Metrization of Spaces Which Have σ-as-finite Bases

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(*Received April 15, 1976*)

1. Introduction

In our previous paper [5], we have introduced the notion of as-finite[1] collections which is a generalization of locally finite collections, and have studied their properties. And now, as a continuation of the study, we consider metrization of spaces which have σ-as-finite bases.

The classical Nagata-Smirnov metrization theorem ([2], [6]) asserts that a regular space is metrizable if and only if it has a σ-locally finite[2] base. Recently, in [1], D. Burke, R. Engelking and D. Lutzer gave a generalization of the Nagata-Smirnov metrization theorem in terms of a hereditarily closure-preserving base.

In this paper, we will prove the following metrization theorem:

THEOREM. Let X be a regular quasi-k-space.[3] X is metrizable if and only if it has a σ-as-finite[2] base.

This is a generalization of the Nagata-Smirnov metrization theorem in quasi-k-spaces.

2. Definitions and notations

In this section, we give the definitions and the notations which are used in this paper.

DEFINITION 1 ([5]). A sequence $\{x_n\}$ of points of X is said to be an *ac-sequence* if each subsequence of $\{x_n\}$ has a cluster point in X.

DEFINITION 2 ([5]). A collection $\mathfrak{F}=\{F_\alpha|\alpha\in A\}$ of subsets of X is *as-finite* if and only if $\{\alpha\in A|F_\alpha\cap S\neq\phi\}$ is finite for every ac-sequence $\{x_n\}$, where $S=\{x_n|n\in\mathbf{N}\}$.

Clearly, every locally finite collection is as-finite. But as-finite collections in a space X may fail to be locally finite, even if they are open collections and X is a Fréchet space ([5, Example 4.3]).

Throughout this paper, topological spaces are assumed to be T_1-spaces. The symbol $\mathbf{N}$ denotes the set of all possitive integers. The notation $\{x_n\}$ (resp. $\{n_k\}$) denotes a

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1) Cf. § 2 Definition 2 (in this paper).

2) A σ-locally finite (resp. σ-as-finite) collection is one which can be written as a countable union of locally finite (resp. as-finite) subcollections.

3) According to Nagata [4], a space X is said to be a quasi-k-space if a set F of X is closed in X if and only if $F\cap C$ is closed in C for every countably compact set C in X.

sequence of points in a space X (resp. of possitive integers), and the notation $\{x_n|n \in \mathbf{N}\}$ denotes the image set of the sequence $\{x_n\}$. As for other terms and symbols in general topology, see [3] and [5, § 2].

3. The proof of the theorem

LEMMA 1. *Let X be a quasi-k-space and let $\mathfrak{H}$ be an as-finite collection of subsets in X. Then $\cap \{H|H \in \mathfrak{H}\}$ is an open subset of X.*

PROOF. For each countably compact subset K of X, the collection $\{H \cap K|H \in \mathfrak{H}\}$ is an as-finite collection in a subspace K. By [5, Corollary 3.2], the collection $\{H \cap K|H \in \mathfrak{H}\}$ contains only finitely many distinct subsets of K. Therefore $K \cap [\cap \{H|H \in \mathfrak{H}\}] = \cap \{H \cap K|H \in \mathfrak{H}\}$ is relatively open in K for each countably compact subset K of X. Since X is a quasi-k-space, $\cap \{H|H \in \mathfrak{H}\}$ is open in X.

LEMMA 2. *Let X be a topological space and suppose that $p \in X$ has a countable nbd base. Let $\mathfrak{H}$ be an as-finite collection of subsets of X and suppose that no member of $\mathfrak{H}$ contains p. Then $\mathfrak{H}$ is locally finite at p.*

PROOF. Let $\mathfrak{V} = \{V_n|n \in \mathbf{N}\}$ be a decreasing nbd base at p. Suppose that each member of $\mathfrak{V}$ meets infinitely many members of $\mathfrak{H}$. Inductively choose members $H_n \in \mathfrak{H}$ for each $n \in \mathbf{N}$ such that $V_n \cap H_n \neq \phi$ for each $n \in \mathbf{N}$. Since $\mathfrak{V}$ is a nbd base at p and no member of $\mathfrak{H}$ contains p, we can choose a sequence $\{n_k\}$ of distinct integers and a sequence $\{x_k\}$ of distinct points in X such that

$$x_k \in V_{n_k} \cap H_{n_k}, \qquad n_1 < n_2 < n_3 < \cdots.$$

Then the sequence $\{x_k\}$ converges to p because $\{V_{n_k}|k \in \mathbf{N}\}$ is a nbd base at p. Therefore $\{x_k\}$ is an ac-sequence. From the construction of the sequence $\{x_k\}$,

$$\{H \in \mathfrak{H}|\{x_k|k \in \mathbf{N}\} \cap H \neq \phi\}$$

is infinite. This contradicts the fact that $\mathfrak{H}$ is as-finite. Consequently, $\mathfrak{H}$ is locally finite at p.

