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Pricing of Reusable Resources under Ambiguous Distributions of Demand and Service Time with Emerging Applications

Xuan Vinh Doan* Xiao Lei† Siqian Shen‡

Abstract

Monopolistic pricing models for revenue management are widely used in practice to set prices of multiple products with uncertain demand arrivals. The literature often assumes deterministic time of serving each demand and that the distribution of uncertainty is fully known. In this paper, we consider a new class of revenue management problems inspired by emerging applications such as cloud computing and city parking, where we dynamically determine prices for multiple products sharing limited resource and aim to maximize the expected revenue over a finite horizon. Random demand of each product arrives in each period, modeled by a function of the arrival time, product type, and price. Unlike the traditional monopolistic pricing, here each demand stays in the system for uncertain time. Both demand and service time follow ambiguous distributions, and we formulate robust deterministic approximation models to construct efficient heuristic fixed-price pricing policies. We conduct numerical studies by testing cloud computing service pricing instances based on data published by the Amazon Web Services (AWS) and demonstrate the efficacy of our approach for managing revenue and risk under various distributions of demand and service time.

Keywords: dynamic pricing; demand and service time uncertainty; distributional ambiguity; robust optimization; cloud computing

1 Introduction

As a central role of revenue management, pricing offers a strategy to increase revenue by intelligently matching limited resource capacities with demand. It allows companies to promptly adjust prices based on demand variation, inventory levels, or production schedules. The literature of pricing strategies dates back to studies of yield management in airlines (see, e.g., Rothstein 1971), and the related revenue management problems that widely arise in most capacity-constrained service industries including car rental, workforce planning, and hotel management.

In this paper, we focus on monopolistic pricing in which prices of products are dynamically set by one company to maximize the company's revenue over a finite sales horizon, as opposed to

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oligopolistic pricing, in which multiple firms set product prices that will affect their competitors' demand and future pricing strategies. Monopolistic pricing models have a wide spectrum of applications that involve setting prices dynamically over a finite sales horizon, shared resource with capacity, random demand arrivals following certain known distribution. We refer to Talluri and van Ryzin (2005) as a comprehensive overview that summarizes different pricing models and solution methods for revenue management. Elmaghraby and Keskinocak (2003), Bitran and Caldentey (2003), McGill and van Ryzin (1999), Chiang et al. (2007), Chen and Chen (2015) all conduct surveys of pricing models for monopolistic revenue management of a single product or multiple perishable products sharing nonrenewable resources.

In the dynamic pricing literature, demand realization is considered following a stochastic process (e.g., a Poisson process) with parameters (e.g., the expected arrival rate) depending on the product price. The existing literature has discussed pricing problems of a single perishable product (Gallego and van Ryzin 1994), of multiple perishable products (Gallego and van Ryzin 1997, Maglaras and Meissner 2006), and of nonperishable products (Kachani and Perakis 2006), given initial inventory and demand functions over a finite sales horizon. Feng and Xiao (2000) develop a dynamic pricing model with discrete price choices and reversible price changes. Federgruen and Heching (1999) and Chen and Simchi-Levi (2004) study how to decide price and inventory simultaneously for a single product, finite time horizon system. Popescu and Wu (2007) study dynamic pricing strategy with customer reference effect, while Liu and Cooper (2015), Chen and Farias (2015) take into account patient customers who are willing to wait. Zhao and Zheng (2000) consider complex model settings with the demand following a non-homogeneous Poisson process and under time-varying distribution of reservation price.

However, exact demand distribution in real-world applications is often unknown and a decision maker can only observe realized demand after releasing product prices in each period. Correspondingly, the state-of-the-art revenue management literature has started investigating problems with unknown demand functions and proposed different models and approaches for handling the demand ambiguity. These approaches include online learning for obtaining demand function parameters over time (see Besbes and Zeevi 2012a, Keskin and Zeevi 2014, Besbes and Zeevi 2012b, Araman and Caldentey 2009, Harrison et al. 2012, den Boer 2014, Cheung et al. 2017), and robust optimization methods using a class of possible demand functions that do not admit any parametric representation (see Besbes and Zeevi 2009, Perakis and Roels 2010, Dokka Venkata Satyanaraya et al. 2018). Lim and Shanthikumar (2007) utilize relative entropy to capture the ambiguity of the distribution of price-dependent customer arrival process and develop models to hedge against the ambiguity. Haviv and Randhawa (2014) focus on pricing problems in queues without exact demand information.

Papier and Thonemann (2010), Levi and Radovanović (2010) consider dynamic pricing for reusable resources, but with deterministic service time, with application mainly in the hotel revenue management. We also study pricing problems with reusable resources in this paper, and assume that random demand of each product arrives in each period, as a function depending on the time of arrival, product type, and current price. It requires a fixed amount of resources to satisfy each unit demand of a product, and the total resource capacity shared by all products is constant throughout the process. Furthermore, demand of all the products arriving in each period cannot be immediately processed, but each demand stays in the system for an uncertain number of periods, which can be viewed as random service time. This differs from the assumptions made in most dynamic pricing literature for manufacturing-to-stock systems, and is motivated by emerging application domains of revenue management, e.g., cloud computing and city parking. For instance, with fixed number of servers and their computing capacity, a manager dynamically sets prices for multiple cloud computing products, each of which has a random number of demand submissions over time. Furthermore, the computation takes a certain amount of machine memory for different products, but the total CPU time for processing each demand unit is uncertain, meaning that a job stays for unknown time periods in the system. We refer the interested readers to Püschel et al. (2015), Kashef et al. (2013) for a broad of revenue management decision-making models for cloud computing service admission problems. Similarly, in city parking, the total number of parking spaces in a parking structure is fixed, and the demand for parking spaces and the required time by each vehicle to stay in a parking structure are random.

Lei and Jasin (2016) considers general models for on-demand pricing with reusable resources and random demand, but they also assume deterministic service time. Besbes et al. (2019) investigate the efficacy of constant-price policy for revenue management of reusable resources, but they assume exponential service time. To the best of our knowledge, the pricing models with both uncertain demand and service time have not been considered in the literature. Under practical concerns, we consider that parameters in the demand functions and service time for processing each demand are unknown, yielding ambiguity associated with the distributions of the two independent uncertainties.

The main contribution of the paper is twofold. First, we develop fixed-price pricing policies using robust deterministic approximation models, which take into account ambiguous distributions of reservation price and random service time taken by each product. We show that the deterministic approximation approach provides good asymptotic bounds for the expected revenue under optimal pricing policies. Second, we provide solution methods to find fixed-price policies when distributional ambiguity of random demand and service time is modeled based on limited historical data. Through comprehensive computational studies of diverse instances, we demonstrate that under the proposed fixed-price policy, the robust approach yields better results in terms of revenue as compared to those of models with fixed distributions of demand and service time. In addition, the proposed fixed-price policy outperforms the commonly-used constant-price policy in out-of-sample tests.

The remainder of the paper is organized as follows. In Section 2, we develop a base model to dynamically optimize products' prices under random demand and service time. In Section 3, we formulate a robust counterpart of the stochastic dynamic program and develop asymptotic bounds for the expected revenue under optimal pricing policies using the approximation. In Section 4, we formulate a robust optimization model to find fixed-price policies derived from the deterministic approximation given a distributional ambiguity model. In Section 5, we demonstrate the performance of the fixed-price policies on a set of instances randomly generated from real data of the Amazon Web Services (AWS). In Section 6, we conclude the paper and propose future research directions.

2 Problem Description and Formulations

In this paper, we formulate a dynamic pricing problem for multiple products subject to a capacity constraint of a single, shared resource over a finite time horizon. The setting of single resource is relevant to most systems of cloud computing and city parking, where we do not differentiate the types of servers and parking spaces, respectively. The consideration of finite time horizon is appropriate given that cloud computing servers usually have finite operating time before being maintained or turned off and parking companies update pricing strategies regularly to incorporate new information about customer preferences. The detailed description of the considered problem is as follow.

A service provider offers n different products to customers, and all the products share a single type of resources. Each product i, $i = 1, \ldots, n$, requires c_i units of resource, $c_i \in \mathbb{Z}_+$, for up to $\tau_i^{\max}$ time units, $\tau_i^{\max} > 0$. There are total C units of resources, $C \in \mathbb{Z}_+$. The problem is formulated over a finite time horizon, $t = 1, \ldots, T$. In each period t, there are random r_i^t units of requests for product i, $i = 1, \ldots, n$, and the service provider seeks an optimal price p_i^t within a price set $\mathcal{P}_i \subseteq [p_i^{\min}, p_i^{\max}]$ with $0 \leq p_i^{\min} \leq p_i^{\max}$ for each product i, $i = 1, \ldots, n$, to maximize the total revenue of all the products over the T periods. We assume that the time period is small enough so that there is at most one product request in each period t, i.e., $\sum_{i=1}^{n} r_i^t \leq 1$ for all $t = 1, \ldots, T$. Under this assumption, $r_i^t \in \{0, 1\}$ and the request arrival of each product i in each period t can be considered as Bernoulli with the probability $q_i^t = \bar{r}_i^t = \mathbb{E}\left[r_i^t\right]$ for all $i = 1, \ldots, n$ and $t = 1 \ldots, T$. The probability that there is no request arrival in period t is $q_0^t = 1 - \sum_{i=1}^{n} \bar{r}_i^t$, which is clearly non-negative under the above assumption for all $t = 1, \ldots, T$. The overall request arrival process can be considered as multinomial with $n + 1$ categories and T trials.

In each period t, given p_i^t, each customer either accepts or rejects the offer based on his/her reservation price. The amount of actual demand, $d_i(p_i^t, t)$, of each product i in period t depends on the arrival rate r_i^t, the price p_i^t and acceptance probability $a_i(p_i^t)$, modeled as

$$d_i(p_i^t, t) = r_i^t \cdot \mathbb{I}\left\{p_i^t \leq \pi_i(t)\right\}. \tag{1}$$

Here $\pi_i(t)$ represents the random reservation price of the actual customer for product i at time t if one arrives; $\mathbb{I}\{\bullet\}$ is an indication function which returns 1 if $\bullet$ is true and 0 otherwise. If $r_i^t = 0$, we can set $\pi_i(t)$ as the random reservation price of an arbitrary customer for product i given that the product with arrival rate r_i^t is always 0 in this case no matter how $\pi_i(t)$ is set. Following the majority of revenue management literature, we assume the same probabilistic acceptance behavior for all customers given price offers for a particular product. More concretely, we consider a single acceptance probability function for each product, i.e., $\mathbb{P}\left(p \leq \pi_i(t)\right) = a_i(p)$ given a price offer p for product i. In order to make sure one can reject a request right away by setting a sufficiently high price when there is not enough capacity of the resource, we assume that there exists a so-called *null* price $\bar{p}_i$ such that $a_i(\bar{p}_i) = 0$. Clearly, $\bar{p}_i \geq p_i^{\max}$ and the actual set of feasible prices is $\bar{\mathcal{P}}_i = \mathcal{P}_i \cup \{\bar{p}_i\}$. In general, it is difficult to know the exact acceptance probability functions. In this paper, we assume that the acceptance probability $a_i(.)$ is unknown to the decision maker and it belongs to an uncertainty set $\mathcal{U}_i^a$ instead. We later use statistical information to construct these uncertainty sets (see, e.g., Bertsimas et al. (2017), Bandi and Bertsimas (2012) for how to construct uncertainty sets using statistical analysis.)

Now, using the actual demand $d_i(p_i^t, t)$, the total revenue in the period t is given by:

$$R(t) = \sum_{i=1}^{n} R_i(t) = \sum_{i=1}^{n} d_i(p_i^t, t) \cdot p_i^t. \tag{2}$$

Note that the revenue is computed when the requests are accepted and we assume that all accepted requests will be served until they finish, which implies that we do not need to consider whether the requests can be served within the finite time horizon T when making accept/reject decisions. We now compute the number of requests of product i that are still in the system at the end of period t, $t = 1, \ldots, T$, as follows:

$$D_i(t) = \sum_{s=1}^{t} d_i(p_i^s, s) \cdot \mathbb{I}\left\{\tau_i(s) > t - s\right\}, \tag{3}$$

where $\tau_i(s)$ represents the random service time of the *actual* demand of product i in period s if there is one. If $d_i(p_i^s, s) = 0$, we can simply assume $\tau_i(s)$ is the random service time of an arbitrary demand of product i given that the product with $d_i(p_i^s, s)$ is always 0 no matter how $\tau_i(s)$ is set. Here an implicit assumption is that the service time of a demand request in period t is considered from the beginning of period $t+1$. The random variables $\tau_i(s)$ follow a probability distribution with a tail probability function $\beta_i(t) = \mathbb{P}(\tau_i(s) > t)$ for each product i, $i = 1, \ldots, n$, and $t = 0, \ldots, \tau_i^{\max}$. Clearly, $\beta_i(0) = 1$ and $\beta_i(\tau_i^{\max}) = 0$. Similar to the acceptance probability function $a_i(\cdot)$, we assume that the tail probability function $\beta_i(\cdot)$ of service time is unknown but belongs to an uncertainty set $\mathcal{U}_i^b$.

The feasibility constraints of the problem are the following capacity constraints:

$$\sum_{i=1}^{n} c_i D_i(t) \leq C, \quad \forall\, t = 1, \ldots, T. \tag{4}$$

A pricing policy is *feasible* if the feasibility constraints are satisfied almost surely given any probability function $a_i(\cdot) \in \mathcal{U}_i^a$ and $\beta_i(\cdot) \in \mathcal{U}_i^b$. Note that these feasibility constraints will affect the decision of whether to accept or reject of a particular request in each period t. More precisely, one indeed needs to reject a request right away if there is not enough available resource to accommodate that request. In this paper, we seek feasible pricing policies, which can be used to determine p_i^t for product i, $i = 1, \ldots, n$, as functions of past demand and actions up to time t, $t = 1, \ldots, T$. This ensures the *non-anticipativity* property of the considered policies.

To characterize the feasible policies, let $y_i^t(\tau)$ be the number of requests of product i appearing in period t that have been in the system for τ time periods, $\tau = 0, \ldots, \tau_i^{\max} - 1$. Under the assumption that there is at most one product request per period, we have $y_i^t(\tau) \in \{0,1\}$ for all τ. In addition, $y_i^t(0) = d_i(p_i^{t-1}, t-1)$ for all $t > 1$. Suppose $y_i^t(\tau) = 1$ for some $t < \tau_i^{\max} - 1$, that product request can remain in the system at the end of period t (or the beginning of period $t+1$), i.e., $y_i^{t+1}(\tau+1) = y_i^t(\tau)$, with the probability

$$b_i(\tau) = \mathbb{P}\left(\tau_i(s) > \tau + 1 \,|\, \tau_i(s) > t\right) = \frac{\beta_i(\tau+1)}{\beta_i(\tau)}. \tag{5}$$

Suppose that for all functions $\beta_i(\cdot) \in \mathcal{U}_i^b$, $\beta_i(\tau) > 0$ for all $\tau < \tau_i^{\max}$, and random service time are independent. Then, there is a positive probability that the number of requests of product i that are still in the system at the end of period t is the maximum possible value of $\sum_{\tau=0}^{\tau_i^{\max}-2} y_i^t(\tau)$. To make sure that the capacity constraints are satisfied almost surely, a request of product i needs to be rejected immediately in period t if there is not enough capacity, i.e.,

$$C - \sum_{k=1}^{n} c_k \left(\sum_{\tau=0}^{\tau_k^{\max}-2} y_k^t(\tau) \right) < c_i.$$

With $\bar{p}_i \in \bar{\mathcal{P}}_i$ and $a_i(\bar{p}_i) = 0$, it is guaranteed that there is always a feasible pricing policy since imposing the null price $\bar{p}_i$ allows us to reject requests right away in those situations. Given the current state $\boldsymbol{Y}_i^t = (y_i^t(0), \ldots, y_i^t(\tau_i^{\max}-1)) \in \{0,1\}^{\tau_i^{\max}}$ for $i = 1, \ldots, n$, the feasible set $\mathcal{A}_i(\boldsymbol{Y}_i^t)$ of prices can then be defined as follows:

$$\mathcal{A}_i(\boldsymbol{Y}_i^t) = \begin{cases} \mathcal{P}_i, & C - \sum_{k=1}^{n} c_k \left(\sum_{\tau=0}^{\tau_k^{\max}-2} y_k^t(\tau) \right) \geq c_i, \\ \{\bar{p}_i\}, & \text{otherwise.} \end{cases} \tag{6}$$

The objective of the problem is to maximize the total expected revenue $\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{n} R_i(t)\right]$. Given that the revenue $R_i(t)$ only depends on the price p_i^t and the action of the concerned customer in period t, it is sufficient to define a feasible pricing policy as a function of the current state, i.e., $p_i^t(\cdot) : \{0,1\}^{\tau^{\max}} \to \bar{\mathcal{P}}_i$ with $\tau^{\max} = \sum_{i=1}^{n} \tau_i^{\max}$, whose domain is restricted as described in (6).

