On Some Generalizations of M-spaces and Σ-spaces

Shōzō SASADA*

(*Received September 10, 1973*)

1. Introduction

In our previous paper [9], we have introduced the notion of as-finite collections which is a generalization of locally finite collections, and studied their properties. And now as a continuation of the study, we will bring in the notions of as-M-spaces (resp. as-Σ-spaces) defined in terms of as-finite closed coverings, which are generalizations of M-spaces (resp. Σ-spaces) introduced by K. Morita [6] (resp. K. Nagami [7]). And we will investigate several properties of these spaces.

For a sequence $\{\mathfrak{U}_n\}$ of open (or closed) coverings of a topological space X, we shall consider the following conditions:

(M) If $\{x_n\}$ is a sequence of points of X such that $x_n \in \mathrm{St}(x_0, \mathfrak{U}_n)$ [1] for each n and for some fixed point x_0 of X, then $\{x_n\}$ has a cluster point in X.

(Σ) If $\{x_n\}$ is a sequence of points of X such that $x_n \in \mathrm{C}(x_0, \mathfrak{U}_n)$ [2] for each n and for some fixed point x_0 of X, then $\{x_n\}$ has a cluster point in X.

A space X is an *M-space* [6] if and only if there exists a normal sequence $\{\mathfrak{U}_n\}$ of open coverings of X satisfying the condition (M). A space X is an *M*-space* [4] if and only if there exists a sequence $\{\mathfrak{U}_n\}$ of locally finite closed coverings of X satisfying the condition (M). A space X is a *Σ-space* [7] if and only if there exists a sequence $\{\mathfrak{U}_n\}$ of locally finite closed coverings of X satisfying the condition (Σ).

In the definitions given above, we can assume without loss of generality that the sequence $\{\mathrm{St}(x, \mathfrak{U}_n)\}$ (or $\{\mathrm{C}(x, \mathfrak{U}_n)\}$) is decreasing. Then a sequence $\{x_n\}$ satisfying the condition (M) (or (Σ)) is certainly an ac-sequence (cf. Definition 1.1). This fact leads us to introduce the following notions of spaces, where the term "locally finite" in the definitions of M*-spaces and Σ-spaces is replaced by "as-finite".

DEFINITION 1.1 ([9]). A sequence $\{x_n\}$ of points of X is said to be an *ac-sequence* if each subsequence of $\{x_n\}$ has a cluster point in X. A collection $\mathfrak{F} = \{F_\alpha | \alpha \in A\}$ of subsets of X is *as-finite* if and only if $\{\alpha \in A | F_\alpha \cap S \neq \phi\}$ is finite for every ac-sequence $\{x_n\}$, where $S = \{x_n | n \in \mathbf{N}\}$.

Evidently, every locally finite collection is as-finite and every as-finite collection is point-finite.

---

* Laboratory of Mathematics, Faculty of Education, Tottori University, Tottori, Japan.

1) $\mathrm{St}(x, \mathfrak{U}) = \cup \{U \in \mathfrak{U} | x \in U\}$.

2) $\mathrm{C}(x, \mathfrak{U}) = \cap \{U \in \mathfrak{U} | x \in U\}$.

DEFINITION 1.2. A space X is an *as-M-space* if and only if there exists a sequence $\{\mathfrak{U}_n\}$ of as-finite closed coverings of X satisfying (M).

DEFINITION 1.3. A space X is an *as-Σ-space* if and only if there exists a sequence $\{\mathfrak{U}_n\}$ of as-finite closed coverings of X satisfying (Σ).

Obviously, as-M-spaces (resp. as-Σ-spaces) include all M-spaces and M*-spaces (resp. Σ-spaces). In quasi-k-spaces, as-M-spaces (resp. as-Σ-spaces) are M*-spaces (resp. Σ-spaces) [9, Corollary 4.1]. And every closed subspace of an as-M-space (resp. as-Σ-space) is also an as-M-space (resp. as-Σ-space).

