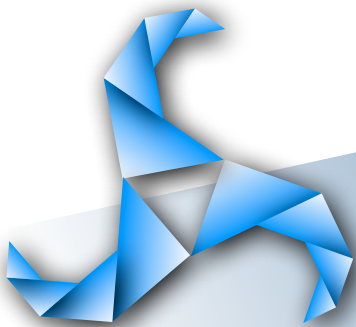


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**FUNCTIONAL GRAPH AS A POWERFUL TOOL IN  
MATHEMATICAL PROBLEM SOLVING: ITS  
CURRICULUM AND INSTRUCTION**

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# FUNCTIONAL GRAPH AS A POWERFUL TOOL IN MATHEMATICAL PROBLEM SOLVING: ITS CURRICULUM AND INSTRUCTION<sup>1), 2)</sup>

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In our research and practice, to solve an equation and to represent a graph are placed as tools for problem solving in the curriculum. We set the new integrated unit called “*Function and Equation*” which is reorganized conventional units in Japan, intended to cultivate students’ ability and attitude to apply those in effectively in each school grade. To realize this, we edited the experimental textbooks along the program that we reconstructed. These are verified through lesson studies.

*Key words: functional graph, problem solving, lesson study, experimental textbook, curriculum*

## PROBLEMATIC OF THE RESEARCH

### *The present situation of Japanese students*

It is pointed out as the present situation of the Japanese junior high school students from various kinds of domestic and international surveys that they cannot make an equation and represent it into a graph necessary for the given problem situation while they can solve the given equation and represent the graph of the given equation solely. In other words, expressions or equations, graphs, and tables cannot be effective tools for their problem solving.

### *The conventional condition in Japanese mathematics curriculum*

In the conventional curriculum condition of the Japanese junior high school mathematics, students usually learn the function after the equation in all school grades. At the 7th grade, they learn linear function limited to proportionality ( $y=ax$ ) and inverse proportionality ( $y=\frac{a}{x}$ ) after linear equation ( $ax+b=0$ ). At the 8th, they learn linear function in general ( $y=ax+b$ ) after simultaneous equations. At the 9th, they learn quadratic function limited to in proportion to the square ( $y=ax^2$ ) after quadratic equation ( $ax^2+bx+c=0$ , limited to real solutions).

However, the arrangement of the curriculum manifests students' obstacles in their problem solving, *i.e.*, the graph cannot function as an effective tool when a student expresses the quantities relation of the problem situation algebraically -in an

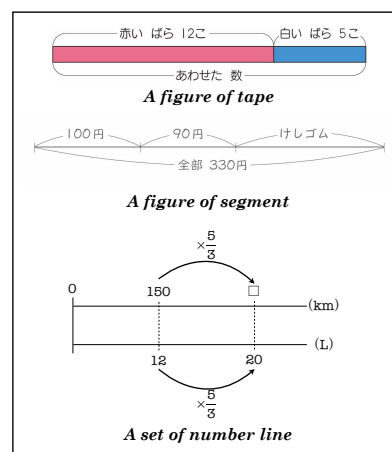
expression-. In addition, from the interaction of the equation and the graph, the student cannot have a standard to judge the validity of the solution of the problem. In the current curriculum, these activities are placed in the last of the function unit. That is irrational and uneconomical.

### **Basic way of thinking about the integration of function and equation**

It is necessary for comprehending a phenomenon mathematically to identify variables inherent in a phenomenon, and to represent the relation into a table, a graph, and a literal expression. This is because it can grasp the characteristic clearly by expressing a phenomenon in a table, a graph, an expression. We think that a table, a graph, and a expression are not skill but tools for mathematical problem solving or decision making. If the relations between variables inherent in a phenomenon are expressed in tables, graphs, and expressions, it is able to solve many problems with utilizing them. We want to think about the curriculum that students can realize this(Sugiyama, et al., 2003).

Particularly, in Japanese elementary school, *a figure of tape* or *a figure of segment* (addition and subtraction), and *a set of number line* (multiplication and division) are used as tools for deciding operation.

