On Radon transform for Minkowski space

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1 Introduction

Let $\Xi_{\boldsymbol{R}^n}$ be the set of all hyperplanes in Euclidean space $\boldsymbol{R}^n$. The Radon transform for $\boldsymbol{R}^n$ is a mapping of a function f on $\boldsymbol{R}^n$ to a function $\hat{f}$ on $\Xi_{\boldsymbol{R}^n}$, where $\hat{f}(\xi)$, $\xi \in \Xi$, is the value of integration of f on ξ. S. Helgason [H] formulated the Radon transform in group-theoretically in more general settings. His formulation is as follows. Let G be a locally compact unimodular group and X and Ξ two left coset spaces of G by closed unimodular subgroups H_X and H_Ξ, respectively:

$$X=G/H_X, \qquad \Xi=G/H_\Xi .$$

Under some more assumptions, he considered the Radon transform for the double fibration:

$$\begin{array}{ccc} & G/(H_X \cap H_\Xi) & \\ \swarrow & & \searrow \\ G/H_X & & G/H_\Xi . \end{array}$$

In the present paper we consider (n+1)-dimensional Minkowski space X. Let $\boldsymbol{M}(1, n)$ be the affine motion group of X, i.e. the semidirect product of the proper Lorentz group $\boldsymbol{SO}_0(1, n)$ with X. Then $X \cong \boldsymbol{M}(1, n)/\boldsymbol{SO}_0(1, n)$. Let Ξ be the set of all hyperplanes in X. Then Ξ is not single homogeneous space of $\boldsymbol{M}(1, n)$ but is the union of three homogeneous spaces of $\boldsymbol{M}(1, n)$. So this gives an example of more general situation than that of Helgason's formulation. However, the results are similar to those of Euclidean cases (cf. [L], [H]). We get the inversion formula for Radon transform and the unitarity of the composition opetator of Radon transform and a certain pseudo-differential operator.

Euclidean space $\boldsymbol{R}^n$ is the tangent space of a Riemannian symmetric space $\boldsymbol{SO}_0(1, n)/\boldsymbol{SO}(n)$ at the origin. On the other hand Minkowski space X is the tangent space of a semisimple symmetric sapce $\boldsymbol{SO}_0(1, n+1)/\boldsymbol{SO}_0(1, n)$ at the origin. Let (G, H) be a semisimple (i.e. an afffine) symmetric pair and $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ be the corresponding Lie algebra decomposition. Then $\mathfrak{q}$ is a pseudo-Euclidean space whose metric is induced by the Killing form of $\mathfrak{g}$ and whose affine Cartan motion group is the semidirect product H with $\mathfrak{q}$. So our study is the first step of reserches on such general cases.

2 Hyperplanes in Minkowski space

Let X be an $n+1$ dimensional real vector space with inner product $\langle\ ,\ \rangle$ of signature $(1, n)$. We fix a Lorentzian orthonormal basis $e_0, e_1, \cdots, e_n$ such that $\langle e_i, e_j\rangle=-1(i=j=0), =1(i=j>0), =0(i\neq j)$. Then $\langle x, y\rangle=-x_0y_0+x_1y_1+\cdots+x_ny_n$ for $x=x_0e_0+x_1e_1+\cdots+x_ne_n$ and $y=y_0e_0+y_1e_1+\cdots\cdots y_ne_n$. We denote by Ξ the set of all hyperplanes in X. We assume that a hyperplane $\xi\in\Xi$ is given by an equation

$$a_0x_0+a_1x_1+\cdots+a_nx_n=c$$

for $a\in\boldsymbol{R}^{n+1}(a\neq 0)$ and $c\in\boldsymbol{R}$. If $\langle a, a\rangle\neq 0$, we put $\omega_0=a_0/\sqrt{|\langle a, a\rangle|}$, $\omega_j=a_j/\sqrt{|\langle a, a\rangle|}$ (j>0) and $p=c/\sqrt{|\langle a, a\rangle|}$. If $\langle a, a\rangle=0$, we put $\omega_0=-a_0/|a_0|$, $\omega_j=a_j/|a_0|$ $(j>0)$ and $p=c/|a_0|$. Then ξ is given by

$$\langle x, \omega\rangle=-x_0\omega_0+x_1\omega_1+\cdots+x_n\omega_n=p,$$

where $\langle\omega, \omega\rangle=\pm 1$ or $\langle\omega, \omega\rangle=0$, $\omega_0=\pm 1$. We denote by $\xi=\xi(\omega, p)$. Note that $\xi(\omega, p)=\xi(-\omega, -p)$ and $\xi(k\omega, 0)=\xi(\omega, 0)$ for $\omega\in X$ and $k\in\boldsymbol{R}$.

Let $X_-^+=\{\omega\in X; \langle\omega, \omega\rangle=-1, \omega_0>0\}$ and $X_-^-=\{\omega\in X; \langle\omega, \omega\rangle=-1, \omega_0<0\}$. $X_-^\pm$ are the spaces of the timelike unit vectors. And we put $X_+=\{\omega\in X; \langle\omega, \omega\rangle=1\}$, $X_0^+=\{\omega\in X; \langle\omega, \omega\rangle=0, \omega_0>0\}$ and $X_0^-=\{\omega\in X; \langle\omega, \omega\rangle=0, \omega_0<0\}$. X_+ is the space of spacelike unit vectors and $X_0^\pm$ are the spaces of lightlike vectors. And we consider subspaces $S_\pm=\{\omega\in X; \langle\omega, \omega\rangle=0, \omega_0=\pm 1\}$. A parameter space of Ξ is $X_-^+\cup(X_+/\boldsymbol{Z}_2)\cup S_+$, where $\boldsymbol{Z}_2=\{\pm 1\}$.

