Note on invariant regular ideals in BP_*

Katsumi SHIMOMURA*

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§1. Introduction

Let BP be the Brown-Peterson spectrum at a prime p. Then its homotopy group $\pi_*(BP)=BP_*$ is the polynomial ring over $Z_{(p)}$ with the Hazewinkel's generators $v_1, v_2, \cdots$, It also gives the Hopf algebroid (BP_*, BP_* BP) with the right and the left units η_R and η_L (cf. [1], [4]). Let J denote an ideal of length n generated by n homogeneous elements a_0, $a_1, \cdots, a_{n-1}$ of BP_* and put $J_k=(a_0, a_1, \cdots, a_{k-1})$ for $k \leqq n$. The ideal J is said to be *regular* if a_0 is a power of the prime p and a_k is not a zero divisor in BP_*/J_k for each k, and to be *invariant* if $\deg a_0 < \deg a_1 < \cdots < \deg a_{n-1}$, $\eta_R a_0 = \eta_L a_0$ and $\eta_R a_k \equiv \eta_L a_k \mod J_k$ for each k.

P. S. Landweber [2] studied some properties of invariant regular ideals and determined all the invariant regular ideals of length 1 and 2. We can read off all the invariant regular ideals of length 3 from the results of H. Miller, D. Ravenel, and S. Wilson [3] for an odd prime p, and from [5] for the prime 2 (see Proposition 2.7). E. Tsukada [7] found all the invariant regular ideals of length $n \geqq 1$ in the case that each generator a_k of J is a some power of the Hazewinkel's v_k for $0 \leqq k \leqq n$ ($v_0=p$). In this note we give a result similar to Tsukada's using the elements $x_{k,i}$ given in [3] instead of v_k (see Proposition 3.8). We note that invariant regular sequences give a periodic family in the E_2-term of Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres ([1], [3]). We also note that there exists the BP-local spectrum Y_J such that $BP_*Y_J=v_n^{-1}BP_*/J$ for each invariant regular ideal J at a large prime p comparing with the length n of J by [6], though we do not know the existence of a spectrum V_J such that $BP_*V_J=BP_*/J$ for J which we have constructed here.

§2. Invariant regular sequences

The coefficient ring BP_* of the Brown-Peterson spectrum BP at a prime p is the polynomial ring $Z_{(p)}[v_1, v_2, \cdots]$ and the BP_*-homology BP_*BP is the polynomial $BP_*[t_1, t_2, \cdots]$, where $\deg v_k = \deg t_k = 2p^k-2$. Then ($BP_*$, BP_*BP) is the Hopf algebroid (cf. [1], [4]), whose right and left units η_R, η_L: $BP_* \to BP_*BP$ are given by the following equalities:

(2.1.1) $$\eta_L v_k = v_k;\ \eta_R l_k = \Sigma_{i+j=k} l_i t_j^{(i)}\ (l_0=1)$$

Department of Mathematics, Faculty of Education, Tottori University, Tottori, 680

for $\eta_R: BP_*\otimes Q \to BP_*BP\otimes Q$, where $BP_*\otimes Q = Q[l_1, l_2, \cdots]$ and

(2.1.2) $$v_k = pl_k - \sum_{i=1}^{n-1} v_{k-i}^{(i)} l_i$$

(in this paper (k) in the exponent denotes p^k). For example, we deduce the following congruences:

(2.1.3) $$\eta_R v_{k+1} \equiv v_{k+1} + v_k t_1^{(k)} - v_k^p t_1 \mod I_k \quad \text{for} \quad k \geqq 1, \text{ and}$$

$$\eta_R v_{k+2} \equiv v_{k+1} t_1^{(k+1)} - v_{k+1}^p t_1 + v_k t_2^{(k)} \mod (I_k, v_k^p) \quad \text{for} \quad k \geqq 2,$$

in which I_k denotes the ideal $(p, v_1, \cdots, v_{k-1})$ of BP_* (cf. [4; p. 145]).