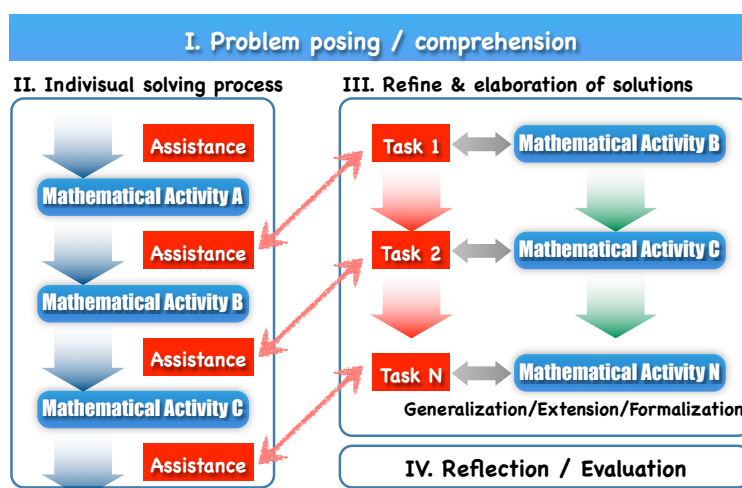
Therefore, in the junior high school, we expect our students to utilize a graph as an effective tool for making an equation.



## **PURPOSE AND METHODOLOGY OF THE RESEARCH**

The purpose of our project is to develop the local curriculum (*Function and Equation*) and the pedagogical mathematical contents for teaching intended that students use equation and graph as effective tools of their problem solving.

We edit the experimental textbooks – **Function and Equation I** (for 7th), **II** (for 8th), **III** (for 9th, *III is not included in this report, under practice now*) – for the junior high school mathematics, and verify the effectiveness and the problems of the textbooks that we developed through *lesson studies*. Our lesson style is called **Problem Solving Lesson** as shown in the figure below(cf. Mizoguchi, 2008).



## PROCESS OF THE RESEARCH

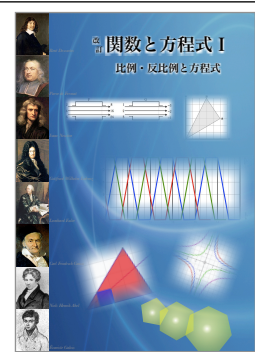
We think that to solve an equation and to represent a graph should be as tools for problem solving in the curriculum. We set the new integrated unit called “*Function and Equation*” that reorganized conventional units, intended to cultivate students’ ability and attitude to apply these in effectively in each school grade. For this, we edited the experimental textbooks along the program that we reconstructed.

In 2010 and 2011 school year, the researcher from Tottori University and the teachers from Junior High School Attached Tottori University carried out above mentioned research collaboratively about the 7th grade and the 8th grade program. In 2012, we work on the 9th grade program likewise, and improve the textbooks of the 7th and the 8th grade.

## EXPERIMENTAL TEXTBOOKS – *Function and Equation I, II* –

*Function and Equation I* contains *proportion, inverse proportion and linear equation*.

- I. Functional way of thinking
  - §1 Problem of crane and tortoise
  - §2 Two quantities relation which vary correspondingly
- II. Proportion
  - §1 Coordinates
  - §2 Graph of the proportion
  - §3 The use of the proportion
- III. Inverse proportion
  - §1 Graph of the inverse proportion
  - §2 The use of the inverse proportion
- IV. How to solve equations
  - §1 Equation
  - §2 Property of the equation
- V. The graph of the function and equation
  - §1 The use of the expression of the graph of the proportion
  - §2 The use of the slope of the graph of the proportion
  - §3 Graph of the proportion and inequality



