The Gedanken Ball-and-Stick Construction Problem: What is the Most Simple Structure that it is Possible to Construct?

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ABSTRACT
A very simple question is posed: Employing a ball-and-stick modelling system, and given a supply of the component balls and rods, then, treating it as a gedanken experiment, what is the most simple structure that it is possible to construct?

Note: This paper is the first of three linked papers, the other two being:

[**2**] Icosahedral Quasicrystalline Structure Modelled as a Dynamically Updating System.

[**3**] A Structuralist Proposal for the Foundations of the Natural Numbers.

1. Introduction

Ball-and-stick models are most commonly associated with applications in chemistry where they are used to construct molecular models (e.g., [**1**]). In this paper, however, we are employing the gedanken ball-and-stick model to investigate "structure" in the abstract. This involves a conventional ball-and-stick model that is assembled in the conventional manner, except that it is specifically a *gedanken* model where an idealised constructor assembles the components as a gedanken experiment, not bound by the constraints of what can actually be physically constructed, but bound by the constraints of what can conceivably be physically constructed.

Deciding whether an object is "simple" or "complicated" can be a matter of interpretation. In this context an informational approach is adopted, meaning that we define the most simple ball-and-stick structure to be that which can be constructed, requiring the minimum amount of prescriptive information. It is known (more formally from algorithmic information theory) that an object generated from minimal information input can yet, in some cases, produce a large information output. In this case we find that the most simple ball-and-stick structure that it is possible to construct, although requiring minimal prescriptive information input, is not a mundane object.

In this paper we claim to answer the question posed in the title. And to the extent that the ball-and-stick modelling system is inherently suited to exemplifying a very simple structure, then the model constructed with this system provides an initial basis from which analogies with respect to fundamental structure, more generally, can be drawn. To quote Shapiro, "the role of concrete and quasi-concrete systems is the motivation of structures and the justification that structures with certain properties exist" [**4**]. The discussion of the broader implications, however, is left to associated paper [**3**]. And for a more detailed description of the geometry of the ball-and-stick construction, we refer to [**2**].

The work presented in this paper is limited to outlining the construction of the ball-and-stick model that is produced in response to the question posed in the title.

2. The Basic Setup

2.1 Components, instruction, objective

There are three main features that constitute the basic setup:

1) *Components*. There are two bins; one is full of identical balls, the other is full of identical rods.
2) *Instruction*. There is the fundamental initiating instruction that tells the constructor to assemble balls and rods.
3) *Objective*. The stated objective is to construct the most simple structure possible.

2.2 Definitions of "most simple" and "structure"

Most simple: We take an informational approach to defining "most simple". The degree to which something is considered to be most simple is proportional to the amount of prescriptive information that is required to produce it (e.g. the amount of instruction required, or the length of the algorithm written). This is an informal approach that deals with information colloquially as semantic content. Instructional information (inputs) and factual information (outputs) are recognized as ordinary language statements.

The statements in subsection 2.1 above constitute the base stratum of information that is required to set up this paper's construction problem.

Structure: We employ the two main common usage definitions of structure: 1) An object that comprises multiple elements; the *structure-object*. 2) The organisation or arrangement of elements such that there are relations between them; the *structure-organisation*.

2.3 The informational ground state

The above two definitions of structure refer to *elements*, and *relations*. In this setup the *elements* are exemplified by the collection of identical balls, and the *relations* are exemplified by the collection of identical rods (that embody the relations between pairs of balls).

The component balls and rods are reproduced so that in each case there are multiple identical copies that individually contribute zero new information content. It is only if the constructor is forced to add special features to the components, requiring prescriptive information input, that they would then be imbued with additional information content – in which case the structure constructed from those components would be no longer be at the maximally information entropic ground state, and would not be optimally most simple.

2.4 Summary

The idealised constructor is supplied with a bin of identical balls and a bin of identical rods, which (along with the instruction to assemble these components) represents the "basic setup". The objective now is to assemble the most simple structure possible. If it becomes necessary to customise the balls or rods, to add special features or specific requirements written over top of their basic description of "identical", that would be to import additional prescriptive information into the assembly process. This would in turn raise the resulting structure above the maximally information entropic ground state.

It is only the model assembled with no additional information input (over and above the "basic setup") that is at the maximally information entropic ground state; and it is only that model that we can assert is definitively the most simple structure.

3. Begin Assembly

First we trial the most obvious assembly method whereby the idealised constructor assembles the ball-and-stick model in a stepwise process, accreting a new ball and linking rods with each step to grow the structure. The first ball is labelled *origin ball O,* and subsequent balls are labelled alphabetically (see Fig. 1).
Note: We adopt some terminology from graph theory.

3.1 Steps 0 to 4

Step 0: There is the origin ball *O* (Fig. 1 (a)). This cannot, in itself, be the most simple structure because, by definition, a single element does not constitute a structure.

Step 1: Ball *A*. The default assembly action attaches *A* to *O* with linking rod *OA* (Fig. 1 (b)). This produces the first example of a structure. In some contexts (and perhaps intuitively) it is considered that the prototypical most simple structure can be defined as two identical coupled elements – which definition would be embodied in the ball-and-stick model at this stage, and thus the stated objective already met.

However, in the case of the informational approach employed here, and given that the core "basic setup" necessarily included the fundamental instruction to "assemble" (subsection 2.1, note 2), then in order to terminate construction of the ball-and-stick model at some specific number of balls it would be necessary to input information prescribing that specific cardinality, along with the explicit termination instruction.

The resulting model would then involve prescriptive information content in addition to the optimal ground state, which would mean that it could no longer be confidently argued that it is definitively the most simple structure. This can be referred to as the "no instruction to terminate" problem. At this stage, then, we allow that the assembly process (at each step) extends, potentially indefinitely.

Step 2: Ball *B:* There are three different configurations in which ball *B* and the associated rod(s) can be attached to the Step 1 structure: One possibility is the cyclic configuration *OAB* shown in Figure 1 (c). The other two possible configurations are acyclic structures in which the rods *AB* or *OB* are pendent rods.

For each of the possible configurations we can, of course, count how many rods attach to each constituent ball (i.e., the degree or valence of each ball) which can now be described as a "feature" that the balls have acquired (the *only* distinguishing feature).

It is now possible to discern between the two types of configuration, cyclic and acyclic; in the cyclic configuration the constituent balls are uniformly degree 2, whereas the two acyclic configurations would produce a non-uniform degree count, 1.2.1.

Non-uniform, acyclic configurations would require specific information inputs to prescribe the number of rods that should attach to each ball for each unique instance. A cyclic configuration, on the other hand, produces uniform degree such that the constituent balls remain identical, meaning that no instruction specific to any unique ball is required. Informationally, a cyclic configuration is the most simple, consequently cyclic configurations are selected for.

Summary: The description of a ball's relations to nearest neighbour balls is the only distinguishing feature that the otherwise featureless, identical balls acquire. And these relations with neighbours are embodied in the linking rods, and can then be expressed as the ball's "degree". On this basis, when selecting for optimal structures a key criterion will be *maximally uniform degree count*.

Note: Balls are labelled alphabetically, however, the order in which they appear is prioritized with respect to an arrangement in the completed construction that becomes relevant later. At this stage ball *B* is followed by ball *G*.