The main results of this paper are as follows:

(I) If a space X is a countable sum of closed as-Σ-spaces, then X is an as-Σ-space (Theorem 2.6)

(II) If $\{X_\alpha|\alpha\in A\}$ is an as-finite closed covering of X and each X_α is an as-M-space (resp. as-Σ-space), then X is an as-M-space (resp. as-Σ-space) (Theorem 2.7).

(III) Let X be a regular as-Σ-space with point-countable base. If X has the property (ω^*) [9], then X is a metrizable space (Theorem 3.4).

(IV) If $f: X\rightarrow Y$ is a quasi-perfect mapping, then X is an as-M-space (resp. as-Σ-space) if and only if Y is an as-M-space (resp. as-Σ-space) (Corollary 4.5).

Throughout this paper, topological spaces are assumed to be T_1-spaces, and mappings to be continuous. And $\mathbf{N}$ denotes the set of positive integers. As for terms and symbols in general topology, see [8] and [9, § 2].

2. Some properties of as-M-spaces and as-Σ-spaces

EXAMPLE 2.1. *An as-M-space need not be an M*-space.*

PROOF. Let X be a subspace $\mathbf{N}\cup\{x^*\}$ of Stone-Čech's compactification $\beta\mathbf{N}$ of integers $\mathbf{N}$, where x^* is a point of $\beta\mathbf{N}-\mathbf{N}$. Then X is a paracompact T_2-space with G_δ-diagonal which is not metrizable. Assume that X is an M*-space. Then X is an M-space since it is paracompact. Therefore X is metrizable; this contradicts the above. Hence X is not an M*-space. To show that X is an as-M-space, put $\mathfrak{F}_n=\{\{x\}|x\in X\}$ for each n. Then $\{\mathfrak{F}_n\}$ is a sequence of closed coverings of X satisfying (M). Since $\{x_n|n\in\mathbf{N}\}$ is finite for every ac-sequence $\{x_n\}$ in X, $\mathfrak{F}_n$ is as-finite. Therefore X is an as-M-space.

PROPOSITION 2.2. *Let X be an as-Σ-space. Then X has a sequence $\{\mathfrak{F}_n\}$ of as-finite closed coverings of X satisfying the following conditions* (a), (b) *and* (c):

(a) $\{\mathfrak{F}_n\}$ *satisfies* (Σ)

(b) $C(x, \mathfrak{F}_{n+1})\subset C(x, \mathfrak{F}_n)$ *for each n and for each $x\in X$.*

(c) $C(x, \mathfrak{F}_n)\in\mathfrak{F}_n$ *for each n and for each $x\in X$.*

PROOF. Let $\{\mathfrak{U}_n\}$ be a sequence of as-finite closed coverings of X satisfying (Σ). And let $\mathfrak{F}_n$ be the collection of all finite intersections of elements of $\bigwedge_{i=1}^{n} \mathfrak{U}_i$, for each n. Then $\{\mathfrak{F}_n\}$ is a sequence of as-finite closed coverings of X satisfying (a), (b) and (c).

PROPOSITION 2.3. *Let X be an as-M-space, and let $\{\mathfrak{F}_n\}$ be a decreasing sequence (in the sense of refinement) of as-finite closed coverings of X satisfying (Σ). Then the followings hold:*

(1) *$S(x)=\bigcap_{n=1}^{\infty} St(x, \mathfrak{F}_n)$ is a countably compact closed set for each point x in X.*

(2) *For every open set U with $S(x)\subset U$, there exists a positive integer n such that $St(x, \mathfrak{F}_n)\subset U$.*

PROOF. (1) Let $\{x_n\}$ be a sequence in S(x). Since $x_n \in \mathrm{St}(x, \mathfrak{F}_n)$ for each n, by the condition (M), $\{x_n\}$ clusters at some point y in X. For each n,

$$y\in\overline{\{x_i | i\geqq n\}}\subset\overline{\mathrm{St}(x, \mathfrak{F}_n)}=\mathrm{St}(x, \mathfrak{F}_n).$$

$$\therefore \qquad y\in \mathrm{S}(x)=\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathfrak{F}_n).$$

Therefore S(x) is a countably compact closed set.