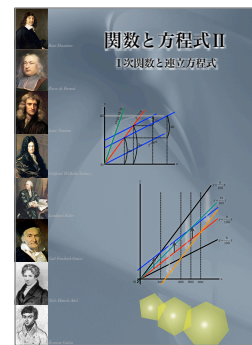
<p>問題16: 長さの異なる2本のロープA, Bがあります。Aは燃えつきるのに8時間、Bは48時間かかります。いま、同時に火をつけ、ある時刻燃えつき、同時に火を消しました。このとき、Aの残りはBの残りの長さの2倍ありました。何時間何分がグラフを無効としたでしょう。</p> <p>はじめのロープの長さの比をもととしておきます。ロープAは8時間でこの長さだけ燃え、ロープBは48時間で燃え、このことをグラフで表すと、右のようになります。これより、ロープAとBの燃焼の様子は、それぞれ</p> $A: y = \frac{x}{8}, \quad B: y = \frac{x}{48}$ <p>と表されることがわかります。このとき、グラフの下限は、それぞれのロープの燃焼した長さを表しますから、グラフの上側からそれぞれ、それぞれの燃え残った長さになります。そこで、ロープA, Bの残った長さが2:1となる時刻を求めればよいので、そのような時刻を右の図のようによみとることができます。これより、次の方程式を立てることができます。</p> $A - \frac{A}{8} = 2(B - \frac{B}{48})$ <p>この方程式を解くには、次のような2つの方法が考えられます。1つは、具体的な数値を代入して解く方法(解法1)です。もう1つは、Aをそのままにして計算し、後で消去する方法(解法2)です。</p>	<p>【解法1】  <math>A=1</math>とすると、  <math>1 - \frac{1}{8} = 2(1 - \frac{1}{48})</math>  <math>\frac{7}{8} = 2(\frac{47}{48})</math>  <math>\frac{7}{8} = \frac{47}{24}</math>  <math>24 \times \frac{7}{8} = 24 \times \frac{47}{24}</math>  <math>21 = 47</math>  <math>21 - 24 = 47 - 24</math>  <math>-3 = 23</math>  <math>\frac{-3}{23} = \frac{24}{x}</math>  <math>x = \frac{24 \times 23}{-3} = -184</math>  <math>x = 184</math>分</p> <p>よって、48時間184分が求める時刻であることがわかります。</p> <p>問題17: ある建物に、1階500人、2階300人、3階200人が入居し、1階1300円で、2階1500円で、3階1700円で、合計5000円を求めたい。このりんごの仕入単価を求めなさい。</p> <p>りんご1個の仕入単価をa円とすると、りんごの代表(x)はaを比例定数として傾斜(a)に定めます。これは1300円で買ったときも同様です。そこで、問題場面をグラフに表すと右図のようになります。</p> <p>500a + 1.3 \times 1300 = 4800</p> <p>題意より、仕入れ総額500aを求めればよいので、500a = 4800 - 1.3 \times 1300</p> <p>したがって、求める仕入単価は、<math>\frac{4800 - 1.3 \times 1300}{500}</math>円</p>	<p>問題21: ある濃度の食塩水が1kgあります。この食塩水から100gをとり、水を100g加えてよく混ぜた後、さらにこの食塩水から200gとり、水を200g加えてよく混ぜたところ、最初の食塩水になりました。はじめの食塩水は何%だったでしょう。</p> <p>★いくつかの段階に分けて、順に考えてみましょう。</p> <p>[1] はじめの食塩水(A)の濃度をa%とすると、食塩水Aに含れた食塩の量(g)は食塩水の量(g)に比例するので、  <math>y = \frac{a}{100}x</math>      と表されます。      食塩水Aから100gをとり、水を100g加えてよく混ぜたところ、濃度は変わらなくなりました。</p> <p>[2] 次に、食塩水Aに水を100g加えた後の食塩水(B)の濃度をa%とすると、明らかにa &gt; a'であり、  <math>y = \frac{a'}{100}x</math>      と表され、食塩水Bから200gをとり、水を200g加えてよく混ぜたところ、濃度は変わらなくなりました。</p>	<p>[3] さらに、食塩水Bに水を200g加えた後の食塩水(C)の濃度がa.64%です。(ここでも明らかに、a &gt; a.64)</p> <p>[1]~[3]によって表された各々のグラフの傾きの比を順に注目すると、  <math>\frac{a}{100} \times 800 = \frac{a.64}{100} \times 1000</math>  <math>8a = 6.4a</math>  <math>8a - 6.4a = 6.4a - 8a</math>  <math>1.6a = -1.6a</math>  <math>1.6a + 1.6a = -1.6a + 1.6a</math>  <math>3.2a = 0</math>  <math>a = 0</math></p> <p>よって、はじめの食塩水(A)の濃度は、10.8%であることがわかります。</p> <p>また、このことから、  <math>\frac{a}{100} \times 800 = \frac{10.8}{100} \times 1000</math>  <math>8a = 10.8</math>  <math>a = 1.35</math>      よって、a=1.35、したがって、はじめの食塩水(A)の濃度は、13.5%であることがわかります。</p>
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*Function and Equation II* contains *linear function and simultaneous equations*.

