THE ASSOCIATED SHEAF FUNCTOR THEOREM IN ALGEBRAIC SET THEORY

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Abstract. We prove a version of the associated sheaf functor theorem in Algebraic Set Theory. The proof is established working within a Heyting pretopos equipped with a system of small maps satisfying the axioms originally introduced by Joyal and Moerdijk. This result improves on the existing developments by avoiding the assumption of additional axioms for small maps and the use of collection sites.

1. Introduction

The associated sheaf functor theorem asserts that the inclusion functor from the category of sheaves over a site into the corresponding category of presheaves has a left adjoint which preserves finite limits, and it implies that categories of sheaves have small colimits [24, Chapter 3]. In topos theory, the construction of internal sheaves provides a method to define new elementary toposes from old ones, analogous to the method of forcing extensions for models of Zermelo-Frankel set theory. Indeed, sheaf toposes have been widely used to prove independence results [8, 9, 10, 12, 33]. Within this context, sites are considered as being internal to an elementary topos, and the notions of presheaf and sheaf are also defined internally, without reference to the category of sets. Categories of internal sheaves have finite colimits since they form elementary toposes, and the topos-theoretic version of the associated sheaf functor theorem [11, 17, 22] provides a way to describe these colimits in terms of those of the ambient topos.

Versions of the associated sheaf functor theorem in Algebraic Set Theory involve replacing elementary toposes by pairs $(\mathcal{E}, \mathcal{S})$ consisting of a category $\mathcal{E}$ equipped with a distinguished family of maps $\mathcal{S}$. The category $\mathcal{E}$ is thought of as a category of classes, and the family $\mathcal{S}$, whose elements are referred to as *small maps*, is thought of as the family of functions between classes whose fibers are sets [20]. Within this context, sites are considered as being internal to $\mathcal{E}$ and therefore may be required to satisfy appropriate smallness conditions. Since the category $\mathcal{E}$ is not generally assumed to be an elementary topos, versions of the associated sheaf functor theorem are essential to prove that categories of internal sheaves inherit the structure that $\mathcal{E}$ is assumed to have, thus making it possible to obtain independence results.

Although Algebraic Set Theory has proved to be a flexible framework to study category-theoretic models of various set theories [3, 4, 6, 7, 13, 21,

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27, 31], a general treatment of sheaf constructions does not seem to have emerged yet. Such a development would have a natural application in the development of a theory of forcing extensions for models of constructive and intuitionistic set theories [2, 16, 23, 32]. We aim to improve on this situation by establishing a new version of the associated sheaf functor theorem in Algebraic Set Theory. Our version is proved working within a Heyting pretopos $\mathcal{E}$ equipped with a class of small maps $\mathcal{S}$ satisfying just the basic axioms for small maps originally introduced in [20]. Therefore, we avoid the assumption of the structure of a ΠW-pretopos on $\mathcal{E}$ or of additional axioms for small maps on $\mathcal{S}$. Furthermore, our proof will be simpler than the existing ones, since we avoid any use of the notion of a collection site, and work instead with Grothendieck sites with small covers.

The notion of a collection site was introduced by Ieke Moerdijk and Erik Palmgren in [27] in order to establish a version of the associated sheaf functor theorem for sites with small covers within a stratified pseudotopos, a possible predicative counterpart of the notion of an elementary topos [26, 27]. Their proof proceeds in two steps. First, they proved a version of the theorem for collection sites with small covers. Secondly, they showed that every site with small covers is equivalent to a collection site with small covers. In [27], the reduction of a site with small covers to a collection site with small covers relies on an application of the Axiom of Multiple Choice, a new axiom for small maps that is assumed to hold in a stratified pseudotopos, which has subsequently been studied also from a set-theoretic perspective [29]. By a result of Benno van den Berg [5], the reduction of sites with small covers to collection sites with small covers can also be carried out with a weaker axiom for small maps, asserting that the universal small map is a collection map, which is equivalent to the Collection Axiom. However, until now it has been an open problem whether it is possible to prove the associated sheaf functor theorem avoiding the use of collection sites. One motivation to have such a proof is that the notion of a collection site is rather complex, and hence difficult to work with.

Here we work with Grothendieck sites with small covers. This notion seems to capture an appropriate level of generality. The assumption that the Grothendieck site is small seems instead to be too restrictive. The reason for this is closely related to the procedure of generating Grothendieck sites by closing off a site that satisfies only the Local Character condition under the Maximality and Transitivity conditions of a Grothendieck site [18, Chapter C.2]. While the generating site can be safely assumed to be small, it does not seem possible to show that the associated Grothendieck site is again small without assuming that W-types of small maps with small codomain are again small [5, 26, 27]. Another advantage of working with Grothendieck sites with small covers is that, by simply adding a further smallness condition, it is possible to obtain a version of the associated sheaf functor theorem that works in the context of Heyting categories with a restricted form of exactness, generalising the version of the associated sheaf functor theorem implicit in the results announced in [7].

Before concluding these introductory remarks, let us point out that we do not consider here the problem of isolating an appropriate notion of small map between sheaves, and leave the treatment of this issue to future research.

2. Heyting pretoposes with small maps

2.1. **Axioms for small maps.** Let $\mathcal{E}$ be a Heyting pretopos [20, 25]. The pullback functor along $f : B \to A$ is denoted here as $f^* : \mathcal{E}/A \to \mathcal{E}/B$. Its restriction to subobjects, written $f^{-1} : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$, has both a left and a right adjoint, written $\exists_f : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ and $\forall_f : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$, respectively. As usual, $A+B$ denotes the coproduct of $A, B \in \mathcal{E}$. The initial and terminal object of $\mathcal{E}$ are written 0 and 1, respectively.

