# Stability test and dominant eigenvalues computation for second-order linear systems with multiple time-delays using receptance method

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**Abstract:** Stability analysis and dominant eigenvalues computation for second-order linear systems with multiple time-delays are addressed by using a reduced characteristic function and the associated characteristic matrix comprised of measured open-loop receptances. This reduced characteristic function is derived from the original characteristic function of the second-order time delayed systems based on a reasonable assumption that eigenvalues of the closed-loop system are distinct from those of the open-loop system. Then a contour integral is used to test the stability and provide the stability chart with respect to different displacement and velocity feedback time-delays, and a Newton-type method to compute the dominant eigenvalues via this characteristic function. The proposed approach also utilizes the spectrum distribution features of the retarded time-delay systems. Finally, numerical examples are given to illustrate the effectiveness of the proposed approach.

**Keywords:** Second-order system; Receptance; Multiple time-delays; Retarded time-delay system; Stability; Dominant eigenvalue

## 1. Introduction

The study of time-delay effects on active vibration control of various mechanical and structural systems has attracted increasing attention in the last decade. In the presence of delays, primarily due to the time it takes in the feedback loop to acquire and process the states information, and to execute the control action, the controlled systems may suffer from significant performance degradation or even destabilization [1-3]. However, time delays have also been shown to be beneficial in many cases, such as stabilizing effect of feedback with delay for unstable systems and using delays to improve existing vibration control techniques [4, 5]. Mathematically, a delay brings extra dynamics into the system and hence makes the control design and analysis more complex.

The stability as well as the estimation of eigenvalues (or characteristic roots) of linear time invariant systems with time delays, modelled by first-order delay differential equations, has been extensively studied in the past [6], and considerable effort has been made to extend existing control techniques to time delayed systems, e.g. [7,8]. For mechanical and structural systems whose equations of motion are naturally formulated in the second-order setting, the time-delay effects on their closed-loop controlled systems are generally analysed by means of first-order approaches without taking advantage of the well-known benefits of the design and analysis directly available in second-order models [9,10]. To overcome the shortcomings of the above approaches, some work has been devoted to the design and analysis of second-order time delayed systems without using a-priori transformation to first-order state space models, e.g., full or partial eigenvalue assignment by the single input and multi-input [11-13]. However, these approaches require good knowledge of mass, damping and stiffness matrices, which undoubtedly involves errors in relation to practical systems and quite often is not available.

Another interesting and useful scheme still in development is to use nonparametric models, i.e., measured receptances, which was originally developed to design linear vibration control [14-16]. There are some works on partial eigenvalue assignment of the second-order time delayed systems using the receptance method [17-22]. Additionally, a simple stability criterion for second-order systems with time-varying delay based on the receptance approach was presented in [23]. The proposed approach used the Small-Gain Theorem and the closed-loop receptance which is directly related to the open-loop one by using the Sherman–Morrison–Woodbury formula.

This paper proposes a stability-testing formula and an approach of computing the dominant eigenvalues for second-order systems with multiple constant time-delays based on the open-loop receptance matrix. Firstly, the reduced characteristic function $f_{\mathrm{m}}(\lambda)$ and the associated characteristic matrix $\mathbf{J}_{\mathrm{m}}(\lambda)$ of the closed-loop system are derived. They have the same eigenvalues (or characteristic roots) as those of the closed-loop system and involve only control gains, time-delay parameters and the measured open-loop receptances at the sensor/actuator coordinates. Utilizing the spectrum distribution features of the resultant retarded time-delay systems, a stability-testing formula in the form of a contour integration of $f_{\mathrm{m}}(\lambda)$ based on the Argument Principle is presented, and a root-finding algorithm of Newton-type, combined with a search strategy to provide an accurate initial guess, is used to compute the dominant eigenvalues of the closed-loop system in a rectangular region centred at the origin of the complex plane. Furthermore, a two-dimensional (2D) map, known as a stability chart, can be obtained using the testing formula. This chart reveals the effects of delay parameters on stability [24].

The paper is organized as follows. In Section 2, the system involved is described. A stability testing formula and the computational procedure of the dominant eigenvalues are presented in Sections 3 and 4, respectively. In Section 5, numerical examples are provided to demonstrate the proposed approach. Conclusions are finally drawn in Section 6.

## 2. Basic statement

A linear second-order controlled system with time-delay is described by:

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{B}\mathbf{u}(t,\tau_1,\tau_2)+\mathbf{f}(t) \tag{1}$$

$$\mathbf{u}(t,\tau_1,\tau_2)=-\mathbf{G}_1\mathbf{y}(t-\tau_1)-\mathbf{G}_2\dot{\mathbf{y}}(t-\tau_2) \tag{2}$$

$$\mathbf{y}(t)=\mathbf{D}\mathbf{x}(t) \tag{3}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are known as the $n\times n$ mass, damping and stiffness matrices respectively; $\mathbf{u}$ is a control force vector and $\mathbf{f}$ is an external applied force vector; $\mathbf{B}$

is the $n \times p$ control input distribution matrix and $\mathbf{D}$ is the $m \times n$ measurement distribution matrix, and they are both elementary matrices; $\mathbf{G}_2$ and $\mathbf{G}_1$ are the $p \times m$ velocity and displacement feedback gain matrices respectively, and $p < n$, $m < n$. $\tau_1$ and $\tau_2$ are displacement and velocity feedback time-delays, respectively. Substituting (2), (3) into (1) gives

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\big(-\mathbf{G}_1\mathbf{D}\mathbf{x}(t-\tau_1) - \mathbf{G}_2\mathbf{D}\dot{\mathbf{x}}(t-\tau_2)\big) + \mathbf{f}(t) \tag{4}$$