- I. Linear function and equation
  - §1 Problem of crane and tortoise(Part 2): What do we learn at this unit?
  - §2 Proportion and linear function

(continued)

- II. Graph of linear function
  - §1 Rate of change
  - §2 Graph of linear function
  - §3 Linear function and linear equation in two variables
- III. Linear function and inequality & simultaneous equations
  - §1 Graph of the linear function and inequality & simultaneous equation
  - §2 Various how to solve simultaneous equations
  - §3 Complicated simultaneous equation
- IV. Various inequalities and simultaneous equations
  - §1 Problem solving(1)
  - §2 Problem solving(2)



**1. 1次関数と方程式**

例題1: 最も魚が合はせてお釣りで、足の数は全部で2本です。鰻と魚はそれぞれ何匹いますか。

問題1は、「関数と方程式」ではじめて考えた問題です。この問題を、これまでで学習してきたグラフを用いて考えましょう。

鰻の足は2本です。魚の足の数(x)は鰻の足(x)に比例して、 $y=2x$ で表されます。同時に、魚の足の数(y)は魚の足(x)に比例して、 $y=4x$ で表されます。そこで、これらの式をグラフに表してみましょう。

鰻と魚は合わせて28匹です。もともと合はせてお釣りの数は50本になり、足の本数は足りません。また魚だけでは足の本数は100本になりすぎます。

ここでグラフを手がかりにすると、問題の解決のためには、魚が1匹(4本)、2匹(8本)、...と増えているところから鰻が1匹(2本)、2匹(4本)、...と増えて、合わせて28匹のときに足の数がちょうど28本になるというので、そのようになるためには、 $y=2x$ のグラフと $y=4x$ のグラフを28本の足に平行移動して、この2つのグラフのyの値が等しいところから、

同じように方程式がたてられ、これを解いて $x=11$ が得られます。すなわち、魚が11匹であることがわかります。これより、鰻は14匹であることがわかります。

同じように、鰻が1匹、2匹、...と増えている、解いてあるところから魚が1匹、2匹、...と増えていくと、グラフから方程式を立て、問題1を解決してみましょう。

このグラフは、 $y=2x$ のグラフをy軸方向に28だけ平行移動したもので、 $y=2x+28$ と表されることがわかります。

そこで、魚が何匹で、どこから鰻を数え始めればよいということ、このグラフも $y=2x$ のグラフの交点として求められます。ここで、この2つのグラフのyの値が等しいところから、

$4x=2x+28$

という方程式がたてられ、これを解いて $x=14$ が得られます。すなわち、魚が14匹であることがわかります。これより、鰻は11匹であることがわかります。

同じように、鰻が1匹、2匹、...と増えている、解いてあるところから魚が1匹、2匹、...と増えていくと、グラフから方程式を立て、問題1を解決してみましょう。

問題2: 各駅が20分おきの水そうに水をいれるのに、A管とB管を使います。はじめA管で2時間入れた後、B管で2時間入れたと清水になります。また、A管で2時間入れた後、B管で2時間入れたとしても清水になります。A管とB管を同時に使って水をいれると、この水そうは何時間何分で満水になるでしょうか。

問題場面をグラフに表現してみましょう。

A管からaL/時、B管からbL/時 水を入るとする。

(1) A管で3時間入れた後、B管で2時間入れたと満水になる。

右のグラフより、 $3a+2b=200$  ①

(2) A管で2時間入れた後、B管で4時間入れたと満水になる。

右のグラフより、 $2a+4b=200$  ②

①、②より、 $(a, b)=(5, 2.5)$

(3) A管とB管を同時に使って水をいれると、この水そうは何時間何分で満水になるか。

$y=(a+b)x$ において、 $(a, b)=(5, 2.5)$  また、 $y=200$ であるから、 $200=(5+2.5)x$   
 $200=7.5x$   
 $x=26\frac{2}{3}$

