Lower estimate of the exponential sums of the sequence $(\alpha n + \beta \log n)$ and its application to the discrepancy

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Abstract

We obtain a lower bound for the discrepancy of the sequence $(\alpha n + \beta \log n)$, where α is an irrational number and β is a non-zero real number. In order to show the result, we estimate the exponential sums by the saddle-point method.

Keywords and Phrases. discrepancy, exponential sums, saddle-point method.

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1 Introduction and the main results

Throughout this paper, $\{x\} = x - [x]$ denotes the fractional part of the real number x, where $[x]$ is the integral part of x, and $C(a, b, c, \dots)$ denotes a constant that depends only on $a, b, c, \dots$. We write $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$ and $e(x) = e^{2\pi i x}$.

For functions $f(x)$ and $g(x)$ defined on $x \geq x_0$ for some x_0, the notation $f(x) = O(g(x))$ means that there exists a positive constant C such that $|f(x)| \leq C|g(x)|$ for all $x \geq x_0$, and the notation $f(x) \ll g(x)$ means $f(x) = O(g(x))$. The notation $f(x) \asymp g(x)$ means that $f(x) \ll g(x)$ and $g(x) \ll f(x)$ and the notation $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.

Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. The *discrepancy* of (x_n) is defined by

$$D_N(x_n) = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,\, b)}(x_n) - (b - a) \right|,$$

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where N is a positive integer and $\chi_{[a,\ b)}(x)$ is the characteristic function mod 1 of $[a,b)$, that is, $\chi_{[a,b)}(x)=1$ for $\{x\}\in[a,\ b)$ and $\chi_{[a,b)}(x)=0$ otherwise (see [6]).

Let α be an irrational number. Let ψ be a non-decreasing positive function that is defined at least for all positive integers. We shall say that α is *of type* $<\psi$ if $h\|\alpha h\|>1/\psi(h)$ holds for each positive integer h. If ψ is a constant function, then α of type $<\psi$ is called of *constant type* (see [6, p.121, Definition 3.3]). Let η be a positive real number. The irrational number α is said to be *of finite type* η if η is infimum of all real numbers τ for which there exists a positive constant $C=C(\tau,\alpha)$ such that α is of type $<\psi$, where $\psi(q)=Cq^{\tau-1}$ (see [6, p.121,Definition 3.4 and Lemma 3.1]).

Tichy and Turnwald [10] obtained an upper estimate of $D_N(\alpha n+\beta\log n)$, but Ohkubo improved the result, that is, Ohkubo [8] obtained an upper estimate of $D_N(\alpha n+\beta\log n)$ as follows:
If α is an irrational number of finite type η and β is a non-zero real number, then for any $\varepsilon>0$ and for all positive integer N

$$D_N(\alpha n+\beta\log n)\le C(\beta,\varepsilon)N^{-\frac{1}{\eta+1/2}+\varepsilon}.$$

If α is of constant type and β is non-zero real, then for all positive integer N

$$D_N(\alpha n+\beta\log n)\le C(\beta)N^{-\frac{2}{3}}\log N. \tag{1.1}$$

See also [2] and [3].

The purpose of this paper is to obtain the lower bounds for the discrepancies of the sequence $(\alpha n+\beta\log n)$ (see [4] for a generalized version). We use a lower bound for the discrepancy that contains an exponential sums (see Lemma 2.6, below). We need the detailed estimates of the exponential sums.

Theorem 1.1. *Let α be an irrational number, β be a non-zero real number, and h be a positive integer. Then*

$$\begin{aligned}\sum_{1\lesssim n\lesssim 2c} e(h(\alpha n+\beta\log n)) =& \frac{|\beta|^{1/2}h^{1/2}}{|k-\alpha h|}e\left(\beta h\log\left(\frac{\beta h}{k-\alpha h}\right)-\beta h-\frac{\operatorname{sgn}(\beta)}{8}\right)\\ &+O\left(\frac{|\beta|^{1/3}h^{1/3}}{|k-\alpha h|}\right)+O\left(\frac{1}{|k-\alpha h|}\right)+O\left(\frac{1}{1-|k-\alpha h|}\right)\\ &+O(|\beta|h)+O\left(\left(|\beta|^{1/2}h^{1/2}+1\right)\log\left(|\beta|h+\frac{1}{2}\right)\right)\\ &+O\left(|\beta|^{1/2}h^{1/2}\right)+O(1),\end{aligned}$$

where $k=[\alpha h]+\frac{1}{2}(\operatorname{sgn}(\beta)+1)$, $c=\dfrac{\beta h}{k-\alpha h}$, and $\operatorname{sgn}(\beta)=1$ if $\beta>0$, $\operatorname{sgn}(\beta)=-1$ if $\beta<0$ and the constants implied by the O's are absolute.

