Solution of Linear Differential Equation by Walsh Functions

by

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(Received September 1, 1990)

An algorithm for solving linear differential equations (DEs) by Walsh functions(WFs) is proposed. In this algorithm, the solutions are given by partial integrals of Walsh series of derivatives of the solutions. Consequently, the solutions are determined in a form of piecewise-linear approximation(PWLA) by means of fast algorithms of inverse Walsh transforms. In this approach, the accuracy of the solutions is improved and hence the number of computations is reduced greatly, compared with that of the conventional stairstep approximations for the same order of the approximations of the solutions.

Key words : Walsh functions, linear differential equations, partial integrals of Walsh series of derivatives of the solutions, piecewise-linear approximations.

1. Introduction

Walsh functions(WFs) have been applied to many fields of engineering because of their significant computational advantages[1],[2]. In applications of WFs to solution of differential equations(DEs), advantage is taken of the fact that operations of integration in problem systems are reduced to algebraic operations through the so called operational matrix for integration. In the conventional Walsh approach, the solutions can be obtained by solving matrix equations and then expressed in the form of stairstep approximation[3],[4]. In general, the solutions of DEs are given/defined in a continuous form. Hence to improve the accuracy of the Walsh approximations, many terms in Walsh series expansion to the solutions are needed. As a consequence, computations using matrix of a large size are performed and this also necessitates a large memory capacity.

To overcome such a difficulty, as we have already described in the solutions of linear DEs by rationalized Haar functions[5], we propose to express the solutions in a form of piecewise-linear approximation(PWLA). For this purpose, we expand first derivatives of the solutions of the problem equations into Walsh series with unknown coefficients. When the unknown coefficients are obtained by the matrix computations, their solutions in a form of PWLA can be determined by means of fast Walsh transformations efficiently. In this approach, the accuracy of the solutions is improved greatly compared with that of the conventional stairstep approximation. This saves a good deal of time and memory locations in matrix computations in the conventional Walsh approach.

2. Walsh Functions and Their Integrals in Terms of Time Variable

Walsh functions(WFs) form an ordered orthonormal set of rectangular waveforms taking only two values +1 and -1. Like the sine-cosine functions, two arguments are required for complete definition, a time t and an ordering number i related to frequency in some way. Put the sequency-ordered WF to Wal(i,t)[1],[2]. The first four WFs are shown in Fig. 1. As for the WF system, we have the orthogonality relation as

$$\int_0^T \mathrm{Wal}(i,t)\mathrm{Wal}(j,t)dt = \begin{cases} T & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \tag{1}$$

Integrals of WFs in terms of the time variable t can be expressed as

$$\int_0^t \mathrm{Wal}(i,t)dt = \frac{T}{N}\sum_{P=0}^{k-1} \mathrm{Wal}(i, p/N) + (t - t_k)\mathrm{Wal}(i, t_k) \tag{2}$$

$$\text{for } t_k \leq t < t_{k+1}$$

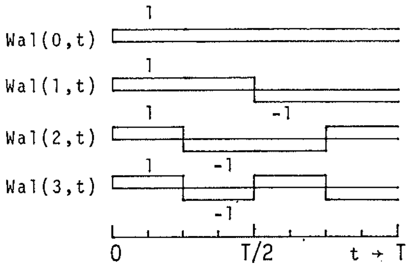

Fig. 1. The first four Walsh functions.

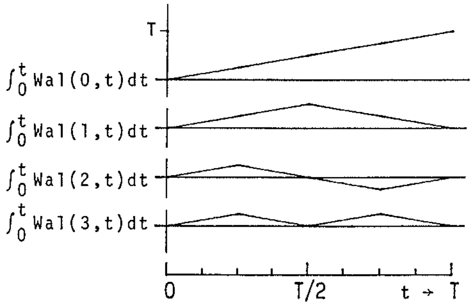

Fig. 2. Integral waveforms of the first four Walsh functions.

where $t_k = kT/N$, $t_{k+1} = (k+1)T/N$, $k = 0, 1, \ldots\ldots, N-1$, $N = 2^n$, $n = 1, 2, \ldots\ldots\ldots$, $i = 0, 1, \ldots\ldots, N-1$, and $p = 0, 1, \ldots\ldots\ldots, N-1$. Integral waveforms for the first four WFs are shown in Fig. 2.