Consider the following BP_*BP-comodules derived from the comodule BP_* defined by:

(2.2) $N_n^0 = BP_*/I_n$, and the exact sequence

$$0 \longrightarrow N_n^k \longrightarrow v_{n+k}^{-1} N_n^k \longrightarrow N_n^{k+1} \longrightarrow 0 \qquad \text{for} \quad k \geqq 0.$$

The coactions of these comodules are the ones induced from the right unit η_R of BP_* and also denoted by η_R. We shall abbreviate N_0^k to N^k. Each homogeneous element x of N_n^k is written by a linear combination of fractions:

(2.3) $$x = w/v \quad \text{for} \quad w \in BP_* \quad \text{and} \quad v = \prod_{i=n}^{n+k-1} a_i, \quad \text{and}$$

$$x = 0 \quad \text{if} \quad w \in I_n \quad \text{or} \quad a_i | w \quad \text{for some} \quad i,$$

where a_i $(i \geqq n)$ are elements of BP_* such that $\deg a_i < \deg a_{i+1}$ and the radical of the ideal $(I_n, a_n, \cdots, a_{n+k-1})$ is I_{n+k}.

Let M denote a comodule difined above. We define

$$H^0 M = \operatorname{Ker} d$$

for $d = \eta_R - \eta_L$. The module $H^0 N^k$ is closely related to the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres (cf. [3]).

Let $J = \{a_k\}_{k \geqq 0}$ be a sequence of infinite elements of BP_* with $\deg a_k < \deg a_{k+1}$ for $k \geqq 0$, and $J_n = \{a_k\}_{0 \leqq k < n}$ denote the subsequence of J. J_n will also be written as $a_0, \cdots, a_{n-1}$. Then the sequence J_n is called *regular* if (J_n) is a proper ideal, $a_0 \neq 0$ and a_k is a non-zero divisor in $BP_*/(J_k)$ for each $k < n$, and called *invariant* if $\eta_R a_0 = \eta_L a_0$ and $\eta_R a_k \equiv \eta_L a_k \mod (J_k)$ for each $k < n$. For an invariant regular sequence J_{n+1}, consider the element

$$x(J, n) = a_n / a_0 \cdots a_{n-1} \in N^n.$$

The following is an easy consequence of (2.3):

LEMMA 2.4. *Let J_{n+1} be an invariant regular sequence, and a an element of BP_*. Then, $ax(J, n) = 0$ if and only if $a \in (J_n)$.*

PROOF. If $a \in (J_n)$, then (2.3) implies $ax(J, n)=0$. On the other hand, $ax(J, n)=0$ implies the equality $aa_n=\Sigma_{j<n}u_j a_j$ by (2.3), and so $aa_n \equiv 0 \mod (J_n)$. Hence $a \equiv 0 \mod (J_n)$ by the regularity of J_{n+1}. q. e. d.

LEMMA 2.5. *Suppose that J_n is an invariant regular sequence for $n \geqq 1$. Then a regular sequence J_{n+1} is invariant if and only if $x(J, n) \in H^0 N^n$.*

PROOF. First we prove the following by the induction on k:

(2.5.1) $$d(1/a_0 \cdots a_k)=0 \text{ on } N^{k+1} \text{ if } a_0, \cdots, a_k \text{ is invariant.}$$

Since $a_0=p^e$ for some e by [2; Prop. 2.5], we have $\eta_R a_0=\eta_L a_0$, and so $d(1/a_0)=0$ on N^1. Both η_R and η_L are algebra maps, which implies

(2.5.2) $$d(a_{k+1}/a_0 \cdots a_k)=\eta_R a_{k+1} d(1/a_0 \cdots a_k)+d(a_{k+1})/a_0 \cdots a_k.$$

It turns into $d(a_{k+1}/a_0 \cdots a_k)=d(a_{k+1})/a_0 \cdots a_k$ by the inductive hypothesis. Besides, $d(a_{k+1}) \equiv 0 \mod J_{k+1}$ implies $d(a_{k+1})=a_0 u_0+\cdots+a_k u_k$ for some $u_i \in BP_*$. Therefore $d(a_{k+1})/a_0 \cdots a_k=0$ by (2.3) and hence $d(a_{k+1}^{-1}/a_0 \cdots a_k)=0$ in $v_{k+1}^{-1}N^k$, which with the exact sequence in (2.2) shows (2.5.1) for $k+1$.