Theorem 1.2. *Let α be an irrational number, β be a non-zero real number, and $\alpha = [a_0, a_1, a_2, \ldots]$ be the continued fraction expansion of α with $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$. For any positive integer m, let*

$$h_m = \begin{cases} q_{2m+1}, & \text{if } \beta > 0; \\ q_{2m}, & \text{if } \beta < 0, \end{cases} \qquad k_m = \begin{cases} p_{2m+1}, & \text{if } \beta > 0; \\ p_{2m}, & \text{if } \beta < 0, \end{cases}$$

and let $N_m = \left[\dfrac{2\beta h_m}{|k_m - \alpha h_m|} \right]$.
Then for any $\varepsilon > 0$ and sufficiently large integer m,

$$D_{N_m}(\alpha n + \beta \log n) \geq \left(\frac{|\beta|^{1/4}}{8} - \varepsilon \right) N_m^{-3/4}.$$

Remark 1.1. *The smallest order of upper bounds for the discrepancy of the sequence $(\alpha n + \beta \log n)$ which we know is $N^{-2/3} \log N$ for the irrational number α of constant type (see (1.1)). On the other hand, as mentioned above, the order of the lower bound is $N^{-3/4}$. A reason why there exists the gap between the orders of the lower bound and the upper bound is that* $(\log n)$ *is not uniformly distributed mod* 1 *(see [6]).*

Throughout this paper, the constants implied by the O's and $\ll$'s are absolute.

2 Preliminary lemmas

The following lemma is a result that is got by combining Lemma 4.6 and Notes for Chapter 4 in [9].

Lemma 2.1 ([9, Lemma 4.6 and Notes]). *Let $g(x)$ be a real-valued function on the interval $[a, b]$ and suppose that $g(x)$ satisfies the following conditions*:

(i) g''' *is continuous on* $[a, b]$;
(ii) *either* $g'' > 0$ *on* $[a, b]$ *or* $g'' < 0$ *on* $[a, b]$;
(iii) *there exists positive number* m_2 *such that* $g''(x) \asymp m_2$ *on* $[a, b]$;
(iv) *there exists positive number* m_3 *such that* $g'''(x) \ll m_3$ *on* $[a, b]$;
(v) $g'(c) = 0$ *for some* $c \in [a, b]$.

Then

$$\int_a^b e(g(x))dx = e\left(g(c) + \frac{1}{8} \operatorname{sgn}(g''(c)) \right) |g''(c)|^{-1/2} + O\left(m_2^{-1} m_3^{1/3} \right) + O\left(\min\left(\frac{1}{|g'(a)|}, m_2^{-1/2} \right) \right) + O\left(\min\left(\frac{1}{|g'(b)|}, m_2^{-1/2} \right) \right).$$

The following lemma is essentially in [11, p.198, Lemma 4.4]. Since we need the more detailed error term, we modify the result and give precisely the proof which runs along the same lines as that of [11, Lemma 4.4].

Lemma 2.2 ([11, Lemma 4.4]). *Let $f(x)$ be a real-valued function and $f'(x)$ be monotone on the interval $[a,b]$ such that $|f'(x)|\le\lambda$ on $[a,b]$ for some $0<\lambda<1$. Then*

$$\left|\int_a^b e(f(x))dx-\sum_{a<n\le b}e(f(n))\right|=O\left(\frac{1}{1-\lambda}\right).$$

Proof. We set $\Psi(x)=[x]+1/2$ and $\chi(x)=x-\Psi(x)$. Then

$$\begin{aligned}\int_a^b e(f(x))dx-\sum_{a<n\le b}e(f(n))&=\int_a^b e(f(x))dx-\int_a^b e(f(x))d\Psi(x)\\&=\int_a^b e(f(x))d(x-\Psi(x))\\&=\int_a^b e(f(x))d\chi(x)\\&=[e(f(x))\chi(x)]_a^b-\int_a^b\chi(x)(e(f(x)))'dx\\&=-\int_a^b(e(f(x)))'\chi(x)dx+R(a,b),\end{aligned}\tag{2.1}$$

where $R(a,b)=e(f(b))\chi(b)-e(f(a))\chi(a)$.

The Fourie expansion of χ is

$$\chi(x)=-\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{\pi n},$$

if x is not an integer, and the series is boundedly convergent, so that we may multiply by an integrable function and integrate term-by-term. Thus

$$\begin{aligned}\int_a^b(e(f(x)))'\chi(x)dx&=-\sum_{n=1}^{\infty}\frac{1}{\pi n}\int_a^b\sin(2\pi nx)(e(f(x)))'dx\\&=-\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_a^b e(f(x)+nx)f'(x)dx-\int_a^b e(f(x)-nx)f'(x)dx\right).\end{aligned}\tag{2.2}$$

We have

$$\int_a^b e(f(x)-nx)f'(x)dx=\int_a^b\frac{f'(x)}{f'(x)-n}d(e(f(x)-nx)).$$

Without loss of generality, we may assume that $\dfrac{f'(x)}{f'(x)-n}$ is non-decreasing. Then, by the second mean value theorem, for some $\xi \in (a,b)$, we have

$$\begin{aligned}\int_a^b \frac{f'(x)}{f'(x)-n} d(e(f(x)-nx)) &= \frac{f'(b)}{f'(b)-n}\int_\xi^b d(e(f(x)-nx)) \\ &= \frac{f'(b)}{f'(b)-n}\left(e(f(b)-nb)-e(f(\xi)-n\xi)\right).\end{aligned}$$

Therefore, for $n \geq 1$, we have

$$\left|\int_a^b e(f(x)-nx)f'(x)dx\right| \leq \frac{2|f'(b)|}{n-|f'(b)|} \leq \frac{2\lambda}{n-\lambda}. \tag{2.3}$$

Similarly, we have

$$\left|\int_a^b e(f(x)+nx)f'(x)dx\right| \leq \frac{2|f'(b)|}{n+|f'(b)|} \leq \frac{2\lambda}{n+\lambda} \leq \frac{2\lambda}{n-\lambda}. \tag{2.4}$$

