Ruin Probability in Dependent Risk Models

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To my parents

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Abstract

This thesis considers one of the most active topics in actuarial mathematics literature, deriving the probability of ruin for the enlarged risk models. In this thesis, the classical Cramér-Lundberg risk process will be extended by several dependent risk processes, including the time dependent risk process, the claim dependent risk process and the surplus dependent risk process. Under these dependent model settings, we investigated the changes in the probabilities of ruin, which provides us with an approach of how to adapt classical risk theory to the contemporary complex financial market. In particular, for claim dependent model, we focused on the discrete binomial risk process and mixed over the parameter of the probability of successful claims. In addition, the inhomogeneous type of Seal's formulae are derived to obtain the finite time ruin probability under the time dependent risk process, which is referred as the inhomogeneous Poisson process model and a number of specific Cox processes. Furthermore, we analyzed the surplus dependent reinsurance contracts and applied ruin probability as the risk measure, which is evaluated by the idea of two barriers model.

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Chapter 1

Introduction

Ruin theory has been one of the most active research topics in actuarial science for more than a hundred years. Considering the arrival time of the claims and their amounts over the insurer's income, we aim to measure the risk as the event/probability the surplus level of an insurer for insurance portfolios falls below 0, which is known as ruin probability. In order to mathematically formulate the behaviour of such a risk process, the classical insurance risk model describes the surplus of an insurance company, assuming that the insurer starts with a non-negative amount of initial capital, collects premiums and pays claims. Therefore, initial surplus, premiums received and claims paid, determine the classical risk model of an insurance surplus process. If the surplus level falls below zero, we say that ruin has occurred.

Due to the complicated nature of the financial market, risk models have been adapted to the real financial market with extreme challenges. Although they have developed and extended since the beginning of the 20^{th} century, they still need further improvements. In this thesis, the main work focuses on modelling and investigating the surplus process under time dependent, claim dependent and surplus dependent models and measuring the risk by their ruin probabilities. The main contributions in this thesis are summerized as below.

- The application of mixing distributions with the more convenient set up of the parameter of success probability over Gamma and Lévy (heavy-tailed) distributions under the setting of discrete binomial risk process. The ultimate probabilities of ruin are given in Corollaries 3.2.1 and 3.2.3 for Gamma and Lévy mixing distributions, respectively, by applying the method given in Proposition 3.2.1,

- The classical Seal's formulae are extended to fit the setting of the inhomogeneous Poisson process (Theorem 4.2.3) by applying the backward martingale (Theorem 4.2.2). In addition, using the fact in Theorem 4.2.5, the infinite time ruin probability for the inhomogeneous Poisson process is derived,

- Ultimate ruin probabilities for the Markov jump process and two states model are derived by applying the backward recursions (4.3.2) on the integro-differential equations and the total probability theorem,

- A number of surplus dependent partial injection models and reinsurance contracts are used to envaluate the risk by measuring their ruin probabilities by applying the idea of two barriers model (Chapter 5).

Literature reviews in the next two sections are organized by risk models and mathematical methods which are closely related to the main contributions in this thesis.

1.1 Literature review of risk models

The foundation of the risk process is the classical compound Poisson model with a constant intensity parameter. It has been described by the summation of initial capital and premium collected with the negative aggregate claim process. According to the assumptions of independence of claim occurrence times and sizes, the inter-arrival time is not related to the amount of the claim.

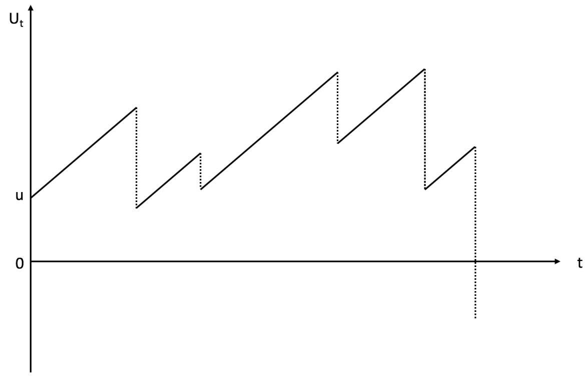


Figure 1.1: Surplus process for classical Cramér-Lundberg risk process

A basic "net profit condition" has to be satisfied, saying that on average the incoming premium rate is greater than the average paid claim rate. The most fundamental risk model was initially introduced in 1903 by Filip Lundberg and his work was republished by Cramér (1930) as the Cramér-Lundberg model, which considers the risk reserve process U_t with an initial capital $U_0 = u$ and a constant premium rate c, such that

$$U_t = u + ct - S_t, \tag{1.1}$$

where S_t, the aggregate claim process, represents the cumulative amount of claims up to time t. In particular, it is presented as

$$S_t = \sum_{i=1}^{N_t} X_i, \tag{1.2}$$

where N_t follows a Poisson point process with an intensity (defined as a constant intensity λ or a functional intensity $\lambda(t)$ or a stochastic process λ_t under the homogeneous, inhomogeneous or Cox Poisson process, respectively) and the X_i for $i = 1, 2, ...$ represent the claims, which are independent and identically distributed random variables independent of N_t, with density function $f_X(x)$, cumulative distribution function $F_X(x)$ and mean $\mu = \mathbb{E}[X_i] = \int_0^\infty x dF_X(x)$.

At the beginning, the primary investigation of risk theory was on the specific claim size distributions (Cramér, 1930). For instance, in the case of sub-exponentially distributed claim sizes, the ruin is asymptotically determined by a single extreme claim (Embrechts et al., 1993), it can be seen as the result of the heavy-tailed distributed claims (Asmussen and Albrecher, 2010). Thorin and Wikstad (1977) analysed the ruin problem when claims are log-normally distributed. Gerber et al. (1987) obtained the ruin probability for mixture Erlang-distributed claim by studying the severity of ruin, as well as the probability of ruin. In addition, Klüppelberg and Stadtmueller (1998) considered the large claims case, where the claim size distribution has a regularly varying tail and their results applied for instance to Pareto, log-gamma, certain Benktander and stable claim size distributions.

Later, Andersen (1957) developed the classical risk model into a more general framework. In the Sparre Andersen model, it is assumed that the inter-arrival times are independent and identically distributed random variables, in other words, the model permits non-exponential inter-arrival times but retains the Cramér-Lundberg assumptions on the claim sizes. For instance, it was shown that for any inter-arrival time distribution, the probability of ruin still has an exponential type of bound (Andersen, 1957). In addition, for particular distributions of the inter-arrival time, the literature of Dickson and Hipp (1998), Borovkov and Dickson (2008), Li and Garrido (2004) and Gerber and Shiu (2005) analysed either the asymptotic behaviours of the probability of ruin or moments of the time of ruin. Furthermore, the first investigation of the ruin problem when the occurrence of claims is described by a Cox process was due to (Ammeter, 1948). Cox (1955) provided a more general stochastic process, which was a generalization of a Poisson process where the intensity of claim frequency is a stochastic process. In order to let the parameter of the intensity process represent the dependency with respect to time, Jesper (2002) provided an application of the shot-noise Cox process, which considers the intensity process as a stochastic process which varies with external events and their occurring times,

$$\lambda_t = \lambda_0 + \sum_{n=1}^{N_\rho(t)} h(t - T_n, Y_n),$$

and is introduced in Section 4.3.1. Grandell (1991b) introduced the Cox process with Markov intensity process, this model described the intensity process in different stages, where the stage changes by independent identically distributed exponential times. According to the time dependent properties of the Cox process, finite time problems became key questions to evaluate the probability of ruin under the setting of the Cox process. In summary, the properties of different types of point processes are given in the next table,

	Homogeneous	Inhomogeneous	Cox
Parameter	λ: Constant	$\lambda(t)$: Function	λ_t: Random process
$\mathbb{P}[N(t)=n]$	$\frac{(\lambda t)^n}{n!}e^{-\lambda t}$	$\frac{(\int_0^t \lambda(s)ds)^n}{n!}e^{-\int_0^t \lambda(s)ds}$	Depends on the case
$\mathbb{E}[N(t)]$	λt	$\mathbb{E}[\int_0^t \lambda(s)ds]$	Depends on the case

Table 1.1: Features of the homogeneous, inhomogeneous and Cox Poisson process

Cox Process	HTI	GBM	DJP	MJP
Position	Ex. 4.3.1	Ex. 4.3.2	Ex. 4.3.3	Ex 4.3.4
$\mathbb{P}[N(t)=n]$	unknown	unknown	known	known
$\mathbb{E}[N(t)]$	$\lambda_0 e^{-\delta t}e^{\rho t(\mathbb{M}_Y(\alpha)-1)}$	$\lambda_0 e^{\mu t}$	$\lambda_0 + e^{-\delta t}e^{\rho t(\mathbb{E}(Y)-1)}$	$\lambda_0 + \frac{n}{\omega}$ as $t \to 0$

Table 1.2: Some Cox process properties

where HTI = Heavy tailed intensity, GBM = Geometric Brownian motion intensity, DJP = Discounted jump process and MJP = Markov jump process can be found in Chapter 4.

Chapter 4 is based on previous results from Willmot (2015) and inspired by Takács (1977). The inhomogeneous Poisson process and the Cox process are fully investigated. Using a different setting of the surplus process in comparison to (1.1), the modified surplus process considers the initial age. Theorem 4.2.2 provides a backward martingale with respect to the surplus level and the intensity function, which is the key to evaluating the probability of ruin and deriving the inhomogeneous type of Seal's formulae. In addition, a number of examples for the Cox process are given in Section 4.3. In particular, under the setting of the Markov jump process, Theorem 4.3.2 shows integro-differential equations which can be used to calculate the ultimate ruin probability by applying backward recursions. Two states model is a special case of the Markov jump process and its ruin probability is derived in Example 4.3.4.

Independence assumptions of claim size and a claim's occurring time can be too restrictive in practical applications and it is natural to look for explicit formulae for the ruin probability and related quantities in the presence of dependence among the risks. Over recent decades a number of dependence structures have been identified that allow for analytical formulae (see e.g. Bühlmann (1970), Asmussen and Albrecher (2010)). Albrecher et al. (2011) provided an additional class of continuous dependent risk models for which explicit expressions for ruin probability can be obtained. In this thesis, Chapter 3 is from our article Constantinescu et al. (2018), which presents basic properties and discusses potential insurance applications of a new class of probability distributions on positive integers with power law tails under the discrete binomial risk process. In particular, the probability of ruin in the compound binomial risk model is obtained where the claims are zero-inflated discrete Pareto and Weibull distributed with correlation induced by mixing distributions. Equation (3.1) shows the relationship between the probability mass function of claims by mixing over ρ and the Laplace transform of Θ. In addition, Proposition 3.2.1 provides the equation to derive the ruin probability. Under the claim's setting of a zero-modified Pareto and Weibull distribution, the explicit ruin probabilities are derived in Corollary 3.2.1 and 3.2.3.

Apart from special claim and time dependent risk processes, Nie et al. (2011) and Nie et al. (2015) calculated the ruin probability under the classical capital injection environment with infinite and finite time horizon. They considered the question of "whether the insurer can reduce the ultimate ruin probability by allocating part of the initial funds to the purchase of a reinsurance contract" (Nie et al., 2011). This reinsurance contract would restore the insurer's surplus to a positive level k every time the surplus level falls between 0 and k. The insurer's objective is to decide if they should raise more capital or purchase the reinsurance agreements to minimize the ultimate ruin probability.

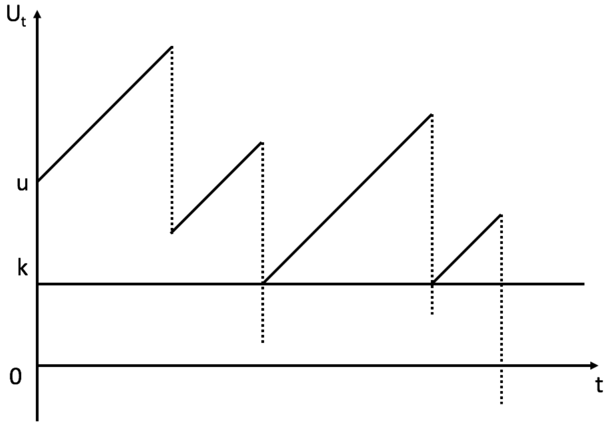


Figure 1.2: Surplus process with capital injection

More examples of the capital injection model can be found in Dickson and Qazvini (2016). The model is represented in Figure 1.2. Dividend strategies for insurance risk models were firstly proposed by De Finetti (1957) to more realistically reflect the surplus cash flows in an insurance portfolio. Barrier strategies for the compound Poisson risk model have been studied in a number of papers and books, such as Albrecher et al. (2005), Gerber (1979), Lin et al. (2003) and Segerdahl (1970).

Chapter 5 provides a number of examples of partial injection models and reinsurance contracts by extending the findings in Lin and Pavlova (2006), Nie et al. (2011) and Dickson and Qazvini (2016). The ruin probability is derived by applying the idea of two barriers model and the joint probability of ruin and deficit, which are introduced in Sections 2.6 and 2.7, probabilities for the barrier model are given in (2.23). The partial injection model is extended from the capital injection model in Chapter 5. Its injection is related to the part of amount/deficit by which the surplus process falls below a fixed compensation level $0 \leq k \leq u$. More precisely, suppose that on the i^{th} occasion that the surplus falls between 0 and k, the insurer's surplus falls to a level $k - y_i$ (such that $0 < y_i < k$), the reinsurer makes an instant payment of the part of the deficit py_i ($p \in (0, 1)$) to the insurer. If any claim leads the insurer's surplus to drop to a level below 0 (or the lower level, e.g. Sections 5.1.2 and 5.1.3), the reinsurer does not make a payment and ruin for the portfolio occurs at the time of this claim. In addition, the relationship between the ruin and the injection is defined and investigated in Section 5.2, which leads ruin probabilities to behave in different ways under the different models' settings. The optimal setting of initial investment is derived by considering the cost of reinsurance contracts in Section 5.4. Furthemore, the premium calculation will be derived in order to construct the capital injection and partial discrete capital injection model as a reinsurance contract.

1.2 Literature review of the mathematical methods

The simplest case of a classical risk model assumes that claim size is exponentially distributed. Cramér (1930) applied a differential argument to derive an expression for the non-ruin probability and solved it by assuming the claim size follows an exponential distribution, resulting in an exponential solution. Since then, many actuarial mathematicians have started to analyse the problem of ruin. One direction is to estimate the ruin probability for some particular claims' distributions by approximations (Beekman, 1969; Kingman, 1962; Bloomfield and Cox, 1972; De Vylder, 1978; Willmot and Lin, 2001) and asymptotic analyses (Klüppelberg et al., 2004; Palmowski and Pistorius, 2009; Albrecher et al., 2012), especially for heavy-tailed claims (Ramsay, 2003). In addition, discrete heavy-tailed distributions are an important and active topic in non-life insurance research and practice (see, e.g., Castanér et al. (2013), Cheng and Shiu (2000), Li and Garrido (2009)).

Ever since the explicit solution of ruin probabilities for exponential claim sizes was established (Cramér, 1930), investigation into the explicit ruin probability for the light-tailed distributions has become an active direction of research, particular in the last 50 years. The literature for deriving explicit expressions for the ultimate ruin probability of the classical Cramér-Lundberg risk model (1.1) with various light-tailed claim distributions is abundant in both methods and results. Cramér (1955) and Feller (1968) both derived the solution of the non-ruin probability by applying a differential argument (2.4), under some conditions, the probability can be solved analytically when the claims are exponentially distributed by either differentiating both sides of the integro-differential equation (the method of solving an ordinary differential equation) or taking the Laplace transform and its inversion. Pakes (1975) derived the relationship between ruin probability and the tail distribution of the claim severity by considering limiting waiting distributions of $GI/G/1$ queue. The maximum of the waiting times is also obtained in a limited theorem.

There exist a number of results for various special risk models and processes. Gerber (1973) analysed the risk process with independent and stationary increments by applying the martingale theorem. Thorin (1973) derived an integral expression for the ruin probability for the classical risk model with special Gamma distributed claims (can also be found in Constantinescu et al. (2017)). Furthermore, Ramsay (2003) applied the inverse Laplace transform over the complex domain to derive a solution of the ruin probability when the claim size follows a special Pareto distribution. For heavy-tailed distributed claims, only asymptotic results were derived in the literature. Embrechts et al. (2017) and Rolski et al. (1999) particularly studied subexponential claim cases. In addition, asymptotic results on ruin probabilities for Lévy insurance risk processes can be found in Klüppelberg et al. (2004).

There is very little literature for explicit finite time ruin probability. For the classical risk model, Asmussen (1984) provided the result of explicit finite time ruin probability with exponential claims. Dickson and Willmot (2005) derived an expression for the density of the time to ruin in the classical risk model by inverting its Laplace transform. Gani and Prabhu (1959) introduced another method which is based on the queueing theory, applying the Laplace transform and its inversion with respect to the initial capital u to derive the formulae for ruin probability. In queuing theory, the Pollaczeck-Khinchine formula is one of the most fundamental tools, which was first introduced by Pollaczek (1930). This formula was applied to risk theory to get the expression for ruin probability (Asmussen and Albrecher, 2010). In addition, Willmot (2015) found a solution for the finite time ruin probability with mixture Erlang claim distribution by applying Seal's type integro-differential equations (also see Klausügman et al. (2013)) and Michna (2011) derived a new version of Seal's formulae for the spectrally positive Lévy process.

Furthermore, Takács (1955) applied the double Laplace transformation on the integro-differential equation for joint waiting time distribution with respect to initial capital and time variable. Borovkov and Dickson (2008) extended the result for the classical case to the Sparre Andersen model with exponential claims by using alternative approaches. In order to generate the explicit probability, Dassios et al. (2015) provided an infinitesimal generator $\mathcal{A}f(u, \lambda, x, t)$ where $f(u, \lambda, x, t)$ denotes a function of the surplus level, value of intensity parameter, time elapsed and time variable. This generator can be applied to obtain the ruin probability by the theorem provided by Paulsen and Gjessing (1997). By this theorem, one could simply let the function $f(\cdot)$ become the ruin probability. However, under some Cox processes with a non-Markovian intensity process (e.g. the shot-noise Cox process), the function $f(\cdot)$ has to combine a new random variable: time elapsed, therefore the explicit ruin probability under this process becomes very difficult to obtain.

The problem of risk processes with the upper barrier under the dividend or reinsurance agreements has been investigated very well. The ruin probability of the two barriers can be found in Dickson and Gray (1986). The optimal calculation for dividend strategy was initially proposed by De Finetti (1957) for a general binomial model. In the classical risk model, literature for dividend strategy problems and more general barrier strategies can be found in Borch (1969), Bühlmann (1970), Segerdahl (1970) and Gerber (1973). Furthermore, Dickson and Drekic (2006) studied the optimal dividend problem under a ruin probability constraint. Albrecher et al. (2005) investigated a barrier strategy with generalized Erlang(n) claim inter-arrival times, e.g. in the Sparre Andersen model.

Bühlmann (1972) started with simple models for which explicit solutions are available and subsequently mixed over involved parameters. The changes made lead the marginal distributions and risks to be measured dependently. "One can then balance the marginal distribution of the risks with their dependent structure in such a way that properties like level crossing probabilities can be studied without direct treatment of the dynamics of the process" (Albrecher et al., 2011). In other words, the mixing of the parameters can be carried over to the mixing of the final quantities under study. This results in a new set of dependent models for which explicit results can be obtained and may serve as a new structure for a larger model class (see Asmussen and Albrecher (2010)).

The theory and applications of such zero-modified discrete distributions is an important area in distribution theory, with applications in manufacturing (Lambert, 1992), econometrics (Mullahy, 1997), economics (Aryal, 2011; Iwunor, 1995; Sharma, 1985), and accident analysis (Miaou, 1994; Shankar et al., 1994), among others. Such modifications, also known as zero-adjusted, zero-altered, or zero-inflated discrete distributions, have been developed for many standard discrete distributions to account for disproportionally large (or small) frequencies of zeroes observed in empirical data, compared with the standard models (Johnson and Kemp, 1994). Popular models of this type include those based upon the Poisson distribution (Goralski, 1977; Greene, 2000; Heilbron, 1994; Min and Agresti, 2005), generalized Poisson distribution (Gupta et al., 1996)), binomial distribution (Greene, 2000), geometric and negative binomial distributions (Greene, 2000; Iwunor, 1995; Min and Agresti, 2005; Sharma, 1985), and logarithmic distribution (Khatri, 1961; Patil, 1964).

1.3 Summary of the thesis

The second chapter introduces preliminaries and models. Definitions, properties and results in the literature are fully provided and the main methodologies for solving the three dependent models are introduced. For the claim dependent model, we suggest the mixing procedures as a general tool for dependent modelling in collective risk theory. For the time dependent model, the point process is modelled by the inhomogeneous

Poisson or the Cox process. In addition, for the surplus dependent model, the approach for solving the two barriers model will be discussed.

Chapter three investigates the claim dependent model. By applying the simple mixing approach, we manage to extend the class of the claim size distributions and aim to derive the ultimate probability of ruin under the discrete binomial risk process. In particular, we introduce the more convenient way of mixing the parameter of the probability of success in the zero-modified claim distribution, resulting in the more tractable claim distributions and explicit expressions for ruin probabilities.

The fourth chapter is dedicated to the time dependent model. A new type of Seal's formulae are derived and applied in order to compute the finite time ruin probability under the inhomogeneous Poisson process risk model. Analyses under some specific Cox process models with a number of statistical properties are introduced and the possibility of deriving the explicit ruin probability under the Cox process is discussed.

Chapter five discusses the surplus dependent model. In fact, the idea of an upper barrier model provides the main inspiration for this model, in which instead of introducing an upper barrier, we bring a compensation level into the original risk process and investigate the behaviour of the surplus process. Furthermore, we provide some reinsurance agreements and measure the risk by the ultimate ruin probability. We aim to answer the following key risk management questions:

- Should a company buy reinsurance or raise more capital?
- What is the optimal initial capital setting?
- How can risk theory help decisions regarding reinsurance?

The last chapter concludes the current findings and proposes further research.

Chapter 2

Preliminaries

This chapter provides the foundation of the concepts to be presented in this thesis. As our goal is to investigate the claim dependent risk model, time dependent risk model and surplus dependent risk model, the basic concepts and preliminaries of the upcoming problems will be introduced. Firstly, Ni (2015) summarized some basic insurance mathematics concepts, which can be seen throughout this thesis.

- An *insurance premium* is referred to as the premium rate in risk theory, denoted by c (if there are no other declarations), which is an amount of money that insurance company collects from policyholders per unit time.

- *Claims* are the amount of losses an insurer needs to pay for an insured product. The value of a claim is referred to as the *claim size* and it is considered as a non-negative random variable, denoted by X, with common distribution function $F_X(x)$.

- The *number of claims* that occur in a certain period is a non-negative integer-valued random variable. The *claim counting process* is often denoted by $\{N_t, t > 0\}$ where N_t is the number of claims up to time t.

- An *epoch of a claim*, or sometimes called a *claim arrival time* is the time at which a claim happens. We denote the epochs by t_1, t_2, $\dots$ and the *inter-arrival times* or *waiting times* by $T_i = t_i - t_{i-1}$, with common distribution function F_T.

- A *risk surplus* denoted here by U_t is the amount of capital an insurance company has at time t. It increases by collecting premiums and drops by the payment of claims.

- The *net profit condition* for a risk model is

$$c \cdot \mathbb{E}(T_i) = (1+\eta)\mathbb{E}(X_i), \tag{2.1}$$

where $\eta > 0$ is called the *safety loading.* It describes the situation where the insurance company can avoid certain ruin. If the net profit condition is not satisfied, i.e., $c \cdot \mathbb{E}(T_i) < \mathbb{E}(X_i)$, then the ruin occurs almost surely irrespective of the large value of the initial surplus.

2.1 Ruin probabilities

We start with the definition of the probability of ruin. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete probability space containing insurance mathematics concepts previously discussed for the model (1.1), where $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a nature filtration. This model describes the amount of surplus U_t of an insurance portfolio at time t. Conditioning on the initial capital u, the finite time ruin probability $\psi(u,T)$ is expressed by the probability of the smallest surplus level being below 0 in the time interval $(0,T)$,

$$\psi(u,T) = \mathbb{P}\left(\inf_{0\leq t<T} U_t < 0 \middle| U_0 = u\right) = \mathbb{P}\left(\tau_u < T\right), \quad u \geqslant 0, T \geqslant 0, \tag{2.2}$$

where τ_u is the first hitting time or the time of ruin

$$\tau_u = \inf\left\{t \geq 0 : \, U_t < 0\right\}. \tag{2.3}$$

Furthermore, we are able to obtain the ultimate ruin probability when taking $T \to \infty$,

$$\lim_{T\to\infty} \psi(u,T) = \psi(u), \quad u \geqslant 0$$

and the non-ruin, or survival probability, is denoted by

$$\phi(u) = 1 - \psi(u), \quad u \geqslant 0.$$

Recall that the risk process U_t in (1.1) is referred to as the classical Cramér -Lundberg risk process if the point process N_t is a homogeneous Poisson process with constant intensity parameter λ. Cramér (1930) used the properties of a Poisson process (can be found in the third definition in 2.2.1) to derive an integro-differential equation for the non-ruin probability by applying differential arguments. In addition, Grandell (1991a) considered U_t in a sufficiently small time interval $(0, \Delta]$ and separated the four possible cases:

1. **No** claim occurs in $(0, \Delta]$;
2. **One** claim occurs in $(0, \Delta]$, but ruin does not happen;
3. **One** claim occurs in $(0, \Delta]$, and ruin happens;
4. **More than one** claim occurs in $(0, \Delta]$.

Then the non-ruin probability satisfies

$$\phi(u) = (1 - \lambda\Delta)\phi(u + c\Delta) + \lambda\Delta \int_0^{u+c\Delta} \phi(u + c\Delta - x)dF_X(x) + o(\Delta), \quad u \geqslant 0.$$

Letting $\Delta \to 0$, one obtains the integro-differential equation with respect to the non-ruin probability and the cumulative distribution function of the claim size

$$\frac{d}{du}\phi(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u - x)dF_X(x), \quad u \geqslant 0. \tag{2.4}$$

The proof can be found in Gerber (1979), Grandell (1991a) and Panjer and Willmot (1992). The idea to solve equation (2.4) is to use the Laplace transformation.

Definition 2.1.1. The Laplace transform of a function $f(t)$, defined for any real number $t > 0$, is the function $f(s)$, defined by

$$\hat{f}(s) = \mathcal{L}_s\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

Properties of the Laplace transform required in this thesis are given by the following proposition.

Proposition 2.1.1. The Laplace transform of a differential function $\frac{d}{dt}f(t)$ is given by

$$\mathcal{L}_s\{\frac{d}{dt}f(t)\} = s\hat{f}(s) - f(0).$$

The Laplace transform of a convolution with $f(t)$ and $g(t)$,

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$$

is given by

$$\mathcal{L}_s\{(f * g)(t)\} = \hat{f}(s)\hat{g}(s).$$

The Laplace transform of a time shifting function $f(t-a)$ is given by

$$\mathcal{L}_s\{f(t-a)\} = e^{-as}\hat{f}(s).$$

Then applying a Laplace transform on the integro-differential equation (2.4) leads to

$$\hat{\phi}(s) = \frac{c\phi(0)}{cs - \lambda + \lambda\hat{f}_X(s)}. \tag{2.5}$$

When the claim sizes are exponentially distributed with parameter β, the Laplace transform of the claim size density equals

$$\hat{f}_X(s) = \frac{\beta}{s+\beta}, \quad \Re(s) > -\alpha.$$

By applying the fractional decomposition and inverse Laplace transformation, the non-ruin probability can be obtained from equation(2.5)

$$\phi(u) = 1 - \frac{\lambda}{\beta c}e^{-(\beta - \frac{\lambda}{c})u}, \quad u \geqslant 0. \tag{2.6}$$

When the claim sizes are Erlang distributed with parameter α and scale n, the Laplace transform of the claim size density equals

$$\hat{f}_X(s) = \left(\frac{\alpha}{s+\alpha}\right)^n, \quad \Re(s) > -\alpha,$$

the expression on the right hand side of (2.5) can be derived by the ratio of two polynomial functions with respect to s. For the Erlang distributed claim case, one can then use the **partial** fraction decomposition and invert (2.5) to obtain a linear combination of exponential functions (Grandell, 1991a; He et al., 2003; Constantinescu et al., 2017).

Notice that for a rational shape parameter $r = m/n \in \mathbb{Q}$ where m and n are both positive integers, with $\Re(s) > \alpha$, one could shift the argument s to obtain

$$\hat{\phi}(s-\alpha) = \frac{c\phi(0)}{c(s-\alpha) - \lambda + \lambda(\frac{\alpha}{s})^{m/n}} = \frac{c\phi(0)s^{m/n}}{c(s-\alpha)s^{m/n} - \lambda + \lambda\alpha^{m/n}},$$

which is a ratio of polynomials of orders m and $(m+1)$ in $t = s^{1/n}$. In this case, an explicit expression and model settings can be found in Zhu (2013) and Constantinescu et al. (2017), using the two parameter (m, n) Mittag-Leffler function

$$\phi(u) = e^{-\alpha u} u^{\frac{1}{n}-1} \sum_{k=0}^{m+n-1} m_k E_{\frac{1}{n},\frac{1}{n}}\left(s_k u^{\frac{1}{n}}\right), \quad u \geqslant 0 \tag{2.7}$$

with s_k and m_k real constants, determined on a case-by-case basis.

As mentioned, the non-ruin probability can be derived by the inversion of the Laplace transformation. For some particular cases, the inversion results on the Laplace transformation are explicit and ready to use. For other general scenarios, a numerical inversion technique might be required. Some references can be found with respect to numerical inversion of Laplace transforms, e.g. Ahn et al. (2000) and Abate and Whitt (1995).

2.2 Inhomogeneous Poisson point process and Cox process

The point process has always been a core research direction in risk theory. For the classical homogeneous Poisson process, the intensity of the claim frequency is defined as a constant, since, it is independent of the time variable. When we consider the aggregate claims size combined with an inhomogeneous Poisson process or Cox process, then the intensity of the point process will be correlated with the time dependent model and is referred to as a function $\lambda(t)$ or a stochastic process λ_t (Parzen, 1962; Cox and Isham, 1980) Then, an inhomogeneous Poisson process is defined in the following way.

Definition 2.2.1. A stochastic point process N_t is called an inhomogeneous Poisson process with intensity function $\lambda(t)$ for all $t \geq 0$ and the integrated process is called the mean value function

$$\Lambda(t) = \mathbb{E}[N_t] = \int_0^t \lambda(s)ds,$$

which has the following key properties:

1. $\mathbb{P}[N_t = n] = \frac{\Lambda(t)^n}{n!} e^{-\Lambda(t)}$, for $t > 0$,
2. $\mathbb{E}[S_t] = \mu\Lambda(t)$, $Var[S_t] = \mu^2\Lambda(t)$,
3. $\lim_{h\to 0} \frac{1-\mathbb{P}[N_{t+h}-N_t=0]}{h} = \lambda(t)$, $\lim_{h\to 0} \frac{\mathbb{P}[N_{t+h}-N_t=1]}{h} = \lambda(t)$.

Then the probability mass function of an inhomogeneous Poisson process is given by

$$\mathbb{P}[N_{t+s} - N_t = k] = e^{-[\Lambda(t+s)-\Lambda(t)]} \frac{[\Lambda(t+s) - \Lambda(t)]^k}{k!}, \text{for } k \geq 0,\ t \geq 0.$$

Now, if we consider the mean value function to be the time $\Lambda(t)$, an inhomogeneous Poisson process with time t can be constructed by a homogeneous Poisson process with time $\Lambda(t)$.

Proposition 2.2.1. (Time shifting) The inhomogeneous Poisson process N_t with intensity function $\lambda(t)$ at time t can be considered as a homogeneous Poisson process $\hat{N}_t$ with constant intensity 1 at time $\Lambda(t)$, where

$$\mathbb{P}[N_t = n] = \frac{[\Lambda(t)]^n}{n!} e^{-\Lambda(t)} = \mathbb{P}[\hat{N}_{\Lambda(t)} = n].$$

Proposition 2.2.2. Assume that N_t follows the inhomogeneous Poisson process with mean value function $\Lambda(t) = \int_0^t \lambda(s)ds$ and any $\theta \neq 0$ and $t_2 \geq t_1 \geq 0$ s.t.

$$\mathbb{E}[\theta^{N_{t_2}-N_{t_1}}] = \mathbb{E}[e^{(\theta-1)(\Lambda(t_2)-\Lambda(t_1))}].$$

Proof. Start with the definition of the inhomogeneous Poisson process,

$$\begin{aligned}
\mathbb{E}[\theta^{N_{t_2}-N_{t_1}}] &= \mathbb{E}[\mathbb{E}[\theta^{N_{t_2}-N_{t_1}} | t_2 \geq t_1 \geq 0]] \\
&= \mathbb{E}[e^{-(\Lambda(t_2)-\Lambda(t_1))} \sum_{k=0}^{\infty} \theta^k \frac{(\Lambda(t_2) - \Lambda(t_1))^k}{k!}] \\
&= \mathbb{E}[e^{-(\Lambda(t_2)-\Lambda(t_1))} \sum_{k=0}^{\infty} \frac{[\theta(\Lambda(t_2) - \Lambda(t_1))]^k}{k!}] \\
&= \mathbb{E}[e^{-(\Lambda(t_2)-\Lambda(t_1))} e^{\theta(\Lambda(t_2)-\Lambda(t_1))}] = \mathbb{E}[e^{(\theta-1)(\Lambda(t_2)-\Lambda(t_1))}];
\end{aligned}$$

See, for example, Dassios and Jang (2005). □

Definition 2.2.2. The stochastic point process N_t is called a $\mathcal{F}_t$-Cox (doubly stochastic) Poisson process with intensity λ_t if we assume N_t to be adapted to a history $\mathcal{F}_t$, λ_t to be $\mathcal{F}_t$-measurable, $t \geq 0$ and that

$$\Lambda_t = \int_0^t \lambda_s ds < \infty.$$

For all $0 \leq t_1 \leq t_2$,

$$\mathbb{E}[e^{\alpha(N_{t_2}-N_{t_1})}|\mathcal{F}_{t_1}] = \exp\left((e^{\alpha}-1)\int_{t_1}^{t_2}\lambda_s ds\right),$$

and the distribution of the point process is defined by a conditional probability.

$$\mathbb{P}[N(t) = n|\mathcal{F}_t] = \frac{(\int_0^t \lambda_s ds)^n}{n!} e^{-\int_0^t \lambda_s ds}.$$

However, under the setting of Cox process, we cannot easily generate the distribution of the point process because it has to be conditioned on the path of history. The reason we investigate the inhomogeneous Poisson process is to construct a stochastic intensity process: the Cox process (specifically for the shot-noise Cox process, see Albrecher and Asmussen (2006)). By applying the total probability theorem, we aim to derive the unconditional probabilities from the distribution of the point process, conditioning on the history $\mathcal{F}_t$, which is also referred to as the conditional distribution of the intensity process. Therefore, the key problem of deriving the distribution of the Cox process can be replaced by generating the unconditional distribution for the intensity process.

