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A Njifenjou, H Donfack, A Kinfack Jeutsa

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A. Njifenjou

*University of Yaounde I*
*National Advanced School of Engineering*
*P.O. Box: 8390 Yaounde - Cameroon*
*and*
*University of Douala*
*Faculty of Industrial Engineering*
*P.O. Box: 2701 Douala - Cameroon*

njifa2000@yahoo.fr (Corresponding author)

H. Donfack

*University of Yaounde I*
*Faculty of Science*
*P.O. Box: 812 Yaounde - Cameroon*

hubertdonfack@yahoo.fr

A. Kinfack Jeutsa

*University of Buea*
*Higher Technical Teachers' Training College*
*P.O. Box: 249 Buea - Cameroon*

jeutsa2001@yahoo.fr

**Abstract**

This paper deals with a finite volume analysis for diffusion phenomena, with prescribed periodic boundary conditions. The problem addressed here is the so-called local problem that arises in multi-scale physics such as fluid-flows in porous materials with spatially periodic microstructures. The so-called local problem is discretized with a finite volume method of new generation, namely Discrete Duality Finite Volumes (DDFV for short). In view to deal with stability analysis of the DDFV solution, an adequate discrete $H_0^1-$norm is defined and a discrete version of Poincaré inequality involving this discrete norm is introduced. The DDFV problem displays the same main features as the continuous one: (i) A compatibility condition is required from the discrete source-term for existence of discrete solutions; (ii) Two orthogonality conditions are required from DDFV solution for uniqueness. Note that only one such condition is required from classical cell-centered finite volumes dealing with *isotropic models.* Error estimate results are obtained for $L^2 - norm$ and for a discrete $H^1-$norm. Numerical tests confirm our theoretical results. Note that these results are in accordance with those from the fifth international conference on Finite Volumes for Complex Applications (FVCA5) held in France in 2008.

**Key words** : Flow problems; Nonhomogeneous anisotropic media; Finite volumes of DDFV type; Discrete energy norms; stability; Error estimates; Test simulations.

# 1 General introduction and description of the model problem

A deep understanding of the behavior of multi-scale heterogeneous physics (as climate changes, multi-phase flow and transport in geological formations, heat diffusion or fissure propagation in composite materials,...) is of great interest today. The mathematical models of these physics involve rapidly oscillating parameters. The computation of solutions to these models requires the use of grids whose size should be of the same order of magnitude as the smallest size of heterogeneities of the structure under consideration. In concrete situations the length-scale of micro-heterogeneities (denoted by $\varepsilon$ in the sequel) is so small that even with the help of high-speed modern computers, addressing the boundary value problems related to such media with a large number of heterogeneities is not an easy task. A natural way to overcome this difficulty is to replace the differential operator (associated with the multi-scale heterogeneous physics under consideration) with a slowly varying one that preserves the large scale properties of the exact solution. This procedure is usually called homogenization process and the large scale properties are named homogenized parameters.

One way for finding these large scale properties is to carry out experimental tests. It is quite evident that because of the volume and cost of the required tests for all possible reinforcement types, experimental measurements are often impracticable. Depending on material types, different alternative approaches could be used for obtaining the homogenized parameters. For instance, in most reservoir engineering codes, empirical algebraic formulas of homogenization have been used. On the other hand, for man-made materials (like composite materials with spatially periodic microstructure), the mathematical theory of homogenization based on microstructure periodicity setting has displayed a capability to lead to exact homogenized coefficients. These homogenized parameters may be obtained from either Formal Asymptotic Analysis (see for instance pioneer works from E. Sanchez-Palencia [S 80], L. Tartar [T 85], A. Bensoussan, J.L. Lions and G. Papanicolaou [BLP 78]) or Two-scale Convergence theory introduced by G. Nguetseng in [N 89] (see also D. Lukkassen, G. Nguetseng and P. Wall [LNW 02], G. Nguetseng [N 04] and G. Allaire [A 92]).

We briefly describe the homogenization process for a second order elliptic operator. The cell problem this homogenization process leads to motivates our interest for the model problem this work is focused on. Let $D \subset \mathbb{R}^d_\xi$ (with $d$ the spatial dimension, $\xi$ the macroscopic spatial coordinates and $\mathbb{R}^d_\xi$ the macroscopic space) be a heterogeneous porous material with a spatially periodic microstructure. An incompressible one-phase flow in $D$ (viewed as a continuum medium) is governed by the following equation :

$$-div\left[a\left(\xi, \frac{\xi}{\varepsilon}, grad\, u_\varepsilon\right)\right] = Q \quad in\ D \tag{1.1}$$

where $a(.,.,.)$ is a given map from $D \times \mathbb{R}^d_x \times \mathbb{R}^d$ into $\mathbb{R}^d$with adequate properties (see [N 04] for more details). Note that $R^d_x$ is a microscopic space and $x$ denotes the microscopic spatial coordinates attached to the macroscopic ones by the relation $x = \xi/\varepsilon$. Recall that the real number $\varepsilon \succ 0$ represents the length-scale of micro-heterogeneities in the porous material $D$ (see Figure 1 below).

When the flow in the porous medium $D$ is slow, one can replace Leray-Lions type operator in the right-hand side of the equation (1.1) by the following operator:

$$-div\left[\Lambda\left(\xi, \frac{\xi}{\varepsilon}\right)\ grad\, u_\varepsilon\right] = Q \quad in\ D \tag{1.2}$$

where $\Lambda(.,.)$ is roughly speaking the permeability tensor of the porous medium $D$, depending on both the macroscopic space coordinates $\xi$ and the microscopic space coordinates $x = \xi/\varepsilon$. Due to spatial periodicity of the microstructure of $D$, the permeability matrix function $\Lambda(.,.)$ is $\Omega - periodic$ with respect to $x$, where $\Omega = \prod_{i=1}^{d} ]0, p_i[$ is a datum. This means that for a fixed $\xi \in D \subset R^d_\xi$ the following identity holds.

$$\forall k = (k_i) \in \mathbb{Z}^d \quad \Lambda(\xi, x + k \otimes p) = \Lambda(\xi, x)$$

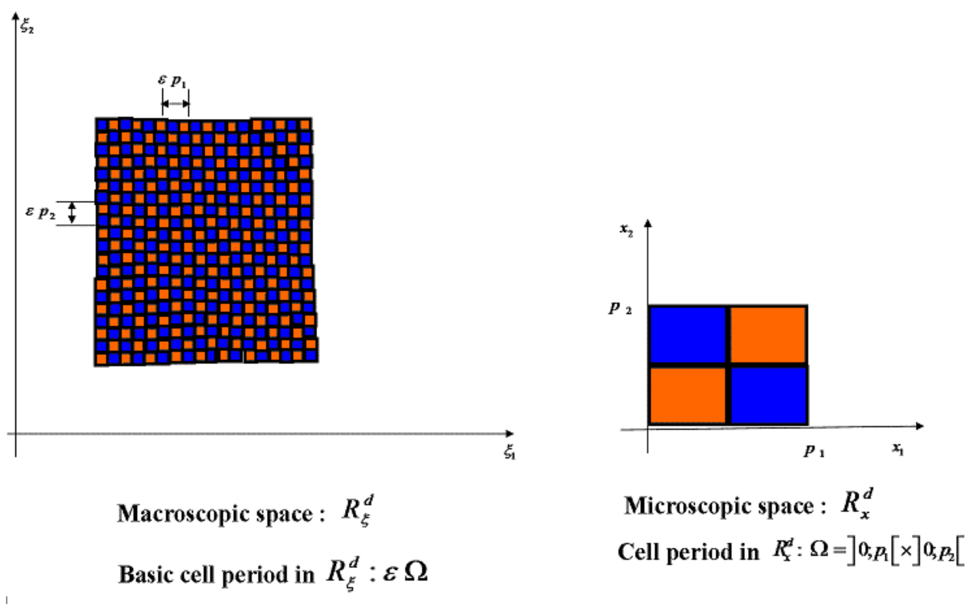


Figure 1: Micro-heterogeneities in a porous material.

where $\mathbb{Z}$ is the set of integers, $p = (p_i)_{i=1}^{d}$ and where the symbol $\otimes$ is the well-known Hadamard vector product defined as:

$$(k \otimes p)_i = k_i p_i \qquad \forall 1 \leq i \leq d$$

The permeability matrix $\Lambda(\xi, x)$ is known to be symmetric and positive definite i.e.

$$\Lambda_{ij}(\xi, x) = \Lambda_{ji}(\xi, x) \qquad \forall 1 \leq i, j \leq d \quad (\textit{symmetry property}) \tag{1.3}$$

$$\exists \alpha \geq \beta \succ 0 \textit{ such that } \forall \varsigma \in \mathbb{R}^d,\ \beta\,|\varsigma|^2 \leq \varsigma^t \Lambda(\xi, x)\, \varsigma \leq \alpha\,|\varsigma|^2 \ a.e.\, in\, D \times \mathbb{R}^d_x \tag{1.4}$$

(uniform ellipticity and boundedness property). One should note that the preceding property ensures that the permeability matrix is an $L^\infty\left(D \times \mathbb{R}^d_x\right) - function.$

By two-scale asymptotic expansion technique combined with energy method (see for instance [S 80], [BLP 78] and references therein) or by two-scale convergence method (see for instance [A 92] [N 04] and references therein) one can show that as $\varepsilon \to 0$ the unique solution of (1.2) lying in $H^1_0(D)$ weakly converges to a function $u^{Hom} \in H^1_0(D)$ such that

$$-div\left[\Lambda^{Hom}(\xi)\ \ grad\, u^{Hom}\right] = Q \quad in\ D$$

where the so-called homogenized permeability matrix $\Lambda^{Hom}(\xi) = \left\{\Lambda^{Hom}_{ij}(\xi)\right\}_{1 \leq i,j \leq d}$ depending exclusively on the macroscopic coordinates is given by the following relation

$$\Lambda_{ij}^{Hom}(\xi) = \int_\Omega \Lambda(\xi,x)\left[grad_x\, w_i(x) + e_i\right]\cdot\left[grad_x\, w_j(x) + e_j\right]dx \;\; \forall 1 \le i,j \le d \quad (1.5)$$

Where $e_i$ (for $i = 1, ..., d$) is the $i^{th}$ vector of the canonical basis of $\mathbb{R}_x^d$ and where $w_i$ (for $i = 1, ..., d$) is the unique solution (up to an additive constant with respect to $x$) of the following so-called local problem:

$$\begin{cases} -div_x\left[\Lambda(\xi,x)\, grad_x\, w_i(x)\right] = \frac{\partial \Lambda_{ki}(\xi,x)}{\partial x_k} \quad in\ \mathbb{R}_x^d \\ w_i \quad being\ \Omega - periodic \end{cases} \quad (1.6)$$

From (1.5) one can easily see that an accurate computation of $grad_x\, w_i$ for $i = 1, ..., d$ is required for a better calculation of the homogenized permeability matrix. The Discrete Duality Finite Volume (DDFV) method has displayed higher accuracy order for performing pressure gradient computations in the framework of second order elliptic problems, with Dirichlet and/or Neumann boundary conditions (see Benchmark FVCA5 [HH 08]). This work aims to carry out the computation and the theoretical analysis (in terms of stability and error estimates) of DDFV solution to a model problem from the same class as the system of equations (1.6). More precisely, let us consider (with $p_1 = a$ and $p_2 = b$ for simplicity of notations) the 2D diffusion problem that consists in finding a function $\varphi$ which satisfies the following partial differential equation associated with prescribed periodic boundary conditions:

$$-div\left(K\, grad\varphi\right) = f \quad in\ \ \Omega = ]0,a[\times]0,b[\,. \quad (1.7)$$

$$\left.\begin{array}{c} \varphi(.,0) = \varphi(.,b) \\ K(.,0)\, grad\varphi(.,0).n_{\Gamma_{So}} + K(.,b)\, grad\varphi(.,b).n_{\Gamma_{No}} = 0 \end{array}\right\} in\ \ [0,a]$$
$$\left.\begin{array}{c} \varphi(0,.) = \varphi(a,.) \\ K(0,.)\, grad\varphi(0,.).n_{\Gamma_{We}} + K(a,.)\, grad\varphi(a,.).n_{\Gamma_{Es}} = 0 \end{array}\right\} in\ \ [0,b] \quad (1.8)$$

where $\Gamma_{No}$ (northern boundary), $\Gamma_{So}$ (southern boundary), $\Gamma_{Es}$ (eastern boundary) and $\Gamma_{We}$ (western boundary) define a partition of the domain boundary denoted by $\Gamma$, and $n_{\Gamma_{No}}, n_{\Gamma_{So}}, n_{\Gamma_{Es}}, n_{\Gamma_{We}}$the corresponding outward unit normal vectors, and where $f$ is a given function (commonly called source term), $K = K(x)$, with $x = (x_1,x_2)^t \in \Omega$, is a full matrix describing the spatial variation of the diffusion coefficient . The entries of $K(.)$ are denoted by $K_{ij}\ (for\ i,j = 1,2)$ and are supposed to satisfy the following assumptions:

$$K_{ij}(x) = K_{ji}(x) \quad \text{a.e.} \quad \text{in} \quad \Omega \qquad for\ \ i,j = 1,2 \quad (1.9)$$

and

$$\begin{array}{lcr} \exists \gamma_{min}, \gamma_{max} \in \mathbb{R}_+^\star & such\ that & \forall \xi \in \mathbb{R}^2 \\ & \gamma_{min}\left|\xi\right|^2 \le \xi^t K(x)\xi \le \gamma_{max}\left|\xi\right|^2 \quad a.e. \quad in \quad \Omega & \end{array} \quad (1.10)$$

where $|.|$ denotes the euclidian norm in $\mathbb{R}^2$.

Due to periodicity conditions on the flux (see equations (1.8)), the source-term $f$ should satisfy the following compatibility condition:

$$\int_{\Omega} f dx = 0 \tag{1.11}$$

# 2 Gridding and Notations

In what follows we fix the framework in terms of gridding and notations.

## 2.1 Gridding

The DDFV theory exposed in this work is inspired from the one developed in [NDM 13]. Therefore our exposition is based upon a primal unstructured mesh, an auxiliary mesh and the associated dual mesh clearly defined in what follows.

### 2.1.1 Definition of a primal mesh

The spatial domain $\Omega$ is split into a finite family of convex open polygons with matching interfaces. This family defines what is called in the sequel an unstructured matching primal mesh denoted by $\mathcal{P}$. Note that the set $\mathcal{V}$ of primal mesh vertices contains the set :

$$\mathcal{V}^{\text{co}} = \{(0,0), (a,0), (0,b), (a,b)\}$$

made of four corner-points. To get advantage of the periodicity setting of the problem, the boundary-vertices are distributed along the domain boundary in such a way that the orthogonal projection of a boundary-vertex $M$ (different from a corner-point ) on the opposite side of $\Gamma$ is also a boundary-vertex (different from a corner-point) denoted by $M^{\perp}$. Figure 2 below illustrates this fact that one can express in other words by:

$$\forall M \in \Gamma \qquad (M \in \mathcal{V} \setminus \mathcal{V}^{co}) \implies (M^{\perp} \in \mathcal{V} \setminus \mathcal{V}^{co}) \tag{2.1}$$

It is clear that $\left(M^{\perp}\right)^{\perp} = M$ for any $M \in \mathcal{V} \setminus \mathcal{V}^{co}$.

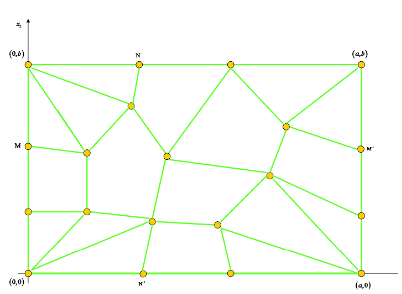


Figure 2: Example of a primal Mesh made up of convex polygons

### 2.1.2 Derivation of auxiliary and dual meshes

Following an original idea from [NDM 13] one can introduce an auxiliary mesh denoted by $\mathcal{A}$. This mesh plays a key role either in the construction of our DDFV scheme or the stability analysis and error estimates developed later.

For deriving an auxiliary mesh from a given primal mesh, one starts with selecting arbitrarily one point per primal edge (different from edge extremities). Let us denote by $\mathcal{E}^{int}$ the set of these edge-points. For exploiting later the periodicity setting of the problem, select one boundary-point per boundary-edge in such a way that the set $\mathcal{E}^{ext}$ of these boundary-edges meets the following property:

$$\forall I \in \Gamma \qquad \left( I \in \mathcal{E}^{ext} \Longrightarrow I^{\perp} \in \mathcal{E}^{ext} \right) \tag{2.2}$$

The figure below illustrates this property.

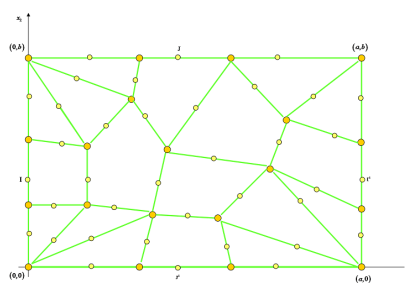


Figure 3: Generation of edge-points (yellow color) in view to build an auxiliary Mesh

For a better understanding of the way the auxiliary mesh should be built, one should know about the concept of neighboring edge-points introduced in [NDM 13]. Two edge-points are neighboring if they share a primal cell and their corresponding edges intersect at some vertex of this primal cell. The auxiliary mesh $\mathcal{A}$ is generated by joining with a straight line all neighboring edge-points (see Figure 4 below).

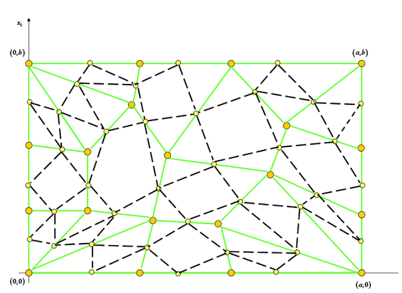


Figure 4: Coupling of a primal Mesh (green lines) with an auxiliary Mesh (black discontinuous lines)

Note that this auxiliary mesh permits to fix perimeters in which should necessarily be located a finite family of primal cellpoints (or briefly cellpoints) for ensuring the regularity of the gridding. Figure 5 below gives an illustration of the situation.