3 Action of the affine motion group

Let $G=\boldsymbol{SO}_0(1, n)$ be the proper Lorentz group, that is, the group of $(n+1, n+1)$ matrices $g=(g_{ij})$, $0\leq i, j\leq n$, which leaves the indefinite inner product $\langle\ ,\ \rangle$ and $\det g=1$, $g_{00}\geq 1$. Let K be the subgroup of G of $k=(k_{ij})$ satisfying $k_{00}=1$. Then $k_{0j}=k_{i0}=0$, $i, j=1,\cdots, n$, and K is isomorphic to $\boldsymbol{SO}(n)$ and is a maximal compact subgroup of G. Let H be the subgroup of G of $h=(h_{ij})$ satisfying $h_{11}=1$. Then $h_{1j}=h_{i1}=0$, $i, j=0, 2,\cdots, n$ and H is isomorphic to $\boldsymbol{SO}_0(1, n-1)$. And we define the subgroups M, A and N as follows.

$$M=\left\{ m=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & m & \\ 0 & 0 & & & \end{pmatrix} ;\ m\in \boldsymbol{SO}(n-1) \right\}$$

$$A=\left\{ a(t)=\begin{pmatrix} \cosh t & \sinh t & 0 & \cdots & 0 \\ \sinh t & \cosh t & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{n-1} & \\ 0 & 0 & & & \end{pmatrix} ;\ t\in \boldsymbol{R} \right\},$$

and

$$N=\left\{ n=\begin{pmatrix} 1+\Delta/2 & -\Delta/2 & y_2 & \cdots & y_n \\ \Delta/2 & 1-\Delta/2 & y_2 & \cdots & y_n \\ y_2 & -y_2 & & & \\ \vdots & \vdots & & I_{n-1} & \\ y_n & -y_n & & & \end{pmatrix} ;\ y_i\in \boldsymbol{R} \right\},$$

where $\Delta=y_2^2+\cdots+y_n^2$. We put $P=MAN$ the minimal parabolic subgroup of G.

The group G acts on X by $x\to gx$, where $x=\sum_{i=0}^n x_i e_i$ and $(gx)_i=\sum_{j=0}^n g_{ij}x_j$. Then G acts on $X_-^{\pm}$ transitively and the subgroup fixing e_0 is K. So we can identify $X_-^{\pm}$ with G/K : $X_-^{\pm}\cong G/K$. In the same way, $X_=^{-}\cong G/K$, $X_+\cong G/H$, $X_0^+\cong X_0^-\cong G/MN$ and $S_+\cong S_-\cong \boldsymbol{S}^{n-1}\cong G/P\cong K/M$ as homogeneous spaces. And we have the following G-orbit space decomposition of X.

$$X=(\bigcup_{t>0} tX_-^+)\cup(\bigcup_{t>0} tX_-^-)\cup(\bigcup_{t\neq 0} tX_+)\cup X_0^+\cup X_0^-\cup\{0\}.$$

Let $\boldsymbol{M}(1, n)$ be the affine motion group on X, i.e. the semidirect product of G with X. The action of $(g, z)\in\boldsymbol{M}(1, n)$ $(g=(g_{ij})\in G,\ z=z_0e_0+z_1e_1+\cdots+z_ne_n\in X)$ on X is $(g, z)x=gx+z\ (x\in X)$. Then as a homogeneous space $\boldsymbol{M}(1, n)/G\cong X$. We identify the subgroup $\{(g, z)\in\boldsymbol{M}(1, n); g_{11}=1,\ z_1=0\}$ with $\boldsymbol{M}(1, n-1)$. And we also identify the subgroup $\{(g, z)\in\boldsymbol{M}(1, n); g_{00}=1,\ z_0=0\}$ with the Euclidean motion group $\boldsymbol{M}(n)$ which is the semidirect product of $\boldsymbol{SO}(n)$ with $\boldsymbol{R}^n$.

Let $\xi=\xi(\omega, p)\in\Xi$. For $x\in\xi(\omega, p)$ and $(g, z)\in\boldsymbol{M}(1, n)$ we put $y=(g, z)x$. Then we have

$$\begin{aligned}\langle y, g\omega\rangle&=\langle g^{-1}y, \omega\rangle=\langle x+g^{-1}z, \omega\rangle\\&=\langle x, \omega\rangle+\langle z, g\omega\rangle=p+\langle z, g\omega\rangle.\end{aligned}$$

Hence $y\in\xi(g\omega, p+\langle z, g\omega\rangle)$. Thus $\boldsymbol{M}(1, n)$ acts on X by

$$(g, z)\xi(\omega, p)=\xi(g\omega, p+\langle z, g\omega\rangle).$$

Therefore, we have the following an $\boldsymbol{M}(1, n)$-orbit decomposition.

$$\Xi=(\boldsymbol{M}(1, n)\xi(e_0, 0))\cup(\boldsymbol{M}(1, n)\xi(e_1, 0))\cup(\boldsymbol{M}(1, n)\xi(e_0+e_1, 0)).$$