Laplace transform of (4) gives

$$(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})\mathbf{x}(s) = -\mathbf{B}(\mathbf{G}_1\mathbf{D}\mathrm{e}^{-s\tau_1} + s\mathbf{G}_2\mathbf{D}\mathrm{e}^{-s\tau_2})\mathbf{x}(s) + \mathbf{f}(s) \tag{5}$$

$$[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} + \mathbf{B}(\mathbf{G}_1\mathbf{D}\mathrm{e}^{-s\tau_1} + s\mathbf{G}_2\mathbf{D}\mathrm{e}^{-s\tau_2})]\mathbf{x}(s) = \mathbf{f}(s) \tag{6}$$

Then the $n \times n$ *full* receptance matrices of the open-loop and closed-loop system are represented by

$$\mathbf{H}_0(s) = (s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})^{-1} \tag{7}$$

$$\mathbf{H}_\mathrm{c}(s) = [\mathbf{H}_0^{-1}(s) + \mathbf{B}(\mathbf{G}_1\mathbf{D}\mathrm{e}^{-s\tau_1} + s\mathbf{G}_2\mathbf{D}\mathrm{e}^{-s\tau_2})]^{-1} \tag{8}$$

The closed-loop receptance matrix $\mathbf{H}_\mathrm{c}(s)$ for the system with delay can be directly related to the open-loop receptance matrix $\mathbf{H}_0(s)$ by using the Sherman–Morrison–Woodbury formula as follows.

$$\mathbf{H}_\mathrm{c}(s) = \mathbf{H}_0(s) - \mathbf{H}_0(s)\,\mathbf{B}\big[\mathbf{I}_\mathrm{p} + (\mathbf{G}_1\mathrm{e}^{-s\tau_1} + s\mathbf{G}_2\mathrm{e}^{-s\tau_2})\mathbf{D}\mathbf{H}_0(s)\,\mathbf{B}\big]^{-1}(\mathbf{G}_1\mathrm{e}^{-s\tau_1} + s\mathbf{G}_2\mathrm{e}^{-s\tau_2})\mathbf{D}\mathbf{H}_0(s) \tag{9}$$

Besides, pre- and post-multiplying both sides of (9) by $\mathbf{D}$ and $\mathbf{B}$, respectively, yields

$$\mathbf{H}_\mathrm{cm}(s) = \mathbf{H}_\mathrm{0m}(s) - \mathbf{H}_\mathrm{0m}(s)\big[\mathbf{I}_\mathrm{p} + (\mathbf{G}_1\mathrm{e}^{-s\tau_1} + s\mathbf{G}_2\mathrm{e}^{-s\tau_2})\mathbf{H}_\mathrm{0m}(s)\big]^{-1}(\mathbf{G}_1\mathrm{e}^{-s\tau_1} + s\mathbf{G}_2\mathrm{e}^{-s\tau_2})\mathbf{H}_\mathrm{0m} \tag{10}$$

where $\mathbf{H}_\mathrm{cm}(s)$ and $\mathbf{H}_\mathrm{0m}(s)$ are the $m \times p$ '*measured*' closed-loop and open-loop receptance matrices, respectively; $\mathbf{H}_\mathrm{0m}(s) = \mathbf{D}\mathbf{H}_0(s)\mathbf{B}$.

The characteristic function of the second-order linear time-delay system (6) is given in the form

$$f(\lambda) = \det\big(\mathbf{Z}(\lambda)\big) = \det\big[\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} + \mathbf{B}\big(\mathbf{G}_1\mathbf{D}\mathrm{e}^{-\lambda\tau_1} + \lambda\mathbf{G}_2\mathbf{D}\mathrm{e}^{-\lambda\tau_2}\big)\big] \tag{11}$$

where $\mathbf{Z}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} + \mathbf{B}\left(\mathbf{G}_1\mathbf{D}\mathrm{e}^{-\lambda\tau_1} + \lambda\mathbf{G}_2\mathbf{D}\mathrm{e}^{-\lambda\tau_2}\right)$ is the so-called dynamic stiffness matrix of the closed-loop system (6), and $\mathbf{Z}(\lambda) = \mathbf{H}_\mathrm{c}^{-1}(\lambda)$ from (8). $f(\lambda)$ is a transcendental function containing some exponential terms, also called quasi-polynomial, which has an infinite number of roots. The roots of $f(\lambda)$ are also known as eigenvalues (poles or characteristic roots) of (6), whose distribution on the complex plane determines the stability and dynamic behaviour of (6).

## 3. Stability testing by using contour integral evaluation

Mikhailov-type stability criterion is based on the Argument Principle or Cauchy Theorem [25, 26]. Let $N$ be the number of the characteristic roots $\lambda$ of $f(\lambda)$ that lie in the open right-half complex plane. Assume that $f(\lambda)$ has no roots on the imaginary axis, it has been proven that $N = 0$ holds if and only if there is a sufficiently large $T > 0$ such that [25]

$$\int_0^T \mathrm{Re}\left(\frac{f'(\omega\mathrm{i})}{f(\omega\mathrm{i})}\right) \mathrm{d}\omega > \frac{(2n-1)\pi}{2} \tag{12}$$

where $\mathrm{Re}(z)$ represents the real part of $z$, $f'(\lambda)$ is the differentiation of $f(\lambda)$ with respect to $\lambda$. It follows from the Trace-Theorem of Devidenko [27] and (11) that

$$\frac{f'(\omega\mathrm{i})}{f(\omega\mathrm{i})} = \mathrm{tr}\left(\mathbf{Z}^{-1}(\omega\mathrm{i})\mathbf{Z}'(\omega\mathrm{i})\right) \tag{13}$$

where $\mathrm{tr}(\,)$ denotes the trace. In consideration of $\mathbf{Z}(\omega\mathrm{i})\mathbf{H}_\mathrm{c}(\omega\mathrm{i}) = \mathbf{I}$ and the following formula

$$\mathbf{Z}'(\omega\mathrm{i}) = -\mathbf{H}_\mathrm{c}^{-1}(\omega\mathrm{i})\mathbf{H}_\mathrm{c}'(\omega\mathrm{i})\mathbf{Z}(\omega\mathrm{i}) \tag{14}$$