$40 \times \frac{2}{3} = 100$ より、求める時間は、**2時間40分**

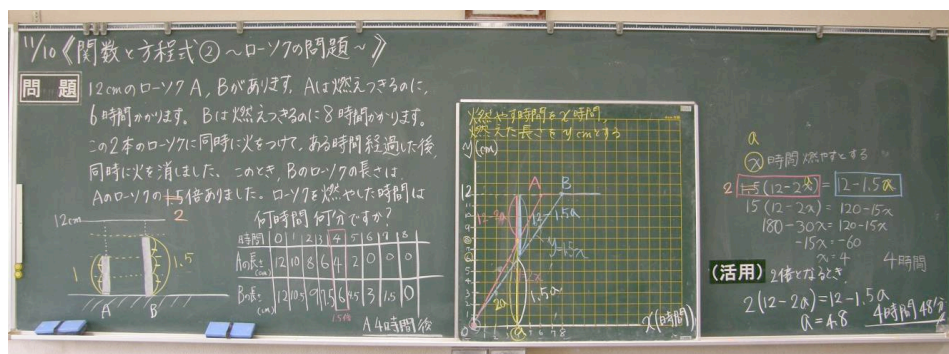
満水だけでなく、何かの割合だったり、単位量あたりの大きさなど、グラフでは様々な考え方がいっしょに!

もう少し考えたら、問題場面をグラフに表すと、方程式を立てやすくなるぞう。

## LESSON STUDIES – partially – A. Problem of Candle (7th grade)

◎ Purpose of the lesson: To understand that to represent a phenomenon by a graph is useful to make an equation.

*Problem: There are candle A and B of 12cm both. Candle A takes eight hours to burn out, and candle B takes six hours. Setting fire at the same time, having passed at a certain time, and putting out fire at the same time. Then the length of A was 1.5 times of B. How long did two candles burn?*



◎ Students' activities are:

- ▶ to investigate by a table;
- ▶ to represent graphs because the length that candles burnt is proportional to time, and to read it from coordinate plane directly;

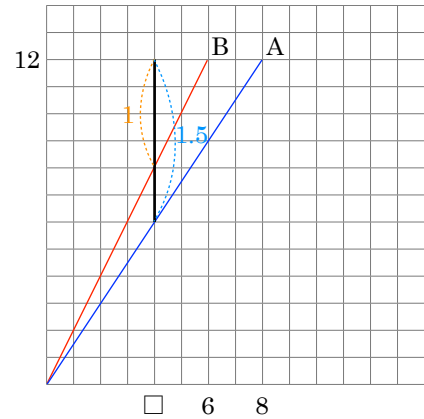
$$A: y = \frac{12}{8}x \quad B: y = \frac{12}{6}x$$

▶ to make an equation from the graphs.

$$1.5 \times \left( 12 - \frac{12}{6}t \right) = 12 - \frac{12}{8}t$$

◎Discussion (excerpted)

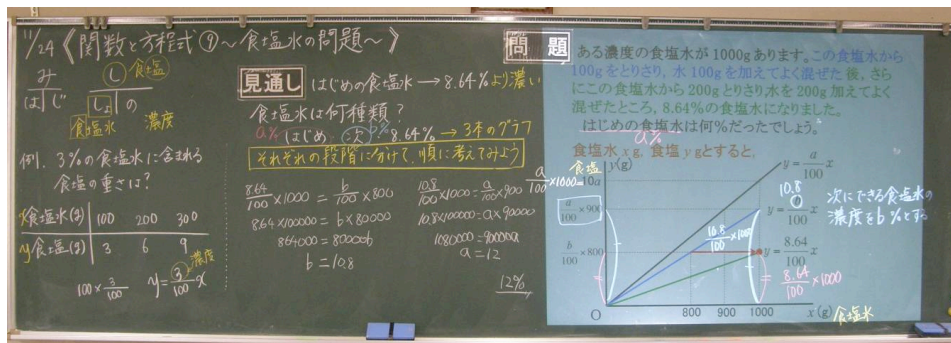
It was difficult for the students to see two graphs at the same time, especially, to compare the length that each candle burns with a change of the time.



B. Problem of Salt Solutions (7th grade)

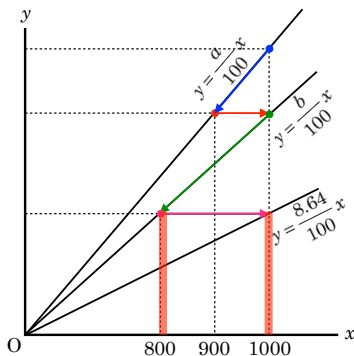
◎ Purpose of the lesson: To represent a problem situation to graph, and make equations from it.