2.3 The idea of mixing distribution

According to Bühlmann (1970), one can use ruin probability formulae of the Cramer-Lundberg risk model (which are explicit for certain classes of claim size distributions, see e.g. Asmussen and Albrecher (2010), Albrecher et al. (2011)) and mix over an involved parameter Θ, which can be considered as the Poisson parameter Λ or claim size parameter $\hat{\beta}$. The resulting ruin probability for the new dependent model is given by

$$\psi(u) = \int_0^\infty \psi_\theta(u) dF_\Theta(\theta). \tag{2.8}$$

Note that this formula was initially given by Bühlmann (1972), where the mixing procedure was used in the context of dynamic credibility-based premiums for the risk process (Dubey, 1977; Bühlmann and Gerber, 1978; Gerber, 1979). If mixing over the Poisson parameter Λ (which is a random variable, representing the intensity of the claims frequency) with the mixing cumulative distribution function $F_\Lambda(x)$, the net profit condition will be violated whenever the realisation of Λ is larger than the threshold value $\lambda_c = c/\mathbb{E}(X_1)$. Hence, a refined version of (2.8) is

$$\psi(u) = \int_0^{\lambda_c} \psi_\lambda(u) dF_\Lambda(\lambda) + \bar{F}_\Lambda(\lambda_c). \tag{2.9}$$

Correspondingly,

$$\lim_{u\to\infty} \psi(u) = \bar{F}_\Lambda(\lambda_c).$$

If one considers the model with exponentially distributed claim size and constant Poisson parameter λ, then clearly the conditional non-ruin probability is given by (2.6).

Example 2.3.1. Pareto Inter-arrival Times

If one considers Λ as a Gamma(α, θ) distributed random variable, the resulting mixing distribution for the marginal inter-occurrence time T_k is Pareto distributed with tail

$$\bar{F}_T(t) = \int_0^\infty e^{-\lambda t} f_\Lambda(\lambda) d\lambda = \left(1 + \frac{t}{\theta}\right)^{-\alpha}.$$

If one considers the case of (2.6) with $\lambda_c = c\beta$, then the explicit formula for the ruin probability is

$$\psi(u) = \frac{\theta^\alpha e^{-\beta u}}{\lambda_c} \left(\theta - \frac{u}{c}\right)^{-1-\alpha} \left(\alpha - \frac{\Gamma(\alpha+1, \lambda_c\theta - \beta u)}{\Gamma(\alpha)}\right) + \frac{\Gamma(\alpha, \lambda_c\theta)}{\Gamma(\alpha)}.$$

In particular, we have

$$\psi(0) = \frac{1}{\lambda_c\theta} \left(\alpha - \frac{\Gamma(\alpha+1, \lambda_c\theta)}{\Gamma(\alpha)}\right) + \frac{\Gamma(\alpha, \lambda_c\theta)}{\Gamma(\alpha)}$$

and

$$\lim_{u\to\infty} \psi(u) = \frac{\Gamma(\alpha, \lambda_c\theta)}{\Gamma(\alpha)}.$$

Example 2.3.2. Weibull Inter-arrival Times

If Λ is stable (1/2) distributed, the resulting mixing distribution for the marginal inter-occurrence times T_k is Pareto distributed with tail

$$\bar{F}_T(t) = \int_0^\infty e^{-\lambda t} f_\Lambda(\lambda) d\lambda = e^{-\alpha\sqrt{t}}.$$

Therefore, if inter-arrival times are Weibull distributed with shape parameter 1/2 then the explicit formula for the ruin probability is given by

$$\psi(u) = \frac{\alpha i e^{-i\alpha\sqrt{u/c}-u\beta}}{4\beta\sqrt{cu}} \left[-1 + \text{Erf}\left(\frac{\alpha}{2\sqrt{\lambda_c}} - i\sqrt{u\beta} \right) + e^{2i\alpha\sqrt{u/c}} \text{Erfc}\left(\frac{\alpha}{2\sqrt{\lambda_c}} + i\sqrt{u\beta} \right) \right] + \text{Erfc}(\frac{\alpha}{2\sqrt{\lambda_c}}),$$

where Erfc and Erf are the error functions, denoted by $\text{Erfc}(x) = 1 - \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy,$. In particular, we have

$$\psi(0) = (1 - \frac{\alpha^2}{2\lambda_c})\text{Erfc}(\frac{\alpha}{2\sqrt{\lambda_c}}) + \frac{\alpha}{\sqrt{\lambda_c \pi}} e^{-\frac{\alpha^2}{4\lambda_c}}$$

and

$$\lim_{u\to\infty} \psi(u) = \text{Erfc}(\frac{\alpha}{2\sqrt{\lambda_c}}).$$

Furthermore, let $\hat{\beta}$ be a positive random variable with distribution $F_{\hat{\beta}}(x)$ and consider the classical compound Poisson risk model (1.1) with exponential claim sizes that fulfil, given $\hat{\beta} = \beta$,

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, ..., X_n > x_n | \hat{\beta} = \beta) = \prod_{i=1}^n e^{-\beta x_i}, \tag{2.10}$$

for each n, where the X_i for $i \geq 1$ are conditionally independent and exponentially distributed. However, in general, the resulting marginal distributions of the X_i will no longer be exponential and the claim sizes will be dependent. Let $\psi_\beta(u)$ denote the ruin probability of the classical compound Poisson risk model with independent exponential claim amounts, given by (2.6). Then for the dependent model (2.10), since the net profit condition is violated (for $\beta \leq \beta_c = \frac{\lambda}{c}$) and consequently $\psi_\beta(u) = 1$ for all $u \geq 0$, the ruin probability can be obtained by applying the total probability theorem,

$$\psi(u) = \int_0^\infty \psi_\beta(u) dF_{\hat{\beta}}(\beta) = F_{\hat{\beta}}(\beta_c) + \int_{\beta_c}^\infty \psi_\beta(u) dF_{\hat{\beta}}(\beta) \tag{2.11}$$

with the limit

$$\lim_{u\to\infty}\psi(u) = F_{\hat{\beta}}(\beta_c),$$

which is positive whenever the random variable $\hat{\beta}$ has probability mass at or below $\beta_c = \frac{\lambda}{c}$. Albrecher et al. (2011) provide the following proposition,

Proposition 2.3.1. The joint distribution of the dependent model characterized by (2.10) $F_{X_1,...,X_n}$ can equivalently be described by the Laplace transformation of $f_{\hat{\beta}}$ with respect to the random variable $\hat{\beta}$.

Proof. For each n, the joint distribution of the tail of $(X_1, ..., X_n)$ can be denoted as

$$\bar{F}_{X_1,...,X_n}(x_1, ..., x_n) = \mathbb{P}[X_1 \geqslant x_1, ..., X_n \geqslant x_n] = \int_0^\infty e^{-\beta(x_1+\cdots+x_n)} dF_{\hat{\beta}}(\beta) = \hat{f}_{\hat{\beta}}(x_1 + \cdots + x_n)$$

and for each of the marginal distribution of X_i, we have

$$F_{X_i}(x_i) = \int_0^\infty e^{-\beta x_i} dF_{\hat{\beta}}(\beta) = \hat{f}_{\hat{\beta}}(x_i).$$

□

Now let us look at some particular examples.

Example 2.3.3. (**Pareto claims**)
If $\hat{\beta}$ is a Gamma(α, θ) random variable with density function

$$f_{\hat{\beta}}(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x},\ \theta > 0,$$

the resulting mixing distribution for the marginal claim size X_i is

$$\bar{F}_X(x) = \int_0^\infty e^{-\beta x} f_{\hat{\beta}}(\beta) d\beta = \left(1 + \frac{x}{\theta}\right)^{-\alpha},\ x \geqslant 0.$$

It shows X_i is Pareto(α, θ) distributed (Klausügman et al., 2013). From equations (2.6) and (2.11), the ruin probability for this model satisfies

$$\psi(u) = 1 - \frac{\Gamma(\alpha, \theta\beta_c)}{\Gamma(\alpha)} + \theta\beta_c e^{\beta_c}\left(1 + \frac{u}{\theta}\right)^{-(\alpha-1)} \frac{\Gamma\left(\alpha - 1, (\theta + u)\beta_c\right)}{\Gamma(\alpha)},$$

where $\Gamma(\alpha, x) = \int_x^\infty y^{\alpha-1} e^{-y} dy$ is the incomplete Gamma function and $\beta_c = \frac{\lambda}{c}$. For the extreme case, using the facts $\lim_{x\to\infty} \frac{\Gamma(s,x)}{x^{s-1}e^{-x}} = 1$ and $e^0 = 1$, we have

$$\lim_{u\to\infty}\psi(u) = 1 - \frac{\Gamma(\alpha, \theta\beta_c)}{\Gamma(\alpha)},\ \ \psi(0) = 1 - \frac{\Gamma(\alpha, \theta\beta_c)}{\Gamma(\alpha)} + \theta\beta_c \frac{\Gamma(\alpha - 1, (\theta + u)\beta_c)}{\Gamma(\alpha)}.$$

Example 2.3.4. (Weibull claims)

If the random variable $\hat{\beta}$ is stable (1/2) distributed (also called Lévy distributed) with density function

$$f_{\hat{\beta}}(x) = \frac{\alpha}{2\sqrt{\pi x^3}} e^{-\frac{\alpha^2}{4x}},$$

then the resulting mixing distribution tail of the claim size random variable X_i is

$$\bar{F}_X(x) = \int_0^\infty e^{-\beta x} f_{\hat{\beta}}(\beta) d\beta = e^{-\alpha\sqrt{x}}, \ x \geqslant 0,$$

so that the claim size follows a Weibull distribution with shape parameter 1/2. From equation (2.11) we can obtain the expression for the ruin probability in terms of the error function.

$$\psi(u) = \mathrm{Erfc}(\frac{\alpha}{2\sqrt{\beta_c}}) + \frac{\beta_c}{\alpha^2} e^{-\frac{\alpha^2}{4\beta_c}} \Big[-\frac{2\alpha}{2\sqrt{\beta_c \pi}} + e^{\frac{(c\alpha - 2\sqrt{\lambda u})^2}{4c\lambda}} (1 + \alpha\sqrt{u}) \mathrm{Erfc}(\sqrt{\beta_c u} - \frac{\alpha}{2\sqrt{\beta_c}})$$
$$+ e^{\frac{(c\alpha - 2\sqrt{\lambda u})^2}{4c\lambda}} (-1 + \alpha\sqrt{u}) \mathrm{Erfc}(\sqrt{\beta_c u} + \frac{\alpha}{2\sqrt{\beta_c}}) \Big].$$

For the extreme case, by using the facts of $\lim_{x\to\infty} \mathrm{Erfc}(x) = 0$ and $e^0 = 1$, we have

$$\lim_{u\to\infty} \psi(u) = \mathrm{Erfc}(\frac{\alpha}{2\sqrt{\beta_c}}), \ \psi(0) = \mathrm{Erfc}(\frac{\alpha}{2\sqrt{\beta_c}}) - \frac{2\sqrt{\beta_c}}{\alpha\sqrt{\pi}} e^{-\frac{u^2}{4\beta_c}} + \frac{2\beta_c}{\alpha^2} \mathrm{Erf}(\frac{\alpha}{2\sqrt{\beta_c}}).$$

Furthermore, the mixing idea can be developed further in many directions (Albrecher et al., 2011).

Example 2.3.5. (Independent parallel mixing). One can mix both inter-arrival times and claim sizes independently at the same time. Then the ruin probability can be calculated by

$$\psi(u) = \int_0^\infty \int_0^\infty \psi_{\beta,\lambda}(u) dF_{\hat{\beta}}(\beta) dF_\Lambda(\lambda),$$

where $\psi_{\beta,\lambda}(u)$ is the conditional probability of ruin given that $\hat{\beta} = \beta$ and $\Lambda = \lambda$. Whenever there is an explicit expression for $\psi_{\beta,\lambda}(u)$, this leads to an explicit expression for $\psi(u)$ in renewal models with both dependent inter-occurrence times and dependent claim sizes.

Example 2.3.6. (Comonotonic mixing). One can also consider mixing dependence between inter-occurrence times and claim sizes and at the same time dependence among claim sizes and among inter-occurrence times. One way to do this is comonotonic mixing, where the realization λ of Λ is a deterministic function of the realisation β of $\hat{\beta}$ in the form

$$\lambda(\beta) = F_{\Lambda}^{-1}(F_{\hat{\beta}}(\beta)).$$

The ruin probability under this model is given by

$$\psi(u) = \int_0^{\infty} \psi_{\beta,\lambda(\beta)}(u) dF_{\hat{\beta}}(\beta),$$

where $\psi_{\beta,\lambda}(u)$ is the conditional probability of ruin given that $\hat{\beta} = \beta$ and $\Lambda = \lambda$.

2.4 Discrete compound binomial risk model

As we know, discrete heavy-tailed distributions are an important and active area in non-life insurance research and practice (Castanér et al., 2013; Cheng and Shiu, 2000; Li and Garrido, 2009). It is well-known that Pareto and Weibull distributions are used in insurance practice for modelling claim sizes. However, their theoretical implementation in collective risk models is non-trivial. We consider the compound binomial risk model

$$U_t = u + t - \sum_{i=1}^{t} X_i, \quad t \in \mathbb{N}_0 = \{0, 1, \ldots\}, \tag{2.12}$$

introduced in Gerber (1988). The ruin probability is defined by

$$\psi(u) = \mathbb{P}[U_t < 0 \text{ for some } t \geq 0 | U_0 = u].$$

Sundt and dos Reis (2007) claimed that the ruin probability $\psi(u)$ admits an explicit form when the claim amounts $\{X_i\}$ have zero-modified geometric (ZMG) distribution ZMG(q, ρ). The latter is given by the probability mass function (PMF) $\mathbb{P}(X_i = k) = g(k)$, where

$$g(k) = q\delta_{k0} + (1 - \delta_{k0})(1 - q)\rho(1 - \rho)^{k-1}, \quad k \in \mathbb{N}_0 \tag{2.13}$$

and δ_{kj} is the Kronecker delta function, which satisfies

$$\delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

In this case we have

$$\psi(u) = \min\left\{\frac{1-q}{\rho}\left(\frac{1-\rho}{q}\right)^{u+1}, 1\right\}, \tag{2.14}$$

Dutang et al. (2013) extend the formula (2.14) by using a mixing approach (Albrecher et al., 2011), assuming that given $\Theta_1 = \theta_1$, where Θ_1 is a mixing random variable on $\mathbb{R}_+$, the claim amounts $\{X_i\}$ are independent and identically distributed zero-modified geometric ZMG(q, ρ_1) with the success probability

$$\rho = e^{-\theta_1}. \tag{2.15}$$

The mixing variable Θ_1 is considered as a random variable, just as in Dutang et al. (2013), however, with this choice of Θ_1, the resulting distribution of the claim amounts will have a very different distribution as follows:

(i) For Θ_1 having exponential distribution with parameter β, given by the probability density function (PDF)

$$f(x) = \beta e^{-\beta x}, \quad x \in \mathbb{R}_+,$$

the claim amounts have a zero-modified Yule distribution with the PMF

$$\mathbb{P}(X = k) = q\delta_k + (1-\delta_k)(1-q)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j\beta}{\beta+j}, \quad k \in \mathbb{N}_0,$$

(ii) For Θ_1 having gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, given by the probability density function

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}, \quad x \in \mathbb{R}_+, \tag{2.16}$$

In this case the claim amounts have the PMF

$$\mathbb{P}(X = k) = q\delta_k + (1-\delta_k)(1-q)\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j\lambda^\alpha}{(\lambda+j)^\alpha}, \quad k \in \mathbb{N}_0,$$

and the probability of ruin can be expressed in terms of incomplete gamma function.
(iii) For Θ_1 having a Lévy stable distribution (stable subordinator with exponent 1/2 and α), given by the probability density function

$$f(x) = \frac{\alpha}{2\sqrt{\pi x^3}}e^{-\frac{\alpha^2}{4x}}, \quad x \in \mathbb{R}_+. \tag{2.17}$$

In this case the claim amounts have the PMF

$$\mathbb{P}(X=k)=(1-q)\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j e^{-\tau\sqrt{j}},\ \ k\in\mathbb{N}_0,$$

and the ruin probability can be expressed in terms of complementary error special function.

In this thesis, we use a more convenient set up

$$\rho = 1-e^{-\theta} \tag{2.18}$$

rather than (2.15) as in Dutang et al. (2013). Thus, while in the set-up above the geometric probability of success is taken as $e^{-\theta_1}$, we use this to express the probability of failure. Let us note that a geometric distribution with the probability of success given by (2.18) is a discrete version of an exponential one, since the geometric PMF can be derived by the difference of two consecutive exponential tails with parameter θ

$$\mathbb{P}(X=k)=e^{-(k-1)\theta}-e^{-k\theta}=\left(1-e^{-\theta}\right)\left(e^{-\theta}\right)^{k-1},\ \ k\in\mathbb{N}=\{1,2,\ldots\}.$$

Throughout the ruin theory literature, the binomial risk model has developed in different directions (Willmot, 1993; Dickson, 1994). These new, zero-modified discrete Pareto and Weibull distributions may provide a useful addition to an actuary's statistical toolbox, going beyond modelling claim amounts of discrete type. In fact, the zero-modified discrete Pareto model may also be a useful heavy-tailed model for the frequency of claim, as it can be extended to a continuous-time, discrete-valued stochastic process in the spirit of the classical Poisson process due to its fundamental property of infinite divisibility.

2.5 Integrated and differential stochastic process

In this section, the method for integrating a stochastic process is discussed, the example of compound Poisson process is displayed in Figure 2.1. Assume the compound Poisson process to be $S_t = \sum_{i=0}^{N_t} X_i$, with jumps X_i and corresponding jump occurrence times T_i.

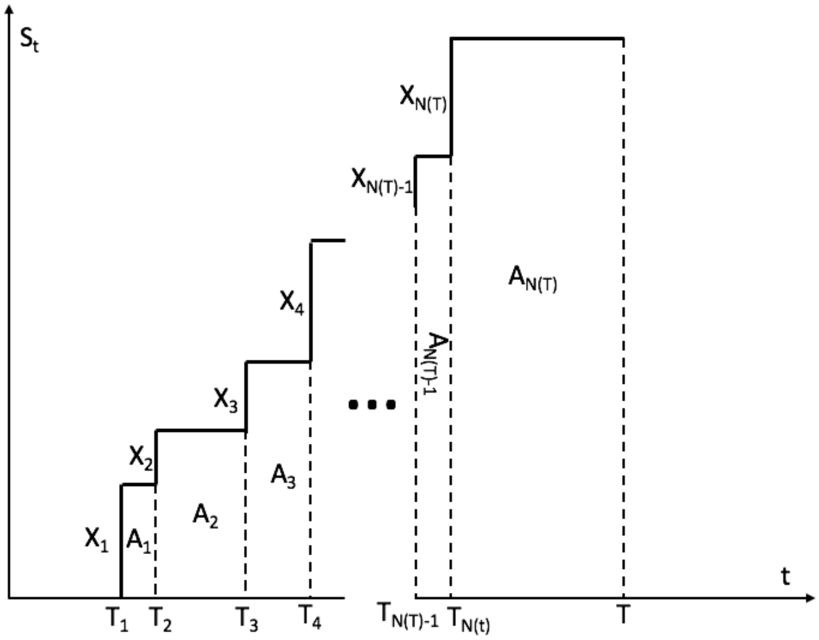


Figure 2.1: Compound Poisson process

The integrated stochastic process is denoted by $Y_t = \int_0^t S_z dz$, which can be calculated as the area of the graphic. We denote A_i, for $i = 1...N_T$, as the areas and $t_i = T_{i+1} - T_i, i = 1...N_T$, $T_0 = 0$, as the inter-arrival times, then we have $Y_T = \sum_{i=0}^{N_T} A_i$. Therefore,

$$\begin{cases} A_1 = X_1 t_2, \\ A_2 = (X_1 + X_2) t_3, \\ A_3 = (X_1 + X_2 + X_3) t_4, \\ \quad \vdots \\ A_{N_t} = \sum_{i=1}^{N_t} X_i (t - \sum_{m=2+i}^{N_t} t_m). \end{cases}$$

It is clear to see there are N_t terms of X_1, $N_t - 1$ terms of X_2... and 1 term of X_{N_t}. Therefore, the summation of the area can be denoted by the following theorem.

Theorem 2.5.1. The integrated compound Poisson process is given by

$$Y_t = \int_0^t S_z dz = \sum_{i=0}^{N_t} A_i = \sum_{i=0}^{N_t} X_i(t - T_i). \tag{2.19}$$

Respectively, if the model has exponential increments i.e. $S_t = \prod_{i=0}^{N_t} e^{X_i} = e^{\sum_{i=0}^{N_t} X_i}$. If we assume the model to be a compound Poisson process with discounted rate δ, say $S_t = \sum_{i=0}^{N_t} X_i e^{-\delta t}$,

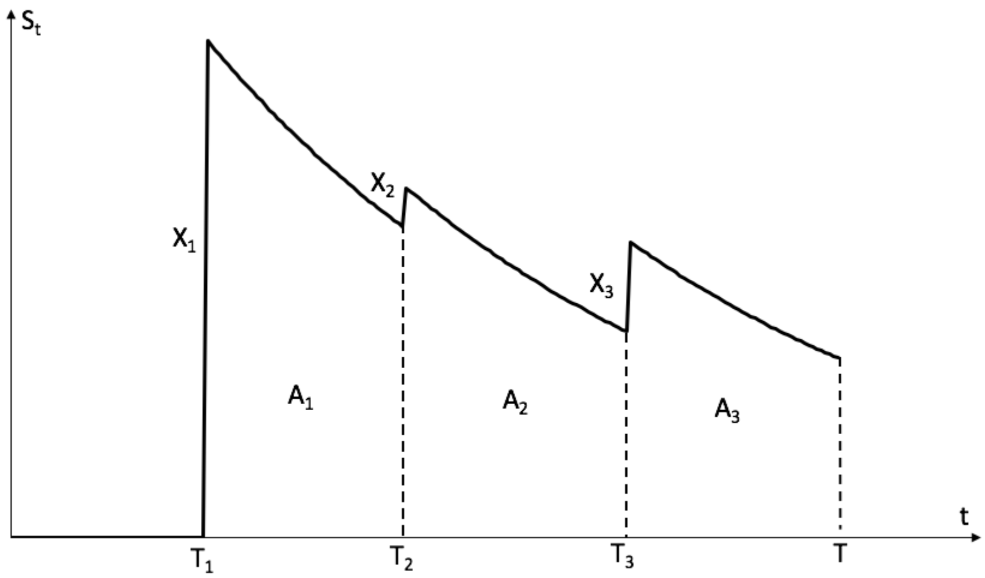


Figure 2.2: Discounted compound Poisson process

then we have the following equations with the fact of an integral $f(t) = \int_0^t e^{-\delta s} ds =$

$\frac{1}{\delta}(1-e^{-\delta t})$,

$$\begin{cases} A_1 = f(t_2)X_1 = \dfrac{1}{\delta}X_1\left(1-e^{-\delta t_2}\right), \\ A_2 = f(t_3)\left(X_1e^{-\delta t_2}+X_2\right) = \dfrac{1}{\delta}\left(X_1e^{-\delta t_2}+X_2\right)\left(1-e^{-\delta t_3}\right), \\ A_3 = f(t_4)\left(X_1e^{-\delta(t_2+t_3)}+X_2e^{-\delta t_3}+X_3\right) = \dfrac{1}{\delta}\left(X_1e^{-\delta(t_2+t_3)}+X_2e^{-\delta t_3}+X_3\right)\left(1-e^{-\delta t_4}\right), \\ \vdots \\ A_{N_t-1} = f(t_{N_t-1})\left(X_1e^{-\delta(t_2+t_3+...+t_{N_t-1})}+X_2e^{-\delta(t_3+...+t_{N(t)-1})}+...+X_{N_t-1}\right) \\ = \left(X_1e^{-\delta(t_2+t_3+...+t_{N_t-1})}+X_2e^{-\delta(t_3+...+t_{N_t-1})}+...+X_{N_t-1}\right)\left(1-e^{-\delta t_{N_t}}\right), \\ A_{N_t} = f(t_{N(t)})\left(X_1e^{-\delta(t_2+t_3+...+t_{N_t})}+X_2e^{-\delta(t_3+...+t_{N_t})}+...+X_{N_t}\right) \\ = \left(X_1e^{-\delta(t_2+t_3+...+t_{N_t})}+X_2e^{-\delta(t_3+...+t_{N_t})}+...+X_{N_t}\right)\left(1-e^{-\delta(t-T_{N_t})}\right). \end{cases}$$

In A_{N_t}, we have $t_2+t_3+...+t_{N_t} = T_{N_t}-T_1$, the summation will remove the majority of the terms . Therefore, we have the following theorem

Theorem 2.5.2. The integrated discounted compound Poisson process is given by

$$Y_t = \int_0^t S_z dz = \sum_{i=0}^{N_t} A_i = \frac{1}{\delta}\sum_{i=0}^{N_t} X_i\left(1-e^{-\delta(t-T_i)}\right). \tag{2.20}$$

In order to generate the derivative of a compound Poisson process, the derivative of a Poisson point process is required which is denoted by an instantaneous vector rate in Cox and Isham (1980),

$$\frac{dN_t}{dt} = \sum_j \Delta(t-T_j), \tag{2.21}$$

where $\Delta(x)$ is a Dirac delta function which can be loosely thought of as a function on the real line which is zero everywhere except at the origin, at which it is infinite

$$\Delta(x) = \begin{cases} +\infty, & x=0, \\ 0, & x\neq 0, \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{+\infty} \Delta(x)dx = 1.$$

In the classical compound Poisson process model,

$$dS_t = d\sum_{i=0}^{N_t} X_i = X_{N_t} dN_t = X_{N_t} \frac{dN_t}{dt} dt = \sum_i X_{N_t} \Delta(t - T_i) dt.$$

Thus, the derivative of a compound Poisson process can be denoted by

$$y_t = \frac{dS_t}{dt} = \sum_i X_{N_t} \Delta(t - T_i)$$

and for the discounted compound Poisson process, we have

$$dS_t = de^{-\delta t} \sum_{i=0}^{N_t} X_n = e^{-\delta t} \sum_i X_{N_t} \Delta(t - T_i) dt - \delta e^{-\delta t} \sum_{i=0}^{N_t} X_i dt.$$

Therefore, the derivative of a discounted compound Poisson process can be derived by

$$y_t = \frac{dS_t}{dt} = e^{-\delta t} \sum_i X_{N_t} \Delta(t - T_i) - \delta e^{-\delta t} \sum_{n=0}^{N_t} X_n.$$

2.6 The deficit at ruin and joint probabilities

Here are some basic deficit concepts which will be seen throughout the thesis. In this section, we will look at the amount of the insurer's deficit at the time of ruin. The first time the surplus process falls below zero is referred to as τ_u in (2.3), then we denote Y_u to be the deficit at ruin from initial surplus u. Gerber et al. (1987) proposed a quantitative measure

$$G(u, y, t) = \mathbb{P}[\tau_u \leq t, Y_u \leq y],$$

defined as the joint probability of ruin by time t with a deficit of at most y at ruin. Letting $t \to \infty$, this becomes

$$G(u, y) = \lim_{t \to \infty} G(u, y, t),$$

which is the joint probability of ultimate ruin and the deficit of at most y at ruin, with the defective density $g(u, y) = \frac{d}{dy} G(u, y)$ (Gerber, 1988). Note that

$$\lim_{y \to \infty} G(u, y) = \mathbb{P}[\tau_u \leq \infty, Y_u \leq \infty] = \psi(u).$$

Bowers et al. (1997) provided an expression for $g(0,y)$ under the classical risk model given by (1.1),

$$g(0,y) = \frac{\lambda}{c}\bar{F}(y), \tag{2.22}$$

where $\bar{F}(y)$ is the tail distribution of the claim., s.t. $\bar{F}(y) = 1 - F(y)$. Using this expression and conditioning on the amount of surplus immediately after the first time the surplus falls below its initial level, we have the following results,

Theorem 2.6.1. (Defective renewal equation for $g(u,y)$) The defective density function $g(u,y)$ satisfies the defective renewal equation under the classical risk process given by (1.1),

$$g(u,y) = \frac{\lambda}{c}\int_0^u g(u-x,y)\bar{F}(x)dx + \frac{\lambda}{c}\bar{F}(u+y);$$

see, for example, Gerber et al. (1987).

Applying the Laplace transform on $g(u,y)$ with respect to u, we have the following corollary.

Corollary 2.6.1. The Laplace transformation of the function $g(u,y)$ with respect to u is given by

$$\hat{g}(s,y) = \frac{\frac{\lambda}{c}[\frac{1}{s} - e^{sy}\hat{F}(s)]}{1 - \frac{\lambda}{c}[\frac{1}{s} - \hat{F}(s)]};$$

see, for example, Gerber et al. (1987).

Hence, if the Laplace transformation of the function $g(u,y)$ can be found, the remaining work in order to obtain the explicit solution of $G(u,y)$ is to invert the transformation. The explicit solutions for $G(u,y)$ have been found for individual claim amount distributions that follow a combination of exponential distributions or a combination of gamma distributions (Gerber et al., 1987).

2.7 The barrier model

Barrier problems originated from the classical risk process, where we seek variations of the surplus process to reflect some real-life scenario of the insurance and reinsurance portfolio. De Finetti (1957) defined the maximum surplus level prior to ruin as

$$M_u = \max(U_t : 0 < t < \tau_u)$$

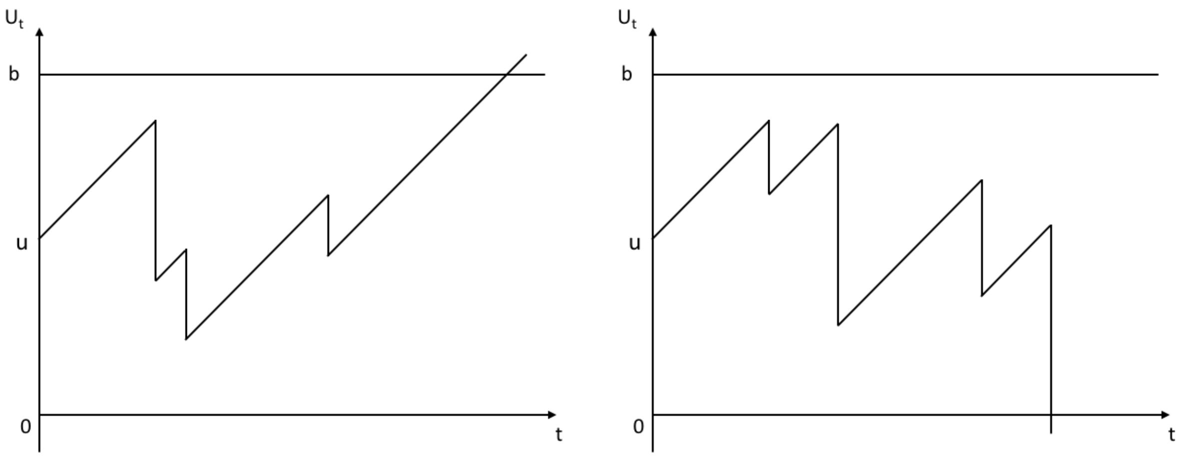
and the ruin time for the two barriers model,

$$\tau_u^b = \inf\{t \geq 0 : U_t = b | U_0 = u\},\ u \leq b$$

to be the first hitting time of a barrier b from initial surplus u. Let

$$\xi(u, b) = \mathcal{P}[M_u < b, \tau_u < \infty] = \mathcal{P}[\tau_u < \tau_u^b], 0 \leq u < b.$$

It is clear that $\xi(u, b)$ is the probability that ruin occurs without the surplus level reaching b and $\bar{\xi}(u, b) = 1 - \xi(u, b)$ denotes the probability of attaining the level b from initial capital u, without ruin occurring. The following figure illustrates the two situations described.



(a) Reaching barrier before ruin $\bar{\xi}(u, b)$ (b) Ruin before reaching barrier $\xi(u, b)$

Figure 2.3: The sample path of the barrier process

Dickson and Gray (1986) showed that ξ and $\bar{\xi}$ can be expressed in terms of the ultimate ruin probability $\psi(u)$,

$$\bar{\xi}(u,b) = \frac{1-\psi(u)}{1-\psi(b)}, \quad \xi(u,b) = \frac{\psi(u)-\psi(b)}{1-\psi(b)}. \tag{2.23}$$

In fact, the results of the absorbing barrier problem are usually referred to as a dividend problem. Whenever the surplus attains the level b, the premium income is paid to shareholders continuously as dividends until the next claim occurs, so that in this modified surplus process, the surplus never attains a level greater than b. In other words, it can be considered as an absorbing barrier with dividend payments.

Asmussen and Taksar (1997) investigated the optimal dividend strategies under a controlled diffusion model, where the dividend is paid at a unfixed rate, depending on the surplus level. Paulsen and Gjessing (1997) extended the claim process with a Brownian motion and defined a stochastic interest rate on reserves by an another independent Brownian motion. In recent years, the dividend strategy problem has been considered in a wide variety of risk models. Lin et al. (2003) investigated the Gerber-Shiu function in the presence of the dividend barrier. In addition, Gerber and Shiu (2004) provided a general recursive formula to obtain moments of the present value of shareholders' income when the surplus process is modelled by a Brownian motion with positive drift.

Chapter 3

Claim Dependent Risk Process

The purpose of the mixing distribution is to provide an additional class of dependence models for which explicit expressions for ruin probability can be obtained (Bühlmann, 1972; Albrecher et al., 2011). To that end, we start with some specific models for which explicit expressions of the ruin probability are available and then mix over involved parameters of claim size. In this chapter, we will introduce the claim dependent model under the discrete compound binomial risk process (2.12) by applying the mixing idea over values of involved parameter. Recall that the claim amounts $\{X_i\}$ are identically, independent distributed zero-modified geometric ZMG(q, ρ) with the new convenient setting of the success probability $\rho = 1 - e^{-\Theta}$, where the mixing variable Θ is considered as a random variable. Equation (3.1) investigates the relationship between the probability mass function of claims by mixing over ρ and the Laplace transform with respect to Θ. In addition, Proposition 3.2.1 provides the equation to derive the ruin probability. Under the claim's setting of a zero-modified Pareto and Weibull distribution, the explicit ruin probabilities are given in Corollary 3.2.1 and 3.2.3.

3.1 Mixing claim distribution

Consider again the compound binomial risk model (2.12) were, given $\Theta = \theta$, the $\{X_i\}$ have ZMG distribution given by the PMF (2.13) with the success probability as in (2.18).