Thus, by (2.1), (2.2), (2.3), and (2.4), we have

$$\begin{aligned}&\left|\int_a^b e(f(x))dx - \sum_{a<n\leq b} e(f(n))\right| \\ &= \left|-\int_a^b (e(f(x)))'\chi(x)dx + R(a,b)\right| \\ &\leq \left|\int_a^b (e(f(x)))'\chi(x)dx\right| + 2 \\ &\leq \sum_{n=1}^\infty \frac{1}{n}\left|\int_a^b e(f(x)+nx)f'(x)dx\right| + \sum_{n=1}^\infty \frac{1}{n}\left|\int_a^b e(f(x)-nx)f'(x)dx\right| + 2 \\ &\leq 4\lambda \sum_{n=1}^\infty \frac{1}{n(n-\lambda)} + 2 \\ &= 4\lambda\left(\frac{1}{1-\lambda} + \sum_{n=2}^\infty \frac{1}{n(n-\lambda)}\right) + 2 \\ &= O\left(\frac{1}{1-\lambda}\right),\end{aligned}$$

which completes the proof. □

We quote Lemma 10.5 in [11, p.226].

Lemma 2.3 ([11, Lemma 10.5]). *Let $\varphi(x)$ be a positive decreasing and differentiable function defined on the interval $[a,b]$. If $g''(x)$ is of constant sign and $\varphi'(x)/g''(x)$ is monotone on $[a,b]$, then*

$$\left|\int_a^b \varphi(x)e(g(x))dx\right| \leq 8 \max_{a\leq x\leq b}\left(\frac{\varphi(x)}{|g''(x)|^{1/2}}\right)+\max_{a\leq x\leq b}\left(\left|\frac{\varphi'(x)}{g''(x)}\right|\right).$$

The following lemma is a saddle-point theorem, which is proved by combining Lemma 2.1 and a part of proof of [5, p.71, Lemma 2].

Lemma 2.4. *Suppose that $g(x)$ and $\varphi(x)$ are real-valued functions on the interval $[a,b]$ which satisfy the following conditions:*

(i) *$g'''(x)$ is continuous and $\varphi''(x)$ exists on $[a,b]$;*

(ii) *either $g''>0$ on $[a,b]$ or $g''<0$ on $[a,b]$;*

(iii) *there exists a number $m_2>0$ such that*

$$g''(x)\asymp m_2 \text{ on } [a,b];$$

(iv) *there exists a number m_3 such that $g'''(x)\ll m_3$;*

(v) *there exist numbers H_0, H_1, and H_2 such that*

$$\varphi(x)\ll H_0,\quad \varphi'(x)\ll H_1,\quad \varphi''(x)\ll H_2;$$

(vi) *$g'(c)=0$ for some $c\in[a,b]$.*

Then

$$\begin{aligned}\int_a^b \varphi(x)e(g(x))dx =& e\left(g(c)+\frac{1}{8}\text{sgn}(g''(c))\right)\varphi(c)|g''(c)|^{-1/2}\\ &+O(H_0m_2^{-1}m_3^{1/3})+O(H_1m_2^{-1})+O(H_2m_2^{-1}(b-a))\\ &+O(H_1m_2^{-2}m_3(b-a))+O(H_2m_2^{-2}m_3(b-a)^2)\\ &+O(H_0\min(m_2^{-1/2},|g'(a)|^{-1}))+O(H_0\min(m_2^{-1/2},|g'(b)|^{-1})).\end{aligned} \tag{2.5}$$

Proof. See [4, Lemma 2.5] on the detailed proof. □

The following lemma is Koksma's inequality (see [6, p.143, Theorem 5.1]).

Lemma 2.5 (Koksma's inequality, [6]). *Let f be a periodic function on $[0,1]$ of bounded variation $V(f)$, and let (x_n) be a sequence of real numbers of $[0,1)$. Then*

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n)-\int_0^1 f(t)dt\right|\leq V(f)D_N(x_n).$$

Applying Lemma 2.5, we obtain the following lemma by the same reasoning as in the proof of [6, p.143, Corollary 5.1] (see also [1, p.95]).

Lemma 2.6 ([6], [1]). *Let (x_n) be a sequence of real numbers. Then, for any positive integer h,*

$$\frac{1}{4h}\left|\frac{1}{N}\sum_{n=1}^{N} e(hx_n)\right| \le D_N(x_n).$$

3 Proof of Theorem 1.1

To prove Theorem 1.1, we need two lemmas.

Lemma 3.1. *Let α be an irrational number, β be a non-zero real number, and h be a positive integer. Then*

$$\sum_{c/2<n\le 2c} e(h(\alpha n+\beta\log n)) = \frac{|\beta|^{1/2}h^{1/2}}{|k-\alpha h|} e\left(\beta h \log\left(\frac{\beta h}{k-\alpha h}\right) - \beta h - \frac{\operatorname{sgn}(\beta)}{8}\right) + O\left(\frac{|\beta|^{1/3}h^{1/3}}{|k-\alpha h|}\right) + O\left(\frac{1}{|k-\alpha h|}\right) + O\left(\frac{1}{1-|k-\alpha h|}\right),$$

where $k = [\alpha h] + \frac{1}{2}(\operatorname{sgn}(\beta)+1)$ and $c = \dfrac{\beta h}{k-\alpha h}$ and the constants implied by the O's are absolute.