(2) If not, then there exists a point $x_n \in St(x, \mathfrak{F}_n)-U$ for each $n\in\mathbf{N}$. Since $\{\mathfrak{F}_n\}$ is decreasing and satisfies (M), $\{x_n\}$ clusters at some point $y\in \mathrm{S}(x)$. This contradicts the fact that $y\in\overline{\{x_n | n\in\mathbf{N}\}}\subset\overline{X-U}=X-U$. The proof is complete.

The proof of the following proposition is similar to that of Proposition 2.3.

PROPOSITION 2.4. *Let X be an as-Σ-space, and let $\{\mathfrak{F}_n\}$ be a sequence of as-finite closed coverings of X satisfying the conditions (a), (b) and (c) in Proposition 2.2. Then the followings hold:*

(1) *$C(x)=\bigcap_{n=1}^{\infty} C(x, \mathfrak{F}_n)$ is a countably compact closed set for each point x in X.*

(2) *For every open set U with $C(x)\subset U$, there exists a positive integer n such that $C(x, \mathfrak{F}_n)\subset U$.*

PROPOSITION 2.5. *Let X be an as-M-space, let C be a countably compact subset of X, and let $\{\mathfrak{F}_n\}$ be a decreasing sequence of as-finite closed coverings of X satisfying (M). If $\{x_n\}$ is a sequence of points in X such that $x_n\in St(C, \mathfrak{F}_n)$ for each n, then $\{x_n\}$ clusters in X.*

PROOF. For each $n\in\mathbf{N}$, there exist an element $F_n\in\mathfrak{F}_n$ and a point y_n of X such that $x_n\in F_n$ and $y_n\in F_n\cap C\neq\phi$. Since C is a countably compact set, $\{y_n\}$ clusters at some point y_0 of C and $\{F\in\mathfrak{F}_n | F\cap C\neq\phi\}$ is finite, by [9, Corollary 3.2]. Put

$$U_n(y_0)=X-\cup\{F\in\mathfrak{F}_n | F\cap C\neq\phi,\ y_0\notin F\}.$$

Then $U_n(y_0)$ is an open nbd of y_0. Now,

$$\{F\in\mathfrak{F}_n|F\cap C\neq\phi,\ y_0\not\in F\}=\{F\in\mathfrak{F}_n|y_0\not\in F\}-\{F\in\mathfrak{F}_n|F\cap C=\phi\}.$$

Therefore, put $A_n(y_0)=X-\cup\{F\in\mathfrak{F}_n|y_0\not\in F\}$, and then

$$U_n(y_0)\subset A_n(y_0)\cup[\cup\{F\in\mathfrak{F}_n|F\cap C=\phi\}].$$

Since y_0 is a cluster point of $\{y_n\}$, there exists an increasing sequence $\{k_n\}$ of integers such that $k_n\geqq n$ and $y_{k_n}\in U_n(y_0)$ for each $n\in\mathbf{N}$. Then $y_{k_n}\not\in\cup\{F\in\mathfrak{F}_n|F\cap C=\phi\}$, because $y_{k_n}\in C$. Consequently, $y_{k_n}\in A_n(y_0)$ for each n; this implies that $y_0\in F$ whenever $y_{n_k}\in F\in\mathfrak{F}_n$. Therefore

$$x_{k_n}\in\mathrm{St}(y_{k_n},\mathfrak{F}_{k_n})\subset\mathrm{St}(y_{k_n},\mathfrak{F}_n)\subset\mathrm{St}(y_0,\mathfrak{F}_n).$$

By the condition (M), $\{x_n\}$ clusters in X. This completes the proof.