Eq.(13) can be rewritten as

$$\frac{f'(\omega\mathrm{i})}{f(\omega\mathrm{i})} = \mathrm{tr}\left(-\mathbf{H}_\mathrm{c}'(\omega\mathrm{i})\mathbf{H}_\mathrm{c}^{-1}(\omega\mathrm{i})\right) \tag{15}$$

Then Mikhailov-type stability criterion (12) can be reformulated in terms of the closed-loop receptance matrix (9) as follows.

$$\int_0^T \mathrm{Re}\left(\mathrm{tr}\left(-\mathbf{H}_\mathrm{c}'(\omega\mathrm{i})\mathbf{H}_\mathrm{c}^{-1}(\omega\mathrm{i})\right)\right) \mathrm{d}\omega > \frac{(2n-1)\pi}{2} \tag{16}$$

Since numerical calculations of $\mathbf{H}_\mathrm{c}'(\omega\mathrm{i})$ and $\mathbf{H}_\mathrm{c}^{-1}(\omega\mathrm{i})$ are inconvenient when the size of these matrices is large, the stability criteria (16) is limited in practical use.

Now the reduced form of characteristic function (11) and the corresponding characteristic equation are presented. Without loss of generality, **assume that the eigenvalues of the closed-loop system are distinct from those of the open-loop system**. Suppose that $\mathbf{A}$ and $\mathbf{Q}$ are nonsingular matrices of appropriate orders. The following determinant formula is given as

$$|\mathbf{A}+\mathbf{EQF}| = |\mathbf{A}||\mathbf{Q}||\mathbf{Q}^{-1}+\mathbf{FA}^{-1}\mathbf{E}| \tag{17}$$

Substituting $\mathbf{A}=\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}$, $\mathbf{E}=\mathbf{B}$, $\mathbf{F}=\mathbf{G}_1\mathbf{D}\mathrm{e}^{-\lambda\tau_1}+\lambda\mathbf{G}_2\mathbf{D}\mathrm{e}^{-\lambda\tau_2}$ and $\mathbf{Q}=\mathbf{I}_\mathrm{p}$ into (17), then $f(\lambda)$ in (11) can be rewritten as follows.

$$f(\lambda)=\det[\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K}]\det\left[\mathbf{I}_\mathrm{p}+\left(\mathbf{G}_1\mathrm{e}^{-\lambda\tau_1}+\lambda\mathbf{G}_2\mathrm{e}^{-\lambda\tau_2}\right)\mathbf{H}_{0\mathrm{m}}(\lambda)\right] \tag{18}$$

with $\mathbf{H}_{0\mathrm{m}}(\lambda)=\mathbf{DH}_0(s)\mathbf{B}=\mathbf{D}(\lambda^2\mathbf{M}+\lambda\mathbf{C}+\mathbf{K})^{-1}\mathbf{B}$. The formula (18) holds for any $\lambda$ except for finite eigenvalues $\lambda_{0i}$ ($i$=1,2,...,2$n$) of the open-loop system. Thus the characteristic roots of (18), i.e. eigenvalues of the closed-loop system (6), satisfy the following reduced characteristic equation

$$f_\mathrm{m}(\lambda)=\det\left[\mathbf{I}_\mathrm{p}+\left(\mathbf{G}_1\mathrm{e}^{-\lambda\tau_1}+\lambda\mathbf{G}_2\mathrm{e}^{-\lambda\tau_2}\right)\mathbf{H}_{0\mathrm{m}}(\lambda)\right]=0 \tag{19}$$

with the following reduced characteristic matrix

$$\mathbf{J}_\mathrm{m}(\lambda)=\mathbf{I}_\mathrm{p}+\left(\mathbf{G}_1e^{-\lambda\tau_1}+\lambda\mathbf{G}_2e^{-\lambda\tau_2}\right)\mathbf{H}_{0\mathrm{m}}(\lambda) \tag{20}$$

Solving eigenvalues of (6) now become finding roots of $f_\mathrm{m}(\lambda)$ in the complex plane, and this can also be considered a nonlinear eigenvalue problem of $\mathbf{J}_\mathrm{m}(\lambda)$ which depends nonlinearly on a single scalar parameter $\lambda$. Either of them is a non-trivial task. It is impossible and unnecessary to determine every root of $f_\mathrm{m}(\lambda)$ due to its transcendental nature. Nonetheless, for retarded time-delay systems (the majority of vibration suppression problems of closed-loop control systems belong to this category, e.g. (1)-(3)), their spectrum distributions have the following 'nice' features [3,28,29]: if there exists a sequence $\{\lambda_k\}$ of eigenvalues of the systems such that $\lim\limits_{k\to\infty}|\lambda_k|\to+\infty$, then $\lim\limits_{k\to\infty}\mathrm{Re}(\lambda_k)\to-\infty$, and thus there are only a finite number of eigenvalues in any given right-half complex plane. This also implies that the eigenvalues with high frequencies tend to be far off the imaginary axis in the left half complex plane. Furthermore, the dominant eigenvalues (i.e. the rightmost characteristic roots in some sense) which lie closest to the imaginary axis have the small modulus and low frequencies, and the overall dynamics and the stability of the retarded system is mainly dominated by these eigenvalues. This is a very interesting property which has consequences in the following stability investigation.