To see why the latter condition is more convenient than the one given by (2.15), we first derive the probability mass function of the claim amount X under the new setting (2.18). Let F_Θ be the cumulative distribution function of the mixing variable Θ and let f_Θ be the corresponding density function. Clearly, $\mathbb{P}(X=0)=q$, while for $k \geqslant 1$, we have

$$\mathbb{P}(X=k)=\int_0^\infty \mathbb{P}(X=k|\Theta=\theta)dF_\Theta(\theta)=\int_0^\infty (1-q)(1-e^{-\theta})(e^{-\theta})^{k-1}dF_\Theta(\theta)$$

$$=(1-q)\left\{\int_0^\infty e^{-\theta(k-1)}dF_\Theta(\theta)-\int_0^\infty e^{-\theta k}dF_\Theta(\theta)\right\}=(1-q)\left\{\hat{f}_\Theta(k-1)-\hat{f}_\Theta(k)\right\},$$

where $\hat{f}_\Theta$ is the Laplace transform of the variable Θ. This leads to a convenient, general formula for the PMF of X:

$$\mathbb{P}(X=k)=q\delta_{k0}+(1-\delta_{k0})(1-q)\left\{\hat{f}_\Theta(k-1)-\hat{f}_\Theta(k)\right\},\quad k\in\mathbb{N}. \tag{3.1}$$

With this choice of Θ, the resulting distributions of the claim amounts can be calculated by (3.1):

1. For Θ having gamma distribution with shape parameter $\alpha>0$ and scale parameter $\beta>0$, with the density function (2.16), its Laplace transformation $\hat{f}_\Theta(k)=(\frac{\beta}{\beta+k})^\alpha$, the PMF of the claim amount X turns into that of the zero-modified discrete Pareto (ZMP) distribution:

$$\mathbb{P}(X=k)=q\delta_{k0}+(1-\delta_{k0})(1-q)\left\{\left(\frac{\beta}{\beta+k-1}\right)^\alpha-\left(\frac{\beta}{\beta+k}\right)^\alpha\right\}\quad k\in\mathbb{N}. \tag{3.2}$$

2. For Θ having a Lévy stable distribution (stable subordinator with exponent 1/2) with the density function (2.17), its Laplace transformation is given as $\hat{f}_\Theta(k)=e^{-\alpha k^{1/2}}$ and the PMF of the claim amount X turns into that of the zero-modified discrete Weibull (ZMW) distribution:

$$\mathbb{P}(X=k)=q\delta_{k0}+(1-\delta_{k0})(1-q)\left\{e^{-\alpha(k-1)^{1/2}}-e^{-\alpha k^{1/2}}\right\},\quad k\in\mathbb{N}. \tag{3.3}$$

When comparing the ZMG, ZMP and ZMW models (see in Figures 3.1 and 3.2), we notice that for the same expectation of claims and the same value of q when the zero claims occured, the PMF displays the heavier tail of the ZMW and ZMP distributions.

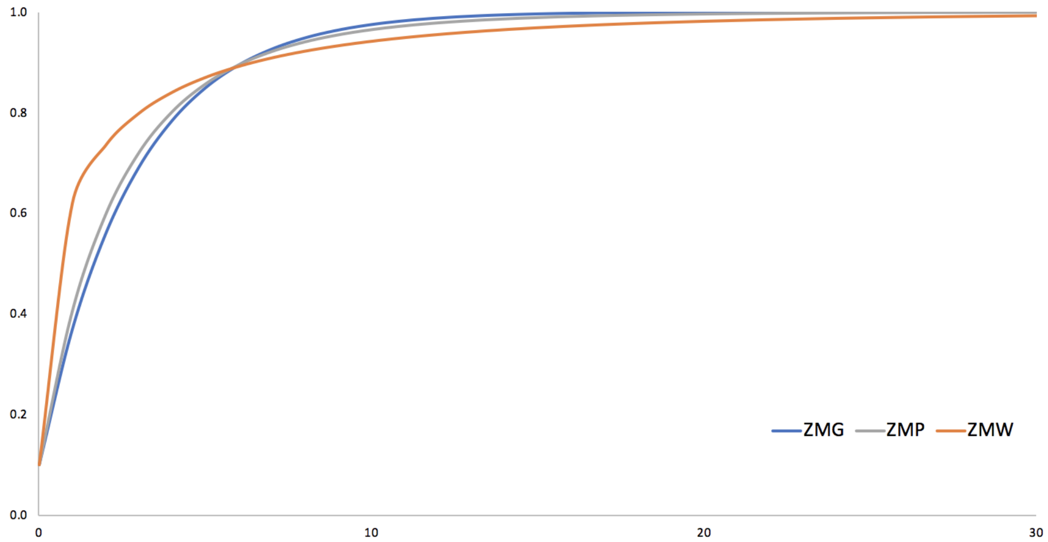


Figure 3.1: The CDFs under ZMG, ZMP and ZMW models

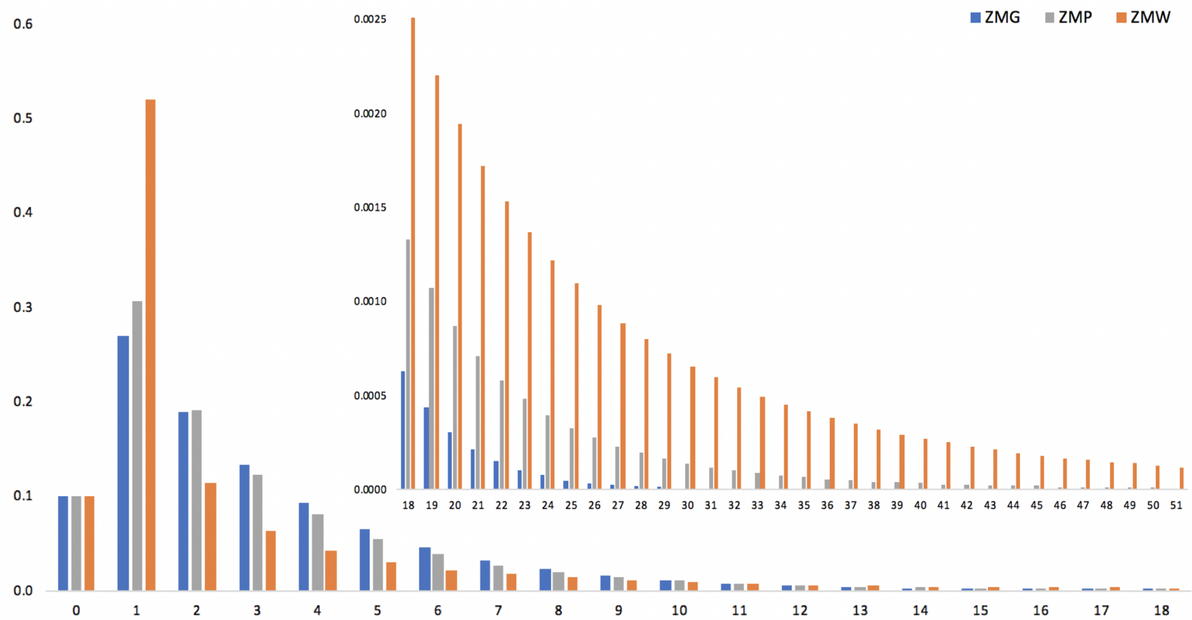


Figure 3.2: The PMFs under ZMG, ZMP and ZMW models

Similar calculations show that the CDF of the claim distribution in our set-up is given by

$$\mathbb{P}(X \leq x) = 1 - (1-q)\hat{f}_{\Theta}(\lfloor x \rfloor), \;\; x \in \mathbb{R}_+,$$

while the survival probability becomes

$$\mathbb{P}(X > x) = (1-q)\hat{f}_\Theta(\lfloor x \rfloor), \quad x \in \mathbb{R}_+,$$

where $\lfloor x \rfloor$ denotes the integer part of x (the floor function). When Θ is either gamma distributed with the PDF (2.16) or is positive stable with the PDF (2.17), then the tail probabilities take on particularly simple forms, given by

$$\mathbb{P}(X > x) = (1-q)\left(\frac{1}{1+\lfloor x \rfloor/\lambda}\right)^\alpha \quad \text{and} \quad \mathbb{P}(X > x) = (1-q)e^{-\tau(\lfloor x \rfloor)^\alpha},$$

respectively. The above formulae should be contrasted with the rather inconvenient integral that appears in the first paragraph of Section 4.2 in Dutang et al. (2013).

3.2 The probability of ruin

Let us now derive the probability of ruin under our set-up. First, let us note that the probability of ruin in (2.14) becomes

$$\psi(u) = \frac{1-q}{\rho}\left(\frac{1-\rho}{q}\right)^{u+1}$$

if and only if $\rho \geq 1-q$ (the net profit condition). To see this, observe that the above holds if and only if

$$\frac{1-q}{\rho}\left(\frac{1-\rho}{q}\right)^{u+1} \leq 1,$$

which is equivalent to

$$\frac{(1-\rho)^{u+1}}{\rho} \leq \frac{q^{u+1}}{1-q}. \tag{3.4}$$

Consider the function $h(\rho) = (1-\rho)^{u+1}/\rho$, $\rho \in (0,1)$. Since

$$\frac{dh(\rho)}{d\rho} = -(1-\rho)^u \frac{(u+1)\rho + 1 - \rho}{\rho^2} < 0,$$

the function h is decreasing on the interval $(0,1)$, and so (3.4) is equivalent to $\rho \geq 1-q$ as desired. Now, if we set $1-\rho = e^{-\theta}$, the net profit condition becomes $\theta > \theta^*$, where $\theta^* = -\log q \in (0,\infty)$. Then, analogously to (10) in Dutang et al. (2013), the probability of ruin can be written as

$$\psi(u) = F_\Theta(\theta^*) + J(u,\theta^*), \tag{3.5}$$

where

$$J(u,\theta^*) = \frac{1-q}{q^{u+1}} \int_{\theta^*}^{\infty} \frac{e^{-\theta(1+u)}}{1-e^{-\theta}} dF_\Theta(\theta). \tag{3.6}$$

One can obtain a compact formula for the above probability, in terms of a geometric random variable $\mathcal{N} \sim \text{Geo}(p)$, given by the PMF

$$\mathbb{P}(\mathcal{N} = k) = p(1-p)^{k-1}, \ \ k \in \mathbb{N}, \tag{3.7}$$

the probability generating function (PGF)

$$\mathbb{E}\left(s^{\mathcal{N}}\right) = \frac{sp}{1-s(1-p)}, \ \ s \in (0,1), \tag{3.8}$$

and the excess random variable

$$\Theta^* \stackrel{d}{=} \Theta - \theta^* | \Theta \geq \theta^*. \tag{3.9}$$

If Θ is absolutely continuous, then the PDF of the latter is

$$f_{\Theta^*}(\theta) = \frac{f_\Theta(\theta+\theta^*)}{1-F_\Theta(\theta^*)}, \ \ \theta \in \mathbb{R}_+. \tag{3.10}$$

The following result provides relevant details.

Proposition 3.2.1. Let Θ have an absolutely continuous distribution on $\mathbb{R}_+$ with the CDF and the PDF denoted by F_Θ and f_Θ, respectively, and suppose that, given $\Theta = \theta$, the variables $\{X_i\}$ of the discrete time risk model (2.12) are independent and identically distributed modified geometric ZMG(q,ρ) with the PMF (2.13) and $\rho = 1 - e^{-\theta}$. Then, the probability of ruin is given by

$$\psi(u) = F_\Theta(\theta^*) + [1 - F_\Theta(\theta^*)]\mathbb{E}\left\{e^{-(u+\mathcal{N})\Theta^*}\right\}, \tag{3.11}$$

where $\theta^* = -\log q$, Θ^* is the excess random variable given by the PDF (3.10), and $\mathcal{N}$ is a geometric random variable (3.7) with parameter $p = 1 - q$, independent of Θ^*.

Proof. Let us work with the quantity $J(u,\theta^*)$ given by (3.6). We have

$$J(u,\theta^*) = [1 - F_\Theta(\theta^*)]\frac{1-q}{q} \int_{\theta^*}^{\infty} \frac{e^{-\theta u} e^{-\theta} q^{-u}}{1-e^{-\theta}} \frac{f_\Theta(\theta)}{[1 - F_\Theta(\theta^*)]} d\theta. \tag{3.12}$$

Note that

$$q^{-u} = e^{-u\log q} = e^{\theta^* u},$$

so that

$$J(u,\theta^*) = [1-F_\Theta(\theta^*)]\frac{1-q}{q}\int_{\theta^*}^{\infty} \frac{e^{-u(\theta-\theta^*)}e^{-\theta}}{1-e^{-\theta}}\frac{f_\Theta(\theta)}{[1-F_\Theta(\theta^*)]}d\theta.$$

Upon the substitution of $x = \theta - \theta^*$ into (3.12) we obtain

$$J(u,\theta^*) = [1-F_\Theta(\theta^*)]\int_0^{\infty} e^{-ux}\frac{(1-q)e^{-x}}{1-qe^{-x}} f_{\Theta^*}(x)dx. \qquad (3.13)$$

We now recognize the term

$$\frac{(1-q)e^{-x}}{1-qe^{-x}}$$

under the integral in (3.13) as the PGF of geometric variable $\mathcal{N}$ with the PMF (3.7) and $p = 1-q$, evaluated at $s = e^{-x}$ (so this is actually the Laplace transform of $\mathcal{N}$), so that we can write the above integral as

$$\mathbb{E}\left\{e^{-u\Theta^*}\mathbb{E}\left(e^{-\Theta^*\mathcal{N}}|\Theta^*\right)\right\} = \mathbb{E}\left\{\mathbb{E}\left(e^{-u\Theta^*}e^{-\Theta^*\mathcal{N}}|\Theta^*\right)\right\} = \mathbb{E}\left\{e^{-(u+\mathcal{N})\Theta^*}\right\},$$

as desired. This completes the proof. □

Routine calculations lead to the following result, describing the special case with gamma-distributed Θ and zero-modified discrete Pareto (3.2) correlated claim amounts. Note that the probability of ruin given below involves the (upper) incomplete gamma function,

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt$$

as it does in an analogous problem considered by Dutang et al. (2013)).

Corollary 3.2.1. Let Θ have a gamma distribution with the PDF (2.16) and suppose that, given $\Theta = \theta$, the variables $\{X_i\}$ in (2.12) are independent and identically distributed modified geometric ZMG(q,ρ) with the PMF (2.13) and $\rho = 1-e^{-\theta}$. Then, the probability of ruin $\psi(u)$ is given by

$$\psi(u) = 1 - \frac{\Gamma(\alpha, -\beta\log q)}{\Gamma(\alpha)} + \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{1-q}{q^{u+1}}\sum_{k=1}^{\infty}\frac{\Gamma\left(\alpha, -(k+u+\beta)\log q\right)}{(k+u+\beta)^\alpha}.$$

Below we present a special case with exponential mixing distribution, where the probability of ruin may take on an explicit form.

Corollary 3.2.2. Let Θ have an exponential distribution with parameter $\beta > 0$ and suppose that, given $\Theta = \theta$, the variables $\{X_i\}$ in (2.12) are independent and identically distributed modified geometric ZMG(q, ρ) with the PMF (2.13) and $\rho = 1 - e^{-\theta}$. Then, if $\beta \in \mathbb{N}$, the probability of ruin is given by

$$\psi(u) = (1-q)\left\{1 - \frac{\beta}{q^{u+1}}\left[\log(1-q) + \sum_{k=1}^{u+\beta} \frac{q^k}{k}\right]\right\}.$$

Remark 3.2.1. As can be seen from the ruin probability formula in the ZMP case, the probability of ruin converges to a non-zero level as $u \to \infty$, which is due to the net profit condition being violated. Therefore, in the ZMP model the ruin probability is more stable for large u compared with its behaviours under the ZMG model. Furthermore, the rate of convergence can vary with the parameters, as can be seen in the example given in Table 3.2, by the parameters 1-4 provided in Table 3.1 below. When comparing Set 1 with Set 2, and Set 2 with Set 3, one can notice that larger β and smaller α lead to a larger probability of ruin and faster convergence (the difference in ruin probabilities between $u = n$ and $u = n+1$ is smaller than 10^{-8}). In other words, larger β and lower α flatten the ruin probability. According to Set 4, one can see that as the probability q of no claims increases, the ruin probability decreases. Moreover, starting with $u = 53$, the probability is already convergent to the level where the net profit condition is violated. We also notice that the decrease is of 9.719% (from $\psi(0) = 54.1\%$ to $\psi(53) = 44.39\%$). This decrease is larger than the one in the case of Set 1, which was only 0.028% (from $\psi(0) = 86.6\%$ to $\psi(20) = 86.36\%$). Thus, the larger the q, the lower the ruin probability, the steeper the decrease, and the slower the convergence.

Set	1	2	3	4
α	2	2	4	2
β	5	10	5	5
q	0.2	0.2	0.2	0.5

Table 3.1: Parameters' coefficients

Set	1	2	3	4
$\psi(0)$	0.86584	0.99264	0.49289	0.54108
$\psi(\infty)$	0.86356	0.99263	0.46225	0.44389
convergent after $u =$	20	15	24	53

Table 3.2: Results for the speed of convergence

The result below provides the ruin probability for the special case where Θ is Lévy stable with index 1/2 and PDF (2.17), in which case we have conditionally independent zero-modified discrete Weibull (ZMW) claim amounts, with the PMF (3.3) and 1/2. As in the analogous problem considered by Dutang et al. (2013), the probability of ruin can be expressed in terms of the complementary error special function

$$\mathrm{Erfc}(z) = 1 - \mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{3.14}$$

Corollary 3.2.3. Let Θ be a stable (1/2) distributed random variable (also called Lévy distributed), the variables $\{X_i\}$ in (2.12) are independent and identically distributed modified geometric ZMG(q, ρ) with the PMF (3.3) and $\rho = 1 - e^{-\theta}$. Then the probability of ruin is given by

$$\psi(u) = \mathrm{Erfc}(\frac{\alpha}{2\sqrt{-\log q}}) + \frac{1-q}{q^{u+1}} \sum_{k=1}^{\infty} [q^{u+k} \mathrm{Erf}(\frac{\alpha}{2\sqrt{-\log q}})$$

$$- \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1} (u+k)^{n+\frac{1}{2}}}{n! \sqrt{\pi} (2n+1) 4^n} \Gamma(-\frac{2n-1}{2}, -(u+k)\log q)],$$

where $\Gamma(\cdot,\cdot)$ and $\mathrm{Erfc}(\cdot)$ are given by (3.2) and (3.14), respectively.

Proof. Let $\theta^* = -\log q$ as before. Then, by taking into account the PDF of Θ given by (2.17) and Proposition 3.2.1, we obtain

$$\psi(u) = F_\Theta(\theta^*) + (1 - F_\Theta(\theta^*)) \mathbb{E}\left\{ e^{-(u+N)\Theta^*} \right\},$$

$$\mathrm{Erfc}\left(\frac{\alpha}{2\sqrt{\theta^*}}\right) + (1 - F_\Theta(\theta^*)) \sum_{k=1}^{\infty} \int_0^{\infty} \frac{f_\Theta(\theta + \theta^*)}{1 - F_\Theta(\theta^*)} e^{-(u+k)\theta} (1-q) q^{k-1} d\theta$$

$$= \mathrm{Erfc}\left(\frac{\alpha}{2\sqrt{\theta^*}}\right) + \sum_{k=1}^{\infty} \int_0^{\infty} f_\Theta(\theta + \theta^*) e^{-(u+k)\theta} (1-q) q^{k-1} d\theta$$

$$= \text{Erfc}\left(\frac{\alpha}{2\sqrt{\theta^*}}\right) + \sum_{k=1}^{\infty}(1-q)q^{k-1}e^{(u+k)\theta^*}\int_{\theta^*}^{\infty} f_\Theta(t)e^{-(u+k)t}dt$$
$$= \text{Erfc}\left(\frac{\alpha}{2\sqrt{\theta^*}}\right) + \sum_{k=1}^{\infty}(1-q)q^{k-1}q^{-(u+k)}\int_{\theta^*}^{\infty} f_\Theta(t)e^{-(u+k)t}dt$$
$$= \text{Erfc}\left(\frac{\alpha}{2\sqrt{\theta^*}}\right) + \frac{1-q}{q^{u+1}}\sum_{k=1}^{\infty}\int_{\theta^*}^{\infty} f_\Theta(t)e^{-(u+k)t}dt,$$

Where in the last equality we used

$$\int_{\theta^*}^{\infty} f_\Theta(t)e^{-(u+k)t}dt = \int_{\theta^*}^{\infty} e^{-(u+k)\theta}d\text{Erfc}(\frac{\alpha}{2\sqrt{\theta}})$$
$$= e^{-(u+k)\theta^*}\text{Erfc}(\frac{\alpha}{2\sqrt{\theta^*}}) + (u+k)\int_{\theta^*}^{\infty} e^{-(u+k)\theta}\text{Erfc}(\frac{\alpha}{2\sqrt{\theta}})d\theta.$$

Finally, the substitution

$$\text{Erfc}\left(\frac{\alpha}{2\sqrt{\theta}}\right) = 1 - \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\alpha}{2})^{2n+1}}{n!(2n+1)}\theta^{-n-\frac{1}{2}}$$

leads to

$$\int_{\theta^*}^{\infty} f_\Theta(t)e^{-(u+k)t}dt =$$
$$e^{-(u+k)\theta^*}\text{Erfc}\left(\frac{\alpha}{2\sqrt{\theta^*}}\right) + (u+k)\int_{\theta^*}^{\infty} e^{-(u+k)\theta}\left(1 - \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\alpha}{2})^{2n+1}}{n!(2n+1)}\theta^{-n-\frac{1}{2}}\right)d\theta$$
$$= e^{-(u+k)\theta^*}\text{Erfc}\left(\frac{\alpha}{2\sqrt{\theta^*}}\right) - \sum_{n=0}^{\infty}\frac{(-1)^n\alpha^{2n+1}(u+k)^{n+\frac{1}{2}}}{n!\sqrt{\pi}(2n+1)4^n}\Gamma\left(-\frac{2n-1}{2},(u+k)\theta^*\right),$$

and the result follows. □

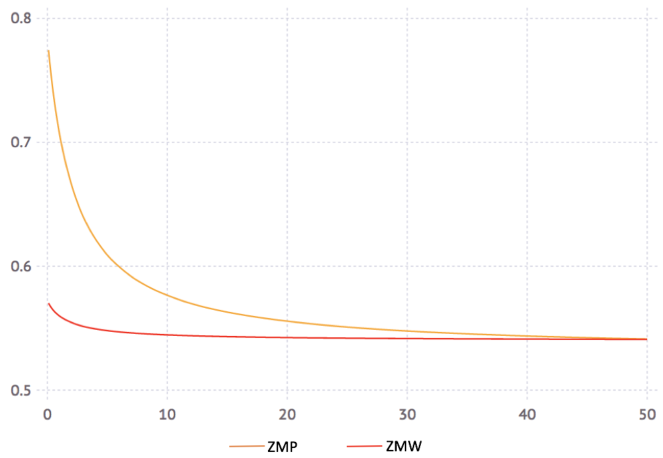


Figure 3.3: Ruin probabilities under the ZMP and ZMW models with a same level of $\lim_{u\to\infty}\psi(u)$

Remark 3.2.2. Let $L = F_{\Theta}(\theta^*)$ be the level at which the net profit condition is violated. In Figure 3.3, one can set up the same level L of $\psi(u)$ as $u \to \infty$ for both, zero modified Pareto and Weibull models (denoted, respectively, by ZMP and ZMW). From Figure 3.3, one can see that the ruin probability curve is steeper under the ZMP model and it starts from a higher initial ruin probability $\psi(0)$.

Remark 3.2.3. Figure 3.4 and Table 3.3 below show that, when we increase the value of τ (the parameter in the ZMW model) from 1 to 1.1, the ruin probability curve decreases by 3% at given level L. This can be observed by increasing the expectation of the claims. Additionally, a smaller α corresponds to a larger ruin probability and faster convergence to level L.

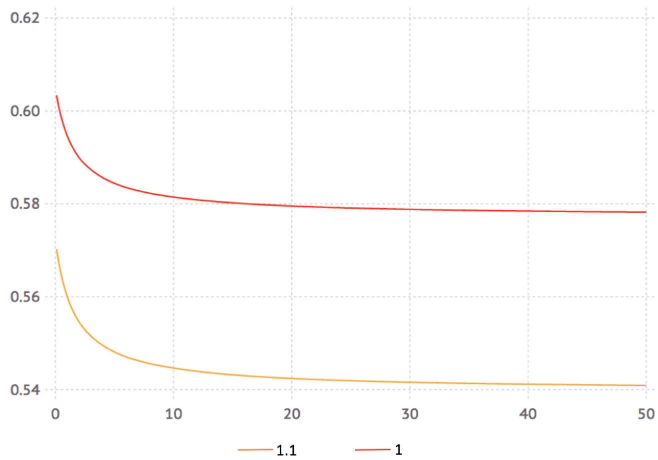


Figure 3.4: Ruin probabilities under the ZMW model with different value of α

Set	$\alpha = 1$	$\alpha = 1.1$
$\psi(0)$	0.60338	0.57028
$\psi(\infty)$	0.57776	0.54037
convergent after $u =$	50	70

Table 3.3: Results for the speed of convergence under the ZMW model

3.3 Illustrative data example

As an illustration, here we fit the three zero-modified models, ZMG, ZMP and ZMW, to a real reinsurance data from a large UK company. The data was skewed and scaled for confidentiality reasons. Claims data span the time period of 11 years. The zero and the non-zero frequencies are shown in Table 3.4 given below. Zero claims refer to accidents that the company paid nothing for, due to deductibles on other contracts considerations.

	Zero claims	Non-zero claims	Total claims
Number	97	348	445

Table 3.4: The structure of the analyzed reinsurance data set

The model frequency q of zero claims is estimated by the corresponding sample frequency, $\hat{q}$, resulting in $\hat{q} = 0.218$. Figure 3.5 illustrates the ruin probabilities under the three models.

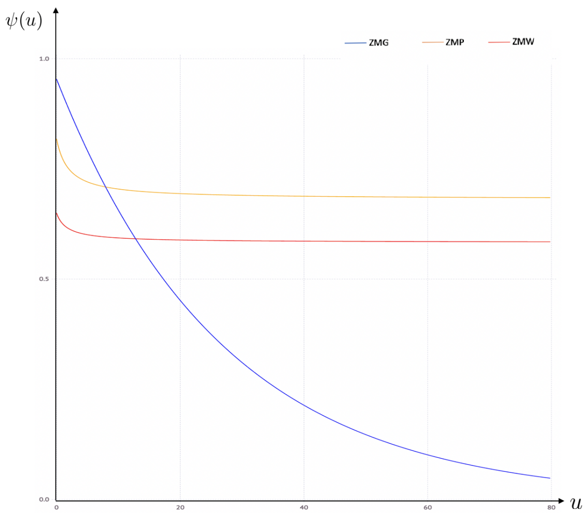


Figure 3.5: Ruin probabilities for the three considered models

Remark 3.3.1. Note that while fitting the data, we will keep the same net profit condition, meaning the same θ^* in (3.11). In the Figure (3.5), the levels of convergence $F(\theta^*)$ are different due to different distributions F.

u	$\psi(u)_{ZMG}$	$\psi(u)_{ZMP}$	$\psi(u)_{ZMW}$
0	0.954	0.818	0.650
1	0.919	0.772	0.625
2	0.885	0.749	0.614
3	0.852	0.736	0.608
4	0.821	0.727	0.603
5	0.791	0.720	0.601
10	0.656	0.704	0.593
15	0.544	0.698	0.590
25	0.374	0.692	0.588
30	0.311	0.690	0.587
40	0.214	0.688	0.586
50	0.147	0.687	0.585
51-100	0.146-0.005	0.687-0.685	0.585-0.584

Table 3.5: Ruin probabilities for three considered models

To measure the goodness-of-fit, we use P-P plots and the sum of the squared errors (SSE), shown in Figure 3.6 and Table 3.6, respectively.

	ZMG	ZMP	ZMW
$\hat{q}$	0.218	0.218	0.218
$\hat{\rho}$	0.79	N/A	N/A
$\hat{\alpha}_{ZMP}$	N/A	1.289	N/A
$\hat{\alpha}_{ZMW}$	N/A	N/A	0.958
$\hat{\beta}$	N/A	0.986	N/A
SSE	1.026	0.023	0.035

Table 3.6: Estimated parameters of the three considered models

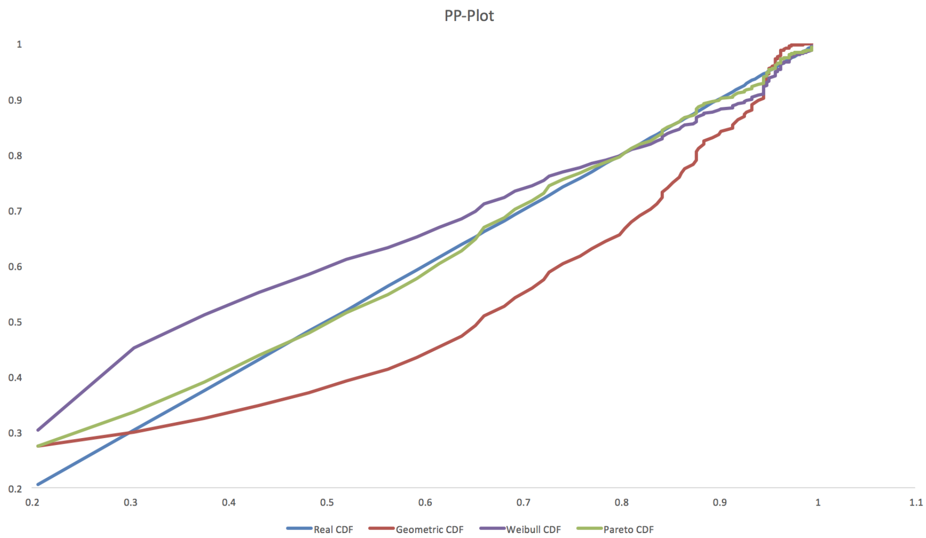


Figure 3.6: PP-plots for the three considered models

It is aparent that the heavy-tailed ZMP model provides the best fit for the data among three fitted models. Furthermore, our data analysis leads to the same conclusion as that provided by our theoretical results. Namely, while the ZMG model has the largest ruin probability when $u = 0$, it decays very quickest as the initial investment increases. As far as the ZMP and ZMW models, the ruin probability under the ZMP model is always larger than that under the ZMW model.

Chapter 4

Time Dependent Risk Process

The main results in this chapter are based on Willmot (2015) and ideas given by Takács (1977). We aim to derive the Seal's formulae for the general inhomogeneous Poisson process and explicit expressions of the finite time and infinite time ruin probability. In particular, we derive the infinite time ruin probability by applying the idea of Usabel (1998), who considered the ruin probability as the summation of the probability of ruin before and after a certain time. Using a different setting of the surplus process in comparison to (1.1), the initial age of the surplus process is considered as in (4.1). Theorem 4.2.2 provides a backward martingale with respect to the surplus level and the intensity function, which is the key to evaluating the probability of ruin and deriving the inhomogeneous type of Seal's formulae. In addition, a number of examples for the Cox process are given in Section 4.3. In particular, under the setting of the Markov jump process, Theorem 4.3.2 shows integro-differential equations which can be used to calculate the ultimate ruin probability by applying backward recursions. Two states model is a special case of the Markov jump process and its ruin probability is derived in Example 4.3.4.

4.1 Model setting

In this chapter, we define a modified stochastic process from (1.1) with initial capital u and duration from initial age $a+s$ to time $a+t$,

$$R_{a+s}^{a+t}(u) = u + C_{a+s}^{a+t} - S_{a+s}^{a+t}, \quad R_{a+s}^{a+t}(0) = R_{a+s}^{a+t}, \quad t \geq s \tag{4.1}$$

with

$$S_{a+s}^{a+t} = \sum_{n=N_{a+s}}^{N_{a+t}} X_n, \quad t \geq s,$$

where $F_{a+s}^{a+t} = F(a+t) - F(a+s)$ for any process F and N_t follows the inhomogeneous Poisson process with intensity function $\lambda(t)$. For the net profit condition, a suitable premium $C(t)$ should first exceed the average paid claims s.t. $C(t) = (1+\theta)\Lambda(t)\mathbb{E}[X]$.

In particular, we assume that

$$\frac{C_{a+s}^{a+t}}{C_{a+s}^{a+v}} = \frac{\Lambda_{a+s}^{a+t}}{\Lambda_{a+s}^{a+v}} \text{ and } \Lambda_a^{a+t} > 0 \text{ for any } t \geq v \geq s \geq 0. \tag{4.2}$$

Furthermore, the distribution of the aggregate claims size process is given by

$$\begin{aligned} K_{a+s}^{a+t}(x) &= \mathbb{P}[S_{a+s}^{a+t} \leq x] = \mathbb{P}[\sum_{n=N_{a+s}}^{N_{a+t}} X_n \leq x] \\ &= \sum_{m=0}^{\infty} \frac{[\Lambda(a+t) - \Lambda(a+s)]^m}{m!} e^{-[\Lambda(a+t)-\Lambda(a+s)]} F_X^{*m}(x). \end{aligned}$$

When $a = s = 0$, we denote $K_0^t(x) = K(x,t)$ and $K_a^a(x) = 1$ and $k(y,t)$ as the density of the aggregate claims size proces, s.t. $\int_0^x k(y,t)dy = K(x,t)$. Now compared to (2.3) and (2.2), the time of ruin is denoted by

$$\tau_{a+s}(u) = \inf\{0 \leq v \leq \infty; R_{a+s}^{a+s+v}(u) < 0\}$$

and the finite time ruin probability from initial age $a+s$ to time $a+t$ is defined as

$$\begin{aligned} \psi_{a+s}^{a+t}(u) &= 1 - \phi_{a+s}^{a+t}(u) = \mathbb{P}[a+s < \tau_{a+s}(u) \leq a+t] \\ &= \mathbb{P}[R_{a+s}^{a+s+v}(u) < 0, \text{for some } v \in (0, t-s)]. \end{aligned}$$

When $a = s = 0$, we denote

$$\psi_0^t(u) = 1 - \phi_0^t(u) = \psi(u,t) = 1 - \phi(u,t)$$

and when t goes to infinity, we have the infinite time ruin probability

$$\lim_{t\to\infty} \psi_{a+s}^{a+t}(u) = \mathbb{P}\left[a + s < \tau_{a+s}(u) < \infty\right] = \psi_{a+s}(u).$$

Seal (1974) provided the classical Seal's formulae in order to derive the general solution for the finite time non-ruin probability under the classical case (1.1).

Theorem 4.1.1. The finite time non-ruin probability for the homogeneous Poisson process is given by

$$\phi(0,t) = \frac{1}{ct}\int_0^{ct} \mathbb{P}[S_t < x]dx = \frac{1}{ct}\int_0^{ct} K(x,t)dx,$$

$$\begin{aligned}\phi(u,t) &= \mathbb{P}[S_t < u + ct] - \int_0^t \phi(0,t-s)\mathbb{P}[S_s \in (u, u+ds)] \\ &= K(u+ct,t) - \int_0^t \phi(0,t-s)k(u+sc,s)ds,\end{aligned}$$

Example 4.1.1. Now construct the model with constant intensity λ, premium rate p and exponentially distributed claims $X \sim \text{Exp}(\beta)$, thus the density function of the aggregate claims size is given by

$$k(x,t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}\mathbb{P}^{*n}(x) = \frac{1}{\lambda}\eta_{\lambda,n+1}(t)\eta_{\beta,n+1}(x),$$

then

$$K(y,t) = \int_0^y k(x,t)dx$$

and the ruin probability can be simply expressed by

$$\begin{aligned}\phi(u,t) = &\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\sum_{i=0}^{m} c_{n,i}\eta_{\beta,m-i+1}(u)\eta_{\lambda+p\beta,n+i+1}(t) \\ &- p\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{i} a_{n,m}b_{i,j}\eta_{\beta,i-j+1}(u)\eta_{\lambda+p\beta,n+m+i+j}(t),\end{aligned}$$

where $a_{n,m}$, $b_{i,j}$ and $c_{n,i}$ are all constants.

4.2 Seal's formulae for the inhomogeneous Poisson process model

In this section, we are going to derive the Seal's formulae for the inhomogeneous Poisson process model. According to the lecture notes given by Schmidli (2017), firstly, we construct a conditional expectation.