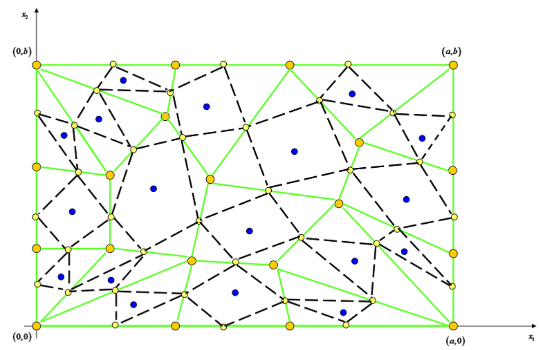


Figure 5: Generation of a finite family of cellpoints (in blue color) inside adequate auxiliary cells.

There is a trivial bijective map between the family of cellpoints and the family of primal cells. So, to any cellpoint $P$ one may associate a unique primal cell denoted by $C_P$, and vice-versa. It is then clear that both of families could be indexed by $\mathcal{P}$. Joining any cellpoint with any edgepoint sharing the same primal cell leads to what is called (in the sequel) a dual mesh (see Figure 6 below).

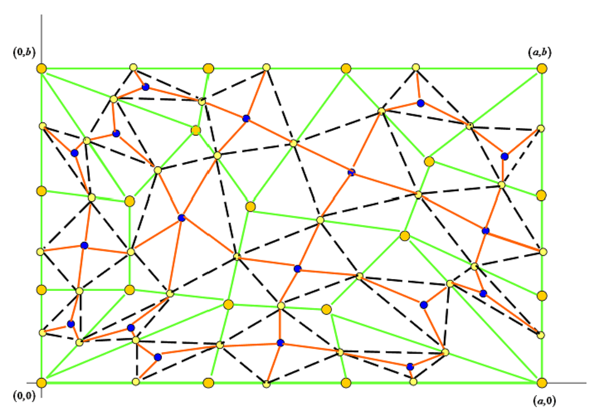


Figure 6: Generation of the dual mesh (red color) associated with the primal and auxiliary meshes.

## 2.2 Notations and definitions

We list in this section the main notations currently used through this work.

- $\Omega$ is a rectangular spatial domain and $\Gamma$ is its boundary;
- $\Gamma_{No}$ is the northern boundary; $\Gamma_{So}$ is the southern boundary; $\Gamma_{Es}$ is the eastern boundary; $\Gamma_{We}$ is the western boundary;
- $\mathcal{P}$ is the primal mesh and more precisely the set of primal cells. We naturally identify $\mathcal{P}$ with the set of cellpoints (to be introduced later);
- $\mathcal{D}$ is the dual mesh and more precisely the set of dual cells. We naturally identify $\mathcal{D}$ sometimes with the set $\mathcal{V}$ of primal vertices;
- $\mathcal{A}$ is the auxiliary mesh. It plays a key role for the definition of a discrete energy norm used for stability and convergence analysis of DDFV solution to the model problem;
- $\mathcal{V}^{co}$ is the subset of $\mathcal{V}$ made of four corner-points: $O_1^\star$, $O_2^\star$, $O_3^\star$ and $O_4^\star$ with respective coordinates $(0,0)$, $(a,0)$, $(a,b)$ and $(0,b)$;
- $C_P$ is the primary (mesh) cell with corresponding cellpoint $P$;
- $C_{A^\star}$ is the dual (mesh) cell associated with the vertex $A^\star \in \mathcal{V}$;
- $C_{E^\star E^{\star\perp}}$ is the recomposed dual (mesh) cell associated with boundary vertices $E^\star$ and $E^{\star\perp}$ supposed different from corner-points;
- $C_\pi$ is the recomposed dual (mesh) cell associated with the four corner-points;
- $\Gamma_P$ is the boundary of the primary cell $C_P$, whereas $\Gamma_{A^\star}$ represents the boundary of the dual cell $C_{A^\star}$;
- $\mathcal{E}$ is the set of selected edgepoints;
- $\mathcal{E}^P$ is the subset of $\mathcal{E}$ made of edgepoints exclusively from $\Gamma_P$;
- $\mathcal{E}^{\text{int}}$ is the subset of $\mathcal{E}$ made of interior edgepoints;
- $\mathcal{E}^{\text{ext}}$ is the subset of $\mathcal{E}$ made of boundary edgepoints;
- $\mathcal{E}^{B^\star}$ is the subset of $\mathcal{E}$ made of edgepoints lying on the boundary of the dual cell $C_{B^\star}$;
- $\mathcal{E}^{O_i^\star}$ is the subset of $\mathcal{E}$ made of edgepoints lying on the boundary of the degenerate dual cell $C_{O_i^\star}$ associated with the corner vertex $O_i^\star$, $i = 1,\ 2,\ 3,\ 4$;
- $\mathcal{E}^{E^\star E^{\star\perp}}$ is the subset of $\mathcal{E}$ made of edgepoints lying on the boundary of the recomposed dual cell $C_{E^\star E^{\star\perp}}$;
- $S_J\left(H^\star, T^\star, V, W\right)$ represents 2D-Lebesgue measure of the diamond cell defined by the cellpoints $V$ and $W$ and the vertices $H^\star$ and $T^\star$;

- $\mathcal{I}_{\mathcal{P}}$ is the set of primary mesh interfaces;

- $\Gamma^{12}$ is the intersection of $\Gamma$ with the two coordinate axis namely $(O, x_1)$ and $(O, x_2)$. In fact, $\Gamma^{12} = \Gamma_{So} \cup \Gamma_{We}$;

- $L^2(\Omega)$ is the usual Lebesgue space made of Lebesgue measurable functions $v$ defined in $\Omega$ such that $\int_\Omega \mid v(x) \mid^2 dx$ is a real number. Usually $\| \, . \, \|_{L^2(\Omega)}$ denotes the standard norm of $L^2(\Omega)$ defined by $\| \, v \, \|^2_{L^2(\Omega)} = \int_\Omega \mid v(x) \mid^2 dx$;

- $H^m(\Omega) = \{v \in L^2(\Omega); D^\alpha v \in L^2(\Omega) \quad \forall \, 0 \leq \mid \alpha \mid \leq m\}$ is the so-called Sobolev space, where $m$ is a positive integer, $\alpha = (\alpha_1, \alpha_2)$ a multi-index with positive integer components, $\mid \alpha \mid = \alpha_1 + \alpha_2$, $D^\alpha$ is the usual differential operator of order $\alpha$, with derivatives in the distributional sense. Adopted is the convention that $D^\alpha v = v$ as soon as $\mid \alpha \mid = 0$. Usually $\| \, . \, \|_{m,\Omega}$ denotes the standard norm of $H^m(\Omega)$ defined by:

$$\| \, v \, \|^2_{m,\Omega} = \sum_{0 \leq |\alpha| \leq m} \| \, D^\alpha v \, \|^2_{L^2(\Omega)};$$

- What follows defines a family of semi-norms on $H^m(\Omega)$, indexed by $k \in \{0, 1, ..., m\}$:

$$\mid v \mid^2_{k,\Omega} = \sum_{|\alpha| = k} \| \, D^\alpha v \, \|^2_{L^2(\Omega)} \qquad \forall \, 0 \leq k \leq m;$$

  Note that $\mid v \mid_{0,\Omega} = \| \, v \, \|_{L^2(\Omega)}$. Therefore the following convention is reasonable: $H^0(\Omega) = L^2(\Omega)$

- $RHS$ means Right-Hand Side.

Definition 2.1 Set:

$$\mathcal{P}^{int} \;=\; \text{the subset of } \mathcal{P} \text{ made up of primary cells } A \text{ such that } \mathcal{E}^A \subset \mathcal{E}^{int}$$

$\mathcal{P}^{ext}$ = the subset of $\mathcal{P}$ made of primary cells $A$ such that $\mathcal{E}^A \cap \mathcal{E}^{ext} \neq \emptyset$ (i.e. different from an empty set). $\square$

In the same order of idea, let us give the following definition.

Definition 2.2 Set:

$$\mathcal{V}^{ext} \;=\; \text{the subset of } \mathcal{V} \text{ made up of vertices lying inside the spatial domain } \Omega;$$

$$\mathcal{V}^{int} \;=\; \text{the subset of } \mathcal{V} \text{ made up of vertices lying on the domain boundary } \Gamma.$$

$\square$

For sake of clarity of the presentation, we need to introduce the following definitions. When there is no risk of confusion, we identify the set of primal cells with the set of cellpoints.

Definition 2.3 Let $A$ be an arbitrarily fixed cellpoint from the set $\mathcal{P}$.

(i) For any $I \in \mathcal{E}^A$, $B^\star(A, I)$ is the generic name of vertices in $\mathcal{V}$ such that the segment $[B^\star(A, I); I]$ is part of $\Gamma_A$.

(ii) For any $I \in \mathcal{E}^A \cap \mathcal{E}^{int}$, $B(A, I)$ is the generic name of cellpoints from $\mathcal{P}$ such that $I \in \mathcal{E}^A \cap \mathcal{E}^{B(A,I)}$.

(iii) For any $I \in \mathcal{E}^A \cap \mathcal{E}^{ext}$, $B(A, I^\perp)$ is the generic name of cellpoints from $\mathcal{P}$ such that $I^\perp \in \mathcal{E}^{B(A,I^\perp)} \cap \mathcal{E}^{ext}$.

Recall that $J^\perp$ is the orthogonal projection of any $J \in \Gamma$ on opposite side of $\Gamma$. $\square$

Proposition 2.4 Assume that $\mathcal{P}$ is a conforming mesh, that is, a mesh such that two adjacent cells display one and only one common edge. Then the following assertions hold:

(i) For any $A \in \mathcal{P}$ and any $I \in \mathcal{E}^A$, there exist exactly two different vertices denoted by $B_1^\star(A, I)$ and $B_2^\star(A, I)$ such that $[B_1^\star, I]$ and $[B_2^\star, I]$ are part of $\Gamma_A$. Moreover, $[B_1^\star, I] \cup [B_2^\star, I] = [B_1^\star, B_2^\star]$ is the edge of (the primal) cell $C_A$, involving the edgepoint $I$;

(ii) $B(A, I)$ exists and is unique for any $A \in \mathcal{P}$ and any $I \in \mathcal{E}^A \cap \mathcal{E}^{int}$;

(iii) $B(A, I^\perp)$ exists and is unique for any $A \in \mathcal{P}$ such that $\mathcal{E}^A \cap \mathcal{E}^{ext} \neq \emptyset$, where $I \in \mathcal{E}^A \cap \mathcal{E}^{ext}$. $\square$

Definition 2.5 A dual cell is said *degenerate* if the corresponding vertex is located on the domain boundary. $\square$

This definition gives rise to the following trivial results.

Proposition 2.6 Let $A^\star$ be a vertex such that $\mathcal{E}^{A^\star} \cap \mathcal{E}^{ext} \neq \emptyset$. Then the corresponding dual cell is degenerate i.e. $A^\star \in \Gamma$. $\square$

Proposition 2.7 Any degenerate dual cell possesses exactly two different edgepoints lying on $\mathcal{E}^{ext}$. $\square$

Proposition 2.8 The map from $\mathcal{V}^{ext} \setminus \mathcal{V}^{co}$ onto $\mathcal{V}^{ext} \setminus \mathcal{V}^{co}$, associating with any boundary-vertex $A^\star$, the boundary-vertex $A^{\star\perp}$ is an involution. $\square$

As immediate consequence of this proposition is what follows.

Corollary 2.9 The map consisting to associate with any degenerate dual cell $C_{A^\star}$ the degenerate dual cell $C_{A^{\star\perp}}$, with $A^\star$ in $\mathcal{V}^{ext} \setminus \mathcal{V}^{co}$, is an involution.

Definition 2.10 Let $A^\star$ be an arbitrarily fixed vertex in $\mathcal{V}$ .
(i) For any $I \in \mathcal{E}^{A^\star} \cap \mathcal{E}^{int}$, $B^\star(A^\star, I)$ defines the vertex such that the segment $[A^\star, B^\star(A^\star, I)]$ is the primal edge involving the edgepoint $I$. In the same order of idea, $P(A^\star, I)$ is the generic name of cellpoints in $\mathcal{P}$ such that the segment $[P(A^\star, I), I]$ is part of $\Gamma_{A^\star}$, where $\Gamma_{A^\star}$ denotes the boundary of the dual cell associated with the vertex $A^\star$.
(ii) For any $I \in \mathcal{E}^{A^\star} \cap \mathcal{E}^{ext}$, $B^\star(A^\star, I)$ and $P(A^\star, I)$ conserve the same definition as in (i), whereas $P(A^{\star\perp}, I^\perp)$ defines cellpoints from $\mathcal{P}$ such that $I^\perp \in \mathcal{E}^{P(A^{\star\perp}, I^\perp)}$ .
□

# 3 DDFV formulation of the model problem

We start with recalling that, having set

$$\mathcal{H} = \left\{ v \in H^1(\Omega);\ v\,(.,0) = v\,(.,b) \quad in \quad [0,a] \text{ and } v\,(0,.) = v\,(a,.) \quad in \quad [0,b] \right\}.$$

a weak (or variational) solution to (1.7)-(1.8) is any function $\varphi \in \mathcal{H}$ such that

$$\int_\Omega K\,(x)\, grad\varphi . gradv\, dx = \int f v\, dx \qquad \forall v \in \mathcal{H}.$$

For $f$ in $L^2(\Omega)$ such that condition ( 1.11) is satisfied, the Lax-Milgram theorem ensures existence and uniqueness (up to just an additive constant) of a weak (or variational) solution to (1.7)-(1.8). Higher regularity of the solution could be achieved under suitable assumptions on the data (see for instance [K 08], pp 25-32). Let us assume that the diffusion matrix $K$ is a piecewise constant matrix function over $\Omega$. This assumption is realistic for engineering problems as reservoir or aquifer simulations.

Definition 3.1 A mesh $\mathcal{M}$ defined over $\Omega$ is compatible with the discontinuities of $K$ in $\Omega$ if these discontinuities are confined to interfaces of $\mathcal{M}$.

**Main assumptions:**

$(\mathcal{A}_1)$ We assume that the primary mesh $\mathcal{P}$ is compatible with the discontinuities of the permeability tensor $K$ defined in $\Omega$.

$(\mathcal{A}_2)$ The permeability discontinuities divide $\Omega$ into a finite number of convex polygonal subsets denoted by $\{\Omega_s\}_{s\in S}$.

We suppose that the restriction over $\overline{\Omega}_s$ of the exact solution $\varphi$ to the model problem (1.7)-(1.8), denoted by $\varphi\,|_{\Omega_s}$, satisfies the following property:

$$\varphi\,|_{\Omega_s} \in C^2(\overline{\Omega}_s) \qquad \forall\, s \in S \tag{3.1}$$

remark 3.2 Note that the *convexity* of subdomains $\Omega_s$ is necessary for getting that the true solution $\varphi$ to the model problem (1.7)-(1.8) is $H^2$ over $\Omega_s$. In this perspective, it is always possible to derive from any family of permeability discontinuity lines defining a polygonal partition of $\Omega$ a new one made up of a finite number of *convex* polygonal subsets of $\Omega$. This is the reason why the assumption $(\mathcal{A}_2)$ is not restrictive. $\square$

In this section we concentrate on building a DDFV formulation of the continuous problem (1.7)-(1.8). Recall that the general principle of the DDFV theory is the discretization of balance equations on a primal mesh and its corresponding dual mesh. The DDFV balance equations on interior cells from primal and dual meshes could be derived following a technique fully detailed in [NDM 13]. However a short survey of this technique will be exposed. In the current section the challenge concerns the DDFV discretization of Balance equation for boundary cells from both primal and dual meshes.

### 3.1 Matrix form of the discrete problem

We derive in this subsection the matrix form of our DDFV finite volume formulation for (1.7)-(1.8). The spatial domain $\overline{\Omega}$ is covered with an unstructured primal mesh $\mathcal{P}$ (see Figure 2 in previous section).

The following additional assumption ensures the uniqueness of a variational solution to the system of equations (1.7)-(1.8):

$$\int_{\Omega} \varphi(x)\, dx \;=\; 0 \tag{3.2}$$

Let us focus on a DDFV formulation of the problem (1.7)-(1.8) in terms of a linear system which involves $\{\varphi_P\}_{P\in\mathcal{P}}$ and $\{\varphi_{D^\star}\}_{D^\star\in\mathcal{D}}$ as discrete unknowns expected to be close approximations of $\{\overline{\varphi}_P\}_{P\in\mathcal{P}}$( cell-point pressures) and $\{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}}$ (vertex pressures) respectively, where $\overline{\varphi}_p = \overline{\varphi}\left(x_1^P, x_2^P\right)$ and $\overline{\varphi}_{D^\star} = \overline{\varphi}\left(x_1^{D^\star}, x_2^{D^\star}\right)$ and where $\mathcal{D}$ represents the dual mesh (Figure 6). Let us give a description of the DDFV procedure leading to the discrete balance equation for any primal cell $C_P$, where P is the corresponding cellpoint. We start with integrating the two sides of the balance equation (1.7) in a primal cell $C_P$. We obtain so an equation where the left-hand side represents some mass of fluid exchanged (for a certain time period) between the cell $C_P$ and its nearest neighborhood. In other words, we obtain:

$$\sum_{I\in\mathcal{E}^P}\int_{[A^\star(P,I);B^\star(P,I)]} K^P\, grad\varphi\cdot \xi^P_{[A^\star(P,I);B^\star(P,I)]}\, ds \;=\; \int_{C_P} f(x)\, dx \qquad \forall\, P\in\mathcal{P} \tag{3.3}$$

where $K^P$ is the permeability tensor of $C_P$, $\xi^P_{[A^\star(P,I);B^\star(P,I)]}$ the unit normal vector to $[A^\star(P,I);B^\star(P,I)]$ oriented outside the triangle $(P, A^\star(P,I), B^\star(P,I))$ (see Figure 7 below where $A^\star \equiv A^\star(P,I)$ and $B^\star \equiv B^\star(P,I)$). Notice that the vertices $A^\star(P,I)$ and $B^\star(P,I)$ are defined in the previous section mainly devoted to Notations.