THEOREM 2.6. *Let $\{X_n|n\in\mathbf{N}\}$ be a countable closed covering of a space X. If each X_n is an as-Σ-space, then X is an as-Σ-space.*

PROOF. Let $\{\mathfrak{U}_{ij}\}_{j=1}^{\infty}$ be a sequence of as-finite closed coverings of X_i satisfying (Σ) ($i=1, 2, \dots$). Put

$$\mathfrak{U}'_{ij}=\{X\}\cup\mathfrak{U}_{ij}\qquad\text{and}\qquad\mathfrak{F}_n=\bigwedge_{i,j\leqq n}\mathfrak{U}'_{ij}.$$

Then $\mathfrak{F}_n$ is an as-finite closed covering of $X(n=1, 2, \dots)$. To prove that $\{\mathfrak{F}_n\}$ satisfying (Σ), let $\{x_n\}$ be a sequence of points of X such that $x_n\in C(x_0,\mathfrak{F}_n)$ for some point x_0 in X and for each $n\in\mathbf{N}$. Choose an integer k with $x_0\in X_k$, and then for each $n\geqq k$

$$x_n\in C(x_0,\mathfrak{F}_n)\subset C(x_0,\mathfrak{U}'_{kn})=C(x_0,\mathfrak{U}_{kn}).$$

Therefore $\{x_n|n\geqq k\}$ clusters in X. The proof is complete.

THEOREM 2.7. *Let $\{X_\alpha|\alpha\in A\}$ be an as-finite closed covering of a space X. If each X_α is an as-M-space (resp. as-Σ-space), X is also an as-M-space (resp. as-Σ-space).*

PROOF. For each $\alpha\in A$, let $\{\mathfrak{F}_{\alpha,n}\}_{n=1}^{\infty}$ be a sequence of as-finite closed coverings of X_α satisfying (M) (resp. (Σ)). Put $\mathfrak{G}_n=\cup\{\mathfrak{F}_{\alpha,n}|\alpha\in A\}$. Then $\mathfrak{G}_n$ is an as-finite closed covering. Since $\{X_\alpha|\alpha\in A\}$ is point-finite, $\{\mathfrak{G}_n\}$ satisfies the condition (M) (resp. (Σ)). The proof is complete.

COROLLARY 2.8. *Let $\mathfrak{F}=\bigcup_{n=1}^{\infty}\mathfrak{F}_n$ be a σ-locally finite closed covering of a space X. If each $F\in\mathfrak{F}$ is an as-Σ-space, then X is an as-Σ-space.*

The following corollaries are derived immediately from Hodel's sum theorems [3; I, II] and Theorem 2.7.

COROLLARY 2.9. *Let* $\mathfrak{G}=\bigcup_{n=1}^{\infty}\mathfrak{G}_n$ *be a* σ*-locally finite open covering of a space* X *such that the closure of each element of* $\mathfrak{G}$ *is an as-M-space (resp. as-*Σ*-space). Then* X *is an as-M-space (resp. as-*Σ*-space).*

COROLLARY 2.10. *Let* $\mathfrak{G}=\bigcup_{n=1}^{\infty}\mathfrak{G}_n$ *be a* σ*-locally finite open covering of a space* X*, each element of which is an as-M-space (resp. as-*Σ*-space) and has compact boundary. Then* X *is an as-M-space (resp. as-*Σ*-space).*

3. Metrization of as-Σ-spaces

In [10], T. Shiraki proved that every Σ-space with a point-countable pseudo-base[3] is a σ-space. By making slight modifications of the proof of this theorem, we can prove the following theorem.

THEOREM 3.1. *Every as-*Σ*-space* X *with a point-countable pseudo-base has a* σ*-as-finite closed net*[4].

For the proof of this theorem we need the following Lemma.

LEMMA 3.2 (A. Miščenko [5]). *Let* $\mathfrak{U}$ *be a point-countable collection of subsets of a set* X*, and* Y *a subset of* X*. Then there are at most countably many finite minimal coverings of* Y *by elements of* $\mathfrak{U}$*, where by a minimal covering we mean a covering which contains no proper subcovering.*