Now turn to the proof of the lemma. If J_{n+1} is invariant, then $d(a_n) \equiv 0 \mod J_n$ and (2.5.1–2) imply $d(x(J, n))=0$. Conversely if $d(x(J, n))=0$, (2.5.1–2) again imply $d(a_n)/a_0 \cdots a_{n-1}=0$, which shows $d(a_n) \equiv 0 \mod J_n$ and J_{n+1} is invariant. q. e. d.

LEMMA 2.6. *Let $J_{n+1}=\{a_k\}_{0 \leqq k \leqq n}$ and $K_{n+1}=\{b_k\}_{0 \leqq k \leqq n}$ be invariant regular sequences. If $(J_{n+1})=(K_{n+1})$, then $(J_m)=(K_m)$ $(1 \leqq m \leqq n+1)$ and $\deg a_i=\deg b_i$ $(0 \leqq i \leqq n)$.*

PROOF. Suppose first that $(J_{m+1})=(K_{m+1})$. Then,

(2.6.1) If a_i of J_m $(i<m)$ satisfies $a_i \notin (K_m)$, then $\deg a_i \geqq \deg b_m$.

In fact, $a_i \equiv ub_m \mod (K_m)$ by the assumption $(J_{m+1})=(K_{m+1})$ for a non-trivial element u of BP_*. Furthermore suppose $(J_m) \neq (K_m)$. If $(J_m) \supset (K_m)$, there exists a_i of J_m so that $a_i \notin (K_m)$ $(i<m)$. Therefore we see that $\deg a_m > \deg a_i \geqq \deg b_m$ by (2.6.1). On the other hand, $b_m \equiv wa_m \mod (J_m)$ for some $w \in BP_*$ by the assumption. These imply $w=0$ and $b_m \in (J_m)$. Then $(J_m) \supset (K_{m+1})=(J_{m+1})$ which contradicts to the regularity of J_{m+1}. Thus $(J_m) \not\supset (K_m)$. Similarly $(K_m) \not\supset (J_m)$. In this case there exist a_i of J_m and b_j of K_m so that $a_i \notin (K_m)$ and $b_j \notin (J_m)$. Then (2.6.1) is also applied to show

$$\deg a_m > \deg a_i \geqq \deg b_m > \deg b_j \geqq \deg a_m,$$

which is again a contradiction. Therefore we have proved that $(J_{m+1})=(K_{m+1})$ implies $(J_m)=(K_m)$. Thus we obtain the first statement.

If $\deg a_i < \deg b_i$, then we have $a_i \in (K_i)$, since $a_i \in (K_{i+1})(=(J_{i+1}))$ and $a_i \not\equiv ub_i \mod (K_i)$ for any $u \in BP_*$. Therefore $(J_{i+1}) \subset (K_i)=(J_i)$. This also contradicts to the regularity of J_{m+1}. Thus $\deg a_i \geqq \deg b_i$. Similarly $\deg a_i \leqq \deg b_i$. q. e. d.

An ideal (K) generated by elements in a sequence $K=\{a_k\}_{0\leqq k\leqq n}$ is said to be *invariant regular* if K is invariant regular (cf. [2; Cor. 2.4]). Let IR_n denote the set of all invariant regular ideals of length n. For a $\mathbf{Z}_{(p)}$-module M, $\{M\}$ denotes the set of the subsets $\{x\}$ for all additive generators $x \in M$, where $\{x\}=\{\lambda x \mid \lambda \in \mathbf{Z}_{(p)}-p\mathbf{Z}_{(p)}\}$. Then we have

PROPOSITION 2.7. *There exists an injective map* f_n: $IR_{n+1}\to\{H^0N^n\}$ $(n>0)$ *assigning* (J_{n+1}) *to* $\{x(J, n)\}$.