Let $\mathcal{F}_i$ be the set of feasible pricing policies $p_i^t(\cdot)$, $i = 1, \ldots, n$. Given a feasible pricing policy $p_i^t(\cdot) \in \mathcal{F}_i$, let $\mathcal{V}_t(\boldsymbol{Y}_1^t, \ldots, \boldsymbol{Y}_n^t)$ be the total expected revenue from period t onwards given the state $\boldsymbol{Y}^t = (\boldsymbol{Y}_1^t, \ldots, \boldsymbol{Y}_n^t)$. We can define this revenue function recursively as follows:

$$\begin{aligned} \mathcal{V}_t(\boldsymbol{Y}_1^t, \ldots, \boldsymbol{Y}_n^t) = \; & \sum_{i=1}^{n} a_i(p_i^t(\boldsymbol{Y}^t)) \cdot \bar{r}_i^t \cdot p_i^t(\boldsymbol{Y}^t) \\ & + \sum_{i=1}^{n} a_i(p_i^t(\boldsymbol{Y}^t)) \cdot \bar{r}_i^t \cdot \mathbb{E}\left[\mathcal{V}_{t+1}\left((0, \overline{\boldsymbol{Y}}_1^{t+1}), \ldots, (1, \overline{\boldsymbol{Y}}_i^{t+1}), \ldots, (0, \overline{\boldsymbol{Y}}_n^{t+1})\right)\right] \\ & + \left(1 - \sum_{i=1}^{n} a_i(p_i^t(\boldsymbol{Y}^t)) \cdot \bar{r}_i^t\right) \cdot \mathbb{E}\left[\mathcal{V}_{t+1}\left((0, \overline{\boldsymbol{Y}}_1^{t+1}), \ldots, (0, \overline{\boldsymbol{Y}}_n^{t+1})\right)\right], \end{aligned} \tag{7}$$

where $\bar{r}_i^t = \mathbb{E}[r_i^t]$ (as previously defined) and $\overline{\boldsymbol{Y}}_i^t = (y_i^t(1), \ldots, y_n^t(\tau_i^{\max}-1)) \in \{0,1\}^{\tau_i^{\max}-1}$, i.e., $\boldsymbol{Y}_i^t = (y_i^t(0), \overline{\boldsymbol{Y}}_i^t)$, with the boundary condition $\mathcal{V}_{T+1}(\cdot) = 0$. The first term is the expected revenue obtained in period t given the price offers $p_i^t(\boldsymbol{Y}^t)$. The second term is the expected revenue from period $t+1$ onwards if the customer accepts the price offer in period t. The remaining term is the expected revenue from period $t+1$ onwards if there is no additional service added in period t. The state $(\overline{\boldsymbol{Y}}_1^{t+1}, \ldots, \overline{\boldsymbol{Y}}_n^{t+1})$ in period $t+1$ depends on $(\boldsymbol{Y}_1^t, \ldots, \boldsymbol{Y}_n^t)$ and the transition probabilities can be computed as follows given the independence of random service times:

$$\mathbb{P}\left(\overline{\boldsymbol{Y}}_i^{t+1} \mid (\underline{\boldsymbol{Y}}_i^t, y_i^t(\tau_i^{\max}-1))\right) = \prod_{\tau \in \mathcal{T}_i(\boldsymbol{Y}_i^t)} [b_i(\tau)]^{y_i^{t+1}(\tau+1)} \cdot [1-b_i(\tau)]^{1-y_i^{t+1}(\tau+1)}, \tag{8}$$

for all $\overline{\boldsymbol{Y}}_i^{t+1} \leq \underline{\boldsymbol{Y}}_i^t$, where $\underline{\boldsymbol{Y}}_i^t = (y_i^t(0), \ldots, y_i^t(\tau_i^{\max}-2)) \in \{0,1\}^{\tau_i^{\max}-1}$ and $\mathcal{T}(\boldsymbol{Y}_i^t) = \{\tau < \tau_i^{\max}-1 \,:\, y_i^t(\tau) = 1\}$. For $\overline{\boldsymbol{Y}}_i^{t+1} \in \{0,1\}^{\tau_i^{\max}-1}$ and $\overline{\boldsymbol{Y}}_i^{t+1} \not\leq \underline{\boldsymbol{Y}}_i^t$, $\mathbb{P}\left(\overline{\boldsymbol{Y}}_i^{t+1} \mid (\underline{\boldsymbol{Y}}_i^t, y_i^t(\tau_i^{\max}-1))\right)$ is set to be 0. The two expected values in (7) can then be computed accordingly using those transition probabilities.

The total expected revenue is $\mathcal{V}_1(\mathbf{0}, \ldots, \mathbf{0})$, which is uncertain given that $a_i(\cdot) \in \mathcal{U}_i^a$ and $\beta_i(\cdot) \in \mathcal{U}_i^b$. We consider the maximin formulation of the problem to maximize the worst-case expected revenue

$$\max_{p_i^t(\cdot)\in\mathcal{F}_i: i=1,\ldots,n} \left\{ \min_{a_i(\cdot)\in\mathcal{U}_i^a, \beta_i(\cdot)\in\mathcal{U}_i^b: i=1,\ldots,n} \mathcal{V}_1(\mathbf{0}, \ldots, \mathbf{0}) \right\}. \tag{9}$$

The difficulty of this problem comes from the recursive definition of the value function $\mathcal{V}_t(\cdot)$ and it also depends on the structure of the uncertainty sets $\mathcal{U}_i^a$ and $\mathcal{U}_i^b$. Under the case when there is no ambiguity in demand and service time distributions, i.e., $\mathcal{U}_i^a = \{\hat{a}_i(\cdot)\}$ and $\mathcal{U}_i^b = \{\hat{\beta}_i(\cdot)\}$ for all i, the problem can be rewritten as a dynamic programming formulation using the interchangeability property of expectation and optimization operators (see, e.g., Wets 2002). The resulting Bellman equation of the problem reads:

$$\begin{aligned} &\mathcal{V}_t(\boldsymbol{Y}_1^t, \ldots, \boldsymbol{Y}_n^t) = \\ &\max_{p_i \in \mathcal{A}_i(\boldsymbol{Y}_i^t): i=1,\ldots,n} \left\{ \sum_{i=1}^n \hat{a}_i(p_i) \cdot \bar{r}_i^t \left(p_i + \mathbb{E}\left[\mathcal{V}_{t+1}\left((0, \overline{\boldsymbol{Y}}_1^{t+1}), \ldots, (1, \overline{\boldsymbol{Y}}_i^{t+1}), \ldots, (0, \overline{\boldsymbol{Y}}_n^{t+1}) \right) \right] \right) \right. \\ &\qquad \left. + \left(1 - \sum_{i=1}^n \hat{a}_i(p_i) \cdot \bar{r}_i^t \right) \mathbb{E}\left[\mathcal{V}_{t+1}\left((0, \overline{\boldsymbol{Y}}_1^{t+1}), \ldots, (0, \overline{\boldsymbol{Y}}_n^{t+1}) \right) \right] \right\}, \end{aligned} \tag{10}$$

with the boundary condition $\mathcal{V}_{T+1}(\cdot) = 0$. The transition probabilities are computed as in (8) with $\beta_i(\cdot) \equiv \hat{\beta}_i(\cdot)$ for all i. The maximum number of feasible values of $\overline{\boldsymbol{Y}}_i^{t+1}$ is $2^{|\mathcal{T}(\boldsymbol{Y}_i^t)|}$, which makes this dynamic programming formulation for fixed demand and service time distributions still difficult to solve for large instances. Instead of focusing on optimal policies for (9) given general uncertainty sets, in the next section, we shall consider a deterministic approximation of the problem, in which random parameters are replaced by their corresponding expected values. This deterministic approximation will be used to generate heuristic-based fixed-price policies for the problem.

3 Deterministic Approximation

The deterministic formulation is constructed by removing the stochastic variability of *actual* demand and service processes following the similar approach used in revenue management literature such as Gallego and van Ryzin (1994), Gallego and van Ryzin (1997), and Maglaras and Meissner (2006). In each time period t, the random realized demand $d_i(p_i^t, t)$ of product i is replaced by its mean value:

$$\begin{aligned}\mathbb{E}\left[d_i(p_i^t, t)\right] &= \mathbb{E}\left[r_i^t \cdot \mathbb{I}\left\{p_i^t \leq \pi_i(t)\right\}\right] \\ &= \mathbb{E}\left[r_i^t\right] \cdot \mathbb{P}\left(p_i^t \leq \pi_i(t)\right) \\ &= \bar{r}_i^t \cdot a_i(p_i^t).\end{aligned}$$

Similarly, the random number of requests of product i remaining in the system in each period t is replaced by its mean

$$\begin{aligned}\mathbb{E}\left[D_i(t)\right] &= \mathbb{E}\left[\sum_{s=1}^{t} d_i(p_i^s, s) \cdot \mathbb{I}\left\{\tau_i(s) > t-s\right\}\right] \\ &= \sum_{s=1}^{t} \mathbb{E}\left[d_i(p_i^s, s)\right] \cdot \mathbb{P}\left(\tau_i(s) > t-s\right) \\ &= \sum_{s=1}^{t} \bar{r}_i^s \cdot a_i(p_i^s) \cdot \beta_i(t-s).\end{aligned}$$

The total revenue with the mean realized demand can be formulated as $\sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot a_i(p_i^t) \cdot p_i^t$ and the deterministic capacity constraints are given by:

$$\sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot a_i(p_i^s) \cdot \beta_i(t-s)\right) \leq C, \quad t = 1, \ldots, T.$$

As $a_i(\cdot) \in \mathcal{U}_i^a$, $i = 1, \ldots, n$, the total revenue is uncertain. Similarly, the capacity constraints depend on $a_i(\cdot) \in \mathcal{U}_i^a$ and $\beta_i(\cdot) \in \mathcal{U}_i^b$ for $i = 1, \ldots, n$. Applying the robust optimization framework, we obtain the following deterministic relaxation of (9) to maximize the worst-case total revenue while satisfying the capacity constraints for any realization of $a_i(\cdot)$ and $\beta_i(\cdot)$, $i = 1, \ldots, n$:

$$\begin{aligned}\max_{\boldsymbol{P}} \quad & \min_{a_i(\cdot)\in\mathcal{U}_i^a : i=1,\ldots,n} \sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot a_i(p_i^t) \cdot p_i^t \\ \text{s.t.} \quad & \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot a_i(p_i^s) \cdot \beta_i(t-s)\right) \leq C, \quad \forall\, a_i(\cdot) \in \mathcal{U}_i^a,\ \beta_i(\cdot) \in \mathcal{U}_i^b,\ i = 1, \ldots, n, \\ & \qquad \forall\, t = 1, \ldots, T, \\ & p_i^t \in \mathcal{P}_i, \quad \forall\, i = 1, \ldots, n,\ t = 1, \ldots, T,\end{aligned} \tag{11}$$

where $\boldsymbol{P}$ denotes the collection of p_i^t, $i = 1, \ldots, n$ and $t = 1, \ldots, T$. Note that, for this deterministic approximation when mean values are considered for demand and service time, we only consider $p_i^t \in \mathcal{P}_i$ with the implicit assumption that the capacity is large enough to handle mean demand and service time. Next, we would like to reformulate this robust formulation and in order to do so, we make the following assumption regarding the uncertainty set $\mathcal{U}_i^a$, $i = 1, \ldots, n$:

Assumption 1. *Both functions* $\underline{a}_i(\cdot)$ *and* $\bar{a}_i(\cdot)$ *defined with* $\underline{a}_i(p) = \min_{a_i(\cdot)\in\mathcal{U}_i^a} a_i(p)$ *and* $\bar{a}_i(p) = \max_{a_i(\cdot)\in\mathcal{U}_i^a} a_i(p)$ *respectively, for all* $p \in \mathcal{P}_i$*, belong to the uncertainty set* $\mathcal{U}_i^a$ *for all* $i = 1, \ldots, n$*.*

The assumption that $\underline{a}_i(\cdot), \bar{a}_i(\cdot) \in \mathcal{U}_i^a$ as described in Assumption 1 is reasonable in general, especially when the uncertainty set $\mathcal{U}_i^a$ is constructed from experiments historical data as discussed later in Section 4. Now, given a fixed pricing solution $\boldsymbol{p}$, the inner minimization objective function of (11) can be restructured as follows:

$$\begin{aligned}
\min_{a_i(\cdot)\in\mathcal{U}_i^a: i=1,\ldots,n} \sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot a_i(p_i^t)\cdot p_i^t \quad &= \quad \sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot \min_{a_i(\cdot)\in\mathcal{U}_i^a} a_i(p_i^t)\cdot p_i^t && (12)\\
&= \quad \sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot \underline{a}_i(p_i^t)\cdot p_i^t, && (13)
\end{aligned}$$

since $\underline{a}_i(\cdot) \in \mathcal{U}_i^a$ for all $i = 1, \ldots, n$. Similarly, the left-hand side of the capacity constraint in (11) has its maximum value

$$\begin{aligned}
&\max_{a_i(\cdot)\in\mathcal{U}_i^a: i=1,\ldots,n} \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot a_i(p_i^s)\cdot \beta_i(t-s)\right)\\
&= \sum_{i=1}^{n} c_i \max_{a_i(\cdot)\in\mathcal{U}_i^a} \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot a_i(p_i^s)\cdot \beta_i(t-s)\right)\\
&= \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot \max_{a_i(\cdot)\in\mathcal{U}_i^a} a_i(p_i^s)\cdot \beta_i(t-s)\right)\\
&= \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot \bar{a}_i(p_i^s)\cdot \beta_i(t-s)\right),
\end{aligned}$$

since $\bar{a}_i(\cdot) \in \mathcal{U}_i^a$ for all $i = 1, \ldots, n$. The deterministic relaxation of (9) can then be reformulated as:

$$\begin{aligned}
\max_{\boldsymbol{P}} \quad & \sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot \underline{a}_i(p_i^t)\cdot p_i^t\\
\text{s.t.} \quad & \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot \bar{a}_i(p_i^s)\cdot \beta_i(t-s)\right) \le C, \quad \forall\, \beta_i(\cdot) \in \mathcal{U}_i^b,\ i = 1,\ldots,n, \qquad (14)\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \forall\, t = 1,\ldots,T,\\
& p_i^t \in \mathcal{P}_i, \quad \forall\, i = 1,\ldots,n,\ t = 1,\ldots,T.
\end{aligned}$$

The deterministic relaxation (14) does not take into account the stochasticity of demand and service time, and its feasible solutions do not represent any feasible pricing policy, which needs to satisfy the feasibility constraints (4) almost surely given actual realizations of demand and service time. However, they can be used to construct fixed-price pricing policies, i.e., policies with state-independent prices if there are enough capacities, following the same approach as discussed in Gallego and van Ryzin (1994). Since both original problem and its deterministic relaxation have the same objective of maximizing the expected revenue, an optimal solution of (14) will be chosen to construct fixed-price pricing policies. Given optimal prices $\boldsymbol{P}^*$ of (14), a fixed-price policy H can be defined as follows:

$$p_i^{t,H}(\boldsymbol{Y}^t) = \begin{cases} p_i^{t,*}, & \boldsymbol{Y}^t \in \mathcal{Y}_H, \\ \bar{p}_i, & \text{otherwise}, \end{cases} \tag{15}$$

for all $\boldsymbol{Y}^t = (\boldsymbol{Y}_1^t, \ldots, \boldsymbol{Y}_n^t)$ with $\boldsymbol{Y}_i^t \in \{0,1\}^{\tau_i^{\max}}$, $i = 1, \ldots, n$, and $t = 1, \ldots, T$, where $\mathcal{Y}_H \subseteq \mathcal{Y}_F = \left\{ \boldsymbol{Y}^t \,|\, C - \sum_{k=1}^n c_k \left(\sum_{\tau=0}^{\tau_k^{\max}-2} y_k^t(\tau) \right) \geq c_i \right\}$ is used to indicate when one should not reject requests under the defined policy. Note that under this setting, $\mathcal{Y}_H$ can be a proper subset of $\mathcal{Y}_F$, i.e., $\mathcal{Y}_H \subsetneq \mathcal{Y}_F$, which allows us to define relevant fixed-price policies used in later proofs. In general, solving (14) is difficult and we are going to discuss its tractability in Section 4 but before that, we shall first consider the relationship between the deterministic approximation and the original stochastic problem.