3. Solution of Linear Differential Equations by Walsh Functions

As a solution of linear DEs by WFs, we consider the following equation in which the solutions are called as the error function :

$$\left.\begin{aligned} &y''(t) + 2ty'(t) = 0, \quad 0 \le t < T, \\ &y'(0) = 2/\sqrt{\pi} \text{ and } y(0) = 0. \end{aligned}\right\} \quad (3)$$

For expressing the solution of (3) in a form of PWLA, we expand the first derivative $y'(t)$ into a truncated Walsh series with unknown coefficients G_r, e.i., $y'(t) \simeq \sum_{r=0}^{N-1} G_r \mathrm{Wal}(r,t)$. If the coefficients $\{G_r\}$, $r = 0, 1, \ldots\ldots, N-1$, are written as $\mathbf{G} = [G_0, \ldots, G_r, \ldots\ldots, G_{N-1}]$, $y(t)$ of (3) is expressed by $(\mathbf{G}\,\mathbf{D} + \mathrm{Y})\mathbf{\Phi}$, where $\mathbf{D}$ denotes an $N \times N$ operational matrix for integration as shown in Figs. 3 and 4, and $\mathbf{\Phi}$ denotes the transpose of the vector $[\mathrm{Wal}(0,t), \ldots\ldots, \mathrm{Wal}(N-1,t)]$ [3],[4]. Also $\mathbf{Y}$ shows an initial value vector of N components, $\mathbf{Y} = [y(0), 0, \ldots\ldots, 0]$. Besides $y''(t)$ of (3) can be expressed in a matrix form using the inverse of the matrix $\mathbf{D}$, $\mathbf{D}^{-1}$, which may be regarded as an operational matrix for differentiation, as $(\mathbf{G} - \mathbf{Y}')\mathbf{D}^{-1}\mathbf{\Phi}$ (see Figs. 5 and 6), where $\mathbf{D} \times \mathbf{D}^{-1} = \mathbf{E}$, $\mathbf{E}$ denotes an $N \times N$ identity matrix. $\mathbf{Y}'$ denotes a vector of N components for corresponding to $y'(0)$ where $\mathbf{Y}' = [y'(0), 0, \ldots\ldots, 0]$.

With these relations, equation(3) can be transformed into a matrix equation as

$$(\mathbf{G} - \mathbf{Y}')\mathbf{D}^{-1} + 2\,\mathbf{G}\mathbf{D}_t \simeq \mathbf{0} \quad (4)$$

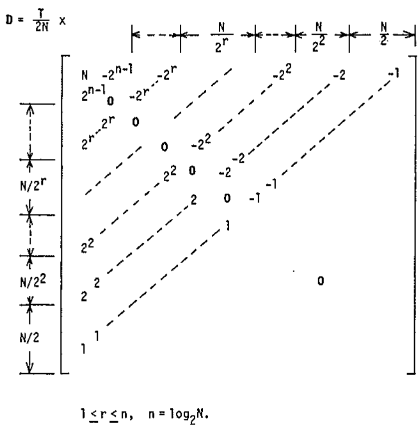

Fig. 3. A matrix **D**.

$$\mathbf{D} = \begin{bmatrix} 1 & -1/2 & 0 & -1/4 & 0 & 0 & 0 & -1/8 \\ 1/2 & 0 & -1/4 & 0 & 0 & 0 & -1/8 & 0 \\ 0 & 1/4 & 0 & 0 & 0 & -1/8 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & -1/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 4. The matrix **D** for N=8.

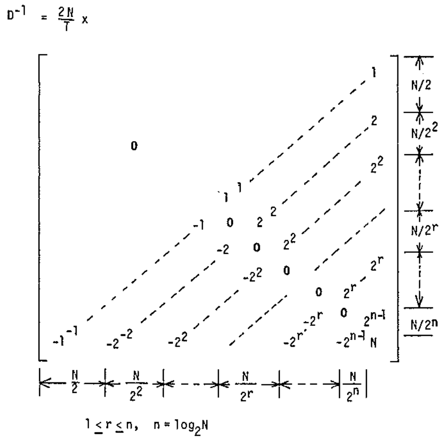

Fig. 5. A matrix $\mathbf{D}^{-1}$.

$$\mathbf{D}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 & 0 & 0 & 16 \\ 0 & 0 & -8 & 0 & 0 & 0 & 16 & 0 \\ 0 & -8 & 0 & 0 & 0 & -16 & 0 & 32 \\ -8 & 0 & 0 & 0 & -16 & 0 & -32 & 64 \end{bmatrix}$$

Fig. 6. The matrix $\mathbf{D}^{-1}$ for N=8.

where $\mathbf{D}_t$ is an $N \times N$ matrix determined by a Walsh series expansion of t, i.e., $\mathbf{D}_t = [D_t(i,j)]$, and its matrix elements are given as

$$D_t(i,j) = \frac{1}{T}\int_0^T t\,\mathrm{Wal}(i,t)\,\mathrm{Wal}(j,t)\,dt$$

$$= \frac{1}{T}\int_0^T t\,\mathrm{Wal}(i_{(2)} \oplus j_{(2)}, t)\,dt. \qquad (5)$$