PROOF. First we shall show that $\{x(J, n)\}=\{x(K, n)\}$ if $(J_{n+1})=(K_{n+1})$ for invariant regular sequences $J_{n+1}=\{a_k\}_{0\leqq k\leqq n}$ and $K_{n+1}=\{b_k\}_{0\leqq k\leqq n}$. Lemma 2.6 and the regularity imply the follwing:

(2.7.1) $\qquad b_k=\lambda_k a_k+\Sigma_{j<k}\, u_j a_j$ for some $\lambda_k \in \mathbf{Z}_{(p)}-p\mathbf{Z}_{(p)}$ and $u_j \in BP_*$.

Then by the definition of N^n, we have $x(J, n)=\lambda x(K, n)$ for some $\lambda \in \mathbf{Z}_{(p)}-p\mathbf{Z}_{(p)}$. Therefore the map f_n is well defined.

Now suppose that $\{x(J, n)\}=\{x(K, n)\}$. Then we see that $(J_n)=(K_n)$ by Lemma 2.4, and we can apply (2.7.1) to show $1/a_0\cdots a_{n-1}=\lambda/b_0\cdots b_{n-1}$ for $\lambda \in \mathbf{Z}_{(p)}-p\mathbf{Z}_{(p)}$. Thus $\lambda a_n\equiv b_n$ mod (J_n) and we have the equality $(J_{n+1})=(K_{n+1})$. q. e. d.

If $n>1$, the map f_n is not surjective. In fact, we can find an element $\{a_n/a_0\cdots a_{n-1}\}$ of $\{H^0N^n\}$ with a_n a zero divisor of $BP_*/(a_0, \cdots, a_{n-1})$. For example, take

$$\{pv_2^{(5)}+v_1^{(3)+(2)}v_2^{(5)-(2)}/p^2v_1^{(3)+(2)+(1)}\} \qquad \text{if} \quad n=2.$$

§3. The elements $x_{n,i}$ for an odd prime

From here on we assume that the prime p is odd. Then the elements $x(n, i)\in v_n^{-1}BP_*$ $(n\geqq 1,\ i\geqq 0)$ $(=x_{n,i}$ in [3]) are defined as follows (cf. [3; p. 494]):

(3.1) $$x(n, 0)=v_n,\ x(n, i)=x(n, i-1)^p-(v_{n-1})^{b(n,i)}y(n, i) \quad \text{for } i\geqq 1.$$

Here $v_0=p$, the elements $y(n, i)$ are given by

(3.1.1) $$y(n, 1)=v_n^{-1}v_{n+1} \qquad \text{if} \quad n\geqq 2;$$

$$y(2, 2)=v_2^{e(1,2)}(v_2+v_1^pv_2^{-p}v_3);\ y(2, i)=2v_2^{e(1,i-1)+1} \qquad \text{if} \quad i\geqq 3;$$

$$y(n, i)=v_n^{e(1,i-1)+1} \qquad \text{if} \quad n\geqq 3, \quad i\equiv 1\ (n-1) \text{ and } i>1; \text{ and}$$

$$y(n, i)=0 \qquad \text{otherwise, for the integers}$$

(3.1.2) $$e(k, j)=kp^i-p^{i-1}$$

and the integers $b(n, i)$ denote p^i for $n=1$ or $i<n$, and

(3.1.3) $$b(n, i)=p^j(p^{k(n-1)}-1)(p^n-1)/(p^{n-1}-1)$$

for $n>1$ and $i=k(n-1)+j+1\geqq n$ with $0\leqq j<n-1$.