3.1 Deterministic Revenue as an Upper Bound

Similar to Gallego and van Ryzin (1997), we shall first attempt to show that the uncertainty stemmed from customers' decisions and service time reduces the expected revenue that one could achieve. More concretely, let $Z^*(\boldsymbol{a}, \boldsymbol{\beta})$ be the expected revenue obtained from an optimal pricing policy of the original stochastic problem, given probability functions $\boldsymbol{a} = \{a_i(\cdot)\}_{i=1,\ldots,n}$ with $a_i(\cdot) \in \mathcal{U}_i^a$ and $\boldsymbol{\beta} = \{\beta_i(\cdot)\}_{i=1,\ldots,n}$ with $\beta_i(\cdot) \in \mathcal{U}_i^b$ for all $i = 1, \ldots, n$. In addition, let Z^D be the optimal revenue obtained from the deterministic approximation problem (14). To establish the relationship between the deterministic revenue Z^D and the expected revenues $Z^*(\boldsymbol{a}, \boldsymbol{\beta})$, similar to Gallego and van Ryzin (1997), we need to consider some *regularity* conditions of (actual) demand functions to guarantee the convexity of the deterministic approximation problem with respect to its decision variables. Under the general setting with uncertain acceptance probability functions, one regularity condition is that two upper and lower acceptance probability functions $\bar{a}_i(\cdot)$ and $\underline{a}_i(\cdot)$ are affine for all $i = 1, \ldots, n$. This condition indicates that uncertain acceptance probability functions are confined in linear confidence bands, which can be considered as reasonable. When acceptance probability functions are fixed, this condition can be further relaxed. The following proposition shows the relationship between Z^D and expected revenues obtained from optimal policies given linear confidence bands of acceptance probability functions.

Proposition 1. *Assuming that confidence bands of acceptance probability functions are linear, i.e., $\bar{a}_i(\cdot), \underline{a}_i(\cdot) \in \mathcal{U}_i^a$ are non-increasing affine functions, and $\bar{\mathcal{P}}_i = [p_i^{\min}, p_i^{\max}]$ for all $i = 1, \ldots, n$, then*

$$\min_{a_i(\cdot) \in \mathcal{U}_i^a, \beta_i(\cdot) \in \mathcal{U}_i^b : i=1,\ldots,n} Z^*(\boldsymbol{a}, \boldsymbol{\beta}) \leq Z^D.$$

Proof. Let $\bar{a}_i(p) = \bar{\alpha}_i \cdot p + \bar{\gamma}_i$ and $\underline{a}_i(p) = \underline{\alpha}_i \cdot p + \underline{\gamma}_i$ with $\bar{\alpha} \leq 0$ and $\underline{\alpha} \leq 0$. Consider an optimal pricing policy that results in $p_i^t(\boldsymbol{Y}^t)$ for all possible states $\boldsymbol{Y}^t = (\boldsymbol{Y}_1^t, \ldots, \boldsymbol{Y}_n^t)$ with $\boldsymbol{Y}_i^t \in \{0,1\}^{\tau_i^{\max}}$, $i = 1, \ldots, n$, and $t = 1, \ldots, T$. We have

$$\mathbb{E}\left[R_i(t)\right] = \mathbb{E}\left[d_i(p_i^t(\boldsymbol{Y}^t), t) \cdot p_i^t(\boldsymbol{Y}^t)\right] = \bar{r}_i^t \cdot \mathbb{E}\left[a_i(p_i^t(\boldsymbol{Y}^t)) \cdot p_i^t(\boldsymbol{Y}^t)\right].$$

The second equality is obtained by taking expectation with respect to random request arrivals and customer reservation price in period t, which are independent of $\boldsymbol{Y}^t$. We then have

$$\min_{a_i(\cdot)\in\mathcal{U}_i^a,\beta_i(\cdot)\in\mathcal{U}_i^b:i=1,\ldots,n} Z^*(\boldsymbol{a},\boldsymbol{\beta}) \le Z^*(\underline{\boldsymbol{a}},\boldsymbol{\beta}) = \sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot \mathbb{E}\left[\underline{a}_i(p_i^t(\boldsymbol{Y}^t))\cdot p_i^t(\boldsymbol{Y}^t)\right].$$

The function $\underline{a}_i(p)\cdot p = \underline{\alpha}_i \cdot p^2 + \underline{\gamma}_i \cdot p$ is a concave (quadratic) function in p given that $\underline{\alpha}_i \le 0$ for all $i = 1,\ldots,n$. Applying Jensen's inequality, we have:

$$\min_{a_i(\cdot)\in\mathcal{U}_i^a,\beta_i(\cdot)\in\mathcal{U}_i^b:i=1,\ldots,n} Z^*(\boldsymbol{a},\boldsymbol{\beta}) \le \sum_{t=1}^{T}\sum_{i=1}^{n} \bar{r}_i^t \cdot \underline{a}_i(\mathbb{E}\left[p_i^t(\boldsymbol{Y}^t)\right])\cdot \mathbb{E}\left[p_i^t(\boldsymbol{Y}^t)\right].$$

Now we have:

$$\mathbb{E}\left[D_i(t)\right] = \sum_{s=1}^{t} \bar{r}_i^s \cdot \mathbb{E}\left[a_i(p_i^s(\boldsymbol{Y}^s))\right]\cdot \beta_i(t-s).$$

The constraint (4) is satisfied almost surely. Taking the expectation, we have: $\mathbb{E}\left[\sum_{i=1}^{n} c_i D_i(t)\right] \le C$ for all $a_i(\cdot)\in\mathcal{U}_i^a$, $\beta_i(\cdot)\in\mathcal{U}_i^b$, and $t = 1,\ldots,T$. Let $\bar{a}_i(\cdot)\in\mathcal{U}_i^a$ for all $i = 1,\ldots,n$, we then have:

$$\sum_{i=1}^{n} c_i \sum_{s=1}^{t} \bar{r}_i^s \cdot \mathbb{E}\left[\bar{a}_i(p_i^s(\boldsymbol{Y}^s))\right]\cdot \beta_i(t-s) \le C, \quad \forall\, t = 1,\ldots,T.$$

The function $\bar{a}_i(\cdot)$ is affine, and thus we can rewrite the above inequalities as follows:

$$\sum_{i=1}^{n} c_i \sum_{s=1}^{t} \bar{r}_i^s \cdot \bar{a}_i(\mathbb{E}\left[p_i^s(\boldsymbol{Y}^s)\right])\cdot \beta_i(t-s) \le C, \quad \forall\, t = 1,\ldots,T.$$

Finally, given that $\mathcal{P}_i = [p_i^{\min}, p_i^{\max}]$, we have $\mathbb{E}\left[p_i^t(\boldsymbol{Y}^t)\right] \in \mathcal{P}_i$ for all $i = 1,\ldots,n$ and $t = 1,\ldots,T$. It shows that $p_i^t = \mathbb{E}\left[p_i^t(\boldsymbol{Y}^t)\right]$, $i = 1,\ldots,n$, $t = 1,\ldots,T$ is a feasible solution of (14), which implies that

$$\min_{a_i(\cdot)\in\mathcal{U}_i^a,\beta_i(\cdot)\in\mathcal{U}_i^b:i=1,\ldots,n} Z^*(\boldsymbol{a},\boldsymbol{\beta}) \le Z^D.$$

□

Proposition 1 shows that the optimal worst-case expected revenue is bounded by Z^D. Affine acceptance probability functions imply that random reservation prices are uniformly distributed. The proof techniques, which are similar to those applied in Gallego and van Ryzin (1997), rely on the concavity of the revenue function as well as the linearity of the capacity constraints, which in turn depend on $\bar{a}_i(\cdot)$ and $\underline{a}_i(\cdot)$ for $i = 1,\ldots,n$. When the acceptance probability function for each product is fixed, i.e., $\mathcal{U}_i^a = \{\hat{a}_i(\cdot)\}$, we have: $\bar{a}_i(\cdot) = \underline{a}_i(\cdot) = \hat{a}_i(\cdot)$. This allows us to relax the conditions in Proposition 1. The following corollary provides bounding results for fixed acceptance probability functions using relaxed conditions, which turn out to be the same as the main regularity conditions set in Gallego and van Ryzin (1997) and Maglaras and Meissner (2006). Note that we still consider the distributional ambiguity of random service time in this corollary.

Corollary 1. *Consider the case when acceptance probability functions are fixed, i.e.,* $\mathcal{U}_i^a = \{\hat{a}_i(\cdot)\}$ *for all* $i = 1, \ldots, n$*. Assuming that* $\hat{a}_i(\cdot)$ *has an inverse function* $\hat{p}_i(\cdot)$ *with the domain* $\mathcal{A}_i = [a_i^{\min}, a_i^{\max}]$ *and the revenue function* $R_i(a) = a \cdot \hat{p}_i(a)$ *is concave in* a *for all* $i = 1, \ldots, n$*, then*

$$\min_{\beta_i(\cdot) \in \mathcal{U}_i^b : i=1,\ldots,n} Z^*(\hat{\boldsymbol{a}}, \boldsymbol{\beta}) \leq Z^D.$$

Proof. Changing the decision variables from p_i^t to a_i^t using the inverse function $\hat{p}_i(\cdot)$ for $i = 1, \ldots, n$ and $t = 1, \ldots, T$, we obtain the following equivalent formulation of the deterministic relaxation:

$$\begin{aligned} \max_{\boldsymbol{A}} \quad & \sum_{t=1}^{T} \sum_{i=1}^{n} \bar{r}_i^t \cdot \hat{p}_i(a_i^t) \cdot a_i^t \\ \text{s.t.} \quad & \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot a_i^s \cdot \beta_i(t-s) \right) \leq C, \qquad \forall\, \beta_i(\cdot) \in \mathcal{U}_i^b,\, i = 1, \ldots, n, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, t = 1, \ldots, T, \\ & a_i^t \in \mathcal{A}_i, \quad \forall\, i = 1, \ldots, n,\, t = 1, \ldots, T, \end{aligned} \tag{16}$$

where $\boldsymbol{A}$ denotes the collection of a_i^t, $i = 1, \ldots, n$, and $t = 1, \ldots, T$. Similar to the arguments used in the proof of Proposition 1 we can claim that $a_i^t = \mathbb{E}\left[a_i(p_i^t(\boldsymbol{Y}^t))\right] \in \mathcal{A}_i$ for $i = 1, \ldots, n$, $t = 1, \ldots, T$ is a feasible solution of (16). The second inequality is then simply due to the concavity of the revenue function $R_i(a) = a \cdot \hat{p}_i(a)$. □

The upper bound established in this section demonstrates the connection between the deterministic approximation and the original stochastic problem in terms of their optimal objective values. In the next section, we will use it to analyze the asymptotic performance of the proposed heuristic fixed-price policies.

3.2 Asymptotic Analysis of Heuristic Fixed-Price Policies

To analyze the asymptotic performance of fixed-price policies, we consider a sequence of problems, indexed by $\theta \in \mathbb{N}$, in which the resource capacity is $\theta \cdot C$ and there are $r_i^t(\theta)$ requests for product $i = 1, \ldots, n$. Similar to the original unscaled problem with $\theta = 1$, we assume that there are at most θ requests in total in each period t, i.e., $\sum_{i=1}^{n} r_i^t(\theta) \leq \theta$, and $\mathbb{E}\left[r_i^t(\theta)\right] = \theta \cdot \bar{r}_i^t$. Furthermore, we assume that we can divide each time period into θ sub-periods with $r_{i,s}^t$ being the number of requests of product i in sub-period s within time period t such that $\sum_{i=1}^{n} r_{i,s}^t \leq 1$, i.e., at most one request per sub-period, and $\sum_{s=1}^{\theta} r_{i,s}^t = r_i^t(\theta)$ with $\mathbb{E}\left[r_{i,s}^t\right] = \bar{r}_i^t$. We also assume that these requests are independent. Under these settings, requests in each sub-period are considered exactly as requests in each period in the original unscaled problem. These assumptions indicate that the demand process in the original problem is duplicated to generate demand $r_i^t(\theta)$ in the scaled problems.

Given the sequence of these problems, we will analyze their feasible pricing policies asymptotically when $\theta \to +\infty$. These pricing policies are used to determine price p_i^t for each product i in each period t as a function of past demand and actions up to time t. To characterize the feasibility of these pricing policies, let $p_{i,s}^t$ be the actual price in each sub-period s within time period t. We impose the condition $p_{i,s}^t \in \{p_i^t, \bar{p}_i\}$ for all s, which makes sure that the prices in sub-periods cannot be changed ($p_{i,s}^t = p_i^t$) unless one has to reject new requests due to insufficient resources ($p_{i,s}^t = \bar{p}_i$). In addition to the state variables $\boldsymbol{Y}_i^t = (y_i^t(0), \ldots, y_i^t(\tau_i^{\max} - 1))$, we need the additional state variable $z_{i,s}^t$ to account for the amount of actual demand for product i in the first $(s-1)$ sub-periods within the time period t, $z_{i,s}^t = \sum_{\sigma=1}^{s-1} d_i(p_{i,\sigma}^t, t, \sigma)$ for $s \geq 2$, where $d_i(p, t, s) = r_{i,s}^t \cdot \mathbb{I}\{p \leq \pi_i(t,s)\}$ for $s = 1, \ldots, \theta$. Here $\pi_i(t,s)$ is the reservation price for product i of the actual customer in sub-period s of period t if one arrives in that sub-period. Clearly, $z_{i,1}^t = 0$, and the feasible set $\mathcal{A}_i^s(\boldsymbol{Y}_i^t, z_{i,s}^t)$ of prices $p_{i,s}^t$ in each sub-period s within time period t can then be defined as follows:

$$\mathcal{A}_i^1(\boldsymbol{Y}_i^t, z_{i,1}^t) = \begin{cases} \mathcal{P}_i, & \theta \cdot C - \sum_{k=1}^{n} c_k \left(\sum_{\tau=0}^{\tau_k^{\max}-2} y_i^t(\tau) \right) \geq c_i, \\ \{\bar{p}_i\}, & \text{otherwise}, \end{cases} \tag{17}$$

and

$$\mathcal{A}_i^s(\boldsymbol{Y}_i^t, z_{i,s}^t) = \begin{cases} \left\{p_{i,1}^t\right\}, & \theta \cdot C - \sum_{k=1}^{n} c_k \left(z_{k,s}^t + \sum_{\tau=0}^{\tau_k^{\max}-2} y_k^t(\tau) \right) \geq c_i, \\ \{\bar{p}_i\}, & \text{otherwise}, \end{cases} \quad \text{for } s = 2, \ldots, \theta. \tag{18}$$

These feasible sets of prices indicate that a single price $p_i^t = p_{i,1}^t$ will be set for product i in period t until new requests cannot be accepted due to insufficient resources. Finally, the total revenue in the period t can be computed as $R(t) = \sum_{i=1}^{n} R_i(t) = \sum_{i=1}^{n} \sum_{s=1}^{\theta} d_i(p_{i,s}^t, t, s) \cdot p_{i,s}^t$ with $p_{i,s}^t \in \mathcal{A}_i^s(\boldsymbol{Y}_i^t, z_{i,s}^t)$ as defined in (17) and (18).

We now consider the deterministic relaxation of these problems. Similar to the original setting, given a price p_i^t for product i in period t, the expected realized demand is $\mathbb{E}\left[r_i^t(\theta)\right] \cdot a_i(p_i^t) = \theta \cdot \bar{r}_i^t \cdot a_i(p_i^t)$. Furthermore, the expected number of requests for product i remaining in the system in period t is $\theta \cdot \sum_{s=1}^{t} \bar{r}_i^s \cdot a_i(p_i^s) \cdot \beta_i(t-s)$. Given the uncertain functions $a_i(\cdot) \in \mathcal{U}_i^a$ and $\beta_i(\cdot) \in \mathcal{U}_i^b$ for $i = 1, \ldots, n$, the deterministic relaxation is given by

$$\begin{aligned} \max_{\boldsymbol{P}} \quad & \min_{a_i(\cdot) \in \mathcal{U}_i^a : i=1,\ldots,n} \sum_{t=1}^{T} \sum_{i=1}^{n} \theta \cdot \bar{r}_i^t \cdot a_i(p_i^t) \cdot p_i^t \\ \text{s.t.} \quad & \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \theta \cdot \bar{r}_i^s \cdot a_i(p_i^s) \cdot \beta_i(t-s) \right) \leq \theta \cdot C, \quad \forall\, a_i(\cdot) \in \mathcal{U}_i^a,\, \beta_i(\cdot) \in \mathcal{U}_i^b,\, i = 1, \ldots, n, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, t = 1, \ldots, T, \\ & p_i^t \in \mathcal{P}_i, \quad \forall\, i = 1, \ldots, n,\; t = 1, \ldots, T, \end{aligned} \tag{19}$$

which is equivalent to

$$
\begin{aligned}
\max_{\boldsymbol{P}} \quad & \theta \cdot \sum_{t=1}^{T} \sum_{i=1}^{n} \bar{r}_i^t \cdot \underline{a}_i(p_i^t) \cdot p_i^t \\
\text{s.t.} \quad & \sum_{i=1}^{n} c_i \left(\sum_{s=1}^{t} \bar{r}_i^s \cdot \bar{a}_i(p_i^s) \cdot \beta_i(t-s) \right) \le C, \quad \forall \beta_i(\cdot) \in \mathcal{U}_i^b, \, i = 1, \dots, n, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t = 1, \dots, T, \\
& p_i^t \in \mathcal{P}_i, \quad \forall i = 1, \dots, n, \, t = 1, \dots, T.
\end{aligned}
\tag{20}
$$

where $\underline{a}_i(\cdot)$ and $\bar{a}_i(\cdot)$ are defined as mentioned previously.