When the open-loop system is stable, the poles of $\mathbf{H}_{0\mathrm{m}}(\lambda)$ all lie in the left half complex plane. In the finite size region of the right half complex plane, $f_{\mathrm{m}}(\lambda)$ can be considered to have no poles within it in view of (19) and (20). Assume also that $f_{\mathrm{m}}(\lambda)$ has no roots on the imaginary axis. Therefore, using the same formulas as (13) and the argument principle based on Cauchy's theorem, a stability testing formula based on the contour integration in terms of the open-loop measured receptance matrix $\mathbf{H}_{0\mathrm{m}}(\lambda)$ yields

$$N = (2\pi\mathrm{i})^{-1} \oint_{\partial C} \mathrm{tr}\left(\mathbf{J}_{\mathrm{m}}^{-1}(\lambda)\mathbf{J}_{\mathrm{m}}^{'}(\lambda)\right) \mathrm{d}\lambda \tag{21}$$

where $\partial C$ is the closed semicircle centred at the origin with a proper radius $R$ in the right half complex plane, as shown in Fig.1. $\mathbf{J}_{\mathrm{m}}^{'}(\lambda)$ has the following form

$$\mathbf{J}_{\mathrm{m}}^{'}(\lambda) = \left(\mathbf{G}_2\mathrm{e}^{-\lambda\tau_2} - \tau_1\mathbf{G}_1 e^{-\lambda\tau_1} - \lambda\tau_2\mathbf{G}_2\mathrm{e}^{-\lambda\tau_2}\right)\mathbf{H}_{0\mathrm{m}}(\lambda) + \left(\mathbf{G}_1 e^{-\lambda\tau_1} + \lambda\mathbf{G}_2 e^{-\lambda\tau_2}\right)\mathbf{H}_{0\mathrm{m}}^{'}(\lambda) \tag{22}$$

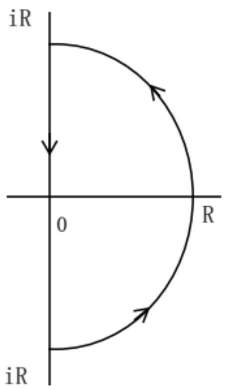



**Fig.1**. The integral contour $\partial C$

The testing formula (21) can determine the number $N$ of unstable dominant eigenvalues within contour $\partial C$, which can be calculated simply by using common numerical integration formulas. If the closed-loop system without delays (i.e. $\tau_1 = \tau_2 = 0$) and with feedback gains $\mathbf{G}_1$ and $\mathbf{G}_2$ is stable (this is not a necessary premise to use (21)), then some stable dominant eigenvalues would cross the imaginary axis to become the unstable dominant eigenvalues (if any) as the values of time-delays of the closed-loop system increase. Therefore, the testing formula (21) can capture the unstable dominant eigenvalues (if any) with small moduli and low frequencies provided a properly chosen radius $R$ is determined. Besides, when $N$ in (21) is zero, no unstable dominant eigenvalues exist within contour $\partial C$ and the closed-loop system with the specific time-delays and the feedback gains $\mathbf{G}_1$ and $\mathbf{G}_2$ can be deemed to be stable. Obviously, the correct determination of radius $R$ is a crucial factor in using the testing formula (21). As a rule of thumb, radius $R$ can be chosen to be slightly larger than the largest modulus among dominant eigenvalues of the open-loop system.

**Remarks:** (i) the dimension of $\mathbf{J}_{\mathrm{m}}(\lambda)$ and $\mathbf{H}_{0\mathrm{m}}(\lambda)$ may be significantly smaller than the dimension of the original system, $n$. (ii) the stability or instability of the closed-loop system can be determined directly by using the testing integral (21), and its implementation does not become more complex in the presence of multiple time-delay parameters. (iii) $\mathbf{H}_{0\mathrm{m}}'(\lambda)$ in $\mathbf{J}_{\mathrm{m}}'(\lambda)$ of the testing integral (21) can be approximated by using numerical difference methods in practice. To reduce computational cost and yet keep high computational accuracy, 8–16 contour points suffice for the numerical integration of (21) using a high-order numerical integration scheme such as Gauss–Legendre quadrature.

## 4. Computing dominant eigenvalues based on $f_{\mathrm{m}}(\lambda)$

For a given initial guess $\lambda^{(0)}$, Newton's method can be used to find the roots of the reduced characteristic equation $f_{\mathrm{m}}(\lambda)$ in any subregion of the complex plane as follows.

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{f_{\mathrm{m}}(\lambda^{(k)})}{f_{\mathrm{m}}'(\lambda^{(k)})}, k = 0,1,2,\ldots \tag{23}$$

where $f_{\mathrm{m}}(\lambda) = \det(\mathbf{J}_{\mathrm{m}}(\lambda))$ in (20). Directly using the reduced characteristic matrix $\mathbf{J}_{\mathrm{m}}(\lambda)$ in each iteration step, several algorithmic variants have been developed for (23), including, e.g., the Newton-trace iteration [30] and the Newton-QR iteration [31].

The former rewrites the Newton iteration (23) as

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{1}{\mathrm{tr}\left(\mathbf{J}_{\mathrm{m}}^{-1}(\lambda^{(k)})\mathbf{J}_{\mathrm{m}}^{'}(\lambda^{(k)})\right)}, k = 0,1,2,\dots \tag{24}$$

Note that only the diagonal entries of $\mathbf{J}_{\mathrm{m}}^{-1}(\lambda^{(k)})\mathbf{J}_{\mathrm{m}}^{'}(\lambda^{(k)})$ are need in each iteration.