If $(g, z)\xi(e_0, 0)=\xi(e_0, 0)$, then $ge_0=e_0$ and $\langle z, e_0\rangle=0$. Hence $g\in K$ and $z_0=0$. So the isotropy subgroup of $\xi(e_0, 0)$ in $\boldsymbol{M}(1, n)$ is $\boldsymbol{M}(n)$. If $(g, z)\xi(e_1, 0)=\xi(e_1, 0)$, then $ge_1=\pm e_1$ and $\langle z, e_1\rangle=0$. Therefore, $\pm g\in H$ and $z_1=0$. Hence the isotropy subgroup of $\xi(e_1, 0)$ in $\boldsymbol{M}(1, n)$ is isomorphic to $\boldsymbol{Z}_2\cdot\boldsymbol{M}(1, n-1)$. If $(g, z)\xi(e_0+e_1, 0)=\xi(e_0+e_1, 0)$, then $g(e_0+e_1)=(e_0+e_1)$ and $\langle z, e_0+e_1\rangle=0$. Let $g=ka(t)n(k\in K,\ a(t)\in A,\ n\in N)$ be the Iwasawa decomposition of g. Then $n(e_0+e_1)=(e_0+e_1)$ and $a(t)(e_0+e_1)=e^t(e_0+e_1)$. Hence $e^tk(e_0+e_1)=(e_0+e_1)$. So we have $t=0$ and $k\in M$. Thus we have $g\in MN$ and $z_0=z_1$. If we identify $ze_0+ze_1+z_2e_2+\cdots+z_ne_n\in X$ with $ze_1+z_2e_2+\cdots+z_ne_n\in\boldsymbol{R}^n$, the isotropy subgroup of $\xi(e_0+e_1, 0)$ in $\boldsymbol{M}(1, n)$ is isomorphic to $MN\times\boldsymbol{R}^n$.

LEMMA 1. *The space* Ξ *of all hyperplanes in* X *is decomposed to* $\boldsymbol{M}(1, n)$*-orbits by*

$$\Xi \cong \boldsymbol{M}(1, n)/\boldsymbol{M}(n) \bigcup \boldsymbol{M}(1, n)/(\boldsymbol{Z}_2 \cdot \boldsymbol{M}(1, n-1)) \bigcup \boldsymbol{M}(1, n)/(MN \times \boldsymbol{R}^n).$$

We define a coordinate system and an Euclidean measure on ξ by the following way. We assume that $\omega_0 \geq 0$.

(i) $\omega = \omega_K \in X^{\pm}_{-}$. There exists an element $g_\omega \in G$ such that $\omega = g_\omega e_0$. We put $\eta_i = g_\omega e_i$, $i = 1, \cdots, n$. Then the system $\omega_K, \eta_1, \cdots, \eta_n$ is a Lorentzian orthonormal system. It is easy to see that $\langle x, \omega_K \rangle = p$ if and only if there exist $t_1, \cdots, t_n \in \boldsymbol{R}$ such that $x = -p\omega_K + t_1\eta_1 + \cdots + t_n\eta_n$. We write $x = x(t_1, \cdots, t_n)$. In this case $\langle x, x \rangle = -p^2 + t_1^2 + \cdots + t_n^2$. We give a Euclidean measure $dm = dm_\xi$ on ξ by $dm(x) = dt_1 \cdots dt_n$ for $x = x(t_1, \cdots, t_n) \in \xi$.

(ii) $\omega = \omega_H \in X_+$. There exists $g_\omega \in G$ such that $\omega_H = g_\omega e_1$. We put $\eta_1 = g_\omega e_0$ and $\eta_i = g_\omega e_i$, $i = 2, \cdots, n$. Then the system $\{\eta_1, \omega_H, \eta_2, \cdots, \eta_n\}$ is a Lorentzian orthonormal system in this order. Then $\langle x, \omega_H \rangle = p$ if and only if there exist $t_1, t_2, \cdots, t_n \in \boldsymbol{R}$ such that $x = p\omega_H + t_1\eta_1 + t_2\eta_2 + \cdots + t_n\eta_n$. The measure on ξ is $dm(x) = dm_\xi(x) = dt_1 dt_2 \cdots dt_n$ for $x = x(t_1, t_2. \cdots, t_n) \in \xi$. In this case $\langle x, x \rangle = p^2 - t_1^2 + t_2^2 + \cdots + t_n^2$.

(iii) $\omega = \omega_P \in X_0^{\pm}$. We put $x^* = x - x_0 e_0$ for $x \in X$. Then $\langle \omega^*, \omega^* \rangle = 1$. There exists $g_\omega \in K$ such that $\omega_P{}^* = g_\omega e_1$. We put $\eta_i = g_\omega e_i$, $i = 2, \cdots, n$. Then $\eta^*{}_i = \eta_i (i = 2, \cdots, n)$ and the system $\{\omega^*, \eta_2, \cdots \eta_n\}$ is orthonormal with respect to $\langle\ ,\ \rangle$. Clearly $\langle \omega^*, \omega \rangle = \langle \eta_2, \omega \rangle = \cdots = \langle \eta_n, \omega \rangle = 0$. If $\langle x, \omega \rangle = p$, then $x_0 = \langle x, \omega^* \rangle - p$. We write $x*$ as a linear combination of ω^*, $\eta_2, \cdots, \eta_n$: $x^* = t_1\omega^* + t_2\eta_2 + \cdots + t_n\eta_n$. Since $\langle x, \omega^* \rangle = \langle x^*, \omega^* \rangle$, $t_1 = x_0 + p$. We put $\eta_1 = \omega$. Thus we have that $\langle x, \omega_P \rangle = p$ if and only if there exist $t_1, \cdots, t_n \in \boldsymbol{R}$ such that $x = -pe_0 + t_1\eta_1 + \cdots + t_n\eta_n$. The measure on ξ is $dm(x) = dm_\xi(m) = dt_1 \cdots dt_n$.