Let $Z_\theta^*(\boldsymbol{a}, \boldsymbol{\beta})$ be the expected revenue obtained from the optimal policy of the θ-scaled problem with the resource capacity $\theta \cdot C$ and the demand $r_i^t(\theta)$, given probability functions $a_i(\cdot) \in \mathcal{U}_i^a$ and $\beta_i(\cdot) \in \mathcal{U}_i^b$. To establish asymptotic results, let us consider a modified pricing problem in which all requests for product i stay in the system for the maximum duration $\tau_i^{\max}$ while the uncertainty sets $\mathcal{U}_i^a$, $i = 1. \dots, n$, remain the same. This modified problem has fixed service time for each product, i.e., $\mathcal{U}_i^b = \{\hat{\beta}(\cdot)\}$ with $\hat{\beta}_i(\cdot) \equiv \beta_i^{\max}(\cdot)$, where $\beta_i^{\max}(t) = 1$ for all $t = 0, \dots, \tau_i^{\max} - 1$, $i = 1, \dots, n$. Let Z^{DD} be the optimal value of the deterministic relaxation of the original unscaled modified problem. Furthermore, let $Z_\theta^m(\boldsymbol{a})$ be the expected revenue obtained from the optimal policy of the modified θ-scaled problem given probability functions $a_i(\cdot) \in \mathcal{U}_i^a$. We are now ready to state the asymptotic results under the same regularity conditions as discussed previously.

Theorem 1. *Assuming that confidence bands of acceptance probability functions are linear, i.e., $\bar{a}_i(\cdot), \underline{a}_i(\cdot) \in \mathcal{U}_i^a$ are non-increasing affine functions, and $\bar{\mathcal{P}}_i = [p_i^{\min}, p_i^{\max}]$ for all $i = 1, \dots, n$, then*

$$
Z^{DD} = \lim_{\theta \to \infty} \frac{1}{\theta} \cdot \min_{a_i(\cdot) \in \mathcal{U}_i^a} Z_\theta^m(\boldsymbol{a}) \le \lim_{\theta \to \infty} \frac{1}{\theta} \cdot \min_{a_i(\cdot) \in \mathcal{U}_i^a, \beta_i(\cdot) \in \mathcal{U}_i^b} Z_\theta^*(\boldsymbol{a}, \boldsymbol{\beta}) \le Z^D.
$$

Proof. Let Z_θ^D and Z_θ^{DD} be the optimal value of the deterministic approximation of the θ-scaled problem and the modified one with fixed service time, respectively. It is clear from (14) and (20) that $Z_\theta^D = \theta \cdot Z^D$ and similarly, $Z_\theta^{DD} = \theta \cdot Z^{DD}$. Any optimal policy of the modified θ-scaled problem is a feasible policy of the original θ-scaled problem given that requests of product i under the modified setting stay in the system exactly $\tau_i^{\max}$ periods, which is the maximum duration, for all $i = 1, \dots, n$. We modify this policy for the original θ-scaled problem by computing the remaining resource capacities under the assumption that all requests of product i stay $\tau_i^{\max}$ periods regardless how service time is actually realized. Clearly, the modified policy is still feasible for the original θ-scaled problem and the expected revenue $Z_\theta(\boldsymbol{a}, \boldsymbol{\beta})$ obtained from this policy given probability functions $a_i(\cdot)$ and $\beta_i(\cdot)$ is the same as $Z_\theta^m(\boldsymbol{a})$. Thus we have:

$$
\min_{a_i(\cdot) \in \mathcal{U}_i^a : i=1,\dots,n} Z_\theta^m(\boldsymbol{a}) = \min_{a_i(\cdot) \in \mathcal{U}_i^a, \beta_i(\cdot) \in \mathcal{U}_i^b : i=1,\dots,n} Z_\theta(\boldsymbol{a}, \boldsymbol{\beta}) \le \min_{a_i(\cdot) \in \mathcal{U}_i^a, \beta_i(\cdot) \in \mathcal{U}_i^b : i=1,\dots,n} Z_\theta^*(\boldsymbol{a}, \boldsymbol{\beta}).
$$

The second inequality that we need to prove then follows.

Next, we will prove that $\min_{a_i(\cdot)\in\mathcal{U}_i^a,\beta_i(\cdot)\in\mathcal{U}_i^b:i=1,\ldots,n} Z_\theta^*(\boldsymbol{a},\boldsymbol{\beta}) \le Z_\theta^D$. Similar to the argument presented in the proof of Proposition 1, we are able to show that

$$p_i^t = \frac{1}{\theta}\sum_{s=1}^{\theta}\mathbb{E}\left[p_{i,s}^t(\boldsymbol{Y}^t,\boldsymbol{z}_s^t)\right],$$

for $i=1,\ldots,n$, $t=1,\ldots,T$, is a feasible solution of (20), where $\boldsymbol{Y}^t=(\boldsymbol{Y}_1^t,\ldots,\boldsymbol{Y}_n^t)$ and $\boldsymbol{z}_s^t=(z_{1,s}^t,\ldots,z_{n,s}^t)$ are all possible states, due to the fact that $\bar{a}_i(\cdot)$ is affine. The concavity of the revenue function $\underline{a}_i(p)\cdot p$ then implies $\min_{a_i(\cdot)\in\mathcal{U}_i^a,\beta_i(\cdot)\in\mathcal{U}_i^b:i=1,\ldots,n} Z_\theta^*(\boldsymbol{a},\boldsymbol{\beta}) \le Z_\theta^D$, which in turn implies the third inequality that we need to prove. Using the same argument for the modified θ-scaled problem, we obtain the inequality

$$\min_{a_i(\cdot)\in\mathcal{U}_i^a:i=1,\ldots,n} Z_\theta^m(\boldsymbol{a}) \le Z_\theta^{DD} = \theta\cdot Z^{DD},$$

which implies

$$\lim_{\theta\to\infty}\frac{1}{\theta}\cdot\min_{a_i(\cdot)\in\mathcal{U}_i^a:i=1,\ldots,n} Z_\theta^m(\boldsymbol{a}) \le Z^{DD}.$$

To prove the first equality, we construct a policy for the θ-scaled problem as follows. Let $p_i^{t,*}$ be an optimal solution to the modified unscaled deterministic relaxation problem with $\mathcal{U}_i^b=\{\beta_i^{\max}(\cdot)\}$. Clearly, it is also an optimal solution of the modified θ-scaled deterministic relaxation problem. The proposed policy uses p_i^t as the price of product i in period t and switch to $\bar{p}_i$ if the accepted demand of product i exceeds $\lfloor\theta\cdot\bar{r}_i^t\cdot\bar{a}_i(p_i^{t,*})\rfloor$, i.e.,

$$\mathcal{Y}_H=\left\{\boldsymbol{Y}^t \mid \sum_{\tau=0}^{\tau_i^{\max}-2} y_i(\tau)\le\lfloor\theta\cdot\bar{r}_i^t\cdot\bar{a}_i(p_i^{t,*})\rfloor,\ i=1,\ldots,n\right\}.$$

This is a feasible policy, i.e., $\mathcal{Y}_H\subset\mathcal{Y}_F$, given that the accepted demand in each period t is at most $\theta\cdot\bar{r}_i^t\cdot\bar{a}_i(p_i^{t,*})$ and $p_i^{t,*}$ satisfies the capacity constraint in (20) with $\mathcal{U}_i^b=\{\beta_i^{\max}(\cdot)\}$ for $i=1,\ldots,n$.

The expected revenue under this policy given probability functions $a_i(\cdot)\in\mathcal{U}_i^a$ is computed as follows:

$$Z_\theta(\boldsymbol{a})=\sum_{t=1}^{T}\sum_{i=1}^{n}p_i^{t,*}\cdot\mathbb{E}\left[\min\left\{\sum_{s=1}^{\theta}d_i(p_i^{t,*},t,s),\lfloor\theta\cdot\bar{r}_i^t\cdot\bar{a}_i(p_i^{t,*})\rfloor\right\}\right].$$

The random demand $d_i(p_i^{t,*},t,s)$ for each product i, $i=1,\ldots,n$, is Bernoulli with the probability $\mathbb{P}\left(d_i(p_i^{t,*},t,s)=1\right)=\bar{r}_i^t\cdot a_i(p_i^{t,*})=q_i$ for all $s=1,\ldots,\theta$. These random demand values are independent, implying that $X_i=\sum_{s=1}^{\theta}d_i(p_i^{t,*},t,s)$ is binomial random variable with the mean $\mu_i=\theta q_i$ and variance $\sigma_i^2=\theta q_i(1-q_i)\le\frac{1}{4}\theta$. Let $d_i=\lfloor\theta\cdot\bar{r}_i^t\cdot\bar{a}_i(p_i^{t,*})\rfloor$, we have:

$$\mathbb{E}\left[\min\{X_i,d_i\}\right]=\frac{1}{2}\mathbb{E}\left[X_i+d_i-|X_i-d_i|\right]=\frac{1}{2}\left(\mu_i+d_i-\mathbb{E}\left[|X_i-d_i|\right]\right).$$

Applying Jensen's inequality, we have: $(\mathbb{E}[|X|])^2 \leq \mathbb{E}\left[X^2\right]$ given the convexity of the quadratic function $f(x) = x^2$. Therefore, we have:

$$\begin{aligned}\mathbb{E}[\min\{X_i, d_i\}] &\geq \frac{1}{2}\left(\mu_i + d_i - \sqrt{\sigma_i^2 + (\mu_i - d_i)^2}\right) \\ &\geq \frac{1}{2}\left[\mu_i + d_i - (\sigma_i + |\mu_i - d_i|)\right] = \min\{\mu_i, d_i\} - \frac{1}{2}\sigma_i.\end{aligned}$$

As a result,

$$Z_\theta(\boldsymbol{a}) \geq \sum_{t=1}^{T}\sum_{i=1}^{n} p_i^{t,*} \cdot \left(\min\{\theta \cdot \bar{r}_i^t \cdot a_i^t(p_i^{t,*}), \lfloor \theta \cdot \bar{r}_i^t \cdot \bar{a}_i^t(p_i^{t,*})\rfloor\} - \frac{1}{4}\sqrt{\theta}\right)$$

Furthermore, taking the minimum over $a_i(\cdot) \in \mathcal{U}_i^a$, $i = 1, \ldots, n$, we have:

$$\begin{aligned}\min_{a_i(\cdot)\in\mathcal{U}_i^a : i=1,\ldots,n} Z_\theta^m(\boldsymbol{a}) &\geq \min_{a_i(\cdot)\in\mathcal{U}_i^a : i=1,\ldots,n} Z_\theta(\boldsymbol{a}) \\ &\geq \min_{a_i(\cdot)\in\mathcal{U}_i^a : i=1,\ldots,n} \sum_{t=1}^{T}\sum_{i=1}^{n} p_i^{t,*} \cdot \left(\min\{\theta \cdot \bar{r}_i^t \cdot a_i^t(p_i^{t,*}), \lfloor \theta \cdot \bar{r}_i^t \cdot \bar{a}_i^t(p_i^{t,*})\rfloor\} - \frac{1}{4}\sqrt{\theta}\right) \\ &= \sum_{t=1}^{T}\sum_{i=1}^{n} p_i^{t,*} \cdot \left(\min\{\theta \cdot \bar{r}_i^t \cdot \underline{a}_i^t(p_i^{t,*}), \lfloor \theta \cdot \bar{r}_i^t \cdot \bar{a}_i^t(p_i^{t,*})\rfloor\} - \frac{1}{4}\sqrt{\theta}\right).\end{aligned}$$

Taking the limit $\theta \to \infty$, we have:

$$\begin{aligned}\lim_{\theta\to\infty} \frac{1}{\theta} \cdot \min_{a_i(\cdot)\in\mathcal{U}_i^a : i=1,\ldots,n} Z_\theta^m(\boldsymbol{a}) &\geq \lim_{\theta\to\infty} \sum_{t=1}^{T}\sum_{i=1}^{n} p_i^{t,*} \cdot \left(\min\left\{\bar{r}_i^t \cdot \underline{a}_i^t(p_i^{t,*}), \frac{1}{\theta} \cdot \lfloor \theta \cdot \bar{r}_i^t \cdot \bar{a}_i^t(p_i^{t,*})\rfloor\right\} - \frac{1}{4\sqrt{\theta}}\right) \\ &= \sum_{t=1}^{T}\sum_{i=1}^{n} p_i^{t,*} \cdot \left(\min\left\{\bar{r}_i^t \cdot \underline{a}_i^t(p_i^{t,*}), \bar{r}_i^t \cdot \bar{a}_i^t(p_i^{t,*})\right\}\right) \\ &= \sum_{t=1}^{T}\sum_{i=1}^{n} p_i^{t,*} \cdot \bar{r}_i^t \cdot \underline{a}_i^t(p_i^{t,*}) = Z^{DD}.\end{aligned}$$

Thus, the first equality is now proved, i.e., $\lim_{\theta\to\infty} \frac{1}{\theta} \cdot \min_{a_i(\cdot)\in\mathcal{U}_i^a : i=1,\ldots,n} Z_\theta^m(\boldsymbol{a}) = Z^{DD}$. □

Similar to Corollary 1, the conditions in Theorem 1 can be relaxed when the acceptance probability functions are fixed, i.e., $\mathcal{U}_i^a = \{\hat{a}_i(\cdot)\}$, $\bar{a}_i(\cdot) = \underline{a}_i(\cdot) = \hat{a}_i(\cdot)$. The following corollary provides asymptotic results with those relaxed conditions.

Corollary 2. *Consider the case when acceptance probability functions are fixed, i.e., $\mathcal{U}_i^a = \{\hat{a}_i(\cdot)\}$ for all $i = 1, \ldots, n$. Assuming that $\hat{a}_i(\cdot)$ has an inverse function $\hat{p}_i(\cdot)$ with the domain $\mathcal{A}_i = [a_i^{\min}, a_i^{\max}]$ and the revenue function $R_i(a) = a \cdot \hat{p}_i(a)$ is concave in a for all $i = 1, \ldots, n$, then*

$$Z^{DD} = \lim_{\theta\to\infty} \frac{1}{\theta} \cdot Z_\theta^m(\hat{\boldsymbol{a}}) \leq \lim_{\theta\to\infty} \frac{1}{\theta} \cdot \min_{\beta_i(\cdot)\in\mathcal{U}_i^b : i=1,\ldots,n} Z_\theta^*(\hat{\boldsymbol{a}}, \boldsymbol{\beta}) \leq Z^D.$$

Remark 1. *Below we present some remarks of the above results.*

i) The asymptotic setting is slightly different from those in Gallego and van Ryzin (1994), Gallego and van Ryzin (1997). Instead of expanding the time horizon in the scaled problems, *we increase the demand rates in each time period. The effect of this setting on the demand remains the same given that sub-periods are used. The advantage of this setting is that it allows us to efficiently analyze the additional time-dependent capacity constraints that are necessary in our models when service time is involved. Note that these asymptotic results can also be extended for problems with multiple resources similar to the setting in Gallego and van Ryzin (1997). We only present here the results for single resource, which are relevant to the applications discussed, for simplicity and readability of the paper.*

ii) The asymptotic results in Theorem 1 as well as the bounds in Proposition 1 can be considered as a generalization of the results obtained by Gallego and van Ryzin (1994), Gallego and van Ryzin (1997) and Maglaras and Meissner (2006) when distributional ambiguity of random demand and especially, service time is considered. Given the ambiguous demand and service time distributions, the asymptotic results only provide an asymptotic $\frac{Z^{DD}}{Z^{D}}$*-approximation guarantee of the heuristic fixed-priced policy used in the proof, not an asymptotic optimality, since* $Z^{DD} < Z^{D}$ *under the general setting. We only achieve the asymptotic optimality result if the service time is fixed, i.e.,* $U_i^b = \{\beta_i^{\max}(\cdot)\}$ *for all* $i = 1, \ldots, n$*, which demonstrates the added complexity of the models in the presence of random service time.*

iii) The proposed regularity conditions requires linear confidence bands for uncertain acceptance probability functions under the general settings. With fixed acceptance probability functions, we can use the same regularity conditions as discussed in Gallego and van Ryzin (1994), Gallego and van Ryzin (1997) and Maglaras and Meissner (2006) even with ambiguous distributions of service time to achieve the bounds and asymptotic results.