Step 3: Ball *G* is attached to the Step 2 equilateral triangle *OAB* with linking rods *OG, AG,* and *BG* (Fig. 1 (d)). Of the various possible arrangements, the uniform degree requirement determines that the resulting configuration is necessarily tetrahedron *OABG,* where every ball is degree 3.

Step 4: Ball *C* is attached to the Step 3 tetrahedron *OABG* with linking rods *OC, BC,* and *GC* to form the triangular bipyramid (Fig. 1 (e)). This configuration produces the best possible uniformity of degree, however, no configuration of five balls, following on from Step 3, can produce uniform degree count across all balls. In this case two balls are degree 3, and three balls are degree 4.

3.2 Steps 5 to 12

Note: Steps 5 to 11 are not individually discussed, nor are the images of the models show, but the initial trial assembly is described here generally below.

Continuing to draw from the bins of identical balls and rods, the constructor carries on assembling the most simple ball-and-stick structure through Steps 5 to 12 on the basis of trialling configurations, primarily to find the structure that produces maximal uniformity of degree across the collection of balls.

In subsection 3.1 it was noted that cyclic configurations produce optimal uniformity and this is evident also through Steps 5 to 12 where, in three dimensions, the criterion of uniform degree count biases the construction toward producing a cluster configuration such that balls form in a shell about central origin ball *O*.

However, from Step 5 onward the assembly produces no configurations that have uniform degree-spread. Even the best possible constructions at each step tend toward a divergence away from uniform degree count, until Step 9, at which point configurations track back toward producing the required uniform degree-spread across all vertices, until the best possible configuration is produced at Step 12. See Figure 2.

Step 12: The process of assembling balls to the cluster formation completes the peripheral shell so that there is the central origin ball *O* and twelve peripheral balls, *A* to *L,* and no further balls and associated linking rods can be fitted. This configuration produces uniform degree across all peripheral vertices (degree 6). Only origin vertex *O* remains non-uniform (degree 12).

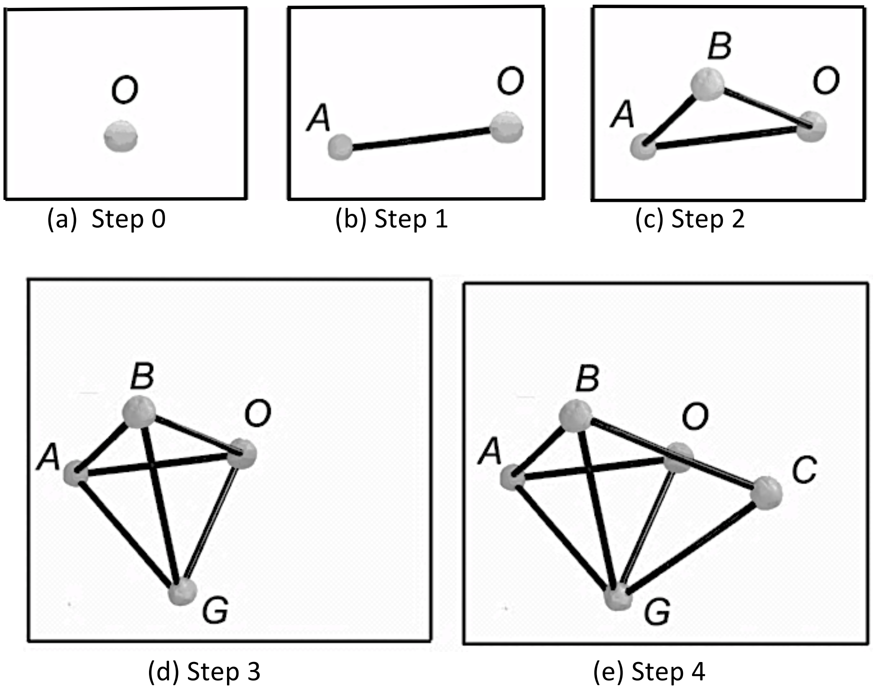

Figure 1. Initiating the ball-and-stick construction. Note that, naturally, some rods appear foreshortened in this 2D image of the 3D model. Also, in (e) rods *OB* and *OC* are obscured from view by rod *BC.* The alphabetical labelling of the balls is prioritized with respect to a circumference route about the complete figure (see Fig. 2)*.* In the assembly process from step 3 onward each new ball and the associated rods attach to the face of a tetrahedron to form a new tetrahedron.

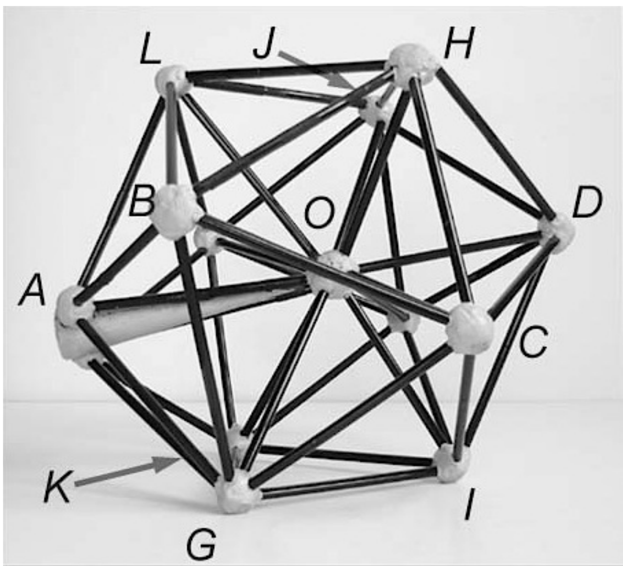

Figure 2. Regularised IQC. The ball-and-stick construction produces a skeletal polyhedron made up of 20 tetrahedra clustered about *O.* 12 rods project radially out from the central origin ball *O* (aligned with the six icosahedral symmetry axes) to the 12 peripheral balls labelled *A* to *L*. In the regularised model the radial rods are shortened slightly so that the 12 peripheral balls (connected by 30 unit length rods) form a regular icosahedron.

4. Two Significant Problems

Looking at the configurations trialled up to this stage, at Step 12 the cluster model is the best possible configuration – however, there are (obviously) two significant problems:

1) **Geometrical frustration.** Contrary to what is shown in Figure 2, the balls and rods in the cluster configuration cannot in fact all connect as required – the construction is geometrically frustrated.
2) **Non-uniform degree count.** For the intermediary Steps between 3 and 12, no configuration of the growth model trialled could produce uniform degree count for all of the constituent balls. Although the situation is improved at Step 12, it remains unacceptable that whole stages of the growth model fail to meet the main criterion of uniformity, meaning that specific instruction would have to be written to cover those examples. (In the worst case, at Step 9 the degree count ranged between 4 and 9.)

   At Step 12 the best possible configuration with respect to uniform degree count is produced (i.e., the completed cluster configuration: all peripheral balls = degree 6), however, a residual non-uniformity persists (origin vertex *O* is degree 12).

The geometrical frustration problem is discussed in the following Sections 5 and 6.
The non-uniform degree count problem is discussed in Section 7.