Proof. We note that $c > 0$. We set $a = c/2$, $b = 2c$ and $f(x) = h(\alpha x + \beta \log x)$. Then

$$\begin{aligned} \sum_{a<n\le b} e(f(n)) &= \sum_{a<n\le b} e(f(n) - kn) \\ &= \sum_{a<n\le b} e((\alpha h - k)n + \beta h \log n) \qquad (3.1)\\ &= \sum_{a<n\le b} e(g(n)), \end{aligned}$$

where $g(x) = (\alpha h - k)x + \beta h \log x$.

Now, we suppose that $\beta > 0$. For $x \in [a, b]$, we have

$$\begin{aligned} g'(x) &= \alpha h - k + \frac{\beta h}{x} \\ &\le \frac{2\beta h}{c} - (k-\alpha h) \\ &= 2\beta h \frac{k-\alpha h}{\beta h} - (k-\alpha h) \\ &= k - \alpha h. \end{aligned}$$

On the other hand, since

$$g'(x)=(\alpha h-k)+\frac{\beta h}{x}\geq -(k-\alpha h)\quad \text{for } x>0,$$

we have

$$|g'(x)|\leq k-\alpha h\quad \text{for } a\leq x\leq b.$$

Similarly, in the case $\beta<0$, we also have

$$|g'(x)|\leq \alpha h-k\quad \text{for } a\leq x\leq b.$$

Anyway we have

$$|g'(x)|\leq |k-\alpha h|\quad \text{for } a\leq x\leq b.$$

Hence, by (3.1) and Lemma 2.2 with $\lambda=|k-\alpha h|$, we obtain

$$\sum_{a<n\leq b} e(f(n))=\sum_{a<n\leq b} e(g(n))=\int_a^b e(g(x))dx+O\left(\frac{1}{1-|k-\alpha h|}\right). \tag{3.2}$$

Applying integration by parts to the integral in (3.2), we obtain

$$\begin{aligned}\int_a^b e(g(x))dx &=\int_a^b e((\alpha h-k)x+\beta h\log x)dx\\ &=\int_a^b e((\alpha h-k)x)e(\beta h\log x)dx\\ &=\int_a^b \left(\frac{e((\alpha h-k)x)}{2\pi i(\alpha h-k)}\right)' e(\beta h\log x)dx\\ &=\left[\frac{e((\alpha h-k)x)}{2\pi i(\alpha h-k)}e(\beta h\log x)\right]_a^b-\int_a^b \frac{e((\alpha h-k)x)}{\alpha h-k}\beta h\frac{1}{x}e(\beta h\log x)dx\\ &=\frac{1}{2\pi i(\alpha h-k)}\left(e((\alpha h-k)b+\beta h\log b)-e((\alpha h-k)a+\beta h\log a)\right)\\ &\quad -\frac{\beta h}{\alpha h-k}\int_a^b \frac{1}{x}e((\alpha h-k)x+\beta h\log x)dx\\ &=\frac{\beta h}{k-\alpha h}\int_a^b \frac{1}{x}e((\alpha h-k)x+\beta h\log x)dx+O\left(\frac{1}{|k-\alpha h|}\right)\\ &=\frac{\beta h}{k-\alpha h}\int_a^b \varphi(x)e(g(x))dx+O\left(\frac{1}{|k-\alpha h|}\right),\end{aligned} \tag{3.3}$$

where $\varphi(x)=1/x$.

We show that the conditions in Lemma 2.4 are satisfied:

$$g'''(x) \text{ is continuous and } \varphi''(x) \text{ exists on } [a,b]$$

by the definitions of $g(x)$ and $\varphi(x)$;
if $\beta > 0$, then for $a \leq x \leq b$,

$$\beta hc^{-2} \ll \beta h \left(\frac{1}{2c}\right)^2 \leq -g''(x) = \beta hx^{-2} \leq \beta h \left(\frac{2}{c}\right)^2 \ll \beta hc^{-2} \ ;$$

if $\beta < 0$, then for $a \leq x \leq b$,

$$-\beta hc^{-2} \ll -\beta h \left(\frac{1}{2c}\right)^2 \leq g''(x) = -\beta hx^{-2} \leq -\beta h \left(\frac{2}{c}\right)^2 \ll -\beta hc^{-2} \ ;$$

$$|g'''(x)| = 2|\beta|hx^{-3} \leq 2|\beta|h \left(\frac{c}{2}\right)^{-3} \ll \beta hc^{-3} \ ;$$

$$|\varphi(x)| = \frac{1}{x} \leq \frac{2}{c} \ll \frac{1}{c} \ ;$$

$$|\varphi'(x)| = \frac{1}{x^2} \leq \left(\frac{2}{c}\right)^2 \ll \frac{1}{c^2} \ ;$$

$$|\varphi''(x)| \leq \frac{2}{x^3} \leq 2\left(\frac{2}{c}\right)^3 \ll \frac{1}{c^3} \ ;$$

$$g'(c) = 0, \quad c \in [a,b].$$

Hence, we can apply Lemma 2.4 with $m_2 = |\beta|hc^{-2}$, $m_3 = |\beta|hc^{-3}$, $H_0 = c^{-1}$, $H_1 = c^{-2}$, and $H_2 = c^{-3}$.