PROOF of Theorem 3.1. Let $\{\mathfrak{F}_n\}$ be a sequence of as-finite closed coverings of X satisfying the conditions (a), (b) and (c) in Proposition 2.2, and let $\mathfrak{U}$ be a point-countable pseudo-base for X. Let us denote by $\mathfrak{F}'_i=\{C_{i\alpha}|\alpha\in A_i\}$ the set of distinct elements of $\{C(x,\mathfrak{F}_i)|x\in X\}$. Then $\mathfrak{F}'_i\subset\mathfrak{F}_i$ and
(1) $\mathfrak{F}'_i$ is an as-finite closed covering of X.
By Lemma 3.2, the collection of finite minimal coverings of each $C_{i\alpha}$ by elements of $\mathfrak{U}$ is at most countable, and it can be denoted by $\{\omega_k^{i\alpha}|k\in\mathbf{N}\}$. Then $\mathfrak{U}_{i\alpha}=\{U|U\in\omega_k^{i\alpha}, k\in\mathbf{N}\}$ is countable and hence the collection $\mathfrak{V}_{i\alpha}$ of all finite unions of sets of $\mathfrak{U}_{i\alpha}$ is also countable. Therefore we can write

$$\mathfrak{V}_{i\alpha}=\{V_1^{i\alpha}, V_2^{i\alpha},\dots, V_j^{i\alpha},\dots\}.$$

Put $\mathfrak{L}_{ij}=\{C_{i\alpha}\cap(X-V_j^{i\alpha})|\alpha\in A_i\}$. Then
(2) $\mathfrak{L}_{ij}$ is an as-finite closed collection of X.
Therefore, by (1) and (2),

---

3) A collection $\mathfrak{U}$ of open subsets of a space X is called a *pseudo-base* of X if $\{x\}=\cap\{U|x\in U\in\mathfrak{U}\}$ for each $x\in X$.

4) A collection $\mathfrak{B}$ of subsets of a space X is called a *net* for X if for each $x\in X$ and open nbd U of x there exists a $B\in\mathfrak{B}$ such that $x\in B\subset U$.

$$\mathfrak{N}=[\bigcup_{n=1}^{\infty}\mathfrak{F}'_n]\cup[\bigcup_{i,j=1}^{\infty}\mathfrak{L}_{ij}]$$

is σ-as-finite closed collection of X.

That $\mathfrak{N}$ is a net for X is proved in the same way as in the proof of [10, Theorem 1.1], by using Proposition 2.4. This completes the proof.

COROLLARY 3.3. *If X is an as-Σ-space with point-countable base, then X is a developable space.*

PROOF. Since X has a point-countable base, X is a first countable space. Therefore, by Theorem 3.1 and [9, Theorem 4.6], X is a σ-space. By [11, Proposition 4], X is a semi-metrizable space, since X is a first countable σ-space. Consequently, by [2, Theorem 1], X is developable. The proof is complete.

THEOREM 3.4. *Let X be a regular as-Σ-space with point-countable base. If X has the property (ω^*) (cf. [9]), then X is a metrizable space.*

PROOF. By Corollary 3.3, X is a developable space, and therefore X is a subparacompact space. Hence, by [9, Proposition 6.2], X is paracompact. Consequently X is metrizable. The proof is complete.

COROLLARY 3.5. *Every collectionwise normal as-Σ-space X with point-countable base is metrizable.*

4. Mapping theorems

THEOREM 4.1. *Let $f: X\to Y$ be a quasi-perfect mapping from X onto Y.*
(a) *If X is an as-M-space, then so is Y.*
(b) *If X is an as-Σ-space, then so is Y.*

PROOF. (a) Let $\{\mathfrak{F}_n\}$ be a decreasing sequence of as-finite closed coverings of X satisfying the condition (M). Put $\mathfrak{L}_n=f(\mathfrak{F}_n)$, and then, by [9, corollary 5.2], $\{\mathfrak{L}_n\}$ is a decreasing sequence of as-finite closed coverings of Y. For each sequence $\{y_n\}$ with $y_n\in \mathrm{St}(y, \mathfrak{L}_n)$, there exists an element F_n of $\mathfrak{F}_n$ for each n such that $\{y, y_n\}\subset f(F_n)$. Pick up a point x_n in $f^{-1}(y_n)\cap F_n\neq\phi$ for each n, and then $x_n\in \mathrm{St}(f^{-1}(y), \mathfrak{F}_n)$. By Proposition 2.5, $\{x_n\}$ clusters in X because $f^{-1}(y)$ is countably compact. Since f is continuous, $\{y_n\}$ also clusters in Y. Hence Y is an as-M-space.