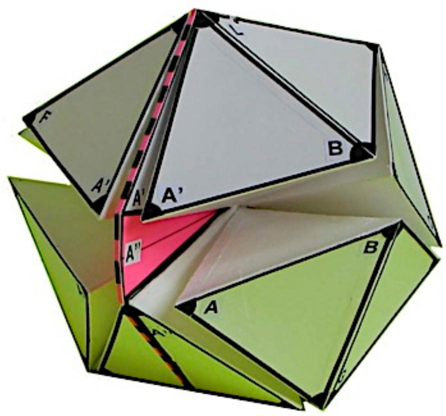

Figure 3. The frustrated IQC

5. The Icosahedral Quasicrystal (IQC) and the Geometrical Frustration Problem

The criterion of maximal information entropy has steered the construction toward producing the cluster configuration that comprises the central origin ball *O* and twelve peripheral balls (and associated rods). These components have arranged naturally in a formation with icosahedral symmetry that is known in materials science as the *icosahedral quasicrystal,* IQC [**5**]. See Figure 2.

In the context of the ball-and-stick construction, assembly of the IQC can be described as a clustering growth model where the rods and balls form twenty tetrahedra that share common central ball *O*. However, the dihedral angle of a tetrahedron is not commensurable with 2π, consequently the first of the problems that was signalled in the previous section is now evident; very simply, in the ball-and-stick construction not all balls and rods connect (contrary to what is shown in Figure 2).

Because it is difficult to clearly show a non-connecting ball-and-stick model, it is constructed here in two versions: The first version, shown in Figure 2, is constructed as a regularised approximant; i.e., the rod lengths of the ball-and-stick model are adjusted so that the twelve peripheral balls and associated rods can connect to form a regular icosahedron. The second version, Figure 3, is constructed as a solid model – this more readily illustrates the non-connecting, frustrated structure in one example of the various configurations that may actually result from trying to assemble identical rods and balls (as the problem properly requires).

Propagating the ball-and-stick model in three-dimensional space is essentially a variant of the age-old sphere-packing problem that can be found as far back as Sanskrit writings 499 AD [**6**]. (In this context here the balls in the model can be thought of as the centres of identical, kissing spheres.) In which case, to find that the assembly process runs into the geometrical frustration problem is not exactly unexpected.

Nevertheless, although the problem is well understood, in the context of the ball-and-stick construction there is yet some naïve residual expectation that it should be possible to assemble the balls and unit length rods to grow the model indefinitely; essentially we expect to propagate congruent tetrahedra to tile the volume of space (as in fact Aristotle famously believed was possible) as a natural extension of the way that the plane can be tiled with equilateral triangles.

In very simple terms, a linear extension of the balls spaced apart by unit length rods can equate with the marks on a ruler; then, looking at the problem in this context is to conclude that if conventional measurement principles hold with respect to certain spatial axes, there are certain other axes in that same space where those same principles are not commensurable. Or, broadly, the measurement system that works fine in one direction in space cannot simultaneously work fine in this other direction in that same space. This is, of course, rooted in the well understood commensurability problems that classical mathematics has long since dealt with; nevertheless, when considered as a very basic measurement problem, un-finessed, the situation can seem untenable.

The problem of filling space with the ball-and-stick construction intuitively feels like a very basic problem that should have a solution in an equally basic, canonical, model. In the context of this paper, what started as a straightforward proposal to assemble the balls and rods into a structure became entropically biased toward cluster formation with icosahedral order, thus producing the IQC that is geometrically frustrated, and where, in the extended structure, periodicity is lost in all three dimensions. So the structure is now difficult to describe in three-dimensional space, and it is challenging to find the correct mathematical formalism to express the ideal model. Current approaches include integrating concepts from Penrose tiling, Fibonacci series, and projections of lattice points from hyperspace to physical space (for overview see [7]), all of which are complicated and there is no canonical model.

6. Resolving the Geometrical Frustration Problem

5.1 Considering the options available

If there was a willingness to relax the informational strictures and concede to axiomatically introducing additional prescriptive information to modify the settings as required, there would then be several approaches to resolving the frustration problem that otherwise stymies the assembly of the ball-and-stick model in three-dimensional space.

For example, we could write instructions into the assembly process that would adjust rod length so that the regularised version of the IQC shown in Figure 2 is constructed. Essentially, rods forming the peripheral shell need to be slightly longer than the radial rods projecting out from the centre. However, we know that for any structure where icosahedral order is imposed locally such that five tetrahedra share a common edge, it is impossible to fill space even if it is allowed that the tetrahedra may be severely distorted. A considerable amount of instruction would have to be written over top of the optimal, maximally information entropic ground state.

Another approach may resort to constructing the ball-and-stick model, configured as the common cubical lattice – essentially producing the conventional mapping of space with orthogonal coordinate surfaces. A cubical lattice, however, lacks rotational symmetry and, essentially, some background notion of orthogonality would have to be written into the system. Clearly this approach would also require inputting a large prescriptive information content.

Rather than capitulating to these types of approach, we maintain the original resolve to construct the ball-and-stick model without increasing the prescriptive information content of the system above the maximally entropic ground state. If successful, this will allow us to confidently argue that the structure produced is the most simple possible.

The gedanken ball-and-stick construction problem can now be viewed as the problem of constructing a concrete model that must resolve the mathematical problem of geometrical frustration and translation asymmetry in the global model of an icosahedrally ordered lattice structure, or, the problem of producing the ideal model for the icosahedral quasicrystal.

The unifying character of the currently favoured approaches to this problem (for an overview, again see [7]) is that generally they involve an ideal model that is theorized to live in an unphysical higher-dimensional space from which Euclidean space can be recovered. Hyperspace is of course outside of the reach of intuitive visualisation or exemplification in any concrete model such as could be constructed with our ball-and-stick modelling system.

6.2 Introducing the ball-and-stick model constructed as a dynamically updating system

This is a critical juncture: In this paper's setting the question that we are seeking to answer is specifically concerned with construction of the ball-and-stick physical model, consequently the preferred conventional approaches that appeal to a non-physical hyperspace are not available. *As an alternative approach we intend to argue that the ideal model is necessarily a dynamically updating system.*

Unlike the various hyperspace approaches, the dynamically updating system is amenable to construction in a gedanken ball-and-stick model. That development is discussed in Section 9. Before dealing with that, however, we will first discuss the non-uniform degree count problem (i.e., the second of the "two significant problems" mentioned in Section 4).

At this stage we signal ahead that the geometrical frustration problem is resolved in the ball-and-stick model constructed as a dynamically updating system. With that provisional understanding in place, in order to conceptualise the ongoing construction we continue to employ the regularised geometric structure that gives the static approximant of the IQC (e.g., Fig 2). In other words, we now have to remember that as the focus returns to the non-uniform degree count problem, the static approximants that are referred to in the following sections are temporarily standing in for the ball-and-stick assemblage that should more correctly be modelled as a dynamically updating system.

7. Returning to the Non-uniform Degree Count Problem

7.1 Quantum growth mode

To recap: The trial construction that was initiated at Section 3 started from origin ball *O* and assembled the model by a stepwise accretion balls and rods to form the cluster configuration that auto-completed (i.e., all available spaces were filled) at the shell of the IQC. However, in that construction process, steps in the assembly of the model involved stages where a large amount of prescriptive information would have to be written to resolve the obvious non-uniform degree count problem.