We compute each term on the right-hand side of (2.5) in Lemma 2.4 as follows:

$$\begin{aligned} e\left(g(c) + \frac{1}{8}\text{sgn}(g''(c))\right) \varphi(c)|g''(c)|^{-1/2} &= \frac{1}{c}(|\beta|hc^{-2})^{-1/2} e\left(g(c) - \frac{\text{sgn}(\beta)}{8}\right) \\ &= |\beta|^{-1/2}h^{-1/2} e\left(g(c) - \frac{\text{sgn}(\beta)}{8}\right), \end{aligned}$$

$$H_0 m_2^{-1} m_3^{1/3} = c^{-1}(|\beta|hc^{-2})^{-1}(|\beta|hc^{-3})^{1/3} = |\beta|^{-2/3}h^{-2/3},$$

$$H_1 m_2^{-1} = c^{-2}(|\beta|hc^{-2})^{-1} = |\beta|^{-1}h^{-1},$$

$$H_2 m_2^{-1}(b-a) = \frac{3}{2}c^{-3}(|\beta|hc^{-2})^{-1}c = \frac{3}{2}|\beta|^{-1}h^{-1},$$

$$H_1 m_2^{-2} m_3(b-a) = \frac{3}{2}c^{-2}(|\beta|hc^{-2})^{-2}|\beta|hc^{-3}c = \frac{3}{2}|\beta|^{-1}h^{-1},$$

$$H_2 m_2^{-2} m_3(b-a)^2 = \frac{9}{4}c^{-3}(|\beta|hc^{-2})^{-2}|\beta|hc^{-3}c^2 = \frac{9}{4}|\beta|^{-1}h^{-1},$$

$$\begin{aligned} H_0\min(m_2^{-1/2}, |g'(a)|^{-1}) &\le H_0|g'(a)|^{-1} \\ &= c^{-1}\left|\alpha h - k + \frac{2\beta h}{c}\right|^{-1} \\ &= \frac{k-\alpha h}{\beta h}\frac{1}{|k-\alpha h|} = |\beta|^{-1}h^{-1}, \end{aligned}$$

and similarly,

$$H_0\min(m_2^{-1/2}, |g'(b)|^{-1}) \le |\beta|^{-1}h^{-1}.$$

Therefore, by Lemma 2.4, we obtain

$$\begin{aligned} \int_a^b \varphi(x)e(g(x))dx =& |\beta|^{-1/2}h^{-1/2}e\left(g(c) - \frac{\operatorname{sgn}(\beta)}{8}\right) \\ &+ O(|\beta|^{-2/3}h^{-2/3}) + O(|\beta|^{-1}h^{-1}). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), it follows that

$$\begin{aligned} &\int_a^b e(g(x))dx \\ =& \frac{\beta h}{k-\alpha h}\left(|\beta|^{-1/2}h^{-1/2}e\left(g(c)-\frac{\operatorname{sgn}(\beta)}{8}\right) + O(|\beta|^{-2/3}h^{-2/3}) + O(|\beta|^{-1}h^{-1})\right) \\ &+ O\left(\frac{1}{|k-\alpha h|}\right) \\ =& \frac{|\beta|^{1/2}h^{1/2}}{|k-\alpha h|}e\left(g(c)-\frac{\operatorname{sgn}(\beta)}{8}\right) + O\left(\frac{|\beta|^{1/3}h^{1/3}}{|k-\alpha h|}\right) + O\left(\frac{1}{|k-\alpha h|}\right). \end{aligned} \tag{3.5}$$

By (3.2) and (3.5), we obtain

$$\sum_{a<n\le b} e(f(n)) = \frac{|\beta|^{1/2}h^{1/2}}{|k-\alpha h|} e\left(\beta h \log\left(\frac{\beta h}{k-\alpha h}\right) - \beta h - \frac{\operatorname{sgn}(\beta)}{8}\right)$$
$$+ O\left(\frac{|\beta|^{1/3}h^{1/3}}{|k-\alpha h|}\right) + O\left(\frac{1}{|k-\alpha h|}\right) + O\left(\frac{1}{1-|k-\alpha h|}\right),$$

which completes the proof of Lemma 3.1. □

Lemma 3.2. *Let α be an irrational number, β be a non-zero real number, and h be a positive integer. Then*

$$\sum_{1\le n\le c/2} e(h(\alpha n + \beta \log n)) \ll |\beta| h + \left(|\beta|^{1/2}h^{1/2}+1\right)\log\left(|\beta|h+\frac{1}{2}\right)$$
$$+ |\beta|^{1/2}h^{1/2} + \frac{1}{|k-\alpha h|} + 1,$$

where $k = [\alpha h] + \frac{1}{2}(\operatorname{sgn}(\beta)+1)$ and $c = \dfrac{\beta h}{k-\alpha h}$ and the constant implied by the $\ll$ is absolute.

Proof. We may suppose that $\beta > 0$; otherwise, we replace α by $-\alpha$ and β by $-\beta$.
We have the trivial equality:

$$\sum_{1\le n\le c/2} e(h(\alpha n + \beta \log n)) = \sum_{1<n\le c/2} e(g(n)) + O(1), \tag{3.6}$$

where $g(x) = \alpha h x + \beta h \log x$. Since $g'(x)$ is decreasing, we can define a_p as follows:

$$g'(a_p) = p - \frac{1}{2} \quad \text{for} \quad p \in \mathbb{Z},\ p > \alpha h + \frac{1}{2}.$$