The Newton-type method is an efficient and accurate method for roots-finding provided that good initial guesses $\lambda^{(0)}$ are made. To make good initial guesses for the simple roots of $f_{\mathrm{m}}(\lambda)$ located in the complex plane region $\mathbb{D}$ with the boundaries $\alpha_{\min} < \mathrm{Re}(\mathbb{D}) < \alpha_{\max}$ and $\omega_{\min} < \mathrm{Im}(\mathbb{D}) < \omega_{\max}$, a search strategy for good initial guesses is given as follows.

**Algorithm 3.1.** Searching good initial guesses

1. a regular mesh grid covering the region $\mathbb{D}$ is presented as

$$\Pi = \begin{bmatrix} \alpha_0 + \omega_0 \mathrm{i} & \cdots & \alpha_{k_{max}} + \omega_0 \mathrm{i} \\ \vdots & \alpha_k + \omega_l \mathrm{i} & \vdots \\ \alpha_0 + \omega_{l_{max}} \mathrm{i} & \cdots & \alpha_{k_{max}} + \omega_{l_{max}} \mathrm{i} \end{bmatrix} \tag{25}$$

$\alpha_k = \alpha_{min} + k\Delta, k = 0, 1, \dots, k_{max}$, $\omega_l = \omega_{min} + l\Delta, l = 0, 1, \dots, l_{max}$ with a grid step $\Delta$.

2. The absolute values $|\mathrm{Re}(f_{\mathrm{m}}(\lambda_{kl}))|$ and $|\mathrm{Im}(f_{\mathrm{m}}(\lambda_{kl}))|$ at each grid point of (25) are evaluated. If $|\mathrm{Re}(f_{\mathrm{m}}(\lambda_{kl}))| < \beta$ and $|\mathrm{Im}(f_{\mathrm{m}}(\lambda_{kl}))| < \beta$ (a given constant, $0.0 < \beta < 1.0$), then the location of the point $(\alpha_k, \omega_l)$ is labelled in an indexing matrix $\mathbf{I}(k, l)$ with $|f_{\mathrm{m}}(\lambda_{kl})|$ being its entry, else with 1 being its entry.
3. The region $\mathbb{D}$ is partitioned by centring the minimum values in the different position of indexing matrix $\mathbf{I}(k, l)$. The grid points in each subdivision region of the region $\mathbb{D}$ can be randomly chosen as multi-starting guesses for finding roots of $f_{\mathrm{m}}(\lambda)$ in the subregion by using a Newton-type method such as (24).

It should be noted that the proposed gridding procedure here is similar to that of the QPmR mapping algorithm [32]. The QPmR finds roots by locating intersection points on the zero level curves $\mathrm{Re}(f_{\mathrm{m}}(\lambda_{kl})) = 0$ and $\mathrm{Im}(f_{\mathrm{m}}(\lambda_{kl})) = 0$ and Newton's iteration method is applied to increase the accuracy of each root. In this paper, however, the gridding procedure is applied to find basins of convergence for

Newton-type methods based on $|\mathrm{Re}(f_{\mathrm{m}}(\lambda_{kl}))|<\beta$ and $|\mathrm{Im}(f_{\mathrm{m}}(\lambda_{kl}))|<\beta$. It was also suggested [32] that the grid step $\Delta = \frac{\pi}{10\,\tau_{\max}}$ would guarantee a sufficiently dense grid ($\tau_{\max}$ is the maximal value among multiple time-delays) and a grid adaptation rule was presented. Additionally, the deflation procedure may be necessary when there are regions with very close roots, in order to prevent the Newton iteration converging to an already computed root [33].

The dominant roots of $f_{\mathrm{m}}(\lambda)$, i.e. the dominant eigenvalues of the closed-loop system (1)-(3), in only the upper quadrant of the rectangular region $\mathbb{D}$ centred at the origin of the complex plane need to be computed because complex roots form conjugate pairs. It is also worth mentioning that solving the nonlinear eigenvalue problem of the reduced characteristic matrix $\mathbf{J}_{\mathrm{m}}(\lambda)$ provides an alternative approaches for computation of the dominant eigenvalues. This could be done by using the contour integration approach and sampling via rational interpolation approach. For details, one can refer to [34,35] and references therein.

## 5. Numerical examples

An example is considered with the following system matrices [12],

$$\mathbf{M} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},\ \mathbf{K} = 100\begin{bmatrix} 15 & -5 & 0 \\ -5 & 6 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Let $\mathbf{B} = \mathbf{I}_{3\times 2}$ and $\mathbf{D} = \mathbf{I}_{3\times 3}$. Its open-loop eigenvalues are: $-0.0344 \pm 2.6775$i, $-0.1366 \pm 6.3592$i and $-0.2290 \pm 13.1266$i.