We recall the axioms for open and small maps [19, 20]. A class of maps $\mathcal{S}$ in $\mathcal{E}$ is said to be a class of *open maps* if it satisfies the axioms (A1)-(A7) stated below.

(A1): The class $\mathcal{S}$ contains isomorphisms and is closed under composition.

(A2): For every pullback square of the form

(1)
$$\begin{array}{ccc} D & \xrightarrow{k} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[h]{} & A \end{array}$$

if $f : B \to A$ is in $\mathcal{S}$, then $g : D \to C$ is in $\mathcal{S}$.

(A3): For every pullback square as (1), if $h : C \to A$ is an epimorphism and $g : D \to C$ is in $\mathcal{S}$, then $f : B \to A$ is in $\mathcal{S}$.

(A4): The maps $0 \to 1$ and $1 + 1 \to 1$ are in $\mathcal{S}$.

(A5): If $f : B \to A$ and $g : D \to C$ are in $\mathcal{S}$, then $f+g : B+D \to A+C$ is in $\mathcal{S}$.

(A6): For every commutative triangle of the form

$$\begin{array}{ccc} B & \xrightarrow{p} & B' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & A & \end{array}$$

if $p : B \to B'$ is an epimorphism and $f : B \to A$ is in $\mathcal{S}$, then $f' : B' \to A$ is in $\mathcal{S}$.

(A7): For every small map $f : B \to A$ and every epimorphism $p : X \to B$, there exists a quasi-pullback diagram of the form

$$\begin{array}{ccccc} D & \longrightarrow & X & \xrightarrow{p} & B \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle f} \\ C & & \xrightarrow[h]{} & & A \end{array}$$

where $g : D \to C$ is in $\mathcal{S}$ and $h : C \to A$ is an epimorphism.

A class of open maps $\mathcal{S}$ is said to be a class of *small maps* if it satisfies also the axioms (S1)-(S2) stated below.

(S1): If $f : B \to A$ is in $\mathcal{S}$, then the pullback functor $f^* : \mathcal{E}/A \to \mathcal{E}/B$ has a right adjoint, which we write $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$.

(S2): There exists a map $u : E \to U$ in $\mathcal{S}$ such that every map $f : B \to A$ in $\mathcal{S}$ fits in a diagram of form

$$\begin{array}{ccccc} B & \longleftarrow & D & \longrightarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ A & \underset{h}{\longleftarrow} & C & \longrightarrow & U \end{array} \tag{2}$$

where $h : C \to A$ is an epimorphism and both squares are pullbacks.

We refer to (A6) as the Quotients Axiom, to (A7) as the Collection Axiom, to (S1) as the Exponentiability Axiom, and to (S2) as the Representability Axiom. As we will see, the Collection Axiom plays an essential role in the proof of the associated sheaf functor theorem.

Let $(\mathcal{E}, \mathcal{S})$ be a Heyting pretopos equipped with a class of small maps. We say that an object A is *small* if the unique map $A \to 1$ is small. A *small subobject* of an object A is a subobject $R \rightarrowtail A$ such that R is a small object. A small map of the form $R \rightarrowtail X \times A \to X$, where $X \times A \to X$ is the first projection, will be referred to an *X-indexed family of small subobjects* of A. As shown in [20, §I.3], indexed families of small subobjects can be classified. This means that for every object A there exists an object, written $\mathcal{P}(A)$ and called the *power-object* of A, and a $\mathcal{P}(A)$-indexed family of small subobjects of A, written $\ni_A \rightarrowtail \mathcal{P}(A) \times A \to \mathcal{P}(A)$ and called the *membership relation* of A, such that for every X-indexed family of small subobjects of A, say $R \rightarrowtail X \times A \to X$, there exists a unique map $\chi_R : X \to \mathcal{P}(A)$ fitting in commutative diagram of form

$$\begin{array}{ccc} R & \longrightarrow & \ni_A \\ \downarrow & & \downarrow \\ X \times A & \longrightarrow & \mathcal{P}(A) \times A \\ \downarrow & & \downarrow \\ X & \underset{\chi_R}{\longrightarrow} & \mathcal{P}(A) \end{array}$$

where both squares are pullbacks.

From now on, we work with a Heyting pretopos $\mathcal{E}$ equipped with a class of small maps $\mathcal{S}$. Let us point out, however, that Benno van den Berg observed that the proof in [20, §I.3] that indexed families of small subobjects can be classified carries over even if the Representability Axiom is replaced by the Weak Representability Axiom, which is obtained from the Representability Axiom by requiring the square on the left-hand side of the diagram in (2) to be a quasi-pullback rather than a pullback. This observation implies that all the results in what follows hold also when $\mathcal{S}$ satisfies the Weak Representability Axiom rather than the Representability Axiom.

2.2. **The internal language.** As in [26, 27] we use extensively the internal language of $(\mathcal{E}, \mathcal{S})$. This allows us to treat $\mathcal{E}$ as if it were a category of 'sets' equipped with a distinguished family of maps that gives rise to a notion of 'small set'. Here, 'sets' will support all of the operations that are part of the structure of a Heyting pretopos. In particular, we can interpret any first-order logical formula and form quotients of arbitrary equivalence relations. The closure properties of 'small sets' are determined by the axioms for small maps. Details of the formulation of the axioms for small maps in the internal language can be found in [4, 27].