We note that

$$a_p = \frac{2\beta h}{2(p-\alpha h)-1},$$

and a_p is decreasing.
Then

$$\sum_{1<n\le c/2} e(g(n)) = \sum_{p=r}^{s-1} \sum_{a_{p+1}<n\le a_p} e(g(n)) + \sum_{1<n\le a_s} e(g(n)) + \sum_{a_r<n\le c/2} e(g(n)), \tag{3.7}$$

where $r=\min\{p\in\mathbb{Z}:\ g'(c/2)\le p-\frac{1}{2}\}$ and $s=\max\{p\in\mathbb{Z}:\ p-\frac{1}{2}<g'(1)\}$. Since

$$k=g'(c)<g'(c/2)\le r-\frac{1}{2}=g'(a_r)<r, \tag{3.8}$$

we have

$$r\ge k+1, \tag{3.9}$$

and so

$$a_r\le a_{k+1}.$$

Since $g'(x)$ is decreasing and $g'(c/2)\le g'(a_r)$ because of (3.8), we have

$$a_r\le\frac{c}{2}.$$

Since

$$g'(a_s)=s-\frac{1}{2}<g'(1)\le s+\frac{1}{2}=g'(a_{s+1})$$

and $g'(x)$ is decreasing, we obtain

$$a_{s+1}\le 1<a_s.$$

Hence it follows that

$$a_{s+1}\le 1<a_s<\cdots<a_{p+1}<a_p<\cdots<a_r\le a_{k+1},\ a_r\le\frac{c}{2},$$

and

$$k+1\le r<r+1<\cdots<p<p+1<\cdots<s.$$

For $r\le p\le s-1$, we have

$$|g'(x)-p|\le\frac{1}{2}\quad\text{on}\quad[a_{p+1},a_p].$$

Then

$$\sum_{a_{p+1}<n\le a_p}e(g(n))=\sum_{a_{p+1}<n\le a_p}e(g(n)-pn)=\sum_{a_{p+1}<n\le a_p}e(\mathfrak{p}(n)), \tag{3.10}$$

where $\mathfrak{p}(x)=g(x)-px$. Since $|\mathfrak{p}'(x)|=|g'(x)-p|\le\frac{1}{2}$ on $[a_{p+1},a_p]$, Lemma 2.2 with $\lambda=\frac{1}{2}$ yields

$$\sum_{a_{p+1}<n\le a_p}e(\mathfrak{p}(n))=\int_{a_{p+1}}^{a_p}e(\mathfrak{p}(x))dx+O(1). \tag{3.11}$$

We have

$$\int_{a_{p+1}}^{a_p} e(\mathfrak{p}(x))dx = \int_{a_{p+1}}^{a_p} e(g(x) - px)dx$$
$$= \int_{a_{p+1}}^{a_p} e((\alpha h - p)x + \beta h \log x)dx$$
$$= \int_{a_{p+1}}^{a_p} e((\alpha h - p)x)e(\beta h \log x)dx$$
$$(3.12) \qquad = \int_{a_{p+1}}^{a_p} \left(\frac{e((\alpha h - p)x)}{2\pi i(\alpha h - p)}\right)' e(\beta h \log x)dx$$
$$= \left[\frac{e(\mathfrak{p}(x))}{2\pi i(\alpha h - p)}\right]_{a_{p+1}}^{a_p} - \frac{\beta h}{\alpha h - p}\int_{a_{p+1}}^{a_p} \varphi(x)e(\mathfrak{p}(x))dx,$$

where $\varphi(x) = x^{-1}$. Since $\mathfrak{p}'(x) = \alpha h - p + \dfrac{\beta h}{x}$ and $\mathfrak{p}''(x) = -\dfrac{\beta h}{x^2} < 0$, we obtain

$$\left|\frac{\varphi'(x)}{\mathfrak{p}''(x)}\right| = \frac{1}{\beta h} \quad \text{and} \quad \frac{\varphi(x)}{|\mathfrak{p}''(x)|^{1/2}} = (\beta h)^{-1/2}.$$

Therefore, applying Lemma 2.3, we obtain

$$(3.13) \qquad \int_{a_{p+1}}^{a_p} \varphi(x)e(\mathfrak{p}(x))dx = O((\beta h)^{-1/2}) + O((\beta h)^{-1}).$$

By (3.12) and (3.13), we obtain

$$(3.14) \qquad \int_{a_{p+1}}^{a_p} e(\mathfrak{p}(x))dx = O\left(\frac{(\beta h)^{1/2}}{|\alpha h - p|}\right) + O\left(\frac{1}{|\alpha h - p|}\right).$$

By (3.10), (3.11), and (3.14), we have

$$(3.15) \qquad \sum_{a_{p+1} < n \leq a_p} e(g(n)) = O\left(\frac{(\beta h)^{1/2}}{|\alpha h - p|}\right) + O\left(\frac{1}{|\alpha h - p|}\right) + O(1)$$

for $r \leq p \leq s - 1$. Similarly, we obatin

$$(3.16) \qquad \sum_{1 < n \leq a_s} e(g(n)) = O\left(\frac{(\beta h)^{1/2}}{|\alpha h - s|}\right) + O\left(\frac{1}{|\alpha h - s|}\right) + O(1).$$

Combining (3.15) and (3.16), we have

$$\begin{aligned}
&\sum_{1<n\le a_s} e(g(n))+\sum_{p=r}^{s-1}\sum_{a_{p+1}<n\le a_p} e(g(n)) \\
(3.17)\qquad &=\sum_{p=r}^{s}\left(O\left(\frac{(\beta h)^{1/2}}{|\alpha h-p|}\right)+O\left(\frac{1}{|\alpha h-p|}\right)+O(1)\right) \\
&\ll\left((\beta h)^{1/2}+1\right)\sum_{p=r}^{s}\frac{1}{|\alpha h-p|}+\sum_{p=r}^{s}1.
\end{aligned}$$