The theoretical asymptotic results developed in this section show the quality of the fixed-price policies in the limit and they require assumptions such as the linearity of confidence bands of acceptance probability functions. As discussed, this is due to the added complexity of our proposed model with ambiguity in acceptance probabilities and especially, the consideration of random service time and its ambiguous distribution as compared to those in Gallego and van Ryzin (1994), Gallego and van Ryzin (1997) and Maglaras and Meissner (2006). However, in practice, instead of the theoretical asymptotic results, one might focus more on how to construct fixed-price policies given a finite time horizon based on the deterministic approximation developed in this section. The approximation results in a non-linear optimization model (14) in general, whose complexity depends on $\underline{a}_i(\cdot)$ and $\bar{a}_i(\cdot)$, as well as the structure of $\mathcal{U}_i^b$, $i = 1, \ldots, n$. In the next section, we shall investigate how to construct relevant uncertainty sets of probability functions $a_i(\cdot)$ and $\beta_i(\cdot)$, $i = 1, \ldots, n$ from experiments and available historical data using data-driven approaches, which might not necessarily result in linear confidence bands for acceptance probability functions needed for the theoretical asymptotic results. We shall instead focus on the tractability of the deterministic approximation problem with those uncertainty sets, which is essential to the construction of fixed-price policies in practice. To that aim, we shall develop a tractable formulation with data-driven uncertainty sets for the deterministic approximation problem in the next section. Furthermore, computational approaches including discrete price approximation will be discussed and the resulting fixed-price policies will be compared with commonly used constant-price policies in Section 5.

4 Data-Driven Formulations

In real-world applications, practitioners often conduct experiments or customer surveys to measure customer willingness-to-pay, i.e., the acceptance probability function $a_i(\cdot)$ for each product i, $i = 1, \ldots, n$ (see, e.g., (Breidert et al. 2006)). Given a finite set of prices, $\mathcal{P}_i^d = \{p_{i,1}, \ldots, p_{i,K_i}\} \subset [p_i^{\min}, p_i^{\max}]$, one can collect samples of customer decisions, i.e., whether the offered price is accepted or rejected, for each price value $p_{i,k}$. The proportion of accepted customer is an estimation for $a_i(p_{i,k})$, and we assume that for $p \in [p_{i,k}, p_{i,k+1}]$, $a_i(p) = a_i(p_{i,k}) + \dfrac{p - p_{i,k}}{p_{i,k+1} - p_{i,k}}(a_i(p_{i,k+1}) - a_i(p_{i,k}))$, i.e., the function is piecewise linear. Without loss of generality, we assume that $p_{i,1} = p_i^{\min} < p_{i,2} < \ldots < p_{i,K_i} < p_i^{\max}$. Given this set of samples, we would like to construct an uncertainty set $\mathcal{U}_i^a$ for the acceptance probability function $a_i : [p_i^{\min}, p_i^{\max}] \to [0, 1]$ under the assumption that $\mathcal{P}_i = [p_i^{\min}, p_i^{\max}]$. According to (14), we would simply need $\underline{a}_i(p)$ and $\bar{a}_i(p)$ for all $p \in \mathcal{P}_i$. From data samples for each price $p_{i,k}$, $k = 1, \ldots, K_i$, we can compute nominal values of the acceptance probability, denoted by $\hat{a}_i(p_{i,k})$. The lower and upper bounds, $\underline{a}_i(p_{i,k})$ and $\bar{a}_i(p_{i,k})$, can be set as confidence intervals. For the rest of this paper, we set $\bar{a}_i(p_{i,k})$ and $\underline{a}_i(p_{i,k})$ according to Wilson's confidence interval (see Wilson and Wilson 1927), specified as:

$$\frac{1}{1 + \frac{1}{m} z_{UL}^2} \left[\hat{a}_i(p_{i,k}) + \frac{1}{2m} z_{UL}^2 \pm z_{UL} \sqrt{\frac{1}{m} \hat{a}_i(p_{i,k})(1 - \hat{a}_i(p_{i,k})) + \frac{1}{4m^2} z_{UL}^2} \right],$$

where m is the number of in-sample instances, UL represents an uncertainty level, such that z_{UL} is the $\frac{1}{2}(1 + UL)$ quantile of a standard Normal distribution. (To ensure that $\bar{a}_i(p_{i,k})$ and $\underline{a}_i(p_{i,k})$ correspond to real distributions, we set $\bar{a}_i(0) = \underline{a}_i(0) = 1$.) For example, for a 95% uncertainty level, $\frac{1}{2}(1 + UL) = 0.975$ and $z_{UL} = 1.96$. According to Agresti and Coull (1998), the Wilson's confidence interval performs better than other types of confidence intervals, especially in terms of coverage probabilities in general settings. Given a vector $\boldsymbol{a} = (a_1, \ldots, a_{K_i}) \in [0, 1]^{K_i}$, we can construct $f_{\boldsymbol{a}}(\cdot)$, the piecewise linear function from $[p_i^{\min}, p_i^{\max}]$ to $[0, 1]$ with $K_i - 1$ pieces such that $f_{\boldsymbol{a}}(p_{i,k}) = a_k$ for all $k = 1, \ldots, K_i$. In this paper, we use $f_{\boldsymbol{a}}(\cdot)$ to define the uncertainty set $\mathcal{U}_i^a$ as follows:

$$\mathcal{U}_i^a = \{f_{\boldsymbol{a}}(\cdot) \,|\, a_k \in [\underline{a}_i(p_{i,k}), \bar{a}_i(p_{i,k})],\ k = 1, \ldots, K_i\}. \tag{21}$$

Given this uncertainty set, it is clear that two functions $\underline{a}_i$ and $\bar{a}_i$ required in (14) are $f_{\underline{\boldsymbol{a}}_i}$ and $f_{\bar{\boldsymbol{a}}_i}$, respectively, where $\underline{a}_{i,k} = \underline{a}_i(p_{i,k})$ and $\bar{a}_{i,k} = \bar{a}_i(p_{i,k})$ for all $k = 1, \ldots, K_i$. In Figure 1 we show an example of the piecewise linear acceptance function $\hat{a}$ and the corresponding $\bar{a}$ and $\underline{a}$ as dashed lines.

Next, we model uncertainty sets of tail probabilities $\beta_i(\cdot)$ of random service time based on the idea of statistical analysis of the empirical distribution function $\hat{F}_i(\cdot)$ (or equivalently, the empirical tail probability function $\hat{\beta}_i(\cdot) = 1 - \hat{F}_i(\cdot)$) given the historical data of service time. The uniform or

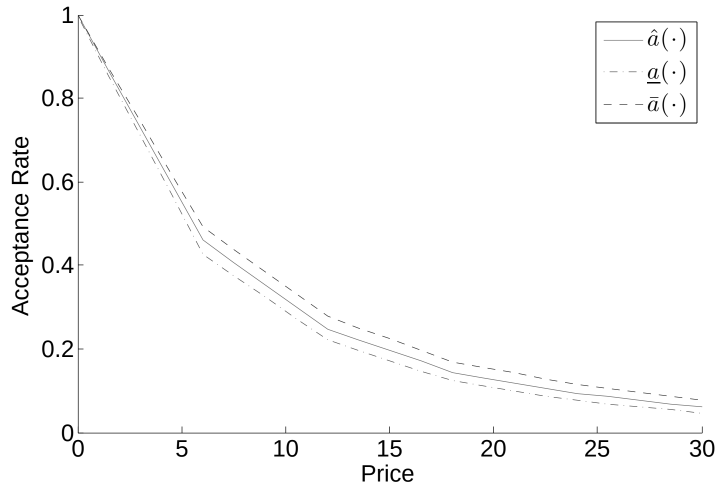

Figure 1: An example of piecewise linear $\hat{a}(\cdot)$, $\bar{a}(\cdot)$ and $\underline{a}(\cdot)$

Kolmogorov distance $d_K(\hat{F}_i, F_i)$ between $\hat{F}_i$ and F_i,

$$d_K(\hat{F}_i, F_i) = \sup_{t\in[0,+\infty)} \left|\hat{F}_i(t) - F_i(t)\right|, \tag{22}$$

is first studied by Kolmogorov (see for example, Shiryayev (1992)), to show the convergence of the empirical distribution function $\hat{F}_i$ to F_i with the convergence rate of $1/\sqrt{N_i}$, where N_i is the number of data samples. More specifically, it is shown by Massart (1990) that $\mathbb{P}\left(\sqrt{N_i} \cdot d_K(\hat{F}_i, F_i) > \lambda\right) \leq 2e^{-2\lambda^2}$ for $\lambda > 0$. Similarly, the Kantorovich-Rubinstein or Wasserstein distance $d_W(\hat{F}_i, F_i)$ between $\hat{F}_i$ and F_i,

$$d_W(\hat{F}_i, F_i) = \int_{t=0}^{+\infty} \left|\hat{F}_i(t) - F_i(t)\right| \mathrm{d}t, \tag{23}$$

is analyzed by del Barrio et al. (1999), which show the convergence of $\sqrt{N_i} \cdot d_W(\hat{F}_i, F_i)$. Fournier and Guillin (2015) show that $\mathbb{P}\left(\sqrt{N_i} \cdot d_W(\hat{F}_i, F_i) > \lambda\right) \leq Ce^{-c\lambda^2}$, where C and c are some positive constants, for $N_i > \lambda > 0$ under some technical conditions. Based on these convergence results and given the fact that $\left|\hat{F}_i(x) - F_i(x)\right| = \left|\hat{\beta}_i(x) - \beta_i(x)\right|$ for all $x \geq 0$, we can now construct an uncertainty set for β_i based on the Kolmogorov and Wasserstein distance measures with some indication of probabilistic guarantee (Bertsimas et al. 2017). For each product i, let $\tau_i^{\max}$ be the maximum service time, which could be determined from historical data. Specifically, for our interested applications such as cloud computing and car parking, $\tau_i^{\max}$ can be set by service providers, and the value may depend on the usage time of past orders, service cost, and the common standard in the industry. Given the discretized time horizon, we are interested in $\beta_i(t)$ for $t = 1, \ldots, \tau_i^{\max}$, knowing that $\beta_i(\tau_i^{\max} + 1) = 0$ and $\beta_i(0) = 1$. The two distances can be computed (approximately) with discretized time points

$$d_K(\hat{F}_i, F_i) \cong \max_{t=1,\ldots,\tau_i^{\max}} \left|\hat{F}_i(t) - F_i(t)\right| = \max_{t=1,\ldots,\tau_i^{\max}} \left|\hat{\beta}_i(t) - \beta_i(t)\right|,$$

and

$$d_W(\hat{F}_i, F_i) \cong \sum_{t=1}^{\tau_i^{\max}} \left|\hat{F}_i(t) - F_i(t)\right| \Delta t = \sum_{t=1}^{\tau_i^{\max}} \left|\hat{\beta}_i(t) - \beta_i(t)\right|.$$

The uncertainty set $\mathcal{U}_i^b$ can then be constructed using the following constraints:

$$\left|\beta_i(t) - \hat{\beta}_i(t)\right| \leq \Gamma_i^K, \quad t = 1, \ldots, \tau_i^{\max}, \tag{24}$$

$$\sum_{t=1}^{\tau_i^{\max}} \left|\beta_i(t) - \hat{\beta}_i(t)\right| \leq \Gamma_i^W, \text{ and} \tag{25}$$

$$0 \leq \beta_i(t) \leq \beta_i(t-1) \leq 1, \quad t = 1, \ldots, \tau_i^{\max}, \tag{26}$$

where Γ_i^K and Γ_i^W can be set as functions of $\sqrt{N_i}$, where N_i is the number of data samples, to make sure these constraints cover all historical data with high probability. For example, the violation probability of each constraint (24) can be bounded by a small $\epsilon > 0$ if Γ_i^K is approximately set to $\sqrt{-1/(2N_i)\ln(\epsilon/2)}$ given the probabilistic inequalities mentioned previously. Note that constraints (26) ensure that $\beta_i(\cdot)$ is a tail probability distribution.

We are now ready to reformulate (11) given these proposed uncertainty sets under the assumption that $p_i \in \mathcal{P}_i = [p_i^{\min}, p_i^{\max}]$ for $i = 1, \ldots, n$.

Proposition 2. *Given the proposed uncertainty sets $\mathcal{U}_i^a$ and $\mathcal{U}_i^b$, the robust deterministic relaxation problem* (11) *is equivalent to*

$$\max_{\boldsymbol{P}} \sum_{t=1}^{T} \sum_{i=1}^{n} \bar{r}_i(t) \left(\sum_{k=1}^{K_i-1} (p_{i,k+1} - p_{i,k}) \left(\underline{a}_i(p_{i,k+1}) - \underline{a}(p_{i,k})\right) \cdot \nu_i^2(t,k) + \right. \tag{27a}$$

$$\left. \sum_{k=1}^{K_i-1} p_{i,k}\underline{a}_i(p_{i,k}) \cdot \zeta_i(t,k) + [p_{i,k}\underline{a}_i(p_{i,k+1}) + p_{i,k+1}\underline{a}_i(p_{i,k}) - 2p_{i,k}\underline{a}_i(p_{i,k})] \cdot \nu_i(t,k) \right) \tag{27b}$$

$$\text{s.t.} \sum_{i=1}^{n} c_i \left(\bar{r}_i(t) \left(\sum_{k=1}^{K_i-1} \bar{a}_i(p_{i,k}) \cdot \zeta_i(t,k) + (\bar{a}_i(p_{i,k+1}) - \bar{a}(p_{i,k})) \cdot \nu_i(t,k) \right) + \right. \tag{27c}$$

$$\Gamma_i^W w_i(t) + \Gamma_i^+(s) y_i^+(t,s) + \Gamma_i^-(s) y_i^-(t,s) +$$

$$\left. \sum_{s=1}^{\tau_i(t)} \hat{\beta}_i(s) \left[x_i^+(t,s) - x_i^-(t,s) + y_i^+(t,s) - y_i^-(t,s)\right] \right) \leq C, \ \forall\, t = 1, \ldots, T, \tag{27d}$$

$$q_i(t,s) = \bar{r}_i(t-s)\bar{a}_i(p_i(t-s)), \quad \forall\, t = 1, \ldots, T, \ s = 1, \ldots, \tau_i(t), \tag{27e}$$

$$\left[x_i^+(t,s) - x_i^-(t,s)\right] + \left[y_i^+(t,s) - y_i^-(t,s)\right] - z_i(t,s) + z_i(t,s-1) \geq q_i(t,s),$$
$$\forall\, t = 1, \ldots, T, \ s = 1, \ldots, \tau_i(t), \tag{27f}$$

$$w_i(t) \geq x_i^+(t,s) + x_i^-(t,s), \quad \forall\, t = 1, \ldots, T, \ s = 1, \ldots, \tau_i(t), \tag{27g}$$

$$\sum_{k=1}^{K_i-1} \zeta_i(t,k) = 1, \ \forall i = 1, \ldots, n, \ t = 1, \ldots, T, \ k = 1, \ldots, K_i - 1 \tag{27h}$$

$$\nu_i(t,k) \leq \zeta_i(t,k), \ \forall i = 1, \ldots, n, \ t = 1, \ldots, T, \ k = 1, \ldots, K_i - 1 \tag{27i}$$

$$w_i(t) \geq 0, \quad \forall t = 1, \ldots, T, \ i = 1, \ldots, n \tag{27j}$$

$$x_i^+(t,s), x_i^-(t,s), y_i^+(t,s), y_i^-(t,s), z_i(t,s) \geq 0, \ z_i(t,0) = z_i(t, \tau_i(t)) = 0,$$
$$\forall\, t = 1, \ldots, T, \ i = 1, \ldots, n, \ s = 1, \ldots \tau_i(t), \tag{27k}$$

$$\zeta_i(t,k) \in \{0,1\}, \nu_i(t,k) \in [0,1], \quad \forall\, i = 1, \ldots, n, \, t = 1, \ldots, T, \ k = 1, \ldots, K_i - 1, \tag{27l}$$

where $\tau_i(t) = \min\{t-1, \tau_i^{\max}\}$, $\Gamma_i^+(t) = \min\{1 - \hat{\beta}_i(t), \Gamma_i^K, \Gamma_i^W\}$ *and* $\Gamma_i^-(t) = \min\{\hat{\beta}_i(t), \Gamma_i^K, \Gamma_i^W\}$ *for* $t = 1, \ldots, \tau_i^{\max}$.