We frequently define subobjects $S \rightarrowtail A$ in $\mathcal{E}$ using explicit definitions of the form $S =_{\text{def}} \{x \in A \mid \phi(x)\}$, where $\phi(x)$ is a formula of the internal language of $\mathcal{E}$ with a free variable x ranging over elements of A. For a generalised element $a \in A$, given by an arrow $a : X \to A$ in $\mathcal{E}$, we then write either $\phi(a)$ or $a \in S$ to express that $a : X \to A$ factors through $S \rightarrowtail X$. This convention allows us to use some standard abbreviations. For example, given two subobjects $S \rightarrowtail A$ and $T \rightarrowtail A$, we abbreviate $(\forall x \in A)(x \in S \Rightarrow x \in T)$ by simply writing $S \subseteq T$.

3. Grothendieck sites

3.1. **Presheaves.** Let $\mathbb{C}$ be a small category in $(\mathcal{E}, \mathcal{S})$. This means that both the 'object of objects' $\mathbb{C}_0$ and the 'object of arrows' $\mathbb{C}_1$ are small objects in $\mathcal{E}$. Furthermore, we assume that $\mathbb{C}$ has small diagonals, which means that the diagonal maps $\Delta_{\mathbb{C}_0} : \mathbb{C}_0 \rightarrowtail \mathbb{C}_0 \times \mathbb{C}_0$ and $\Delta_{\mathbb{C}_1} : \mathbb{C}_1 \rightarrowtail \mathbb{C}_1 \times \mathbb{C}_1$ are small maps. We write $\text{Psh}_{\mathcal{E}}(\mathbb{C})$ for the category of internal presheaves over $\mathbb{C}$, defined as in [24, §V.7]. It is well-known that $\text{Psh}_{\mathcal{E}}(\mathbb{C})$ is a Heyting pretopos. For a presheaf F, the result of the action on $x \in F(a)$ of an arrow $f : b \to a$ in $\mathbb{C}$ will be written as $x \cdot f \in F(b)$. Thus, the associativity and unit axioms for presheaves can be written as follows:

$$(x \cdot f) \cdot g = x \cdot (fg)\,, \qquad x \cdot 1_a = x\,,$$

where $1_a : a \to a$ is the identity map on a, and $f : b \to a$, $g : c \to b$ are composable maps in $\mathbb{C}$. The Yoneda embedding of an object $a \in \mathbb{C}$ is denoted as $\text{y}_{\mathbb{C}}(a) \in \text{Psh}_{\mathcal{E}}(\mathbb{C})$.

3.2. **Covering sieves.** For $a \in \mathbb{C}$, a *sieve* on a is a subobject $P \rightarrowtail \text{y}_{\mathbb{C}}(a)$. It will be convenient to identify a sieve $P \rightarrowtail \text{y}_{\mathbb{C}}(a)$ with a subobject of the object of arrows of $\mathbb{C}$, which we denote also by P, whose elements are arrows with codomain a and such that for every $f : b \to a$ and every $g : c \to b$, if $f \in P$ then $fg \in P$. For a sieve $P \rightarrowtail \text{y}_{\mathbb{C}}(a)$ and an arrow $f : b \to a$, we write $P \cdot f \rightarrowtail \text{y}_{\mathbb{C}}(b)$ for the sieve defined by letting

$$P \cdot f =_{\text{def}} \{g : c \to b \mid f\,g : c \to a \in P\}\,. \tag{3}$$

A *small sieve* is a sieve $S \rightarrowtail \text{y}_{\mathbb{C}}(a)$ for which $S(b)$ is small for every $b \in \mathbb{C}$. Since $\mathbb{C}_0$ is small and has a small diagonal, this holds if and only S is small as an object of $\mathcal{E}$. Since $\mathbb{C}_1$ has a small diagonal, the operation defined in (3) restricts to an operation on small sieves, and therefore the definition $\Omega(a) =_{\text{def}} \{S \rightarrowtail \text{y}_{\mathbb{C}}(a) \mid S \text{ small sieve}\}$, for $a \in \mathbb{C}$, determines a presheaf Ω.

A site consists of a small category with small diagonals and of a coverage. Here we consider coverages that are sifted, in the sense that we work with covering sieves and not with covering families. Furthermore, we consider coverages that satisfy not only the Local Character property (L), but also the Maximality (M) and Transitivity (T) properties. Following [18, Chapter C.2], we refer to them as Grothendieck coverages. For each object $a \in \mathbb{C}$, we write $M_a \rightarrowtail \mathrm{y}_{\mathbb{C}}(a)$ for the maximal sieve on a, which is given by the identity map.

Definition 3.1. Let $\mathbb{C}$ be a small category with small diagonals in $\mathcal{E}$. A *Grothendieck coverage with small covers* on $\mathbb{C}$ consists of a family $(\mathrm{Cov}(a) \mid a \in \mathbb{C})$ such that elements of $\mathrm{Cov}(a)$ are small sieves, and the following hold:

- (M) $M_a \in \mathrm{Cov}(a)$.
- (L) If $f : b \to a$ and $S \in \mathrm{Cov}(a)$, then $S \cdot f \in \mathrm{Cov}(b)$.
- (T) If $S \in \mathrm{Cov}(a)$, T is a small sieve on a, and for all $f : b \to a \in S$ we have $T \cdot f \in \mathrm{Cov}(b)$, then $T \in \mathrm{Cov}(a)$.

A *Grothendieck site with small covers* is a pair $(\mathbb{C}, \mathrm{Cov})$, where $\mathbb{C}$ is a small category with small diagonals and Cov is a Grothendieck coverage with small covers on $\mathbb{C}$.