*Case 1*: $\mathbf{G}_1$ and $\mathbf{G}_2$ are taken from [12] except that the original entry 6.2047 is changed to 6.1047 in $\mathbf{G}_1$ and –0.4836 to –0.5836 in $\mathbf{G}_2$. The reason for the slight changes on feedback matrices of [12] is that the original feedback matrices are determined from the algorithm of partial eigenvalue assignment, but concerned major assumption in this paper is that eigenvalues of the closed-loop system are distinct from those of the open-loop system. Then feedback matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ are given by

$$\mathbf{G}_1 = \begin{bmatrix} 2.0078 & 6.1142 & 22.7881 \\ 6.1047 & 18.8949 & 70.4229 \end{bmatrix},\ \mathbf{G}_2 = \begin{bmatrix} -0.5836 & -1.3621 & -4.6610 \\ -1.4946 & -4.2094 & -14.4039 \end{bmatrix}.$$

The time delay in the displacement and velocity feedback loop is taken as $\tau_1 = \tau_2 = 0.1$. Fig.2 and Table 1 show part of the closed-loop eigenvalues and the dominant eigenvalues (in this and the following figures, (o) denotes a root that is obtained from a spectral method [36], and (+) the one that further be corrected using a Newton iteration). They are determined from a Matlab package for computing all the characteristic roots of delay differential equations in a given right half plane using the spectral method, which requires the knowledge of system matrices. There are a pair of unstable dominant eigenvalues 0.2818$\pm$2.9699i in this case.

Now the testing formula (21) is used to check the stability of this case. The radius of the integral contour $R$ = 7.0, which is slightly larger than the modulus of the second pair of open-loop eigenvalues –0.1366$\pm$6.3592i. The result is $N = 2.0012 - 0.0000\mathrm{i}$, which indicates a pair of unstable eigenvalues within the contour. This result is in good agreement with Fig.2.

Fig.3 and Table 1 show part of the closed-loop eigenvalues and the dominant eigenvalues for $\tau_1 = 1.0,\ \tau_2 = 0.5$. There are a pair of unstable dominant eigenvalues 0.2861$\pm$2.2977i too. The testing formula with $R$ = 7.0 gives $N = 2.0024 - 0.0000\mathrm{i}$. The two sets of results are also consistent.

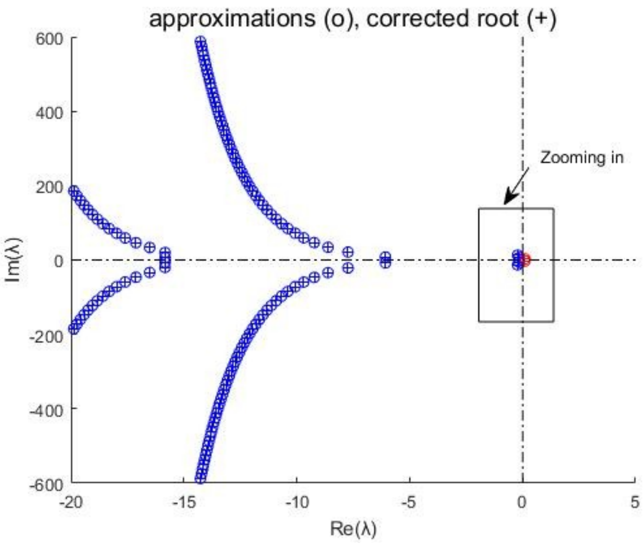
*Case 2*: $\mathbf{G}_1$ and $\mathbf{G}_2$ are taken as $\mathbf{G}_1 = 5 \times \mathbf{I}_{2\times 3}$, $\mathbf{G}_2 = 2 \times \mathbf{I}_{2\times 3}$. For $\tau_1 = 1.0$, $\tau_2 = 0.5$, the testing formula with $R$ = 7.0 gives $N = -0.0036 + 0.0000\mathrm{i}$, which means no eigenvalues within the contour. Fig.4 and Table 1 confirm this result.

*Case 3*: A stability chart can be obtained with respect to two delay parameters, $\tau_1$ and $\tau_2$ using the testing formula (20), as shown in Fig.5 (a). The time delay axis partition in computing the chart is $\tau_1$ and $\tau_2 = [0.0{:}\,0.05{:}\,3.0\,]$. $\mathbf{G}_1$ and $\mathbf{G}_2$ here are taken as the same as in Case 1. Fig.5 (b) is obtained from a Matlab package using the spectral method mentioned above. In spite of some subtle differences, they reveal nearly identical stable/unstable regions.

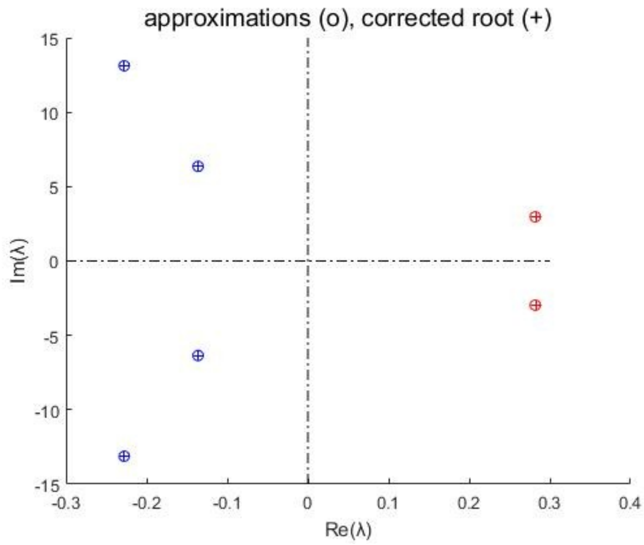
*Case 4*: $\mathbf{G}_1$ and $\mathbf{G}_2$ here are taken as the same as in Case 1. $\tau_1 = 1.0,\ \tau_2 = 0.5$. The dominant roots of $f_{\mathrm{m}}(\lambda)$ in the following region of the complex plane are to be searched and determined.

$$\mathbb{D}=\{\lambda \in \mathbb{C}:\ -5 \leq \mathrm{Re}(\lambda) \leq 5,\ -15 \leq \mathrm{Im}(\lambda) \leq 15\ \}$$

The search parameters are taken to be: grid step $\Delta = 0.02$ and $\beta = 0.7$. Fig.6 shows five pairs of possible convergence basins in the region $\mathbb{D}$ (marked by red arrows) that as initial guesses could allow solution of the roots of $f_{\mathrm{m}}(\lambda)$ using the Newton iteration. The subsequent computation exactly yields the corresponding five pairs of eigenvalues, i.e., 0.2861±2.2977i, −0.1534±6.3348i, −0.2244±13.1255i, −2.6994 ±5.6284i, −4.2480±11.5231i, as shown in Fig.7.