To develop computational techniques for integrals of the left-hand side of the previous equation is the main task in finite volume modeling of flows. For this purpose, one may separately treat the case $\mathcal{E}^P\cap\mathcal{E}^{ext} = \emptyset$ and the case $\mathcal{E}^P\cap\mathcal{E}^{ext} \neq \emptyset$.

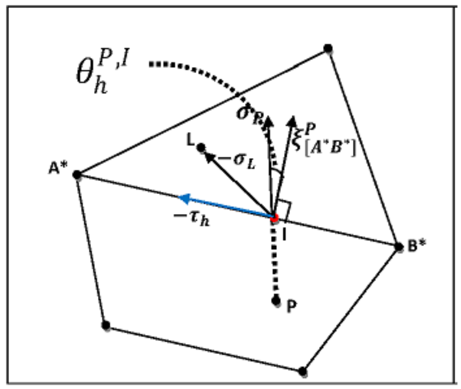


Figure 7: A molecule for DDFV computations of the flux across the edge $[A^\star B^\star]$

• Let us first suppose that $C_P$ is such that $\mathcal{E}^P \cap \mathcal{E}^{ext} = \emptyset$. Therefore any edge $[A^\star(P,I); B^\star(P,I)]$ of $C_P$ is also that of a certain primal cell, let us say $C_L$, with $L = L(P,I)$ according to notations introduced in Section 2. Thus, for ensuring the conservativity of a finite volume approximation of the following flux

$$F^P_{[A^\star(P,I);B^\star(P,I)]} = \int_{[A^\star(P,I);B^\star(P,I)]} K^P\, grad\varphi \cdot \xi^P_{[A^\star(P,I);B^\star(P,I)]}\, ds \tag{3.4}$$

we impose that this approximation should be the opposite of that of

$$F^L_{[A^\star(L,I);B^\star(L,I)]} = \int_{[A^\star(L,I);B^\star(L,I)]} K^L\, grad\varphi \cdot \xi^L_{[A^\star(L,I);B^\star(L,I)]}\, ds \tag{3.5}$$

where $A^\star(L,I) = A^\star(P,I)$, $B^\star(L,I) = B^\star(P,I)$, and where $\xi^L_{[A^\star(L,I);B^\star(L,I)]} = -\xi^P_{[A^\star(P,I);B^\star(P,I)]}$. Recall that $K^P$ and $K^L$ denote respectively the permeability tensor of primal cells $C_P$ and $C_L$.

For the sake of simplicity of notations, we will use in the sequel $A^\star$ and $B^\star$ instead of $A^\star(P,I)$ and $B^\star(P,I)$ whenever there is no risk of confusion. Let us set, for $I \in \mathcal{E}^P \cap \mathcal{E}^L$ :

$$\overrightarrow{PI} = \left\|\overrightarrow{PI}\right\| \sigma_P, \quad \overrightarrow{A^\star B^\star} = \left\|\overrightarrow{A^\star B^\star}\right\| \tau_h \tag{3.6}$$

$$\begin{aligned} a_h\left(K^P\right) &= \tfrac{1}{cos(\theta_h^{P,I})} \left(\xi^P_{[A^\star B^\star]}\right)^t K^P \left(\xi^P_{[A^\star B^\star]}\right) \\ b_h\left(K^P\right) &= \tfrac{1}{cos(\theta_h^{P,I})} \left(\xi^{B^\star}_{[PI]}\right)^t K^P \left(\xi^P_{[A^\star B^\star]}\right) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \hat{a}_h\left(K^L\right) &= \tfrac{1}{cos(\theta_h^{L,I})} \left(\xi^P_{[A^\star B^\star]}\right)^t K^L \left(\xi^P_{[A^\star B^\star]}\right) \\ \hat{b}_h\left(K^L\right) &= \tfrac{1}{cos(\theta_h^{L,I})} \left(\xi^{B^\star}_{[IL]}\right)^t K^L \left(\xi^P_{[A^\star B^\star]}\right) \end{aligned} \tag{3.8}$$

where $h = max\left\{size(\mathcal{P})\,, size\,(\mathcal{D})\right\}$, $\theta_h^{P,I}$ is the angle defined by the vectors $\sigma_P$ and $\xi^P_{[A^\star B^\star]}$, $\theta_h^{L,I}$ is the angle defined by the vectors $-\sigma_L$ and $\xi^P_{[A^\star B^\star]}$, and where $\xi^{B^\star}_{[IL]}$

denotes the unit normal vector to $[I\,;\,L]$ pointing out of the triangle $(I,P,B^\star)$: Figure 7 above gives an illustration of previous comment. Note that $0 \leq \theta_h^{P,I}, \theta_h^{L,I} < \frac{\pi}{2}$ and therefore $0 < cos\theta_h^{P,I}, cos\theta_h^{L,I} \leq 1$.

Definition 3.3 The system $(\mathcal{P}\,;\,\mathcal{D})$ defines an *eligible* system of meshes if the following conditions are fulfilled:
(i) There exists $\theta \in ]0\,,\,\frac{\pi}{2}\,]$, mesh independent, such that:

$$0 \,\leq\, \theta_h^{P,I} \,\leq\, \frac{\pi}{2} - \theta \qquad \forall\, P \in \mathcal{P} \quad \forall\, I \in \mathcal{E}^P \tag{3.9}$$

(ii) There exists $0 < \varpi \leq 1$, mesh independent, such that:

$$\forall\, P \in \mathcal{P} \qquad \forall\, I \in \mathcal{E}^P \qquad \varpi\, h \,\leq\, h_{PI}\;, h_{A^*B^*} \,\leq h \tag{3.10}$$

where $h_{PI} = \| \overrightarrow{PI} \|$ and $h_{A^\star B^\star} = \| \overrightarrow{A^\star B^\star} \|$ . □

Following ideas we have exposed in [NDM 13], one easily see that the flux $F^P_{[A^\star;B^\star]}$ can be approximated, with conservatitvity guarantee, as it follows :

Proposition 3.4 Under the only assumption (3.1) we have:

$$F^P_{[A^*B^*]} \approx \left[\frac{a_h(K^P)\widehat{a}_h(K^L)h_{A^*B^*}}{a_h(K^P)h_{IL} + \widehat{a}_h(K^L)h_{PI}}\right][\varphi_P - \varphi_L]$$

$$+\left[\frac{\widehat{a}_h(K^L)b_h(K^P)h_{PI} + a_h(K^P)\widehat{b}_h(K^L)h_{IL}}{\widehat{a}_h(K^L)h_{PI} + a_h(K^P)h_{IL}}\right][\varphi_{B^*} - \varphi_{A^*}] \tag{3.11}$$

In addition, if the system of meshes $(\mathcal{P};\mathcal{D})$ is eligible in the sense of Definition 3.3, the truncation error $T^P_{[A^*B^*]}$ (also denoted by $T^{P,I}$) associated with this flux approximation satisfies the following inequality

$$|\, T^{P,I} \,| \quad \equiv \quad |\, T^P_{[A^*B^*]} \,| \quad \leq \quad \mathbf{C}\, h^2 \tag{3.12}$$

where $A^\star = A^\star(P,I)$, $B^\star = B^\star(P,I)$, $L = L(P,I)$ and where $\mathbf{C}$ is a mesh independent positive number. □

Recall that only the case $\mathcal{E}^P \cap \mathcal{E}^{ext} = \emptyset$, with $P \in \mathcal{P}$, is concerned by the previous result. Summing the two sides of relation (3.11) on $I \in \mathcal{E}^P$ leads to the following result.

Proposition 3.5 Let us suppose that the system of meshes $(\mathcal{P};\mathcal{D})$ is eligible in the sense of Definition 3.3. Under the assumption (3.1), the discrete balance equation in any primary cell $C_P$, with $\mathcal{E}^P \cap \mathcal{E}^{ext} = \emptyset$, reads as:

$$\sum_{I\in\mathcal{E}^P} \left\{T^{P,I} + \left[\tfrac{a_h(K^P)\hat{a}_h(K^L)h_{A^\star B^\star}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\varphi_P - \varphi_L] + \left[\tfrac{a_h(K^P)\widehat{b}_h(K^L)h_{IL}+b_h(K^P)\widehat{a}_h(K^L)h_{IP}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\varphi_{B^\star} - \varphi_{A^\star}]\right\} = \int_{C_P} f dx \tag{3.13}$$

where $A^\star = A^\star(P,I)$, $B^\star = B^\star(P,I)$ and $L = L(P,I)$ (see Setion 2 devoted mainly to Notations).
Moreover, the truncation error $T^{P,I}$ satisfies the inequality (3.12). $\square$

- Let us examine the less classical case (in the DDFV framework) $\mathcal{E}^P \cap \mathcal{E}^{ext} \neq \emptyset$ i.e. there exists (at least) one boundary edgepoint, let say $I$, such that $I \in \mathcal{E}^P \cap \mathcal{E}^{ext}$. Remarking that $I$ cannot be a corner point, we deduce that $I^\perp$ exists and lives in the domain boundary $\Gamma$. Thanks to periodicity considerations and gridding properties (relevant to these periodicity considerations), namely (2.1) and (2.2), we may identify the boundary edge associated with $I$ and the one associated with $I^\perp$. In other words, if $A^\star = A^\star(P,I)$ and $B^\star = B^\star(P,I)$ are endpoints of the boundary edge associated with $I$, we have (making use of notations introduced in Section 2) the following identification :

$$[A^\star; B^\star] \equiv [A^{\star\perp}; B^{\star\perp}] \tag{3.14}$$

Thanks to this identification, the primary cells $C_P$ and $C_{Q(P,I^\perp)}$ are called *adjacent cells in the periodic sense*, with common edge $[A^\star; B^\star]$. We should underline the fact that gridding properties (2.1) and (2.2) play a key role here.

Combining this approach with ideas we have developed in [NDM 13] leads to the following result.

PROPOSITION 3.6 Let us suppose that the system of meshes $(\mathcal{P};\mathcal{D})$ is eligible in the sense of Definition 3.3. Under the assumption (3.1), the discrete balance equation in any primary cell $C_P$, with $\mathcal{E}^P \cap \mathcal{E}^{ext} \neq \emptyset$, reads as :

$$\sum_{I\in\mathcal{E}^P\cap\mathcal{E}^{int}} \Bigg\{ T^{P,I} + \left[\frac{a_h(K^P)\hat{a}_h(K^L)h_{A^\star B^\star}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\varphi_P-\varphi_L] +$$
$$\left[\frac{a_h(K^P)\widehat{b}_h(K^L)h_{IL}+b_h(K^P)\widehat{a}_h(K^L)h_{IP}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\varphi_{B^\star}-\varphi_{A^\star}]\Bigg\} +$$
$$\sum_{I\in\mathcal{E}^P\cap\mathcal{E}^{ext}} \Bigg\{ T^{P,I} + \left[\frac{a_h(K^P)\hat{a}_h\left(K^{Q(P,I^\perp)}\right)h_{A^\star B^\star}}{a_h(K^P)h_{I^\perp Q(P,I^\perp)}+\widehat{a}_h\left(K^{Q(P,I^\perp)}\right)h_{IP}}\right]\left[\varphi_P-\varphi_{Q(P,I^\perp)}\right] +$$
$$\left[\frac{a_h(K^P)\hat{b}_h\left(K^{Q(P,I^\perp)}\right)h_{I^\perp Q(P,I^\perp)}+b_h(K^P)\hat{a}_h\left(K^{Q(P,I^\perp)}\right)h_{IP}}{a_h(K^P)h_{I^\perp Q(P,I^\perp)}+\widehat{a}_h\left(K^{Q(P,I^\perp)}\right)h_{IP}}\right]\times$$
$$[\varphi_{B^\star}-\varphi_{A^\star}]\Bigg\} = \int_{C_P} f(x)\,dx \tag{3.15}$$

where $A^\star = A^\star(P,I)$, $B^\star = B^\star(P,I)$ and $L = L(P,I)$, and where we mean by $Q\left(P, I^\perp\right)$ the cellpoint of a primary boundary-cell close (i.e. adjacent in the sense that (3.14) holds) to the primary cell $C_P$ trough the periodic setting, when $\mathcal{E}^P \cap \mathcal{E}^{ext} \neq \emptyset$ (see Section 2 devoted to Notations for more details).
Moreover, the truncation error $T^{P,I}$ satisfies the inequality (3.12). $\square$

REMARK 3.7 Notice one may use the convention that the second summation in (3.15) is defined and equal to zero for any primal cell $C_B$ such that $\mathcal{E}^B \cap \mathcal{E}^{ext} = \emptyset$. Thanks to this convention, equation (3.13) becomes a particular case of equation (3.15). We consequently consider that in what follows equation (3.15) is the discrete balance equation for any primary cell. $\square$

It is clear that the number of discrete unknowns $\{\varphi_P\}_{P\in\mathcal{P}}$ and $\{\varphi_{A^\star}\}_{A^\star\in\mathcal{D}}$ is greater than the number of discrete balance equations given by the system of equations (3.15) valid for all $P \in \mathcal{P}$ ( accounting with the convention introduced in Remark 3.7). We naturally should close this system with discrete equations obtained from mass balance equations over dual cells. It is our purpose now to look for discrete balance equations over dual cells $C_{A^\star}$. So, we integrate the two sides of (1.7) in $C_{A^\star}$. The left-hand side is the flow exchanged between $C_{A^\star}$ and outside of this cell, whereas the right-hand side is the term-source contribution (for a fixed time period). Therefore, we can rewrite this balance equation as:

$$\sum_{I\in\mathcal{E}^{A^\star}} \int_{[P(A^\star,I);I;L(A^\star,I)]} K\, grad\varphi \cdot \xi^{A^\star}_{[P(A^\star,I);I;L(A^\star,I)]}\, ds = \int_{C_{A^\star}} f(x)\, dx \qquad \forall A^\star \in \mathcal{D} \tag{3.16}$$

where the points $P(A^\star,I)$ and $L(A^\star,I)$ follow notations defined in Section 2 and where $\xi^{A^\star}_{[P(A^\star,I);I;L(A^\star,I)]}$ is the unit normal vector to $[P(A^\star,I);I;L(A^\star,I)]$ oriented outside the quadrangle $\{P(A^\star,I);I;L(A^\star,I);A^\star\}$; we define $[P(A^\star,I);I;L(A^\star,I)]$ as the union of the segments $[P(A^\star,I);I]$ and $[I;L(A^\star,I)]$, where $\overrightarrow{P(A^\star,I)I}$ and $\overrightarrow{IL(A^\star,I)}$ are not necessarily aligned vectors. Note that the particular case where these vectors are aligned is widely analyzed in the literature with DDFV formulations based on definition of discrete gradient (in the so-called diamond cells) and discrete divergence (in primal and dual cells) that meet a discrete integration by parts (see for instance [DO 05][O 09][O 11][BH 08][D 14][DEH 15] and references therein). Note that this approach is proven to be very powerful as it allows to address nonlinear elliptic operators including Leray-Lions type operators (see for instance [DEGGH 16][ABH 07][BH 08]). For the sake of clarity of the exposition, we write $P$ and $L$ instead of $P(A^\star,I)$ and $L(A^\star,I)$.

- Let us start with interior dual cells i.e. dual cells $C_{A^\star}$ such that $\mathcal{E}^{A^\star} \cap \mathcal{E}^{ext} = \emptyset$ or equivalently $A^\star$ does not belong to $\Gamma$, the domain boundary. Following an approach we have developed in [NDM 13] the exact flux

$$F^{A^\star}_{[PIL]} = \int_{[P;I;L]} K\, grad\varphi \cdot \xi^{A^\star}_{[P;I;L]}\, ds \tag{3.17}$$

can be rewritten as sum of flux integrals over $[P;I]$ and $[I;L]$. Next, with the help of adequate quadratures combined with Taylor expansion (based up on assumption (3.1) ) we obtain:

PROPOSITION 3.8 Under the assumption (3.1), the flux $F^{A^\star}_{[PIL]}$ satisfies the following identity :

$$\begin{aligned} F^{A^\star}_{[PIL]} = & \Big\{ T^{A^\star,I} + \left[\tfrac{b_h(K^P)\hat{a}_h(K^L)h_{IP}+a_h(K^P)\hat{b}_h(K^L)h_{IL}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\varphi_P-\varphi_L]+ \\ & \left[\tfrac{W_h(P,L,I)}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\varphi_{A^\star}-\varphi_{B^\star}]\Big\} \end{aligned} \tag{3.18}$$

where $T^{A^\star,I}$ is the truncation error, $B^\star = B^\star(A^\star, I)$ (using notations defined in Section 2), and where we have set

$$\begin{aligned} W_h(P,L,I) = & \left[a_h\left(K^P\right)h_{IL}+\hat{a}_h\left(K^L\right)h_{IP}\right]\left[d_h\left(K^P\right)h_{IP}+\widehat{d}_h\left(K^L\right)h_{IL}\right]+ \\ & h_{IP}h_{IL}\left[\hat{b}_h\left(K^L\right)-b_h\left(K^P\right)\right]\left[b_h\left(K^P\right)-\hat{b}_h\left(K^L\right)\right] \end{aligned} \tag{3.19}$$

□

Summing the two sides of the equation (3.18) on $I \in \mathcal{E}^{A^\star}$ leads to what follows.