From now on, we fix a Grothendieck site with small covers $(\mathbb{C}, \mathrm{Cov})$. For $a \in \mathbb{C}$, an element $S \in \mathrm{Cov}(a)$ will be referred to as a *small covering sieve* on a. For our development, it is essential to define what it means for a sieve that is not necessarily small to be a covering sieve: we will say that a sieve $P \rightarrowtail \mathrm{y}_{\mathbb{C}}(a)$ on a is a *covering sieve* if there exists $S \in \mathrm{Cov}(a)$ such that $S \subseteq P$. We introduce a minor abuse of notation and write $P \in \mathrm{COV}(a)$ to mean that P is a covering sieve on a. Formally, this is defined by letting

$$P \in \mathrm{COV}(a) =_{\mathrm{def}} (\exists S \in \mathrm{Cov}(a))\ S \subseteq P\,. \tag{4}$$

Note that if $S \rightarrowtail \mathrm{y}_{\mathbb{C}}(a)$ is a small sieve, we have that $S \in \mathrm{Cov}(a)$ is equivalent to $S \in \mathrm{COV}(a)$. A key ingredient in the proof of the associated sheaf functor theorem is the fact, stated in Proposition 3.3, that general covering sieves satisfy Maximality, Local Character, and Transitivity properties analogous to those in Definition 3.1. In order to prove Proposition 3.3, we need the following technical lemma.

Lemma 3.2. *Let $S \in \mathrm{Cov}(a)$ and Q be a sieve on a. If*

$$(\forall f : b \to a \in S)\ Q \cdot f \in \mathrm{COV}(b)$$

then there exists a family of small sieves $(V_f \mid f : b \to a \in S)$ such that

$$(\forall f : b \to a \in S)\big[V_f \subseteq Q \cdot f\,,\ V_f \in \mathrm{Cov}(b)\big]\,.$$

Proof. See Appendix A. □

Let us point out that the proof of Lemma 3.2 makes essential use of the Collection Axiom. In fact, the proof concentrates all the uses of the Collection Axiom necessary to establish the associated sheaf functor theorem. We use Lemma 3.2 to establish that general covering sieves satisfy the Transitivity property (T) of Proposition 3.3.

Proposition 3.3. *The following properties hold.*

- *(M)* $M_a \in \mathrm{COV}(a)$,
- *(L) If $f : b \to a$ and $P \in \mathrm{COV}(a)$, then $P \cdot f \in \mathrm{COV}(b)$.*
- *(T) If $P \in \mathrm{COV}(a)$, Q is a sieve on a, and for all $f : b \to a \in P$ we have $Q \cdot f \in \mathrm{COV}(b)$, then $Q \in \mathrm{COV}(a)$.*

Proof. Both (M) and (L) are immediate consequences of the definition in (4) and the corresponding properties of the Grothendieck site $(\mathbb{C}, \mathrm{Cov})$. To prove (T), let $P \in \mathrm{COV}(a)$ and let Q be a sieve on a such that

$$(\forall f : b \to a \in P)\ Q \cdot f \in \mathrm{COV}(b)\,.$$

Since $P \in \mathrm{COV}(a)$, there exists $S \in \mathrm{Cov}(a)$ such that $S \subseteq P$. Lemma 3.2 implies that there exists a family of small sieves $(V_f \mid f : b \to a \in S)$ such that

$$(\forall f : b \to a \in S)\big[V_f \subseteq Q \cdot f\,,\ V_f \in \mathrm{Cov}(b)\big]\,.$$

Note that the elements of the family $(V_f \mid f : b \to a \in S)$ are small covering sieves. Using the Quotients Axiom, we define a small sieve V by letting

$$V =_{\mathrm{def}} \{h : c \to a \mid (\exists f : b \to a \in S)(\exists g : c \to b \in V_f)\ h = fg\}\,.$$

Once we prove that $V \in \mathrm{Cov}(a)$ and that $V \subseteq Q$, the definition in (4) implies that $Q \in \mathrm{COV}(a)$, and the proof will be complete. To show that $V \in \mathrm{Cov}(a)$, we use Transitivity of the Grothendieck site. We know that $S \in \mathrm{Cov}(a)$ and that V is a sieve, so it suffices to show that

$$(\forall f : b \to a \in S)\ \big[V \cdot f \in \mathrm{Cov}(b)\big]\,.$$

This follows because $V_f \subseteq V \cdot f$ for every $f : b \to a \in S$. Finally, to check that $V \subseteq Q$, it suffices to recall that for $f : b \to a \in S$, we have that $V_f \subseteq Q \cdot f$. □

Since the proof of the Transitivity property (T) in Proposition 3.3 makes use of Lemma 3.2, it relies essentially on the Collection Axiom. The idea of extending the notion of a covering sieve from small sieves to general sieves as done here generalises, and is inspired by, the idea of extending a nucleus operator from small lower sections to general lower sections of a poset, which arose originally in the study of formal topology within constructive set theories [14, 15]. Indeed, as explained in [28], the notion of a formal topology [30] is essentially a special case of that of a Grothendieck site.

4. The associated sheaf functor theorem

4.1. **Sheaves.** The notion of a sheaf will be formulated as usual in topos theory. However, we require only the existence of amalgamations for matching families of elements indexed by small covering sieves. In order to make this precise, let us fix a presheaf F and a small covering sieve $S \in \mathrm{Cov}(a)$. A family $\mathbf{x} = (x_f \mid f : b \to a \in S)$, where $x_f \in F(b)$ if $f : b \to a \in S$, is said to be *matching* if it satisfies the following compatibility condition: for every $f : b \to a \in S$ and every $g : c \to b$ it holds that

$$x_f \cdot g = x_{fg}\,.$$

An *amalgamation* for a matching family $\mathbf{x}$ as above is an element $x \in F(a)$ such that for all $f : b \to a \in S$ we have $x \cdot f = x_f$. We say that a presheaf F is *separated* if every matching family admits at most one amalgamation, and that it is a *sheaf* if every matching family has a unique amalgamation. Thus, a separated presheaf is a sheaf if and only if every matching family admits at least one amalgamation. It should be noted that here, as elsewhere, satisfaction of these conditions is understood as validity of the corresponding expressions in the internal language of $(\mathcal{E}, \mathcal{S})$, which can be formulated in terms of equivalent elementary diagrammatic conditions in the familiar way. We write $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov})$ for the full subcategory of $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ whose objects are sheaves. The goal of the reminder of this section is to prove the following result.