Since $\alpha h<k\le r-1$, we have

$$(3.18)\qquad |\alpha h-p|=p-\alpha h=r-\alpha h+(p-r)>p-r+1 \quad \text{for} \quad r\le p\le s.$$

Since $s<f'(1)+\frac{1}{2}=\alpha h+\beta h+\frac{1}{2}$, $r\ge k+1$, and $k-\alpha h>0$, we obtain

$$(3.19)\qquad s-r<\alpha h+\beta h+\frac{1}{2}-k-1=\beta h-\frac{1}{2}-(k-\alpha h)<\beta h-\frac{1}{2}.$$

From (3.17), (3.18), and (3.19), it follows that

$$\begin{aligned}
&\sum_{1<n\le a_s} e(g(n))+\sum_{p=r}^{s-1}\sum_{a_{p+1}<n\le a_p} e(g(n)) \\
(3.20)\qquad &\ll\left((\beta h)^{1/2}+1\right)\sum_{n=1}^{s-r+1}\frac{1}{n}+(s-r+1) \\
&\ll\beta h+\left(\beta^{1/2}h^{1/2}+1\right)\log\left(\beta h+\frac{1}{2}\right)+\beta^{1/2}h^{1/2}+1.
\end{aligned}$$

Next we will estimate $\sum_{a_r<n\le c/2} e(g(n))$. In view of (3.9), we distinguish two cases
(i) $r=k+1$ and (ii) $r\ge k+2$.

Case (i): $r=k+1$.

We have

$$(3.21)\qquad \sum_{a_r<n\le c/2} e(g(n))=\sum_{a_r<n\le c/2} e(g(n)-rn)=\sum_{a_r<n\le c/2} e(\mathfrak{r}(n)),$$

where $\mathfrak{r}(x)=g(x)-rx$. Since $f'(x)$ is decreasing, we obtain

$$f'(\frac{c}{2})\le f'(x)\le f'(a_r)=r-\frac{1}{2} \quad \text{for} \quad a_r\le x\le c/2.$$

Thus, for $a_r \le x \le c/2$, we obtain

$$\frac{1}{2} \le -\mathfrak{r}'(x) = r - f'(x) \le r - f'(c/2) = r - 2k + \alpha h = 1 - k + \alpha h < 1,$$

and so

$$|\mathfrak{r}'(x)| \le 1 - k + \alpha h < 1 \quad \text{for} \quad a_r \le x \le c/2.$$

Hence, from (3.21) and Lemma 2.2 with $\lambda = 1 - k + \alpha h$, it follows that

$$\sum_{a_r < n \le c/2} e(g(n)) = \int_{a_r}^{c/2} e(\mathfrak{r}(x))dx + O\left(\frac{1}{k - \alpha h}\right). \tag{3.22}$$

We have

$$\begin{aligned}\int_{a_r}^{c/2} e(\mathfrak{r}(x))dx &= \int_{a_r}^{c/2} e((\alpha h - r)x + \beta h \log x)dx \\ &= \int_{a_r}^{c/2} \left(\frac{e((\alpha h - r)x)}{2\pi i(\alpha h - r)}\right)' e(\beta h \log x)dx \\ &= \left[\frac{e((\alpha h - r)x)}{2\pi i(\alpha h - r)} e(\beta h \log x)\right]_{a_r}^{c/2} - \frac{\beta h}{\alpha h - r} \int_{a_r}^{c/2} \frac{1}{x} e(\mathfrak{r}(x))dx.\end{aligned}$$

In a similar manner as that of (3.13), we obtain

$$\int_{a_r}^{c/2} \frac{1}{x} e(\mathfrak{r}(x))dx = O((\beta h)^{-1/2}) + O((\beta h)^{-1}).$$

Hence we have

$$\int_{a_r}^{c/2} e(\mathfrak{r}(x))dx = O\left(\frac{1}{|\alpha h - k - 1|}\right) + O\left(\frac{(\beta h)^{1/2}}{|\alpha h - k - 1|}\right).$$

Using $|\alpha h - k - 1| = 2 - \{\alpha h\} > 1$, we obtain

$$\int_{a_r}^{c/2} e(\mathfrak{r}(x))dx = O(\beta^{1/2} h^{1/2}) + O(1). \tag{3.23}$$

By (3.22) and (3.23), we have

$$\sum_{a_r < n \le c/2} e(g(n)) = O(\beta^{1/2} h^{1/2}) + O\left(\frac{1}{k - \alpha h}\right) + O(1). \tag{3.24}$$

Case (ii): $r \geq k+2$.
In a similar manner as that of (3.15), we obtain

$$\sum_{a_r<n\leq c/2} e(g(n)) = O\left(\frac{(\beta h)^{1/2}}{|\alpha h - r+1|}\right) + O\left(\frac{1}{|\alpha h - r+1|}\right) + O(1).$$

Since $|\alpha h - r + 1| = r - 1 - \alpha h \geq k + 1 - \alpha h \geq 1$, we have

$$\sum_{a_r<n\leq c/2} e(g(n)) = O(\beta^{1/2}h^{1/2}) + O(1). \tag{3.25}$$

From (3.6), (3.7), (3.20), (3.24), and (3.25), it follows that

$$\sum_{1\leq n\leq c/2} e(g(n)) \ll \beta h + \left(\beta^{1/2}h^{1/2}+1\right)\log\left(\beta h + \frac{1}{2}\right) + \beta^{1/2}h^{1/2} + \frac{1}{k-\alpha h} + 1,$$

which completes the proof of Lemma 3.2. □

Theorem 1.1 follows immediately from Lemma 3.1 and Lemma 3.2.