LEMMA 2. *Let* $\xi \in \Xi$ *and* $x \in \xi$. *If we put* $\xi' = (g, z)\xi$ *and* $y = (g, z)x$ *for* $(g, z) \in \boldsymbol{M}(1, n)$, *then we have*

$$dm_{\xi'}(y) = dm_\xi(x).$$

PROOF. We put

$$\xi' = (g, z)\xi(\omega, p)$$

and

$$y = y(t_1', t_2', \cdots, t_n') = (g, z)x(t_1, t_2, \cdots, t_n).$$

(i) Since $y \in \xi(g\omega, p + \langle\omega, z\rangle)$,

$$\begin{aligned} y &= -(p + \langle\omega, z\rangle)g\omega + t_1' g g_\omega e_1 + \cdots + t_n' g g_\omega e_n \\ &= -(p + \langle\omega, z\rangle)g g_\omega e_0 + t_1' g g_\omega e_1 + \cdots + t_n' g g_\omega e_n. \end{aligned}$$

On the other hand,

$$y = gx + z = -pg\omega + t_1 g g_\omega e_1 + \cdots + t_n g g_\omega e_n + z.$$

Hence $(t_1', t_2', \cdots, t_n')$ is a translation in $\boldsymbol{R}^n$ of $(t_1, t_2, \cdots, t_n)$. So we have the $\boldsymbol{M}(1, n)$-invariance of the measure dm : $dm_{\xi'}(y) = dm_\xi(x)$.

(ii) Since

$$\begin{aligned} y &= (p + \langle\omega, z\rangle)g\omega + t_1' g g_\omega e_0 + t_2' g g_\omega e_2 + \cdots + t_n' g g_\omega e_n \\ &= pg\omega + t_1 g g_\omega e_0 + t_2 g g_\omega e_2 + \cdots + t_n g g_\omega e_n + z, \end{aligned}$$

we have $dm_{\xi'}(y) = dm_\xi(x)$.

(iii) Since

$$\begin{aligned} y &= -(p + \langle\omega, z\rangle)e_0 + t_1' g g_\omega e_1 + t_2' g g_\omega e_2 + \cdots + t_n' g g_\omega e_n \\ &= -pge_0 + t_1 g g_\omega e_0 + t_2 g g_\omega e_2 + \cdots + t_n g g_\omega e_n + z, \end{aligned}$$

we have $dm_{\xi'}(y) = dm_\xi(x)$.

Remark that in each case we have

$$\det \left| \frac{\partial (x_0, x_1, \cdots, x_n)}{\partial (p, t_1, \cdots, t_n)} \right| = 1$$

and so $dx_0 dx_1 \cdots dx_n = dp dt_1 \cdots dt_n$.

4 Radon transform

We put $\varphi(\omega, p) = \varphi(\xi(\omega, p))$ for any function φ on Ξ. Let f be a function on X, integrable on each hyperplane in X. As in the Euclidean space, we define the Radon transform $\hat{f} = Rf$ of f by

$$\begin{aligned}
\hat{f}(\xi) &= \hat{f}(\omega, p) = (Rf)(\xi) \\
&= \int_\xi f(x)\, dm(x) \\
&= \int_{\langle x, \omega \rangle = p} f(x)\, dm(x) \\
&= \int_X f(x)\, \delta(p - \langle x, \omega \rangle)\, dx,
\end{aligned}$$

where $dm = dm_\xi$ is the Euclidean measure on ξ and δ is Dirac's delta function.

Let $d\mu_-(\omega)$ and $d\mu_+(\omega)$ be the G-invariant measures on $X^+_- \cup X^-_-$ and X_+, respectively, normalized so that

$$\begin{aligned}
&\int_X f(x)\, dx \\
&= \int_0^\infty \int_{X^+_-} f(t\omega)\, t^n dt d\mu_-(\omega) + \int_0^\infty \int_{X^-_-} f(t\omega)\, t^n dt d\mu_-(\omega) + \int_0^\infty \int_{X_+} f(t\omega)\, t^n dt d\mu_+(\omega) \\
&= \int_{-\infty}^\infty \int_{X^+_-} f(t\omega)\, |t|^n dt d\mu_-(\omega) + \int_{-\infty}^\infty \int_{X^+_+} f(t\omega)\, |t|^n dt d\mu_+(\omega) \\
&= \int_{-\infty}^\infty \int_{X^-_-} f(t\omega)\, |t|^n dt d\mu_-(\omega) + \int_{-\infty}^\infty \int_{X^-_+} f(t\omega)\, |t|^n dt d\mu_+(\omega)
\end{aligned}$$

$$=\frac{1}{2}\left\{\int_{-\infty}^{\infty}\int_{X_-}f(t\omega)\,|\,t\,|^n dtd\mu_-(\omega)+\int_{-\infty}^{\infty}\int_{X_+}f(t\omega)\,|\,t\,|^n dtd\mu_+(\omega)\right\}.$$

Then

$$d\mu_\pm(\omega)=\frac{1}{|\,\omega_i\,|}d\omega_0\cdots d\omega_i\cdots d\omega_n$$

in a neibourhood where $\omega_i\neq 0$.

Let $\partial X=X_-^+\cup X_-^-\cup X_+\cup S_+\cup S_-$, the 'boundary' of X. We define the measure $d\mu(\omega)$ on ∂X by

$$\int_{\partial X}\psi(\omega)\,d\mu(\omega)=\int_{X_-^+\cup X_-^-}\psi(\omega)\,d\mu_-(\omega)+\int_{X_+}\psi(\omega)\,d\mu_+(\omega),$$

where $\psi\in C_0(\partial X)$.

We identify a function $\varphi(\xi)$ on Ξ with a function $\varphi(\omega,\ p)$ on $\partial X\times \boldsymbol{R}$ satisfying $\varphi(-\omega,\ -p)=\varphi(\omega,\ p)$. Then the measure $d\mu(\xi)$ defines a G-invariant measure $d\sigma_x$ on $\check{x}=\{\xi\in\Xi:\ \xi\ni x\}$ by

$$\int_{\xi\ni x}\varphi(\xi)\,d\sigma_x(\xi)=\int_{\partial X}\varphi(\omega,\ \langle x,\ \omega\rangle)\,d\mu(\omega).$$

Now we define the dual Radon transform $\check{\varphi}=R^*\varphi$ of an integrable function φ on Ξ by

$$\check{\varphi}(x)=(R^*\varphi)(x)=\int_{\xi\ni x}\varphi(\xi)\,d\sigma_x(\xi)=\int_{\partial X}\varphi(\omega,\ \langle x,\ \omega\rangle)\,d\mu(\omega).$$

LEMMA 3.

$$\int_X f(x)\overline{R^*\varphi(x)}\,dx=\int_{\partial X}\int_{-\infty}^{\infty}(Rf)(\omega,\ p)\overline{\varphi(\omega,\ p)}\,d\mu(\omega)\,dp \tag{4.1}$$

for f in $C_0(X)$ and $\varphi \in C_0(\Xi)$.