Theorem 4.1. *Let $(\mathcal{E}, \mathcal{S})$ be a Heyting pretopos with a class of small maps. For every Grothendieck site with small covers $(\mathbb{C}, \mathrm{Cov})$ in $\mathcal{E}$, the inclusion functor $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov}) \to \mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ has a left adjoint which preserves finite limits.*

The proof of Theorem 4.1 uses Grothendieck's double-plus construction [24, Chapter 3], but exploits in a crucial way the properties of general covering sieves established in Proposition 3.3.

4.2. **Grothendieck's double-plus construction.** Let F be a presheaf. We define an equivalence relation on matching families of F by letting, for $\mathbf{x} = (x_f \mid f : b \to a \in S)$ and $\mathbf{x}' = (x'_f \mid f : b \to a \in S')$,

$$\mathbf{x} \sim \mathbf{x}' =_{\mathrm{def}} \big(\exists V \in \mathrm{Cov}(a)\big)\big(V \subseteq S \cap S' \,,\ (\forall f : b \to a \in V)\, x_f = x'_f\big)\,. \tag{5}$$

For a matching family $\mathbf{x} = (x_f \mid f : b \to a \in S)$, we write $[\mathbf{x}]$ for its equivalence class under the equivalence relation in (5). We define $F^+(a)$ as the object of equivalence classes of the equivalence relation defined in (5). This object can be given as a quotient of the object of matching families of $F(a)$, and exists since $\mathcal{E}$ is an exact category. In turn, the object of matching families of $F(a)$ can be constructed using the Exponentiability Axiom, since we are considering matching families indexed by small covering sieves.

Observe that objects of matching families admit an evident presheaf structure. Given a matching family $\mathbf{x} = (x_f \mid f : b \to a \in S)$, and an arrow $f : b \to a$, we obtain a new matching family by letting $\mathbf{x} \cdot f =_{\mathrm{def}} (x_{gf} \mid g : c \to b \in S \cdot f)$. Note that $S \cdot f \in \mathrm{Cov}(b)$ by the Local Character property of the Grothendieck site. This action is clearly compatible with the equivalence relation defined in (5) and hence it determines a presheaf structure on F^+.

Lemma 4.2. *For every presheaf F, the presheaf F^+ is separated.*

Proof. Let $\mathbf{x} = (x_f \mid f : b \to a \in S)$, where $S \in \mathrm{Cov}(a)$, and $\mathbf{x}' = (x'_f \mid f : b \to a \in S')$, where $S' \in \mathrm{Cov}(a)$, be matching families of elements of F. Assuming that $[\mathbf{x}]$ and $[\mathbf{x}']$ are amalgamations of a matching family of elements of $F^+(a)$, we need to show that $\mathbf{x} \sim \mathbf{x}'$. By the assumption, there exists $V \in \mathrm{Cov}(a)$ such that for all $f : b \to a \in V$, we have $\mathbf{x} \cdot f \sim \mathbf{x}' \cdot f$. We define a sieve Q by letting

$$Q =_{\mathrm{def}} \{f : b \to a \in S \cap S' \mid x_f = x'_f\}\,.$$

Observe that Q need not be small since the diagonal map $\Delta_F : F \rightarrowtail F \times F$ is not assumed to be small. We prove that $Q \in \mathrm{COV}(a)$ using the Transitivity property of Proposition 3.3. Since $V \in \mathrm{Cov}(a)$ and Q is a sieve, it is sufficient to show that $Q \cdot f \in \mathrm{COV}(b)$ for every $f : b \to a \in V$. If $f : b \to a \in V$, we have $\mathbf{x} \cdot f \sim \mathbf{x}' \cdot f$, and so there exists $W \in \mathrm{Cov}(b)$ such that $W \subseteq (S \cdot f) \cap (S' \cdot f)$ and for all $g : c \to b \in W$ it holds that $x_{fg} = x'_{fg}$.

Hence, we have found $W \in \mathrm{Cov}(b)$ such that $W \subseteq Q \cdot f$. By (4) we obtain that $Q \cdot f \in \mathrm{COV}(b)$, as required.

Since we have shown that $Q \in \mathrm{COV}(a)$, we can apply again the definition in (4) and derive that there exists $T \in \mathrm{Cov}(a)$ such that $T \subseteq Q$. Therefore we have $T \subseteq S \cap S'$ and we have $x_f = x'_f$ for every $f : b \to a \in T$. By the definition in (5), we have $\mathbf{x} \sim \mathbf{x}'$, as required. □

Lemma 4.3. *For every separated presheaf F, the presheaf F^+ is a sheaf.*

Proof. Let $S \in \mathrm{Cov}(a)$ and let $(\sigma_f \mid f : b \to a \in S)$ be a matching family of elements of F^+. We wish to show that this family admits an amalgamation. We define a covering sieve $Q \in \mathrm{COV}(a)$ by letting

$$\begin{aligned} Q =_{\mathrm{def}} \Big\{ h : c \to a \mid \exists f : b \to a \in S\,,\ \exists V \in \mathrm{Cov}(b)\,, \\ \exists\, \mathbf{y} = (y_g \mid g : c \to b \in V) \text{ matching family such that} \\ \sigma_f = [\mathbf{y}] \text{ and } (\exists g : c \to b \in V)\ h = fg \Big\}\,. \end{aligned}$$