Theorem 4.2.1. Given by the setting of surplus process (4.1) and (4.2), we have for any $s \leq t$,

$$\mathbb{E}[S_a^{a+s}|S_a^{a+t} = y] = \frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}}y.$$

Proof. Denote the permutations of $\{1, 2, ..., n\}$ by σ, we have

$$\begin{aligned}
&\mathbb{E}[S_a^{a+s}|S_a^{a+t} = y, N_{a+t} - N_a = n, N_{a+s} - N_a = m] \\
&= \mathbb{E}[\sum_{i=0}^{m} x_i|S_a^{a+t} = y, N_{a+t} - N_a = n, N_{a+s} - N_a = m] \\
&= \frac{1}{n!}\mathbb{E}[\sum_{\sigma}\sum_{i=0}^{m} x_{\sigma(i)}|S_a^{a+t} = y, N_{a+t} - N_a = n, N_{a+s} - N_a = m] \\
&= \frac{m(n-1)!}{n!}\mathbb{E}[\sum_{i=0}^{n} x_i|S_a^{a+t} = y, N_{a+t} - N_a = n, N_{a+s} - N_a = m] = \frac{my}{n},
\end{aligned}$$

because of the independent increment property of the inhomogeneous Poisson process, for $0 < s < t$, we have

$$\begin{aligned}
&\mathbb{P}[N_{a+s} - N_a = m|N_{a+t} - N_a = n] = \frac{\mathbb{P}[N_{a+s} - N_a = m, N_{a+t} - N_a = n]}{\mathbb{P}[N_{a+t} - N_a = n]} \\
&= \frac{\mathbb{P}[N_{a+s} - N_a = m, N_{a+t} - N_{a+s} = n - m]}{\mathbb{P}[N_{a+t} - N_a = n]} = \frac{\mathbb{P}[N_{a+s} - N_a = m]\mathbb{P}[N_{a+t} - N_{a+s} = n - m]}{\mathbb{P}[N_{a+t} - N_a = n]} \\
&= \binom{n}{m}(\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}})^m(\frac{\Lambda_{a+s}^{a+t}}{\Lambda_a^{a+t}})^{n-m} = \binom{n}{m}(\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}})^m(1 - \frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}})^{n-m}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbb{E}[S_a^{a+s}|S_a^{a+t} = y, N_{a+t} - N_a = n] = \sum_{m=0}^{n}\frac{my}{n}\binom{n}{m}(\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}})^m(1 - \frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}})^{n-m} \\
&= \sum_{m=1}^{n}\binom{n-1}{m-1}(\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}})^m(1 - \frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}})^{n-m}y = \frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}}y,
\end{aligned}$$

which is independent of n, thus

$$\mathbb{E}[S_a^{a+s}|S_a^{a+t}=y]=\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}}y.$$

□

Now we consider backward martingales,

Theorem 4.2.2. Given by the setting of surplus process (4.1) and (4.2) and Theorem 4.2.1, for $0<s\leq t$, the processes

$$M_s^t(a)=\frac{y-S_a^{a+s}}{\Lambda_{a+s}^{a+t}},$$
$$N_s^t(a)=\frac{y-R_a^{a+s}}{\Lambda_{a+s}^{a+t}}$$

are backward martingales with limits of

$$\lim_{s\to t}M_s^t(a)=0,\ \lim_{s\to t}N_s^t(a)=c.$$

Proof. According to Theorem 4.2.1, we obtain

$$\mathbb{E}[R_a^{a+s}|R_a^{a+t}=y]=\mathbb{E}[C_a^{a+s}-S_a^{a+s}|S_a^{a+t}=C_a^{a+t}-y]=C_a^{a+s}+\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}}(y-C_a^{a+t})$$
$$=\mathbb{E}[S_a^{a+s}|S_a^{a+t}=y]+C_a^{a+s}-\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}}C_a^{a+t}.$$

Due to the assumption given by (4.2), we then have

$$\mathbb{E}[R_a^{a+s}|R_a^{a+t}=y]=\mathbb{E}[S_a^{a+s}|S_a^{a+t}=y]=\frac{\Lambda_a^{a+s}}{\Lambda_a^{a+t}}y,$$

therefore,

$$\mathbb{E}[\frac{y-S_a^{a+s}}{\Lambda_{a+s}^{a+t}}|S_a^{a+v},S_a^{a+t}=y]=\frac{y-S_a^{a+v}-\mathbb{E}[S_a^{a+s}-S_a^{a+v}|S_a^{a+v},S_a^{a+t}=y]}{\Lambda_{a+s}^{a+t}}$$
$$=\frac{y-S_a^{a+v}-\frac{\Lambda_{a+v}^{a+s}}{\Lambda_{a+v}^{a+t}}(y-S_a^{a+v})}{\Lambda_{a+s}^{a+t}}=\frac{\frac{\Lambda_{a+s}^{a+t}}{\Lambda_{a+v}^{a+t}}(y-S_a^{a+v})}{\Lambda_{a+s}^{a+t}}=\frac{y-S_a^{a+v}}{\Lambda_{a+v}^{a+t}}=M_v^t(a)$$

and

$$\mathbb{E}[\frac{y-R_a^{a+s}}{\Lambda_{a+s}^{a+t}}|R_a^{a+v},R_a^{a+t}=y]=\frac{y-R_a^{a+v}}{\Lambda_{a+v}^{a+t}}=N_v^t(a).$$

Thus $M_v^t(a)$ and $N_v^t(a)$ are backward martingales. Furthermore, the second equivalence can be proved by assuming $\lim_{s\to t} \frac{C_{a+s}^{a+t}}{\Lambda_{a+s}^{a+t}} = c$,

$$\lim_{s\to t} M_s^t(a) = 0, \lim_{s\to t} N_s^t(a) = \lim_{s\to t} \frac{C_{a+s}^{a+t} - S_{a+s}^{a+t}}{\Lambda_{a+s}^{a+t}} = c.$$

□

Now we aim to find the conditional finite non-ruin probability given the terminal R_a^{a+t} or $R_a^{a+t}(u)$ for some fixed t with the following proposition,

Proposition 4.2.1. Let t be fixed, $u = 0$ and $0 < y \leq ct$. According to assumption given by (4.2), we have

$$\mathbb{P}[R_a^{a+s} \geq 0, 0 < s < t | R_a^{a+t} = y] = \frac{y}{c\Lambda_a^{a+t}}.$$

Proof. Now let $T(y) = \inf\{t \geq s \geq 0 : R_a^{a+s} = y\}$, according to theorem 4.2.2, we have

$$\lim_{s\to t} \mathbb{E}[N_s^t(a)\mathbf{1}_{\{T(y)<s\}} | R_a^{a+t} = y] = c\mathbb{P}[T(y) = t | R_a^{a+t} = y].$$

Note that $\{N_{T(y)\wedge 0}^t(a)\}$ is a bounded martingale, according to $N_{T(y)}^t(a) = 0$ on $\{T(y) < t\}$, we have

$$N_0^t(a) = \frac{y}{\Lambda_a^{a+t}} = c\mathbb{P}[T(y) = t | R_a^{a+t} = y],$$

where $\mathbb{P}[T(y) = 0 | R_a^{a+t} = y] = \mathbb{P}[R_a^{a+s} \geq 0, 0 < s < t | R_a^{a+t} = y]$. □

Now we obtain the following theorem for Seal's formulae under the general inhomogeneous Poisson process model.

Theorem 4.2.3. Given by the setting of surplus process (4.1) and (4.2) and Theorem 4.2.1, we have the inhomogeneous type of Seal's formulae from initial age $a + s$ to time $a + t$. For initial capital $u = 0$,

$$\phi_{a+s}^{a+t}(0) = \frac{1}{c\Lambda_{a+s}^{a+t}} \int_0^{C_{a+s}^{a+t}} K_{a+s}^{a+t}(y)dy.$$

For $u > 0$,

$$\phi_{a+s}^{a+t}(u) = K_{a+s}^{a+t}(u + C_{a+s}^{a+t}) - c\int_s^t \phi_{a+v}^{a+t}(0)d_v K_{a+s}^{a+v}(u + C_{a+s}^{a+v})$$

and when $t \to \infty$

$$\psi_a(u) = \lim_{t\to\infty} c\int_0^t \phi_{a+s}^{a+t}(0)dK_a^{a+s}(u + C_a^{a+s}).$$

Proof. For $u = 0$,

$$\begin{aligned}
\phi_{a+s}^{a+t}(0) &= \mathbb{E}[R_{a+s}^{a+t} \vee 0]\frac{1}{c\Lambda_{a+s}^{a+t}} = \mathbb{E}[C_{a+s}^{a+t} - S_{a+s}^{a+t}) \vee 0]\frac{1}{c\Lambda_{a+s}^{a+t}} \\
&= \frac{1}{c\Lambda_{a+s}^{a+t}}\int_0^{C_{a+s}^{a+t}} C_{a+s}^{a+t} - x dK_{a+s}^{a+t}(x) = \frac{1}{c\Lambda_{a+s}^{a+t}}\int_0^{C_{a+s}^{a+t}}\int_x^{C_{a+s}^{a+t}} dy dK_{a+s}^{a+t}(x) \\
&= \frac{1}{c\Lambda_{a+s}^{a+t}}\int_0^{C_{a+s}^{a+t}}\int_0^y dK_{a+s}^{a+t}(x)dy = \frac{1}{c\Lambda_{a+s}^{a+t}}\int_0^{C_{a+s}^{a+t}} K_{a+s}^{a+t}(y)dy.
\end{aligned}$$

For $u > 0$,

$$\phi_{a+s}^{a+t}(u) = \mathbb{P}[R_{a+s}^{a+t}(u) > 0] - \mathbb{P}[\exists s \le v < t : R_{a+s}^{a+v}(u) = 0, R_{a+s}^{a+z}(u) > 0 \text{ for v} < z \le t].$$

Now let $\tau_{a,s}(t) = \inf\{s \le v \le t : R_{a+s}^{a+v}(u) = 0\}$, set $\tau_{a,s}(t) = \infty$ if $R_{a+s}^{a+v}(u) > 0$ for all $v \in [s,t]$, then we have

$$\begin{aligned}
&\mathbb{P}\Big[\tau_{a,s}(t) \in [a+s, a+v+dv)\Big] \\
&= \mathbb{P}\Big[R_{a+s}^{a+v}(u) \in (-C_{a+s}^{a+v+dv}, 0], R_{a+s}^{a+z}(u) > 0 \text{ for } z \in [v+dv, t-s]\Big] \\
&= \Big[K_{a+s}^{a+v}(u + C_{a+s}^{a+v+dv}) - K_{a+s}^{a+v}(u + C_{a+s}^{a+v})\Big]\phi_{a+v+dv}^{a+t}(0) \\
&= c\int_s^t \phi_{a+v}^{a+t}(0)\lambda(a+v)d_v K_{a+s}^{a+v}(u + C_{a+s}^{a+v}).
\end{aligned}$$

Thus

$$\phi_{a+s}^{a+t}(u) = K_{a+s}^{a+t}(u + C_{a+s}^{a+t}) - c\int_s^t \phi_{a+v}^{a+t}(0)\lambda(a+v)d_v K_{a+s}^{a+v}(u + C_{a+s}^{a+v}).$$

Besides, we have $\lim_{t\to\infty} K_a^{a+t}(u + C_a^{a+t}) = 1$, thus

$$\psi_a(u) = \lim_{t\to\infty} c\int_0^t \phi_{a+s}^{a+t}(0)\lambda(a+s)d_s K_a^{a+s}(u + C_a^{a+s}).$$

□

Remark 4.2.1. In fact, we could apply the idea of the time shifting property from proposition 2.2.1 to obtain the inhomogeneous type of Seal's formulae under this setting 4.1.

Proposition 4.2.2. For the classical homogeneous Poisson process with constant intensity λ, theorem 4.2.3 is able to fit the classical Seal's formulae given by theorem 4.1.1. For $u = 0$,

$$\begin{aligned}\phi_{a+s}^{a+t}(0) &= \frac{1}{(1+\theta)\mu\lambda(t-s)}\int_0^{(1+\theta)\mu\lambda(t-s)} K_{a+s}^{a+t}(y)dy \\ &= \frac{1}{p(t-s)}\int_0^{p(t-s)} K_{a+s}^{a+t}(y)dy = \phi(0,t-s),\end{aligned}$$

where $p = (1+\theta)\mu\lambda$. Then for the non-ruin probability with initial capital u,

$$\begin{aligned}\phi_a^{a+t}(u) &= K_a^{a+t}(u+(1+\theta)\mu\lambda t) - p\int_0^t \phi(0,t-s)d_sK_a^{a+s}(u+(1+\theta)\mu\lambda s]) \\ &= K(u+pt,t) - p\int_0^t \phi(0,t-s)dK(u+ps,s)\end{aligned}$$

are all independent of the initial age a, due to the renewal property.

Proof. Under the homogeneous Poisson process condition, we have $\Lambda(t) = \lambda t$ and $\lambda(\cdot) = \lambda$, then the distribution of the aggregate claims is defined by

$$K_a^{a+s}(x) = \mathbb{P}[S_a^{a+s} < x] = \sum_{m=0}^{\infty} \frac{(\lambda s)^m}{m!} e^{-\lambda s} F_X^{*m}(x) = K(x,s),$$

which is independent of a. Then for $u = 0$,

$$\begin{aligned}\phi_{a+s}^{a+t}(0) &= \frac{1}{(1+\theta)\mu\lambda(t-s)}\int_0^{(1+\theta)\mu\lambda(t-s)} K_{a+s}^{a+t}(y)dy \\ &= \frac{1}{p(t-s)}\int_0^{p(t-s)} K_{a+s}^{a+t}(y)dy = \phi(0,t-s),\end{aligned}$$

where $p = (1+\theta)\mu\lambda$. For the non-ruin probability with initial capital u, according to the renewal property, we have

$$\begin{aligned}\phi_a^{a+t}(u) &= \mathbb{P}[R_a^{a+t}(u) > 0] - \mathbb{P}\Big[\exists 0 \le s \le t : R_a^{a+s}(u) = 0, R_a^{a+v}(u) > 0 \text{ for } 0 \le v < s\Big] \\ &= K_a^{a+t}(u+(1+\theta)\mu\lambda t) - p\int_0^t \phi(0,t-s)d_sdK_a^{a+s}(u+(1+\theta)\mu\lambda s]) \\ &= K(u+pt,t) - p\int_0^t \phi(0,t-s)dK(u+ps,s).\end{aligned}$$

□

Example 4.2.1. Assume the claim follows an exponential distribution $X \sim exp(\beta)$. Then the aggregate claims follow an Erlang distribution. The density function of an Erlang random variable with parameter β and n is defined by

$$\eta_{\beta,n+1}(x) = \frac{\beta^{n+1}x^n}{n!}e^{-\beta x}.$$

Then the distribution of the aggregate claims size is

$$K_a^{a+s}(x) = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\frac{\big[\Lambda(a+s)-\Lambda(a)\big]^n}{\beta n!}e^{-\big[\Lambda(a+s)-\Lambda(a)\big]}\eta_{\beta,m+1}(x).$$

The non-ruin probability with zero initial can be calculated by theorem (4.2.3),

$$\begin{aligned}\phi_{a+s}^{a+t}(0) &= \frac{1}{c\big[\Lambda(a+t)-\Lambda(a+s)\big]}\int_0^{C_{a+s}^{a+t}} K_{a+s}^{a+t}(y)dy \\ &= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{k=m+1}^{\infty}\frac{\big[\Lambda(a+t)-\Lambda(a+s)\big]^{n-1}}{n!c\beta^2}e^{-\big[\Lambda(a+t)-\Lambda(a+s)\big]}\eta_{\beta,k+1}(C_{a+s}^{a+t}).\end{aligned}$$

Now we derive the decomposition derivative,

$$\begin{aligned}&d_s\big[K_a^{a+s}(u+c\Lambda_a^{a+s})\big] \\ &= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{k=0}^{m}\lambda(a+s)\eta_{\beta,k+1}(u)\boldsymbol{d_s}\frac{k!(m-k)!\big[\Lambda(a+s)-\Lambda(a)\big]^n}{\beta^2 n!}e^{-\big[\Lambda(a+s)-\Lambda(a)\big]}\eta_{\beta,m-k+1}(C_a^{a+s}),\end{aligned}$$

then we define

$$H_a(t,n,m,k) = \int_0^t \phi_{a+s}^{a+t}(0)\lambda(a+s)d_s\frac{k!(m-k)!\big[\Lambda(a+s)-\Lambda(a)\big]^n}{\beta^2 n!}e^{-\big[\Lambda(a+s)-\Lambda(a)\big]}\eta_{\beta,m-k+1}(C_a^{a+s})$$

and

$$\begin{aligned}K_a^{a+t}(u+ct) &= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\frac{\big[\Lambda(a+t)-\Lambda(a)\big]^n}{n!}e^{-\big[\Lambda(a+t)-\Lambda(a)\big]}\eta_{\beta,m+1}(u+C_a^{a+t}) \\ &= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{k=0}^{m}h_a(t,n,m,k)\eta_{\beta,k+1}(u).\end{aligned}$$

Therefore, the non-ruin probability can be derived as

$$\phi_a^{a+t}(u) = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{k=0}^{m}\big[h_a(t,n,m,k)-H_a(t,n,m,k)\big]\eta_{\beta,k+1}(u). \tag{4.3}$$

We now consider the infinite time probability. Referring to Garrido et al. (1996), an analogous Volterra integral equation in terms of $\psi_a(u)$ for the model (4.1) is given by the following theorem,

Theorem 4.2.4. The infinite time probability of ruin beginning with initial reserve u, from initial age a to time $a+t$, satisfies the integral equation:

$$\begin{aligned}\psi_a(u) = &\int_0^{\infty} \lambda(a+s)e^{-\Lambda_a^{a+s}} \int_0^{u+cs} \psi_{a+s}^{a+t}(u+cs-x)dF_X(x)ds \\ &+ \int_0^{\infty} \lambda(a+s)e^{-\Lambda_a^{a+s}} \bar{F}_X(u+cs)ds.\end{aligned}$$

According to the previous results, we aim to figure out the connection between the infinite time and finite time ruin probability. We now introduce the following theorem,

Theorem 4.2.5. For any $h \geq 0$,

$$\psi_a(u) = \psi_a^{a+t}(u) + \sum_{i=0}^{\infty} {}_{t+hi}|\psi_a^{a+t+h(i+1)}(u)$$

and

$${}_t|\psi_a(u) +_t |\phi_a(u) = K_a^{a+t}(u + C_a^{a+t}).$$

Therefore, the ultimate ruin probability can be derived by the infinite sum

$$\psi_a(u) = \psi_a^{a+t}(u) + \sum_{i=0}^{\infty} \int_0^{u+c\Lambda_a^{a+t_i}} \psi_{a+t_i}^{a+t_{i+1}}(u + C_a^{a+t_i} - y)k_a^{a+t_i}(y)dy, \tag{4.4}$$

where $t_i = t+hi$, ${}_{t_i}|\psi_a^{a+t_{i+1}}(u) = \mathbb{P}[a+t_i < \tau_a(u) < a+t_{i+1}]$ and ${}_{t_i}|\phi_a^{a+t_{i+1}}(u) = \mathbb{P}[\tau_a(u) \notin (a+t_i, a+t_{i+1})]$.

Proof. According to Usabel (1998), we have

$$\psi_a(u) = \psi_a^{a+t}(u) +_t |\psi_a(u),$$

by applying the total probability theorem,

$$
\begin{aligned}
&{}_{t|}\psi_a(u) = \mathbb{P}[a+t<\tau_a(u)<\infty] \\
=&\int_0^{u+C_a^{a+t}} \mathbb{P}\big[a+t<\tau_{a+t}(R_a^{a+t}(u))<\infty | R_a^{a+t}(u)=y\big]\mathbb{P}[R_a^{a+t}(u)=y]dy \\
=&\int_0^{u+C_a^{a+t}} \mathbb{P}[a+t<\tau_{a+t}(y)<\infty]\mathbb{P}[R_a^{a+t}(u)=y]dy \\
=&\int_0^{u+C_a^{a+t}} \psi_{a+t}(y)\mathbb{P}[R_a^{a+t}(u)=y]dy = \int_0^{u+C_a^{a+t}} \psi_{a+t}(u+C_a^{a+t}-y)k_a^{a+t}(y)dy \\
=&\int_0^{u+C_a^{a+t}} 1-\phi_{a+t}(u+c\Lambda_a^{a+t_0}-y)k_a^{a+t}(y)dy \\
=&K_a^{a+t}(u+C_a^{a+t}) -_t |\phi_a(u).
\end{aligned}
$$

Therefore, the infinite time ruin probability can be denoted as

$$
\psi_a(u) = \psi_a^{a+t}(u) + \int_0^{u+C_a^{a+t}} \psi_{a+t}(u+C_a^{a+t}-y)k_a^{a+t}(y)dy.
$$

In addition, for any $h \geq 0$, ${}_{t|}\psi_a(u)$ can be expressed as

$$
\begin{aligned}
&{}_{t|}\psi_a(u) = \mathbb{P}[a+t<\tau_a(u)<\infty] = \sum_{i=0}^{\infty} \mathbb{P}[a+t+hi<\tau_a(u)<a+t+h(i+1)] \\
=&\sum_{i=0}^{\infty}\int_0^{u+C_a^{a+t+hi}} \mathbb{P}\big[a+t+hi<\tau_{a+t+hi}(y)<a+t+h(i+1)\big]\mathbb{P}[R_a^{a+t+hi}(u)=y]dy \\
=&\sum_{i=0}^{\infty}\int_0^{u+C_a^{a+t+hi}} \psi_{a+t+hi}^{a+t+h(i+1)}(u+C_a^{a+t+hi}-y)k_a^{a+t+hi}(y)dy \\
=&\sum_{i=0}^{\infty} {}_{t+hi|}\psi_a^{a+t+h(i+1)}(u).
\end{aligned}
$$

Now we let $t_i = t+hi$, therefore

$$
{}_{t|}\psi_a(u) = \sum_{i=0}^{\infty} {}_{t_i|}\psi_a^{a+t_i+h}(u).
$$

□

Example 4.2.2. The model satisfies example 4.2.1, according to (4.3) and (4.4), we have

$$
\phi_{a+t_i}^{a+t_{i+1}}(u) = \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{k=0}^{m}\big[h_{a+t_i}(h,n,m,k) - H_{a+t_i}(h,n,m,k)\big]\eta_{\beta,k+1}(u),
$$

then

$$\begin{aligned}
&\int_0^{u+c\Lambda_a^{a+t+hi}} \eta_{\beta,k+1}(u+c\Lambda_a^{a+t_i}-y)k_a^{a+t_i}(y)dy \\
=&\sum_{j=0}^{\infty}\frac{[\Lambda_a^{a+t_i}]^j}{j!}e^{-\Lambda_a^{a+t_i}}\int_0^{u+c\Lambda_a^{a+t_i}} \eta_{\beta,k+1}(u+c\Lambda_a^{a+t_i}-y)\eta_{\beta,j}(y)dy \\
=&\sum_{j=0}^{\infty}\sum_{p=0}^{k}\frac{[\Lambda_a^{a+t_i}]^j}{j!}e^{-\Lambda_a^{a+t_i}}\frac{\beta^{k+j+1}(-1)^p}{(k-p)!p!(j-1)!}(u+c\Lambda_a^{a+t_i})^{k-p}e^{-\beta(u+c\Lambda_a^{a+t_i})}\int_0^{u+c\Lambda_a^{a+t_i}} y^{p+j-1}dy \\
=&\sum_{j=0}^{\infty}\sum_{p=0}^{k} G_a^{a+t_i}(k,j,p)\eta_{\beta,k+j+1}(u+c\Lambda_a^{a+t_i}),
\end{aligned}$$

where $G_a^{a+t_i}(k,j,p) = \frac{[\Lambda_a^{a+t_i}]^j}{j!}e^{-\Lambda_a^{a+t_i}}\frac{(-1)^k(k+j)!}{(k-p)!p!(j-1)!(p+j)}$. Therefore, we have the ultimate ruin probability expression,

$$\begin{aligned}
\psi_a(u) =&\psi_a^{a+t}(u) + \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\sum_{k=0}^{m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{p=0}^{k}[h_{a+t_i}(h,n,m,k)-H_{a+t_i}(h,n,m,k)] \\
&G_a^{a+t_i}(k,j,p)\eta_{\beta,k+j+1}(u+c\Lambda_a^{a+t_i}).
\end{aligned}$$

4.3 Cox process model

In this section, we are going to construct the Cox process by letting the intensity parameter be a stochastic process which is defined in particular as the shot-noise Cox process or the other specific processes. The reason why we need the Cox process is that the homogeneous and inhomogeneous Poisson processes do not adequately explain the phenomena of catastrophes (Dassios and Jang, 2003). In addition, Cox process can provide a more stochastic setting of the occurrence of claims (Ammeter, 1948). In order to investigate the properties of the cox process and ruin probabilities, we start with the investigation of the claim occurrence process and the distribution of the aggregate claims size.

4.3.1 Shot-noise Cox process

Recall that $S_t = \sum_{i=1}^{N_t} X_i$ denotes the aggregate claims process up to time t. In the classical risk model, S_t represents the compound Poisson process with constant intensity λ. In this section, N_t becomes a Cox process with a Poisson shot-noise intensity which can be found in Albrecher and Asmussen (2006),

$$\lambda_t = \lambda_0 + \sum_{n=1}^{N_\rho(t)} h(t - T_n, Y_n), \tag{4.5}$$

where $\{T_n\}_{n\in\mathbb{N}}$ is the sequence of occurrences of a homogeneous Poisson process with rate ρ and it represents the occurring times of external events. $\{Y_n\}_{n\in\mathbb{N}}$ is a sequence of positive independent and identically distributed random variables (with distribution function F_Y) which are independent of the Poisson process $N_\rho(t)$. In addition, the function $h(t, x)$ represents the non-negative response function (shot function) in the shot-noise Cox process.

The shot-noise process has also been investigated in a more general form. If the function (4.5) is simplified to a multiplicative shot function by splitting $h(t - T_i, Y_i) = Y_i g(t - T_i)$, then we obtain

$$\lambda_t = \lambda_0 + \sum_{n=1}^{N_\rho(t)} Y_n g(t - T_n), \tag{4.6}$$

where $g(t)$ is a non-negative function with following properties:

1. $g(t) \geq 0$ for all $t \geq 0$ and $g(t) = 0$ for $t < 0$,

2. $G(t) = \int_0^t g(s)ds$, $g(\infty) = 0, H(Y, t) = \int_0^t h(Y, s)ds$,

3. The mean value function is defined as an integrated stochastic process $\mathbb{E}[N(t)] = \mathbb{E}[\Lambda_t|\mathcal{F}_t] = \mathbb{E}[\int_0^t \lambda_s ds|\mathcal{F}_t]$,

4. There exists $\theta < \min\{1, \alpha\}$ such that $\int_0^\infty g(t)^\theta dt < \infty$.

Generalisations of the shot-noise Cox process which allow the intensity of the claim frequency to depend on time are given by Dassios and Jang (2003). Furthermore, from Campbell's formula, the expectation of the random process λ_t can be denoted as

$$\mathbb{E}(\lambda_t) = \lambda_0 + \rho \int_0^t \mathbb{E}[g(t-s)Y_1]ds = \lambda_0 + \rho \int_0^t \mathbb{E}[g(s)Y_1]ds = \lambda_0 + \rho G(t)\mathbb{E}[Y_1].$$

Then $\lim_{t\to\infty} \mathbb{E}(\lambda_t) = \lambda_0 + \rho\mathbb{E}[G(\infty)Y_1] = \omega$ which is assumed to be finite. Also, we need to assume the net profit condition $c > \mu$ and $\mu = \omega\mu_X$ ($\mu_X = \mathbb{E}(X)$). Assuming that $h(y,t) = ye^{-\delta t}$, Cox and Isham (1980) have shown the Laplace transformation of $Z_t = \sum_{n=1}^{N_\rho(t)} Y_n g(t-T_n)$ is given by

$$\mathcal{L}_{Z_t}(s) = e^{-\rho\int_0^t \left(1-L_Y(se^{-\delta z})\right)dz}.$$

Assume that all moments of X_t exist, then

$$\mathbb{E}[Z_t^i] = (-1)^i \frac{d^i\mathcal{L}_{Z_t}(s)}{ds^i}\Big|_{s=0},$$

therefore, we could have

$$\mathbb{E}[Z_t] = \frac{\rho\mathbb{E}[Y]}{\delta}(1-e^{-\delta t}), \quad Var[Z_t] = \frac{\rho\mathbb{E}[Y^2]}{2\delta}(1-e^{-2\delta t}).$$

Corollary 4.3.1. The stochastic integrated process $\Lambda_t = \int_0^t \lambda_s ds$ can be written as

$$\Lambda_t = \lambda_0 t + \sum_{n=0}^{N_\rho(t)} Y_n G(t-T_n).$$

Proof. We have

$$\Lambda_t = \int_0^t \lambda_s ds = \lambda_0 t + \int_0^t \int_0^{t_1} Y_{N_\rho(s)} g(t_1 - s) dN_\rho(s) dt_1,$$

after changing the order of the integral by applying Fubini's theorem,

$$= \lambda_0 t + \int_0^t \int_s^t Y_{N_\rho(s)} g(t_1-s)dt_1 dN_\rho(s) = \lambda_0 t + \int_0^t Y_{N_\rho(s)} G(t-s) dN_\rho(s)$$
$$= \lambda_0 t + \sum_{n=0}^{N_\rho(t)} Y_n G(t-T_n).$$

□

From Albrecher and Asmussen (2006), by assuming $g(t) = e^{-\delta t}$ and $G(t) = \int_0^t g(s)ds = \frac{1}{\delta}\left(1 - e^{-\delta t}\right)$, the stochastic integrated process can be rearranged as

$$\Lambda_t = \int_0^t \lambda_s ds = \lambda_0 t + \sum_{i=0}^{N_\rho(t)} Y_i G(t - T_i) = \lambda_0 t + \frac{1}{\delta} \sum_{i=0}^{N_\rho(t)} Y_i \left(1 - e^{-\delta(t-T_i)}\right).$$

by applying Taylor expansion,

$$\Lambda_t = \lambda_0 t + \frac{1}{\delta} \sum_{i=0}^{N_\rho(t)} Y_i \left(1 - e^{-\delta(t-T_i)}\right) = \lambda_0 t + \sum_{i=0}^{N_\rho(t)} Y_i \left[t - T_i - (t - T_i)^2 + (t - T_i)^3 ...\right].$$

Then we have

$$\mathbb{E}[\Lambda_t] = \lambda_0 t + \frac{\rho \mathbb{E}[Y]}{\delta} \left[t - \frac{1}{\delta}(1 - e^{-\delta t})\right], \ \mathbf{V}ar[\Lambda_t] = \lambda_0 t + \frac{\rho \mathbb{E}[Y^2]}{2\delta} \left[t - \frac{1}{2\delta}(1 - e^{-2\delta t})\right].$$

Dassios et al. (2015) introduced some doubts on the application of the shot-noise Cox process: In the case of small ρ (the rate of shot event arrival), they use the shot-noise process as an intensity function for catastrophic events. However, if the parameter becomes large, it means that the shot events are no longer considered to be catastrophes. Therefore, we can consider the shot-noise process to be an intensity function to generate the number of claims due to common events of high frequency, such as car accidents or accidents from a large collective insurance portfolio.

According to the following corollary, we are able to find the moment generating function of the intensity of the shot-noise Cox process.

Corollary 4.3.2. Let $N_\rho(t)$ be a Poisson process with parameter $\rho > 0$ and $Z_t = \sum_{n=1}^{N_\rho(t)} Y_n g(t - T_n)$. Then for any $s > 0$, the moment generating function of Z_t is given by

$$M_{Z_t}(s) = \mathbb{E}[e^{sZ_t}] = e^{\rho \int_0^t [M_Y(sg(t-v))-1]dv}. \tag{4.7}$$

We then have the following theorem,

Theorem 4.3.1. Let λ_t be a shot-noise Cox process with exponential(ω) events Y_i and decay function $g(t) = e^{-\delta t}$, the moment generating function of λ_t is given by

$$M_{\lambda_t}(s) = e^{\lambda_0 s} \left[\frac{\omega - s e^{-\delta t}}{\omega - s}\right]^{\frac{\rho}{\delta}}.$$

Let $t \to \infty$, it is clear to see

$$\lim_{t\to\infty} M_{Z_t}(s) = \left(\frac{\omega}{\omega - s}\right)^{\frac{\rho}{\delta}} = \left(1 - \frac{s}{\omega}\right)^{-\frac{\rho}{\delta}},$$

where $\lim_{t\to\infty} Z_t \sim \Gamma(\frac{\rho}{\delta}, \frac{1}{\omega})$ (Dassios et al., 2015).

Proof. Assuming $Y_i \sim Exp(\omega)$ (its moment generating function is $\frac{\omega}{\omega+s}$) and $g(t) = e^{-\delta t}$, then we have

$$M_{\lambda_t}(s) = e^{\lambda_0 s} e^{\rho \int_0^t \left[\frac{\omega}{se^{-\delta(t-v)}} - 1\right] dv} = e^{\lambda_0 s} e^{\frac{\rho}{\delta} \ln \frac{w - se^{-\delta t}}{\omega - s}}$$
$$= e^{\lambda_0 s} \left[\frac{w - se^{-\delta t}}{\omega - s}\right]^{\frac{\rho}{\delta}} = M_{\lambda_0}(s) M_{Z_t}(s).$$

Let $t \to \infty$ we have $\lim_{t\to\infty} M_{Z_t}(s) = \left(1 - \frac{s}{\omega}\right)^{-\frac{\rho}{\delta}}$ s.t. $\lim_{t\to\infty} Z_t \sim \Gamma(\frac{\rho}{\delta}, \omega)$. Therefore, $\mathbb{P}\left(\lambda_\infty \leq \lambda_0 + a\right) = F_\Lambda(\lambda_0 + a) = \frac{\gamma(\frac{\rho}{\delta}, \frac{a}{\omega})}{\Gamma(\frac{\rho}{\delta})}$, where $a \in (0, +\infty)$ and we use λ_∞ to denote $\lim_{t\to\infty} \lambda_t$. □

4.3.2 Other Cox processes

In this section, we aim to construct some special Cox processes and provide their statistical properties, in addition to providing the brief expression of the ruin probability if it is possible. We are going to provide some examples of the Cox process in order to cover the following fields: heavy-tailed intensity (compound Poisson process with Pareto distributed jumps), correlation with stock price (geometric Brownian motion, applications can be found in Asmussen and Albrecher (2010)), discounted jump process and the Markov jump process.

Example 4.3.1. (Heavy-tailed intensity)
In probability theory, heavy-tailed distributions are probability distributions whose tails are not exponentially bounded (Albrecher and Boxma, 2004). The definition can be given as the distribution of a random variable X with distribution function F_X is said to have a heavy right tail if

$$\lim_{x\to\infty} e^{\lambda x} \mathbb{P}[X > x] = \infty \quad \text{for all } \lambda > 0$$

This is also written in terms of the tail distribution function

$$\bar{F}_X(x) \equiv \mathbb{P}[X > x],$$

as

$$\lim_{x\to\infty} e^{\lambda x}\bar{F}_X(x) = \infty \quad \text{for all } \lambda > 0.$$

This is equivalent to the statement that the moment generating function of $F_X(x)$, $M_X(t)$, is infinite for all $t > 0$.

Here, the heavy-tailed intensity process is given by

$$\lambda_t = \lambda_0 \prod_{n=0}^{N(t)} e^{\alpha Y_n} e^{-\delta t}, \tag{4.8}$$

where $Y_n \sim Exp(\omega)$ and $N(t)$ is a homogeneous process with parameter ρ.