Given the above, as an alternative assembly mode it is proposed that the idealised constructor must assemble the model as a quantum jump from Step 0 (origin ball *O)* directly to Step 12 (origin ball *O* and the full array of twelve balls that complete the peripheral shell). This bypasses the steps that most egregiously exhibit the non-uniform degree problem, bringing the model construction to the completed IQC that, although not yet perfect, is the best-case configuration.

Intuitively, of course, our notion as to what constitutes the most simple method of assembling the ball-and-stick model is influenced by our human constructor's physical ability to construct the model, recognizing that to assemble the twelve peripheral balls in unison would be physically difficult. Superficially then, it may also seem that writing the instruction for the idealized constructor to perform the quantum jump mode of assembly would require a longer algorithm than would be the case for the stepwise assembly process. That intuition brings with it a sense that the stepwise assembly is the default mode over top of which the quantum jump mode would have to be written-in, requiring additional information input.

None of that intuition is, of course, necessarily correct. The basic setup includes the initial instruction to assemble the components. But it is logical to presume that an unbiased constructor will assemble components undiscerningly across all degrees of freedom available, as opposed to selecting specific orientations. The latter would imply an overt action requiring information input.

The only instruction that determines the construction mode is stated in the objective as initially given in subsection 2.1 note 3. The objective states that the structure produced should be the most simple, thus requiring the least prescriptive information input, which implies the growth mode that involves only uniform components – specifically in this case balls with uniform degree. And the assembly mode that best satisfies that criterion is that which proceeds as a quantum jump from Step 0 to Step 12.

7.2 The fundamental building block

At subsection 3.2, Step 12, the stepwise assembly process completed the peripheral shell of the IQC, indicating that a phase of the construction had auto-completed (there were no more ball positions available within the shell). With respect to an entropically driven growth model that is thus biased toward clustering about a central origin ball, the IQC is the resulting structure.

Growth is now revised so that the previous stepwise construction process is now realised, more correctly, to be a quantum growth mode that, in a single step, assembles the peripheral balls to complete the first shell – and as such, that phase of the assembly process can be considered to have auto-completed. These considerations define the IQC as the first "quantum" building block.

To summarise: The assembly necessarily progressed directly from the origin ball *O* to the IQC completed unit as the first building block.

7.3 Penetration twin assembly mode

At this stage we have concluded that for origin ball *O* there is not a first successor ball in the assembly process, but rather, the successor step necessarily jumps to the completed IQC. And given the requirement of uniformity of characteristics across all of the balls in the model, we can also conclude that every ball is equally a central ball about which the model assembles in this quantum growth mode. This argument tells us, first, that *the growth model is to be thought of as an accretion of IQC building blocks,* and second, it also gives us the assembly mode of those building blocks.

Perhaps the first intuition is to assemble the IQC building blocks in non-overlapping contiguous junction, however, the above arguments tell us that in fact they must self-assemble (i.e., assemble in the maximally information entropic formation mode; see also [**5**]) in conjoined, penetration twin junction – or more specifically, in *multiply* conjoined, penetration twinned junctions.

Each of the twelve peripheral balls that make up the first shell of the first IQC that formed about central origin ball *O* is now, in a second phase of growth, the new central origin ball about which a new IQC forms in penetration twin junction with the first central IQC and with each of the neighbouring IQCs. Just as the first phase of the model assembly proceeded as a quantum growth step, so also a second phase of growth proceeds as a quantum growth step in which twelve IQCs are accreted to the initial, central IQC, multiply penetration twinned, forming the second shell about the origin ball *O*.

As discussed above, the penetration twin growth mode produces the required identicalness, or uniformity, across all of the balls. In the second phase of growth, the balls that make up the first shell about the first origin ball *O* are no longer peripheral balls, but they become the origin balls of the penetration twinned IQCs, and are thus now, like the origin ball *O,* also degree 12. This is the first part of the solution to the non-uniform degree count problem.

The multiply penetration twinned growth that accretes twelve new IQCs to the structure resolves the problem of non-uniform degree count with respect to the initial, central IQC; but it also produces, of course, a new peripheral shell filled with balls that are degree 6. So although the initial problem, as strictly defined, has been resolved, it is also the case that the problem has just been deferred. Any cluster configuration with an edge will result in a disparity of degree count between interior balls and boundary balls. This will only be resolved if a structure can be constructed (in this setting of the concrete ball-and-stick model) that is without an edge or boundary.

8. Quasicrystaline Pentakis Dodecahedron (QPD)

The construction of the ball-and-stick model can be described as three distinct phases:

Phase I: Assembly of the ball-and-stick model began at Step 0 with the origin ball *O*.

Phase II: There is the "quantum" jump from Step 0 to Step 12, which completes the assembly of the first shell that is composed of twelve peripheral balls in icosahedral configuration, thus producing the IQC (Fig. 2). The IQC is the minimum unit (i.e., wherever there is a ball, there is necessarily an IQC) and is the fundamental building block for all further construction.

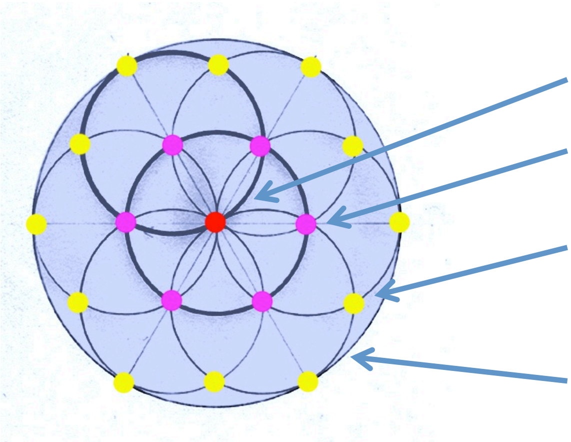

Figure 4. QPD (Note: This is obviously an illustration, not true projection)

Phase III: The Phase II configuration produced icosahedrally ordered structure, and the idealized constructor continued to assemble the structure, attaching in a quantum growth mode twelve penetration twinned IQC building blocks (i.e., each of the central IQC's twelve peripheral balls become the origin ball for the twelve new, penetration twinned IQCs) to form the second shell. As was the case with the first shell that completed Phase II, now the second shell auto-completes Phase III construction.

At the completion of Phase III the structure comprises 13 balls from Phase II, and 32 balls from the Phase III penetration twinned IQCs, for a total of 45 balls. The 32 surface balls form a pentakis dodecahedron.

In geometry, a pentakis dodecahedron is a dodecahedron with a pentagonal pyramid covering each face, and this Phase III figure is characterised as "quasicrystaline" because it is not a geometric solid, but rather, it includes the interior construction that causes it to be geometrically frustrated, and it has icosahedrally ordered structure. So the completed figure (that includes the surface balls as well as those of the interior structure) is referred to here as the *quasicrystaline pentakis dodecahedron*, or QPD.