PROOF.

$$\int_{\partial X}\int_{-\infty}^{\infty}(Rf)(\omega, p)\overline{\varphi(\omega, p)}\,d\mu(\omega)\,dp$$

$$=\int_{\partial X}\int_{-\infty}^{\infty}\int_{X}f(x)\,\delta(p-\langle x, \omega\rangle)\,dx\overline{\varphi(\omega, p)}\,d\mu(\omega)\,dp$$

$$=\int_{X}f(x)\int_{\partial X}\overline{\varphi(\omega, \langle x, \omega\rangle)}\,d\mu(\omega)\,dx.$$

Let π be the quasi-regular representation of $\boldsymbol{M}(1, n)$ on X : $(\pi((g, z))f)(x)=f((g, z)^{-1}x)=f(g^{-1}x-g^{-1}z)$. Moreover, we put $(\hat{\pi}((g, z))\varphi)(\xi)=\varphi((g, z)^{-1}\xi)$.

LEMMA 4. *For any $(g, z)\in \boldsymbol{M}(1, n)$ we have*

$$R\pi((g, z))=\hat{\pi}((g, z))R$$

and

$$R^{*}\hat{\pi}((g, z))=\pi((g, z))R^{*}.$$

PROOF.

$$(\pi((g, z))f)\hat{\ }(\omega, p)=\int_{\langle x, \omega\rangle=p}f(g^{-1}x-g^{-1}z)\,dm(x)$$

$$=\int_{\langle gy, \omega\rangle=p-\langle z, \omega\rangle}f(y)\,dm(y)$$

$$=\hat{f}(g^{-1}\omega, p-\langle z, \omega\rangle)$$

$$=\hat{f}((g, z)^{-1}\xi(\omega, p))$$

$$=(\hat{\pi}((g, z))\hat{f})(\omega, p).$$

On the other hand,

$$
\begin{aligned}
(\hat{\pi}((g, z))\varphi)^{\vee}(x) &= \int_{\partial X} \hat{\pi}((g, z))\varphi(\omega, \langle x, \omega\rangle)\, d\mu(\omega) \\
&= \int_{\partial X} \varphi(g^{-1}\omega, \langle x, \omega\rangle - \langle z, \omega\rangle)\, d\mu(\omega) \\
&= \int_{\partial X} \varphi(\omega, \langle x-z, g\omega\rangle)\, d\mu(\omega) \\
&= \int_{\partial X} \varphi(\omega, \langle g,^{-1}x - g^{-1}z\rangle)\, d\mu(\omega) \\
&= (\pi((g, z))\varphi)^{\vee}(x).
\end{aligned}
$$

This shows that both the Radon transform and the dual Radon transform are intertwining operators between π and $\hat{\pi}$.

We denote by ∂_i the differential operator $\partial/\partial x_i$.

LEMMA 5. *For* $f \in C_0^\infty(X)$ *we have*

$$
\langle e_i, \omega\rangle \frac{\partial}{\partial p} \hat{f}(\omega, p) = (\partial_i f)\hat{\ }(\omega, p)
$$

and

$$
\langle e_i, \omega\rangle \frac{\partial}{\partial \omega_i} \hat{f}(\omega, p) = -\{(\langle x, e_i\rangle \partial_i + \partial_i(\langle x, e_i\rangle))f\}\hat{\ }(\omega, p).
$$

PROOF. If $t = \langle x, \omega\rangle - p$, then we have

$$
\frac{\partial}{\partial p}\{\delta(\langle x, \omega\rangle - p)\} = -\left(\frac{d}{dt}\delta\right)(\langle x, \omega\rangle - p),
$$

$$
\partial_i(\delta(\langle x, \omega\rangle - p)) = \langle e_i, \omega\rangle \left(\frac{d}{dt}\delta\right)(\langle x, \omega\rangle - p)
$$

and

$$\frac{\partial}{\partial \omega_i}(\delta(\langle x, \omega\rangle - p)) = \langle x, e_i\rangle \left(\frac{d}{dt}\delta\right)(\langle x, \omega\rangle - p).$$

We can get our results from these relations by integration by part.

Let $\square = -\partial_0^2 + \partial_1^2 + \cdots + \partial_n^2$ be the pseudo-Laplacian on X. We define the operator L by

$$(L\varphi)(\omega, p) = \langle \omega, \omega\rangle \left(\frac{\partial^2}{\partial p^2}\varphi\right)(\omega, p).$$

Then

$$(\square f)\hat{}\,(\omega, p) = (L\hat{f})(\omega, p).$$

$$(L\varphi)^{\vee}(x)$$
$$= -\int_{X_-}\frac{\partial^2}{\partial p^2}\varphi(\omega, \langle x, \omega_K\rangle)\, d\omega_K + \int_{X_+}\frac{\partial^2}{\partial p^2}\varphi(\omega, \langle x, \omega_H\rangle)\, d\omega_H.$$

On the other hand $\square\ (\varphi(\omega, \langle x, \omega\rangle)) = \langle \omega, \omega\rangle \frac{\partial^2}{\partial p^2}\varphi(\omega, \langle x, \omega\rangle)$. Hence

$$(L\varphi)^{\vee}(x) = \square\ (\check{\varphi})(x).$$

Thus we have the following proposition.