Proof. As previously discussed, the robust deterministic relaxation problem (11) can be rewritten as in (14) or equivalently,

$$\begin{aligned} J_r^D = \max_{\boldsymbol{P}} \quad & \sum_{t=1}^{T} \sum_{i=1}^{n} \bar{r}_i(t) \underline{a}_i(p_i(t)) p_i(t) \\ \text{s.t.} \quad & \sum_{i=1}^{n} c_i \left(\bar{r}_i(t) \bar{a}_i(p_i(t)) + \max_{\beta_i(\cdot) \in \mathcal{U}_i^b} \sum_{s=1}^{t-1} \bar{r}_i(s) \bar{a}_i(p_i(s)) \beta_i(t-s) \right) \leq C, \quad \forall\, t = 1, \ldots, T, \\ & p_i(t) \in [p_i^{\min}, p_i^{\max}], \quad \forall\, i = 1, \ldots, n, \, t = 1, \ldots, T, \end{aligned}$$

given the separability of the uncertainty sets $\mathcal{U}_i^b$, $i = 1, \ldots, n$. The main constraint involves the following optimization problem:

$$\max_{\boldsymbol{\beta}_i(\cdot) \in \mathcal{U}_i^b : i = 1, \ldots, n} \sum_{s=1}^{\tau_i(t)} q_i(t,s) \beta_i(s),$$

where $\tau_i(t) = \min\{t-1, \tau_i^{\max}\}$ and $q_i(t,s) = \bar{r}_i(t-s) \bar{a}_i(p_i(t-s))$ for $s = 1, \ldots, \tau_i(t)$. The setting of $\tau_i(t)$ is appropriate given the fact that if $t \geq \tau_i^{\max} + 1$, we only need to consider $s \geq t - \tau_i^{\max}$ since $\beta_i(s) = 0$ for all $s \geq \tau_i^{\max} + 1$. On the other hand, if $t < \tau_i^{\max} + 1$, variables $\beta_i(s)$ for $s = t, \ldots, t_i^{\max}$ can be set to be the nominal values $\hat{\beta}_i(s)$ without affecting the optimal objective value. Specifying constraints proposed for $\mathcal{U}_i^b$, we have

$$\begin{aligned} \max \quad & \sum_{s=1}^{\tau_i(t)} q_i(t,s) \beta_i(s) \\ \text{s.t.} \quad & \sum_{s=1}^{\tau_i(t)} \left| \beta_i(s) - \hat{\beta}_i(s) \right| \leq \Gamma_i^W, \\ & \beta_i(s) \leq \hat{\beta}_i(s) + \Gamma_i^+(s), \ \forall\, s = 1, \ldots, \tau_i(t), \\ & \beta_i(s) \geq \hat{\beta}_i(s) - \Gamma_i^-(s), \ \forall\, s = 1, \ldots, \tau_i(t), \\ & \beta_i(s) - \beta_i(s+1) \geq 0, \ \forall\, s = 1, \ldots, \tau_i(t) - 1, \end{aligned}$$

where $\Gamma_i^+(s) = \min\{1 - \hat{\beta}_i(s), \Gamma_i^K, \Gamma_i^W\}$ and $\Gamma_i^-(s) = \min\{\hat{\beta}_i(s), \Gamma_i^K, \Gamma_i^W\}$ for $s = 1, \ldots, \tau_i(t)$ as mentioned previously. These modified parameters guarantee that $0 \leq \beta_i(s) \leq 1$ for all $s = 1, \ldots, \tau_i(t)$.

We define an auxiliary variable $\gamma_i(s)$ for replacing each $\left| \beta_i(s) - \hat{\beta}_i(s) \right|$, and reformulate the above model as a linear program:

$$\max \sum_{s=1}^{\tau_i(t)} q_i(t,s)\beta_i(s) \tag{28a}$$

$$\text{s.t.} \sum_{s=1}^{\tau_i(t)} \gamma_i(s) \le \Gamma_i^W, \tag{28b}$$

$$\beta_i(s) - \gamma_i(s) \le \hat{\beta}_i(s),\ \forall\, s = 1,\dots,\tau_i(t), \tag{28c}$$

$$\beta_i(s) + \gamma_i(s) \ge \hat{\beta}_i(s),\ \forall\, s = 1,\dots,\tau_i(t), \tag{28d}$$

$$\beta_i(s) \le \hat{\beta}_i(s) + \Gamma_i^+(s),\ \forall\, s = 1,\dots,\tau_i(t), \tag{28e}$$

$$\beta_i(s) \ge \hat{\beta}_i(s) - \Gamma_i^-(s),\ \forall\, s = 1,\dots,\tau_i(t), \tag{28f}$$

$$\beta_i(s) - \beta_i(s+1) \ge 0,\ \forall\, s = 1,\dots,\tau_i(t)-1, \tag{28g}$$

Taking the dual of formulation (28), for each product i and time t, we obtain:

$$\begin{aligned}
\min \quad & \Gamma_i^W w_i + \sum_{s=1}^{\tau_i(t)} \hat{\beta}_i(s)\left[x_i^+(s) - x_i^-(s)\right] + \hat{\beta}_i(s)\left[y_i^+(s) - y_i^-(s)\right] + \Gamma_i^+(s) y_i^+(s) + \Gamma_i^-(s) y_i^-(s) \\
\text{s.t.} \quad & \left[x_i^+(s) - x_i^-(s)\right] + \left[y_i^+(s) - y_i^-(s)\right] - z_i(s) + z_i(s-1) \ge q_i(t,s),\ \forall\, s = 1,\dots,\tau_i(t), \\
& w_i \ge x_i^+(s) + x_i^-(s),\ \forall\, s = 1,\dots,\tau_i(t), \\
& w_i, x_i^+(s), x_i^-(s), y_i^+(s), y_i^-(s), z_i(s) \ge 0,\ z_i(0) = z_i(\tau_i(t)) = 0,
\end{aligned}$$

where dual variables $w_i, x_i^+(s), x_i^-(s), y_i^+(s), y_i^-(s), z_i(s)$ correspond to constraints (28b), (28c), (28d), (28e), (28f), and (28g), respectively. The dual constraints are associated with variables $\beta_i(s)$ and $\gamma_i(s)$ for all $s = 1,\dots,\tau_i(t)$ of each product i in period t, for $i = 1,\dots,n$ and $t = 1,\dots,T$.

Finally, we rewrite the robust deterministic relaxation problem as

$$\max_{\boldsymbol{P}} \sum_{t=1}^{T} \sum_{i=1}^{n} \bar{r}_i(t) p_i(t) \underline{a}_i(p_i(t)) \tag{29a}$$

$$\begin{aligned}
\text{s.t.} \sum_{i=1}^{n} c_i \Bigg(& \bar{r}_i(t)\bar{a}_i(p_i(t)) + \Gamma_i^W w_i(t) + \Gamma_i^+(s) y_i^+(t,s) + \Gamma_i^-(s) y_i^-(t,s) + \\
& \sum_{s=1}^{\tau_i(t)} \hat{\beta}_i(s)\left[x_i^+(t,s) - x_i^-(t,s) + y_i^+(t,s) - y_i^-(t,s)\right] \Bigg) \le C,\ \forall\, t = 1,\dots,T,
\end{aligned} \tag{29b}$$

$$q_i(t,s) = \bar{r}_i(t-s)\bar{a}_i(p_i(t-s)), \quad \forall\, t = 1,\dots,T,\ s = 1,\dots,\tau_i(t), \tag{29c}$$

$$\begin{aligned}
& \left[x_i^+(t,s) - x_i^-(t,s)\right] + \left[y_i^+(t,s) - y_i^-(t,s)\right] - z_i(t,s) + z_i(t,s-1) \ge q_i(t,s), \\
& \qquad \forall\, t = 1,\dots,T,\ s = 1,\dots,\tau_i(t),
\end{aligned} \tag{29d}$$

$$w_i(t) \ge x_i^+(t,s) + x_i^-(t,s), \quad \forall\, t = 1,\dots,T,\ s = 1,\dots,\tau_i(t), \tag{29e}$$

$$w_i(t) \ge 0, \quad \forall t = 1,\dots,T,\ i = 1,\dots,n \tag{29f}$$

$$\begin{aligned}
& x_i^+(t,s), x_i^-(t,s), y_i^+(t,s), y_i^-(t,s), z_i(t,s) \ge 0,\ z_i(t,0) = z_i(t,\tau_i(t)) = 0, \\
& \qquad \forall\, t = 1,\dots,T,\ i = 1,\dots,n,\ s = 1,\dots \tau_i(t),
\end{aligned} \tag{29g}$$

$$p_i(t) \ge 0, \quad \forall\, i = 1,\dots,n,\ t = 1,\dots,T, \tag{29h}$$

Recall the piecewise linear assumption of the acceptance probability function. For each $i = 1,\dots,n,\ t = 1,\dots,T$, we define binary decision variables $\zeta_i(t,k)$, such that $\zeta_i(t,k) = 1$ if the price p_i^t lies in $[p_{i,k}, p_{i,k+1}]$, and $\zeta_i(t,k) = 0$ if not, for each $k = 1,\dots,K_i - 1$. We have: $\sum_{k=1}^{K_i-1} \zeta_i(t,k) = 1$ for all $i = 1,\dots,n$. We also define continuous variables $\eta_i(t,k) \in [0,1]$, for $i = 1,\dots,n,\ t = 1,\dots,T,\ k = 1,\dots,K_i - 1$, to compute the value of p_i^t using a convex combination of two end points of the interval k assuming that $\zeta_i(t,k) = 1$. We can then compute p_i^t and $a_i(p_i^t)$, and the expected revenue as follows:

$$p_i^t = \sum_{k=1}^{K_i-1} \zeta_i(t,k) \cdot \left[p_{i,k} + (p_{i,k+1} - p_{i,k}) \cdot \eta_i(t,k)\right], \tag{30}$$

$$a_i(p_i^t) = \sum_{k=1}^{K_i-1} \zeta_i(t,k) \cdot \left[a_i(p_{i,k}) + (a_i(p_{i,k+1}) - a(p_{i,k})) \cdot \eta_i(t,k)\right], \tag{31}$$

and

$$p_i^t a(p_i^t) = \sum_{k=1}^{K_i-1} \zeta_i(t,k) \cdot \left[p_{i,k} + (p_{i,k+1} - p_{i,k}) \cdot \eta_i(t,k)\right] \cdot \left[a_i(p_{i,k}) + (a_i(p_{i,k+1}) - a(p_{i,k})) \cdot \eta_i(t,k)\right]. \tag{32}$$

Given that $\zeta_i(t,k) \in \{0,1\}$ and $\eta_i(t,k) \in [0,1]$, we can linearize the above formulation with a new decision variable $\nu_i(t,k) \in [0,1]$ to replace $\zeta_i(t,k) \cdot \eta_i(t,k)$ using the additional constraint $\nu_i(t,k) \le \zeta_i(t,k)$ for all $i = 1,\dots,n,\ t = 1,\dots,T,\ k = 1,\dots,K_i - 1$. We then have:

$$p_i^t = \sum_{k=1}^{K_i-1} p_{i,k} \cdot \zeta_i(t,k) + (p_{i,k+1} - p_{i,k}) \cdot \nu_i(t,k), \tag{33}$$

$$a_i(p_i^t) = \sum_{k=1}^{K_i-1} a_i(p_{i,k}) \cdot \zeta_i(t,k) + (a_i(p_{i,k+1}) - a(p_{i,k})) \cdot \nu_i(t,k), \tag{34}$$

and

$$\begin{aligned} p_i^t a(p_i^t) = \ & \sum_{k=1}^{K_i-1} p_{i,k} a_i(p_{i,k}) \cdot \zeta_i(t,k) + \left[p_{i,k} a_i(p_{i,k+1}) + p_{i,k+1} a_i(p_{i,k}) - 2 p_{i,k} a_i(p_{i,k})\right] \cdot \nu_i(t,k) + \\ & \sum_{k=1}^{K_i-1} (p_{i,k+1} - p_{i,k}) \left(a_i(p_{i,k+1}) - a(p_{i,k})\right) \cdot \nu_i^2(t,k). \end{aligned} \tag{35}$$

Now, we can substitute $a_i(p_i^t)$ from (34) to (27d) and (27e) as well as $p_i^t a(p_i^t)$ from (35) to (29) with additional decision variables $\zeta_i(t,k) \in \{0,1\}$ and $\nu_i(t,k) \in [0,1]$ and additional constraints, $\sum_{k=1}^{K_i-1} \zeta_i(t,k) = 1$ and $\nu_i(t,k) \le \zeta_i(t,k)$ for all $i = 1,\dots,n,\ t = 1,\dots,T,\ k = 1,\dots,K_i - 1$ to get the final formulation. □

Remark 2. *The choice of additional decision variables $\zeta_i(t,k)$ and $\eta_i(t,k)$ (and $\nu_i(t,k)$) maintains the concavity of the objective function in terms of these new decision variables given the monotonicity of the acceptance probability functions. It shows that the resulting formulation (27) is a* *mixed-integer quadratic programming (MIQP) problem, whose continuous relaxation is* convex. *The problem can be solved by branch-and-bound methods with a convex sub-problem at each node.*

5 Numerical Studies

In this section, we consider a cloud computing service pricing application, and use public available data by the Amazon Web Services (AWS), which is recognized as "a collection of cloud computing services that make up the on-demand computing platform offered by Amazon.com," to generate all the instances to test our approaches. Al-Roomi et al. (2013), Cheng et al. (2016), Xu and Li (2013), Chen et al. (2019) made recent progress on heuristic-based cloud computing pricing strategies, where the authors also summarize different pricing schemes companies use in practice. We investigate the proposed distributionally robust fixed-price policy with $\mathcal{Y}_H = \mathcal{Y}_F$ and compare it with common pricing strategies such as constant-price policies. We use Python 2.7 and Gurobi 7.0 for solving all the optimization models. The solver terminates when the optimality gap reaches 0.01% or the solving time reaches one hour. We perform all the tests on a 2.2 GHz Intel Core i7 CPU with 16GB RAM.

5.1 Experimental Setup

We arbitrarily select three products, named "m1.small," "c1.medium," and "m1.large," from the complete list of Amazon Elastic Compute Cloud (EC2) products (see Javadi et al. 2013). We extract the resource (storage) and the mean price information from Table 1 in Javadi et al. (2013), and the mean and standard deviation of service time from Table 5 in the same literature. Also, in Table 5 we observe that the demand for these three products are 3278, 3642 and 2033, respectively, so follow the proportion, we assume that the mean arrival probabilities are 0.32, 0.36 and 0.20, respectively. All the information are presented in Table 1.

Table 1: Parameters of three AWS products from Javadi et al. (2013)

Product i	m1.small	c1.medium	m1.large
Storage (GB) c_i	160	350	850
Mean arrival (per period) $\bar{r}_i(t)$	0.32	0.36	0.20
On-demand price (in cent) η_i	10	20	40
Mean of service time (in period) μ_i	22.2	20.0	35.8
Std of service time (in period) σ_i	35.3	20.9	186.0

In each period t, we generate samples of the random service demand $r_i(t)$ of the three products by following a categorical distribution: 32% probability for Product 1, 36% probability for Product 2, 20% probability for Product 3, and 12% probability for non-arrival. (The values are presented in the Row "Mean arrival (per period)" in Table 1.) We generate i.i.d. samples of $\pi_i(t)$, the reservation price, by letting $\pi_i(t) = \rho_i(t)\epsilon_i(t)$, where $\rho_i(t) \sim \text{Uniform}[0.2, 1.8]$ is a scale parameter, and $\epsilon_i(t) \sim \text{Exp}(\eta_i)$ follows a commonly used Exponential distribution (for acceptance probabilities) with parameters η_i being 10, 20, and 40 cents as the on-demand prices of the three products in Table 1. The exponential reservation price corresponds to exponential acceptance probability, which is common in dynamic pricing and economic literature, e.g., (Gallego and van Ryzin 1994). The random scale parameter is used to represent the distributional ambiguity to some extent with deviations from the commonly used Exponential distribution. Similarly, to generate random samples of the service time $\tau_i(t)$, we consider the commonly used Log-Normal distributions in the appointment scheduling literature, and set $\tau_i(t) = \nu_i(t)W_i(t)$, where $\nu_i(t) \sim \text{Uniform}[0.2, 1.8]$ scales the realizations of the random variable $W_i(t) \sim \text{Log-Normal}(\mu_i, \sigma_i)$, with μ_i and σ_i being the means and standard deviations of the service durations of product i in Table 1. The log-normal distribution fits well to service time, see (Gualandi and Toscani 2018). In our computational studies, we set $\tau_i^{\max} = 40$ and truncate all the service time values that are larger than 40. For each product i, we set $a_i(\cdot)$ as a piecewise linear function with five intervals, which are evenly distributed over interval $[0, 3\eta_i]$. The nominal distribution of reservation price and service time are then constructed using 5000 i.i.d. samples of the random variables $r_i(t)$, $\pi_i(t)$, and $\tau_i(t)$. We use $T = 100$ periods in all our instances. We use 99% uncertainty level for the reservation price distribution. Given the numbers of orders of three product types in the 5000 samples, we set $\Gamma_K = 0.04$ for all products so that approximately, the violation probability of each constraint (24) is less than 1%. We set $\Gamma_W = 5\Gamma_K = 0.20$ for all products. Note that, experimentally, we did vary the ratio Γ_W/Γ_K in a wide range from 5 to 25 and results are not significantly affected by the change.