(b) Let $\{\mathfrak{F}_n\}$ be a sequence of as-finite closed coverings of X which has been constructed in the proof of Proposition 2.2. Put $\mathfrak{L}_n=f(\mathfrak{F}_n)$, and then $\{\mathfrak{L}_n\}$ is a sequence of as-finite closed coverings of Y by [9, Corollary 5.2]. To show that $\{\mathfrak{L}_n\}$ satisfies (Σ), let $\{y_n\}$ be a sequence of points of Y such that $y_n\in C(y, f(\mathfrak{F}_n))$ for some point y in Y and for each n. Put

$$L_n=\overline{\{y_i|i\geqq n\}}.$$

Then $L_n\subset C(y, f(\mathfrak{F}_n))$ because $\{C(y, f(\mathfrak{F}_n))\}$ is decreasing. Now let x be a fixed point of $f^{-1}(y)$. Then

(3) $$f^{-1}(L_n)\cap C(x)\neq\phi, \text{ for each } n\in\mathbf{N}.$$

The reason is as follows: Assume that $f^{-1}(L_k)\cap C(x)=\phi$ for some $k\in\mathbf{N}$. Since $f^{-1}(L_k)$ is a closed set, by Proposition 2.4, there exists an integer l such that $f^{-1}(L_k)\cap C(x, \mathfrak{F}_l)=\phi$. Put $m=\max\{k, l\}$, and then

$$f^{-1}(L_m)\cap C(x, \mathfrak{F}_m)=\phi.$$

By Proposition 2.2 (c), we can put $F=C(x, \mathfrak{F}_m)\in\mathfrak{F}_m$, and therefore $L_m\cap f(F)=\phi$. Since $y\in f(F)$,

$$L_m\cap C(y, f(\mathfrak{F}_m))=\phi.$$

This contradicts the fact that $L_n\subset C(y, f(\mathfrak{F}_n))$ for each n. Hence (3) is valid.

From (3) and countable compactness of $C(x)$, we obtain

$$[\bigcap_{n=1}^{\infty} f^{-1}(L_n)]\cap C(x)\neq\phi.$$

Therefore $\bigcap_{n=1}^{\infty} L_n\neq\phi$. This implies that $\{y_n\}$ clusters in Y. Hence Y is an as-Σ-space. The proof is complete.

Recently, J. M. Atkins and F. G. Slaughter, Jr. ([1], [12]) established pull-back theorems for several spaces, such as metrizable spaces, Σ-spaces and M-spaces, etc. So, in the same way, we shall establish pull-back theorems for as-M-spaces and as-Σ-spaces.

According to J. M. Atkins and F. G. Slaughter, Jr. [1], a continuous mapping f from X onto Y is said to be *decomposable* provided that $Y=Y_0\cup[\cup\{Y_j|j=1, 2, \dots\}]$ where $f^{-1}(y)$ is countably compact for $y\in Y_0$ and Y_j is discrete as a set of points in Y for $j\in N$, and a closed mapping f from X onto Y is said to be *almost quasi-perfect* if f is decomposable and also Bdry $f^{-1}(y)$ is countably compact for y in Y. It follows immediately from the definitions that a quasi-perfect mapping is almost quasi-perfect.

THEOREM 4.2. *Let f: $X\to Y$ be an almost quasi-perfect mapping from X onto Y. If Y and all the fiberes of f are as-M-spaces, then X is an as-M-space.*

PROOF. Since Y is an as-M-space, there is a decreasing sequence $\{\mathfrak{F}_n\}$ of as-finite closed coverings of Y satisfying (M). Let $Y=Y_0\cup[\cup\{Y_j|j\in\mathbf{N}\}]$ illustrate that f is decomposable. We can assume without loss of generality that Y_j's are pairwise disjoint for $j\geqq 0$ and Int $f^{-1}(y)\neq\phi$ for $y\in Y_j (j\geqq 1)$. And, for $j\geqq 1$ and $y\in Y_j$, there exists a decreasing sequence $\{\mathfrak{D}_{j,y,k}\}_{k=1}^{\infty}$ of as-finite closed coverings of $f^{-1}(y)$ satisfying