By using the technique of the proof of [1, Theorem 5], we obtain the proof of the theorem.

PROOF of THEOREM. The necessity follows directly from the Nagata-Smirnov theorem.

To prove the sufficiency, let $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ be a σ-as-finite base for X. Here we can assume without loss of generality that X belongs to $\mathfrak{B}_n$ for each $n \in \mathbf{N}$. Let p be a nonisolated point of X and put

$$\mathfrak{C}_n = \{C = B - \{p\}|B \in \mathfrak{B}_n\}.$$

Then $\mathfrak{C}_n$ is as-finite. By Lemma 1, $B_n = \cap \{B | B \in \mathfrak{B}_n, p \in B\}$ is an open nbd of p because X is a quasi-k-space and $\mathfrak{B}_n$ is as-finite. Since $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ is a base for X, $\{B_n | n \in \mathbf{N}\}$ is a nbd base at p. Therefore, by Lemma 2, $\mathfrak{C}_n$ is locally finite at p. Since p is a nonisolated point,

$$\mathrm{ord}(V, \mathfrak{B}_n) = \mathrm{ord}(V, \mathfrak{C}_n)^{4)}$$

for each nbd V of p. Consequently, $\mathfrak{B}_n$ is locally finite at p.

Put

$$X_n = \{x \in X | \mathfrak{B}_n \text{ is locally finite at } x\}$$

for each $n \in \mathbf{N}$, then each set X_n is an open set and contains all nonisolated points of X. Let

$$\mathfrak{B}'_n = \{B \cap X_n | B \in \mathfrak{B}_n\}$$

for each $n \in \mathbf{N}$. Then each $\mathfrak{B}'_n$ is locally finite in X and $\mathfrak{B}' = \bigcup_{n=1}^{\infty} \mathfrak{B}'_n$ contains a nbd base at each nonisolated point of X.

Let

$$\mathfrak{B}''_n = \{\{x\} | \{x\} \in \mathfrak{B}_n\}$$

for each $n \in \mathbf{N}$. Then $\mathfrak{B}''_n$ is an as-finite collection of open and closed subsets in X. Since X is a quasi-k-space, by [5, Corollary 4.11] $\mathfrak{B}''_n$ is locally finite. Also $\mathfrak{B}'' = \bigcup_{n=1}^{\infty} \mathfrak{B}''_n$ contains a nbd base at each isolated point of X. Therefore $\mathfrak{B}' \cup \mathfrak{B}''$ is a σ-locally finite base for X. According to the Nagata-Smirnov theorem, the space X is metrizable. The proof is complete.

Remark. Every locally finite collection is as-finite, but an as-finite collection $\mathfrak{F}$ of subsets of a space X may fail to be locally finite even if $\mathfrak{F}$ is a collection of open subsets of X and X is a Fréchet space ([5, Example 4.3]). Therefore the theorem is a generalization of the Nagata-Smirnov metrization theorem in quasi-k-spaces.

References

[1] D. Burke, R. Engelking and D. Lutzer, Hereditarily closure-preserving collections and metrization, Proc. Amer. Math. Soc., **51** (1975), 483–488.

[2] J. Nagata, On a necessary and sufficient condition of metrizability, J. Inst. Polytech. Osaka City Univ. Ser. A Math., **1** (1950), 93–100.

[3] ———, Modern general topology, Wiley (Interscience), New York (1968).

[4] ———, Quotient and bi-quotient spaces of M-spaces, Proc. Japan Acad., **45** (1969), 25–29.

[5] S. Sasada, A generalization of locally finite collections, J. Fac. Educ. Tottori Univ., Nat. Sci., **23** (1972), 164–178.

[6] J. M. Smirnov, A necessary and sufficient condition for metrizability of a topological space, Dokl. Akad. Nauk SSSR, **77** (1951), 197–200. (MR 12, 845.)

4) ord(V, $\mathfrak{F}$) denotes the cardinal number of $\{F \in \mathfrak{F} | V \cap F \neq \phi\}$.