9. Returning to the Ball-and-stick Model as a Dynamically Updating System

To recap: In subsection 3.2, Step 12, the assembly of the ball-and-stick model completed the shell of the IQC. Section 4 acknowledged two significant problems – one being the inherent geometrical frustration. Section 6 discussed optional approaches to resolving the problem. The favoured conventional approaches commonly theorise unphysical constructions that live in hyperdimensional spaces. Those approaches are obviously not accessible to the physical ball-and-stick construction.

As an alternative approach, at subsection 6.2 the concept of the ball-and-stick model as a dynamically updating system was introduced. In fact, all of the structure produced in the previous sections has been based on the IQC building block for which the static regularised approximant has been standing in for the actual dynamically updating model, ahead of the discussion in this section.

The concept of the model as a dynamically updating system is straightforward: Figure 3 showed the IQC constructed as a solid model, clearly demonstrating the geometrical frustration that prevents the ball-and-stick model from being assembled with the balls and rods completely connecting in the manner required.

The general concept of the dynamically updating system applied to this model, works on the basis that where there is the frustration gap, e.g., Figure 3, at position *A,* rods can be pulled together such that a ball can be fixed at that position – but that is only possible if it is granted that rods are, as a consequence of that action, also pulled apart from a ball at a neighbouring position. That ball can no longer be fixed at that neighbouring position until the gap there is pulled together, and the domino effect perpetuates this attaching and detaching action cyclically around the configuration. We refer to this action as having the effect of "creating" and "annihilating" balls.

The proposal put forward here is that the geometrically frustrated, asymmetric configuration that the static construction has produced (e.g., Fig. 3) is smoothed out in the dynamically updating system so that the IQC and the QPD are, in the mean, centrally symmetric structures. The ball-and-stick construction becomes an animated model in which, triggered by a sequential domino effect, balls and rods are clicking together and pulling apart, created and annihilated, in rapid succession throughout the complicated system so that, as the idealised constructor views it, the components average out to the blurred image of a centrally symmetric structure.

That is a reasonable image of the model, however it is not the only one. Especially, it is not implied in the ball-and-stick model that the update process is necessarily dynamical in the traditional sense that it is evolving over time, either continuous or discrete time. The information in the system has not implied or required the notion of some background metronome that the process conforms to; nor has it implied that the update process is inherently continuous.

The concept of the dynamical update system requires only that where there are frustration gaps in the existing structure, the idealised constructor can pull rods and ball together to fix the ball in that position; but accepting that this is only possible at the expense of pulling neighbouring ball and rods apart so that that ball is no longer fixed in place. It subsequently follows that the constructor must at some later stage then repair the connection to that second ball; however, there is no sense of "when". The domino update process is causally driven, not temporally driven.

The above outline has broadly introduced the mechanics of the dynamically updating ball-and-stick construction. However, it is also a requirement that the system acquires this capacity to update for free (i.e., for zero prescriptive information input); whereas, superficially at least, it would appear that introducing the update system will require considerable prescriptive information input.

As the idealised constructor stands at his workbench and assembles the balls and rods, commonly, the first intuition is that the static model is the default most simple setup over top of which considerable prescriptive information input would be required to animate this construction to produce the dynamically updating model described above.

It is true that the concept of the dynamically updating system has been introduced in order to resolve the geometrical frustration problem, but it is not one proposal, or philosophical approach, hypothesised from a wide range of available approaches. For any proposed system, the constructor is in fact, in all cases, forced to decide on one of two alternatives – the system is static, or it is dynamical.

The idealised constructor may well ask, in what sense, actually, or conceptually, is the ball-and-stick construction on the workbench inherently static? It is of course composed of dynamically updating atomic and subatomic constituents that are all together rotationally shifted on the surface of the Earth that is orbiting the sun that races around the galactic centre that is moving toward some distant point in the universe. The concept of the static model was, of course, contingent upon an artificially contrived frame of reference. And conceptually, it is not essentially static either.

So we cannot make the traditional assumption that our model is by default static, but rather, we are forced to make the argument equally for either static or dynamical; and – colloquially at this stage – there is every reason to argue that the default ground state model is a dynamical system from which the static model can be abstracted at the expense of some information input.

Just as it was necessary to revise the initial intuition and consider that the default construction mode necessarily propagated structure with respect to all degrees of freedom equally in quantum growth, so also any initial intuition that the default most simple structure is necessarily a static structure, is revised in favour of the dynamical alternative.

10. The Boundary Problem Resolved in the Dynamically Updating QPD

Recap: Phase I consisted of the origin ball *O*. Phase II proceeded as a quantum jump that filled in all ball positions available in the first shell, thus producing the fundamental building block, the IQC. Phase III proceeded as the quantum jump that accreted a ring of penetration twinned IQC building blocks, filling in the second shell that gave us the QPD.

At the completion of Phase II the non-uniform degree problem persisted residually, evident in the degree-disparity between the central origin ball *O* (degree 12) and peripheral balls (degree 6), prompting the ensuing Phase III construction – which did solve the immediate problem, but only, of course, by transferring it from the periphery of the IQC, to the periphery of the QPD. The balls at the new periphery are once again degree 6, while interior balls are degree 12. Essentially, it is clear that the most simple ball-and-stick construction must have no edge.

If we wish to construct the QPD as a finite space that has no edge, we have, of course examples of such – a common one being the 3-sphere. However, all such topological constructions are hyperdimensional and require the abstract notion of gluing, both of which features have no direct physical analogue and as such are obviously not available to the concrete ball-and-stick construction. The proposal now is that, just as the concept of the dynamically updating system has provided a physical model at that point where conventional approaches might appeal to higher dimensions, so also treating the ball-and-stick construction as a dynamically updating system offers a "physical" solution to the present boundary problem.

This can be conceptualised as a gedanken experiment in which the intrinsic idealised observer (as distinct from the "constructor") takes a (graph-theoretic) walk through the ball-and-stick construction. Starting at the central origin ball *O,* the idealised observer's first step arrives at a ball on the first shell, or on the periphery of the IQC. The observer's walk can logically only include balls that are fully connected to rods, or, can include only "created" balls but not "annihilated" balls. It is conjectured that the idealised observer's walk is necessarily correlated with ball creation – the walk can only proceed on the basis that the inherently dynamical system updates the approached ball-rod connection to "created" status.

This progression of the idealised observer, transitioning from the centre toward the boundary, involves a system wherein the geometric frustration gaps are closed as balls and rods are pulled together ahead (ball creation) causing gaps to open where balls and rods are pulled apart behind (ball annihilation), with the effect that in relation to the overall QPD structure, the idealised observer remains always at a central origin vertex *O.* It is proposed that when the mechanics of the idealised observer's transition through the structure are fully developed, it will be seen that this process will necessarily induces a correlated churn of balls and rods being zipped together ahead and unzipped behind such that the observer appears to drag the QPD structure along, while always remaining at the centre.

11. The Quintessential Most Simple Ball-And-Stick Structure

11.1 Review

The "basic setup" (as first outlined in Section 2) provided the base stratum of information necessary to initiate this paper's construction program.

In the previous sections we've seen that the assembly process organises into three phases:

Phase I

- Step 0 introduced the origin ball *O,* the first ball – this does not meet the definition of "structure".

- Step 1 introduced the second ball so that the structure now comprises two identical balls coupled with a rod. This is not, however, the most simple structure because the “no instruction to terminate” problem (subsection 3.1) cannot be resolved for this construction; therefore at this stage the assembly process is necessarily indefinitely extending.