In (5), $\oplus$ denotes binary addition without carry, and $i_{(2)}$ and $j_{(2)}$ denote binary representation of i and j, respectively. A numerical example of $\mathbf{D}_t$ for $N=8$ is shown in Fig. 7. A simple arrangement of (4) leads to

$$D_t = \begin{bmatrix} 1 & -1/2 & 0 & -1/4 & 0 & 0 & 0 & -1/8 \\ -1/2 & 1 & -1/4 & 0 & 0 & 0 & -1/8 & 0 \\ 0 & -1/4 & 1 & -1/2 & 0 & -1/8 & 0 & 0 \\ -1/4 & 0 & -1/2 & 1 & -1/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/8 & 1 & -1/2 & 0 & -1/4 \\ 0 & 0 & -1/8 & 0 & -1/2 & 1 & -1/4 & 0 \\ 0 & -1/8 & 0 & 0 & 0 & -1/4 & 1 & -1/2 \\ -1/8 & 0 & 0 & 0 & -1/4 & 0 & -1/2 & 1 \end{bmatrix}$$

Fig. 7. A numerical example of $\mathbf{D}_t$ for N=8.

$$\mathbf{G}(\mathbf{D}^{-1} + 2\mathbf{D}_t) = \mathbf{Y}'\,\mathbf{D}^{-1} \qquad (6)$$

Equation (6) represents simultaneous linear equations with respect to unknown coefficients G_r. Values of G_r can be determined by means of a numerical method of linear equations.

Upon using the numerical values of **G** in (6), solutions of (3) can be expressed in a form of piecewise-linear approximation such that

$$y(t) \simeq A_k t + B_k, \quad t_k \leq t < t_{k+1} \qquad (7)$$

$$k = 0,\ 1,\ \ldots\ldots,\ N-1.$$

where

$$\left.\begin{aligned} A_k &= \sum_{r=0}^{N-1} G_r \mathrm{Wal}(r,\ t_k), \\ B_k &= \frac{T}{N}\sum_{r=0}^{N-1} G_r \Big[\sum_{p=0}^{N-1} \mathrm{Wal}(r,\ pT/N) - k\,\mathrm{Wal}(r,\ t_k)\Big] + y(0). \end{aligned}\right\} \qquad (8)$$

Also, at the equal-space points t_k (= kT/N), we have

$$y(t_k) = A_k t_k + B_k \qquad (9)$$

(Coefficients A_k and B_k are detailed in the reference [6].). Values of A_k and B_k are given in Table 1, where $N = 8$ and $0 \leq t < 2$. With values of A_k and B_k, the solutions y(t) can be obtained as seen in Fig. 8 in a form of PWLA. In Fig. 8, numerical values of y(t) is drawn with its stairstep approximation.

Other examples of the solutions are shown in Table 2 with their numerical results of Figs. 9 and 10, where the numerical solutions are plotted with their true values, respectively. Further, these numerical solutions yield the Dawson integral and the Bessel functions $J_0(t)$ and $J_1(t)$, respectively.

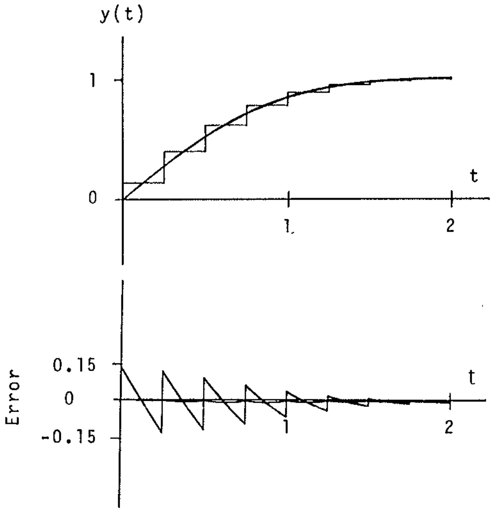

Fig. 8. Solutions of $y''(t)+2ty'(t)=0$, $y'(0)=2/\sqrt{\pi}$, $y(0)=0$, N=8, in forms of PWLA and stair-step approximation and their corresponding errors, respectively.

Table 1. Numerical Values of G_r, A_k and B_k for the error function.

r/k	G_r	A_k	B_k
0	0.4942	1.0942	0.0000
1	0.3430	0.9691	0.0313
2	0.0498	0.7596	0.1360
3	0.1447	0.5259	0.3113
4	-0.0009	0.3207	0.5165
5	-0.0263	0.1715	0.7030
6	0.0202	0.0800	0.8402
7	0.0694	0.0324	0.9236

$N = 8,\ 0 \leq t < 2$

Table 2. Examples of Different Differential Equations.