(M), because $f^{-1}(y)$ is an as-M-space. Now, set

(4) $$R_k = X - \cup\{\operatorname{Int} f^{-1}(y) | y \in Y_j,\ 1 \leqq j \leqq k\},$$

(5) $$\mathfrak{E}_k = \{R_k\} \cup [\cup\{\mathfrak{D}_{j,y,k} | y \in Y_j,\ 1 \leqq j \leqq k\}].$$

Then $\mathfrak{E}_k$ is an as-finite closed covering of X, because $\{f^{-1}(y) | y \in Y_j\}$ is a discrete collection in X. Therefore,

$$\mathfrak{G}_k = f^{-1}(\mathfrak{F}_k) \wedge \mathfrak{E}_k,$$

is an as-finite closed covering of X.

To show that $\{\mathfrak{G}_n\}$ satisfies the condition (M), let $\{x_n\}$ be a sequence of points of X such that $x_n \in \operatorname{St}(x, \mathfrak{G}_n)$ for some fixed point x in X and for each $n \in \mathbf{N}$. Then,

(6) $$x_n \in \operatorname{St}(x, f^{-1}(\mathfrak{F}_n)) \cap \operatorname{St}(x, \mathfrak{E}_n) \qquad \text{for each } n \in \mathbf{N}.$$

Case 1: There is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(x_{n_k})$'s are distinct for distinct k. Therefore,

$$f(x_{n_k}) \in f[\operatorname{St}(x, f^{-1}(\mathfrak{F}_{n_k}))] = \operatorname{St}(f(x), \mathfrak{F}_{n_k})$$
$$\subset \operatorname{St}(f(x), \mathfrak{F}_k) \qquad (k = 1, 2, \dots).$$

By the condition (M), $\{f(x_{n_k})\}$ has a cluster point in Y. Since f is closed mapping and $f(x_{n_k})$'s are distinct, $\{x_{n_k}\}$ clusters in X. Therefore $\{x_n\}$ has a cluster point in X.

Case 2: Case 1 does not hold, that is, there are a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point y in Y with $f(x_{n_k}) = y$ for each $k \in \mathbf{N}$. If $y \in Y_0$, then $\{x_{n_k}\}$ has a cluster point in $f^{-1}(y)$ because $f^{-1}(y)$ is countably compact. So, let $y \notin Y_0$. Then $y \in Y_j$ for some $j \geqq 1$. If $x_{n_k} \in \operatorname{Bdry} f^{-1}(y)$ for infinitely many integers k, then $\{x_{n_k}\}$ has a cluster point in $\operatorname{Bdry} f^{-1}(y)$ because $\operatorname{Bdry} f^{-1}(y)$ is countably compact. Therefore we may assume that

(7) $$x_{n_k} \in \operatorname{Int} f^{-1}(y) \qquad (j \leqq n_1 < n_2 < \cdots).$$

From (4), (5), (6) and (7), we obtain

$$x \in \operatorname{St}(x_{n_k}, \mathfrak{E}_{n_k}) = \operatorname{St}(x_{n_k}, \mathfrak{D}_{j,y,n_k}) \qquad (\text{for each } k \in \mathbf{N}).$$

Consequently, $x \in f^{-1}(y)$ and $x_{n_k} \in \operatorname{St}(x, \mathfrak{D}_{j,y,n_k})$. Hence $\{x_{n_k}\}$ has a cluster point in $f^{-1}(y) \subset X$. This shows that X is an as-M-space. The proof is complete.