PROPOSITION 3.9 Under the assumption (3.1), the flux balance equation for any *interior dual cell* $C_{A^\star}$ reads,:

$$\begin{aligned} & \sum_{I\in\mathcal{E}^{A^\star}} \Big\{ T^{A^\star,I} + \left[\tfrac{b_h(K^P)\hat{a}_h(K^L)h_{IP}+a_h(K^P)\hat{b}_h(K^L)h_{IL}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\varphi_P-\varphi_L]+ \\ & \left[\tfrac{W_h(P,L,I)}{h_{A^\star B^\star}(a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP})}\right][\varphi_{A^\star}-\varphi_{B^\star}]\Big\} = \int_{C_{A^\star}} f(x)dx \qquad \forall A^\star\in\mathcal{V}\backslash\mathcal{V}^{ext} \end{aligned} \tag{3.20}$$

where $W_h(P,L,I)$ is defined by (3.19) and where we have set $P = P(A^\star, I)$, $L = L(A^\star, I)$ and $B^\star = B^\star(A^\star, I)$ following notations defined in Section 2.
In addition, if the system of meshes $(\mathcal{P};\mathcal{D})$ is eligible in the sense of Definition 3.3, the truncation error $T^{A^\star,I}$ obeys the following inequality :

$$\mid T^{A^\star,I} \mid \leq \mathbf{C}\, h^2 \tag{3.21}$$

where $\mathbf{C}$ is a mesh independent positive number. □

The next step concerns *balance equation in degenerate dual cells* i.e. dual cells with corresponding vertexpoints on the domain boundary. Due to $\Omega-$*periodicity* considerations and its implication on gridding structure, the orthogonal projection of any boundary vertexpoint $E^\star \in \mathcal{V}\setminus\mathcal{V}^{co}$ on opposite side of the domain boundary is also a boundary vertexpoint in $\mathcal{V}\setminus\mathcal{V}^{co}$ denoted by $E^{\star\perp}$. The same assertion is valid for the edgepoints. Let us recall that if $I$ is a boundary edgepoint, we denote by $I^{\perp}$ its orthogonal projection on opposite side of the domain boundary.

Our strategy for obtaining discrete balance equations associated with degenerate dual cells accounts with periodicity setting of the model problem we are dealing with. In this connection, we consider on one hand that the dual cells $C_{O_i^\star}$, for $i \in \{1, ..., 4\}$, could be brought together for giving rise to what we call a recomposed dual cell of first kind $C_\pi$ defined as :

Definition 3.10

$$C_\pi = \bigcup_{i=1}^{4} C_{O_i^\star} \tag{3.22}$$

On the other hand, for any $E^\star \in \mathcal{V}^{ext} \setminus \mathcal{V}^{co}$, we have $E^{\star\perp} \in \mathcal{V}^{ext} \setminus \mathcal{V}^{co}$ and we bring together the dual cells $C_{E^\star}$ and $C_{E^{\star\perp}}$ for defining a recomposed dual of second kind $C_{E^\star E^{\star\perp}}$ as :

Definition 3.11

$$C_{E^\star E^{\star\perp}} = C_{E^\star} \cup C_{E^{\star\perp}} \tag{3.23}$$

Remark that Definition 3.10 and Definition 3.11 combined with periodicity setting of the model problem allow to treat recomposed dual cells $C_{E^\star E^{\star\perp}}$ and $C_{\{O_i^\star\}}$ as an ordinary dual cell. Therefore, the flux across any part $[PIL]$ of the boundary of such a dual cell ia actually approximated using a technique combining numerical integration (of the flux across subedges $[PI]$ and $[IL]$ of $C_{E^\star E^{\star\perp}}$) and local Taylor's expansion as the true solution $\varphi$ is locally $C^2$ according to (3.1). This is the way we obtain the following relations.

Proposition 3.12 Let us suppose that the system of meshes $(\mathcal{P};\mathcal{D})$ is eligible in the sense of Definition 3.3. Under the assumption (3.1), the balance equation for recomposed dual cells of the form $C_{E^\star E^{\star\perp}}$ reads as :

$$\sum_{Q^\star \in \left\{E^\star, E^{\star\perp}\right\}} \sum_{I \in \mathcal{E}^{Q^\star} \cap \mathcal{E}^{int}} \Bigg\{ T^{Q^\star, I} + \left[\varphi_{A(Q^\star,I)} - \varphi_{B(Q^\star,I)}\right] \times$$

$$\left[\frac{b_h\left(K^{A(Q^\star,I)}\right)\hat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)} + a_h\left(K^{A(Q^\star,I)}\right)\hat{b}_h\left(K^{B(Q^\star,I)}\right)h_{IB(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)} + \hat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}\right]$$

$$+ \left[\frac{W_h(A(Q^\star,I),B(Q^\star,I),I)}{h_{Q^\star F^\star(Q^\star,I)}\left(a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)} + \widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}\right)}\right]\left[\varphi_{Q^\star} - \varphi_{F^\star(Q^\star,I)}\right]\Bigg\}$$

$$+\sum_{I\in\mathcal{E}^{E^\star E^{\star\perp}}\cap\mathcal{E}^{ext}}\Big\{T^{E^\star E^{\star\perp},I}+\Big[\varphi_{A(E^\star,I)}-\varphi_{B\left(E^\star,I^\perp\right)}\Big]\times$$

$$\left[\frac{b_h\left(K^{A(E^\star,I)}\right)\hat{a}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{IA(E^\star,I)}+a_h\left(K^{A(E^\star,I)}\right)\hat{b}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{I^\perp B(E^\star,I^\perp)}}{a_h\left(K^{A(E^\star,I)}\right)h_{I^\perp B(E^\star,I^\perp)}+\widehat{a}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{IA(E^\star,I)}}\right]$$

$$+\left[\frac{W_h(A(E^\star,I),B\left(E^\star,I^\perp\right),I)}{h_{E^\star F^\star(E^\star,I)}\left(a_h\left(K^{A(E^\star,I)}\right)h_{I^\perp B(E^\star,I^\perp)}+\widehat{a}_h\left(K^{B(E^\star,I)}\right)h_{IA(E^\star,I)}\right)}\right]\times$$

$$\left[\varphi_{E^\star}-\varphi_{F^\star(E^\star,I)}\right]\Big\}=\int_{C_{E^\star E^{\star\perp}}}f(x)\,dx \tag{3.24}$$

where $A(Q^\star,I)$, $B(Q^\star,I)$, $F^\star(Q^\star,I)$, $A(E^\star,I)$, $B(E^\star,I)$, $F^\star(E^\star,I)$ are notations defined in Section 2, and where $T^{Q^\star,I}$, $T^{E^\star E^{\star\perp},I}$ named truncation errors satisfy the inequalities:

$$|\,T^{Q^\star,I}\,|\leq \mathbf{C}\,h^2,\qquad |\,T^{E^\star E^{\star\perp},I}\,|\leq \mathbf{C}\,h^2. \tag{3.25}$$

where $\mathbf{C}$ is a mesh independent positive number. $\square$

Before dealing with the balance equation in the recomposed dual cell $C_{\{O_i^\star\}}$, we should introduce the following notations needed in the sequel.

$$\mathfrak{R}=\Big\{\{k,j\}\subset\{1,2,3,4\},\ with\ k\neq j;\exists I_k\in\mathcal{E}^{O_k^\star}\cap\mathcal{E}^{ext}\ \exists I_j\in\mathcal{E}^{O_j^\star}\cap\mathcal{E}^{ext},\text{with } I_j=I_k^\perp\Big\}$$

Set:

$$\Gamma^1=\Gamma\cap[O\,x_1),\qquad \Gamma^2=\Gamma\cap[O\,x_2)\qquad and\qquad \Gamma^{12}=\Gamma^1\cup\Gamma^2$$

and recall that $\Gamma$ is the domain boundary.
Let us ultimately deal with the balance equation for the degenerate dual $C_\pi$. We start with integrating the two sides of the balance equation (1.7) in

$$C_\pi\ \underset{def}{=}\ \underset{i=1,\dots,4}{\cup}C_{O_i^\star}$$

where $O_1^\star$, $O_2^\star$, $O_3^\star$ and $O_4^\star$ are four corner-vertices with respective coordinates $(0,0)$, $(a,0)$, $(a,b)$ and $(0,b)$, and where $C_{O_i^\star}$ is the degenerate dual cell associated with $O_i^\star$. It is then natural to split this integral into four integrals as follows:

$$-\int_{C_\pi}div[K\,grad\,\varphi]dx\equiv-\sum_{i=1}^{4}\int_{C_{O_i^\star}}div[K\,grad\,\varphi]dx=\int_{C_\pi}f(x)dx \tag{3.26}$$

From Green's theorem, we know that the integral over $C_{O_i^\star}$ could be rewritten in terms of boundary integral as :

$$\int_{C_{O_i^\star}}div[K\,grad\,\varphi]dx=$$

$$\sum_{I\in\mathcal{E}^{O_i^\star}\cap\mathcal{E}^{int}} \int_{[P(O_i^\star,I),I,L(O_i^\star,I)]} [K\, grad\,\varphi]\cdot\nu_I ds + \sum_{I\in\mathcal{E}^{O_i^\star}\cap\mathcal{E}^{ext}} \int_{[P(O_i^\star,I),I]} [K\, grad\,\varphi]\cdot\nu_I ds \tag{3.27}$$

where $\nu_I$ is the unit normal vector to $[P(O_i^\star,I),I,L(O_i^\star,I)]$ or $[P(O_i^\star,I),I]$, pointing out of $C_{O_i^\star}$, and where $P(O_i^\star,I)$ and $L(O_i^\star,I)$ are, according to notations defined in Section 2, cellpoints from primary cells sharing the subedge $[O_i^\star,I]$. Inspired by ideas developed in [NDM 13], we perform computation of integrals involved in equation (3.27) with the help of adequate quadrature formulas and Taylor expansion. This leads to what follows.

Proposition 3.13 Let us suppose that the system of meshes $(\mathcal{P};\mathcal{D})$ is eligible in the sense of Definition 3.3. Under the assumption (3.1), the discrete balance equation for $C_\pi$ reads as:

$$\begin{aligned}
&\sum_{Q^\star\in\{O_i^\star\}_{i=1}^4}\ \sum_{I\in\mathcal{E}^{Q^\star}\cap\mathcal{E}^{int}} \Bigg\{\Bigg[\frac{b_h\left(K^{A(Q^\star,I)}\right)\hat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}} + \\
&+\frac{a_h\left(K^{A(Q^\star,I)}\right)\hat{b}_h\left(K^{B(Q^\star,I)}\right)h_{IB(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}\Bigg]\times\left[\varphi_{A(Q^\star,I)}-\varphi_{B(Q^\star,I)}\right] + \\
&+\left[\frac{W_h(A(Q^\star,I),B(Q^\star,I),I)}{h_{Q^\star F^\star(Q^\star,I)}\left(a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}\right)}\right]\times \\
&\times\left[\varphi_{Q^\star}-\varphi_{F^\star(Q^\star,I)}\right] + T^{Q^\star,I}\Bigg\} + \\
&+\sum_{I\in\Gamma^{12}\cap\mathcal{E}^{ext}}\ \sum_{\{i,j\}\in\Re, I\in\mathcal{E}^{O_i^\star}, I^\perp\in\mathcal{E}^{O_j^\star}} \Bigg\{\Bigg[\frac{b_h\left(K^{A(O_i^\star,I)}\right)\hat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}{a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}} + \\
&+\frac{a_h\left(K^{A(O_i^\star,I)}\right)\hat{b}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}}{a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}\Bigg]\times\left[\varphi_{A(O_i^\star,I)}-\varphi_{B\left(O_j^\star,I^\perp\right)}\right] + \\
&+\left[\frac{W_h\left(A\left(O_i^\star,I\right),B\left(O_j^\star,I^\perp\right),I\right)}{h_{O_i^\star T^\star\left(O_i^\star,I\right)}\left(a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}\right)}\right]\times \\
&\times\left[\varphi_{O_i^\star}-\varphi_{T^\star\left(O_i^\star,I\right)}\right] + T^{O_i^\star,O_j^\star,I}\Bigg\} = \sum_{i=1}^4\int_{C_{O_i^\star}} f(x)\,dx
\end{aligned} \tag{3.28}$$

where $T^{Q^\star,I}$ and $T^{O_i^\star,O_j^\star,I}$ are given real numbers named truncation errors, and where in the second double summation, $A(O_i^\star,I)$ and $B\left(O_j^\star,I^\perp\right)$ are cellpoints of two primary boundary-cells involving respectively boundary subedges $[O_i^\star,I]$ and $\left[O_j^\star,I^\perp\right]$.

Moreover, the truncation errors $T^{Q^\star,I}$ and $T^{O_i^\star,O_j^\star,I}$ satisfy the inequalities:

$$\mid T^{Q^\star,I}\mid\leq \mathbf{C}\,h^2, \qquad \mid T^{O_i^\star,O_j^\star,I}\mid\leq \mathbf{C}\,h^2 \tag{3.29}$$

where $\mathbf{C}$ is a mesh independent positive number. $\square$

Following Remark 3.7, the equation (3.15) may be used as balance equation for both interior and boundary primal cells. Therefore, when neglecting truncation errors in this equation and in the equations (3.20), (3.24), (3.28) as well, we get the following discrete problem, named *discrete duality finite volume* formulation of the model problem (1.7) – (1.8) :

Find real numbers $\{\overline{\varphi}_P\}_{P\in\mathcal{P}}$ and $\{\overline{\varphi}_{A^\star}\}_{A^\star\in\mathcal{D}}$ such that

$$\sum_{I\in\mathcal{E}^P\cap\mathcal{E}^{int}}\left\{\left[\frac{a_h\left(K^P\right)\hat{a}_h\left(K^L\right)h_{A^\star B^\star}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\overline{\varphi}_P-\overline{\varphi}_L]+\right.$$

$$\left.\left[\frac{a_h\left(K^P\right)\widehat{b}_h\left(K^L\right)h_{IL}+b_h\left(K^P\right)\widehat{a}_h\left(K^L\right)h_{IP}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\overline{\varphi}_{B^\star}-\overline{\varphi}_{A^\star}]\right\}+$$

$$\sum_{I\in\mathcal{E}^P\cap\mathcal{E}^{ext}}\left\{\left[\frac{a_h\left(K^P\right)\hat{a}_h\left(K^{Q\left(P,I^\perp\right)}\right)h_{A^\star B^\star}}{a_h(K^P)h_{I^\perp Q(P,I^\perp)}+\widehat{a}_h\left(K^{Q(P,I^\perp)}\right)h_{IP}}\right]\left[\overline{\varphi}_P-\overline{\varphi}_{Q(P,I^\perp)}\right]\;+\right. \tag{3.30}$$

$$\left[\frac{a_h\left(K^P\right)\hat{b}_h\left(K^{Q\left(P,I^\perp\right)}\right)h_{I^\perp Q(P,I^\perp)}+b_h\left(K^P\right)\hat{a}_h\left(K^{Q\left(P,I^\perp\right)}\right)h_{IP}}{a_h(K^P)h_{I^\perp Q(P,I^\perp)}+\widehat{a}_h\left(K^{Q(P,I^\perp)}\right)h_{IP}}\right]\times$$

$$\left.[\overline{\varphi}_{B^\star}-\overline{\varphi}_{A^\star}]\right\}\;=\;\int_{C_P}f\,(x)\,dx\qquad\qquad\forall\,P\in\mathcal{P}$$

$$\sum_{I\in\mathcal{E}^{A^\star}}\left\{\left[\frac{b_h\left(K^P\right)\hat{a}_h\left(K^L\right)h_{IP}+a_h\left(K^P\right)\hat{b}_h\left(K^L\right)h_{IL}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right][\overline{\varphi}_P-\overline{\varphi}_L]+\right.$$

$$\left.\left[\frac{W_h(P,L,I)}{h_{A^\star B^\star}(a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP})}\right][\overline{\varphi}_{A^\star}-\overline{\varphi}_{B^\star}]\right\}=\int_{C_{A^\star}}f(x)dx\qquad\qquad\forall A^\star\in\mathcal{V}^{int} \tag{3.31}$$

$$\sum_{Q^\star\in\left\{E^\star,E^{\star\perp}\right\}}\ \sum_{I\in\mathcal{E}^{Q^\star}\cap\mathcal{E}^{int}}\left\{\left[\overline{\varphi}_{A(Q^\star,I)}-\overline{\varphi}_{B(Q^\star,I)}\right]\times\right.$$

$$\left[\frac{b_h\left(K^{A(Q^\star,I)}\right)\hat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}+a_h\left(K^{A(Q^\star,I)}\right)\hat{b}_h\left(K^{B(Q^\star,I)}\right)h_{IB(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}\right]$$

$$\left.+\left[\frac{W_h(A(Q^\star,I),B(Q^\star,I),I)}{h_{Q^\star F^\star(Q^\star,I)}\left(a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}\right)}\right]\left[\overline{\varphi}_{E^\star}-\overline{\varphi}_{F^\star(Q^\star,I)}\right]\right\}$$

$$+\sum_{I\in\mathcal{E}^{E^\star E^{\star\perp}}\cap\mathcal{E}^{ext}}\left\{\left[\overline{\varphi}_{A(E^\star,I)}-\overline{\varphi}_{B\left(E^\star,I^\perp\right)}\right]\times\right.$$

$$\left[\frac{b_h\left(K^{A(E^\star,I)}\right)\hat{a}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{IA(E^\star,I)}+a_h\left(K^{A(E^\star,I)}\right)\hat{b}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{I^\perp B\left(E^\star,I^\perp\right)}}{a_h\left(K^{A(E^\star,I)}\right)h_{I^\perp B\left(E^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{IA(E^\star,I)}}\right]$$

$$+\left[\frac{W_h(A(E^\star,I),B\left(E^\star,I^\perp\right),I)}{h_{E^\star F^\star(E^\star,I)}\left(a_h\left(K^{A(E^\star,I)}\right)h_{I^\perp B\left(E^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B(E^\star,I)}\right)h_{IA(E^\star,I)}\right)}\right]\times$$