Differential Equations	G
$y' - 1 + 2ty = 0$ * $y(0) = 0$	$G = K(E + 2DD_t)^{-1}$ $K = [1,0,0,....,0]$
$t^2y'' + ty' + (t^2 - n^2)y = 0$ ** $y'(0) = y_0'$ and $y(0) = y_0$	$G = (Y'D^{-1}D_{t^2} - YD_{t^2-n^2})(D^{-1}D_{t^2} + D_t + DD_{t^2-n^2})^{-1}$ $Y' = [y'(0),0,0,...,0]$ $Y = [y(0),0,0,....,0]$ $D_{t^2-n^2} = D_{t^2} - n^2E$

* ** Solutions are called as the Dawson integral and the Bessel functions respectively.

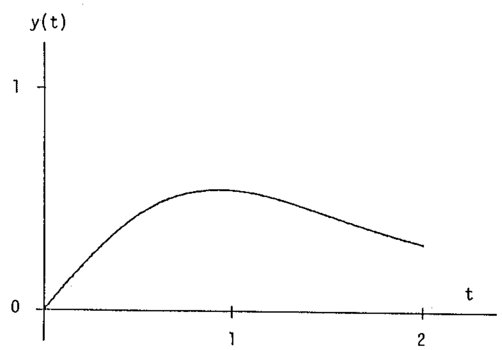

Fig. 9. Numerical solutions of $y'(t)+2ty(t)-1=0$, $y(0)=0$, $N=16$ and $0 < t \leq 2$.

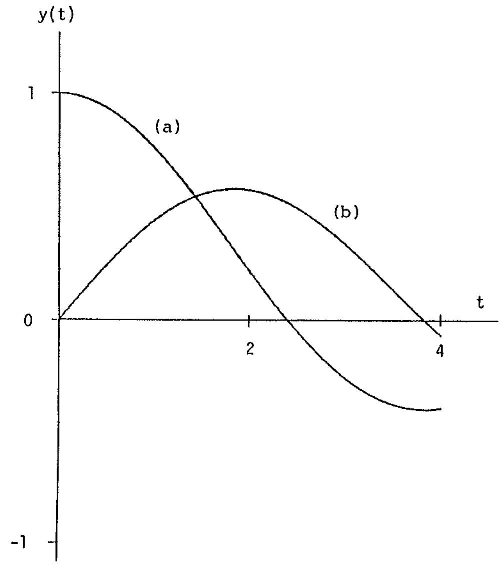

Fig. 10. Numerical solutions of $t^2y''(t)+ty'(t)+(t^2-n^2)y(t)=0$, $N=32$ and $0 < t \leq 4$, (a) $n=0$: $y'(0)=0$ and $y(0)=1$, (b) $n=1$: $y'(0)=1/2$ and $y(0)=0$.

4. Discussion

In the PWLA of the solution, values of the approximations, in principle, equal the theoretical solutions at the equally spaced points t_k, $k=0, 1, \ldots, N-1$[5]. Hence, the approximation yields inscribed- and/or circumscribed- ones. In this approximation, maximum magnitudes of the errors can be estimated as $\frac{1}{8}(\frac{T}{N})^2 |y''(t)|_{max}$[6], where $|y''(t)|_{max}$ denotes the maximum absolute value of $y''(t)$ within the defined interval. Thus the error caused by this method is reduced at the rate of $\frac{T}{4N}\cdot\frac{|y''(t)|_{max}}{|y'(t)|_{max}}$ compared with that by the stairstep approximation, where $|y'(t)|_{max}$ denotes the maximum absolute value of $y'(t)$ within the defined interval.

As for the solutions of the equation : $y''(t) + 2ty'(t) = 0$, $y'(0) = 2/\sqrt{\pi}$, $y(0) = 0$, $N = 8$ and $0 \leq t < 2$, errors caused by their piecewise-linear- and stairstep- approximations are less than 0.0126 and 0.1368, respectively(see Fig. 8). In this example, the necessary order of the stairstep approximation corresponding to the accuracy of PWLA for $N = 8$ is estimated as $2^7 = 128$. Hence, this serves to reduce the number of computations greatly in the conventional Walsh approach with a little additional computational procedures of fast algorithms of inverse Walsh transforms.

5. Conclusions

Solutions of linear DEs have been expressed in a form of piecewise-linear approximation(PWLA). From the view point of expression of the solutions, this algorithm is different from that of the conventional Walsh approach. This algorithm serves to reduce greatly the number of computations required in that Walsh approach for the same order of the approximations to the solutions.

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