PROPOSITION. *We have*

$$R\square = LR \qquad \textit{and} \qquad R^*L = \square R^*.$$

5 The Inversion formula

Let $\mathscr{S}(\boldsymbol{R}^{n+1})$ be the usual Schwartz space of C^∞ rapidly decreasing functions on X as

Euclidean space $\boldsymbol{R}^{n+1}$. Let $\mathcal{F}f=\tilde{f}$ be the Fourier transform of $f\in\mathcal{S}(\boldsymbol{R}^{n+1})$:

$$\tilde{f}(u)=\int_X f(x)e^{-i\langle x,\, u\rangle}dx\ (u\in X).$$

We know that $\mathcal{F}$ is an isomorphism of $\mathcal{S}(\boldsymbol{R}^{n+1})$ onto $\mathcal{S}(\boldsymbol{R}^{n+1})$. If $t\in\boldsymbol{R}$ and $\omega\in\partial X$, then

$$\begin{aligned}\tilde{f}(t\omega)&=\int_X f(x)e^{-it\langle x,\,\omega\rangle}dx\\&=\int_{-\infty}^{\infty}\int_{\langle x,\,\omega\rangle=p}f(x)e^{-itp}dp\,dm(x)\\&=\int_{-\infty}^{\infty}\hat{f}(\omega,\,p)e^{-itp}dp.\end{aligned}$$

Hence

$$\hat{f}(\omega,\,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(t\omega)e^{itp}dt. \tag{5.1}$$

We denote by $\boldsymbol{N}$ the set of all non-negative integers. To consider the dual Radon transform of $\hat{f}$ we set a condition of f so that $\hat{f}(\omega,\,\langle x,\,\omega\rangle)$ is rapidly decreasing on ∂X. Let $\mathcal{S}(X)$ be a subspace of $\mathcal{S}(\boldsymbol{R}^{n+1})$ of functions f which decrease rapidely at light cone too, i.e. of $f\in C^\infty(X)$ satisfying the following condition: For any $k=(k_0,\cdots,k_n)\in\boldsymbol{N}^{n+1}$, $l=(l_0,\cdots,l_n)\in\boldsymbol{N}^{n+1}$ and $m\in\boldsymbol{N}$ there exists a constant $C_{k,l}^m>0$ such that

$$|x_0^{k_0}\cdots x_n^{k_n}\partial_0^{l_0}\cdots\partial_n^{l_n}f(x)|\le C_{k,l}^m|\langle x,\,x\rangle|^m\qquad(x\in X). \tag{5.2}$$

And we put $\mathcal{S}(X)=\mathcal{F}^{-1}(\mathcal{S}(X))$.

Let $\mathcal{S}(\Xi)$ be the space of C^∞ functions ψ on $\partial X\times\boldsymbol{R}$ such that

(1) $\psi(-\omega,\,-t)=\psi(\omega,\,t)$

(2) For any $k=(k_0,\cdots,\,k_n)\in\boldsymbol{N}^{n+1}$, $\boldsymbol{l}=(\boldsymbol{l}_0,\cdots,\,\boldsymbol{l}_n)\in\boldsymbol{N}^{n+1}$ and $m,\,a,\,b\in\boldsymbol{N}$ there exists a constant $C_{k,l,a,b}^m>0$ such that

$$|\omega_0^{k_0}\cdots\omega_n^{k_n}t^a\left(\frac{\partial}{\partial\omega_0}\right)^{l_0}\cdots\left(\frac{\partial}{\partial\omega_n}\right)^{l_n}\left(\frac{\partial}{\partial t}\right)^{b}\psi(\omega,\ t)|\leq C^m_{k,l,a,b}t^{2m}$$
$$((\omega,\ t)\in\partial X\times\boldsymbol{R}).$$

We denote by $\mathscr{S}(\Xi)$ the Fourier inverse image of $\mathscr{S}(\Xi)$ with respect to t:

$$\mathscr{S}(\Xi)=\left\{\varphi(\omega,\ p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\omega,\ t)e^{itp}dt;\ \psi\in\tilde{S}(\Xi)\right\}.$$

Lemma 6. *If* $f\in\mathscr{S}(X)$, *then* $\hat{f}\in\mathscr{S}(\Xi)$.

Proof. By the relation (5.1) if $\omega\in S_+\cup S_-$, then $\hat{f}(\omega,\ p)=0$. Hence we assume that $\omega\in X^{\pm}_{-}\cup X^{-}_{-}\cup X_+$. We choose coordinate neibouhoods $X^{\pm}_{-}$ and $N^{\pm}_{i}=\{\omega\in X_+;\ |\omega_j|>1/\sqrt{n}\}$. To prove the smoothness it is enough to show that in each neibouhood where $\omega_j\neq0$

$$t^a\left(\frac{\partial}{\partial\omega_0}\right)^{l_0}\cdots\widehat{\left(\frac{\partial}{\partial\omega_j}\right)^{l_j}}\cdots\left(\frac{\partial}{\partial\omega_n}\right)^{l_n}\tilde{f}(t\omega)$$

is integrable with respect to t for any $l\in\boldsymbol{N}^{n+1}$, $a\in\boldsymbol{N}$ and $0\leq j\leq n$. Since $|(\partial\omega_j)/(\partial\omega_i)|\leq const.|\omega_i|$, the absolute value of this function is dominated by a linear combination of such functions as

$$|\omega_0^{k_0}\cdots\widehat{\omega_j^{k_j}}\cdots\omega_n^{k_n}t^a(\partial_0^{l_0}\cdots\partial_n^{l_n}\tilde{f})(t\omega)|$$
$$=|t|^{a-(\widehat{k_0}+\cdots+k_i+\cdots+k_n)/2}|(t\omega_0)^{k_0}\cdots\widehat{(t\omega_j)}^{k_j}\cdots(t\omega_n)^{k_n}(\partial_0^{l_0}\cdots\partial_n^{l_n}\tilde{f})(t\omega)|.$$

Then the integrability is clear from the rapidly dicreasing property. Rapid decreasingness of $\hat{f}$ can be prove by the same way.