COROLLARY 4.3. *Let $f\colon X \to Y$ be a quasi-perfect mapping from X onto Y. Then X is an as-M-space if and only if Y is an as-M-space.*

THEOREM 4.4. *Let $f\colon X \to Y$ be an almost quasi-perfect mapping from X onto Y. If Y and all the fibres of f are as-Σ-spaces, then X is an as-Σ-space.*

PROOF. Case 1: f is a quasi-perfect mapping. Let $\{\mathfrak{F}_n\}$ be a sequence of as-finite closed coverings of Y satisfying the conditions (a), (b) and (c) in Proposition 2.2. Put $\mathfrak{L}_n = f^{-1}(\mathfrak{F}_n)$, and then $\{\mathfrak{L}_n\}$ is a sequence of as-finite closed coverings of X by [9, Theorem 5.3]. To show that $\{\mathfrak{L}_n\}$ satisfies (Σ), let $\{x_n\}$ be a sequence of points of X such that $x_n \in C(x, f^{-1}(\mathfrak{F}_n))$ for some point x in X and for each $n \in \mathbf{N}$. Then $f(x_n) \in C(f(x), \mathfrak{F}_n)$, and therefore $\{f(x_n)\}$ has a cluster point in Y. Since f is a closed mapping and $f^{-1}(y)$ is countably compact for each $y \in Y$, $\{x_n\}$ has a cluster point in X.

Case 2: general case. (The proof of this case is the same as that of [1, Theorem 5.2 (d)]. Let $Y = Y_0 \cup [\cup \{Y_j | j = 1, 2, \ldots\}]$ illustrate that f is decomposable, where Y_j's are pairwise disjoint for $j \geqq 0$ and $\mathrm{Int} f^{-1}(y) \neq \phi$ for $y \in Y_j$ $(j \geqq 1)$. For each $y \in Y_j$ $(j \geqq 1)$, we pick up a point x_y of $\mathrm{Int}\ f^{-1}(y)$, and set

$$X_0 = X - \cup \{\mathrm{Int} f^{-1}(y) - \{x_y\} | y \in Y_j, j \geqq 1\}.$$

Then X_0 is a closed subset of X and $f|X_0 : X_0 \to Y$ is a quasi-perfect mapping from X_0 onto Y. By Case 1, X_0 is an as-Σ-space. Put $X_j = f^{-1}(Y_j) = \cup \{f^{-1}(y) | y \in Y_j\}$ for $j \geqq 1$. Then X_j is an as-Σ-space by virtue of the fact that X_j is a discrete union of as-Σ-spaces. By Theorem 2.6, $X = \bigcup_{j=0}^{\infty} X_j$ is an as-Σ-space. The proof is complete.

COROLLARY 4.5. *Let $f: X \to Y$ a quasi-perfect mapping from X onto Y. Then X is an as-Σ-space if and only if Y is an as-Σ-space.*

References

[1] J. M. ATKINS and F. G. SLAUGHTER, Jr., Pull-back theorems for closed mappings, to appear.
[2] R. W. HEATH, On spaces with point-countable bases, Bull. Acad. Polon. Sci. Ser. Sci. Math., **13** (1965), 393–395.
[3] R. E. HODEL, Sum theorems for topological spaces, Pacific J. Math., **30** (1969), 59–65.
[4] T. ISHII, On closed mappings and M-spaces I, Proc. Japan Acad., **43** (1967), 752–756.
[5] A. MIŠČENKO, Spaces with point-countable base, Soviet Math. Dokl. **3** (1962), 855–858.
[6] K. MORITA, Products of normal spaces with metric spaces, Math. Ann., **154** (1964), 365–382.
[7] K. NAGAMI, Σ-spaces, Fund. Math., **65** (1969), 169–192.
[8] J. NAGATA, Modern general topology, Wiley (Interscience), New York (1968).
[9] S. SASADA, A generalization of locally finite collections, J. Fac. Educ. Tottori Univ., Nat. Sci., **23** (1972), 164–178.
[10] T. SHIRAKI, M-spaces, their generalizations and metrization theorems, Sci. Rep. Tokyo Kyoiku Daigaku, Ser. A, **11** (1971), 57–67.
[11] F. SIWIEC and J. NAGATA, A note on nets and metrization, Proc. Japan Acad., **44** (1968), 623–627.
[12] F. G. SLAUGHTER, Jr., Some new results on inverse images of closed mappings, Pacific J. Math., to appear.