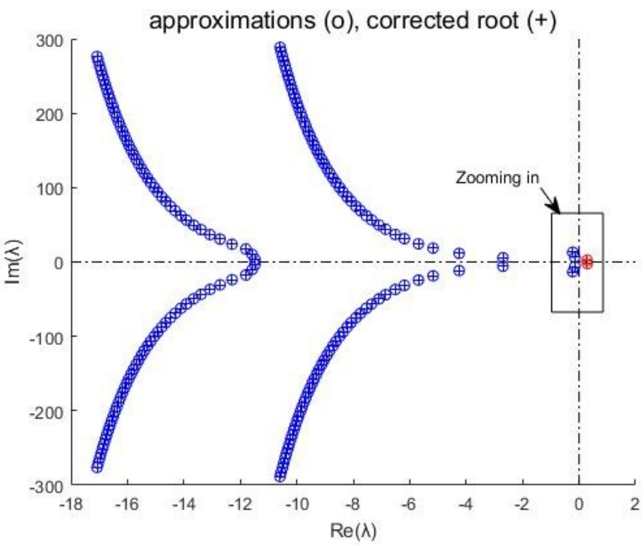



(a) part of closed-loop eigenvalues.

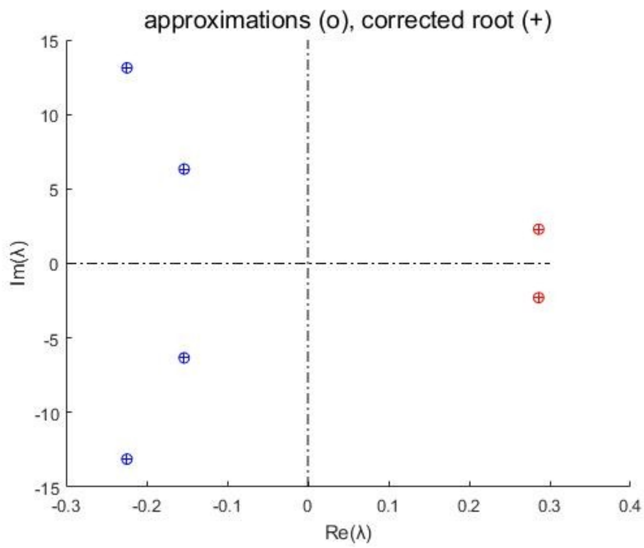



(b) a zoomed-in view of dominant eigenvalues of (a).

**Fig.2.** The closed-loop eigenvalues (Case 1 for $\tau_1 = \tau_2 = 0.1$).

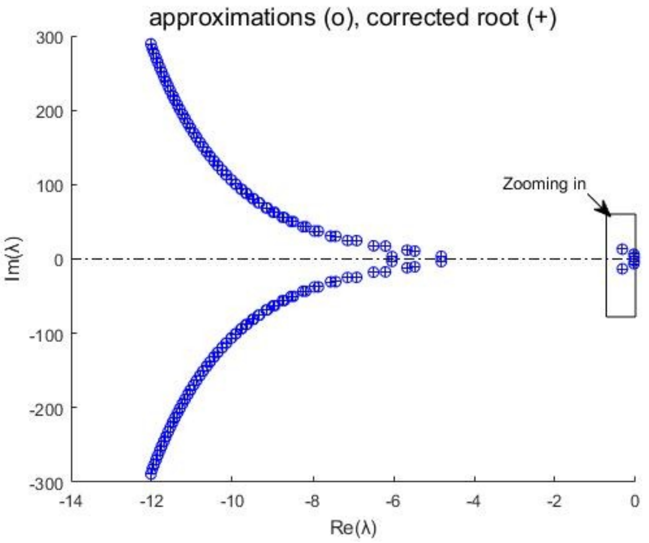



(a) part of closed-loop eigenvalues.

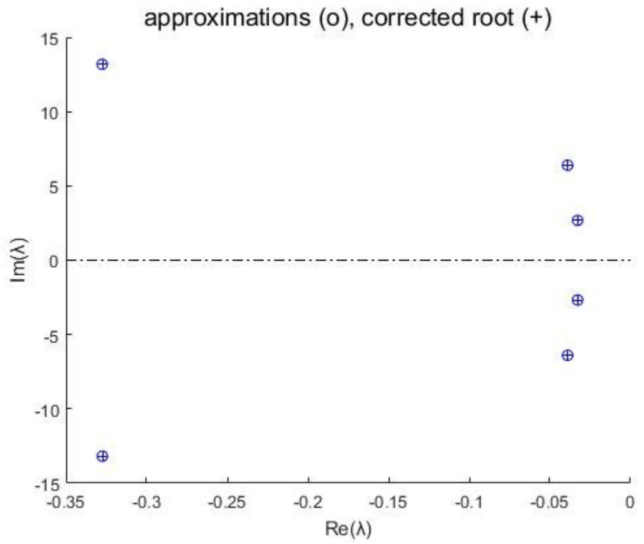



(b) a zoomed-in view of dominant eigenvalues of (a).