We vary the total capacity C as 1500GB, 2500GB and 3500GB shared by all three products over the finite time horizon. Note that when $C = 3500$GB, the capacity is in fact unlimited, as the optimal prices for all periods are the same. We use $C = 1500$GB and $C = 2500$GB to represent moderate and tight capacity limits, respectively. Under these two capacity settings which reflect situations usually faced by small companies with limited resources, the optimal prices are time-varying and we shall demonstrate that the resulting policy is better than the common constant-price policies.

5.2 Computational Results

To construct fixed-price pricing policies, we need to solve the deterministic approximation problem. In particular, we solve instances of Model (27) directly using an optimization solver Gurobi. Table 2 shows the computational results, where the "Optimality Gap" is provided by Gurobi solver at the end of the one-hour computational time limit.

Table 2: Optimality gaps and best objective values for different MIQP instances

	$C = 1500$GB	$C = 2500$GB	$C = 3500$GB
Best Obj. Value	453.70	534.12	571.27
Optimality Gap	14.50%	27.05%	25.54%

The results in Table 2 show that the optimality gap obtained from the Gurobi solver is more than 14% for all the instances we compute. It indicates that even the convex continuous relaxation of the MIQP formulation in (27) is very difficult to optimize. Given this computational drawback, we aim to develop an approximation model with better computational performance. Motivated by the idea of discrete prices in Gallego and van Ryzin (1994), we now consider the pricing problem with discrete prices, which can be considered as an approximation to Model (27) by discretizing the continuous prices. This can be well justified since in practice, prices are usually not be set arbitrarily but belong to a set of rounded prices. (For example, one can round prices to 0.01-cent intervals when using this approximation.) The model with discrete prices can be reformulated as a mixed-integer linear programming (MILP) problem instead of an MIQP formulation as Model (27). The details of the model with discrete prices are presented in Appendix A as Model (A-1). We next evaluate the performance of this discrete price approximation both in terms of computational efficiency and the quality of its solutions.

5.2.1 Performance of Discrete Price Approximation

To investigate the performance of the model with discrete prices as compared to that of Model (27), we discretize the continuous prices with N price choices for each product, where N varies. The discrete prices are evenly distributed over the interval $[0, 3\eta_i]$, and the corresponding acceptance probabilities are computed from the original continuous acceptance probability functions. In Figure 2, we show acceptance probabilities of 20 discrete prices obtained from the discretization of the acceptance probability functions previously shown in Figure 1.

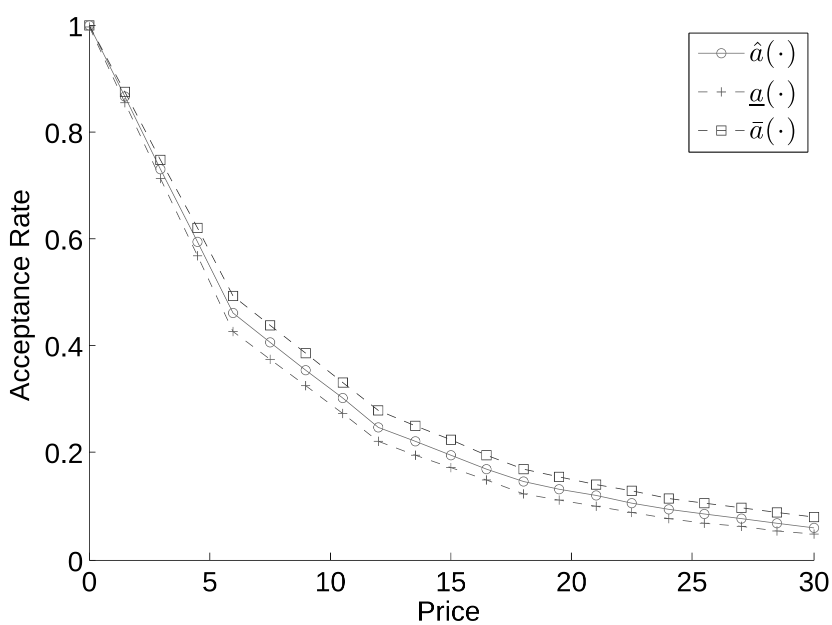

Figure 2: An example of discretized $\hat{a}(\cdot)$, $\bar{a}(\cdot)$ and $\underline{a}(\cdot)$ with 20 price points

We report the computational times to solve instances of Model (A-1) when N varies from $N = 10$ to $N = 100$ in Table 3. It shows that the model with discrete prices can be solved to optimality (with optimality gap below 0.01%) within the time limit with the maximum solving time of less than 40 seconds. It shows that the discrete price approximation is much more computationally efficient than the original MIQP model (27).

Table 3: Computational times for Model (A-1) with different numbers of discrete prices

N	10	20	50	100
Average time (sec)	11.43	12.42	20.39	38.90
Maximum time (sec)	16.91	18.35	33.56	65.44

The performance of Model (A-1) is compared to that of Model (27) using the ratio of their objectives, Z_N/Z, where Z_N is the optimal objective value of instances with N discrete prices and Z is the best objective value achieved for instances of Model (27). Figure 3 shows these ratios for different instances when N is increased from 10 to 100. We can see that the model with discrete prices approximate well the model with continuous prices with the ratios ranging from 99.50% (for $N = 10$) to 102.69% with better solutions (for $N = 100$). The ratios almost stay the same for $N > 50$, which shows the discretization with $N = 100$ is good enough for these instances. In practice, 100 price choices are reasonable for the ranges of prices considered in these instances. It shows that in terms of the quality of solutions, the discrete price approximation with practically enough price choices also performs well as compared to the original MIQP model.

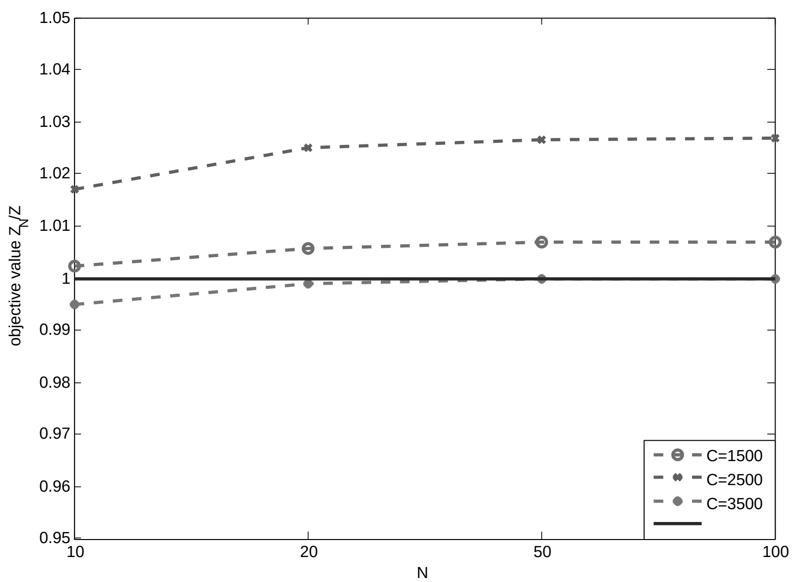

Figure 3: Objective ratios between Model (A-1) and Model (27) for different instances

Given the performance of the discrete price approximation in terms of both computational efficiency and solution quality, we are going to use Model (A-1) with $N = 100$ for numerical results for the rest of this section. Before discussing the performance of the proposed pricing policy, we briefly analyze the prices obtained from the deterministic approximation.

5.2.2 Price Dynamics from Deterministic Approximation

Solving the deterministic approximation problem will provide us with the optimal prices $p_i^{t,*}$ for all $i = 1, \ldots, n$ and $t = 1, \ldots, T$. In Figure 4 we show the price dynamics, i.e., the sequences $\{p_i^{t,*}\}_{t=1,\ldots,T}$ of optimal prices for different products $i = 1, \ldots, n$, under different time horizons when $C = 1500$GB. When T ranges from 40 to 100, the price solutions are generally consistent for the early stage, which implies the effect of finite time horizons is limited in these numerical tests.

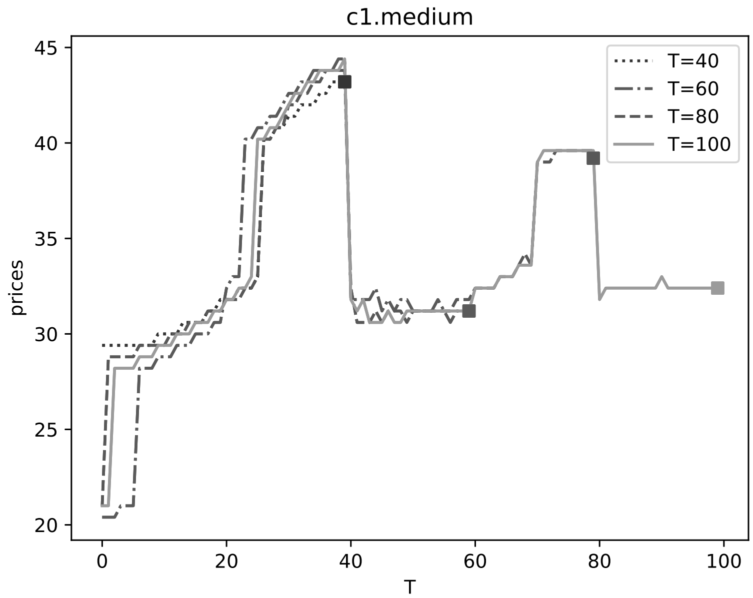

Figure 4: The price dynamics when T ranges from 40 to 100

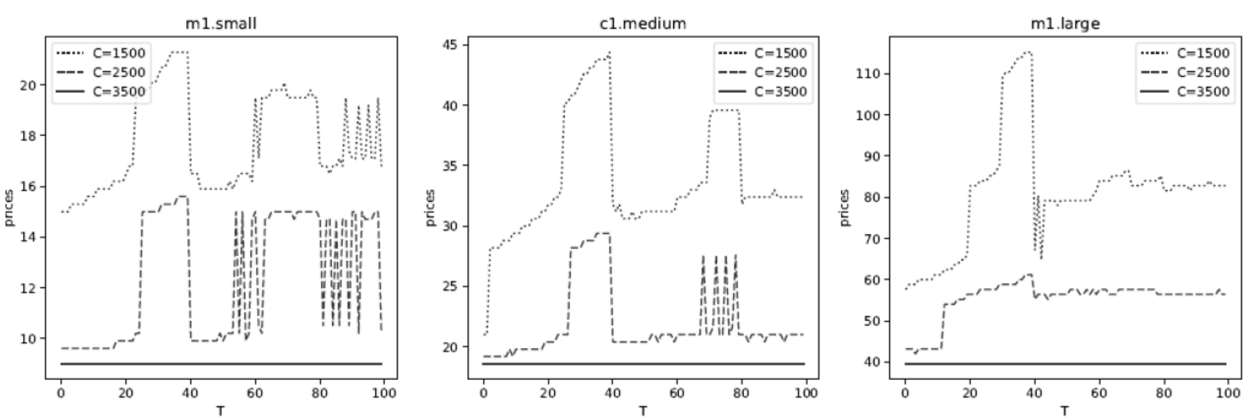

Figure 5: The price dynamics for $C = 1500$GB, $C = 2500$GB, and $C = 3500$GB

In Figure 5, we fix $T = 100$ and show the price dynamics under different capacity constraints. As the capacity increases, the prices lower down and become smoother. For $C = 3500$, we could see that the prices stay unchanged during the whole time horizon, which indicates that the capacity is unlimited. These results demonstrate the effect of capacity on the proposed model in which some capacities should be reserved for demand in later periods by setting appropriate prices in early periods to make sure the revenue is maximized.

5.2.3 Nominal vs. Robust Policy Performance

Our proposed models consider the ambiguity of acceptance probabilities and distributions of random service time. Given the parametric uncertainty sets developed in Section 4, the nominal model with fixed acceptance probabilities and service time distributions can be defined with $UL = 0\%$ and $\Gamma_K = \Gamma_W = 0$. We analyze the effect of distributional ambiguity by compare the results obtained from the nominal model with fixed acceptance probabilities and service time distributions and those of robust ones for which distributional ambiguity is considered. We set $UL = 99\%$, $\Gamma_K = 0.04$ and $\Gamma_W = 0.20$ for the uncertainty sets used in the robust model. For out-of-sample tests, we uses potential worst-case distributions to generate the testing samples, i.e., the reservation prices are sampled by using $\underline{a}_i(\cdot)$ and the service time values are generated by using the optimal solution $\beta_i(\cdot)$ of (28) at $t = T$, where the parameters $q_i(t,s) = \bar{r}_i(t-s)\bar{a}_i(p_i(t-s))$ are computed based on the optimal solution of (A-1).

Table 4: Comparison of policies obtained from robust and nominal models under different capacity limits

	$C = 1500$GB		$C = 2500$GB		$C = 3500$GB	
	Robust	Nominal	Robust	Nominal	Robust	Nominal
Objective Z^D	493.21	556.13	548.52	616.87	571.17	622.41
Mean Revenue Z_W	**320.23**	316.70	**400.39**	385.17	475.67	**481.1544**

Table 4 shows Z^D, the optimal objectives of the deterministic approximations, and Z_W, the mean revenues obtained from out-of-sample tests using fixed-price policies for both nominal and robust models. The robust model yields smaller optimal objective values as it produces more conservative pricing solutions to hedge against the worst case. Regarding the mean revenue, when the capacity is limited ($C = 1500$GB and $C = 2500$GB), the robust model outperforms the nominal model in the out-of-sample test. It shows that the consideration of distribution ambiguity enhances the fixed-price policies in these instances with limited capacity. On the other hand, when the capacity is unlimited ($C = 3500$GB), the fixed-price policy obtained from nominal model performs better. One explanation could be that when the capacity is unlimited, the distributional ambiguity of acceptance probabilities does not have much impact on accept-or-reject decisions of customers given the simple pricing policy based on (constant) greedy prices.

Figure 6 shows the histograms of random revenues obtained from fixed-price policies using results of both nominal ($\text{UL} = 0\%, \Gamma_K = 0, \Gamma_W = 0$) and robust ($\text{UL} = 99\%, \Gamma_K = 0.04, \Gamma_W = 0.20$) models when $C = 2500$GB in the out-of-sample test. The random revenue obtained from the robust model has lower frequencies for all low revenue intervals (< 400), while having higher frequencies

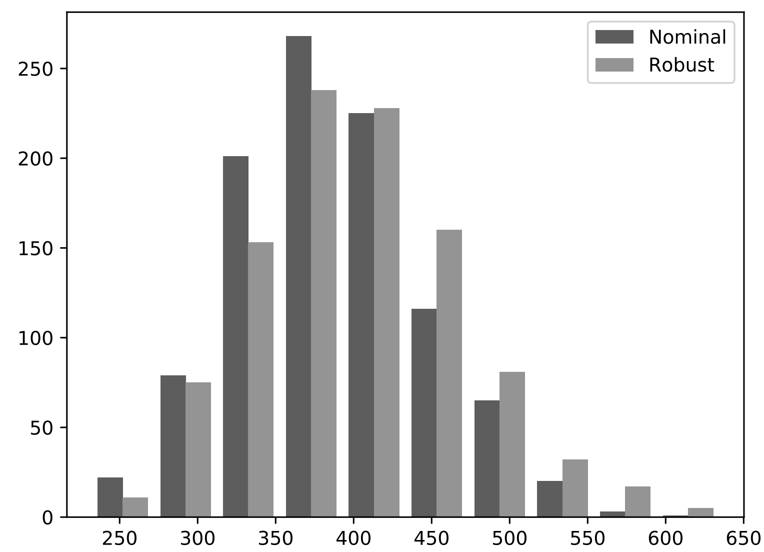

Figure 6: Histograms of random revenues from the out-of-sample test with $C = 2500$GB

for all the high revenue intervals (> 400). The plots show that robust setting results in higher revenues systematically in these tests.

5.2.4 Fixed-Price vs. Constant-Price Policy Performance

In this section, we compare our proposed fixed-price policy with the common constant-price policy with the same out-of-sample test addressed in Section 5.2.3. For constant-price policy, we solve Model (A-1) with one additional constraint that $u_i(t, k) = u_i(0, k)$ for all t. For both formulation, we use the same set of parameters as mentioned above (UL=99%, $\Gamma_K = 0.04$ and $\Gamma_W = 0.20$).