$$\left.\left[\overline{\varphi}_{E^\star}-\overline{\varphi}_{F^\star(E^\star,I)}\right]\right\}=\int_{C_{E^\star E^{\star\perp}}}f\left(x\right)dx\qquad\forall\,E^\star\in\left(\mathcal{V}^{ext}\backslash\mathcal{V}^{co}\right)\cap\Gamma^{12}\tag{3.32}$$

$$\sum_{Q^\star\in\left\{O_i^\star\right\}_{i=1}^{4}}\ \sum_{I\in\mathcal{E}^{Q^\star}\cap\mathcal{E}^{int}}\left\{\left[\frac{b_h\left(K^{A(Q^\star,I)}\right)\hat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}+\right.\right.$$

$$\left.+\frac{a_h\left(K^{A(Q^\star,I)}\right)\hat{b}_h\left(K^{B(Q^\star,I)}\right)h_{IB(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}\right]\times\left[\overline{\varphi}_{A(Q^\star,I)}-\overline{\varphi}_{B(Q^\star,I)}\right]+$$

$$+\quad\left[\frac{W_h(A(Q^\star,I),B(Q^\star,I),I)}{h_{Q^\star F^\star(Q^\star,I)}\left(a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)}+\widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}\right)}\right]\times$$

$$\left.\times\left[\overline{\varphi}_{O_1^\star}-\overline{\varphi}_{F^\star(Q^\star,I)}\right]\right\}\quad+$$

$$+\sum_{I\in\Gamma^{12}\cap\mathcal{E}^{ext}}\ \sum_{\{i,j\}\in\Re,I\in\mathcal{E}^{O_i^\star},I^\perp\in\mathcal{E}^{O_j^\star}}\left\{\left[\frac{b_h\left(K^{A(O_i^\star,I)}\right)\hat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}{a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\hat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}+\right.\right.$$

$$\left.+\frac{a_h\left(K^{A(O_i^\star,I)}\right)\hat{b}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}}{a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\hat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}\right]\times\left[\overline{\varphi}_{A(O_i^\star,I)}-\overline{\varphi}_{B(O_j^\star,I^\perp)}\right]+$$

$$+\left[\frac{W_h(A(O_i^\star,I),B(O_j^\star,I^\perp),I)}{h_{O_i^\star T^\star\left(O_i^\star,I\right)}\left(a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\hat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}\right)}\right]\times$$

$$\left.\times\left[\overline{\varphi}_{O_1^\star}-\overline{\varphi}_{T^\star\left(O_i^\star,I\right)}\right]\right\}=\sum_{i=1}^{4}\int_{C_{O_i^\star}}f\left(x\right)dx \tag{3.33}$$

with (discrete periodic conditions):

$$\overline{\varphi}_{E^\star}=\overline{\varphi}_{E^{\star\perp}}\quad\forall\,E^\star\in\left(\mathcal{V}^{ext}\setminus\mathcal{V}^{co}\right)\cap\Gamma^{12} \tag{3.34}$$

and

$$\overline{\varphi}_{F^\star(A^\star,I)}=\overline{\varphi}_{O_1^\star}\quad if\quad F^\star(A^\star,I)\in\{O_2^\star,\,O_3^\star,\,O_4^\star\} \tag{3.35}$$

Notice that :
In equations (3.30) and (3.31), $A\left(Q^\star,I\right)$, $B\left(Q^\star,I\right)$, $F^\star\left(Q^\star,I\right)$, $A\left(E^\star,I\right)$, $B\left(E^\star,I\right)$, $F^\star\left(E^\star,I\right)$ are notations defined in Section 2;
In equation (3.32), $A^\star=A^\star(P,I)$, $B^\star=B^\star(P,I)$ and $L=L(P,I)$ and we mean by $Q\left(P,I^\perp\right)$ the cellpoint of a primary boundary-cell close (i.e. adjacent in the sense that (3.14) holds) to the primary cell $C_p$ trough the periodic setting, when $\mathcal{E}^P\cap\mathcal{E}^{ext}\neq\emptyset$ (see Section 2 devoted to Notations for more details);
In equation (3.33), $B(O_j^\star,I^\perp)$ and $T^\star(O_i^\star,I)$ are notations defined within Section 2. Recall that $C_{O_i^\star}$ is the degenerate dual cell associated with the boundary vertex $O_i^\star$.

REMARK 3.14 It is natural for someone not familiar with the Discrete Duality Finite Volumes (DDFV) to wonder why the discrete system (3.30)-(3.35) is called DDFV formulation of the model problem $(1.7)-(1.8)$. In fact, in the discrete system (3.30)-(3.33) is hidden a combination of discrete divergence and discrete gradient. These discrete operators are in duality as they meet a discrete Green formula. This is the reason why the finite volume scheme (3.30)-(3.35) is called DDFV. Notice that in our DDFV approach exposed above, an explicit definition of discrete divergence and gradient does not appear. We do not worry about that because our strategy for carrying out stability and error analysis of the DDFV scheme (3.30)-(3.35) will not be based upon a finite element type technique i.e. a variational formulation involving finite-dimensional function spaces. The concepts of discrete divergence and discrete gradient for finite volume analysis of diffusion problems were first introduced by K. Domelevo and P. Omnès in [DO 05] (see also [DDO 07][O 11][O 09][O 12]). An important feature of the variational form of DDFV is its ability to tackle nonlinear operators of Leray-Lions type as shown in [ABH 07][BH 08]. An extension of DDFV to 3D nonlinear elliptic problems is proposed in [CH 11]. To the best of our knowledge, the pioneer works on DDFV accounting with interface flux continuity are those from [NM 01], [H 03] and [DO 05]. However the work inside [H 00] has prepared the ground for [H 03]. □

## 3.2 Existence of discrete solutions and conditions for uniqueness

Let us start this subsection with some preliminary remarks and results. First of all, we assume that the cellpoints and the vertexpoints from the primary grid are numbered. The numbering is performed in such a way that any boundary-vertex (different from a corner-point) and its orthogonal projection on the opposite side get the same number. On the other hand, the four corner-points $O_1^*$, $O_2^*$, $O_3^*$ and $O_4^*$ are given the same number. This way of numbering accounts with the periodicity setting of the discrete problem (see equations (3.34)-(3.35)).

Proposition 3.15 Let $M_h$ be the total number of discrete unknowns according to the previous numbering of cell-points and vertex-points. Set that: $\mathcal{D}_* = \mathcal{D}^{int} \cup \mathcal{D}(\Gamma_{So}) \cup \mathcal{D}(\Gamma_{We}) \cup \{D_{O_1^\star}\}$, where $\mathcal{D}^{int}$ is the set that consists of dual cells associated with interior vertexpoints and where $\mathcal{D}(\Gamma_{So})$ and $\mathcal{D}(\Gamma_{We})$ respectively denote the sets of dual cells associated with boundary vertexpoints different from cornerpoints and located in the $x_1$-axis and the $x_2$-axis. Of course $D_{O_1^\star}$ is the dual cell associated with $O_1^\star$. The two following discrete problems are equivalent:

$(DP_1)$ : Find $(\{\overline{\varphi}_P\}_{P\in\mathcal{P}}\ ;\ \{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}}) \in \mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{D}}$ such that equations (3.30)-(3.35) are satisfied

and

$(DP_2)$ : Find $(\{\overline{\varphi}_P\}_{P\in\mathcal{P}}\ ;\ \{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}_*}) \in \mathbb{R}^{M_h} \equiv \mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{D}_*}$ such that equations (3.30)-(3.33) are satisfied

where, for a given set of mesh elements $\mathcal{K}$, one has set:

$$\mathbb{R}^{\mathcal{K}} = \{ \quad \{v_K\}_{K\in\mathcal{K}}; \qquad \forall\, K \in \mathcal{K} \quad v_K \in \mathbb{R}\} . \quad \Box \tag{3.36}$$

For the sake of clarity of the exposition, $P \in \mathcal{P}$ will be used either for denoting a cellpoint or its associated number, idem for $D^\star \in \mathcal{V} \equiv \mathcal{D}$ concerning primal vertexpoints.

### 3.2.1 Some useful vector spaces and preliminary results

In view to algebraically address the linear discrete system (3.30)-(3.33), we introduce some adequate vector spaces. Recall that $\mathcal{A}$ denotes the auxiliary mesh i.e.

an intermediate mesh between the primal mesh $\mathcal{P}$ and the dual mesh $\mathcal{D}$ (see Figure 6, Section 2, for illustration). The main feature of $\mathcal{A}$ is that each auxiliary cell involves either one only cellpoint or one only vertexpoint. Consequently, we have the following relation between $\mathcal{A}$, $\mathcal{P}$ and $\mathcal{D}$:

$$\mathcal{A} = \{\Omega_P\}_{P\in\mathcal{P}} \cup \{\Omega_{D^\star}\}_{D^\star\in\mathcal{D}} \tag{3.37}$$

where $\Omega_P$ and $\Omega_{D^\star}$ denote the two kinds of auxiliary cells emerging from the definition of the mesh $\mathcal{A}$. These cells will play a key role for the definition of DDFV solutions to the system of equations (1.7), (1.8) and (3.2). Let us introduce the following vector spaces:

$$\mathbb{R}^{\mathcal{A}} = \{\ (\{v_P\}_{P\in\mathcal{P}}; \{v_{D^\star}\}_{D^\star\in\mathcal{D}})\ ; \quad \forall\, P\in\mathcal{P} \quad \forall\, D^\star\in\mathcal{D} \quad v_P,\ v_{D^\star}\in\mathbb{R}\} \tag{3.38}$$

$$\begin{aligned}\mathbb{R}^{\mathcal{A}}_{Perio} = \{\ (\{v_P\}; \{v_{D^\star}\}) \in \mathbb{R}^{\mathcal{A}}; \quad \forall\, D^\star \in \mathcal{D}(\Gamma_{So}) \cup \mathcal{D}(\Gamma_{We}) \quad v_{D^\star} \ = \ v_{D^{\star\perp}} \\ \text{and} \quad v_{O_1^\star} = v_{O_2^\star} = v_{O_3^\star} = v_{O_4^\star}\}\end{aligned} \tag{3.39}$$

Recall that $\mathcal{D}(\Gamma_{So})$ and $\mathcal{D}(\Gamma_{We})$ respectively denote the sets of boundary vertexpoints different from cornerpoints and located in the $x_1$-axis and the $x_2$-axis. In the sequel, we will sometimes do the following identification:

$$\mathbb{R}^{\mathcal{A}} \equiv \mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{D}}.$$

Note that $\mathbb{R}^{\mathcal{A}}_{Perio}$ is an $M_h-$dimensional subspace of $\mathbb{R}^{\mathcal{A}}$. So, the following identification will be sometimes considered:

$$\mathbb{R}^{\mathcal{A}}_{Perio} \equiv \mathbb{R}^{M_h} \equiv \mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{D}_*}.$$

Proposition 3.16 The matrix $\mathbb{M}_h$ associated with the discrete system $(DP_2)$ is singular. $\square$

**Proof.** The proof is elementary. Indeed it suffices to remark from the discrete system (3.30)-(3.33) that

$$\sum_{1\le j\le M_h} (\mathbb{M}_h)_{i\,j} = 0 \qquad\qquad \forall\, 1 \le i \le M_h\,.$$

Thus, the kernel $Ker\,(\mathbb{M}_h)$ of $\mathbb{M}_h$ defined by

$$Ker\,(\mathbb{M}_h) = \big\{ V_h \in \mathbb{R}^{M_h}\ ; \qquad \mathbb{M}_h\, V_h \ = \ 0 \big\}$$

involves a nonzero vector, namely $\mathbb{I}_h = (1, 1, ..., 1) \in \mathbb{R}^{M_h}$, and therefore $\mathbb{M}_h$ is singular. $\blacksquare$

We know from the previous proposition that the discrete problem $(DP_2)$ could get either no solution or an infinite number of solutions. Indeed, existence of solutions to this problem depends on whether the right-hand side to $(DP_2)$ is in the orthogonal of the kernel of $\mathbb{M}_h$. Our purpose now is to give a characterization of $Ker\,(\mathbb{M}_h)$, the kernel of $\mathbb{M}_h$ . Before that, we recall a result we need for proving the characterization of $Ker\,(\mathbb{M}_h)$.

Lemma 3.17 (see [NDM 13], Proposition 2.12)
Let $P$ and $L$ be cellpoints of two adjacent primary cells (in the DDFV framework exposed above) sharing $[E^\star; F^\star]$ as common edge. We mean by $I$ the edgepoint from $\mathcal{E}$ associated with $[E^\star; F^\star]$. Assume that $(\mathcal{P}; \mathcal{D})$ is an eligible system of meshes (in the sense of Definition 3.3) and that the conditions (1.9)-(1.10) are fulfilled. Then the permeability tensor $\mathbb{K}^{PL}$ of homogenization for the porous block $C_P \cup C_L$ defined as:

$$\mathbb{K}_{11}^{PL} = \left[\frac{a_h(K^P)\hat{a}_h(K^L)h_{E^\star F^\star(E^\star,I)}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right]$$

$$\mathbb{K}_{22}^{PL} = \left[\frac{W_h(P,L,I)}{h_{E^\star F^\star(E^\star,I)}(a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP})}\right]$$

$$\mathbb{K}_{21}^{PL} = \mathbb{K}_{12}^{PL} = \left[\frac{a_h(K^P)\widehat{b}_h(K^L)h_{IL}+b_h(K^P)\widehat{a}_h(K^L)h_{IP}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right]$$

is positive definite. Moreover, the least eigenvalue of $\mathbb{K}^{PL}$ is a mesh-independent strictly positive number. $\square$

Notice that in the reference [NDM 13], the positive definiteness of $\mathbb{K}^{PL}$ is shown in the second step of the proof of Proposition 2.12 while the property related to the least eigenvalue of $\mathbb{K}^{PL}$ is given in Lemma 3.4 therein.

Proposition 3.18 (Characterization of $Ker(\mathbb{M}_h)$)

- $Ker\,(\mathbb{M}_h) = \left\{V_h \in \mathbb{R}^{M_h} \equiv \mathbb{R}^{\mathcal{A}}_{Perio}\,; \quad (V_h)^t\,\mathbb{M}_h\,V_h = 0\right\};$

- $(V_h)^t\,\mathbb{M}_h\,V_h = 0 \quad \Leftrightarrow \quad V_h = \sigma_{\mathcal{P}}\,\mathbb{I}_{\mathcal{P}} + \sigma_{\mathcal{D}_*}\,\mathbb{I}_{\mathcal{D}_*} \qquad \forall\, V_h \in \mathbb{R}^{M_h} \equiv \mathbb{R}^{\mathcal{A}}_{Perio},$

where we have set:

$$(\mathbb{I}_{\mathcal{P}})_i = \begin{cases} 1 & if \quad i \in \mathcal{P} \\ 0 & otherwise \end{cases} \qquad and \qquad (\mathbb{I}_{\mathcal{D}_*})_i = \begin{cases} 1 & if \quad i \in \mathcal{D}_* \\ 0 & otherwise. \end{cases}$$

and where $\sigma_{\mathcal{P}}$ and $\sigma_{\mathcal{D}_*}$ are given real numbers. $\square$

**Proof.** The inclusion of $Ker(\mathbb{M}_h)$ in $\left\{V_h \in \mathbb{R}^{M_h}\ ;\quad (V_h)^t\, \mathbb{M}_h\, V_h \ =\ 0\right\}$ is trivial. Let us concentrate on the proof of the inclusion of $\left\{V_h \in \mathbb{R}^{M_h}\ ;\quad (V_h)^t\, \mathbb{M}_h\, V_h \ =\ 0\right\}$ in $Ker(\mathbb{M}_h)$. For this purpose, let $V_h = (\{V_P\}_{P\in\mathcal{P}}; \{V_{D^\star}\}_{D^\star\in\mathcal{D}_*}) \in \mathbb{R}^{M_h} \ \equiv\ \mathbb{R}^{\mathcal{P}\cup\mathcal{D}_*}$ be a vector such that

$$(V_h)^t\, \mathbb{M}_h\, V_h \ =\ 0.$$

where one has set

$$\mathcal{D}_* = \mathcal{D}^{int} \cup \mathcal{D}(\Gamma_{So}) \cup \mathcal{D}(\Gamma_{We}) \cup \{D_{O_1^\star}\}.$$