Problem: There are 1000g of salt solutions of a certain density. At first, removing 100g from this salt solution, and having added 100g of water, second, removing 200g from this salt solution, and having added 200g of water, then it becomes to the salt solution of 8.64%. What percentage the salt solution of the beginning would be?

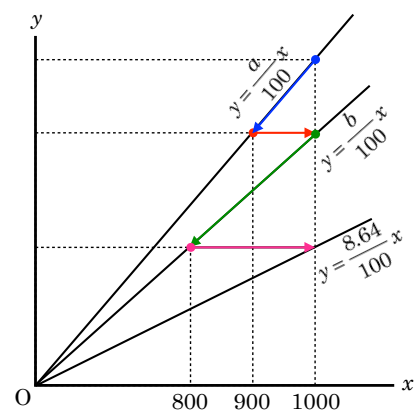


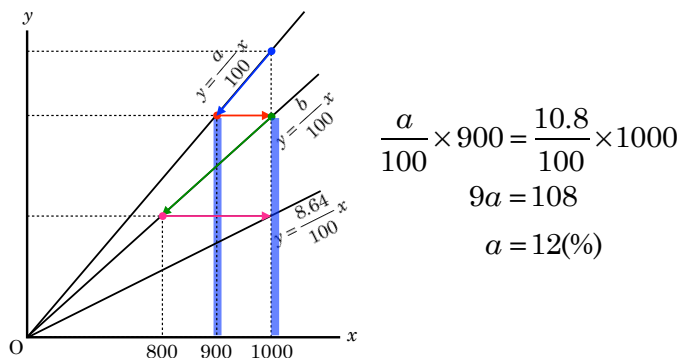
◎Students' activities are:

- ▶ to reproduce the problem situation by the movement of the point on graphs;
- ▶ to make equations from the graphs with focusing to quantity of the salt.



$$\begin{aligned} \frac{b}{100} \times 800 &= \frac{8.64}{100} \times 1000 \\ 8b &= 86.4 \\ b &= 10.8(\%) \end{aligned}$$





● *Discussion (excerpted)*

Many students were able to solve by oneself with observing the movement of the point.

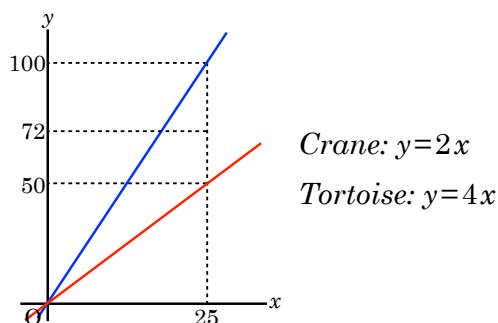
**C. Problem of Crane and Tortoise (8th grade)**

● *Purpose of the lesson:* To understand that to represent a phenomenon by graphs is useful to make equations.

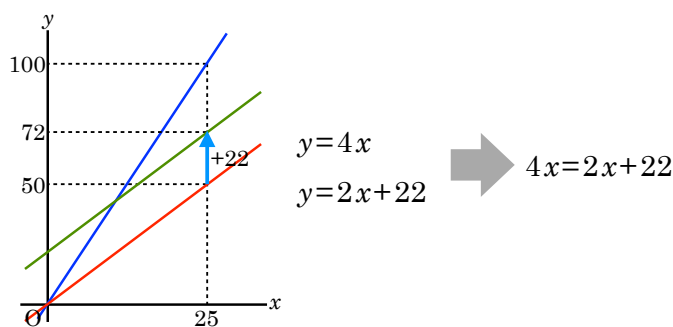
*Problem: There are cranes and a tortoises in total 25, and the number of feet is 72 in total. How many is the number of crane and tortoise respectively?*

● *Students' activities* are as follows:

- ▶ If the number of crane is 25, the number of feet is 50, then it is fewer than 72. If the number of tortoise is 25, the number of feet is 100, then it is more than 72;



- ▶ If several tortoises are there at the beginning and several cranes come afterwards...  
(Translation of the graph of tortoise, also similar of the case of crane)



● *Discussion (excerpted)*

Students were able to find the graph of the linear function from the graph of the proportion.