Calculations with the equalities (2.1.3) and $\eta_R v_1=v_1+pt_1$ given by (2.1.1–2) show us that these elements satisfy the following

PROPOSITION 3.2([3; pp. 492–495]). *Let n and i be positive integres. For the differential* $d=\eta_R-\eta_L$: $v_n^{-1}BP_*\to v_n^{-1}BP_*BP$, $dx(n, i)$ is computed to be:

$$dx(n, 0)\equiv v_{n-1}t_1^{(n-1)} \bmod (I_{n-1}, v_{n-1}^2)\ (v_0=p);$$
$$dx(1, i)\equiv p^{i+1}v_1^e t_1 \bmod (p^{i+2}) \quad for \quad e=p^i-1;$$
$$dx(2, 1)\equiv v_1^p v_2^{p-1}t_1 \bmod (p, v_1^{p+1});$$
$$dx(2, i)\equiv 2v_1^{a(2,i)}v_2^{e(1,i-1)}t_1 \bmod (p, v_1^{1+a(2,i)});\ and$$
$$dx(n, i)\equiv v_{n-1}^{a(n,i)}v_n^{e(1,i-1)}t_1^{(j)} \bmod (I_{n-1}, v_{n-1}^{1+a(n,i)})$$

for $n\geqq 3$, $i=k(n-1)+j+1$ *with* $0\leqq j<n-1$, *and the integres*

(3.2.1)
$$a(2, i)=b(2, i)+p \qquad (n=2,\ i>1)$$
$$a(n, i)=b(n, i) \qquad (n>2,\ i<n)$$
$$a(n, i)=b(n, i)+p^{j+1} \quad (n>2,\ i\geqq n)$$

CONVENTION 3.3. Since $v_i^{-1}BP_*=\mathbf{Z}_{(p)}[v_i^{-1}, v_1, \cdots]$ contains $BP_*=\mathbf{Z}_{(p)}[v_1, \cdots]$ canonically, each element x of $v_i^{-1}BP_*$ is uniquely written as:

$$x=x^{\sim}+x! \quad \text{for } x^{\sim}\in BP_* \text{ and } x!\in v_i^{-1}BP_*$$

such that $x=x!$ in $v_i^{-1}BP_*/BP_*$. Then a sequence J: $a_0, a_1, \cdots$ with $a_0\in BP_*$ and $a_i\in v_i^{-1}BP_*$ $(i\geqq 1)$ is considered to be the sequence of BP_* by replacing a_i with $a_i^{\sim}$, and so we have the ideal (J_n) of BP_*.

Consider the sequence of positive integers

$$S\colon e, s_1, \cdots, s_k, \cdots$$

with $s_k=e_k p^{i_k}$ and $p\nmid e_k$ for $k>0$. We call the sequence S *pre-MRW* if it satisfies

(3.4) $0<e\leqq i_1+1$, $u_k=i_k-i_{k-1}-e+1\geqq 0$, $0<e_{k-1}\leqq a(k, u_k)$, and $e_{k-1}\leqq p^{u_k}$ if $e_k=1$.

A subsequence S_n: $e, s_1, \cdots, s_{n-1}$ of a pre-MRW sequence S is also called pre-MRW. For a pre-MRW sequence S, we have the sequence $J(S)=\{a_k\}_{k\geqq 0}$ of BP_* given by

$$a_0=p^e,\ a_k=x(k, u_k)^f \text{ for } k>0,\ f=e_k p^u>p^u\ (u=i_{k-1}+e-1), \text{ and}$$
$$a_k=v_k^{s_k} \text{ for } k>0 \text{ if } e_k=1.$$

A subsequence of $J(S)$ is said to be a *BT-sequence* if the every entry a_k is a power of v_k. Notice that $J(S)_n$ is regular for any n. The following is a result of [7].

PROPOSITION 3.5. *Let $n>0$ and S be a pre-MRW sequence. If $e_k=1$ for all k with $0<k<n$, then $J(S)_n$ is an invariant regular BT-sequence.*

LEMMA 3.6. *Let S be a pre-MRW sequence and $n>0$. Then we have*

$$pv_n^s\equiv 0 \bmod (J(S)_{n+1}) \text{ if } s\geqq s_n, \text{ and } v_n^s\equiv 0 \bmod (J(S)_{n+1}) \text{ if } s\geqq s_n',$$

for the integers $s_n=e_n p^{i_n}$ and $s_n'=s_n$ $(+p^{i_n-e+1}+p^{i_n-e}$ if $e_n>1)$.