To show that $Q \in \mathrm{COV}(a)$ we use the Transitivity property of Proposition 3.3. Since $S \in \mathrm{Cov}(a)$ and Q is a sieve on a, we need to show that $Q \cdot f \in \mathrm{COV}(b)$ for every $f : b \to a \in S$. Given $f : b \to a \in S$, let $\mathbf{y} = (y_g \mid g : c \to b \in V)$, for $V \in \mathrm{Cov}(b)$, such that $\sigma_f = [\mathbf{y}]$. Since $Q \cdot f = \{g : c \to b \mid f\,g \in Q\}$, it follows that $V \subseteq Q \cdot f$. By the definition (4), we get that $Q \cdot f \in \mathrm{COV}(b)$, as required. Having shown that $Q \in \mathrm{COV}(a)$, we can apply again the definition in (4) and deduce that there exists $T \in \mathrm{Cov}(a)$ such that $T \subseteq Q$.

To define an amalgamation for the matching family $(\sigma_f \mid f : b \to a \in S)$, consider the matching family $\mathbf{x} = (x_h \mid h : c \to a \in T)$ defined by

$$x_h =_{\mathrm{def}} y_g\,, \tag{6}$$

where $g : c \to b$ is any arrow for which there exist an arrow $f : b \to a \in S$, a small covering sieve $V \in \mathrm{Cov}(b)$, and a matching family $\mathbf{y} = (y_g \mid g : c \to b \in V)$, such that $g : c \to b \in V$, $\sigma_f = [\mathbf{y}]$, and $h = fg$. Such a $g : c \to b$ exists since $h : c \to a \in T$ and $T \subseteq Q$. We now use the assumption that F is separated to verify that the family $\mathbf{x}$ is well-defined. In order to do so, we need to show that

$$y_g = y'_{g'}\,. \tag{7}$$

where $\mathbf{y} = (y_g \mid g : c \to b \in V)$ and $\mathbf{y}' = (y'_{g'} \mid g' : c' \to b' \in V')$ are matching families defined on covering sieves $V \in \mathrm{Cov}(b)$ and $V' \in \mathrm{Cov}(b')$, respectively, such that $g : c \to b \in V$, $g' : c' \to b' \in V'$, and we have that $\sigma_f = [\mathbf{y}]$ and $\sigma_{f'} = [\mathbf{y}']$, and that both $h = fg$ and $h = f'g'$, for arrows $f : b \to a \in S$ and $f' : b' \to a \in S$. Since $fg = f'g'$, we have a commutative diagram of the form

$$\begin{array}{ccc} c & \xrightarrow{\ g\ } & b \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f} \\ b' & \xrightarrow[\ f'\ ]{} & a \end{array} \tag{8}$$

Since F is separated, in order to prove (7) it suffices to exhibit both y_g and $y'_{g'}$ as amalgamations for a matching family of elements of F. First, observe that the commutativity of the diagram in (8) and the compatibility of the family $(\sigma_f \mid f : b \to a \in S)$ implies that

$$[\mathbf{y}] \cdot g = \sigma_f \cdot g = \sigma_{fg} = \sigma_{f'g'} = \sigma_{f'} \cdot g' = [\mathbf{y}'] \cdot g' \, .$$

Hence, there exists $W \in \mathrm{Cov}(c)$, such that $W \subseteq (V \cdot g) \cap (V' \cdot g')$ and such that for all $h : d \to c \in W$ we have

$$y_{gh} = y'_{g'h} \, .$$

It is now clear that we have a matching family $(y_{gh} \mid h : d \to c \in W)$. Next, we prove that both y_g and $y'_{g'}$ are amalgamations for this family. This follows from the observation that for $h : d \to c \in W$ we have

$$y_g \cdot h = y_{gh} = y'_{g'h} = y_{g'} \cdot h \, .$$

The final steps of the proof involve the verification that the family $\mathbf{x}$ defined by $\mathbf{x} = (x_h \mid h : c \to a \in T)$ is indeed matching, and that $\sigma =_{\mathrm{def}} [\mathbf{x}]$ is an amalgamation for the given matching family $(\sigma_f \mid f : b \to a \in S)$. We provide the details for completeness. To show that $(x_h \mid h : c \to a \in T)$ is a matching family, we need to consider $h : c \to a \in T$ and $k : d \to c$ and prove that

$$x_h \cdot k = x_{hk} \, .$$

The left-hand side equals $y_g \cdot k$, where $\mathbf{y} = (y_g \mid g : c \to b \in V)$ is a matching family defined on a covering sieve $V \in \mathrm{Cov}(b)$ such that $g : c \to b \in V$ and $\sigma_f = [\mathbf{y}]$ for some $f : b \to a \in S$ such that $h = fg$. The right-hand side equals $y_{g'}$, where $\mathbf{y}' = (y'_{g'} \mid g' : c' \to b' \in V')$ is a matching family defined on a covering sieve $V' \in \mathrm{Cov}(b')$ such that $g' : d \to b' \in V'$ and $\sigma_{f'} = [\mathbf{y}']$ for some $f' : b' \to a \in S$ such that $hk = f'g'$. We show that $y_g \cdot k = y'_{g'}$ using again that F is separated. Indeed, both $y_g \cdot k$ and $y'_{g'}$ are amalgamations for the matching family