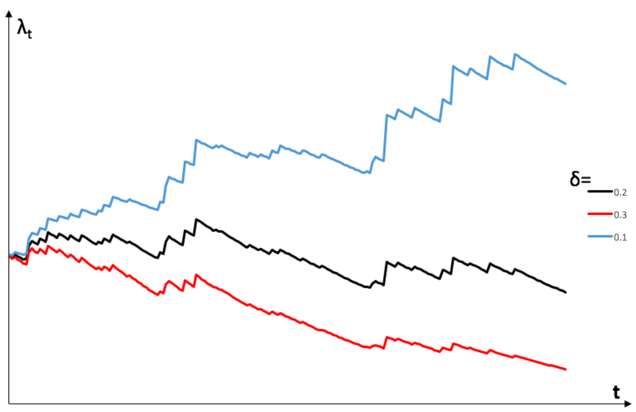


Figure 4.1: Intensity of λ_t with different levels of decay functions under heavy-tailed intensity

Its distribution can be derived by

$$\mathbb{P}[\lambda_t \leq x] = \sum_{n=0}^{\infty} \mathbb{P}\left[\lambda_0 e^{\sum_{i=0}^{n}\alpha Y_i} e^{-\delta t} \leq x\right] \mathbb{P}[N(t) = n]$$

$$= \sum_{n=0}^{\infty} \mathbb{P}\left[\sum_{i=0}^{n} \alpha Y_i \leq \log\frac{x}{\lambda_0} + \delta t\right] \mathbb{P}[N(t) = n]$$

$$= \sum_{n=0}^{\infty} F_{Z_n}\left(\log\frac{x}{\lambda_0} + \delta t\right) \mathbb{P}[N(t) = n],$$

where $\sum_{i=0}^{n} \alpha Y_i = Z_n \sim \text{Erlang}(\omega/\alpha, n)$ and $\mathbb{P}[N(t) = n] = \frac{(\rho t)^n}{n!} e^{\rho t}$. Then its density function is given by $f_{\lambda_t}(x) = \sum_{n=0}^{\infty} \frac{f_{Z_n}(\log \frac{x}{\lambda_0} + \delta t)}{\lambda_0 x} \frac{(\rho t)^n}{n!} e^{\rho t}$. The statistical properties can also be given by

$$\begin{aligned}
\mathbb{E}[\lambda_t] &= \lambda_0 e^{-\delta t} \sum_{n=0}^{\infty} \mathbb{E}[e^{\sum_{i=0}^{n} \alpha Y_i}] \mathbb{P}[N(t) = n] \\
&= \lambda_0 e^{-\delta t} \sum_{n=0}^{\infty} [\mathbb{M}_Y(\alpha)]^n \mathbb{P}[N(t) = n] = \lambda_0 e^{-\delta t} e^{\rho t(\mathbb{M}_Y(\alpha) - 1)}, \\
\mathbf{V}ar[\lambda_t] &= \lambda_0 e^{-\delta t} \sum_{n=0}^{\infty} \mathbf{V}ar[e^{\sum_{i=0}^{n} \alpha Y_i}] \mathbb{P}[N(t) = n] \\
&= \lambda_0 e^{-\delta t} \sum_{n=0}^{\infty} \left\{ \mathbb{E}[(e^{\sum_{i=0}^{n} \alpha Y_i})^2] - \mathbb{E}[e^{\sum_{i=0}^{n} \alpha Y_i}]^2 \right\} \mathbb{P}[N(t) = n] \\
&= \lambda_0 e^{-\delta t} \sum_{n=0}^{\infty} \mathbb{P}[N(t) = n] \left[\mathbb{M}_Y(2\alpha)^n - \mathbb{M}_Y(\alpha)^{2n} \right].
\end{aligned}$$

If we assume the jump $Y \sim N(0, 1)$, it is possible to generate the log-normal intensity process (two directions jump process).

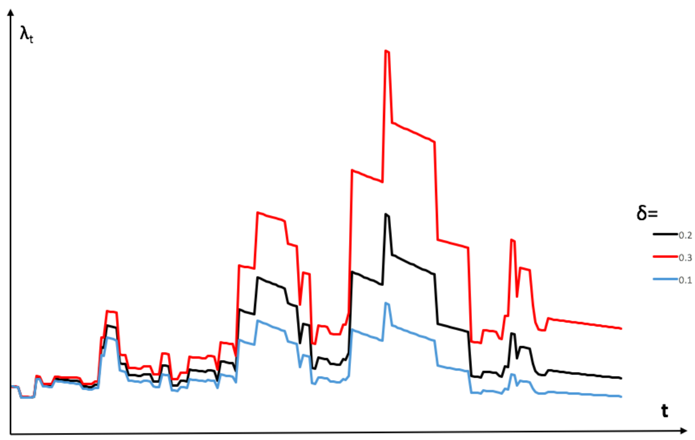


Figure 4.2: Intensity of λ_t with different levels of decay functions under log-normal intensity

Example 4.3.2. Geometric Brownian motion

A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift (Gerber and Shiu, 2004). The intensity process is defined by

$$\lambda_t = \lambda_0 Y_t, \tag{4.9}$$

where $Y_t = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$ and W_t is a Wiener process.

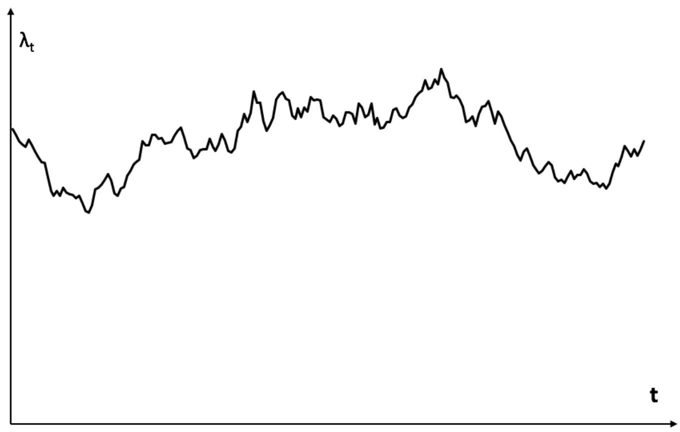


Figure 4.3: Intensity of λ_t under geometric Brownian motion

Its probability density function is given by

$$f_{\lambda_t}(x) = \frac{1}{x\sigma\sqrt{2\pi t}} \exp\left(-\frac{\left(\ln \frac{x}{\lambda_0} - \left(\mu - \delta - \frac{1}{2}\sigma^2\right) t\right)^2}{2\sigma^2 t} \right)$$

and the statistical properties can be computed as

$$\mathbb{E}[\lambda_t] = \lambda_0 e^{\mu t},$$
$$\mathbf{V}ar[\lambda_t] = \lambda_0^2 e^{2\mu t} \left(e^{\sigma^2 t} - 1 \right).$$

Example 4.3.3. Discounted jumps process

The intensity process is defined in a similar way to the shot-noise Cox process, it is given by

$$\lambda_t = \lambda_0 + \sum_{n=0}^{N(t)} Y_n e^{-\delta t}, \tag{4.10}$$

where $Y_n \sim \exp(\omega)$ and $N(t)$ is a homogeneous process with the parameter ρ.

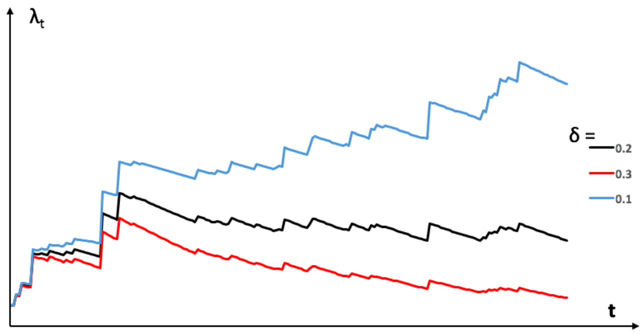


Figure 4.4: Intensity of λ_t with different levels of decay functions under discounted jumps process

Its distribution can be derived by

$$\mathbb{P}[\lambda_t \leq x] = \sum_{n=0}^{\infty} \mathbb{P}\big[\lambda_0 + \sum_{i=0}^{n} Y_i e^{-\delta t} \leq x\big]\mathbb{P}[N(t) = n]$$
$$= \sum_{n=0}^{\infty} \mathbb{P}\big[\sum_{i=0}^{n} Y_i \leq (x-\lambda_0)e^{\delta t}\big]\mathbb{P}[N(t) = n]$$
$$= \sum_{n=0}^{\infty} F_{Z_n}\big((x-\lambda_0)e^{\delta t}\big)\mathbb{P}[N(t) = n],$$

where $\sum_{i=0}^{n} Y_i = Z_n \sim \text{Erlang}(\omega, n)$ and $\mathbb{P}[N(t) = n] = \frac{(\rho t)^n}{n!}e^{-\rho t}$. Then, its density function is given by $f_{\lambda_t}(x) = \sum_{n=0}^{\infty} e^{\delta t} f_{Z_n}[(x-\lambda_0)e^{\delta t}]\frac{(\rho t)^n}{n!}e^{-\rho t}$. Also, the statistical properties can be given by

$$\mathbb{E}[\lambda_t] = \lambda_0 + e^{-\delta t}\sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=0}^{n} Y_i]\mathbb{P}[N(t) = n]$$
$$= \lambda_0 + e^{-\delta t}\sum_{n=0}^{\infty} [\mathbb{E}(Y)]^n \mathbb{P}[N(t) = n] = \lambda_0 + e^{-\delta t}e^{\rho t(\mathbb{E}(Y)-1)},$$
$$\mathbf{V}ar[\lambda_t] = \lambda_0 + e^{-\delta t}\sum_{n=0}^{\infty} \mathbf{V}ar[\sum_{i=0}^{n} Y_i]\mathbb{P}[N(t) = n]$$
$$= \lambda_0 + e^{-\delta t}\sum_{n=0}^{\infty} \frac{n}{\omega^2}\mathbb{P}[N(t) = n].$$

Furthermore, if we assume the integrated process $\Lambda_t = \lambda_0 t + \sum_{i=0}^{N(t)} Y_i e^{-\delta t}$, the intensity of the point process is given by $\lambda_t = e^{-\delta t}\sum_i Y_{N(t)}\delta(t - T_i) - \delta e^{-\delta t}\sum_{i=0}^{N(t)} Y_i$, thus the distribution of the point process is given by

$$\mathbb{P}[N(t)=n]=\mathbb{E}[\frac{\Lambda_t^n}{n!}e^{-\Lambda_t}]=$$
$$\int_0^\infty \frac{x^n}{n!}e^{-x}\sum_{m=0}^{\infty}e^{-(\rho-m\delta)t}\frac{(\rho t)^m}{m!}\frac{\omega^m(x-\lambda_0 t)^{m-1}}{(m-1)!}e^{-\omega}dx$$
$$=\sum_{m=0}^{\infty}\sum_{i=0}^{m-1}\frac{(-\lambda_0 t)^{m-i-1}(\omega\rho t)^m e^{-(\rho-m\delta)t-\omega}}{n!m!i!(m-i-1)!}\int_0^\infty x^{n+i}e^{-x}dx$$
$$=\sum_{m=0}^{\infty}\sum_{i=0}^{m-1}\frac{(-\lambda_0)^{m-i-1}(\omega\rho)^m e^{-\omega}(2m-i-1)!(n+i)!}{n!m!i!(m-i-1)!(\rho-m\delta)^{m+1}}\eta_{\rho-m\delta,2m-i}(t).$$

Example 4.3.4. Markov jump process

We arrange an intensity process as a Markov jump process,

$$\lambda_t=\begin{cases}\lambda_0=\lambda_0, t<T_1\to t\in A_0,\\ \lambda_1=\lambda_0+Y_1, T_1\le t<T_2\to t\in A_1,\\ \lambda_2=\lambda_1+Y_2, T_2\le t<T_3\to t\in A_2,\\ \vdots\\ \lambda_{n-1}=\lambda_{n-2}+Y_{n-1}, T_{n-1}\le t<T_n\to t\in A_{n-1},\\ \lambda_n=\lambda_{n-1}+Y_n, T_n\le t\to t\in A_n.\end{cases}$$

which can be arranged as an infinite stages Markov process,

Stage 0	Stage 1	...	Stage n	...	Stage n+k	...
λ_0	λ_1	...	λ_n	...	λ_n	...
A_0	A_1	...	A_n	...	A_{n+k}	...

Table 4.1: Markov jump process

When $t\ge T_n$, it holds that $\lambda_t=\lambda_n$, therefore the expectation of the intensity can be expressed by

$$\mathbb{E}[\lambda_t]=\sum_{m=0}^{n-1}\mathbb{E}[\lambda_t|t\in A_m]\mathbb{P}[t\in A_m]+\sum_{m=n}^{\infty}\mathbb{E}[\lambda_t|t\in A_m]\mathbb{P}[t\in A_m]$$
$$=\sum_{m=0}^{n-1}(\lambda_0+\frac{m}{\omega})\frac{(\rho t)^m}{m!}e^{-\rho t}+\sum_{m=n}^{\infty}(\lambda_0+\frac{n}{\omega})\frac{(\rho t)^m}{m!}e^{-\rho t}$$
$$=\lambda_0+\frac{n}{\omega}-\sum_{m=0}^{n}(\lambda_0+\frac{n-m}{\omega})\frac{(\rho t)^m}{m!}e^{-\rho t}.$$

We have the stationary case when we take $t \to \infty$,

$$\lim_{t\to\infty} \mathbb{E}[\lambda_t] = \lambda = \lambda_0 + \frac{n}{\omega}. \tag{4.11}$$

According to the safety loading condition, we have

$$c \geq \mathbb{E}[\lambda]\mathbb{E}[X] = (\lambda_0 + \frac{n}{\omega})\frac{1}{\beta}$$
$$n \leq c\omega\beta - \lambda_0\omega.$$

Thus, we let the safety loading as $\theta \geq 0$ and let $\frac{c\omega\beta}{1+\theta} - \lambda_0\omega$ be an integer, then the maximum value of n can be denoted by

$$n = \frac{c\omega\beta}{1+\theta} - \lambda_0\omega, \tag{4.12}$$

where c is the premium rate which assume the insurance company can accept a maximum of n events from intensity jump process. Then, considering an infinitesimal time interval $[0, h)$, there are four possible situations:

1. **No** jump from claim process and intensity process;
2. **One** jump from claim process and **no** jump from intensity process;
3. **No** jump from claim process and **one** jump from intensity process;
4. **More than one** jump from claim process and intensity process.

Then, we could have

$$\phi^*(u, \lambda_k) = \phi(u, \lambda_k|\mathcal{F}_n), \tag{4.13}$$

where the filtration $\mathcal{F}_n$ contains the information of λ_i for $i = 1, 2, ..., n$ and k denotes the states. Therefore, we could have the integro-differential equation in the following theorem

Theorem 4.3.2. Under the setting fo the intensity process λ_t, the non-ruin probability satisfies the following integro-differential equations. For $k = 1$,

$$c\frac{\partial\phi^*(u, \lambda_0)}{\partial u} = (\lambda_0 + \rho)\phi^*(u, \lambda_0) - \lambda_0 \int_0^u \phi^*(u - x, \lambda_0)dF_X(x) - \rho\phi^*(u, \lambda_1).$$

For $k < n$,

$$c\frac{\partial \phi^*(u,\lambda_k)}{\partial u} = (\lambda_k+\rho)\phi^*(u,\lambda_k) - \lambda_k\int_0^u \phi^*(u-x,\lambda_k)dF_X(x) - \rho\phi^*(u,\lambda_{k+1}).$$

For $k \geq n$,

$$c\frac{\partial \phi^*(u,\lambda_n)}{\partial u} = \lambda_n\phi^*(u,\lambda_n) - \lambda_n\int_0^u \phi^*(u-x,\lambda_n)dF_X(x). \tag{4.14}$$

Thus, (4.14) can be considered as a classical case,

$$\phi^*(u,\lambda_n) = 1 - \min\left\{\frac{\lambda_n}{c\beta}\exp[-(\beta-\frac{\lambda_n}{c})u], 1\right\}. \tag{4.15}$$

We could then apply the recursive method to obtain $\phi^*(u,\lambda_{n-1})$, which is denoted by the integro-differential equation

$$c\frac{\partial \phi^*(u,\lambda_{n-1})}{\partial u} = (\lambda_{n-1}+\rho)\phi^*(u,\lambda_{n-1}) - \lambda_{n-1}\int_0^u \phi(u-x,\lambda_{n-1})dF_X(x) - \rho\phi^*(u,\lambda_n). \tag{4.16}$$

Applying the Laplace transform on both sides, we have

$$sc\hat{\phi}^*(s,\lambda_{n-1}) - c\phi^*(0,\lambda_{n-1}) = (\lambda_{n-1}+\rho)\hat{\phi}^*(s,\lambda_{n-1}) - \lambda_{n-1}\hat{\phi}(s,\lambda_{n-1})\frac{\beta}{s+\beta} - \rho\hat{\phi}^*(s,\lambda_n),$$

$$\hat{\phi}^*(s,\lambda_{n-1}) = \frac{c\phi^*(0,\lambda_{n-1}) - \rho\hat{\phi}^*(s,\lambda_n)}{cs - \lambda_{n-1} - \rho + \lambda_{n-1}\frac{\beta}{s+\beta}} = \frac{(s+\beta)[c\phi^*(0,\lambda_{n-1}) - \rho\hat{\phi}^*(s,\lambda_n)]}{cs^2 - (c\beta - \lambda_{n-1} - \rho)s - \rho\beta}$$

$$= (\frac{s^+_{n-1}+\beta}{s^+_{n-1}-s^-_{n-1}}\frac{1}{s-s^+_{n-1}} + \frac{s^-_{n-1}+\beta}{s^-_{n-1}-s^+_{n-1}}\frac{1}{s-s^-_{n-1}})[c\phi^*(0,\lambda_{n-1}) - \rho\hat{\phi}^*(s,\lambda_n)]$$

$$= \hat{f}_{n-1}(s)[c\phi^*(0,\lambda_{n-1}) - \rho\hat{\phi}^*(s,\lambda_n)],$$

where $\hat{f}_{n-1}(s) = (\frac{s^+_{n-1}+\beta}{s^+_{n-1}-s^-_{n-1}}\frac{1}{s-s^+_{n-1}} + \frac{s^-_{n-1}+\beta}{s^-_{n-1}-s^+_{n-1}}\frac{1}{s-s^-_{n-1}})$. Thus we can substitute the expression of $\hat{\phi}^*(s,\lambda_{n-1})$ into the next recursion,

$$\begin{aligned}\hat{\phi}^*(s,\lambda_{n-2}) &= \hat{f}_{n-2}(s)[c\phi^*(0,\lambda_{n-2}) - \rho\hat{\phi}^*(s,\lambda_{n-1})]\\ &= \hat{f}_{n-2}(s)[c\phi^*(0,\lambda_{n-2}) - \rho\hat{f}_{n-1}(s)[c\phi^*(0,\lambda_{n-1}) - \rho\hat{\phi}^*(s,\lambda_n)]]\\ &= c\hat{f}_{n-2}(s)\phi^*(0,\lambda_{n-2}) - \rho\hat{f}_{n-1}(s)\hat{f}_{n-2}(s)[c\phi^*(0,\lambda_{n-1}) - \rho\hat{\phi}^*(s,\lambda_n)].\end{aligned}$$

Iterating from step 1 to step k, we have

$$\begin{aligned}&\hat{\phi}^*(s,\lambda_{n-k}) = c\hat{f}_{n-k}(s)\phi^*(0,\lambda_{n-k})\\ &- c\sum_{i=1}^{k-1}\prod_{j=0}^{i}\hat{f}_{n-k+j}(s)\rho^{i-1}\phi^*(0,\lambda_{n-k+i}) - \prod_{j=0}^{k-1}\rho^k\hat{f}_{n-k+j}(s)\hat{\phi}^*(s,\lambda_n).\end{aligned}$$

Eventually, letting $k = n$ gives

$$\hat{\phi}^*(s,\lambda_0) = c\hat{f}_0(s)\phi^*(0,\lambda_0) - c\sum_{i=1}^{n-1}\prod_{j=0}^{i}\hat{f}_j(s)\rho^{i-1}\phi^*(0,\lambda_i) - \prod_{j=0}^{n-1}\rho^n\hat{f}_j(s)\hat{\phi}^*(s,\lambda_n), \quad (4.17)$$

then applying the inverse Laplace transform, we have

$$\phi^*(s,\lambda_0) = cf_0(u)\phi^*(0,\lambda_0) - c\sum_{i=1}^{n-1}\rho^{i-1}\phi^*(0,\lambda_i)\prod_{j=0}^{i}f_j^{*i}(u) - \rho^n\prod_{j=0}^{n-1}f_j^{*n-1} * \phi^*(:,\lambda_n)(u), \quad (4.18)$$

where $\prod_{j=0}^{n-1} f_j^{*n-1}(u)$ is the $n-1$ fold convolution of the terms $f_j(:)$ for $j = 0, ..., n-1$ and $\prod_{j=0}^{n-1} f_j^{*n-1} * \phi^*(:,\lambda_n)(u)$ is the n fold convolution with respect to u. According to the definition of ϕ^*, the probability condition on $\mathcal{F}_n$, we could apply the total probability theorem in order to obtain $\phi(u,\lambda_0)$. Before the calculation, we denote $f_n(u) = f(u,\lambda_n)$, then

$$\begin{aligned}
\phi(u,\lambda_0) &= \oint_0^{c\beta} \phi^*(u,\lambda_0)f(y_1)f(y_2)\dots f(y_n)dy_n\dots dy_2dy_1 \\
&= \oint_0^{c\beta} cf_0(u)\phi^*(0,\lambda_0) - c\sum_{i=1}^{n-1}\rho^{i-1}\phi^*(0,\lambda_j)\prod_{j=0}^{i}f^{*i}(:,\lambda_j)(u) \\
&- \rho^n\prod_{j=0}^{n-1}f^{*n-1}(:,\lambda_j) * \phi^*(:,\lambda_n)(u)f(\lambda_1)f(\lambda_2)\dots f(\lambda_n)d\lambda_n\dots d\lambda_2 d\lambda_1,
\end{aligned}$$

where $\oint_{\lambda_0}^{c\beta} = \int_{\lambda 0}^{c\beta}\int_{\lambda_1}^{c\beta}\cdots\int_{\lambda_{n-1}}^{c\beta}$.

Considering the case of two states ($n = 2$) under exponential claim distribution with density function $f_X(x) = \beta e^{-\beta x}$ and assuming Y_1 is a constant, we have

$$c\frac{\partial\phi_0(u)}{\partial u} = (\lambda_0+\rho)\phi_0(u) - \lambda_0\int_0^u \phi_0(u-x)dF_X(x) - \rho\phi_1(u)$$

and

$$c\frac{\partial\phi_1(u)}{\partial u} = \lambda_1\phi_1(u) - \lambda_1\int_0^u \phi_1(u-x)dF_X(x).$$

We now apply the Laplace transform with respect to u on both upon functions,

$$\hat{\phi}_0(s) = \frac{c\phi_0(0) - \rho\hat{\phi}_1(s)}{cs - \lambda_0 - \rho + \lambda_0\hat{f}_X(s)}$$

and

$$\hat{\phi}_1(s) = \frac{c\phi_1(0)}{cs - \lambda_1 + \lambda_1 \hat{f}_X(s)}.$$

Therefore we have

$$\begin{aligned}
\hat{\phi}_0(s) &= \frac{c\phi_0(0) - \rho \frac{c\phi_1(0)}{cs-\lambda_1+\lambda_1\hat{f}_X(s)}}{cs - \lambda_0 - \rho + \lambda_0 \hat{f}_X(s)} \\
&= \frac{c\phi_0(0)}{cs - \lambda_0 - \rho + \lambda_0 \hat{f}_X(s)} - \frac{c\rho\phi_1(0)}{\Big(cs - \lambda_0 - \rho + \lambda_0 \hat{f}_X(s)\Big)\Big(cs - \lambda_1 + \lambda_1 \hat{f}_X(s)\Big)},
\end{aligned}$$

which can be calculated by two part. Firstly, we have

$$\frac{c\phi_0(0)}{cs - \lambda_0 - \rho + \lambda_0 \hat{f}_X(s)} = \phi_0(0)\big(\frac{a_1}{s - s_1} + \frac{a_2}{s - s_2}\big),$$

where

$$s_{1,2} = \frac{\frac{\lambda_0+\rho}{c} - \beta \pm \sqrt{(\frac{\lambda_0+\rho}{c} - \beta)^2 - 4\frac{\rho\beta}{c}}}{2} < 0$$

and coefficients $a_{1,2}$ satisfy

$$\begin{cases} a_1 = \dfrac{s_1 - \beta}{s_1 - s_2}, \\ a_2 = \dfrac{\beta - s_2}{s_1 - s_2}. \end{cases}$$

The second part can be derived as

$$\begin{aligned}
&\frac{c\rho\phi_1(0)}{\Big(cs - \lambda_0 - \rho + \lambda_0 \hat{f}_X(s)\Big)\Big(cs - \lambda_1 + \lambda_1 \hat{f}_X(s)\Big)} \\
&= \frac{\rho}{c}\phi_1(0)\left(\frac{b_0}{s} + \frac{b_1}{s - s_1} + \frac{b_2}{s - s_2} + \frac{b_3}{s - s_3}\right),
\end{aligned}$$

where

$$s_3 = \frac{\lambda_1}{c} - \beta < 0$$

and coefficients $b_{0,1,2,3}$ satisfy that

$$\begin{cases} b_0 + b_1 + b_2 + b_3 = 0, \\ (s_1 + s_2 + s_3)b_0 + (s_2 + s_3)b_1 + (s_1 + s_3)b_2 + (s_1 + s_2)b_3 = 0, \\ (s_1 s_2 + s_2 s_3 + s_1 s_3)b_0 + s_2 s_3 b_1 + s_1 s_3 b_2 + s_1 s_2 b_3 = 1, \\ s_1 s_2 s_3 b_0 = -\beta. \end{cases}$$

Thus we have

$$\begin{cases} b_0 = -\dfrac{\beta}{s_1 s_2 s_3}, \\ b_1 = \dfrac{\beta + s_1}{s_1(s_2 - s_1)} \dfrac{1}{s_3 - s_1}, \\ b_2 = \dfrac{\beta + s_2}{s_2(s_1 - s_2)} \dfrac{1}{s_3 - s_2}, \\ b_3 = \dfrac{\beta + s_3}{s_3(s_3 - s_1)(s_3 - s_2)}. \end{cases}$$

We could now obtain the ruin probability by applying the inverse Laplace transformation,

$$\phi_0(u) = \phi_0(0)\left(a_1 e^{s_1 u} + a_2 e^{s_2 u}\right) - \frac{\rho}{c}\phi_1(0)\left(b_0 + b_1 e^{s_1 u} + b_2 e^{s_2 u} + b_3 e^{s_3 u}\right). \tag{4.19}$$

Now consider Y_1 be a random variable, the probability of ruin can be derived by the idea of mixing distribution (Bühlmann, 1970; Albrecher et al., 2011; Constantinescu et al., 2018), i.e. the ultimate ruin probability conditions on the value of first jump Y_1 from the intensity process ,

$$\psi_1(u|Y_1 = y) = \min\left\{\frac{\lambda_0 + y}{c\beta}\exp[-(\beta - \frac{\lambda_0 + y}{c})u], 1\right\}.$$

According to the net profit condition, $Y_1 < c\beta - \lambda_0 = y^*$, we could obtain the ruin probability for the stage 1,

$$\psi_1(u) = \int_0^{y^*} \frac{\lambda_0 + y}{c\beta} e^{-(\beta - \frac{\lambda_0 + y}{c})u} f_{Y_1}(y)dy + \bar{F}_{Y_1}(y^*).$$

Assume the $Y_1 \sim Exp(\omega)$, thus

$$\psi_1(u) = e^{-\omega(c\beta - \lambda_0)} + \frac{\lambda_0 \omega}{c\beta(\omega - \frac{u}{c})}\left(1 - e^{-(\omega - \frac{u}{c})(c\beta - \lambda_0)}\right)$$
$$+ \frac{\omega}{c\beta}\sum_{k=0}^{\infty} \frac{(\omega - \frac{u}{c})^{k-1}(c\beta - \lambda_0)^k}{k!} e^{-(\omega - \frac{u}{c})(c\beta - \lambda_0)}.$$

Furthermore, (4.19) is also condidered as a conditional probability. We can derive the unconditional probability by applying the total probability theorem. According to the fact of integral

$$\int \frac{1}{a + bx} e^{-cx} = -\frac{E_1(cx + \frac{ac}{b})e^{\frac{ac}{b}}}{b} + C,$$

where $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt$ is an exponential integral, we can separate the b_3 into

$$b_3 = \frac{p_0}{\frac{\lambda_1}{c} - \beta} + \frac{p_1}{\frac{\lambda_1}{c} - \beta - s_1} + \frac{p_2}{\frac{\lambda_1}{c} - \beta - s_2},$$

where

$$\begin{cases} p_0 = \dfrac{\beta}{s_1 s_2}, \\ p_1 = \dfrac{\beta + s_1}{s_1(s_2 - s_1)}, \\ p_2 = \dfrac{\beta + s_2}{s_2(s_1 - s_2)}. \end{cases}$$

Eventually we could obtain the result,

$$\begin{aligned}
\phi_0(u) &= \int_0^{y^*} \phi_0(u,y) f_{Y_1}(y)dy + \bar{F}_{Y_1}(y^*) \\
&= [\phi_0(0)\big(a_1 e^{s_1 u} + a_2 e^{s_2 u}\big) - \frac{\rho}{c}\phi_1(0)\big(b_1 e^{s_1 u} + b_2 e^{s_2 u}\big)]F_{Y_1}(y^*) + \bar{F}_{Y_1}(y^*) \\
&+ \frac{E_1\left(\frac{\omega(s_1 s_2 \lambda_0 - c\beta)}{s_1 s_2}\right) - E_1\left(\omega y^* + \frac{\omega(s_1 s_2 \lambda_0 - c\beta)}{s_1 s_2}\right)}{s_1 s_2} c e^{\frac{\omega(s_1 s_2 \lambda_0 - c\beta)}{s_1 s_2}} \\
&+ \frac{E_1\left(\frac{(c\beta - \lambda_0)(u - c\omega)}{\lambda_0}\right) - E_1\left((\omega - \frac{u}{c})y^* + \frac{(c\beta - \lambda_0)(u - c\omega)}{\lambda_0}\right)}{\lambda_0} c e^{\frac{(c\beta - \lambda_0)(u - c\omega)}{\lambda_0} - (\beta - \frac{\lambda_0}{c})u} \\
&+ \frac{E_1\left(\frac{(c(\beta + s_1) - \lambda_0)(u - c\omega)}{\lambda_0}\right) - E_1\left((\omega - \frac{u}{c})y^* + \frac{(c(\beta + s_1) - \lambda_0)(u - c\omega)}{\lambda_0}\right)}{\lambda_0} c e^{\frac{(c(\beta + s_1) - \lambda_0)(u - c\omega)}{\lambda_0} - (\beta - \frac{\lambda_0}{c})u} \\
&+ \frac{E_1\left(\frac{(c(\beta + s_2) - \lambda_0)(u - c\omega)}{\lambda_0}\right) - E_1\left((\omega - \frac{u}{c})y^* + \frac{(c(\beta + s_2) - \lambda_0)(u - c\omega)}{\lambda_0}\right)}{\lambda_0} c e^{\frac{(c(\beta + s_2) - \lambda_0)(u - c\omega)}{\lambda_0} - (\beta - \frac{\lambda_0}{c})u}.
\end{aligned}$$

In this section, we managed to construct Seal's formulae for the general inhomogeneous Poisson process. We then applied the results to generate expressions for the finite and infinite time ruin probability under the inhomogeneous Poisson process. In addition, we investigated the properties of some special Cox processes and discussed the possibility of computing their ruin probabilities. As long as we manage to obtain the distributions of the intensity processes by applying the law of total probability, we can then derive the ruin probabilities by applying Seal's type integro-differential equation for the inhomogeneous Poisson process. However, we are not able to find the distribution of the integrated random process of intensity processes in examples 4.3.1 and 4.3.2. We could apply a similar approach given by Albrecher and Asmussen (2006) in order to obtain the asymptotic solutions under such intensity processes.

Chapter 5

Surplus Dependent Risk Process

Based on the fundamental work of the classical risk process and its applications in insurance and reinsurance portfolio modelling, we made a modification to the original surplus process which contains a lower barrier (the compensation level) $k \geq 0$. This would provide some funds when the surplus level is below a certain compensation level, the funds would be determined by the current surpus level.

In this chepter, this kind of reinsurance strategy being purchased is not a traditional type of contract. It is neither a proportional type nor an excess of loss reinsurance that have traditionally been discussed in most of the actuarial literature. In contrast, it relates to both individual claim or/and the surplus level. More precisely, it relates to the amount by which the surplus process falls below a fixed compensation level $0 \leq k \leq u$. Suppose that on the i^{th} occasion that the surplus falls between 0 and k, the insurer's surplus falls to a level $k - y_i$ (such that $0 < y_i < k$), the reinsurer makes an instant payment of the deficit y_i or the part of the deficit py_i ($p \in (0, 1)$) to the insurer. If any claim leads the insurer's surplus to drop to a level below 0 (or the lower level, e.g. Sections 5.1.2 and 5.1.3), the reinsurer does not make a payment and ruin for the portfolio occurs at the time of this claim. Therefore, the modified surplus process U_t^{Δ} is given by a combination of the original surplus process U_t and the injection process J_t, which is defined as

$$U_t^{\Delta} = U_t + J_t.$$

Denote the injection function as J_t which is considered under some specific conditions (denoted by the indicators). Now define $\psi_k(u) = \mathbb{P}[T_{u,k} < \infty]$ to be the ultimate ruin probability for the modified surplus process with the compensation level at k and let $T_{u,k}$ denote the time to ruin of the modified process. In this chapter we will discuss the following capital injection strategies:

- Classical capital injection with $J_t = \sum_{i=0}^{N_t}(k - U_{t_i}^{\Delta})\mathbb{I}\{0 < U_{t_i^-}^{\Delta} < k\}$ with instant payment $Q(u,k)$,
- Partial discrete capital injection with $J_t = \sum_{i=0}^{N_t} p(k - U_{t_i}^{\Delta})\mathbb{I}\{0 < U_{t_i^-}^{\Delta} < k, U_{t_i^+}^{\Delta} \geq k\}$ with instant payment $Q(u,k)|_P$,

where $Q(u,k)$ and $Q(u,k)|_P$ denote the amount of the instant payment to reinsure of the capital injection and partial discrete capital injection models, which will be introduced in the premium calculation section. Furthermore, we construct the reinsurance strategies as

- Partial discrete compensation with $J_t = \sum_{i=0}^{N_t} p(k - U_{t_i}^{\Delta})\mathbb{I}\{0 < U_{t_i^-}^{\Delta} < k, U_{t_i^+}^{\Delta} \geq k\}$, with reinsurance payment rate $(1-q)c$,
- Partial discrete compensation with $J_t = p(k - U_t)\mathbb{I}\{0 < U_t < k\}$, with reinsurance payment rate $c(1-p)\mathbb{I}\{0 < U_t < k\}$.