Therefore,

$$0 \overset{Assumption}{=} (V_h)^t\, \mathbb{M}_h\, V_h = \sum_{I\in\mathcal{E}^P\cap\mathcal{E}^{int}} V_P \left\{ \left[ \frac{a_h\big(K^P\big)\hat{a}_h\big(K^L\big)h_{A^\star B^\star}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}} \right] [V_P - V_L] + \right.$$

$$\left. \left[ \frac{a_h\big(K^P\big)\widehat{b}_h\big(K^L\big)h_{IL}+b_h\big(K^P\big)\widehat{a}_h\big(K^L\big)h_{IP}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}} \right] [V_{B^\star} - V_{A^\star}] \right\} +$$

$$+ \sum_{I\in\mathcal{E}^P\cap\mathcal{E}^{ext}} V_P \left\{ \left[ \frac{a_h\big(K^P\big)\hat{a}_h\Big(K^{Q(P,I^\perp)}\Big)h_{A^\star B^\star}}{a_h(K^P)h_{I^\perp Q(P,I^\perp)}+\widehat{a}_h\Big(K^{Q(P,I^\perp)}\Big)h_{IP}} \right] \left[V_P - V_{Q(P,I^\perp)}\right] + \right.$$

$$\left[ \frac{a_h\big(K^P\big)\hat{b}_h\Big(K^{Q(P,I^\perp)}\Big)h_{I^\perp Q(P,I^\perp)}+b_h\big(K^P\big)\hat{a}_h\Big(K^{Q(P,I^\perp)}\Big)h_{IP}}{a_h(K^P)h_{I^\perp Q(P,I^\perp)}+\widehat{a}_h\Big(K^{Q(P,I^\perp)}\Big)h_{IP}} \right] \times$$

$$\left. \times \quad [V_{B^\star} - V_{A^\star}] \right\} \ +$$

$$+ \sum_{I\in\mathcal{E}^{A^\star}} V_{A^\star} \left\{ \left[ \frac{b_h\big(K^P\big)\hat{a}_h\big(K^L\big)h_{IP}+a_h\big(K^P\big)\hat{b}_h\big(K^L\big)h_{IL}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}} \right] [V_P - V_L] + \right.$$

$$\left. \left[ \frac{W_h(P,L,I)}{h_{A^\star B^\star}(a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP})} \right] [V_{A^\star} - V_{B^\star}] \right\} \quad +$$

$$+ \sum_{Q^\star \in \left\{E^\star, E^{\star\perp}\right\}} \sum_{I \in \mathcal{E}^{Q^\star} \cap \mathcal{E}^{int}} V_{E^\star} \left\{ \left[ V_{A(Q^\star,I)} - V_{B(Q^\star,I)} \right] \times \right.$$

$$\left[ \frac{b_h\left(K^{A(Q^\star,I)}\right)\hat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)} + a_h\left(K^{A(Q^\star,I)}\right)\hat{b}_h\left(K^{B(Q^\star,I)}\right)h_{IB(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)} + \widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}} \right]$$

$$\left. + \left[ \frac{W_h(A(Q^\star,I),B(Q^\star,I),I)}{h_{Q^\star F^\star(Q^\star,I)}\left(a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)} + \widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}\right)} \right] \left[ V_{E^\star} - V_{F^\star(Q^\star,I)} \right] \right\}$$

$$+ \sum_{I \in \mathcal{E}^{E^\star E^{\star\perp}} \cap \mathcal{E}^{ext}} V_{E^\star} \left\{ \left[ V_{A(E^\star,I)} - V_{B\left(E^\star,I^\perp\right)} \right] \times \right.$$

$$\left[ \frac{b_h\left(K^{A(E^\star,I)}\right)\hat{a}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{IA(E^\star,I)} + a_h\left(K^{A(E^\star,I)}\right)\hat{b}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{I^\perp B\left(E^\star,I^\perp\right)}}{a_h\left(K^{A(E^\star,I)}\right)h_{I^\perp B\left(E^\star,I^\perp\right)} + \widehat{a}_h\left(K^{B\left(E^\star,I^\perp\right)}\right)h_{IA(E^\star,I)}} \right]$$

$$+ \left[ \frac{W_h(A(E^\star,I),B\left(E^\star,I^\perp\right),I)}{h_{E^\star F^\star(E^\star,I)}\left(a_h\left(K^{A(E^\star,I)}\right)h_{I^\perp B\left(E^\star,I^\perp\right)} + \widehat{a}_h\left(K^{B(E^\star,I)}\right)h_{IA(E^\star,I)}\right)} \right] \times$$

$$\left. \left[ V_{E^\star} - V_{F^\star(E^\star,I)} \right] \right\} +$$

$$+ \sum_{Q^\star \in \left\{O_i^\star\right\}_{i=1}^4} \sum_{I \in \mathcal{E}^{Q^\star} \cap \mathcal{E}^{int}} V_{O_1^\star} \left\{ \left[ \frac{b_h\left(K^{A(Q^\star,I)}\right)\hat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)} + \widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}} + \right. \right.$$

$$\left. + \frac{a_h\left(K^{A(Q^\star,I)}\right)\hat{b}_h\left(K^{B(Q^\star,I)}\right)h_{IB(Q^\star,I)}}{a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)} + \widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}} \right] \times \left[ V_{A(Q^\star,I)} - V_{B(Q^\star,I)} \right] +$$

$$+ \left[ \frac{W_h(A(Q^\star,I),B(Q^\star,I),I)}{h_{Q^\star F^\star(Q^\star,I)}\left(a_h\left(K^{A(Q^\star,I)}\right)h_{IB(Q^\star,I)} + \widehat{a}_h\left(K^{B(Q^\star,I)}\right)h_{IA(Q^\star,I)}\right)} \right] \times$$

$$\left. \times \left[ V_{O_1^\star} - V_{F^\star(Q^\star,I)} \right] \right\} +$$

$$+\sum_{I\in\Gamma^{12}\cap\mathcal{E}^{ext}}\ \sum_{\{i,j\}\in\Re, I\in\mathcal{E}^{O_i^\star}, I^\perp\in\mathcal{E}^{O_j^\star}} V_{O_1^\star}\Bigg\{\Bigg[\frac{b_h\left(K^{A(O_i^\star,I)}\right)\hat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}{a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}+$$

$$+\frac{a_h\left(K^{A(O_i^\star,I)}\right)\hat{b}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}}{a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}}\Bigg]\times\left[V_{A(O_i^\star,I)}-V_{B(O_j^\star,I^\perp)}\right]+$$

$$+\left[\frac{W_h(A(O_i^\star,I),B(O_j^\star,I^\perp),I)}{h_{O_i^\star T^\star\left(O_i^\star,I\right)}\left(a_h\left(K^{A\left(O_i^\star,I\right)}\right)h_{I^\perp B\left(O_j^\star,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(O_j^\star,I^\perp\right)}\right)h_{IA\left(O_i^\star,I\right)}\right)}\right]\times$$

$$\times\left[V_{O_1^\star}-V_{T^\star\left(O_i^\star,I\right)}\right]\Bigg\} \tag{3.40}$$

Developing and reordering the terms of the right-hand side of the previous equality yields

$$0=\sum_{I\,\equiv\,P\,|L\,\in\,\mathcal{E}^{int}}\left\{\left[\frac{a_h(K^P)\hat{a}_h(K^L)h_{E^\star(P,I)F^\star(P,I)}}{a_h(K^B)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right](V_P-V_L)^2+\right.$$

$$2\left[\frac{a_h(K^P)\widehat{b}_h(K^L)h_{IL}+b_h(K^P)\widehat{a}_h(K^L)h_{IP}}{a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP}}\right]\left(V_{E^\star(P,I)}-V_{F^\star(P,I)}\right)(V_P-V_L)+$$

$$\left.\left[\frac{W_h(P,L,I)}{h_{E^\star(P,I)F^\star(P,I)}(a_h(K^P)h_{IL}+\widehat{a}_h(K^L)h_{IP})}\right]\left(V_{E^\star(P,I)}-V_{F^\star(P,I)}\right)^2\right\}+$$

$$+\sum_{I\,\equiv\,A\|B(A,I^\perp)\in\mathcal{E}^{ext}}\left\{\left(V_A-V_{B\left(A,I^\perp\right)}\right)^2\times\right.$$

$$\times\left[\frac{a_h(K^A)\hat{a}_h\left(K^{B\left(A,I^\perp\right)}\right)h_{M^\star(A,I)N^\star(A,I)}}{a_h(K^A)h_{I^\perp B(A,I^\perp)}+\widehat{a}_h\left(K^{B(A,I^\perp)}\right)h_{IA}}\right]+$$

$$+2\left[\frac{a_h(K^A)\hat{b}_h\left(K^{B\left(A,I^\perp\right)}\right)h_{I^\perp B(A,I^\perp)}+b_h(K^A)\widehat{a}_h\left(K^{B\left(A,I^\perp\right)}\right)h_{IA}}{a_h(K^A)h_{I^\perp B(A,I^\perp)}+\widehat{a}_h\left(K^{B(A,I^\perp)}\right)h_{IA}}\right]\times$$

$$\times\left(V_A-V_{B\left(A,I^\perp\right)}\right)\left(V_{M^\star(A,I)}-V_{N^\star(A,I)}\right)+$$

$$\left.+\left[\frac{W_h(A,B\left(A,I^\perp\right),I)}{h_{N^\star(A,I)M^\star(A,I)}\left(a_h(K^A)h_{I^\perp B(A,I^\perp)}+\widehat{a}_h\left(K^{B(A,I^\perp)}\right)h_{IA}\right)}\right]\left(V_{M^\star(A,I)}-V_{N^\star(A,I)}\right)^2\right\}$$

where $I\equiv P\mid L$ means that $I$ is an edgepoint from $\mathcal{E}^{int}$, located at the interface between primary cells $C_P$ and $C_L$, and where $I\equiv A\parallel B(A,I^\perp)$ means that $I$ is a boundary edgepoint from $\mathcal{E}^{int}$ located on a boundary side $[M^\star; N^\star]$ of primary cell $C_A$, with $I^\perp$ naturally on the boundary side $[M^{\star\perp}; N^{\star\perp}]$ of primary cell $C_{B(A,I^\perp)}$. Notice that $[E^\star(P,I); F^\star(P,I)]$ materializes the interface between primary cells $C_P$ and $C_L$.

In the previous equation is made the convention that the orthogonal projection of a corner vertexpoint is well defined whenever it is seen as a point of a given side of $\Omega$. For example, $O_3^{\star\perp} = O_2^\star$ when $O_3^\star$ is considered as a point from the side $[(0,b);(a,b)]$ of $\Omega$. However, $O_3^{\star\perp} = O_4^\star$, when $O_3^\star$ is viewed as a point from the side $[(a,0);(a,b)]$ of $\Omega$. From the previous equation, it follows that

$$\begin{aligned}
0 = \sum_{I\,\equiv\,P\,|L\in\mathcal{E}^{int}} & \Big\{ K_{11}^{P,L(P,I)} \left[V_P - V_{L(P,I)}\right]^2 + K_{22}^{P,L(P,I)} \left[V_{E^\star(P,I)} - V_{F^\star(P,I)}\right]^2 \quad + \\
& + \quad 2K_{12}^{P,L(P,I)} \left(V_{E^\star(P,I)} - V_{F^\star(P,I)}\right)\left(V_P - V_{L(P,I)}\right)\Big\} \quad + \\
& + \sum_{I\,\equiv\,A\|B(A,I^\perp)\in\mathcal{E}^{ext}} \Big\{ K_{11}^{A,B\left(A,I^\perp\right)} \left[V_A - V_{B\left(A,I^\perp\right)}\right]^2 \quad + \\
& + \quad 2\,K_{12}^{A,B\left(A,I^\perp\right)} \left(V_A - V_{B\left(A,I^\perp\right)}\right)\left(V_{M^\star(A,I)} - V_{N^\star(A,I)}\right) \quad + \\
& + \quad K_{22}^{A,B\left(A,I^\perp\right)} \left(V_{M^\star(A,I)} - V_{N^\star(A,I)}\right)^2 \Big\}
\end{aligned} \tag{3.41}$$

where we have set

$$K_{11}^{P,L(P,I)} = \left[\frac{a_h\left(K^P\right)\hat{a}_h\left(K^{L(P,I)}\right)h_{E^\star(P,I)F^\star(P,I)}}{a_h(K^P)h_{IL(P,I)}+\widehat{a}_h\left(K^{L(P,I)}\right)h_{IP}}\right]$$

$$K_{22}^{P,L(P,I)} = \left[\frac{W_h(P,L(P,I),I)}{h_{E^\star(P,I)F^\star(P,I)}\left(a_h(K^P)h_{IL(P,I)}+\widehat{a}_h\left(K^{L(P,I)}\right)h_{IP}\right)}\right]$$

$$\begin{aligned}
K_{21}^{P,L(P,I)} &= K_{12}^{P,L(P,I)} = \\
&= \left[\frac{a_h\left(K^P\right)\widehat{b}_h\left(K^{L(P,I)}\right)h_{IL(P,I)}+b_h\left(K^P\right)\widehat{a}_h\left(K^{L(P,I)}\right)h_{IP}}{a_h(K^P)h_{IL(P,I)}+\widehat{a}_h\left(K^{L(P,I)}\right)h_{IP}}\right]
\end{aligned}$$

$$K_{11}^{A,B\left(A,I^\perp\right)} = \left[\frac{a_h\left(K^A\right)\hat{a}_h\left(K^{B\left(A,I^\perp\right)}\right)h_{M^\star N^\star}}{a_h(K^A)h_{I^\perp B\left(A,I^\perp\right)}+\widehat{a}_h\left(K^{B\left(A,I^\perp\right)}\right)h_{IA}}\right]$$

$$K_{22}^{A,B(A,I^{\perp})} = \left[\frac{W_h(A,B(A,I^{\perp}),I)}{h_{M^{\star}N^{\star}}\left(a_h(K^A)h_{I^{\perp}B(A,I^{\perp})}+\widehat{a}_h\left(K^{B(A,I^{\perp})}\right)h_{IA}\right)}\right]$$

$$K_{12}^{A,B(A,I^{\perp})} = K_{21}^{A,B(A,I^{\perp})} =$$

$$= \left[\frac{a_h(K^A)\hat{b}_h\left(K^{B(A,I^{\perp})}\right)h_{I^{\perp}B(A,I^{\perp})}+b_h(K^A)\widehat{a}_h\left(K^{B(A,I^{\perp})}\right)h_{IA}}{a_h(K^A)h_{I^{\perp}C(A,I^{\perp})}+\widehat{a}_h\left(K^{B(A,I^{\perp})}\right)h_{IA}}\right]$$

According to Lemma 3.17, $K^{P,L(P,I)}$ and $K^{A,B(A,I^{\perp})}$ are (symmetric) positive definite matrices and moreover, there exists $\lambda_{min}$ a mesh-independent, strictly positive real number that minimizes both of least eigenvalues of $K^{P,L(P,I)}$ and $K^{A,B(A,I^{\perp})}$. We then deduce from equation (3.41) that

$$0 \geq \sum_{I\equiv P\,|L\in\mathcal{E}^{int}} \lambda_{min}\left\{[V_P-V_L]^2+\left[V_{E^{\star}(P,I)}-V_{F^{\star}(P,I)}\right]^2\right\} +$$

$$+ \sum_{I\equiv A\|B(A,I^{\perp})\in\mathcal{E}^{ext}} \lambda_{min}\left\{\left[V_A-V_{B(A,I^{\perp})}\right]^2+\left[V_{M^{\star}(A,I)}-V_{N^{\star}(A,I)}\right]^2\right\} \quad (3.42)$$

It is obvious from the previous inequality that the space $\{V_h\in\mathbb{R}^{M_h}\,;\quad (V_h)^t\mathbb{M}_hV_h = 0\}$ is included in the one spanned by the vectors $\mathbb{I}_{\mathcal{P}}$ and $\mathbb{I}_{\mathcal{D}_*}$ defined by

$$(\mathbb{I}_{\mathcal{P}})_i = \begin{cases}1 & if \quad i\in\mathcal{P} \quad i.e. \quad i \text{ } is\ a\ cellpoint\ number\\ 0 & otherwise\end{cases}$$

and

$$(\mathbb{I}_{\mathcal{D}_*})_i = \begin{cases}1 & if \quad i\in\mathcal{D}_*\\ 0 & otherwise.\end{cases}$$

Remarking that $\mathbb{I}_{\mathcal{P}}$ and $\mathbb{I}_{\mathcal{D}_*}$ are two (linearly independent) eigenvectors with zero as corresponding eigenvalue with respect to the matrix $\mathbb{M}_h$, the proof of Proposition 3.18 is completed. ■

REMARK 3.19 Note that the dimension of the space $Ker(\mathbb{M}_h)$ is equal to 2. This information plays a key-role for uniqueness conditions investigated in the next section. □

### 3.2.2 Existence and conditions for uniqueness of a solution to $(DP_2)$

Let us start with an existence result for the discrete problem $(DP_2)$ which involves the system of equations (3.30)-(3.33).

PROPOSITION 3.20 (Existence)

Under the assumption (1.11), the discrete system $(DP_2)$ possesses an infinite number of solutions. □

**Proof.** Just remark that the condition (1.11) ensures that the right-hand side of the system of equations (3.30)-(3.33) is a vector of $\mathbb{R}^{M_h}$ orthogonal to $Ker(\mathbb{M}_h)$. ■

Recall that for the continuous problem (1.7)-(1.8), the assumptions (1.9)-(1.11) ensure the existence of a family of variational solutions i.e. set of functions $\varphi$ living in the space

$$\mathcal{H} = \left\{ v \in H^1(\Omega);\ v\,(.,0) = v\,(.,b) \ \ in \ \ [0,a] \text{ and } \ v\,(0,.) = v\,(a,.) \ \ in \ \ [0,b] \right\},$$

and such that

$$\int_\Omega K(x) grad\varphi(x).gradv(x)\, dx = \int_\Omega f(x)v(x)\, dx \qquad \forall v \in \mathcal{H}.$$

The uniqueness of a solution to this continuous problem is got from the subspace of $\mathcal{H}$ made up of functions $v$ satisfying the following condition:

$$\int_\Omega v(x)\, dx \quad = \quad 0. \tag{3.43}$$

This condition makes obvious the necessity of associating a discrete function $v_h$ (defined almost everywhere in $\Omega$) with any vector $(\{v_P\}_{P\in\mathcal{P}}; \{v_{D^\star}\}_{D^\star\in\mathcal{D}})$ from $\mathbb{R}^{\mathcal{A}}_{perio}$.