$$(y_{gkj} \mid j : e \to d \in W) = (y'_{g'j} \mid j : e \to d \in W) \, ,$$

where $W \in \mathrm{Cov}(d)$ is a covering sieve such that $W \subseteq (V \cdot gk) \cap (V' \cdot g')$ and for which all $j : e \to d \in W$ satisfy $y_{gkj} = y'_{g'j}$. Such $W \in \mathrm{Cov}(d)$ exists because $(\sigma_f \mid f : b \to a \in S)$ is a matching family, and we have $fgk = hk = f'g'$. Finally, to prove that σ is an amalgamation of $(\sigma_f \mid f : b \to a \in S)$, we show that if $\sigma_f = [\mathbf{y}]$, where $\mathbf{y} = (y_g \mid g : c \to b \in V)$ for some $V \in \mathrm{Cov}(b)$, then

$$[\mathbf{x}] \cdot f = \sigma_f \, . \tag{9}$$

Define $R =_{\mathrm{def}} V \cap (T \cdot f)$ and observe that $R \in \mathrm{COV}(b)$, since $V \in \mathrm{Cov}(b)$ and $T \cdot f \in \mathrm{Cov}(b)$. By (4), there exists $W \in \mathrm{Cov}(b)$ such that $W \subseteq R$. In particular, $W \subseteq V$ since $W \subseteq R \subseteq V$. We claim that for every $g : c \to b \in W$, we have

$$x_{fg} = y_g \, . \tag{10}$$

This holds by the definition in (6), since we have an arrow $f : b \to a \in S$, a small covering sieve $V \in \mathrm{Cov}(b)$, and a matching family $\mathbf{y} = (y_g \mid g : c \to b \in V)$ such that $g : c \to b \in V$ and $\sigma_f = [\mathbf{y}]$. Note that $g : c \to b \in V$ follows from $W \subseteq V$. We have therefore proved that $\mathbf{x} \cdot f \sim \mathbf{y}$, which implies (9), as required. □

The rest of the proof follows the same steps as the standard proof of the associated sheaf functor theorem [24, Chapter III]. In particular, we have the following lemma, where we use $\eta_F : F \to F^+$ for the natural transformation whose component $(\eta_F)_a : F(a) \to F^+(a)$ maps $x \in F(a)$ into the equivalence class of the matching family $(x \cdot f \mid f : b \to a \in M_a)$.

Lemma 4.4. *Let F and G be presheaves. If G is a sheaf, every natural transformation $\phi : F \to G$ factors uniquely through $\eta_F : F \to F^+$, making the following diagram commute*

$$\begin{array}{ccc} F & \xrightarrow{\eta_F} & F^+ \\ & \searrow^{\phi} & \downarrow{\scriptstyle \bar{\phi}} \\ & & G \end{array}$$

Proof. Given $\sigma \in F^+(a)$, let $\mathbf{x} = (x_f \mid f : b \to a \in S)$ such that $\sigma = [\mathbf{x}]$. Define $\bar{\phi}_a(\sigma) \in G(a)$ as the unique amalgamation of the matching family $\mathbf{y} =_{\text{def}} (\phi_b(x_f) \mid f : b \to a \in S)$. The compatibility of the family $\mathbf{y}$ follows from the compatibility of $\mathbf{x}$ and the naturality of $\phi : F \to G$. This definition can be shown to be independent of the choice of $\mathbf{x}$ such that $\sigma = [\mathbf{x}]$. The diagram commutes since, for $x \in F(a)$, $\phi_a(x) \in G(a)$ provides an amalgamation for the matching family $(\phi_b(x \cdot f) \mid f : b \to a \in M_a)$. □

We have an adjunction of the form

$$\mathrm{Psh}_{\mathcal{E}}(\mathbb{C}) \underset{\perp}{\overset{\mathbf{a}}{\rightleftarrows}} \mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov}) \,.$$

The right adjoint is the inclusion and the left adjoint, called the *associated sheaf functor*, is defined by letting $\mathbf{a}(F) =_{\text{def}} (F^+)^+$. The unit is the natural transformation with components given by the composites

$$F \xrightarrow{\eta_F} F^+ \xrightarrow{\eta_{F^+}} (F^+)^+ \,.$$

To complete the proof of Theorem 4.1, it remains to show that the associated sheaf functor preserves finite limits. This follows from the fact that the plus construction preserves finite limits, which can be easily proved via a direct calculation.

Corollary 4.5. *Let $(\mathcal{E}, \mathcal{S})$ be a Heyting pretopos with a class of small maps. For every Grothendieck site with small covers $(\mathbb{C}, \mathrm{Cov})$ in $\mathcal{E}$, the category $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov})$ is a Heyting pretopos.*

Proof. Finite limits in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov})$ are computed as in $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$. Finite sums in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov})$ are obtained by applying the associated sheaf functor to sums in $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$, so for example the coproduct of two sheaves F and G is obtained by applying the associated sheaf functor to the coproduct of F and G in $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$. Quotients of equivalence relations are computed similarly. The definition of universal quantification is standard [24, §III.8]. □

4.3. **Small-presentable Grothendieck sites.** The version of the associated sheaf functor theorem stated in Theorem 4.1 can be readily adapted to the setting considered in [7]. That setting can be obtained by making the following changes to the one considered here.

(1) Assume that $\mathcal{E}$ has stable quotients only of equivalence relations $R \rightarrowtail X \times X$ given by small monomorphisms, rather than of arbitrary ones. In particular, $\mathcal{E}$ is a Heyting category, not a Heyting pretopos. Accordingly, replace epimorphisms by regular epimorphisms in the formulation of the axioms for small maps.
(2) Assume that the class $\mathcal{S}$ of small maps satisfies the additional property that all diagonal maps $\Delta_A : A \rightarrowtail A \times A$ are small.
(3) Assume that for every small map $f : B \to A$, the functor $\forall_f : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ preserves smallness of monomorphisms.