**Fig.3.** The closed-loop eigenvalues (Case 1 for $\tau_1 = 1.0$, $\tau_2 = 0.5$).

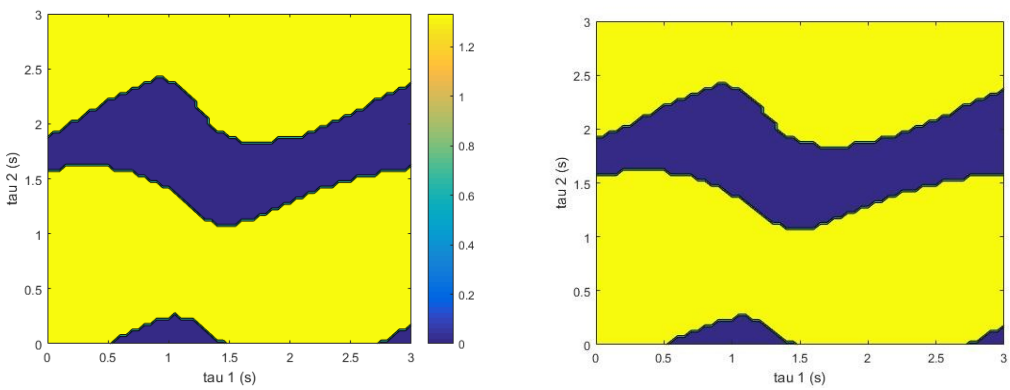



(a) part of closed-loop eigenvalues.




(b) a zoomed-in view of dominant eigenvalues of (a).

**Fig.4.** The closed-loop eigenvalues (Case 3).



(a) (b)

**Fig.5.** A stability chart with respect to two delay parameters, $\tau_1$ and $\tau_2$ (Case 3). Yellow regions: unstable; Dark blue regions: stable. (a) using the testing formula (21); (b) using a spectral method [36].

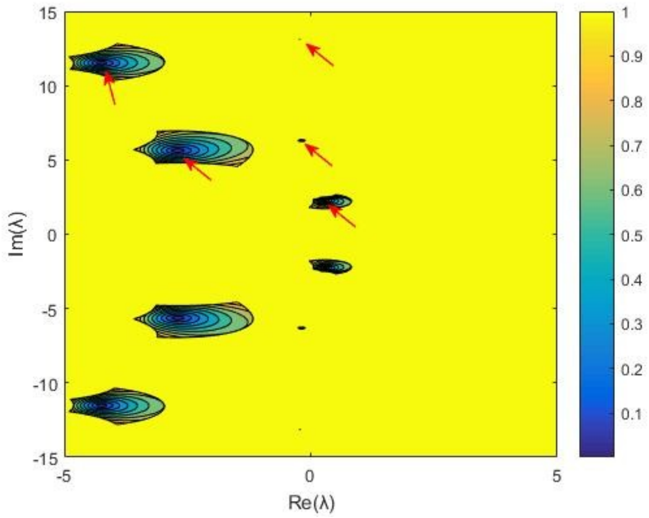



**Fig.6.** Five pairs of basins that provide good initial guesses of dominant eigenvalues in the region $\mathbb{D}$ (Case 4).

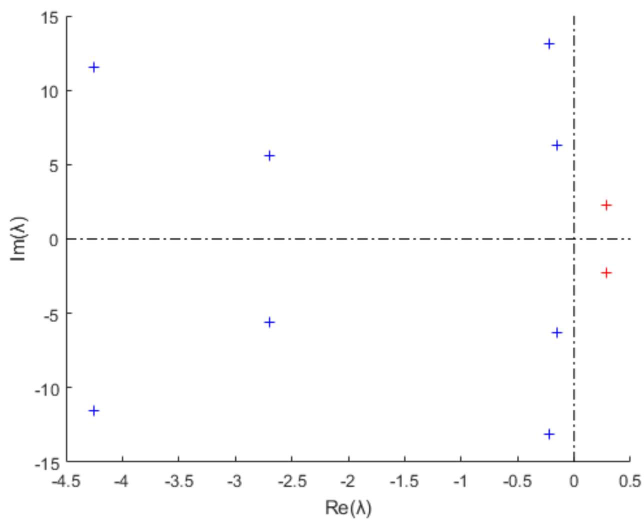



**Fig.7.** Five pairs of eigenvalues in the region $\mathbb{D}$ (Case 4).

**Table 1.**
First three pairs of closed-loop eigenvalues and the contour integration

| Case | Dominant eigenvalues | $N$ (Rounding off) |
|---|---|---|
| 1. | 0.2818 ± 2.9699i | 2 |
| $\tau_1 = \tau_2 = 0.1$ | −0.1362± 6.3585i | |
| | −0.2278±13.1227i | |
| 1. | 0.2861± 2.2977i | 2 |
| $\tau_1 = 1.0, \tau_2 = 0.5$ | −0.1534± 6.3348i | |
| | −0.2244±13.1255i | |
| 2. | −0.0397± 2.6871i | 0 |
| $\tau_1 = 1.0,\ \tau_2 = 0.5$ | −0.1909± 6.4440i | |
| | −0.2380± 13.2337i | |

## 6. Conclusions

The present paper proposes an approach for testing stability and computing the dominant eigenvalues of second-order linear systems with multiple time-delays. It is based on a reduced characteristic function and the associated characteristic matrix, and measured open-loop receptances (hence there is no need to know the system matrices). This approach has other advantages such as fast convenience in computation and easy implementation. Based on the current work, further will be directed to developing a more rigorous formula for stability test and the feedback control design, e.g., via assignment of dominant eigenvalues of retarded systems.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