4 Proof of Theorem 1.2

Proof of Theorem 1.2. The cases $\beta > 0$ and $\beta < 0$ are analogous, so only the former is considered.

Let m be a positive integer. By our assumptions, we have $h_m = q_{2m+1} > 2$ and $k_m = p_{2m+1}$ with $p_n/q_n = [a_0, a_1, \ldots, a_n]$. The sequence (h_m) is strictly increasing. Since $0 < k_m - \alpha h_m = p_{2m+1} - \alpha q_{2m+1} < \dfrac{1}{q_{2m+1}} = \dfrac{1}{h_m} < \dfrac{1}{2}$, we have

$$k_m = [\alpha h_m] + 1$$

and

$$0 < k_m - \alpha h_m < \frac{1}{h_m} \tag{4.1}$$

(see [7, Theorem 5, p.8.]). We set $c_m = \dfrac{\beta h_m}{k_m - \alpha h_m}$, so that $N_m = [2c_m]$. We also set $x_n = \alpha n + \beta \log n$.

Applying Theorem 1.1 with $N = N_m$, $h = h_m$, $k = k_m$, and $c = c_m$, we have

$$
\begin{aligned}
&\frac{1}{h_m N_m}\sum_{n=1}^{N_m} e(h_m x_n) \\
=&\frac{1}{h_m N_m}\sum_{1\le n\le 2c_m} e(h_m x_n) \\
=&\frac{\beta^{1/2}}{h_m^{1/2} N_m (k_m-\alpha h_m)} e\left(\beta h_m \log\left(\frac{\beta h_m}{k_m-\alpha h_m}\right) - \beta h_m - \frac{1}{8}\right) \\
&+O\left(\frac{\beta^{1/3}}{h_m^{2/3} N_m (k_m-\alpha h_m)}\right) + O\left(\frac{1}{h_m N_m (k_m-\alpha h_m)}\right) \\
&+O\left(\frac{1}{h_m N_m (1-(k_m-\alpha h_m))}\right) + O\left(\frac{\beta}{N_m}\right) \qquad (4.2)\\
&+O\left(\frac{\beta^{1/2}}{h_m^{1/2} N_m}\log\left(\beta h_m + \frac{1}{2}\right)\right) + O\left(\frac{1}{h_m N_m}\log\left(\beta h_m + \frac{1}{2}\right)\right) \\
&+O\left(\frac{\beta^{1/2}}{h_m^{1/2} N_m}\right) + O\left(\frac{1}{h_m N_m}\right).
\end{aligned}
$$

Since $\beta h_m^2 > 1$ for sufficiently large m, we have

$$c_m = \frac{\beta h_m}{k_m - \alpha h_m} > \beta h_m^2 > 1,$$

so that

$$N_m = [2c_m] > 2c_m - 1 > c_m.$$

Hence

$$N_m > c_m = \frac{\beta h_m}{k_m - \alpha h_m} > \beta h_m^2, \tag{4.3}$$

and so

$$h_m < \beta^{-1/2} N_m^{1/2}. \tag{4.4}$$

Since $N_m(k_m - \alpha h_m) > c_m(k_m - \alpha h_m) = \beta h_m$ and $N_m(k_m - \alpha h_m) \le 2c_m(k_m - \alpha h_m) = 2\beta h_m$, we have

$$\beta h_m < N_m(k_m - \alpha h_m) \le 2\beta h_m. \tag{4.5}$$

Let ε be any positive number. From (4.2), (4.3), (4.4), and (4.5), it follows that, for $\delta=\frac{\varepsilon}{\beta^{3/4}}$,

$$\begin{aligned}
&\frac{1}{4h_m}\left|\frac{1}{N_m}\sum_{n=1}^{N_m}e(h_m x_n)\right| \\
\geq&\frac{1}{8\beta^{1/2}h_m^{3/2}} \\
&+O\left(\frac{1}{\beta^{2/3}h_m^{5/3}}\right)+O\left(\frac{1}{\beta h_m^2}\right)+O\left(\frac{1}{\beta h_m^3}\right)+O\left(\frac{1}{h_m^2}\right) \\
&+O\left(\frac{1}{\beta^{1/2}h_m^{5/2}}\log\left(\beta h_m+\frac{1}{2}\right)\right)+O\left(\frac{1}{\beta h_m^3}\log\left(\beta h_m+\frac{1}{2}\right)\right) \\
&+O\left(\frac{1}{\beta^{1/2}h_m^{5/2}}\right)+O\left(\frac{1}{\beta h_m^3}\right) \\
\geq&\left(\frac{1}{8\beta^{1/2}}-\delta\right)h_m^{-3/2} \\
\geq&\left(\frac{1}{8\beta^{1/2}}-\delta\right)\left(\beta^{-1/2}N_m^{1/2}\right)^{-3/2} \\
=&\left(\frac{\beta^{1/4}}{8}-\varepsilon\right)N_m^{-3/4}
\end{aligned}$$

for sufficiently large m.
By Lemma 2.6, we arrive at the desired inequality, which complets the proof. □

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