Within this context, it does not seem possible to define the quotient of the equivalence relation in (5) without further assumptions on the Grothendieck site. Let us call a Grothendieck site $(\mathbb{C}, \mathrm{Cov})$ *small-presentable* if there exists a family $(\mathrm{BCov}(a) \mid a \in \mathbb{C})$ such that $\mathrm{BCov}(a)$ is small for every $a \in \mathbb{C}$ and for every small sieve $P \rightarrowtail \mathrm{y}_{\mathbb{C}}(a)$ we have

$$P \in \mathrm{Cov}(a) \Leftrightarrow (\exists U \in \mathrm{BCov}(a))\big[U \subseteq P\big] .$$

Assuming the Grothendieck site to be small-presentable allows us to form the quotient of the equivalence relation in (5), since the formula defining it is equivalent to one defining a small monomorphism. The rest of the proof of the associated sheaf functor theorem carries over unchanged. Furthermore, the small sites considered in [7] give rise to small-presentable Grothendieck sites, and hence we derive a version of the associated sheaf functor for them. The notion of a small-presentable Grothendieck site on partially ordered sets dates back essentially to [16]. Variants of it have been formulated and considered in the study of formal topology within constructive set theory [1, 2, 14].

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Appendix A. Proof of a technical lemma

We prove Lemma 3.2.

Proof. Let $S \in \mathrm{Cov}(a)$, let Q be a sieve on a, and assume that

$$(\forall f : b \to a \in S)\; Q \cdot f \in \mathrm{COV}(b) \tag{11}$$

We need to show that there exists a family of small sieves $(V_f \mid f : b \to a \in S)$ such that

$$(\forall f : b \to a \in S)\big[V_f \subseteq Q \cdot f \, , \; V_f \in \mathrm{Cov}(b)\big] \, .$$

Note that we have

$$(12) \quad (\forall f : b \to a \in S)(\forall T \, , T' \in \Omega(b)) \\ \Big[\big(T \subseteq T' \subseteq Q \cdot f \, , \; T \in \mathrm{Cov}(b)\big) \Rightarrow T' \in \mathrm{Cov}(b)\Big] \, .$$

For $f : b \to a \in S$ and $u \in \Sigma_{f' : b' \to a \in S} \Omega(b')$, define

$$(13) \qquad \phi(f, u) =_{\mathrm{def}} (\exists T \in \Omega(b))\big[u = (f, T) \, , \; T \subseteq Q \cdot f \, , \; T \in \mathrm{Cov}(b)\big] \, .$$

By (11) we know that

$$\big(\forall f : b \to a \in S\big)\big(\exists u \in \Sigma_{f' : b' \to a \in S} \, \Omega(b)\big)\phi(f, u) \, .$$

The Collection Axiom implies that there exists $P \in \mathcal{P}\big(\Sigma_{f' : b' \to a \in S} \, \Omega(b')\big)$ such that

$$(14) \quad (\forall f : b \to a \in S)(\exists u \in P)\phi(f, u) \, , \; (\forall u \in P)(\exists f : b \to a \in S)\phi(f, u) \, .$$

Using the Quotients Axiom, we define the family $(\tau_f \mid f : b \to a \in S)$ by letting

$$\tau_f =_{\mathrm{def}} \{T \in \Omega(b) \mid (f, T) \in P\} \, .$$

We claim that for all $f : b \to a \in S$, it holds that

$$(15) \;\; \tau_f \subseteq \Omega(b) \, , \quad \exists T \in \Omega(b)(T \in \tau_f) \, , \quad \forall T \in \tau_f\big(T \subseteq Q \cdot f \, , \; T \in \mathrm{Cov}(b)\big) \, .$$

For $f : b \to a \in S$, by (14), we have that there exists $u \in P$ such that $\phi(f, u)$ holds. By the definition of $\phi(f, u)$ in (13), it follows that there is $T \in \Omega(b)$ such that $u = (f, T)$, $T \subseteq Q \cdot f$ and $T \in \mathrm{Cov}(b)$. Since $u = (f, T)$ and $u \in P$, it follows that $(f, T) \in P$ and so $T \in \tau_f$, as required. To conclude the verification of (15), let $T \in \tau_f$. We need to show that $T \subseteq Q \cdot f$ and $T \in \mathrm{Cov}(b)$. In order to do so, define $u =_{\mathrm{def}} (f, T)$. Since $T \in \tau_f$, we have $u \in P$. By (14) there exists $f' : b' \to a$ such that $\phi(f', u)$. By the definition of ϕ in (13), we must have that $f : b \to a$ and $f' : b' \to a$ are equal, and so $T \subseteq Q \cdot f$ and $T \in \mathrm{Cov}(b)$, as required.

Finally, the required family $(V_f \mid f : b \to a \in S)$ is defined by letting, for $f : b \to a \in S$

$$V_f =_{\mathrm{def}} \{g : c \to b \mid (\exists T \in \tau_f)(g : c \to b \in T)\} \, .$$

First, we show that $V_f \subseteq Q \cdot f$. For $g : c \to b \in V_f$, there exists $T \in \tau_f$ such that $g : c \to b \in T$. Since $T \in \tau_f$, by (15) we have $T \subseteq Q \cdot f$, and thus $V_f \subseteq Q \cdot f$. Secondly, we show that $V_f \in \mathrm{Cov}(b)$. By (15), we know that there exists $T \in \tau_f$ such that $T \subseteq Q \cdot f$ and that $T \in \mathrm{Cov}(b)$. But we have also $T \subseteq V_f \subseteq Q \cdot f$ and so, by (12), we get $V_f \in \mathrm{Cov}(b)$, as required. □

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