Phase II

- The initial trial of the stepwise assembly process is revised in favour of a quantum growth mode in which twelve balls complete the first shell, producing the IQC. Note: This is not a case of “prescribing” quantum growth; rather, it is a case of realising that, informationally, centrally symmetric quantum growth is the default growth model.
- It is hypothesised that in order to resolve the geometrical frustration problem, the structure is necessarily, by default, a dynamically updating system.
- The IQC fails to be the most simple structure because of the “non-uniform degree” problem and the “no instruction to terminate” problem.
- The IQC is, however, identified as the fundamental building block.

Phase III

- Twelve penetration twinned IQC building blocks are assembled to the model in a quantum jump, with the peripheral balls forming the second shell. The twelve IQCs auto-complete the second shell, producing the QPD (Fig. 4).
- The ball-and-stick QPD structure, as a dynamically updating system, resolves the geometrical frustration problem.
- It is also conjectured that the dynamical update feature produces an evolution of the structure such that for an idealised intrinsic observer the informationally finite QPD structure has no edge or boundary (Section 10); thereby resolving the “non-uniform degree” problem.
- This leaves only the “no instruction to terminate” problem outstanding.

11.2 The “no instruction to terminate” problem

The problem that the constructor is faced with is that the default instruction is to assemble the ball-and-stick model; consequently, to stop assembling requires an instruction input to that effect, probably requiring also a statement to prescribe the cardinality. This would increase the information content above the maximally entropic ground state, in which case we could no longer be certain that the model is the most simple.

It follows that the idealised constructor is necessarily bound to continue assembling the model indefinitely, which presents the obvious problem that the undertaking is never completed. In a conventional growth model the constructor could never point to a definitive model that is the most simple structure that it is possible to construct.

11.3 The appearance of fractal structure

Reluctance to capitulate to writing the additional prescriptive information that would be required to terminate the construction process means that, obviously, assembly of the ball-and-stick model continues on the basis laid out up to this point. Fortunately, however, the idealised constructor observes (without, obviously, inputting any directing instruction) that a fractal character naturally begins to appear in the structure of the ball-and-stick model.

The completion of the second shell that produced the QPD initiates a phase transition with the effect that further assembly of the ball-and-stick model does not produce a third shell, does not produce novel structure, but rather, further construction iterates the QPD structure through successive fractal layers. The iterative fractal character of the indefinitely extensible construction process means that structure that is produced at any future stage is fully represented in the first QPD.

11.4 Summary

At this stage it can now be claimed that the dynamically updating QPD configuration, as reviewed above, and with the fractal character of the structure resolving the "no instruction to terminate" problem, is the definitive most simple ball-and-stick structure that it is possible to construct.

The following section will elaborate on the fractal character of the structure, and additional features of the QPD model become evident (most notably a ubiquitous helicity) as the extended, fractally-layered ball-and-stick model assembly is continued.

12. The Extended Ball-and-Stick Model

In the previous Sections the idealised constructor has completed the assembly of the QPD. Now the assembly of the three-dimensional ball-and-stick structure beyond the first QPD is outlined, referring to the illustration in Figure 5.

Note: For a more detailed description see also [**2**] Section 8.

12.1 Fractal structure

First, growth up to the stage of the QPD is here reviewed. Referring to Figure 5, the notes below are keyed to numbers on the diagram:

1. Origin ball *O* (Phase I).
2. The first IQC, i.e., the first shell (Phase II). This construction has icosahedral symmetry (see also Fig. 2). We can think of the six icosahedral symmetry axes that intersect at origin ball *O* as twelve rays emanating from *O,* aligned with those axes. One of those rays is highlighted in red, Note **7**.

3. To the first, central IQC, Phase III growth accretes twelve penetration twinned IQC building blocks, aligned with the twelve rays in a centrally symmetric quantum jump, to produce the QPD (the second shell). That construction, the QPD, comprises 45 balls and 204 rods (see also Fig. 4).

The above growth has completed the QPD.

4. From the surface of the QPD, the next growth stage accretes another twelve penetration twinned IQC building blocks along each of the twelve rays that radiate from *O* aligned with the symmetry axes. Note **4** sits on the shell of one of those IQCs.
5. We see that further growth does not produce a third shell that enshrouds the QPD, but rather, the only IQC building block accretion option available is for them to stack anisotropically to form rods aligned with the symmetry axes.
6. Shell of typical QPD.
7. Typical symmetry axis (red).
8. Gaps between the rods cannot be filled, but this is not indicative of a geometric frustration-type defect, but rather it is the case that there is a phase transition so that the growth is at this stage producing the second fractal layer of structure.
9. The first IQC, Note **2**, is iterated at the second fractal layer. The shell of the second fractal layer IQC is indicated, Note **9**. From this stage, ongoing construction will assemble those second fractal layer IQC building blocks.

As outlined above, growth of the ball-and-stick construction continues past the QPD (Fig. 5, Note **3**), but at that point there is a phase transition to second fractal layer growth. This is a *seamless* transition in which the idealised constructor continues uninterruptedly (i.e., without extra instruction input), to assemble penetration twinned IQC building blocks in centrally symmetric quantum jumps, adding to the growth (Fig. 5, Notes **4** and **5**) that stacks the IQCs to form rods that project out along each of the twelve rays that radiate from *O*. This is a seamless continuation of prior growth. However, beyond the shell of the first QPD (Note **3**), the growth of IQCs does not coalesce into an encasing third shell, but rather, the stacked IQCs project out as rods that are aligned with the symmetry axes. Now the QPD (Fig. 5, Note **3**) can be thought of as *the second fractal layer origin ball O,* and the stacked IQCs (Fig. 5, Notes **4** and **5**) can be thought of as *the second fractal layer rods*.

The growth of the ball-and-stick model self-organised into Phases I, II, and III; but there is no Phase IV. Rather, the continued growth seamlessly reproduces Phases I, II, and III at the second, third, fourth, etc. fractal layers.

The above analysis of the fractal structure describes the assembly process in terms of growth that emanates outward from *O* in centrally symmetric quantum jumps, and goes on to reproduce the essential QPD structure at ever-larger scale. But it is also the case, naturally, that the fractal structure extends similarly in the other direction. As well as looking at the QPD as the result of growth out from origin ball *O,* we can also zoom in on *O* so that we observe that it resolves to the QPD. That is to say, we