- **Basic discrete function spaces.** Denote by $E(\mathbb{R}^{\mathcal{A}}_{perio})$ the space of such discrete functions $v_h$ and let $\mathbb{I}_{\mathbf{S}}$ be the characteristic function of a subset $\mathbf{S}$ of $\Omega$ i.e. $\mathbb{I}_{\mathbf{S}}(x) = 1$ if $x \in \mathbf{S}$ and 0 otherwise. We define the space $E(\mathbb{R}^{\mathcal{A}}_{perio})$ as :

$$\begin{aligned} E(\mathbb{R}^{\mathcal{A}}_{Perio}) \ &= \ \{v : \Omega \to \mathbb{R};\quad \exists\,(\{v_P\}_{P\in\mathcal{P}}; \{v_{D^\star}\}_{D^\star\in\mathcal{D}}) \in \mathbb{R}^{\mathcal{A}}_{Perio} \ \ such\ that \\ v(x) \ &= \ \sum_{P\in\mathcal{P}} v_P \mathbb{I}_{\Omega_P}(x) \quad + \quad \sum_{D^\star\in\mathcal{D}} v_{D^\star}\mathbb{I}_{\Omega_{D^\star}}(x) \qquad a.e.\ \ in\ \ \Omega\} \end{aligned} \tag{3.44}$$

where $\Omega_P$ and $\Omega_{D^\star}$ are two generic notations of the two kinds of auxiliary cells from $\mathcal{A}$. Let us introduce the following mappings (viewed as projections in some sense):

$$v_h \in E(\mathbb{R}^{\mathcal{A}}_{Perio}) \quad \longmapsto \quad \Pi^{\mathcal{P}}[v_h](x) \equiv v_h^{\mathcal{P}}(x) = \sum_{P\in\mathcal{P}} v_P \mathbb{I}_{C_P}(x) \ \in \ E(\mathbb{R}^{\mathcal{P}}) \tag{3.45}$$

$$v_h \in E(\mathbb{R}^{\mathcal{A}}_{Perio}) \quad \longmapsto \quad \Pi^{\mathcal{D}}[v_h](x) \equiv v_h^{\mathcal{D}}(x) = \sum_{D^\star\in\mathcal{D}} v_{D^\star} \mathbb{I}_{C_{D^\star}}(x) \ \in \ E(\mathbb{R}^{\mathcal{D}}) \tag{3.46}$$

where $E(\mathbb{R}^{\mathcal{P}})$ and $E(\mathbb{R}^{\mathcal{D}})$ are function spaces respectively associated with vector spaces $\mathbb{R}^{\mathcal{P}}$ and $\mathbb{R}^{\mathcal{D}}$ (see relation (3.36)) for the definition of these vector spaces).

- **Looking for uniqueness conditions.** Since the dimension of $Ker(\mathbb{M}_h)$ is equal to 2, we should look for two discrete analogues of (3.43) (namely quadrature formulas), linearly independent, that should ensure uniqueness for a solution to the discrete problem $(DP_2)$. These discrete analogues are defined as :

$$\int_\Omega v_h^{\mathcal{P}}(x)\,dx \;=\; 0 \qquad \textit{and} \qquad \int_\Omega v_h^{\mathcal{D}}(x)\,dx \;=\; 0$$

In other words,

$$\sum_{P\in\mathcal{P}} mes(C_P)\,v_P \;=\; 0 \qquad \textit{and} \qquad \sum_{D^\star\in\mathcal{D}} mes(C_{D^\star})\,v_{D^\star} \;=\; 0$$

PROPOSITION 3.21 Define two quadrature expressions $\mathbb{Q}^{\mathcal{P}}(.)$ and $\mathbb{Q}^{\mathcal{D}}(.)$ on the discrete function space $E(\mathbb{R}^{\mathcal{A}})$ as follows: for all $v_h = (\{v_P\};\{v_{D^\star}\}) \in E(\mathbb{R}^{\mathcal{A}})$

$$\mathbb{Q}^{\mathcal{P}}(v_h) \;=\; \sum_{P\in\mathcal{P}} mes(C_P)\,v_P \quad \textit{and} \quad \mathbb{Q}^{\mathcal{D}}(v_h) \;=\; \sum_{D^\star\in\mathcal{D}} mes(C_{D^\star})\,v_{D^\star}\;.$$

Then, $\mathbb{Q}^{\mathcal{P}}(.)$ and $\mathbb{Q}^{\mathcal{D}}(.)$ are linearly independent linear forms on $E(\mathbb{R}^{\mathcal{A}})$. $\square$

**Proof.** Let $\lambda_{\mathcal{P}}$ and $\lambda_{\mathcal{D}}$ be two real numbers such that $\lambda_{\mathcal{P}}\mathbb{Q}^{\mathcal{P}}(.) + \lambda_{\mathcal{D}}\mathbb{Q}^{\mathcal{D}}(.) = 0$ on $E(\mathbb{R}^{\mathcal{A}})$, that is,

$$\lambda_{\mathcal{P}}\,\mathbb{Q}^{\mathcal{P}}(v_h) \;+\; \lambda_{\mathcal{D}}\,\mathbb{Q}^{\mathcal{D}}(v_h) \;=\; 0 \qquad \forall\, v_h \in E(\mathbb{R}^{\mathcal{A}}).$$

Taking $v_h$ to be successively $\sum\limits_{P\in\mathcal{P}} \mathbb{I}_{\Omega_P}$ and $\sum\limits_{D^\star\in\mathcal{D}} \mathbb{I}_{\Omega_{D^\star}}$, one easily see that $\lambda_{\mathcal{P}} = \lambda_{\mathcal{Q}} = 0$. This proves the proposition. ■

We have gathered all the ingredients for existence and uniqueness of a solution to the discrete problem $(DP_2)$.

PROPOSITION 3.22 (Existence and Uniqueness)
The problem that consists in finding $(\{\overline{\varphi}_P\}_{P\in\mathcal{P}};\{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}}) \in \mathbb{R}^{\mathcal{A}}_{Perio}$ such that the equations (3.30)-(3.33) are satisfied possesses a unique solution if condition (1.11) is fulfilled and if

$$\sum_{P\in\mathcal{P}} mes(C_P)\,\overline{\varphi}_P \;=\; \sum_{D^\star\in\mathcal{D}} mes(C_{D^\star})\,\overline{\varphi}_{D^\star} \quad = \quad 0. \tag{3.47}$$

$\square$

**Proof.** We already know from Proposition 3.20 that under the condition (1.11), the discrete system (3.30)-(3.33) possesses an infinite number of solutions in $\mathbb{R}^{\mathcal{A}}_{Perio}$. Moreover, in the proof of Proposition 3.18, we have shown the fact that

$$(V_h)^t\mathbb{M}_h\,V_h \;\geq\; 0 \qquad \forall\, V_h \in \mathbb{R}^{\mathcal{A}}_{Perio} \tag{3.48}$$

So, to end the proof of Proposition 3.22 , we just need to prove the positive definiteness of $\mathbb{M}_h$ over the subspace $\mathbb{R}^{\mathcal{A}}_{Perio,\mathbf{0}}$ of $\mathbb{R}^{\mathcal{A}}_{Perio}$ made up of vectors $V_h$ satisfying the conditions (3.47). For this purpose, it suffices to show the assertion:

$$(V_h)^t\mathbb{M}_h\,V_h \;=\; 0 \quad \Rightarrow \quad V_h = 0 \qquad \forall\, V_h \in \mathbb{R}^{\mathcal{A}}_{Perio,\mathbf{0}}.$$

The second part of Proposition 3.18 lets this assertion be trivial. ■

## 4 Stability result and error estimations

We start with some preliminaries as discrete version of mean Poincaré inequality that plays a key role for stability and error estimate investigations.

### 4.1 Preliminaries and a Stability Result

We start with endowing the vector spaces $\mathbb{R}^{\mathcal{P}} \equiv E(\mathbb{R}^{\mathcal{P}})$ and $\mathbb{R}^{\mathcal{D}} \equiv E(\mathbb{R}^{\mathcal{D}})$ with the following discrete norms.

$$\| v_h^{\mathcal{P}} \|_{\mathcal{P}}^2 = \sum_{I\equiv P\,|L\in\mathcal{E}^{int}} [v_P - v_L]^2 \; + \sum_{I\equiv A\|B\in\mathcal{E}^{ext}} [v_A - v_B]^2 \; + \\ + \int_\Omega [v_h^{\mathcal{P}}(x)]^2\,dx \tag{4.1}$$

$$\| v_h^{\mathcal{P}} \|_{\mathcal{D}}^2 = \sum_{I\equiv P\,|L\in\mathcal{E}^{int}} \left[v_{E^\star(P,I)} - v_{F^\star(P,I)}\right]^2 + \sum_{I\equiv A\|B\in\mathcal{E}^{ext}} \left[v_{M^\star(A,I)} - v_{N^\star(A,I)}\right]^2 + \\ + \int_\Omega [v_h^{\mathcal{D}}(x)]^2\,dx \tag{4.2}$$

Recall that $I \equiv P \mid L \in \mathcal{E}^{int}$ means $I$ is an interior edgepoint from $\mathcal{E}$, shared by adjacent primal cells $P$ and $L$. Recall also that $I \equiv A \parallel B \in \mathcal{E}^{ext}$ means $I$ is a boundary edgepoint from $\mathcal{E}$, shared by boundary primal cells $A$ and $B$ in the periodic sense (see relation (3.14) and comment that follows).

Definition 4.1 For $v_h \in E(\mathbb{R}^{\mathcal{A}})$ and $\phi \in L^1(\Omega)$, we set :

$$| v_h^{\mathcal{P}} |_{\mathcal{P}}^2 = \sum_{I\equiv P\,|L\in\mathcal{E}^{int}} [v_P - v_L]^2 \; + \sum_{I\equiv A\|B\in\mathcal{E}^{ext}} [v_A - v_B]^2 \; ;$$

$$| v_h^{\mathcal{D}} |_{\mathcal{D}}^2 = \sum_{I\equiv P\,|L\in\mathcal{E}^{int}} \left[v_{E^\star(P,I)} - v_{F^\star(P,I)}\right]^2 + \sum_{I\equiv A\|B\in\mathcal{E}^{ext}} \left[v_{M^\star(A,I)} - v_{N^\star(A,I)}\right]^2 \; ;$$

$$< \phi > = \frac{1}{mes(\Omega)} \int_\Omega \phi(x)\,dx. \quad \square$$

We now give a key ingredient for stability analysis of our DDFV solution.

Lemma 4.2 (Discrete mean Poincaré inequality)
Let $v_h \in E(\mathbb{R}^{\mathcal{A}})$. If the system $(\mathcal{P};\mathcal{D})$ is an eligible system of meshes in the sense of Definition 3.3, then the following inequalities hold:

$$\| v_h^{\mathcal{P}} - < v_h^{\mathcal{P}} > \|_{L^2(\Omega)}^2 \leq \; C \mid v_h^{\mathcal{P}} \mid_{\mathcal{P}}^2 \tag{4.3}$$

$$\| v_h^{\mathcal{D}} - < v_h^{\mathcal{D}} > \|_{L^2(\Omega)}^2 \leq \; C \mid v_h^{\mathcal{D}} \mid_{\mathcal{D}}^2 \tag{4.4}$$

where $C$ is a strictly positive, mesh independent, real number. $\square$

**Proof.** Follow very closely the proof of Theorem 3.4 in [AO 12] (pages 10-11). ■

An immediate consequence of the previous lemma is that

$$||| v_h^{\mathcal{P}} |||_{\mathcal{P}} = \sqrt{| v_h^{\mathcal{P}} |_{\mathcal{P}}^2 + < v_h^{\mathcal{P}} >^2} \quad and \quad ||| v_h^{\mathcal{D}} |||_{\mathcal{D}} = \sqrt{| v_h^{\mathcal{D}} |_{\mathcal{D}}^2 + < v_h^{\mathcal{D}} >^2}$$

define discrete norms on $\mathbb{R}^{\mathcal{P}}$ and $\mathbb{R}^{\mathcal{D}}$ respectively. Identifying $\mathbb{R}^{\mathcal{A}}$ with $\mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{D}}$ allows to consider

$$\| v_h \|_{\mathcal{A}} = \sqrt{||| v_h^{\mathcal{P}} |||_{\mathcal{P}}^2 + ||| v_h^{\mathcal{D}} |||_{\mathcal{D}}^2} \tag{4.5}$$

as a discrete norm in $\mathbb{R}^{\mathcal{A}}$. Therefore, it follows that:

Proposition 4.3 Recall that $\mathbb{R}^{\mathcal{A}}_{Perio,\mathbf{0}}$ is defined as the subspace of $\mathbb{R}^{\mathcal{A}}_{Perio} \subset \mathbb{R}^{\mathcal{A}} \equiv E(\mathbb{R}^{A})$, made up of functions $v_h = (v_h^{\mathcal{P}}; v_h^{\mathcal{D}})$ such that $< v_h^{\mathcal{P}} > = < v_h^{\mathcal{D}} > = 0$. If the system $(\mathcal{P};\mathcal{D})$ is an eligible system of meshes in the sense of Definition 3.3, then:

- $$| v_h |_{\mathcal{A}} = \sqrt{| v_h^{\mathcal{P}} |_{\mathcal{P}}^2 + | v_h^{\mathcal{D}} |_{\mathcal{D}}^2} \qquad \forall v_h \in \mathbb{R}^{\mathcal{A}}_{Perio,\mathbf{0}}$$

defines a discrete norm on $\mathbb{R}^{\mathcal{A}}_{Perio,\mathbf{0}}$. Moreover, the following inequality holds:

- $$|| v_h^{\mathcal{P}} ||_{L^2(\Omega)}^2 + || v_h^{\mathcal{D}} ||_{L^2(\Omega)}^2 \leq C \ | v_h |_{\mathcal{A}}^2 \qquad \forall v_h \in \mathbb{R}^{\mathcal{A}}_{Perio,\mathbf{0}}$$

where $C$ is a strictly positive, mesh-independent, real number. □

Now we can give one of the main results of this section.

Proposition 4.4 (Stability result)
Keep in mind the main assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$ stated after Definition 3.1. On the other hand, assume that $(\mathcal{P};\mathcal{D})$ is an eligible system of meshes (in the sense of Definition 3.3) and that the conditions (1.9)-(1.10) are fulfilled.
Then, the unique solution $\overline{\varphi}_h \equiv (\{\overline{\varphi}_P\}_{P\in\mathcal{P}}; \{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}}) \in \mathbb{R}^{\mathcal{A}}_{Perio,\mathbf{0}}$ of the discrete system (3.30)-(3.33) satisfies the following inequality:

$$| \overline{\varphi}_h |_{\mathcal{A}} \leq \mathbf{C} \, \|f\|_{L^2(\Omega)} \tag{4.6}$$

where $\mathbf{C}$ is a strictly positive real number not depending on the spatial discretization. □

**Proof.** We start with setting (in vector notations): $\overline{\varphi}^{\mathcal{P}} = \{\overline{\varphi}_P\}_{P\in\mathcal{P}}$ and $\overline{\varphi}^{\mathcal{D}_*} = \{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}_*}$. So, $[\varphi^{\mathcal{P}}; \varphi^{\mathcal{D}_*}]^t \in \mathbb{R}^{M_h} \equiv \mathbb{R}^{\mathcal{P}} \times \mathbb{R}^{\mathcal{D}_*}$.

Multiplying (3.30) by $\overline{\varphi}_P$, (3.31) by $\overline{\varphi}_{A^\star}$, (3.32) by $\overline{\varphi}_{E^\star}$, (3.33) by $\overline{\varphi}_{O_1^\star}$ and summing leads to:

$$[\ \overline{\varphi}^{\mathcal{P}} \quad \overline{\varphi}^{\mathcal{D}_*}\ ]\, \mathbb{M}_h \begin{bmatrix} \overline{\varphi}^{\mathcal{P}} \\ \overline{\varphi}^{\mathcal{D}_*} \end{bmatrix} = [\ \overline{\varphi}^{\mathcal{P}} \quad \overline{\varphi}^{\mathcal{D}_*}\ ] \begin{bmatrix} F^{\mathcal{P}} \\ F^{\mathcal{D}_*} \end{bmatrix}$$

where $\mathbb{M}_h$ is the discrete system matrix and where $F^{\mathcal{P}}$ and $F^{\mathcal{D}_*}$ are defined as:

$$F^{\mathcal{P}} = \left\{ \int_{C_P} f(x)dx \right\}_{P\in\mathcal{P}}$$

and

$$F^{\mathcal{D}_*} = \left\{ \int_{C_{D^\star}} f(x)dx \right\}_{D^\star\in\mathcal{D}^{int}} \cup \left\{ \int_{C_{D^\star D^{\star\perp}}} f(x)dx \right\}_{D^\star\in\mathcal{D}(\Gamma_{So})\cup\mathcal{D}(\Gamma_{We})} \cup \left\{ \int_{\{C_{O_i^\star}\}_{i=1}^4} f(x)dx \right\}$$

Let us set

$$LHS = [\ \overline{\varphi}^{\mathcal{P}} \quad \overline{\varphi}^{\mathcal{D}_*}\ ]\,\mathbb{M}_h \begin{bmatrix} \overline{\varphi}^{\mathcal{P}} \\ \overline{\varphi}^{\mathcal{D}_*} \end{bmatrix} \quad and \quad RHS = [\ \overline{\varphi}^{\mathcal{P}} \quad \overline{\varphi}^{\mathcal{P}_*}\ ] \begin{bmatrix} F^{\mathcal{P}} \\ F^{\mathcal{D}_*} \end{bmatrix}$$

It follows from the proof of Proposition 3.18 that there exists a strictly positive number $\lambda_{min}$, mesh-independent, such that:

$$\lambda_{min} \mid \overline{\varphi}_h \mid_{\mathcal{A}}^2 \leq\ LHS\ =\ RHS \tag{4.7}$$

where

$$\overline{\varphi}_h(x)\ =\ \sum_{P\in\mathcal{P}} \overline{\varphi}_P \mathbb{I}_{\Omega_P}(x) \quad + \quad \sum_{D^\star\in\mathcal{D}} \overline{\varphi}_{D^\star} \mathbb{I}_{\Omega_{D^\star}}(x) \qquad a.e.\ \ in\ \ \Omega.$$

On the other hand, we have

$$\tfrac{1}{2}\{RHS\}^2 \leq \left| \sum_{P\in\mathcal{P}} \int_{C_P} f(x)\overline{\varphi}_P \right|^2 +$$

$$\left| \sum_{\substack{A^\star\in\mathcal{D} \\ such\ that\ \Gamma\cap C_{A^\star}=\emptyset}} \int_{C_{A^\star}} f(x)\overline{\varphi}_{A^\star} + \sum_{\substack{E^\star\in\mathcal{D} \\ such\ that\ \Gamma\cap C_{E^\star}\neq\emptyset}} \int_{C_{E^\star E^{\star\perp}}} f(x)\overline{\varphi}_{E^\star} + \sum_{i=1}^4 \int_{C_{O_i^\star}} f(x)\overline{\varphi}_{O_i^\star} \right|^2$$

The Cauchy-Schwartz's inequality ensures that:

$$\left| \sum_{P\in\mathcal{P}} \int_{C_P} f(x)\overline{\varphi}_P \right|^2 \leq \|f\|^2_{L^2(\Omega)} \left\|\overline{\varphi}_h^{\mathcal{P}}\right\|^2_{L^2(\Omega)}$$

and that

$$\left| \sum_{\substack{A^\star\in\mathcal{D} \\ such\ that\ \Gamma\cap C_{A^\star}=\emptyset}} \int_{C_{A^\star}} f(x)\overline{\varphi}_{A^\star} + \sum_{\substack{E^\star\in\mathcal{D} \\ such\ that\ \Gamma\cap C_{E^\star}\neq\emptyset}} \int_{C_{E^\star E^{\star\perp}}} f(x)\overline{\varphi}_{E^\star} + \sum_{i=1}^4 \int_{C_{O_i^\star}} f(x)\overline{\varphi}_{O_i^\star} \right|^2 =$$

$$\left| \sum_{D^\star\in\mathcal{D}} \int_{C_{D^\star}} f(x)\overline{\varphi}_{D^\star} \right|^2 \leq \|f\|^2_{L^2(\Omega)} \left\|\overline{\varphi}_h^{\mathcal{D}}\right\|^2_{L^2(\Omega)}$$

Therefore

$$RHS \leq \sqrt{2}\,\|f\|_{L^2(\Omega)} \left\{ \left\|\overline{\varphi}_h^{\mathcal{P}}\right\|^2_{L^2(\Omega)} + \left\|\overline{\varphi}_h^{\mathcal{D}}\right\|^2_{L^2(\Omega)} \right\}^{\frac{1}{2}} \tag{4.8}$$

It results from (4.7) and (4.8) that the proof is completed with the help of Proposition 4.3. ■

### 4.2 Error estimate results

We start with defining the notions of Error-function and Error-vector.