Lemma 7. *For each* $f\in\mathscr{S}(X)$ *the Radon transform* $\hat{f}(\omega,\ p)$ *satisfies the following homogeneity property: For* $k\in\boldsymbol{N}$ *the integral*

$$\int_{-\infty}^{\infty} \hat{f}(\omega, p) p^k dp$$

can be written as a k-th degree homogeneous polynomial in $\omega_0, \cdots, \omega_n$.

Proof. From

$$\int_{-\infty}^{\infty} \hat{f}(\omega\ p) p^k dp = \int_{-\infty}^{\infty} p^k dp \int_{\langle x,\ \omega\rangle = p} f(x)\, dm(x)$$

$$= \int_X f(x) \langle x,\ \omega\rangle^k dx$$

we have the lemma immediately.

We denote by $\mathscr{S}_H(\Xi)$ the subspce of $\psi \in \mathscr{S}(\Xi)$ which satisfies the above homogeneity property.

Theorem 1. *The Radon transform* $f \rightarrow \hat{f}$ *is a linear one-to-one mapping of* $\mathscr{S}(X)$ *onto* $\mathscr{S}_H(\Xi)$.

Proof. It is enough to prove that Radon transform is surjective. Let $\varphi \in \mathscr{S}_H(\Xi)$. We put

$$\psi(\omega,\ t) = \int_{-\infty}^{\infty} \varphi(\omega,\ p) e^{-itp} dp.$$

Then $\psi \in \mathscr{S}(\Xi)$. We define a function F on X by

$$F(t\omega) = \psi(\omega,\ t).$$

When $u \in X$ is light vector, then $F(u) = 0$, that is, it is identically zero on light cone. Hence it is smooth and rapidly decreasing.

Next, we consider when u is a timelike vector. Let

$$u = t\omega \qquad (\omega \in X_-^+,\ t \in \boldsymbol{R} \setminus \{0\}).$$

Suppose that $t>0$. By the condition of ψ if $u\to 0$, then $F\to 0$ uniformly. We use the locally coordinate system $\{\omega_1,\cdots,\omega_n\}$ on X^{+}_{-}.
Then

$$u_0=t(1+u_1^2+\cdots+u_n^2)^{1/2},\ u_1=t\omega_1,\cdots,\ u_n=t\omega_n.$$

Then

$$\frac{\partial}{\partial u_i}=\sum_{j=1}^{n}\frac{\partial\omega_j}{\partial u_i}\frac{\partial}{\partial\omega_j}+\frac{\partial t}{\partial u_i}\frac{\partial}{\partial t}\quad (0\leq n)$$

and

$$\frac{\partial\omega_j}{\partial u_0}=\frac{u_0u_j}{t^3}\qquad (1\leq j\leq n),$$

$$\frac{\partial\omega_j}{\partial u_i}=\frac{1}{t}\left(\delta_{ij}-\frac{u_iu_j}{t^2}\right)\qquad (1\leq i,\ j\leq n)$$

and

$$\frac{\partial t}{\partial u_0}=(1+\omega_1^2+\cdots+\omega_n^2)^{1/2},$$

$$\frac{\partial t}{\partial u_i}=-\omega_i\qquad (1\leq i\leq n).$$

Hence

$$\frac{\partial}{\partial u_0}=(1+\omega_1^2+\cdots+\omega_n^2)^{1/2}\left(\frac{1}{t}\sum_{j=1}^{n}\omega_j\frac{\partial}{\partial\omega_j}+\frac{\partial}{\partial t}\right),$$

$$\frac{\partial}{\partial u_i}=\frac{1}{t}\frac{\partial}{\partial\omega_i}-\omega_i\left(\frac{1}{t}\sum_{j=1}^{n}\omega_j\frac{\partial}{\partial\omega_j}+\frac{\partial}{\partial t}\right)\qquad (1\leq i\leq n).$$

Therefore, for any $m\in \boldsymbol{N}$ and $i=0,\cdots, n$ there exist constants $C^m_{i,1}$ and $C^m_{i,0}$ such that

$$\left|\frac{\partial}{\partial u_i}F(u)\right| \leq (C^m_{i,1}\ |\langle u,\ u\rangle|^{1/2}+C^m_{i,0})\ |\langle u,\ u\rangle|^{m}.$$

This shows that

$$\frac{\partial}{\partial u_i}F(u)\rightarrow 0$$

uniformly when $\langle u, u\rangle\rightarrow 0$. By repeating the same method we can prove that all devivatives of $F(u)$ with respect to $u_0,\cdots, u_n$ goes to zero uniformly when $\langle u, u\rangle\rightarrow 0$. This holds also for negative t. We can get the same conclusion on spacelike vectors by slight modifiations. Thus we showed that F is smooth on X.

By the above we can easily prove the inequalities (5.2). Thus $F\in \mathscr{S}(X)$. Finally, if f is the function in $\mathscr{S}(X)$ whose Fourier transform is F, then $\hat{f}=\varphi$ by (5.1).

Remark that Lemma 3 and Lemma 5 hold for $f\in \mathscr{S}(X)$.

Let $f\in \mathscr{S}(X)$. By the inversion formula of the Fourier transform we have the following.

