 5.5, λ_2^- never crosses zero because of the simplicity of the lowest eigenvalue of elliptic second order operator. Moreover, since $L_+ - L_- = \sigma 2p|\psi_1|^{2p} > 0$, λ_1^+ and λ_2^+ must be greater than λ_1^- and λ_2^- resp., which eliminates the possibility that λ_1^+ and λ_2^+ cross zero.

□

Remark 6.1. The above result shows not only that the ground state branch reaches $E = 0$, see also [32] but also that there are no bifurcations along it. Now at $E = 0$, the linearization $D_\phi(\psi_{E=0}, E = 0)$ is no longer Fredholm as both L_+ and L_- have the continuous spectrum starting at 0, i.e., spec of $L_\pm = [0, \infty)$. What happen with the ground state at this point i.e., whether it becomes an embedded bound state or a metastable state, is left for another paper.

CHAPTER 7

CONCLUSIONS

This thesis gives several results about bound states, especially the ground state and the first excited state with double well potentials with attractive and repelling nonlinearities. In attractive cases, we attain the result that with large separations, the ground states (ψ_E, E) must undergo a secondary symmetric breaking bifurcation at some finite $E_{*,s}$. In addition, we prove that the ground state is unique up to rotation, orbitally stable and symmetric before bifurcation, and once the bifurcation occurs, it is divided into three branches (up to rotation), one of whom is orbitally unstable and symmetric while the others are asymmetric and orbitally stable for $p < p_*$ and orbitally unstable for $p > p_*$ where the nonlinearity power p_*:

$$p_* = \frac{3+\sqrt{13}}{2}.$$

Similar results have been obtained in the particular cases $n = 1$, see [13], respectively $n \geq 1$ but $p = 2$, see [14]. The latter makes crucial use of the real analyticity of the nonlinearity when $p = 2$ which is not available for our cases. [13] uses both the fact that all one dimensional bound states are real valued (up to a rotation) and that $H^1(\mathbb{R})$ embeds in $L^\infty(\mathbb{R})$. In the $n > 1$ case, we show that bound states which cannot be rotated into real valued ones do not appear near branches we study by employing a Lyapunov-Schmidt decomposition with symmetry which consistently modes out rotations. Moreover, we employ an elliptic regularity type argument to overcome the fact that $H^1(\mathbb{R}^n)$ might not embed into $L^\infty(\mathbb{R}^n)$. The techniques developed in this thesis form the basis to understanding the effect of general nonlinearities on bound states and their bifurcations e.g. (1.2) with $|f(y)| \leq C_1 y^{2p_1} + C_2 y^{2p_2},\ y > 0,\ p_1 \leq p_2$

and $\lim_{y\to 0} f(y)/y^{2p_1} = \sigma$, see [10].

The other important result in the attractive case is that the first excited state branch can be uniquely continued to large amplitude, where their L^2 and H^2 norm are at least of order 1, while the ground state already bifurcates at small amplitude. To determine whether further bifurcations occur in the large amplitude regimes requires non-perturbative techniques as opposed to the perturbative ones developed in this thesis, see the recent progress in this direction [15] and [34].

In the case of repelling nonlinearity we obtain the existence of secondary bifurcation along the first excited state at small amplitude and the non-existence of secondary bifurcations along the ground state. The difference from attractive case is that we can show that the ground state branch can be uniquely continued until it reaches the boundary of the Fredholm domain i.e., until $E = 0$. The behavior of the ground state near $E = 0$ where 0 is in the essential spectrum of the linearization remains an open problem.

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