PROOF. For $n=1$, $J(S)_2$ is a sequence of the form p^e, v_1^s for $s=kp^{e-1}$ $(k>0)$, and so $v_1^s\equiv 0 \bmod (J(S)_2)$. In case $e_n=1$, the lemma is clear. Now suppose that the lemma holds for n, $e_n>1$. Then the ideal $K_n=(p^e, pv_n^{s_n}, v_n^{s_n'})$ is contained in $(J(S)_{n+1})$. We also consider the ideal $L_{n+1}=(K_n, a_{n+1})$. Put $i=i_{n+1}$ and $l=i_n$. By the definition (3.1), we obtain the congruence

$$x(n+1, k)\equiv(v_{n+1})^{(k)}-v_n^{(k)-(k-n-1)}v_{n+1}^{-(k-1)}y \bmod (p) \tag{3.6.1}$$

in $v_{n+1}^{-1}BP_*$ for some $y\in BP_*$, where y is a multiple of $v_n^{(n+1-k)}$ if $k<n+1$. The congruence (3.6.1) implies

$$(v_{n+1})^{s_{n+1}}\equiv sp^{e-1}v_n^{(k)-(k-n-1)}v_{n+1}^{s_{n+1}-p^k-p^{k-1}}\, y \bmod L_{n+1}$$

in BP_* with Convention 3.3. Since $L_{n+1}\subset(J(S)_{n+2})$ and $(v_n^{(k)-(k-n-1)})^2\equiv 0 \bmod L_{n+1}$, we have the lemma. q. e. d.

LEMMA 3.7. *Let $n>1$ and S be a pre-MRW sequence. Put $i=i_{n-1}$ and consider the sequence K_{n-1}: p^e, $v_1^{(i)}$, $\cdots$, $v_{n-3}^{(i)}$, v_{n-2}^c, where $c=p^i$ if* e>1, *and* $=2p^i$ *if $e=1$. Then $(K_{n-1})\subset(J(S)_n)$.*

PROOF. Put $l=i_k$ for $0<k<n-1$. If $e_k=1$, then $a_k=v_k^{(l)}$ for $l=i_k$ $(0<k<n-1)$. Therefore we have $v_k^{(i)}\in(J(S)_n)$ since $i\geqq l$ by the inequality $i-l-e+1\geqq 0$. Notice that $a(m, 0)=1$ for $m>1$. Thus if $e_k>1$, then $i-l-e+1\geqq u_{k+1}>0$, and so $i>l$. Therefore, $p^i>a(k+1, u_{k+1})p^l+p^l+p^{l-1}\geqq s_k'$ if $e>1$ or $k<n-2$ by the assumption, and hence $v_k^{(i)}\in(J(S)_n)$ by Lemma 3.6. Similarly we see that $v_{n-2}^{2(i)}\in(J(S)_n)$ in the case $e=1$.

q. e. d.

A pre-MRW sequence S is said to be an *MRW-sequence* if S satisfies the following conditions a) or b) for each k, and c) if $e=1$.

a) $e_k>1$ and $e_{k-1}<a(k, u_k)(-1$ if $e=1)$.

b) $e_k=1$ and $e_{k-1}<p^u(-1$ if $e=1)$ $(u=u_k=i_k-i_{k-1}-e+1)$.

c) $k\leqq 2$ or $1<u_k\not\equiv 1$ $(k-1)$.

A subsequence S_n: e, s_1, $\cdots$, s_{n-1} of an MRW-sequence S is said to be an *MRW-sequence of length n*.