We aim to derive the ruin probabilities for the above strategies respectively and apply sensitive analysis with respect to all parameters. The premium calculation will be derived in order to construct the capital injection and partial discrete capital injection model as a reinsurance contract. In addition, we manage to find the optimal capital allocation for both models to obtain the minimum ruin probabilities and introduce the equivalent continuous reinsurance payment rate against the fixed instant payment. Finally, we will answer the key risk management questions and propose suggestions of how to choose a reinsurance contract under specific situations.

5.1 The main results

5.1.1 Classical capital injection model

Nie et al. (2011) modified the classical Cramér-Lundberg risk process (1.1) with capital injection model which contains a compensation level k, where $0 \leq k \leq u$. The modification is that if the surplus drops below k but not below 0, an injection of funds will immediately restore the surplus back to k, so that the surplus instantly starts from level k after the claim leads the surplus into $(0, k)$. The company only gets ruined once the claim makes the surplus fall below 0. Thus, the modified surplus process satisfies the following figure.

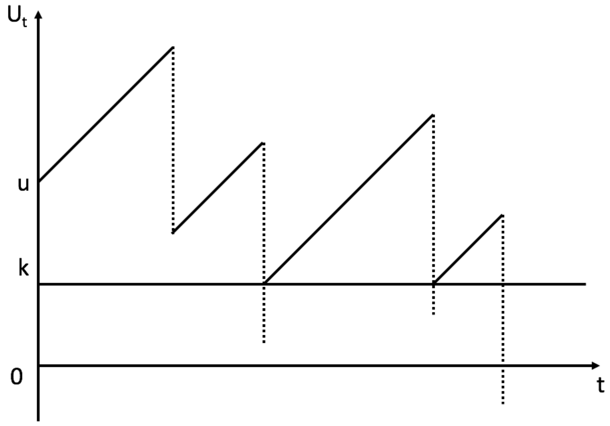


Figure 5.1: Surplus process with capital injection

Obviously, the surplus with the capital injection model will never be in the interval $(0, k)$, due to the full instant compensation restoring the surplus level to k every time a claim leads the surplus level into $(0, k)$. According to Nie et al. (2011), we have the following theorem.

Theorem 5.1.1. When the initial surplus $u = k$, we have

$$\psi_k(k) = \frac{\psi(0) - G(0, k)}{1 - G(0, k)}.$$

When the initial surplus $u > k$, we have

$$\psi_k(u) = \psi(u - k) - G(u - k, k)\phi_k(k).$$

When the initial surplus $0 \leq u < k$, we have

$$\psi_k(u) = \psi_k(k).$$

Proof. Two situations may occur when the surplus starts with $u = k$, either a claim leads the surplus into $(0, k)$ or below 0. Thus, we have

$$\psi_k(k) = \int_0^k g(0, y)\psi_k(k)dy + \int_k^\infty g(0, y)dy.$$

Then we consider the case $u > k$, thus conditioning on the amount of the first drop below level k, for $u \geq k$, we have

$$\phi_k(u) = \phi(u - k) + G(u - k, k)\phi_k(k).$$

□

Example 5.1.1. We assume the claims follow exponential distribution, s.t. $F_X(x) = 1 - e^{-\beta x}$ (Dickson, 2005), It is well known that

$$\psi(u) = \frac{\lambda}{c\beta} e^{-(\beta - \frac{\lambda}{c})u}, \; g(u, y) = \psi(u)\beta e^{-\beta y}, \; G(u, y) = \psi(u)(1 - e^{-\beta y}),$$

thus we have

$$\psi_k(k) = \frac{\psi(0) - G(0, k)}{1 - G(0, k)} = \frac{\lambda e^{-\beta k}}{c\beta - \lambda + \lambda e^{-\beta k}}$$

and

$$\psi_k(u) = \psi(u - k) - G(u - k, k)\bar{\psi}_k(k) = \psi(u - k)[\psi_k(k) + e^{-\beta k}\bar{\psi}_k(k)].$$

5.1.2 Partial discrete capital injection and partial discrete compensation reinsurance contract

According to the risk process given in the capital injection model, there are full injections when the surplus is between 0 and k, always leading the capital back to k. In this section, we aim to construct a partial discrete capital injection, which happens if the claims lead the surplus process to drop below the compensation level k, with partial injections $p(k - U^{\Delta}_{t_i})$, which depend on the deficit below the compensation level k. Furthermore, the injection will not happen when the surplus level remains in the interval between 0 and k. Therefore, the injection J_t can be denoted as

$$J_t = \sum_{i=1}^{N(t)} p(k - U^{\Delta}_{t_i})\mathbb{I}\{0 < U^{\Delta}_{t_i^+} < k, U^{\Delta}_{t_i^-} \geq k\}.$$

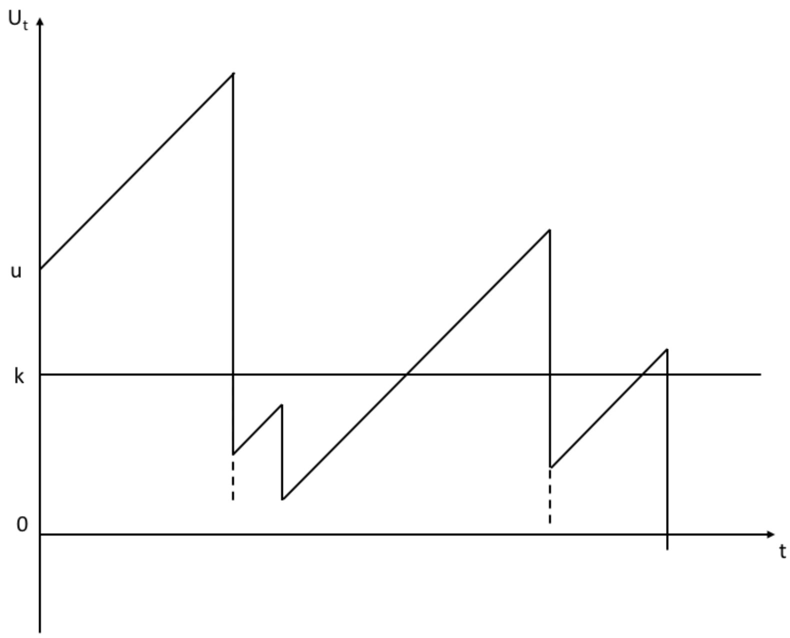


Figure 5.2: Surplus process for partial discrete capital injection model

Remark 5.1.1. One thing needs to be mentioned: due to the definition of the partial injection model, the surplus level is able to remain in the interval $(0, k)$ for a while. Therefore, the additional situation compared to the capital injection model is the surplus between 0 and k.

The probability of ruin in the partial injection model can be solved by applying the idea of Dickson and Gray (1986). In addition, the ruin probability under the compensation level k (for any $0 < u < k$) satisfies the following integro-differential equation,

$$c\psi'(u) = \lambda\psi(u) - \lambda \int_0^u \psi(u - x) dF_X(x) - \lambda\bar{F}(u).$$

Under the exponential claim the ruin probability has the explicit solution (2.6). According to the idea of the classical capital injection and the approach for investigating the two barriers model, we could apply a similar idea to obtain the following theorem

Theorem 5.1.2. When the initial capital $u = k$, we have

$$\psi_k(k) = \int_0^k g(0,y)\big[\xi(k-y+py,k) + \bar{\xi}(k-y+py,k)\psi_k(k)\big]dy + \int_k^\infty g(0,y)dy.$$

When the initial capital $u > k$, we have

$$\psi_k(u) = \psi(u-k) - \phi_k(k)\int_0^k g(u-k,y)\bar{\xi}(k-y+py,k)dy.$$

When the initial capital $0 \le u < k$, we have

$$\psi_k(u) = 1 - \phi_k(k)\bar{\xi}(u,k).$$

Example 5.1.2. We assume the claims follow exponential distribution, s.t. $F_X(x) = 1 - e^{-\beta x}$, we could obtain

$$\bar{\xi}(u,b) = \frac{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})u}}{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})b}}, \quad \xi(u,b) = \frac{\lambda(e^{-(\beta-\frac{\lambda}{c})u} - e^{-(\beta-\frac{\lambda}{c})b})}{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})b}}.$$

According to theorem 5.1.2, we could have

$$\psi_k(k) = \frac{\psi(0) - \psi(0)f_p(k)}{1 - \psi(0)f_p(k)}$$

where

$$f_p(k) = \frac{c\beta}{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})k}}(1 - e^{-\beta k}) - \frac{\lambda\beta e^{-(\beta-\frac{\lambda}{c})k}(1 - e^{-(p\beta+(1-p)\frac{\lambda}{c})k})}{(p\beta + (1-p)\frac{\lambda}{c})(c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})k})}.$$

Then we could have

$$\psi_k(u) = \psi(u-k)[1 - \phi_k(k)f_p(k)] \text{ for all } u \ge k.$$

Inspired by the discrete capital injection model, the corresponding reinsurance contract would provide an injection in the same way, but the payment of the contract become the continuous payment with rate $(1-q)c$, rather than an instant payment $Q(u,k)|_P$ at the beginning. Therefore, the reinsurance compensation is given by $J_t =$

$\sum_{i=0}^{N_t} p(k-U_{t_i}^{\Delta})\mathbb{I}\{0<U_{t_i}^{\Delta}<k, U_{t_i^-}^{\Delta}\geq k\}$, in addition to the original surplus process with a modified drift,

$$U_t = u + qc - S_t.$$

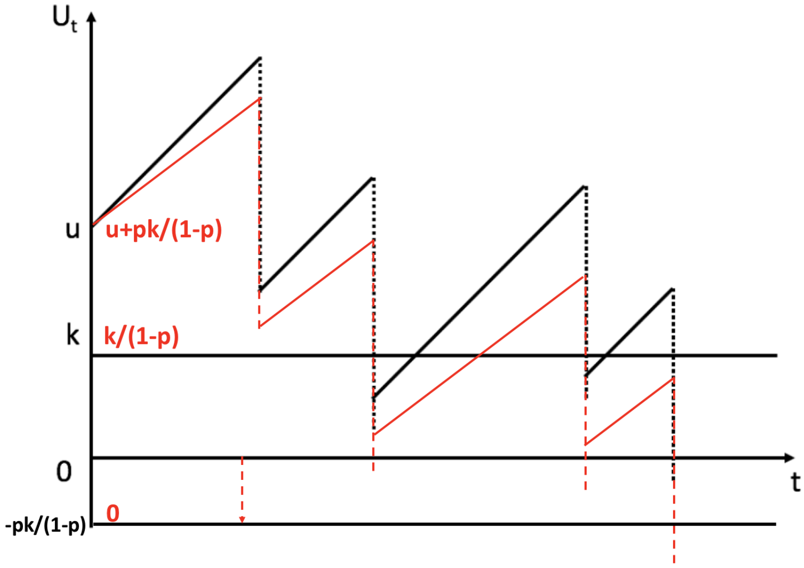


Figure 5.3: Surplus process for discrete compensation reinsurance contract

Remark 5.1.2. The red line represents the modified process under the discrete partial compensation reinsurance contract. As shown by the sample paths for the partial injection and reinsurance model in Figure 5.3, although the premium rate $qc\,(0<q<1)$ is lower than the surplus process under the discrete capital injection model, the reinsurance contract allows the company to survive when the surplus is greater than $-\frac{pk}{1-p}$ rather than 0, which means the company has the additional safe position $(-\frac{pk}{1-p}, 0)$. Therefore we can simply move the x-axis to $-\frac{pk}{1-p}$ and set this as 0. The initial capital and compensation level will be $u' = u + \frac{p}{1-p}k$ and $k' = \frac{1}{1-p}k$ respectively, we then have the following theorem.

Theorem 5.1.3. For the discrete partial compensation reinsurance contract, when the initial capital $u' = k'$, we have

$$\psi_k^q(k) = \int_0^{k'} g(0,y)\big[\xi(k'-y+py,k') + \bar{\xi}(k'-y+py,k')\psi_{k'}^q(k')\big]dy + \int_{k'}^{\infty} g(0,y)dy.$$

When the initial capital $u' > k'$, we have

$$\psi_k'^q(u') = \psi(u'-k') - \phi_{k'}^q(k')\int_0^{k'} g(u'-k',y)\bar{\xi}(k'-y+py,k')dy.$$

When the initial capital $0 \le u' < k'$, we have

$$\psi_k^q(u) = 1 - \phi_{k'}^q(k')\bar{\xi}(u',k').$$

Example 5.1.3. We assume the claims follow exponential distribution, s.t. $F_X(x) = 1 - e^{-\beta x}$, we could have

$$\bar{\xi}(u,b) = \frac{qc\beta - \lambda e^{-(\beta-\frac{\lambda}{qc})u}}{qc\beta - \lambda e^{-(\beta-\frac{\lambda}{qc})b}}, \quad \xi(u,b) = \frac{\lambda(e^{-(\beta-\frac{\lambda}{qc})u} - e^{-(\beta-\frac{\lambda}{qc})b})}{qc\beta - \lambda e^{-(\beta-\frac{\lambda}{qc})b}}$$

then we could derive

$$\psi_k^q(k) = \frac{\psi^q(0) - \psi^q(0)f_p^q(k')}{1-\psi^q(0)f_p^q(k')},$$

where

$$f_p^q(k) = \frac{qc\beta}{qc\beta - \lambda e^{-(\beta-\frac{\lambda}{qc})k}}(1-e^{-\beta k}) - \frac{\lambda\beta e^{-(\beta-\frac{\lambda}{qc})k}(1-e^{-(p\beta+(1-p)\frac{\lambda}{qc})k})}{(p\beta+(1-p)\frac{\lambda}{qc})(qc\beta - \lambda e^{-(\beta-\frac{\lambda}{qc})k})}.$$

Then we could have

$$\psi_k^q(u) = \psi^q(u'-k')[1-\phi_{k'}^q(k')f_p^q(k')], \text{ for all } u' \ge k',$$

where $\psi^q(u) = \frac{\lambda}{qc\beta}e^{-(\beta-\frac{\lambda}{qc})u}$.

5.1.3 Discrete partial compensation reinsurance contract with payment when surplus under level k

According to the partial discrete compensation model, the insurer pays the premium to the reinsurer at the beginning of business. We now aim to construct the situation that the payment occurs only when the surplus level lies in the interval $(0,k)$. The injection process is given by

$$J_t = p(k-U_t)\mathbb{I}\{0 < U_t < k\}.$$

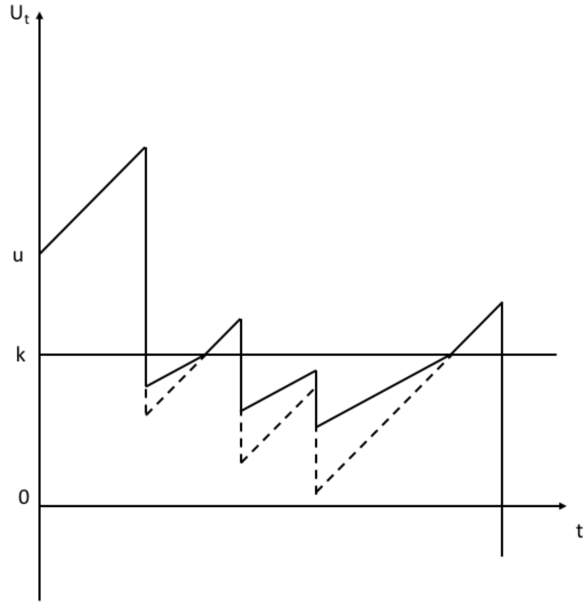


Figure 5.4: Surplus process for discrete partial compensation reinsurance contract with payment when surplus is lower than k

Figure (5.4) illustrates the surplus process can be considered as the shifted process when $U_t < k$. The modified process $U_t^{\Delta} = pk + (1-p)U_t$ has a shifted premium rate $(1-p)c$ and a shifted amount of aggregate claims size $(1-p)S_t$. Considering in an small interval $(0, h)$, there are four possible situations under this process:

1. **no** claim occurs in $(0, h)$ with probability $1 - \lambda h + o(h)$,
2. **one** claim occurs in $(0, h)$ with probability $\lambda h + o(h)$,
3. **more than one** claim occurs in $(0, h)$ with probability $o(h)$.

Note that: currently, we denote the premium rate as c in order to provide a more convenient form. Therefore, the non-ruin probability satisfies

$$\phi^{\Delta}(u) = (1-\lambda h)\phi^{\Delta}(u+ch) + \lambda h \int_0^{\frac{u+ch}{1-p}} \phi^{\Delta}(u+ch-x+px)f(x)dx + o(h),$$

$$c\frac{\phi^{\Delta}(u+ch) - \phi^{\Delta}(u)}{ch} = \lambda\phi^{\Delta}(u+ch) - \lambda \int_0^{\frac{u+ch}{1-p}} \phi^{\Delta}(u+ch-x+px)f(x)dx + \frac{o(h)}{h}.$$

Let $h \to 0$, we have the following integro-differential equation

$$c\frac{d}{du}\phi^{\Delta}(u) = \lambda\phi^{\Delta}(u) - \lambda \int_0^{\frac{u}{1-p}} \phi^{\Delta}(u-(1-p)x)f(x)dx. \tag{5.1}$$

Now apply the Laplace transform with respect to u on both sides,

$$cs\hat{\phi}^{\Delta}(s) - c\phi(0) = \lambda\hat{\phi}^{\Delta}(s) - \lambda\mathcal{L}\Big\{\int_0^{\frac{u}{1-p}} \phi^{\Delta}(u-(1-p)x)f(x)dx\Big\}, \tag{5.2}$$

where the Laplace transformation of the integral part is derived by

$$\mathcal{L}\Big\{\int_0^{\frac{u}{1-p}} \phi^{\Delta}(u-(1-p)x)f(x)dx\Big\} = \int_0^{\infty} e^{-su}\int_0^{\frac{u}{1-p}} \phi^{\Delta}(u-(1-p)x)f(x)dxdu$$
$$= \int_0^{\infty} f(x)\int_{(1-p)x}^{\infty} e^{-su}\phi^{\Delta}(u-(1-p)x)\,du\,dx.$$

We get the last equation by changing the order of the integrals. Now changing the variable using the substitution $z = u-(1-p)x$, then the above becomes

$$\int_0^{\infty} f(x)\int_0^{\infty} e^{-s(z+(1-p)x)}\phi^{\Delta}(z)\,dz\,dx = \hat{\phi}^{\Delta}(s)\int_0^{\infty} f(x)e^{-s(1-p)x}dx = \hat{\phi}^{\Delta}(s)\hat{f}((1-p)s).$$

Substitute the equation into (5.2) and we have

$$cs\hat{\phi}^{\Delta}(s) - c\phi(0) = \lambda\hat{\psi}^{\Delta}(s) - \lambda\hat{\phi}^{\Delta}(s)\hat{f}((1-p)s).$$

Rearranging the above equation gives

$$\hat{\phi}^{\Delta}(s) = \frac{c\phi(0)}{cs-\lambda+\lambda\hat{f}((1-p)s)}.$$

Example 5.1.4. Assume the density of the claims is given by $f(x) = \beta e^{-\beta x}$, its Laplace transform with respect to x is $\frac{\beta}{s+\beta}$. Then we have

$$\hat{f}((1-p)s) = \frac{\beta}{(1-p)s+\beta} = \frac{\beta/(1-p)}{s+\beta/(1-p)}.$$

Thus the case above can be considered under the classical case with enlarged exponential parameter $\frac{\beta}{1-p}$. According to (2.6), the ruin probability can be expressed by

$$\psi^{\Delta}(u) = \frac{\lambda(1-p)}{c\beta}e^{-(\frac{\beta}{1-p}-\frac{\lambda}{c})u}.$$

It is clear to see that when we let $p \to 0$, the model becomes the classical risk model and $\psi^{\Delta}(u) = \psi(u)$, where $\psi(u) = \frac{\lambda}{c\beta}e^{-(\beta-\frac{\lambda}{c})u}$.

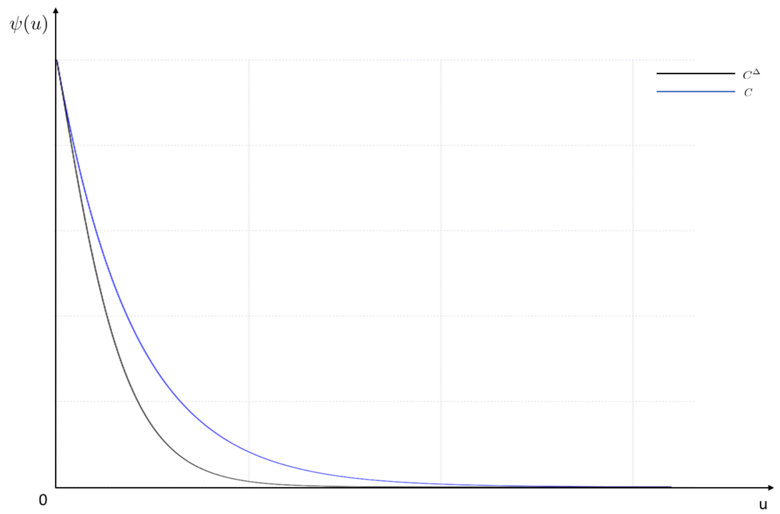


Figure 5.5: The sample path for the new process with $c = 2, \beta = 1, \lambda = 1, p = 0.5$ and $u = 0 - 20$

Clearly, the ruin probability under the compensation process is lower than the classical case's and both of them converge to 0 as $u \to \infty$. Furthermore, for the case given in this section, we replace the premium rate with $(1-p)c$, then the ruin probability becomes

$$\psi^{\Delta}(u) = \frac{\lambda}{c\beta} e^{-(\beta - \frac{\lambda}{c})\frac{u}{1-p}}.$$

Therefore, we have the following theorem.

Theorem 5.1.4. Under the model given in this section, we have

$$\psi_k(k) = \int_0^k g(0,y)\big[\xi^{\Delta}(k-y+py,k) + \bar{\xi}^{\Delta}(k-y+py,k)\psi_k(k)\big]dy + \int_k^{\infty} g(0,y)dy.$$

When the initial capital $u > k$, we have

$$\psi_k(u) = \psi(u-k) - \phi_k(k)\int_0^k g(u-k,y)\bar{\xi}^{\Delta}(k-y+py,k)dy.$$

When the initial capital $0 \le u < k$, we have

$$\psi_k(u) = 1 - \phi_k(k)\bar{\xi}^{\Delta}(pk + (1-p)u, k),$$

where $\xi^{\Delta}(u,b) = \frac{\psi^{\Delta}(u) - \psi^{\Delta}(b)}{1-\psi^{\Delta}(b)}$, $\bar{\xi}^{\Delta}(u,b) = 1 - \xi^{\Delta}(u,b)$.

Example 5.1.5. We assume the claims follow exponential distribution, s.t. $F_X(x) = 1 - e^{-\beta x}$, we could obtain

$$\bar{\xi}^{\Delta}(u,b) = \frac{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})\frac{b}{1-p}}}{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})\frac{u}{1-p}}},\ \xi^{\Delta}(u,b) = \frac{\lambda e^{-(\beta-\frac{\lambda}{c})\frac{b}{1-p}} - \lambda e^{-(\beta-\frac{\lambda}{c})\frac{u}{1-p}}}{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})\frac{u}{1-p}}}.$$

We could derive

$$\psi_k(k) = \frac{\psi(0) - \psi(0) f_p^{\Delta}(k)}{1 - \psi(0) f_p^{\Delta}(k)},$$

with

$$f_p(k) = \frac{c\beta}{c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})\frac{k}{1-p}}}(1 - e^{-\beta\frac{k}{1-p}}) - \frac{\lambda\beta e^{-(\beta-\frac{\lambda}{c})\frac{k}{1-p}}(1 - e^{-(p\beta+(1-p)\frac{\lambda}{c})\frac{k}{1-p}})}{(p\beta + (1-p)\frac{\lambda}{c})(c\beta - \lambda e^{-(\beta-\frac{\lambda}{c})\frac{k}{1-p}})},$$

resulting in

$$\psi_k(u) = \psi(u-k)[1 - \phi_k(k) f_p^{\Delta}(k)] \text{ for all } u \geq k.$$

5.1.4 Continuous capital injection

According to the classical capital injection model, the insurer receives instant compensation as long as any claim leads the surplus level into the interval between 0 and k. Under the continuous capital injection model's setting, we aim to construct the strategy which provides a continuous compensation with rate ac to the insurer, when the surplus level is under the compensation level k. Besides, Li et al. (2018) investigated this process as a refracted risk process, where the surplus process consisted of two parts,

$$dU_t = \begin{cases} cdt - S_t, U_t \geq k, \\ acdt - S_t, 0 < U_t < k. \end{cases}$$

This leads to the following probability of ruin setting, given by Lin and Pavlova (2006),

$$\psi(u) = \begin{cases} \psi_1(u), u \geq k, \\ \psi_2(u), 0 < u < k. \end{cases}$$

Then the joint Laplace transform ψ satisfies the following integro-differential equations:

$$\begin{cases} c\psi_1'(u) = \lambda\psi_1(u) - \lambda\big[\int_0^{u-b} \psi_1(u-x) f(x)dx + \int_{u-b}^{u} \psi_2(u-x) f(x)dx\big] - \lambda\bar{F}(u), u \geq k, \\ ac\psi_2'(u) = \lambda\psi_2(u) - \lambda\int_0^{u} \psi_2(u-x) f(x)dx - \lambda\bar{F}(u), 0 < u < k. \end{cases}$$

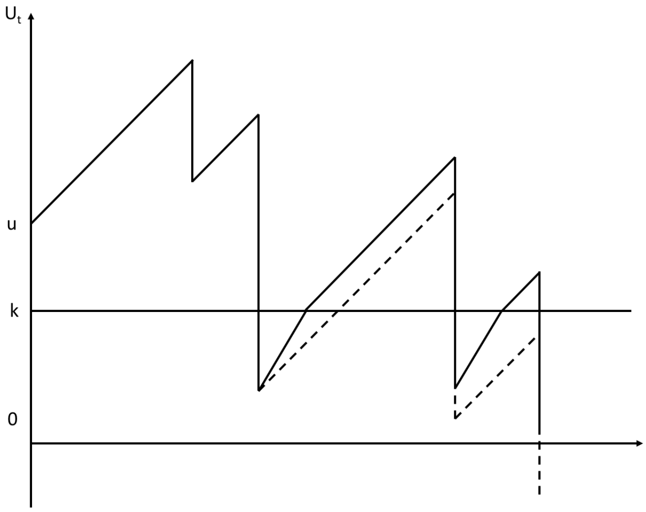


Figure 5.6: Surplus process with continuous injection

Remark 5.1.3. It is clear to see that when $a \to \infty$, the model will be the same as the classical capital injection model. When $a \to 1$, it becomes the classical surplus process. Therefore, we can obtain the following theorem, by applying the idea of the two barriers model.

Theorem 5.1.5. When the initial capital $u = k$, we have

$$\psi_k(k) = \int_0^k g(0,y)\big[\xi^\Delta(k-y,k) + \bar{\xi}^\Delta(k-y,k)\psi_k(k)\big]dy + \int_k^\infty g(0,y)dy.$$

When the initial capital $u > k$, we have

$$\psi_k(u) = \psi(u-k) - \phi_k(k)\int_0^k g(u-k,y)\bar{\xi}^\Delta(k-y,k)dy.$$

When the initial capital $0 \leq u < k$, we have

$$\psi_k(u) = 1 - \phi_k(k)\bar{\xi}^\Delta(u,k),$$

where $\xi^\Delta(u,b) = \frac{\psi^\Delta(u)-\psi^\Delta(b)}{1-\psi^\Delta(b)}$.

Example 5.1.6. We assume the claims follow exponential distribution, s.t. $F_X(x) = 1 - e^{-\beta x}$, we could have

$$\bar{\xi}^\Delta(u,b) = \frac{ac\beta - \lambda e^{-(\beta-\frac{\lambda}{ac})u}}{ac\beta - \lambda e^{-(\beta-\frac{\lambda}{ac})b}}, \quad \xi^\Delta(u,b) = \frac{\lambda(e^{-(\beta-\frac{\lambda}{ac})u} - e^{-(\beta-\frac{\lambda}{ac})b})}{ac\beta - \lambda e^{-(\beta-\frac{\lambda}{ac})b}}.$$

Then we could have

$$\psi_k(k) = \frac{\psi(0) - \psi(0) f_p^{\Delta}(k)}{1 - \psi(0) f_p^{\Delta}(k)},$$

where

$$f_p^{\Delta}(k) = \frac{ac\beta(1 - e^{-\beta k}) - ac\beta e^{-(\beta - \frac{\lambda}{ac})k}(1 - e^{-\frac{\lambda}{ac}k})}{ac\beta - \lambda e^{-(\beta - \frac{\lambda}{ac})k}}.$$

Then the ruin probability can be calculated by

$$\psi_k(u) = \psi(u - k)[1 - \phi_k(k) f_p^{\Delta}(k)] \text{ for all } u \geq k.$$

5.1.5 Partial discrete capital injection for all claims occurred if the surplus process is below k

Now we take the inspiration from Section 5.1.2 and 5.1.3. We aim to define that the injection happens for all claims occurred, as long as the surplus level is below the compensation level k (similar to the combination of the models in Section 5.1.2 and 5.1.3). The injection process is given by

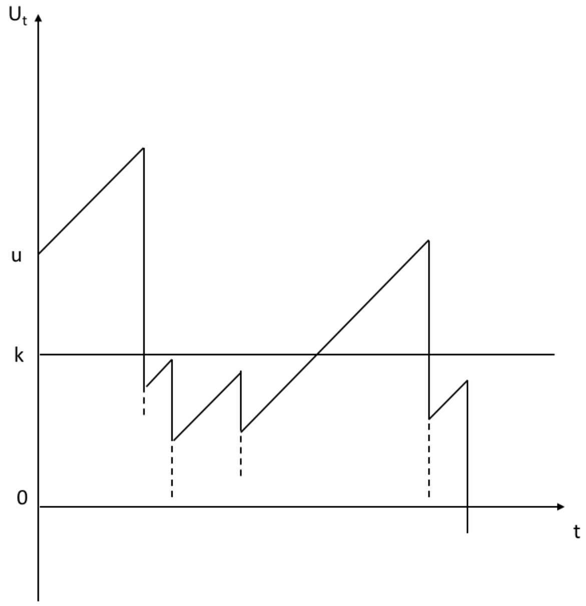


Figure 5.7: Surplus process for PDCIA

$$J_t = \sum_{i=1}^{N(t)} p(k - U_{t_i}^{\Delta})\mathbb{I}\{0 < U_{t_i}^{\Delta} < k\}.$$

Consider an small interval $(0, h)$, there are four possible situations under this process:

- **no** claim occurs in $(0, h)$ with probability $1 - \lambda h + o(h)$ and the surplus level rises by ch,
- **one** claim occurs in $(0, h)$ with probability $\lambda h + o(h)$ and the surplus level rises by ch, drops by X, the claim amount, and a compensation $p(k-u+ch-X)$ is received if $u + ch - X > 0$ or the claim leads the surplus level below 0 s.t. $u + ch - X < 0$,
- **more than one** claim occurs in $(0, h)$ with probability $o(h)$.

Therefore, the non-ruin probability satisfies

$$\phi^{\Delta}(u) = (1 - \lambda h)\phi^{\Delta}(u + ch) + \lambda h \int_0^{u+ch} \phi^{\Delta}(u + ch - x + p(k - u + ch - x))f(x)dx + o(h),$$

$$c\frac{\phi^{\Delta}(u + ch) - \phi^{\Delta}(u)}{ch} = \lambda\phi^{\Delta}(u + ch) - \lambda \int_0^{u+ch} \phi^{\Delta}(pk + (1 - p)(u + ch - x))f(x)dx + \frac{o(h)}{h}.$$

Let $h \to 0$, then we have the following integro-differential equation

$$c\frac{d}{du}\phi^{\Delta}(u) = \lambda\phi^{\Delta}(u) - \lambda \int_0^u \phi^{\Delta}(pk + (1 - p)(u - x))f(x)dx.$$

Now one applies the Laplace transform with respect to u on both sides

$$cs\hat{\phi}^{\Delta}(s) - c\phi^{\Delta}(0) = \lambda\hat{\psi}^{\Delta}(u) - \lambda\mathcal{L}\{\int_0^u \phi^{\Delta}(pk + (1 - p)(u - x))f(x)dx\}.$$

The integral part can be seperatly treated

$$\mathcal{L}\{\int_0^u \phi^{\Delta}(pk + (1 - p)(u - x))f(x)dx\} = \int_0^{\infty} e^{-su} \int_0^u \phi^{\Delta}(pk + (1 - p)(u - x))f(x)dxdu.$$

Changing the variable and using the substitution $pk + (1 - p)(u - x) = z$, thus we have $x = \frac{pk-z}{1-p} + u$, therefore, the above equation becomes

$$\int_0^{\infty} e^{-su} \int_{pk}^{pk+(1-p)u} \phi^{\Delta}(z)f(\frac{pk - z}{1 - p} + u)dzdu.$$

Now changing the order of the integrals, we could obtain

$$\int_{pk}^{\infty} \phi^{\Delta}(z) \int_{\frac{z-pk}{1-p}}^{\infty} e^{-su} f(\frac{pk-z}{1-p} + u) du dz$$
$$= \hat{f}(s) \int_{pk}^{\infty} \phi^{\Delta}(z) e^{-\frac{z-pk}{1-p}s} dz.$$

However, this is impossible to calculate the ruin probability by applying the Laplace transform.

5.2 Special process inspired by model of PDRP

Recall the process given by PDRP in Section (5.1.3), the non-ruin probability conditioning the lower barrier k satisfies (5.1). The ruin of PDRP will be predicated after the injection, in other word, the company will get ruined only if the surplus is below zero when consider the claim as $(1-p)X$. Now, we aim to construct a new reinsurance strategy, which leads the prior ruin predication rather than the injections.