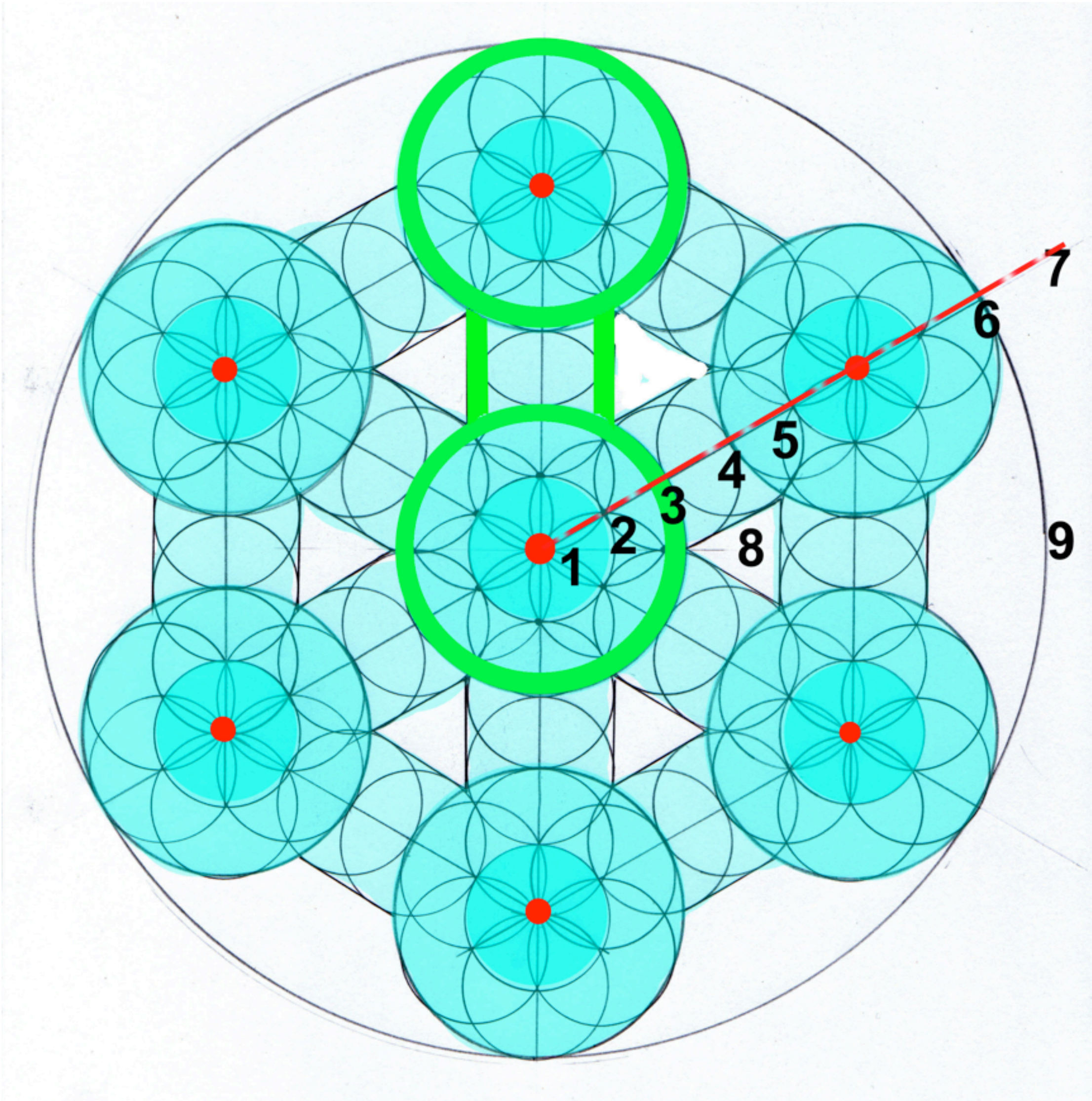

Figure 5. This shows the fractally layered extended structure (illustration of 2D section through 3D structure; i.e., not true projection). The notes below are keyed to the numbers in diagram.

Note: Fractal layer number is given in italicised square brackets.

1). Origin ball *O* (red).
2). Shell of first IQC.
3). Penetration twinned IQCs in cluster formation, forming the second shell = QPD = ball [2] (see also Fig. 4).
4). Shell of 1st IQC attached to surface of QPD.
5). Penetration twinned IQCs in anisotropic formation aligned with symmetry axis = rod *[2].*
6). Shell of typical QPD = ball *[2]* (see also Note 3 above).
7). Typical symmetry axis (red).
8). Gap between rods.
9). Shell of IQC *[2].*

can magnify the ball-and-stick construction to reveal that the original balls are also QPDs, and the original rods are made up of linearly stacked IQCs. Figure 5 shows an example (highlighted with green outline) of the second fractal layer ball-rod-ball assembly (clearly, the actual ball-and-stick modelling components initially shown at Fig. 2 did not anticipate the proportions very accurately).

Summary: The IQC is the fundamental building block from which the ball-and-stick model is constructed. However, the QPD is the fundamental "atom" at every fractal magnification. It is the first element. Each of the balls in the bin in the original basic setup are QPDs. And the complete global structure of the indefinitely extended ball-and-stick construction is equally the QPD.

12.2 The fundamental underlying icosahedral quasicrystalline lattice

Considering the fractal character that has become evident in the overall structure that has been produced thus far, it is now clear that with the first ball in place there was also already implied a fundamental, icosahedrally ordered quasicrystalline lattice structure, a structural chassis that underlies the entire ball-and-stick construction. The idealised constructor, charged with producing the most simple structure (i.e., the maximally information entropic structure) was always ineluctably bound to assemble the three-dimensional ball-and-stick model along those lattice lines through space.

12.2 The dynamically updating ball-and-stick extended model

The idealised constructor is, of course, not constrained by practical considerations in the same way that a human constructor would be. And it has to be remembered that, contrary to the limitations of this presentation (i.e., the static diagrams and descriptions) the model we have been discussing is the network of ball-rod connections that is dynamically updating. In this dynamical model balls are "created" and "annihilated" in the causal-domino effect that ripples throughout the entire icosahedral quasicrystalline lattice structure of the ball-and-stick construction.

The idealised constructor's ability to produce the above described dynamical model, we can well enough conceptualise, but those same construction capabilities do not, of course, extend to a human constructor. In fact, although we are arguing in this paper that the *dynamically updating model is the default most simple idealised gedanken model*, in contrast to that, the default most simple model for the *human* constructor is always the *static* model. (This dichotomy between the dynamical subject matter and the human predilection for converting that to the static model is discussed in [**3**].)

At this stage we anticipate that even a very reduced dynamically updating ball-and-stick model is not only physically difficult for the human constructor to produce, but also presents a combinatorially intractable problem. Where the idealised constructor readily assembles the dynamically updating ball-and-stick model as the default self-perpetuating system that requires minimal informational input, the human constructor on the other hand best resorts to constructing an inferior static model that can at least capture a representative snapshot of that system.

An aspect of the ball-and-stick assemblage that hasn't been discussed yet is the *migration effect* that is anticipated to result from the dynamical update feature. The structural tension throughout the quasicrystaline ball-and-stick model manifests as the geometrical frustration gaps, which are only resolved at one location where a ball and associated rods are pulled together so that the ball is "created", at the expense of a neighbouring ball being pulled apart from the rods, then "annihilated". This update process can perpetuate as a type of domino effect. However, as balls alternate between "created" and "annihilated" states, they do not reappear, re-created, in the same position – but rather, their position is shifted slightly so that the effect over multiple updates causes a *migration* of the overall system.

12.3 The ubiquitous helicity mapped out in the static approximant

For the idealised constructor's dynamically updating ball-and-stick model that has been described above, the human constructor can at best represent that by assembling a static version as an imperfect regularised approximant (in which rods are not identical, but lengths are adjusted). Taking account of the migration effect described above, once that initial approximant ball-and-stick model is constructed, it is then straightforward to produce the geometry that describes the migration pathway between any two neighbouring balls – i.e., that pathway that captures the dynamical evolution of the structure with respect to that segment. In paper [**2**], Section 13, the geometry for the three-dimensional model of that pathway is detailed, revealing it to be a tubular extension, referred to there as the "range-tube pathway".