Table 5: Comparison of fixed-price (F-P) and constant-price (C-P) policies under different capacity limits

	$C = 1500$GB		$C = 2500$GB		$C = 3500$GB	
	F-P	C-P	F-P	C-P	F-P	C-P
Objective Z^D	493.21	449.34	548.52	546.21	571.17	571.17
Mean Revenue Z_W	**320.23**	315.46	**400.39**	400.14	475.67	475.67

In Table 5 we show the objective values and the mean revenues obtained from the two pricing policies in the out-of-sample test under different capacity limits. When the capacity is low, the fixed-price policy has a clear advantage, but such advantage is reduced when the capacity increases. When $C = 3500$GB, the solutions from both policies are the same. In Figure 7 we show histograms of random revenues obtained from these two pricing policies in the out-of-sample test with $C =$

1500GB. These results show that the fixed-priced policies outperform the common constant-price policies when the capacity is limited and they should be adopted under these circumstances.

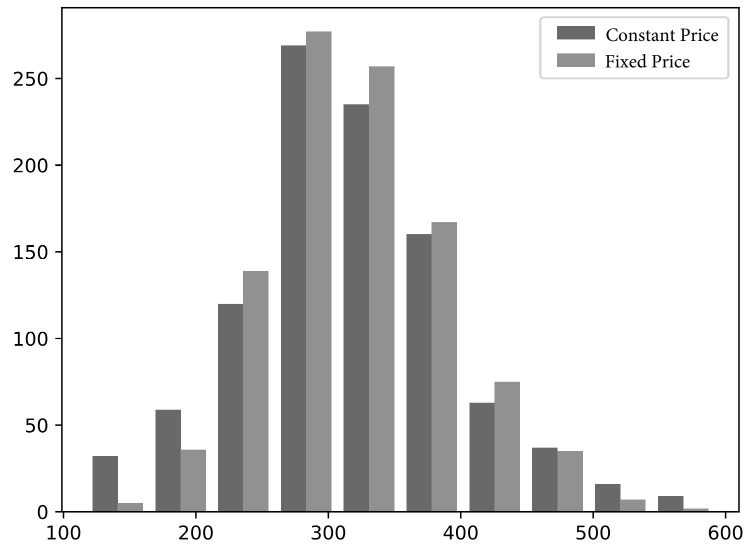

Figure 7: Histogram of random revenues from the out-of-sample test with $C = 1500\text{GB}$

6 Conclusion and Future Research

In this paper, we investigate a new revenue management problem, where we dynamically determine prices for multiple products sharing fixed capacities and aim to maximize the expected revenue over a finite horizon. Both demand quantities of each product and the service time for completing each demand arrival are random, inspired by the emerging cloud computing industry, where prices for different cloud computing products are updated dynamically to meet the random demand and each demand unit takes random computing time on the servers. Moreover, we recognize the distributional ambiguity of the distributions of the two uncertainties, for which we formulate robust optimization models to guarantee the worst-case revenue for any values of the two uncertainties realized in designed uncertainty sets based on data. Via testing instances generated based on data of the AWS, we demonstrate the computational efficacy of the fixed-price policies obtained from the robust approach and compare its results with those from the nominal model under various parameter settings. In general, the robust approach yields better average revenue in the out-of-sample tests under limited resource capacity. We also demonstrate that under limited resource capacity, the proposed fixed-priced policies perform better than the commonly-used constant-price policies in out-of-sample tests, which shows the relevance of our approach.

For future research, one direction is to consider distributionally robust or robust continuous-time control and pricing models for the proposed revenue management problem with relevant uncertainty set design to reflect the practical applications more closely.

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References

Agresti, A. and Coull, B. A. (1998). Approximate is better than "exact" for interval estimation of binomial proportions. *The American Statistician*, 52(2):119–126.

Al-Roomi, M., Al-Ebrahim, S., Buqrais, S., and Ahmad, I. (2013). Cloud computing pricing models: a survey. *International Journal of Grid and Distributed Computing*, 6(5):93–106.

Araman, V. F. and Caldentey, R. (2009). Dynamic pricing for nonperishable products with demand learning. *Operations Research*, 57(5):1169–1188.

Bandi, C. and Bertsimas, D. (2012). Tractable stochastic analysis in high dimensions via robust optimization. *Mathematical programming*, 134(1):23–70.

Bertsimas, D., Gupta, V., and Kallus, N. (2017). Robust sample average approximation. *Mathematical Programming*, pages 1–66.

Besbes, O., Elmachtoub, A. N., and Sun, Y. (2019). Static pricing: Universal guarantees for reusable resources. *arXiv preprint arXiv:1905.00731*.

Besbes, O. and Zeevi, A. (2009). Dynamic pricing without knowing the demand function: Risk bounds and near-optimal algorithms. *Operations Research*, 57(6):1407–1420.

Besbes, O. and Zeevi, A. (2012a). Blind network revenue management. *Operations Research*, 60(6):1537–1550.

Besbes, O. and Zeevi, A. (2012b). On the (surprising) sufficiency of linear models for dynamic pricing with demand learning. Columbia Business School Research Paper.

Bitran, G. and Caldentey, R. (2003). An overview of pricing models for revenue management. *Manufacturing & Service Operations Management*, 5(3):203–229.

Breidert, C., Hahsler, M., and Reutterer, T. (2006). A review of methods for measuring willingness-to-pay. *Innovative Marketing*, 2(4):8–32.

Chen, M. and Chen, Z.-L. (2015). Recent developments in dynamic pricing research: multiple products, competition, and limited demand information. *Production and Operations Management*, 24(5):704–731.

Chen, S., Lee, H. L., and Moinzadeh, K. (2019). Pricing schemes in cloud computing: Utilization-based versus reservation-based. *Production and Operations Management*.

Chen, X. and Simchi-Levi, D. (2004). Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research*, 52(6):887–896.

Chen, Y. and Farias, V. F. (2015). Robust dynamic pricing with strategic customers. In *EC*, page 777.

Cheng, H. K., Li, Z., and Naranjo, A. (2016). Research note—cloud computing spot pricing dynamics: Latency and limits to arbitrage. *Information Systems Research*, 27(1):145–165.

Cheung, W. C., Simchi-Levi, D., and Wang, H. (2017). Dynamic pricing and demand learning with limited price experimentation. *Operations Research*, 65(6):1722–1731.

Chiang, W.-C., Chen, J. C., and Xu, X. (2007). An overview of research on revenue management: Current issues and future research. *International Journal of Revenue Management*, 1(1):97–128.

del Barrio, E., Giné, E., and Matrán, C. (1999). Central limit theorems for the wasserstein distance between the empirical and the true distributions. *Annals of Probability*, pages 1009–1071.

den Boer, A. V. (2014). Dynamic pricing with multiple products and partially specified demand distribution. *Mathematics of Operations Research*, 39(3):863–888.

Dokka Venkata Satyanaraya, T., Jacko, P., and Aslam, W. (2018). Non-parametric dynamic pricing: a non-adversarial robust optimization approach. Working paper.

Elmaghraby, W. and Keskinocak, P. (2003). Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science*, 49(10):1287–1309.

Federgruen, A. and Heching, A. (1999). Combined pricing and inventory control under uncertainty. *Operations Research*, 47(3):454–475.

Feng, Y. and Xiao, B. (2000). A continuous-time yield management model with multiple prices and reversible price changes. *Management Science*, 46(5):644–657.

Fournier, N. and Guillin, A. (2015). On the rate of convergence in wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, 162(3-4):707–738.

Gallego, G. and van Ryzin, G. (1994). Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Marketing Science*, 40(8):999–1020.

Gallego, G. and van Ryzin, G. (1997). A multiproduct dynamic pricing problem and its applications to network yield management. *Operations Research*, 45(1):24–41.

Gualandi, S. and Toscani, G. (2018). Call center service times are lognormal: A fokker–planck description. *Mathematical Models and Methods in Applied Sciences*, 28(08):1513–1527.

Harrison, J. M., Keskin, N. B., and Zeevi, A. (2012). Bayesian dynamic pricing policies: Learning and earning under a binary prior distribution. *Management Science*, 58(3):570–586.

Haviv, M. and Randhawa, R. S. (2014). Pricing in queues without demand information. *Manufacturing & Service Operations Management*, 16(3):401–411.

Javadi, B., Thulasiram, R. K., and Buyya, R. (2013). Characterizing spot price dynamics in public cloud environments. *Future Generation Computer Systems*, 29(4):988–999.

Kachani, S. and Perakis, G. (2006). Fluid dynamics models and their applications in transportation and pricing. *European Journal of Operational Research*, 170(2):496–517.

Kashef, M. M., Uzbekov, A., Altmann, J., and Hovestadt, M. (2013). Comparison of two yield management strategies for cloud service providers. In *International Conference on Grid and Pervasive Computing*, pages 170–180. Springer.

Keskin, N. B. and Zeevi, A. (2014). Dynamic pricing with an unknown demand model: Asymptotically optimal semi-myopic policies. *Operations Research*, 62(5):1142–1167.

Lei, Y. and Jasin, S. (2016). Real-time dynamic pricing for revenue management with reusable resources and deterministic service time requirements. Available at SSRN: `https://ssrn.com/abstract=2816718orhttp://dx.doi.org/10.2139/ssrn.2816718`.

Levi, R. and Radovanović, A. (2010). Provably near-optimal lp-based policies for revenue management in systems with reusable resources. *Operations Research*, 58(2):503–507.

Lim, A. E. and Shanthikumar, J. G. (2007). Relative entropy, exponential utility, and robust dynamic pricing. *Operations Research*, 55(2):198–214.

Liu, Y. and Cooper, W. L. (2015). Optimal dynamic pricing with patient customers. *Operations Research*, 63(6):1307–1319.

Maglaras, C. and Meissner, J. (2006). Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing & Service Operations Management*, 8(2):136–148.

Massart, P. (1990). The tight constant in the dvoretzky-kiefer-wolfowitz inequality. *The Annals of Probability*, 18(3):1269–1283.

McGill, J. I. and van Ryzin, G. J. (1999). Revenue management: Research overview and prospects. *Transportation Science*, 33(2):233–256.

Papier, F. and Thonemann, U. W. (2010). Capacity rationing in stochastic rental systems with advance demand information. *Operations research*, 58(2):274–288.

Perakis, G. and Roels, G. (2010). Robust controls for network revenue management. *Manufacturing & Service Operations Management*, 12(1):56–76.

Popescu, I. and Wu, Y. (2007). Dynamic pricing strategies with reference effects. *Operations Research*, 55(3):413–429.

Püschel, T., Schryen, G., Hristova, D., and Neumann, D. (2015). Revenue management for cloud computing providers: Decision models for service admission control under non-probabilistic uncertainty. *European Journal of Operational Research*, 244(2):637–647.

Rothstein, M. (1971). An airline overbooking model. *Transportation Science*, 5(2):180–192.

Shiryayev, A. (1992). On the empirical determination of a distribution law. In *Selected Works of AN Kolmogorov*, pages 139–146. Springer.

Talluri, K. and van Ryzin, G. (2005). *The Theory and Practice of Revenue Management*. Springer, New York.

Wets, R. (2002). Stochastic programming models: Wait-and-see versus here-and-now. In Greengard, C. and Ruszczynski, A., editors, *Decision Making Under Uncertainty*, pages 1–15. Springer.

Wilson, E. B. and Wilson, E. B. (1927). Probable inference, the law of succession and statistical inference. *Journal of the American Statistical Association*, 22(158):209–212.

Xu, H. and Li, B. (2013). Dynamic cloud pricing for revenue maximization. *IEEE Transactions on Cloud Computing*, 1(2):158–171.

Zhao, W. and Zheng, Y.-S. (2000). Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management Science*, 46(3):375–388.

APPENDIX

A Reformulation of Model (14) With Discrete Prices

Consider the case when every product has a discrete set of allowable prices and for notation simplicity, let $\mathcal{P}_i = \mathcal{P}_i^d, \ \forall i = 1, \ldots, n$. We define binary variables $u_i(t, k) \in \{0, 1\}$ for all $t = 1, \ldots, T$, $k = 1, \ldots, K_i$ to model the pricing decisions, such that $u_i(t, k) = 1$ if the price p_i^t is set to $p_{i,k}$ in period t, and $u_i(t, k) = 0$ otherwise.

The deterministic model (14) can be reformulated as follows:

$$\max \sum_{t=1}^{T}\sum_{i=1}^{n}\bar{r}_i^t \sum_{k=1}^{K_i} \underline{a}_i(p_{i,k})p_{i,k}u_i(t,k) \tag{A-1a}$$

$$\text{s.t.} \sum_{i=1}^{n} c_i \left(\bar{r}_i^t \sum_{k=1}^{K_i} a_i(p_{i,k})u_i(t,k) + \max_{\beta_i(\cdot)\in\mathcal{U}_i^b} \sum_{s=1}^{t-1} \bar{r}_i^s \sum_{k=1}^{K_i} \bar{a}_i(p_{i,k})u_i(s,k)\beta_i(t-s) \right) \leq C, \ \forall\, t=1,\dots,T, \tag{A-1b}$$

$$\sum_{k=1}^{K_i} u_i(t,k) = 1, \quad \forall\, i=1,\dots,n,\ t=1,\dots,T, \tag{A-1c}$$

$$u_i(t,k) \in \{0,1\}, \quad \forall\, i=1,\dots,n,\ k=1,\dots,K_i,\ t=1,\dots,T. \tag{A-1d}$$

The additional constraints $(A-1c)$ of binary decision variables $u_i(k,t)$ ensure that we select one price for each product in each period from the discrete allowable price sets. Here we only need to compare K_i values, $a_i(p_{i,k})$ for $k=1,\dots,K_i$, instead of optimizing over the functions $a_i(\cdot)$ for $i=1,\dots,n$.

Proposition 3. *When $\mathcal{P}_i = \mathcal{P}_i^d$, given the proposed uncertainty sets $\mathcal{U}_i^a$, $\mathcal{U}_i^b$, the robust deterministic relaxation problem* (11) *with discrete prices can be reformulated as follows:*

$$\max \sum_{t=1}^{T}\sum_{i=1}^{n}\bar{r}_i(t) \sum_{k=1}^{K_i} p_{i,k}\underline{a}_i(p_{i,k})u_i(t,k) \tag{A-2a}$$

$$\begin{aligned} \text{s.t.} \sum_{i=1}^{n} c_i \Bigg(& \bar{r}_i(t)\sum_{k=1}^{K_i}\bar{a}_i(p_{i,k})u_i(t,k) + \Gamma_i^W w_i(t) + \Gamma_i^+(s)y_i^+(t,s) + \Gamma_i^-(s)y_i^-(t,s) + \\ & \sum_{s=1}^{\tau_i(t)} \hat{\beta}_i(s)\left[x_i^+(t,s) - x_i^-(t,s) + y_i^+(t,s) - y_i^-(t,s)\right] \Bigg) \leq C, \ \forall\, t=1,\dots,T, \end{aligned} \tag{A-2b}$$

$$q_i(t,s) = \bar{r}_i(t-s)\sum_{k=1}^{K_i}\bar{a}_i(p_{i,k})u_i(t-s,k), \ \forall\, t=1,\dots,T,\ s=1,\dots,\tau_i(t), \tag{A-2c}$$

$$\sum_{k=1}^{K_i} u_i(t,k) = 1, \quad \forall\, i=1,\dots,n,\ t=1,\dots,T, \tag{A-2d}$$

$$\begin{aligned} & \left[x_i^+(t,s) - x_i^-(t,s)\right] + \left[y_i^+(t,s) - y_i^-(t,s)\right] - z_i(t,s) + z_i(t,s-1) \geq q_i(t,s), \\ & \qquad \forall\, t=1,\dots,T,\ s=1,\dots,\tau_i(t), \end{aligned} \tag{A-2e}$$

$$w_i(t) \geq x_i^+(t,s) + x_i^-(t,s), \quad \forall\, t=1,\dots,T,\ s=1,\dots,\tau_i(t), \tag{A-2f}$$

$$w_i(t) \geq 0, \quad \forall t=1,\dots,T,\ i=1,\dots,n, \tag{A-2g}$$

$$\begin{aligned} & x_i^+(t,s), x_i^-(t,s), y_i^+(t,s), y_i^-(t,s), z_i(t,s) \geq 0,\ z_i(t,0) = z_i(t,\tau_i(t)) = 0, \\ & \qquad \forall\, t=1,\dots,T,\ i=1,\dots,n,\ s=1,\dots\tau_i(t), \end{aligned} \tag{A-2h}$$

$$u_i(t,k) \in \{0,1\}, \quad \forall\, i=1,\dots,n,\ k=1,\dots,K_i,\ t=1,\dots,T, \tag{A-2i}$$

where $\tau_i(t) = \min\{t-1, \tau_i^{\max}\}$, $\Gamma_i^+(t) = \min\{1-\hat{\beta}_i(t), \Gamma_i^K, \Gamma_i^W\}$ and $\Gamma_i^-(t) = \min\{\hat{\beta}_i(t), \Gamma_i^K, \Gamma_i^W\}$ for $t=1,\dots,\tau_i^{\max}$.

The proof of Proposition 3 is identical to the one of Proposition 2, except that for each product i at period t, we replace the price variables p_i^t with the sum of the products of the discrete K_i prices and binary variables $u_i(t, k)$.