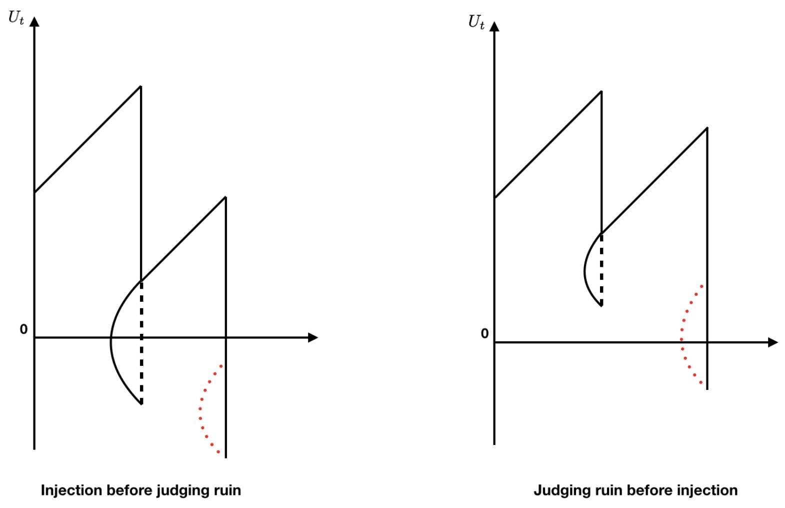


Figure 5.8: Ruin or injection

Then its non-ruinprobability satisfies the following the integro-differential equation,

$$c\phi'(u) = \lambda\phi(u) - \lambda \int_0^u \phi(u-(1-p)y)F(dy).$$

Integrating on both sides over $[0, u]$ gives

$$\begin{aligned}
c(\phi(u) - \phi(0)) &= \lambda\Big(\int_0^u \phi(z)\,dz - \int_0^u \int_0^x \phi(x-(1-p)y)F(dy)\,dx\Big) \\
&= \lambda \int_0^u \phi(z)\,dz - \lambda \int_0^u F(dy) \int_y^u \phi(x-(1-p)y)\,dx \\
&= \lambda \int_0^u \phi(z)\,dz - \lambda \int_0^u F(dy) \int_{py}^{u-(1-p)y} \phi(z)\,dz \\
&= \lambda \int_0^u \phi(z)\,dz - \lambda \int_0^u \phi(z)\,dz \int_0^{\frac{u-z}{(1-p)} \wedge \frac{z}{p}} F(dy) \\
&= \lambda \int_0^u \phi(z)\big(1 - F(\frac{u-z}{1-p} \wedge \frac{z}{p})\big)\,dz.
\end{aligned}$$

Thus

$$\phi(u) = \phi(0) + \frac{\lambda}{c}\int_0^u \phi(z)\big(1 - F(\frac{u-z}{1-p} \wedge \frac{z}{p})\big)\,dz. \tag{5.3}$$

Let $H(u) = \frac{\phi(u)}{\phi(0)}$ on $\mathbb{R}^+$ which satisfies

$$H(u) = 1 + \frac{\lambda}{c}\int_0^u H(z)\overline{F}\Big(\frac{u-z}{1-p} \wedge \frac{z}{p}\Big)\,dz.$$

Let $h_0 = 1$, define for $n \geq 0$

$$\begin{aligned}
h_{n+1}(u) &= \frac{\lambda}{c}\int_0^u h_n(z)\overline{F}\Big(\frac{u-z}{1-p} \wedge \frac{z}{p}\Big)\,dz \\
&= \frac{\lambda}{c}\int_0^{pu} h_n(z)\overline{F}\big(\frac{z}{p}\big)\,dz + \frac{\lambda}{c}\int_0^{(1-p)u} h_n(u-z)\overline{F}\big(\frac{z}{1-p}\big)\,dz \\
&= \frac{\lambda p}{c}\int_0^u h_n(pz)\overline{F}(z)\,dz + \frac{\lambda(1-p)}{c}\int_0^u h_n(u-pz)\overline{F}(z)\,dz.
\end{aligned}$$

Therefore, the general solution ican be denoted by an infinite summation,

$$H(u) = \sum_{n=0}^{\infty} h_n(u). \tag{5.4}$$

5.2.1 Exponential claims

Consider the case $\overline{F}(z) = e^{-\beta z}$ for $z > 0$. Now define an operator as the integral part in (5.3), s.t.

$$\mathcal{T}\{h_n(u)\} = h_{n+1}(u),$$

then we could obtain some facts of calculation. For a constant k,

$$\mathcal{T}\{k\} = \frac{\lambda k}{c\beta} - \frac{\lambda k}{c\beta}e^{-\beta u}. \tag{5.5}$$

For $f(z) = e^{-\gamma\beta z}$ for any $\gamma \geq 0$,

$$\begin{aligned}
&\frac{\lambda p}{c}\int_0^u f(pz)\overline{F}(z)\,dz + \frac{\lambda(1-p)}{c}\int_0^u f(u-(1-p)z)\overline{F}(z)\,dz \\
=&\frac{\lambda p}{c}\int_0^u e^{-p\gamma\beta z}e^{-\beta z}\,dz + \frac{\lambda(1-p)}{c}\int_0^u e^{-\gamma\beta(u-(1-p)z)}e^{-\beta z}\,dz \\
=&\frac{\lambda p}{c\beta}\frac{1}{1+p\gamma}\big(1-e^{-(1+p\gamma)\beta u}\big) + \frac{\lambda(1-p)}{c\beta}\frac{1}{1-(1-p)\gamma}\big(e^{-\gamma\beta u} - e^{-(1+p\gamma)\beta u}\big).
\end{aligned}$$

Therefore, we can obtain the general form of operation for an exponential function,

$$\mathcal{T}\{e^{-\gamma\beta u}\} = \frac{\lambda}{c\beta}\frac{p}{1+p\gamma}\big(1-e^{-(1+p\gamma)\beta u}\big) + \frac{\lambda}{c\beta}\frac{1-p}{1-(1-p)\gamma}\big(e^{-\gamma\beta u} - e^{-(1+p\gamma)\beta u}\big).$$

Furthermore, we let $f(\gamma, u) = 1 - e^{-\beta\gamma u}$ and denote

$$\gamma_n = \sum_{m=0}^{n} p^m, \gamma_0 = 1, \gamma_\infty = \frac{1}{1-p}.$$

The facts can be represented as

$$\begin{aligned}
(1-p)\gamma_n &= (1-p)\sum_{m=0}^{n} p^m = 1 - p^{n+1}, \\
1 + p\gamma_n &= 1 + \sum_{m=1}^{n+1} p^m = \gamma_{n+1}.
\end{aligned}$$

Now we denote $\mathcal{T}^m\{f(\gamma_n, u)\} = \mathcal{T}_n^m$, where $\mathcal{T}^0\{f(\gamma_n, u)\} = f(\gamma_n, u)$, thus

$$\mathcal{T}_n^m = \frac{\lambda}{c\beta}\mathcal{T}_0^{m-1} + \frac{\lambda}{c\beta}\frac{1-p}{p^{n+1}}\mathcal{T}_n^{m-1} - \frac{\lambda}{c\beta}\frac{1}{p^{n+1}\gamma_{n+1}}\mathcal{T}_{n+1}^{m-1}.$$

In particular

$$\begin{aligned}
\mathcal{T}_0^n &= \frac{\lambda}{pc\beta}\mathcal{T}_0^{n-1} - \frac{\lambda}{pc\beta}\frac{1}{\gamma_1}\mathcal{T}_1^{n-1}, \\
\frac{\lambda}{pc\beta}\mathcal{T}_0^{n-1} &= (\frac{\lambda}{pc\beta})^2\mathcal{T}_0^{n-2} - (\frac{\lambda}{pc\beta})^2\frac{1}{\gamma_1}\mathcal{T}_1^{n-2}.
\end{aligned}$$

Then we have

$$\mathcal{T}_0^n = (\frac{\lambda}{pc\beta})^n\mathcal{T}_0^0 - \frac{1}{\gamma_1}\sum_{i=1}^{n}(\frac{\lambda}{pc\beta})^i\mathcal{T}_1^{n-i}. \tag{5.6}$$

Furthermore, we have

$$\mathcal{T}_n^m = \frac{\lambda}{c\beta}\mathcal{T}_0^{m-1} + \frac{\lambda}{c\beta}\frac{1-p}{p^{n+1}}\mathcal{T}_n^{m-1} - \frac{\lambda}{c\beta}\frac{1}{p^{n+1}\gamma_{n+1}}\mathcal{T}_{n+1}^{m-1},$$
$$\frac{\lambda}{c\beta}\frac{1-p}{p^{n+1}}\mathcal{T}_n^{m-1} = (\frac{\lambda}{c\beta})^2\frac{1-p}{p^{n+1}}\mathcal{T}_0^{m-2} + (\frac{\lambda}{c\beta}\frac{1-p}{p^{n+1}})^2\mathcal{T}_n^{m-2} - (\frac{\lambda}{p^{n+1}c\beta})^2\frac{1-p}{\gamma_{n+1}}\mathcal{T}_{n+1}^{m-2}.$$

It results in

$$\mathcal{T}_n^m = (\frac{\lambda}{c\beta}\frac{1-p}{p^{n+1}})^n\mathcal{T}_n^0 + \sum_{i=1}^{m}(\frac{\lambda}{c\beta})^i(\frac{1-p}{p^{n+1}})^{i-1}\mathcal{T}_0^{m-i} - \sum_{i=1}^{m}(\frac{\lambda}{p^{n+1}c\beta})^i\frac{(1-p)^{i-1}}{\gamma_{n+1}}\mathcal{T}_{n+1}^{m-i}.$$

We now have the following theorem.

Theorem 5.2.1. For each $n > 0$, $h_n(u)$ consists of $f(\gamma_0, u), f(\gamma_2, u), \ldots f(\gamma_{n-1}, u)$ for n terms. For the further operation, $h_{n+1}(u) = \mathcal{T}\{h_n(u)\}$ would generate the extra term $f(\gamma_n, u)$,which would lead the number of components of $h_{n+1}(u)$ be $n+1$. Then according to (5.4), we can obtain the general form for the solution

$$H(u) = \sum_{n=0}^{\infty} h_n(u) = 1 + \sum_{n=1}^{\infty} a_n f(\gamma_n, u) \tag{5.7}$$

under the boundary condition

$$\lim_{u\to\infty} H(u) = \frac{1}{\phi(0)} = 1 + \sum_{n=0}^{\infty} a_n. \tag{5.8}$$

Proof.

$$\mathcal{T}\{f(\gamma_n, u)\} = \mathcal{T}\{1 - e^{-\gamma_n\beta u}\} = \mathcal{T}\{1\} - \mathcal{T}\{e^{-\gamma_n\beta u}\}$$
$$= \frac{\lambda}{c\beta}\big(f(\gamma_0, u) - \frac{p}{1+p\gamma_n}f(1+p\gamma_n, u) - \frac{1-p}{1-(1-p)\gamma_n}[f(1+p\gamma_n, u) - f(\gamma_n, u)]\big)$$
$$= \frac{\lambda}{c\beta}\big(f(\gamma_0, u) + \frac{1-p}{p^{n+1}}f(\gamma_n, u) - \frac{f(\gamma_{n+1}, u)}{p^{n+1}\gamma_{n+1}}\big).$$

Therefore, we could obtain the pattern of the operator. □

In addition, the solution holds the fact of

$$1 + \mathcal{T}\{H(u)\} = H(u), \tag{5.9}$$

then we could have

$$\mathcal{T}\{H(u)\} = 1 + \mathcal{T}\{1 + \lim_{n\to\infty}\sum_{m=0}^{n} a_m f(\gamma_m, u)\} = 1 + \mathcal{T}\{1\} + \lim_{n\to\infty}\sum_{m=0}^{n} a_m \mathcal{T}\{f(\gamma_m, u)\}.$$

Recall that $\mathcal{T}\{f(\gamma_m, u)\} = \frac{\lambda}{c\beta}\big(f(\gamma_0, u) + \frac{1-p}{p^{m+1}} f(\gamma_m, u) - \frac{f(\gamma_{m+1}, u)}{p^{m+1}\gamma_{m+1}}\big)$, we then could obtain

$$\mathcal{T}\{H(u)\} = \frac{\lambda}{c\beta}\big(\sum_{m=1}^{\infty} a_m + \frac{1}{p}a_0\big)f(\gamma_0, u) + \frac{\lambda}{c\beta}\sum_{m=1}^{\infty}\big(a_m\frac{1-p}{p^{m+1}} - a_{m-1}\frac{1}{p^m\gamma_m}\big)f(\gamma_m, u) - \lim_{n\to\infty}\frac{\lambda}{c\beta}\frac{a_n}{p^n\gamma_n}f(\gamma_n, u).$$

Now we have

$$\begin{cases} a_0 = \dfrac{\lambda}{c\beta}\big(\displaystyle\sum_{m=1}^{\infty} a_m + \frac{1}{p}(a_0 + 1)\big), \\ a_m = \dfrac{\lambda}{c\beta}\big(a_m\dfrac{1-p}{p^{m+1}} - a_{m-1}\dfrac{1}{p^m\gamma_m}\big), \quad \text{for } m \geq 1. \end{cases} \tag{5.10}$$

From second equation of (5.10), we can obtain

$$\begin{aligned} \frac{p^{m+1}}{1-p}a_m &= \frac{\lambda}{c\beta}\big(a_m - a_{m-1}\frac{p}{(1-p)\gamma_m}\big), \\ a_m\frac{\lambda(1-p) - p^{m+1}c\beta}{c\beta(1-p)} &= a_{m-1}\frac{\lambda}{c\beta}\frac{p}{(1-p)\gamma_m}, \\ a_m &= a_{m-1}\frac{\lambda p/\gamma_m}{\lambda(1-p) - c\beta p^{m+1}}. \end{aligned}$$

Then we have

$$a_m = a_0\big(\frac{p}{1-p}\big)^m\prod_{i=1}^{m}\frac{\lambda/\gamma_i}{\lambda - \gamma_i c\beta}. \tag{5.11}$$

Now we aim to prove the convergence of the series by applying the Leibniz's rule.

Theorem 5.2.2. A series of the form (5.7), which can be written by

$$H(u) = 1 + \sum_{n=0}^{\infty}(-1)^n b_n f(\gamma_n, u),$$

where

$$b_m = a_0\big(\frac{p}{1-p}\big)^m\prod_{i=1}^{m}\frac{\lambda/\gamma_i}{\gamma_i c\beta - \lambda},$$

this alternating series converges.

Proof. By conditioning on $c > \frac{(1-p)\lambda}{\beta}$, we notice that $\lambda - c\beta\gamma_m < 0$ and $\gamma_m < \gamma_{m+1}$ for any $m \geq 1$ and $a_m \geq 0$ for any even m, $a_m \leq 0$ for any odd m. Eventually

$$\lim_{m\to\infty} |a_m| = a_0 \lim_{m\to\infty} |(\frac{p}{1-p})^m \prod_{i=1}^{m} \frac{\lambda/\gamma_i}{\lambda - \gamma_i c\beta}| < a_0 \lim_{m\to\infty} |(\frac{\lambda p}{\lambda - \frac{c\beta}{1-p}})^m| = 0.$$

According to

$$\lambda - \frac{c\beta}{1-p} - \lambda p = (1-p)\lambda - \frac{c\beta}{1-p} < -\lambda p < 0,$$

thus $|a_m|$ converges to 0 when $m \to \infty$. Furthermore,

$$\lim_{m\to\infty} |a_m f(\gamma_m, u)| = a_0 \lim_{m\to\infty} f(\gamma_m, u)|(\frac{p}{1-p})^m \prod_{i=1}^{m} \frac{\lambda/\gamma_i}{\lambda - \gamma_i c\beta}|$$
$$< a_0 \lim_{m\to\infty} f(\gamma_m, u)|(\frac{\lambda p}{\lambda - c\beta(1-p)})^m| = 0,$$

where $|a_m f(\gamma_m, u)|$ converges to 0 when $m \to \infty$. We then have

$$b_{m+1} - b_m = b_m\big(\frac{p}{1-p}\frac{\lambda/\gamma_{m+1}}{\gamma_{m+1}c\beta - \lambda} - 1\big) = b_m \frac{p\lambda - \gamma_{m+1}(1-p)(\gamma_{m+1}c\beta - \lambda)}{\gamma_{m+1}(1-p)(\gamma_{m+1}c\beta - \lambda)} < 0$$

and

$$f(\gamma_{m+1}, u) - f(\gamma_m, u) > 0.$$

Therefore,

$$b_{m+1}f(\gamma_{m+1}, u) - b_m f(\gamma_m, u) < b_{m+1}f(\gamma_{m+1}, u) - b_m f(\gamma_{m+1}, u) < 0.$$

□

Then substitute (5.11) into first equation of (5.10), we obtain

$$a_0 = \frac{1}{1 + p\sum_{m=1}^{\infty}(\frac{p}{1-p})^m \prod_{i=1}^{m} \frac{\lambda/\gamma_i}{\lambda - \gamma_i c\beta}}.$$

Then we could have

$$\sum_{n=0}^{\infty} a_n = \frac{1 + \sum_{m=1}^{\infty}(\frac{p}{1-p})^m \prod_{i=1}^{m} \frac{\lambda/\gamma_i}{\lambda - \gamma_i c\beta}}{1 + p\sum_{m=1}^{\infty}(\frac{p}{1-p})^m \prod_{i=1}^{m} \frac{\lambda/\gamma_i}{\lambda - \gamma_i c\beta}} = \frac{1+\xi}{1+p\xi},$$

where

$$\xi = \sum_{m=1}^{\infty} (\frac{p}{1-p})^m \prod_{i=1}^{m} \frac{\lambda/\gamma_i}{\lambda - \gamma_i c\beta}.$$

Therefore we have the coefficients of the solution (5.7),

$$\begin{cases} a_0 = \dfrac{1}{1+p\xi}, \\ a_m = (-1)^m b_m, \ \text{for } m \geq 1. \end{cases} \tag{5.12}$$

Eventually, the general solution can be written as

$$H(u) = 1 + a_0 f(\gamma_0) + \sum_{m=1}^{\infty} (-1)^m b_m f(\gamma_m, u),$$

where a_0 is given by (5.12). Then we could obtain the non-ruin probability

$$\phi(u) = \phi(0) \left[1 + a_0 f(\gamma_0) + \sum_{m=1}^{\infty} (-1)^m b_m f(\gamma_m, u) \right], \tag{5.13}$$

where

$$\phi(0) = \frac{1}{1 + \frac{1+\xi}{1+p\xi}} \text{ and } f(\gamma_m, u) = 1 - e^{-\beta u \sum_{i=0}^{m} p_i}.$$

5.2.2 Mixture exponential claims

Consider the case $\overline{F}_n(z) = \sum_{i=1}^{n} \omega_i e^{-\beta_i z}$ for $z > 0$ as the tail distribution of the mixture n exponential claims. Then we notice the operator becomes

$$\mathcal{T}\{1\} = \frac{\lambda}{c} \sum_{i=1}^{n} \frac{\omega_i}{\beta_i} f(\beta_i, u),$$

where

$$f(z, u) = 1 - e^{-zu}.$$

Furthermore, we have

$$\mathcal{T}\{\sum_{i=1}^{n} f_i(\gamma_m)\} = \frac{\lambda}{c\beta} \sum_{i=1}^{n} \big(f_i(\gamma_0, u) + \frac{1-p}{p^{m+1}} f_i(\gamma_m, u) - \frac{f_i(\gamma_{m+1}, u)}{p^{m+1}\gamma_{m+1}} \big).$$

For $f(z) = e^{-\alpha z}$ for any $\gamma \geq 0$,

$$
\begin{aligned}
&\frac{\lambda p}{c}\int_0^u f(pz)\overline{F}(z)\,dz + \frac{\lambda(1-p)}{c}\int_0^u f(u-(1-p)z)\overline{F}(z)\,dz\\
=&\frac{\lambda p}{c}\int_0^u e^{-p\alpha z}\sum_{j=0}^{n}\omega_j e^{-\beta_j z}\,dz + \frac{\lambda(1-p)}{c}\int_0^u e^{-\alpha(u-(1-p)z)}\sum_{j=0}^{n}\omega_j e^{-\beta_j z}\,dz\\
=&\frac{\lambda p}{c}\sum_{j=1}^{n}\frac{\omega_j}{p\alpha+\beta_j}\big(1-e^{-(\beta_j+p\alpha)u}\big) + \frac{\lambda(1-p)}{c}\sum_{j=0}^{n}\frac{\omega_j}{\beta_j-(1-p)\alpha}\big(e^{-\alpha u}-e^{-(\beta_j+p\alpha)u}\big)\\
=&\frac{\lambda}{c}\sum_{j=1}^{n}[\frac{p\omega_j}{p\alpha+\beta_j}f(\beta_j+p\alpha,u) + \frac{(1-p)\omega_j}{\beta_j-(1-p)\alpha}\big(f(\beta_j+p\alpha,u)-f(\alpha,u)\big)]\\
=&\frac{\lambda}{c}\sum_{j=1}^{n}\big(\frac{\omega_j\beta_j}{(\beta_j+p\alpha)(\beta_j-(1-p)\alpha)}f(\beta_j+p\alpha,u) - \frac{(1-p)\omega_j}{\beta_j-(1-p)\alpha}f(\alpha,u)\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathcal{T}\{f(\beta_i,u)\} = \mathcal{T}\{1-e^{-\beta_i u}\} = \mathcal{T}\{1\} - \mathcal{T}\{e^{-\gamma_m\beta_i u}\}\\
=&\frac{\lambda}{c}\sum_{j=1}^{n}\big(\frac{\omega_j}{\beta_j}f(\beta_j,u) - \frac{\omega_j\beta_j}{(\beta_j+p\beta_i)(\beta_j-(1-p)\beta_i)}f(\beta_j+p\beta_i,u) - \frac{(1-p)\omega_j}{\beta_j-(1-p)\beta_i}f(\beta_i,u)\big).
\end{aligned}
$$

Then the solution is in terms of the summation of $1, 2, ...$ until infinity sums of the function f, thus let k be the number of sums, we denote the operator of sums

$$
\mathcal{S}_k = \sum_{i_1=1}^{n}\frac{\omega_{i_1}}{\beta_{i_1}}\sum_{i_2=1}^{n}\frac{\omega_{i_2}}{\beta_{i_2}}\cdots\sum_{i_k=1}^{n}\frac{\omega_{i_n}}{\beta_{i_n}},\ \mathcal{S}_0 = 1 \,\text{and}\, \mathcal{S}_k = 0 \,\text{for any}\ k < 1,
$$

the solution consists of

$$
\sum_{k=0}^{\infty}\mathcal{S}_k\sum_{i=1}^{n}a_{i,1,k}f(\beta_i,u) + \sum_{k=1}^{\infty}\mathcal{S}_k\sum_{i=1}^{n}a_{i,2,k}f(\beta_{i_1}+p\beta_i,u) + \cdots + \sum_{k=j-1}^{\infty}\mathcal{S}_k\sum_{i=1}^{n}a_{i,j,k}f(\sum_{m=0}^{j-1}p^m\beta_{i_{j-m-1}},u).
$$

where we denote $i_0 = i$. Therefore, we could obtain the solution of (5.4) under the mixture n exponential claims distribution,

$$
H(u) = 1 + \sum_{i=1}^{n}\sum_{j=1}^{\infty}\sum_{k=j-1}^{\infty}\mathcal{S}_k a_{i,j,k} f(\sum_{m=0}^{j-1}p^m\beta_{i_{j-m-1}},u) \tag{5.14}
$$

with boundary condition

$$
\lim_{u\to\infty}H(u) = \frac{1}{\phi(0)} = 1 + \sum_{i=1}^{n}\sum_{j=1}^{\infty}\sum_{k=j-1}^{\infty}\mathcal{S}_k a_{i,j,k}.
$$

Then we could apply the same method from (5.9),

$$
\begin{aligned}
a_{i,j,k}\mathcal{T}\{f(\sum_{m=0}^{j-1} p^m \beta_{i_{j-m-1}}, u)\} &= a_{i,j,k}\frac{\lambda}{c}\sum_{z=1}^{n}\big(\frac{\omega_z}{\beta_z} f(\beta_z, u) \\
&- \frac{\omega_z\beta_z}{(\beta_z + \sum_{m=0}^{j-1} p^{m+1}\beta_{i_{j-m-1}})(\beta_z - (1-p)\beta_i)} f(\beta_z + \sum_{m=0}^{j-1} p^{m+1}\beta_{i_{j-m-1}}, u) \\
&- \frac{(1-p)\omega_z}{\beta_z - (1-p)\sum_{m=0}^{j-1} p^m \beta_{i_{j-m-1}}} f(\sum_{m=0}^{j-1} p^m \beta_{i_{j-m-1}}, u)\big).
\end{aligned}
$$

Therefore we notice that

$$
\begin{cases}
a_{i,1,k} = \frac{\lambda}{c}\frac{\omega_i}{\beta_i} + \frac{\lambda}{c}\sum_{j=1}^{\infty}\sum_{m=j}^{\infty}\mathcal{S}_m a_{i,j,m} - \frac{\lambda}{c}\mathcal{S}_k a_{i,1,k}\sum_{z=1}^{n}\frac{(1-p)\omega_z}{\beta_z - (1-p)\beta_i}, \\
a_{i,j+1,k} = -\frac{\lambda}{c}[\mathcal{S}_k a_{i,j,k}\sum_{z=1}^{n}\frac{\omega_z\beta_z}{(\beta_z + p\sum_{m=0}^{j} p^m\beta_{i_{j-m-1}})(\beta_z - (1-p)\sum_{m=0}^{j-1} p^m\beta_{i_{j-m-1}})} \\
\qquad - \mathcal{S}_k a_{i,j+1,k}\sum_{z=1}^{n}\frac{(1-p)\omega_z}{\beta_z - (1-p)\sum_{m=0}^{j} p^m\beta_{i_{j-m}}}], \text{ for } m \geq 1.
\end{cases}
\tag{5.15}
$$

According to the second equation of (5.15), we have

$$
\begin{aligned}
a_{i,j+1,k} &= -a_{i,j,k}\frac{\lambda}{c}\frac{\mathcal{S}_k\sum_{z=1}^{n}\frac{\omega_z\beta_z}{(\beta_z+p\sum_{m=0}^{j} p^m\beta_{i_{j-m-1}})(\beta_z-(1-p)\sum_{m=0}^{j-1} p^m\beta_{i_{j-m-1}})}}{1+\mathcal{S}_k\frac{\lambda}{c}\frac{(1-p)\omega_z}{\beta_z-(1-p)\sum_{m=0}^{j} p^j\beta_{i_{j-m}}}} \\
&= a_{i,1,k}(-\frac{\lambda}{c})^j\prod_{h=1}^{j}\frac{\mathcal{S}_h\sum_{z=1}^{n}\frac{\omega_z\beta_z}{(\beta_z+p\sum_{m=0}^{h} p^j\beta_{i_{h-m-1}})(\beta_z-(1-p)\sum_{m=0}^{h-1} p^m\beta_{i_{h-m-1}})}}{1+\mathcal{S}_h\frac{\lambda}{c}\frac{(1-p)\omega_z}{\beta_z-(1-p)\sum_{m=0}^{h} p^m\beta_{i_{h-m}}}}.
\end{aligned}
$$

Therefore, we could obtain the coefficients of (5.14)

$$
\begin{cases}
a_{i,1,k} = \frac{\lambda}{c}\frac{\frac{\omega_i}{\beta_i} + \sum_{j=1}^{\infty}\sum_{m=j}^{\infty}\mathcal{S}_m a_{i,j,m}}{1\frac{\lambda}{c}\mathcal{S}_k a_{i,1,k}\sum_{z=1}^{n}\frac{(1-p)\omega_z}{\beta_z-(1-p)\beta_i}}, \\
a_{i,j+1,k} = a_{i,1,k}(-\frac{\lambda}{c})^j\prod_{h=1}^{j}\frac{\mathcal{S}_h\sum_{z=1}^{n}\frac{\omega_z\beta_z}{(\beta_z+p\sum_{m=0}^{h} p^j\beta_{i_{h-m-1}})(\beta_z-(1-p)\sum_{m=0}^{h-1} p^m\beta_{i_{h-m-1}})}}{1+\mathcal{S}_h\frac{\lambda}{c}\frac{(1-p)\omega_z}{\beta_z-(1-p)\sum_{m=0}^{h} p^m\beta_{i_{h-m}}}}, \text{ for } m \geq 1.
\end{cases}
\tag{5.16}
$$

5.3 Numerical illustrations

Now we provide the numerical analysis which is arranged along with all of the parameters. In this section, there are 6 plots of ruin probability against initial capital u, premium rate c, claim size β, claim frequency intensity λ, the ratio of compensation p and the reinsurance payment rate q, respectively. The notations in the plots are given by

- Classical surplus process: C
- Classical capital injection: CI
- Partial discrete capital injection: PDCI
- Partial discrete reinsurance contract: PDR
- Partial discrete reinsurance contract with specific payment periods: PDRP
- Continuous capital injection: CCI

The first picture plots the relationship between the ruin probability and initial capital u, considering the net profit conditions. Table 5.1 provides the parameters' coefficients for the $\psi(u), u$ plot.

Parameters	u	c	β	λ	k	p	q	a
	0-20	3	1	1	2	$0.5, 0.4^*$	0.8	10

Table 5.1: Parameters' coefficients for Figure 5.9

It is clear that

- According to the compensation strategies, the classical surplus process seems to have the largest ruin probability for all ranges of initial surplus (discuss later). When the insurer has a small amount of initial capital (which is far smaller than the k), the CI model has the best protection for the insurer for a small interval of u. With the exception of this, the PDRP model provides the lowest ruin probability (discuss later). Furthermore, the ruin probabilities under all models will converge to 0 eventually, when initial capital is large, although convergence speed differs between the models.

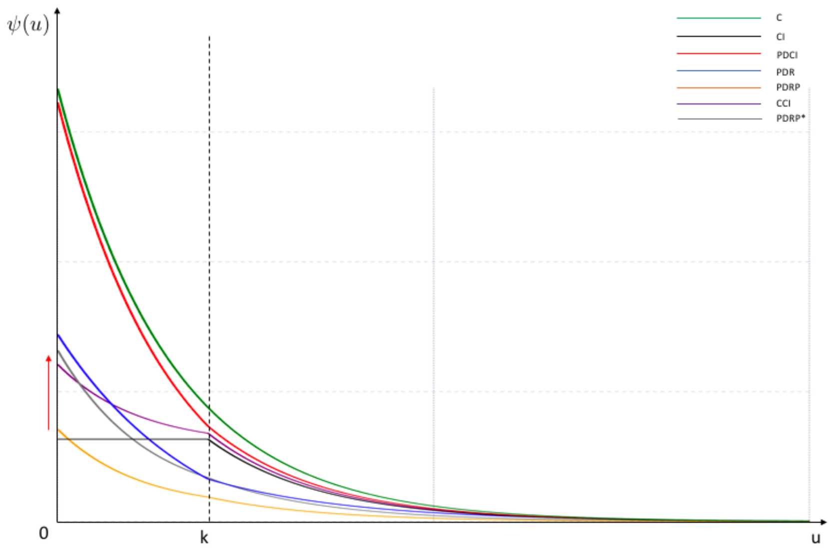


Figure 5.9: The movement of ruin probability with respect to u

- The curves of the ruin probabilities have flatter trends in the interval of initial capital $u \in [0, k)$ in comparison to when $u \geq k$, for most of the compensation strategies (except for PDCI and PDR, which have sharper trends according to an instant injection at the very beginning for initial capital smaller than the compensation level. Besides, the ruin probability of the classical surplus process has the same curve trend for any $u \geq 0$ according to the independence of k). In particular, for the capital injection model, the ruin probability remains constant in the interval of initial capital $u \in [0, k)$, $\psi_k(k)$, because the reinsurer provides a fund which restores the surplus level to the compensation level k for any initial surplus $u \in [0, k)$.

- The lowest curve of ruin probability is given by the model of PDRP for $p = 0.5$ in the above plot. When we decrease the value of p from $p = 0.5$ to $p^* = 0.4$, the orange curve moves up to the grey line, this leads to a greater ruin probability, due to the lower reinsurance payment rate $(0, 4 * c)$, claim amount covered rate $(0.4 * \mu)$ and less additional space ($k^* = \frac{5}{6}k$ and $u^* = u - \frac{1}{3}k$). In other words, PDRP reinsurance contract sufficiently helps to reduce the risks. In general, the ruin probability of the PDRP will converge to the classical case ones when it is assumed that $p = 0$. In fact, for all of the compensation strategies, as long as we reduce the benefits of the compensation, the ruin probability moves to the classical case's. Now if we zoom up the plot from the interval $u > 5$ and let q be the only variable by assuming the other parameters remain as in table 5.1, we have

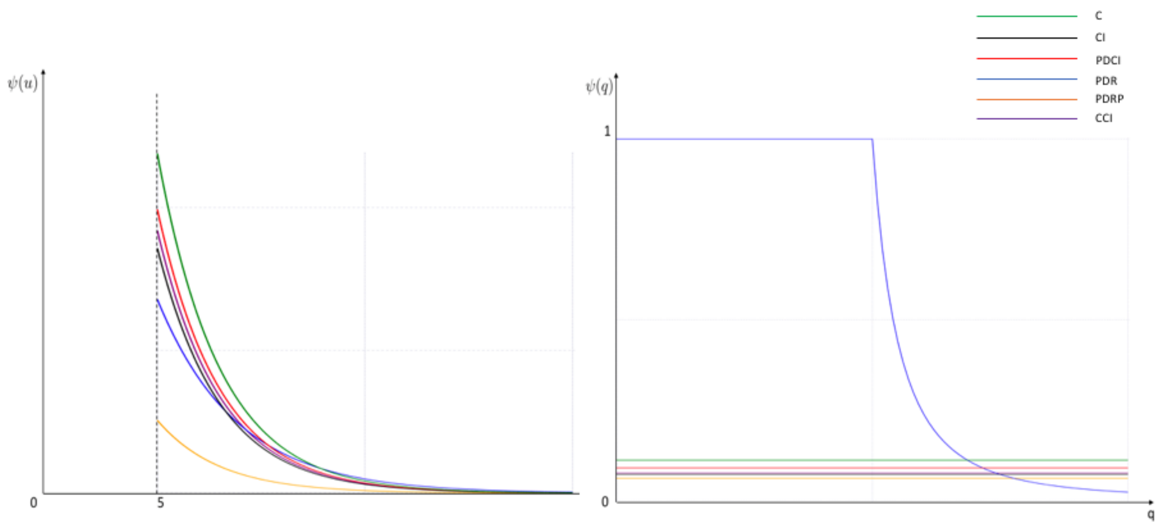


Figure 5.10: The movement of ruin probability with respect to q

	CI	PCI	PDRP	CCI	C
q	0.74-0.75	0.71-0.72	0.77-0.78	0.76-0.77	0.7-0.71

Table 5.2: The ruin probability intersections of q

- Recall the first finding in Figure 5.9, " the classical surplus process has the largest ruin probability for all ranges of initial surplus", this is not true. It is surprising to see that the PDR has the largest ruin probability when initial capital is much bigger than the compensation level k, although it has smaller ruin probability than the capital injection models' at the beginning of the left plot in Figure 5.10. Thus, the ruin probability curve of PDR model intersects the ruin probabilities of all models for the current parameter coefficients. Therefore, the PDR model would have the best performance for small business. Besides, the second observation is correct, the ruin probability given by the model of PDRP remains the smallest. Now, if we compare the CI, PDR and PDRP models, according to the sensitivity tests so far, we would rank them as PDRP > PDR > CI for small business and PDRP = CI > PDR for big business ('>' represents better).

- For the right plot in Figure 5.10, except for the PDR model, the ruin probabilities of all other models remain constant due to their independence of q. The ruin probability under the PDR model remains at 1 for a while due to the net profit condition. After we reduce the payment rate to the reinsurer (q goes bigger), the ruin probability drops very fast. When q is close to 1, the PDR model provides the smallest ruin probability in comparison to other models. In addition, when $q = 1$, the PDR has the same compensation strategy as the PDCI, except for the additional safe position $\frac{p}{1-p}k$. This is the reason why PDR has a lower ruin probability than PDCI.

Besides, the parameter p determines the intensity of the compensation for PDCI, PDR and PDRP and the additional safe position for PDR.

Parameters	u	c	β	λ	k	p	q	a
	5	3	1	1	2	0-1	0.8	10

Table 5.3: Parameters' coefficients for Figure 5.11

We notice that, the ruin probability of the PDCI model converges to the ruin probability of the capital injection model, and the ruin probability of the PDR and PDRP moves to 0 when $p \to 1$ (because when $p \to 1$, the additional space $\frac{p}{1-p}k$ for the ruin tolerance becomes very large). The reason the PDCI model has the greatest ruin probability for small p is due to the reinsurance payment rate q. As long as $q = 1$, PDCI will be equal to the classical case when $p = 0$.