That is, for a representative sample of the structure, the approximant model assembles all of the balls, each located within a range of position. And for any particular ball, the migration hypothesis says that over multiple dynamical updates that ball will be annihilated and re-created. Included in this is a geometry that describes the pathway in the static model that gives the range of position that any annihilation and creation of that ball is restricted to, spanning between all pairwise adjacent balls. Furthermore, there is a geometry that concatenates the pathway segments between pairwise adjacent balls to produce extended pathways throughout the static model of the entire lattice structure. These pathways are essentially describing, with respect to one static snapshot of the structure, the evolution of the overall system, or the macrostate of the system (see [**2**] Sections 16 and 17, the range-tube pathways).

The description given above is saying that for the interval between adjacent balls we can map the possible positions of balls that will be produced by the migration effect. In the full development (e.g., [**2**]) the static model is only able to define the positions and states of these balls probabilistically, and as such they are referred to as *virtual* balls. The geometry that delineates that extension produced by the migration effect is constructed of multiple compound curves so that the overall shape is a curvilinear tubular extension.

When we apply the geometry that concatenates those segments of curved tubular extension, it becomes evident that the extended composite range-tube pathways take on a helical configuration. Every extension that defines the interval between neighbouring balls is curved, and every extended concatenation of those intervals produces a helically coiled pathway. Yet the rays that project out from origin ball *O* aligned with the icosahedral symmetry axes map out the growth of the structure along the underlying chassis of straight lattice lines such that the overall space can be described as flat.

All of this is detailed in [**2**]. There we see that the range-tube pathways that traverse the extended structure can be constructed, originating at origin ball *O* and radiating in centrally symmetric growth outward, exiting the QPD shell and forming the rod extensions in multiply helically coiled paths. Those helices in the first instance coil about a substructure that is a triple helix strand, where that triple helix strand is itself entwined in a helical coil with five others strands.

Referring back to Figure 5, Notes **4** and **5** indicate the rod structure that grows out from the surface of the first QPD. We see that the rod extends a short distance before it is absorbed into the successor QPD. This demonstrates a second fractal layer ball-rod-ball construction, and we can say that the rod defines the interval between the two QPD/balls. Initially this could appear to present another case of the "no instruction to terminate" problem. It is not at first obvious what information there is to determine the rod length, before it terminates at the successor ball?

This is where the dynamical update system provides the auto-termination mechanism. In associated paper [**2**] Section 17, this is developed more fully. There we see that in the static model the dynamical update process is explained in terms of the range-tube pathways that map that update process as a helical pathway through the structure. For example, a typical pathway can be tracked from an origin at *O,* to where it moves through the rod structure before, at a certain point, the helically coiled configuration of the path returns to a point on the symmetry axis that is a reflection of the original starting point, thus signalling that one iteration of rod has terminated at the central origin *O2* of the successor ball – therefore a ball-rod-ball section of the structure is completed without information input.

Note: Every extension that defines the interval between neighbouring balls is curved. And every most simple extended pathway that traverses the overall structure is intrinsically helical. Generally, the defining feature of the model is a ubiquitous helicity that runs throughout the entire icosahedrally ordered, quasicrystalline lattice structure of the ball-and-stick construction that has been here identified as the most simple structure that it is possible to construct.

13. Conclusions

Consider that an observer's view ranges over the entire universe but begins to zoom in, with the Milky Way coming into focus as it moves toward Andromeda, and within the

Milky Way our solar system becomes visible, tracing its orbital path around the galactic centre. The lateral traverse of the galaxy combined with its rotational spin means that if a line is plotted behind our solar system, tracking its pathway, that line is of course a large helix drawn through space.

This helical pathway is a thickened line drawn through space by the motion of the entire solar system within which, naturally, there is the motion of the Earth orbiting the Sun. So the first path, the galactic path, is that thickened helical strand the width of the solar system that is itself, secondly, composed of the helical sub-strands drawn by the paths that the planets are tracing through space. So the rather limited “flat” diagrams of the usual heliocentric model are expanded here to produce the helical pathway.

Of those helical sub-strands there is one, of course, that is carved out by the Earth. And somewhere on the Earth’s surface, the idealised constructor stands at his workbench assembling the ball-and-stick model. The Earth’s movement through space, tracking along its multiply-helical path, combined with its rotational spin, means that within that system the idealised constructor is, of course, also tracing out his own helical pathway. And the laterally transported orbits of atomic and sub-atomic particles that compose the idealised constructor are producing further sub-strands of helical pathways streaming through space (including the helically configured molecular DNA strands that carry the idealised constructor’s own set of assembly instructions).

This is painting the picture of an overwhelmingly helically configured and dynamical milieu within which the idealised constructor assembles the ball-and-stick model. That very simple process, taking from the bin of identical rods and the bin of identical balls and assembling them as outlined in this paper, proceeded without any instruction that was biased toward producing helical structure – in fact, as far as it was possible, the components were of course assembled according to “no rules” whatsoever. Yet we have seen the default, maximally information entropic, most simple structure propagate in quantum jumps of centrally symmetric growth, radiating out along an icosahedrally ordered lattice where every extension is helically configured, composed of multiples of helical strands within helical strands.

The original question was, “What is the most simple structure that it is possible to construct?” We claim that this question has been at least broadly answered with the construction of the ball-and-stick QPD model summarised at Section 11, and where Section 12 describes the extended model, and associated paper [**2**] provides more detail. This work is describing the indefinitely extensible, dynamically updating, icosahedrally ordered, fractally layered and helically configured QPD structure that the idealised constructor has, as outlined in this paper, assembled.

Intuitively, the model presented here may not at first align with the common conception of a most simple structure. However, the conclusion of this paper’s investigation is that when “most simple” is defined in terms of prescriptive information content, and conventional bias that favours a static model as the default model is overridden, then the QPD model described here definitively answers the question posed in the title of this paper.

The QPD model is not presented here as an interesting structure among other interesting structures, but to the extent that the ball-and-stick modelling system is inherently suited to exemplifying, at least colloquially, a very simple structure, then the model constructed with this system is a fundamentally simple object that provides an initial basis from which broader analogies with respect to fundamental structure can be drawn (for which we reference associated paper [**3**]).

Essentially, there is an argument that says that if the QPD model is indeed pointing toward a new structure that is, in some critical sense, *the* fundamental underlying structure, then it may be appropriate to raise the question as to how explicit, or not, the current foundations of mathematics are with respect to even the notion of such a model, particularly in the context of current problems in physics?

Obviously this paper gives only an introductory outline rather than a full exposition, in particular with respect to quantifying the information content; and neither do the associated papers [**2**] and [**3**] yet provide the full, formal, information-theoretic development – that remains an area for future work. The aim has been that this paper should provide a useful introduction to the concept of a most simple concrete model that has potential to analogise more significant proposals around the notion of a fundamental underlying structure (of everything).

Acknowledgment: To follow.

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