PROPOSITION 3.8. *Let p be an odd prime and S_n an MRW-sequence of length $n>1$. Then the sequence $J(S)_n$ is invariant regular.*

PROOF. If $n \leqq 3$, then the results of [3] with Lemma 2.5 lead us to the proposition. Suppose that $n \geqq 3$ and $J(S)_n$ is an invariant regular sequence. It is enough to show that $\eta_R a_n \equiv \eta_L a_n \bmod (J(S)_n)$. We put $i=i_n$ and $l=i_{n-1}$. We first show it in the case $e_n>1$. Now we notice the following:

(3.8.1) If $dx \equiv 0 \bmod (p, a_1, \cdots, a_n)$, then $dx^{(k)} \equiv 0 \bmod (p, a_1^{(k)}, \cdots, a_n^{(k)})$, and

(3.8.2) If $dx \equiv 0 \bmod (p, I)$, then $dx^{(k)} \equiv 0 \bmod (p^{k+1}, I)$.

Here $d=\eta_R-\eta_L$. Consider the invariant regular sequence J'_n: $p, v_1, \cdots, v_{n-3}, v_{n-2}^c, v_{n-1}^a$ and the element $x=x(n, i-l-e+1)$, where $c=3-\min\{2, e\}$ and $a=a(n, i-l-e+1)$. The condition c) guarantees that J'_n is invariant even if $e=1$ since $p|a$. Then we have $dx^s \equiv 0 \bmod (J'_n)$ for $s>1$ by Proposition 3.2, and by the condition c) if $e=1$. Therefore $dx^t \equiv 0 \bmod (J''_n)$ for $t=sp^{l+e-1}$ and the sequence J''_n: $p^e, v_1^{(l)}, \cdots, v_{n-3}^{(l)}, v_{n-2}^{c(l)}, v_{n-1}^{a'}$ with $a'=ap^l$ by (3.8.1–2). Since $J(S)_n$ satisfies a), $a' \geqq e_{n-1}p^l+p^l(+p^l$ if $e=1)>s'_{n-1}$. Thus we have $(J''_n) \subset (J(S)_n)$ by Lemmas 3.6–7, and $dx^t \equiv 0 \bmod (J(S)_n)$. Take now $s=e_n$, and we see that $J(S)_{n+1}$ is invariant.

Next suppose $e_n=1$. In this case we have $dv_n^{(i)} \equiv 0 \bmod (J^{\sim}_n)$ for the sequence $J^{\sim}_n$: p^e, $v_1^{(k)}, \cdots, v_{n-1}^{(k)} (k=i-e+1)$ by (3.8.1–2), since $dv_n \equiv 0 \bmod I_n$. The assumption b) and Lemmas 3.6–7 show that $(J^{\sim}_n) \subset (J(S)_n)$. Thus we prove the case $e_n=1$. q. e. d.

PROPOSITION 3.9. *Let p be an odd prime and S a sequence of integers. If the subsequence S_n is not pre-MRW, then $J(S)_n$ is not invariant regular.*

PROOF. If S_n is not pre-MRW, then we have a positive integer $k \leqq n$ such that S_k is pre-MRW and S_{k+1} is not. If $k=1$ or 2, then the proposition is the corollary of Proposition 2.7 by virtue of the results on H^0N^k $(k=1, 2)$ of [3]. Now suppose $k>2$. Consider the sequence of integers S': $e, s'_1, \cdots$ with $s'_i=p^{ie-i}$ for $i>0$. Then $J(S')_k$ is invariant regular by [7], and the ideal $(J(S')_{k-1}, v_{k-1}^s)$ $(s=s_{k-1})$ contains the ideal $(J(S)_k)$ since $u_i=i_i-i_{i-1}-e+1 \geqq 0$ for $i<k$ and $a_{k-1} \equiv v_{k-1}^s \bmod (J(S')_{k-1})$. If S_k does not satisfy the condition (3.4), then Proposition 3.2 implies $da_k \not\equiv 0 \bmod (J(S')_k)$ and so $J(S)_k$ is not invariant. q. e. d.

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