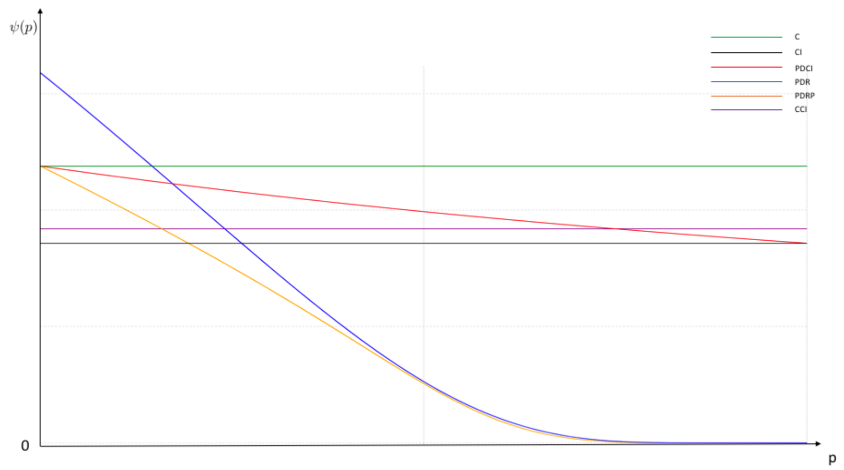


Figure 5.11: The movement of ruin probability with respect to p

Here we provide the ruin probability curves along with the compensation level k, where the ruin probability of the classical surplus process remains constant according to its independence of k. The other parameters' coefficients are given by

Parameters	u	c	β	λ	k	p	q	a
	5	3	1	1	0-5	0.5	0.9	10

Table 5.4: Parameters' coefficients for Figure 5.12

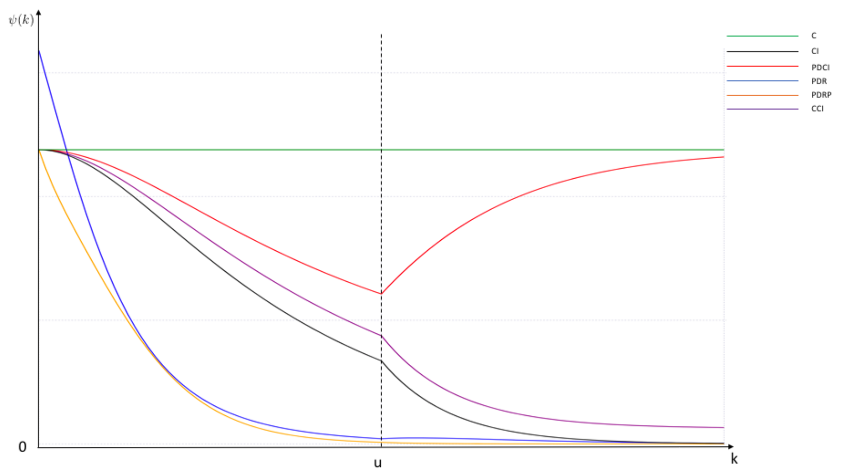


Figure 5.12: The movement of ruin probability with respect to k

It is clear to see that

- The dashed line is the value of initial capital, where $k = u$. For the two intervals $u \geq k$ and $u < k$, the curves have different trends. Eventually, the ruin probabilities of all of the compensation strategies move to 0 (except for PDCI) with increasing k. This can be considered to be due to an instant injection given for the initial surplus. Furthermore, when $k = 0$, the ruin probabilities of most of the models will be the same as the classical process's (except for the PDR model, because of the reinsurance payment rate q). For the model of PDCI, there is no instant compensation when initial surplus is below k. In addition, increasing k leads to the higher ruin probability of PDCI, because the compensation happens only if there is a claim which leads the surplus level to drop below $U_{t_i} < k$ from $U_{t_i^-} > k$, thus if k is very far from the initial capital u, it is very difficult for a compensation event to happen.

- The PDRP and PDR models have the highest sensitivity with respect to k when $k < u$, their ruin probabilities drop very fast at the beginning as k increases. Besides, the ruin probabilities of the PDRP and PDR models will converge to a small number when k is quite larger than u and it will eventually be greater than the CI's ruin probability when k is extreme large.

- Now we know that the model of PDCI has the best performance when initial capital equals the compensation level k, however this is not reliable in practice. Therefore, it is obvious to say that most of the reinsurance agreements perform better than the classical case, in other words, those strategies work properly for reducing the ultimate ruin probability. In the real reinsurance market, most of the compensation level will be set lower than the initial surplus, thus under consideration of the sensitivity test on k, we should choose PDRP or PDR.

According to the net profit condition, the ruin probabilities will be 1 if the premium rate c is too small.

Parameters	u	c	β	λ	k	p	q	a
	5	0.5-5	1	1	2	0.5	0.9	10

Table 5.5: Parameters' coefficients for Figure 5.13

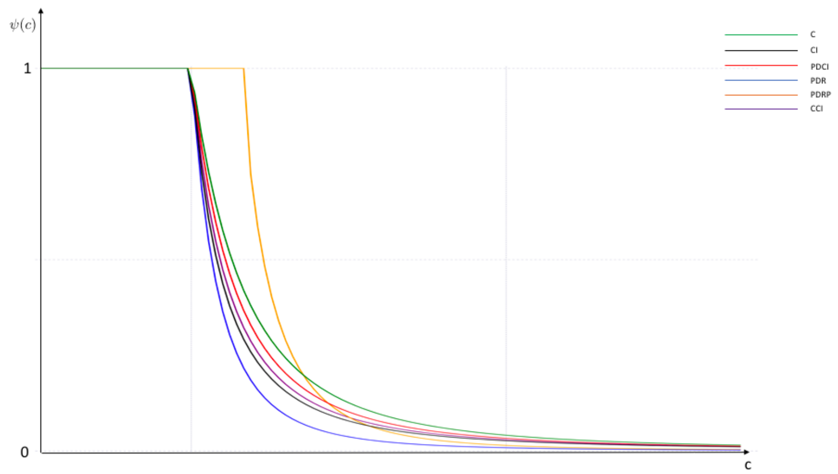


Figure 5.13: The movement of ruin probability with respect to c

When the premium c is very large, it is very hard for the company to become ruined, unless the first large claim occurs at the beginning of the business. According to the additional space $\frac{k}{1-p}$ in the PDR model, its ruin probability would be even lower than the other models' (same situation as PDRP model, for the large premium rate). Furthermore, the sensitivity of c for the model PDRP is delayed, because of the strict net profit condition. In fact, the discussion of the sensitivity of c, β and λ should occur at the same time, because the net profit conditions are determined by these three variables (sometimes must also be considered p, q and a for the models of PDR, PDRP and CCI). Then we have:

Parameters	u	c	β	λ	k	p	q	a
	5	3	0.4-2	1	2	0.5	0.9	10

Table 5.6: Parameters' coefficients for Figure 5.14

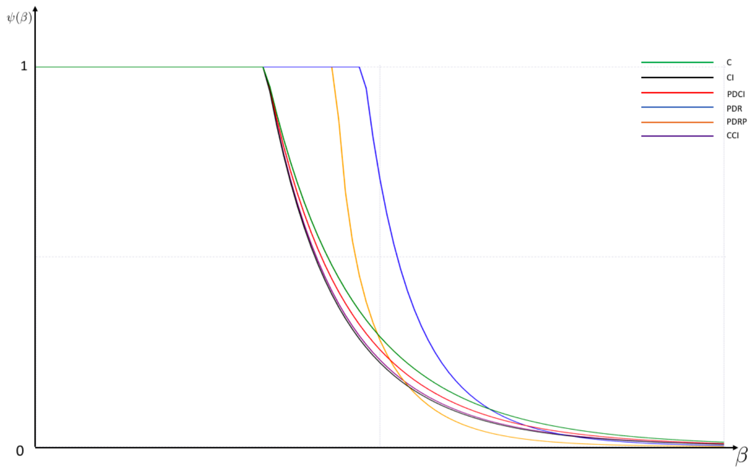


Figure 5.14: The movement of ruin probability with respect to β

- According to the exponential claim distribution, the expectation value of a claim is $\mu = \frac{1}{\beta}$, which has the negative correlation with β. In Figure 5.14, we notice that the PDR model is more sensitive than the other models with respect to β. It has the lowest ruin probability when the mean of claims is very small, however, this becomes the largest when the expected claim amount is quite large.

- The ruin probability of the PDRP model has the second highest convergence speed and drops very fast when the claim amount becomes smaller. On the other hand, CI has the best performance when the expectation of the claims is quite large.

- The company should only apply the PDR and PDRP strategies when it normally faces small claim amount. If the insurer is expected to have large claims, the capital injection model has the best performance among all of the compensation strategies.

For the claims frequency test, we have

Parameters	u	c	β	λ	k	p	q	a
	5	2	1	1-2	2	0.5	0.9	10

Table 5.7: Parameters' coefficients for Figure 5.15

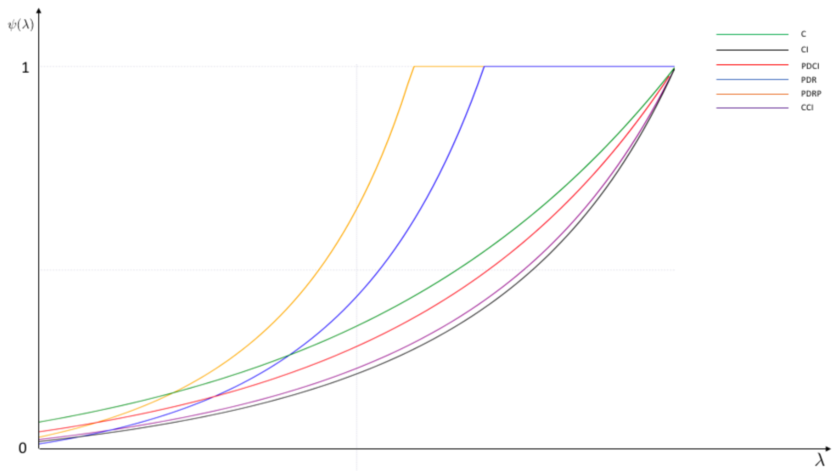


Figure 5.15: The movement of ruin probability with respect to λ

- According to the net profit condition, when a company has extreme high claims frequency, the ruin probability of all models remains constant, equals to 1. In particular, the ruin probabilities of PDRP and PDR have been already remained at 1 for some period before the other models'. In other words, in comparison to the other models, the PDRP and PDR models have stricter net profit condition.

- For the claims frequency plot, the PDR model has a similar conclusion to that for the claim size plot. It has the greatest sensitivity with respect to the claim frequency, therefore tolerance of high frequency for the PDR and PDRP models is very low. PDR has the smallest ruin probability for low claim frequency and the largest ruin probability for high claim frequency, respectively. In addition, the capital injection model has the best performance for high claims frequency.

5.4 The premium calculation for the reinsurer

In this section, we aim to formulate the premium payments collected by the reinsurer for the different strategies. Suppose that the insurer applies a reinsurance agreement under which the reinsurer provides the funds needed to restore the surplus level to a known level every time the surplus falls between 0 (can also be $-\frac{k}{1-p}$ in the case of PDR) and the compensation level k. We denote the premium required by the reinsurer as $Q(u,k)$ for the model of capital injection and $Q(u,k)|_P$ for PDCI, which is a function of the insurers initial capital u and the compensation level k.

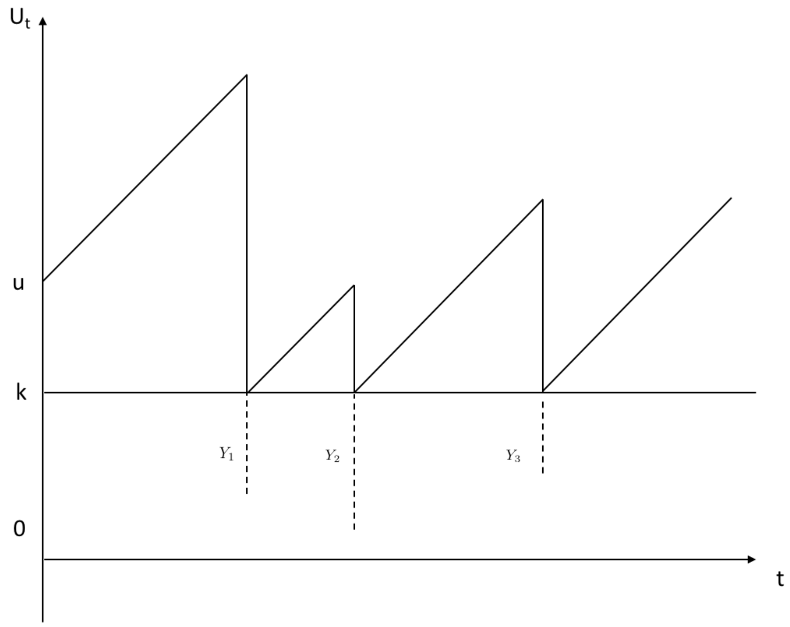


Figure 5.16: The payment of the reinsurer under the capital injection model

For the capital injection model, our aim is to investigate whether the insurer can reduce this ultimate ruin probability by splitting initial capital U into two parts. The first of these, $u \le U$, will be the initial surplus held for the portfolio. The second part is a reinsurance premium which we denote by $Q(u,k)$ or $Q(u,k)|_P$. Let the aggregate amount needed to restore the modified surplus process to some levels up to time t, given initial surplus u, be $S_{t,u,k}|_P$. The simplest case is the capital injection model, the reinsurer provides the funds to recovery the surplus level to constant compensation level k every time when surplus drops into $(0,k)$. Then for the capital injection model, the aggregate amount needed to restore the modified surplus process is denoted as $S_{t,u,k}$. In this paper, we let $Q(u,k) = 1.5\mathbb{E}(S_{t,u,k})$, called the expected value principle reinsurance premium (more applications can be found in Nie et al. (2011) and Nie (2012)).

Let $T_{u,k}$ denote the time of ruin under the modified process with initial surplus u and compensation level k and denote $S_{u,k}|_P = S_{T_{u,k},u,k}|_P$, i.e. the expected total claim amount for the reinsurer up to the time of ruin. First, we consider the capital injection model. Using the idea of Pafumi (1998), when the surplus is below k for the first time, the reinsurer has to make an immediate payment of Y_1 and reserve the amount $S_{k,k}$ for the future payments $Y_{i\geq 2}$, then we have the following theorem.

Theorem 5.4.1. The expected total claim amount for the reinsurer up to the time of ruin under the capital injection model is given by when $u = k$,

$$\begin{aligned}\mathbb{E}(S_{k,k}) &= \int_0^k (y + \mathbb{E}(S_{k,k}))g(0,y)dy \\ &= \int_0^k yg(0,y)dy + \mathbb{E}(S_{k,k})G(0,k) \\ &= \frac{\int_0^k yg(0,y)dy}{1 - G(0,k)},\end{aligned}$$

when $u > k$,

$$\begin{aligned}\mathbb{E}(S_{u,k}) &= \int_0^k (y + \mathbb{E}(S_{k,k}))g(u-k,y)dy \\ &= \int_0^k yg(u-k,y)dy + \mathbb{E}(S_{k,k})G(u-k,k).\end{aligned}$$

Clearly, for any $u < k$,

$$\mathbb{E}(S_{u,k}) = k - u + \frac{\int_0^k yg(0,y)dy}{1 - G(0,k)}$$

and the second order moment can be derived by the same idea, s.t.

$$\begin{aligned}\mathbb{E}(S_{k,k}^2) &= \int_0^k y^2 g(0,y)dy + \mathbb{E}(S_{k,k}^2)G(0,k) + 2\mathbb{E}(S_{k,k})\int_0^k yg(0,y)dy \\ &= \frac{\int_0^k y^2 g(0,y)dy + 2\mathbb{E}(S_{k,k})\int_0^k yg(0,y)dy}{1 - G(u-k,k)},\end{aligned}$$

hence,

$$\mathbb{E}(S_{u,k}^2) = \int_0^k y^2 g(u-k,y)dy + \mathbb{E}(S_{k,k}^2)G(u-k,k) + 2\mathbb{E}(S_{k,k})\int_0^k yg(u-k,y)dy;$$

see, for example, Nie et al. (2011).

Example 5.4.1. We assume the claims follow exponential distribution, s.t. $F_X(x) = 1 - e^{-\beta x}$, we could obtain

$$\mathbb{E}(S_{k,k}) = \frac{\int_0^k yg(0,y)dy}{1 - G(0,k)} = \frac{\frac{\lambda}{c\beta}\gamma(2,\beta k)}{\beta - \frac{\lambda}{c}(1 - e^{-\beta k})}$$

and

$$\begin{aligned}\mathbb{E}(S_{u,k}) &= \int_0^k yg(u-k,y)dy + \mathbb{E}(S_{k,k})G(u-k,k)\\ &= \psi(u-k)[\frac{\gamma(2,\beta k)}{\beta} + \frac{\frac{\lambda}{c\beta}\gamma(2,\beta k)}{\beta - \frac{\lambda}{c}(1 - e^{-\beta k})}(1 - e^{-\beta k})].\end{aligned}$$

For the second order moment,

$$\mathbb{E}(S^2_{u,k}) = \psi(u-k)[\frac{\gamma(3,\beta k)}{2\beta^2} + \mathbb{E}(S^2_{k,k})(1 - e^{-\beta k}) + 2\mathbb{E}(S_{k,k})\frac{\gamma(2,\beta k)}{\beta}]$$

with

$$\mathbb{E}(S^2_{k,k}) = \frac{\psi(0)[\frac{\gamma(3,\beta k)}{2\beta^2} + 2\mathbb{E}(S_{k,k})\frac{\gamma(2,\beta k)}{\beta}]}{1 - G(0,k)}.$$

For PCI, the situation is more complex. According to the partial discrete compensations, the funds given by the reinsurer restore the surplus level back to the random level m, which is in relation to the deficit below k, rather than to the compensation level k. Furthermore, according to the definition of the compensation strategies, the reinsurer will not provide the funds for the claims when the surplus level is lower than the compensation level k.

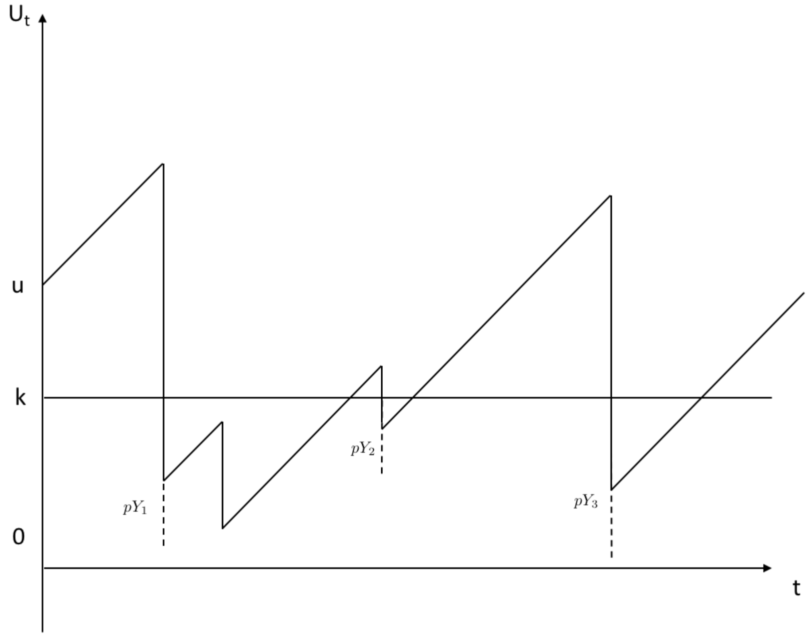


Figure 5.17: The payment of the reinsurer under the partial discrete compensation strategy

Similar to the process under the capital injection model, the payments of the reinsurer are given by the parts of the deficit below the compensation level k, which are $pY_{i\geq 0}$. Then there are two situations after the first deficit Y_1 has occurred:

1. The surplus process gets ruined before it recovers to the compensation level k with probability $\xi(k-(1-p)Y_1, k)$,

2. The surplus process restores to the compensation level k before it gets ruined with probability $\bar{\xi}(k-(1-p)Y_1, k)$,

hence,

Theorem 5.4.2. The expected total claim amount for the reinsurer up to the time of ruin under the partial discrete compensation model is given by for any $u \geq 0$, when

$u = k$,

$$\begin{aligned}\mathbb{E}(S_{k,k}|_P) &= \int_0^k \xi(k-(1-p)y,k)py + \bar{\xi}(k-(1-p)y,k)(py+\mathbb{E}(S_{k,k}|_P))g(0,y)dy \\ &= \int_0^k (py+\bar{\xi}(k-(1-p)y,k)\mathbb{E}(S_{k,k}|_P))g(0,y)dy \\ &= \int_0^k pyg(0,y)dy + \mathbb{E}(S_{k,k}|_P)\psi(0)f_p(k) = p\frac{\int_0^k yg(0,y)dy}{1-\psi(0)f_p(k)},\end{aligned}$$

when $u > k$,

$$\mathbb{E}(S_{u,k}|_P) = \int_0^k (py+\bar{\xi}(k-(1-p)y,k)\mathbb{E}(S_{k,k}|_P))g(u-k,y)dy.$$

For any $u < k$,

$$\mathbb{E}(S_{u,k}|_P) = k-u+p\frac{\int_0^k yg(0,y)dy}{1-\psi(0)f_p(k)}.$$

For the second order moment,

$$\begin{aligned}\mathbb{E}(S^2_{k,k}|_P) &= \int_0^k py^2g(0,y)dy + \bar{\xi}(k-(1-p)y,k)\mathbb{E}(S^2_{k,k}|_P)f_p(k) \\ &+ \int_0^k 2py\bar{\xi}(k-(1-p)y,k)\mathbb{E}(S_{k,k}|_P)g(0,y)dy \\ &= p\frac{\int_0^k y^2g(0,y)dy + \int_0^k 2y\bar{\xi}(k-(1-p)y,k)\mathbb{E}(S_{k,k}|_P)g(0,y)dy}{1-\psi(0)f_p(k)}\end{aligned}$$

and for $u > k$,

$$\mathbb{E}(S^2_{u,k}|_P) = \int_0^k (py^2+\bar{\xi}(k-(1-p)y,k)\mathbb{E}(S^2_{k,k}|_P)+2p\bar{\xi}(k-(1-p)y,k)y\mathbb{E}(S_{k,k}|_P))g(u-k,y)dy.$$

Example 5.4.2. We assume the claims follow exponential distribution, s.t. $F_X(x) = 1-e^{-\beta x}$, we could have

$$\mathbb{E}(S_{k,k}|_P) = p\frac{\int_0^k yg(0,y)dy}{1-\psi(0)f_p(k)} = p\frac{\frac{\lambda}{c\beta}\gamma(2,\beta k)}{\beta-\frac{\lambda}{c}f_p(k)},$$

where the expression of $f_p(k)$ is given in the example 5.1.2. Then for $u > k$,

$$\mathbb{E}(S_{u,k}|_P) = \psi(u-k)[p\frac{\gamma(2,\beta k)}{\beta} + \mathbb{E}(S_{k,k}|_P)f_p(k)].$$

For the second order moment,

$$\mathbb{E}(S_{u,k}^2|_P) = \psi(u-k)[p\frac{\gamma(3,\beta k)}{2\beta^2} + \mathbb{E}(S_{k,k}^2)f_P(k) + 2p\mathbb{E}(S_{k,k})\int_0^k y\bar{\xi}(k-(1-p)y,k)g(0,y)dy]$$

with

$$\mathbb{E}(S_{k,k}^2|_P) = \frac{\psi(0)[p\frac{\gamma(3,\beta k)}{2\beta^2} + 2\mathbb{E}(S_{k,k})\int_0^k y\bar{\xi}(k-(1-p)y,k)g(0,y)dy]}{1-\psi(0)f_p(k)},$$

where

$$\begin{aligned}&\int_0^k y\bar{\xi}(k-(1-p)y,k)g(0,y)dy \\ =&\frac{c}{c\beta-\lambda e^{-(\beta-\frac{\lambda}{c})k}}\gamma(2,\beta k) - \frac{\lambda\beta e^{-(\beta-\frac{\lambda}{c})k}}{(p\beta+(1-p)\frac{\lambda}{c})^2(c\beta-\lambda e^{-(\beta-\frac{\lambda}{c})k})}\gamma(2,p\beta+(1-p)\frac{\lambda}{c}).\end{aligned}$$

Now we can understand the difference between the CI, PDCI and PDR model. Recall that the cost of the reinsurance contract for the CI and PDCI model is an instant payment at the beginning, denoted by $Q(u,k)$ and $Q(u,k)|_P$, thus their initial capital is equal to the sum of surplus for the portfolio and the capital for the contract cost, denoted by $U = u + Q(u,k)$ or $U = u + Q(u,k)|_P$, respectively. Now we set up the parameters' coefficients as $c = 3$, $\beta = 0.4$, $\lambda = 1$, the following table provides the optimal initial surplus, the setting of compensation level and the optimal ruin probability under some levels of initial capital.

			$p=0.1$			$p=0.5$			$p=0.9$		
U	C	CI	PDCI	%/C	%/CI	PDCI	%/C	%/CI	PDCI	%/C	%/CI
10	42.78%	40.79%	42.43%	0.82%	-4.02%	41.38%	3.27%	-0.15%	40.86%	4.49%	-0.17%
15	30.66%	25.10%	30.02%	2.09%	-19.60%	27.49%	10.34%	-9.52%	25.40%	17.16%	-1.20%
20	21.97%	12.69%	21.22%	3.41%	-67.20%	17.81%	18.94%	-40.35%	13.69%	37.69%	-7.88%

Table 5.8: Optimal ruin probabilities

Note that, %/C means $1 - \frac{\psi(u)|_{PDCI}}{\psi(u)|_C}$ and %/CI means $1 - \frac{\psi(u)|_{PDCI}}{\psi(u)|_{CI}}$. It is clear to see that the CI model has the best performance for all range of initial capital and its advantage becomes more obvious as the total amount of initial capital increases. Besides, the strategy for the PDCI model has ruin probability between the ruin probabilities for C and CI, it converges to the CI's when $p \to 1$, where it represents the intensity of the compensation. The next table provides the details of the optimal capital and compensation level setting.

	CI		PDCI, $p = 0.1$		PDCI, $p = 0.5$		PDCI, $p = 0.9$	
U	u^*	k^*	u^*	k^*	u^*	k^*	u^*	k^*
10	6.865	4.128	9.513	7.003	7.883	5.484	6.994	4.345
15	8.863	6.128	14.410	9.767	11.730	8.246	9.288	6.515
20	11.050	8.316	19.426	12.089	16.440	10.840	12.0834	8.923

Table 5.9: Optimal capital and compensation level setting

The intensity of the compensation determines the capital setting for the initial surplus and reinsurance cost. We can consider the PDCI model as the CI model when $p = 1$, then a greater p leads to a decrease in the level of initial surplus, since the high intensity of the compensation requires more capital allocation for the reinsurance contract to obtain the minimum ruin probability.

However, in the model of PDR, the instant payment is replaced by the continuous payment with rate $(1 - q)c$, we now set up the equivalent reinsurance payment rate q in order to match the ruin probabilities under the PDCI model.

U=20			U=15			U=10		
p	q	$1 - q$	p	q	$1 - q$	p	q	$1 - q$
0.1	0.984	0.016	0.1	0.982	0.018	0.1	0.981	0.019
0.5	0.917	0.083	0.5	0.911	0.089	0.5	0.908	0.092
0.9	0.844	0.156	0.9	0.839	0.161	0.9	0.839	0.161

Table 5.10: Equivalent reinsurance payment rate q

The model with higher intensity compensation requires more payment for the reinsurance contract. Furthermore, the process with higher initial capital shows the lower request for the reinsurance payment rate, because the injections rarely happen when the initial surplus is fairly large in comparison to the compensation level.

Chapter 6

Concluding Remarks and Future Work

In this thesis, we have constructed several dependent risk models, including the time dependent model (inhomogeneous Poisson process and Cox process), claim dependent model (mixing over the parameter of the claim intensity process in the classical Cramér-Lundberg risk process and discrete binomial risk process) and surplus dependent model (capital injection and other surplus dependent reinsurance model). Under these dependent models' settings, we have investigated changes in the ruin probabilities (finite and ultimate), which provides us with an approach of how to adapt classical risk theory to the contemporary complex financial market.

In Chapter 3, we applied the idea of mixing distributions over values of involved parameters to extend the class of classical risk processes, focusing on the claim intensity parameter. In fact, in the classical risk process, Albrecher et al. (2011) showed the resulting dependent structure was an Archimedean copula. For the discrete binomial risk process, we introduced a more convenient way of structuring the probability of success $\rho = 1 - \mathrm{e}^{-\Theta}$, in comparison to the results given by Dutang et al. (2013), we obtained much more tractable expressions of both for the claims distributions and the ultimate ruin probabilities. In addition, equation (3.5) shows an interesting fact, $\lim_{u\to\infty} \psi(u) =$

$\bar{F}_\Theta(\theta^*)$, it means the refined ruin probability converges to a non-zero level as $u \to \infty$, which is due to the not profit condition being violated. Besides, apart from dependent modelling, the mixing can also account for parameter uncertainty, which may have a better fit to the real financial market.

In Chapter 4, we managed to combine the classical compound Poisson process with the inhomogeneous Poisson process, which leads the original process to be a non-renewal and non-stationary process. Thus, the approach of computing the ruin probability is not the same as in the classical case. The Volterra integral equations for both the finite and infinite time ruin probability are given by Garrido et al. (1996), however this mathematical interpretation cannot be explicitly expressed for most of continuous processes. Therefore, we derived a new type of the Seal's formulae for the inhomogeneous Poisson process in order to generate the expression for the finite time ruin probability. Furthermore, applying the idea of Usabel (1998), we constructed the ultimate ruin probability using the infinite summation of the finite time ruin probabilities for the infinite intervals of time slots. The ruin probability of the Cox process is very difficult to obtain. Even for the distribution of the point process, it has to be expressed by a conditional probability by conditioning on $\mathcal{F}_t$. Fortunately, we are able to compute the model under the last two examples in Chapter 4 using this approach, due to the existence of the unconditional distribution of the point processes.

In Chapter 5, we set up 5 type of reinsurance contracts, including the capital injection model (CI), partial discrete capital injection model (PDCI), partial discrete reinsurance contract (PDR), partial discrete reinsurance with special settlement (PDRP) and continuous capital injection model (CCI). We then derive the expressions for the ultimate ruin probability for all models respectively, by applying the idea of two-barrier models. We now answer 3 key risk management questions.

1. Should a company buy reinsurance or raise more capital?
2. What is the optimal initial capital setting?

3. How can risk theory help decisions regarding reinsurance?

For the capital injection model, the following table provides the data set for the optimal ruin probabilities under different levels of initial capital, with parameters $c = 3$, $\beta = 0.4$, $\lambda = 1$. In the table, u^*, k^* and $\psi_{k^*}(u^*)$ represent the optimal pure investment in the business, compensation level and ruin probability, where $U = u^* + Q(u^*, k^*)$ and we set $Q(u, k) = 1.2\mathbb{E}(S_{u,k})$ (it is defined in Section 5.4).

U	$\psi(U)$	u^*	k^*	$\psi_{k^*}(u^*)$	%
5	59.71%	4.459	1.722	59.59%	2%
10	42.78%	6.865	4.128	40.79%	4.7%
15	30.66%	8.863	6.128	25.10%	18%
20	21.97%	11.05	8.316	12.69%	42.23%

Table 6.1: Optimal ruin probability under capital injection model

It is clear that for each level of total initial capital, the capital injection model provides a lower ruin probability with lower initial u^* than the classical process with higher initial investment $u^* + Q(u^*, k^*)$. Besides, when the amount of the initial investment gets larger, the benefit from the capital injection will become more significant. For the other reinsurance models, as long as we increase the value of p or a, the ruin probability will definitely be smaller. We then suggest the company should buy a reinsurance contract rather than raise more capital. The method of choosing the optimal capital setting is given in Section 5.4 for the model of CI and PDCI. It depends only on the level of u and k since $U = u + Q(u, k)$. The distance between u and k determines the risk and their relationships can be found in Figures 5.9 and 5.12.

It is obvious that, risk theory plays a significant role in measuring risk and provides mathematical instruments for determining the reinsurance setting. We notice from Figures 5.13 and 5.15 that although the ruin probabilities are reduced by reinsurance contracts, the net profit conditions change with the different strategies. In other words, a company must adjust their management policies when choosing a reinsurance agreements.

Future work will focus on Section 4 and 5. For the time dependent model, where we have the premium rate assumption as shown in equation (4.2), our goal is to eliminate this assumption in order to allow the premium rate to be any from, i.e. a constant premium rate c. However, it cannot be fitted to Theorem 4.2.2, where we construct a conditional martingale with respect to R_a^{a+s}, under the assumption that the premium rate is a constant c. Therefore, $\mathbb{E}[R_a^{a+s}|R_a^{a+t} = y] = \mathbb{E}[S_a^{a+s}|S_a^{a+t} = y]$ does not hold when the premium rate can take any other form.

In Section 5.1.5, the ruin probability of the new risk process cannot be derived in the classical way. Up to now, it can only be computed numerically. Furthermore, this process can be considered as a shifted process when the surplus level is below the compensation level k (for instance, for any $0 < U_t < k$, we have $U_t^\Delta = U_t + p(k - U_t) = pk + (1-p)U_t$. Therefore, the injection from the reinsurance company is given by $Y_i' = pX_i$ for $i \geq 1$. According to the two barriers model, there are then two possible situations to be discussed.

- When a claim Y_i' occurs and leads the surplus level into $(0, k)$, **ruin happens** before the surplus recovers to the compensation level k with probability $\xi(pk + (1-p)u, k)$.

- When a claim Y_i' occurs and leads the surplus level into $(0, k)$, the company will recover to the compensation level k and **ruin dose not happen** with probability $\bar{\xi}(pk + (1-p)u, k)$.

Then, one considers the total amount of claims when $0 < U_t^\Delta < k$ during a period T, where $T \leq t$. We denote

- $M_k(y)$ as an expectation of the sum of claims, when $0 < U_t^\Delta < k$ and the surplus process restores to the compensation level k before the company is ruined, except for the first claim.

- $M_0(y)$ as an expectation of the sum of claims, when $0 < U_t^\Delta < k$ and the surplus process is ruined before it recovers to the compensation level k, except for the first claim and the last claim, which leads the company ruin.

- Let $M(y) = M_k(y) + M_0(y)$

Then the expected total claim amount for the reinsurer up to the time of ruin under the PDRP model is given by when $u = k$,

$$\begin{aligned}\mathbb{E}(S_{k,k}^{PDRP}) &= \int_0^k \left(py + M(y) + \bar{\xi}(k-(1-p)y,k)\mathbb{E}(S_{k,k}^{PDRP})\right) g(0,y)dy \\ &= \int_0^k (py + M(y))g(0,y)dy + \mathbb{E}(S_{k,k})f_p^\Delta \\ &= \frac{\int_0^k (py + M(y))g(0,y)dy}{1-f_p^\Delta},\end{aligned}$$

when $u > k$,

$$\mathbb{E}(S_{k,k}) = \int_0^k \left(py + M(y) + \bar{\xi}(k-(1-p)y,k)\mathbb{E}(S_{k,k}^{PDRP})\right) g(u-k,y)dy.$$

Clearly, for any $u < k$,

$$\mathbb{E}(S_{u,k}) = p(k-u) + \frac{\int_0^k (py + M(y))g(0,y)dy}{1-f_p^\Delta}.$$

Thus $M(y)$ is the key needs to be investigated.

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