**D.Problem of Salt Solution -another- (8th grade)**

● *Purpose of the lesson:* To represent a phenomenon in simultaneous equations and to solve the problem by using the translation of the graphs of the linear function.

*Problem:* There are two container A and B. The salt solution of  $a\%$  is in A, and the salt solution of  $b\%$  is in B and both salt solutions are 400g. The following operations are carried out for A and B.

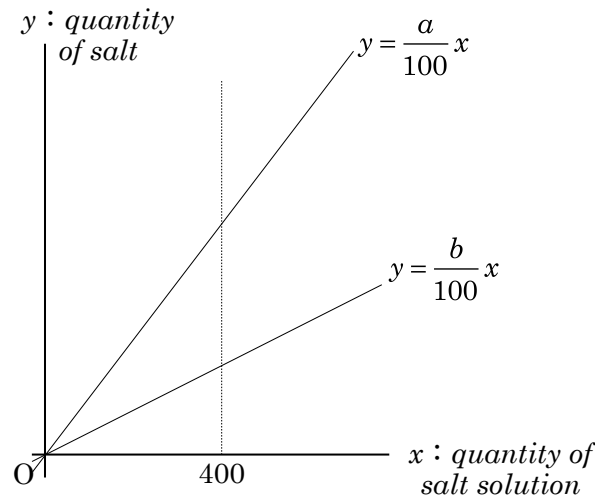
[Operation 1] Adding half quantity of the salt solution in A to B, and having mixed them.

[Operation 2] Adding half quantity of the salt solution in B which was made with the former operation to A.

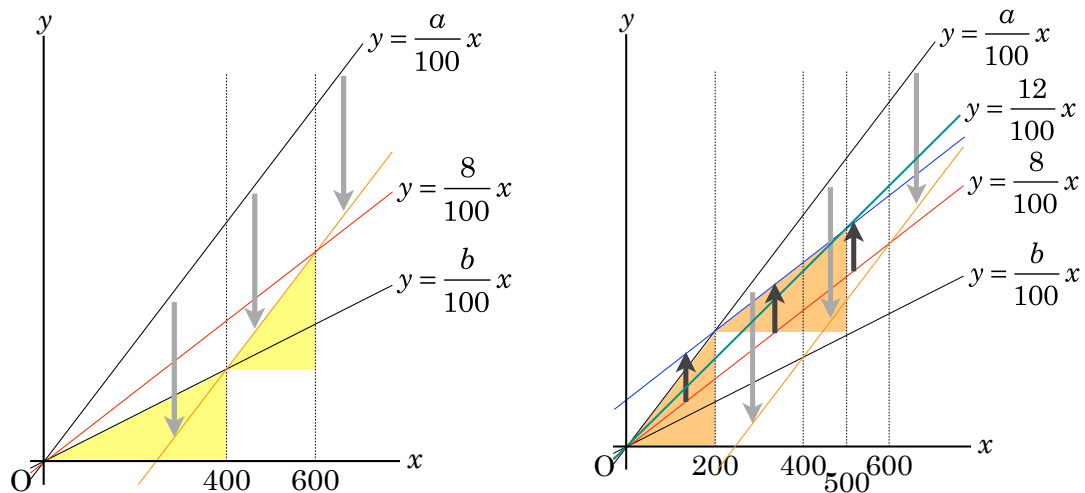
After the operations, the density of salt solution of A is 12%, the density of salt solution of B is 8%. Find the each density of salt solution of A and B before the operations.

● *Students' activities* are almost as follows:

▶  $y$  is proportional to  $x$ .



▶ Representing the operations by the movement of the graphs.



$$\begin{cases} \frac{b}{100} \times 400 + \frac{a}{100} \times 200 = \frac{8}{100} \times 600 \\ \frac{a}{100} \times 200 + \frac{8}{100} \times 300 = \frac{12}{100} \times 500 \end{cases}$$

◎*Discussion (excerpted)*

Students were almost able to use graphs and to translate them for comprehending and solving the problem with making simultaneous equations.

**Notes**

- 1) This report is a revised version of our handout for the 12th International Congress on Mathematical Education(8 July – 15 July, 2012, COEX, Seoul, Korea).
- 2) This research project is funded by Japan Association of Universities of Education.

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(投稿原稿の内容に応じて、外部編集委員を招聘することがあります)

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- ❖ 本誌は、次の稿を対象とします。
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