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^{n+1}}\int_X \tilde{f}(u)\, e^{i\langle x,\ u\rangle} du \\ &= \frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{(\partial X)/Z_2} \tilde{f}(t\omega)\, e^{it\langle x,\ \omega\rangle}\ |\ t\ |^n dt d\mu(\omega) \\ &= \frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{(\partial X)/Z_2}\int_{-\infty}^{\infty} \hat{f}(\omega,\ p)\, e^{-it(p-\langle x,\ \omega\rangle)}\ |\ t\ |^n dp d\mu(\omega)\, dt \end{aligned}$$

If n is even,

$$f(x)=\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{(\partial X)/Z_2}\int_{-\infty}^{\infty}\left(\frac{\partial}{i\partial p}\right)^n\left(\hat{f}(\omega,\ p)\right)e^{-it(p-\langle x,\ \omega\rangle)}dpdtd\mu(\omega)$$

$$\frac{1}{(2\pi)^n}\int_{(\partial X)/Z_2}\left(\frac{\partial}{\partial p}\right)^n\hat{f}(\omega,\ p)\Big|_{p=\langle x,\ \omega\rangle}d\mu(\omega)$$

$$=\frac{1}{2(2\pi)^n}\int_{\partial X}\left(\left(\frac{\partial}{i\partial p}\right)^n\hat{f}\right)(\omega,\ \langle x,\ \omega\rangle)d\mu(\omega)$$

$$=\frac{1}{2(2\pi)^n}\left(\left(\frac{\partial}{i\partial p}\right)^n\hat{f}\right)^{\vee}(x).$$

Suppose n is odd. Let $\mathcal{H}$ be the Hilbert transform, which is, by definition,

$$(\mathcal{H}F)(t)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{F(p)}{t-p}dp.$$

Then

$$(\mathcal{H}F)^{\sim}(s)=\mathrm{sgn}\ s\ F(s)$$

(cf., [H] p. 114), where sgn $s=1(s\geq 0),=-1(s<0)$.

$$f(x)=\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{(\partial X)/Z_2}(\mathrm{sgn}t)\left(\int_{-\infty}^{\infty}\left(\frac{\partial}{i\partial p}\right)^n\hat{f}(\omega,\ p)e^{-itp}dp\right)e^{it\langle x,\ \omega\rangle}dtd\mu(\omega)$$

$$=\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{(\partial X)/Z_2}\left(\int_{-\infty}^{\infty}\mathcal{H}p\left(\frac{\partial}{i\partial p}\right)^n\hat{f}(\omega,\ p)e^{-itp}dp\right)e^{it\langle x,\ \omega\rangle}dtd\mu(\omega)$$

$$=\frac{1}{(2\pi)^n}\int_{(\partial X)/Z_2}\mathcal{H}p\left(\frac{\partial}{i\partial p}\right)^n\hat{f}(\omega,\ p)\Big|_{p=\langle x,\ \omega\rangle}d\mu(\omega)$$

$$=\frac{1}{2(2\pi)^n}\left(\mathcal{H}p\frac{\partial}{i\partial p}\right)^n\hat{f}\Big)^{\vee}(\omega,\ \langle x,\ \omega\rangle).$$

We define $\Lambda\varphi$ by

$$(\Lambda\varphi)(\omega, p)=\begin{cases}\dfrac{1}{2(2\pi)^n}\left(\dfrac{\partial}{i\partial p}\right)^n\varphi(\omega, p), & \text{for even } n,\\[2ex] \dfrac{1}{2(2\pi)^n}\mathcal{H}_p\left(\dfrac{\partial}{i\partial p}\right)^n\varphi(\omega, p), & \text{for odd } n.\end{cases}$$

Then we have the following theorem.

THEOREM 2. *For any* $f\in\mathcal{S}(X)$ *we have*

$$f=(\Lambda\hat{f})^{\vee}. \tag{5.3}$$

As a special case of Theorem 2 we have the following corollary.

COROLLARY. *We assume that* $n\in 4\boldsymbol{N}$. *For any* $f\in\mathcal{S}(X)$ *we have*

$$f=\frac{1}{2(2\pi)^n}(L^{n/2}\hat{f})^{\vee}=\frac{1}{2(2\pi)^n}\square^{n/2}(\hat{f})^{\vee}=\frac{1}{2(2\pi)^n}((\square^{n/2}f)\hat{\ })^{\vee}.$$

Since the operator Λ corresponds to multiplication of tha Fourier transform by $(1/(2(2\pi)^n)$ $|r^n|$, Λ is a positive symmetric operator. So we can associate an operator $\sqrt{\Lambda}$ defined by

$$(\sqrt{\Lambda}h)^{\sim}(r)=\frac{1}{\sqrt{2(2\pi)^n}}|r|^{n/2}\tilde{h}(r).$$

If n is even, we have

$$(\sqrt{\Lambda}h)(p)=\frac{1}{\sqrt{2(2\pi)^n}}\left(\frac{1}{i}\frac{d}{dp}\right)^{n/2}h(p).$$

THEOREM 3. *For* $f\in\mathcal{S}(X)$ *we have*

$$\int_X|f(x)|^2dx=\int_{\partial X}\int_{-\infty}^{\infty}|\sqrt{\Lambda}\hat{f}(\omega, p)|^2dp\,d\mu(\omega).$$

PROOF. Using (5.3) and (4.1), we have

$$\int_X f(x)\overline{f(x)}\,dx = \int_X (R^*\Lambda Rf)(x)\overline{f(x)}\,dx$$

$$= \int_{\partial X}\int_{-\infty}^{\infty} (\Lambda Rf)(\omega, p)\overline{(R^*f)(\omega, p)}\,d\mu(\omega)\,dp$$

$$= \int_{\partial X}\int_{-\infty}^{\infty} |\sqrt{\Lambda}\hat{f}(\omega, p)|^2 dp\,d\mu(\omega).$$

References

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