Definition 4.5 Let us define the Error-function $E_h$ as:

$$E_h(x) = \varphi(x) - \overline{\varphi}_h(x) \qquad a.e. \quad in \quad \Omega \tag{4.9}$$

where $\overline{\varphi}_h(.)$ is the DDFV approximate solution defined as

$$\overline{\varphi}_h(x) \;=\; \sum_{P\in\mathcal{P}} \overline{\varphi}_P \mathbb{I}_{\Omega_P}(x) \quad + \quad \sum_{D^\star\in\mathcal{D}} \overline{\varphi}_{D^\star} \mathbb{I}_{\Omega_{D^\star}}(x) \qquad a.e. \;\; in \;\; \Omega$$

and $\varphi(.)$ the exact solution to (1.7)-(1.8) such that $\int_\Omega \varphi(x)\, dx = 0$.

With this Error-function is associated the Error-vector denoted by $E^{\mathcal{A}}$ and defined by its components $(\{E_P = \varphi_P - \overline{\varphi}_P\}_{P\in\mathcal{P}}; \{E_{D^\star} = \varphi_{D^\star} - \overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}_\star})$. $\square$

#### 4.2.1 Estimates for the Error-Vector $E^{\mathcal{A}}$

Notice that if $E^{\mathcal{A}}$ is zero-vector, there is nothing to estimate as the approximate solution coincides with the exact solution at all cellpoints and vertexpoints. So, we suppose that $E^{\mathcal{A}} \neq 0$ in the sequel. In what follows we aim to look for estimates of $E^{\mathcal{A}}$ distinguishing two cases: (i) $E^{\mathcal{A}}$ belongs to $Ker[\mathbb{M}]$, where $\mathbb{M}$ is the matrix associated with the discrete problem $(DP_2)$; (ii) $E^{\mathcal{A}}$ belongs to the orthogonal of $Ker[\mathbb{M}]$.

- We first analyze the case $E^{\mathcal{A}} \in Ker[\mathbb{M}]$.

It follows from Proposition 3.18 that there exist two real constants $C^{\mathcal{P}}$ and $C^{\mathcal{D}}$ such that

$$E_P = C^{\mathcal{P}} \quad \forall\, P \in \mathcal{P} \qquad and \qquad E_{D^\star} = C^{\mathcal{D}} \quad \forall\, D^\star \in \mathcal{D}.$$

This means that

$$\varphi_P - \overline{\varphi}_P = C^{\mathcal{P}} \quad \forall\, P \in \mathcal{P} \quad and \quad \varphi_{D^\star} - \overline{\varphi}_{D^\star} = C^{\mathcal{D}} \quad \forall\, D^\star \in \mathcal{D}.$$

Since $(\{\overline{\varphi}_P\}_{P\in\mathcal{P}}; \{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}})$ lies in $\mathbb{R}^{\mathcal{A}}_{Perio,0}$ we can see that

$$mes(\Omega)\, C^{\mathcal{P}} \;=\; \sum_{P\in\mathcal{P}} mes(C_P)\varphi_P \tag{4.10}$$

and

$$mes(\Omega)\, C^{\mathcal{D}} \;=\; \sum_{D^\star\in\mathcal{D}} mes(C_{D^\star})\varphi_{D^\star}. \tag{4.11}$$

We assert that:

$$| C^{\mathcal{P}} | \leq C\, h \qquad and \qquad | C^{\mathcal{D}} | \leq C\, h.$$

Indeed, the Taylor expansion based upon the assumption (3.1) leads to

$$\int_{C_P} \varphi(x)\, dx \;=\; meas(C_P)\, \varphi_P \quad + \quad \int_{C_P} grad\, \varphi(\theta_{x,x_P}).(x - x_P)\, dx$$

Summing the two sides of this equality over $P \in \mathcal{P}$ and accounting with (4.10) and (3.2) yields

$$meas(\Omega) \mid C^{\mathcal{P}} \mid = \mid \sum_{P \in \mathcal{P}} \int_{C_P} grad\, \varphi(\theta_{x,x_P}).(x - x_P)\, dx \mid$$

We get from assumption (3.1) that:

$$| C^{\mathcal{P}} | \leq C\, h$$

where $C$ is a mesh-independent strictly positive constant, defined as

$$C^2 \;=\; \max_{s \in S}\{\max_{x \in \overline{\Omega}_s}[| \frac{\partial \varphi}{\partial x_1} |^2 + | \frac{\partial \varphi}{\partial x_2} |^2]\} \tag{4.12}$$

A similar reasoning shows that

$$| C^{\mathcal{D}} | \leq C\, h.$$

To conclude, we can say that if $E^{\mathcal{A}} \in Ker[\mathbb{M}]$ than

$$\| E^{\mathcal{A}} \|_\infty \leq \; C\, h, \tag{4.13}$$

where $C$ is a mesh-independent strictly positive real number.

- Let us now deal with the case $E^{\mathcal{A}}$ is a nonzero vector lying in the orthogonal of $Ker[\mathbb{M}]$.

Combining judiciously the equations (3.15), (3.20), (3.24), (3.28) with respectively the equations (3.30)-(3.33) leads to the following **Error System of equations**:

$$\mathbb{M}_h \begin{bmatrix} E^{\mathcal{P}} \\ E^{\mathcal{D}_*} \end{bmatrix} \;=\; \begin{bmatrix} T^{\mathcal{P}} \\ T^{\mathcal{D}_*} \end{bmatrix} \tag{4.14}$$

where $\mathbb{M}_h$ is the discrete system matrix and where $T^{\mathcal{P}}$ and $T^{\mathcal{D}_*}$ are Truncation Error sub-vectors defined as:

$$T^{\mathcal{P}} \;=\; \{T^P\}_{P \in \mathcal{P}} \qquad and \qquad T^{\mathcal{D}_*} \;=\; \{T^{D^\star}\}_{D^\star \in \mathcal{D}_\star}$$

with (according to (3.12), (3.21), (3.25) and (3.29)):

$$| T^P | \leq \; C\, h^2 \quad \forall\, P \in \mathcal{P} \qquad and \qquad | T^{D^\star} | \leq \; C\, h^2 \quad \forall\, D^\star \in \mathcal{D}_\star.$$

We assume that

$$\exists\, 0 < \alpha < \beta \quad \textit{such that} \quad \alpha\, h^2 \leq meas(M) \leq \beta\, h^2 \qquad \forall\, M \in \mathcal{M}. \tag{4.15}$$

Now all the ingredients are gathered for applying to the analysis of (4.14)) the same technique as the one used in [NDM 13] (see Proposition 3.9 in this reference). Of course is required an adaptation of this technique to periodic setting of the continuous and discrete problems analyzed in this work. This is the way we see that the error-vector $E^{\mathcal{A}}$ satisfies the following inequality

$$\mid E^{\mathcal{A}} \mid_{\mathcal{A}}^2 \leq C\, h \mid E^{\mathcal{A}} \mid_{\mathcal{A}} \tag{4.16}$$

where $C$ is a mesh-independent strictly positive real number. Remark that $\mid E^{\mathcal{A}} \mid_{\mathcal{A}} \neq 0$ as $E^{\mathcal{A}}$ is supposed to be a nonzero vector from the orthogonal of $Ker[\mathbb{M}]$. So, we have $\mid E^{\mathcal{A}} \mid_{\mathcal{A}} \leq C\, h$.

Let us summarize what precedes in what follows.

Proposition 4.6 (Estimates for the Error-vector and its interpolate)
The Error-vector $E^{\mathcal{A}} = (\{E_P\}_{P\in\mathcal{P}}; \{E_{D^\star}\}_{D^\star\in\mathcal{D}}) \in \mathbb{R}^{\mathcal{A}}$ meets the following estimates:

▷ If $E^{\mathcal{A}} \in Ker[\mathbb{M}]$ than

$$\| E^{\mathcal{A}} \|_\infty \leq C\, h \qquad \textit{and} \qquad \| \mathcal{I}(E^{\mathcal{A}}) \|_{L^P(\Omega)} \leq C\, h \quad \forall\, 1 \leq P \leq +\infty; \tag{4.17}$$

▷ If $E^{\mathcal{A}} \in (Ker[\mathbb{M}])^\perp$ than

$$\mid E^{\mathcal{A}} \mid_{\mathcal{A}} \leq C\, h \qquad \textit{and} \qquad \| \mathcal{I}(E^{\mathcal{A}}) \|_{L^2(\Omega)} \leq C\, h \tag{4.18}$$

where $C$ is a mesh-independent strictly positive real number and where $\mathcal{I}(E^{\mathcal{A}})$ is the interpolate of $E^{\mathcal{A}}$ defined as:

$$[\mathcal{I}(E^{\mathcal{A}})](x) = \sum_{P\in\mathcal{P}} E_P \mathbb{I}_{\Omega_P}(x) + \sum_{D^\star\in\mathcal{D}} E_{D^\star} \mathbb{I}_{\Omega_{D^\star}}(x) \;\; a.e. \;\; in \;\; \Omega \quad \square$$

Note that the last estimate is a consequence of the discrete Poincaré inequality given by Lemma 4.2.

### 4.2.2 $L^2$-Estimate for the Error-function

Let us start with the following preliminary results.

Lemma 4.7 Recall that the primal cells are convex polygons while the dual cells are connected are star-sharped with respect to their associated vertex-points. Under the assumption (3.1), the exact solution $\varphi$ to the model problem satisfies the following inequalities:

$$\| \varphi - \Pi^{\mathcal{P}} \varphi \|_{L^2(\Omega)} \leq C\, h \tag{4.19}$$

$$\| \varphi - \Pi^{\mathcal{D}} \varphi \|_{L^2(\Omega)} \leq C\, h \tag{4.20}$$

where $C$ denotes diverse mesh-independent strictly positive constants. $\square$

**Proof.** Since the exact solution $\varphi$ is in $C^2$ for any primal cell $C_P$ supposed to be convex (see assumption (3.1)), we get by Taylor expansion what follows:

$$| \varphi(x) - \varphi(x_P) |^2 = | \, grad \, \varphi(\theta_{x,x_P}) \, |^2 \, . \, | \, x - x_P \, |^2 \leq \, \widehat{C} \, h^2 \qquad \forall \, x \in C_P$$

where $\widehat{C}$ is defined as in (4.12). Integrating the two sides of the previous inequality over $C_P$ and summing over $P \in \mathcal{P}$ yields

$$\| \, \varphi - \Pi^P \varphi \, \|^2_{L^2(\Omega)} \leq \, C \, h^2$$

The proof of (4.19) is ended. The proof of (4.20) is similar and is carried out on all the dual cells $C_{D^\star}$ which are connected and star-shaped with respect to corresponding vertex-points $D^\star$. ■

LEMMA 4.8 Consider the linear operators $\Pi^{\mathcal{P}}$, $\Pi^{\mathcal{D}}$, $T^{\mathcal{P}}$ and $T^D$ introduced in what precedes. Under the assumption (3.1), the exact solution $\varphi$ to the model problem satisfies the following inequalities:

$$\| \, \Pi^{\mathcal{P}} \varphi - T^{\mathcal{P}} \varphi \, \|_{L^2(\Omega)} \leq \, C \, h \tag{4.21}$$

$$\| \, \Pi^{\mathcal{D}} \varphi - T^{\mathcal{D}} \varphi \, \|_{L^2(\Omega)} \leq \, C \, h \tag{4.22}$$

where $C$ denotes diverse mesh-independent strictly positive constants. □

**Proof.** Recall that

$$\Pi^{\mathcal{P}} \varphi(x) \; = \; \sum_{P \in \mathcal{P}} \varphi_P \mathbb{I}_{C_P}(x) \qquad a.e. \quad in \quad \Omega$$

and define:

$$T^{\mathcal{P}} \varphi(x) \; = \; \sum_{P \in \mathcal{P}} [\varphi_P - C^{\mathcal{P}}] \mathbb{I}_{C_P}(x) \qquad a.e. \quad in \quad \Omega$$

From these two identities, it is clear that

$$\Pi^{\mathcal{P}} \varphi(x) \; - \; T^{\mathcal{P}} \varphi(x) \; = \; C^{\mathcal{P}} \qquad a.e. \quad in \quad \Omega$$

We deduce that

$$\int_{C_P} | \, \Pi^{\mathcal{P}} \varphi(x) \; - \; T^{\mathcal{P}} \varphi(x) \, |^2 \; dx \; = \; meas(C_P) \, [C^{\mathcal{P}}]^2$$

Summing over $P \in \mathcal{P}$ leads to

$$\int_{\Omega} | \, \Pi^{\mathcal{P}} \varphi(x) \; - \; T^{\mathcal{P}} \varphi(x) \, |^2 \; dx \; = \; meas(\Omega) \, [C^{\mathcal{P}}]^2$$

The inequality (4.21) obviously follows from the same arguments as those developed in the lines between equations (4.10) and (4.13). Note that the inequality (4.22) is got with a similar reasoning. ■

Let us give now the last (but not the least) theoretical result of this work.

PROPOSITION 4.9 (Function-Error Estimate)
Recall that $\Omega_P$ and $\Omega_{D^\star}$ denote the two kind of auxiliary mesh elements (see Subsection 2.1.2 for the definition of the auxiliary mesh $\mathcal{A}$). We consider the cellwise constant function $\overline{\varphi}_h$ defined almost everywhere in $\Omega$ as:

$$\overline{\varphi}_h(x) = \sum_{P\in\mathcal{P}} \overline{\varphi}_P \mathbb{I}_{\Omega_P}(x) + \sum_{D^\star\in\mathcal{D}} \overline{\varphi}_{D^\star} \mathbb{I}_{\Omega_{D^\star}}(x)$$

where $\{\overline{\varphi}_P\}_{P\in\mathcal{P}}$ and $\{\overline{\varphi}_{D^\star}\}_{D^\star\in\mathcal{D}}$ are components of the vector-solution to the Discrete Duality Finite Volume model represented by equations (3.30)-(3.35). Recall that $\varphi$ denotes the exact solution to the model problem (1.7)-(1.8) such that $\int_\Omega \varphi(x)\, dx = 0$. Then the error-function $E_h(x) = \varphi(x) - \overline{\varphi}_h(x)$ a.e. in $\Omega$ satisfies the following $L^2-$estimate:

$$\| E_h \|_{L^2(\Omega)} = \| \varphi - \overline{\varphi}_h \|_{L^2(\Omega)} \leq C\, h \tag{4.23}$$

where $C$ is a mesh-independent strictly positive real number. $\square$

**Proof.** Set:

$$\Omega^{\mathcal{P}} = \bigcup_{P\in\mathcal{P}} \Omega_P \qquad and \qquad \Omega^{\mathcal{D}} = \bigcup_{D^\star\in\mathcal{D}} \Omega_{D^\star}$$

It is clear that $\Omega^{\mathcal{P}}$ and $\Omega^{\mathcal{D}}$ define a partition of $\Omega$ in the sense that

$$\Omega^{\mathcal{P}} \cap \Omega^{\mathcal{D}} = \emptyset \qquad and \qquad \overline{\Omega}^{\mathcal{P}} \cup \overline{\Omega}^{\mathcal{D}} = \overline{\Omega}.$$

$$\| \varphi - \overline{\varphi}_h \|^2_{L^2(\Omega)} = \int_{\Omega^{\mathcal{P}}} | \varphi - \overline{\varphi}_h |^2 \ dx + \int_{\Omega^{\mathcal{D}}} | \varphi - \overline{\varphi}_h |^2 \ dx \leq$$

$$\leq \int_\Omega | \varphi - \Pi^{\mathcal{P}}\overline{\varphi}_h |^2 \ dx + \int_\Omega | \varphi - \Pi^{\mathcal{D}}\overline{\varphi}_h |^2 \ dx$$

The triangle inequality ensures that the first integral of the right-hand side of the previous inequality obeys what follows

$$\frac{1}{3}\int_\Omega | \varphi - \Pi^{\mathcal{P}}\overline{\varphi}_h |^2 \ dx \leq \| \varphi - \Pi^{\mathcal{P}}\varphi \|^2_{L^2(\Omega)} + \| \Pi^{\mathcal{P}}\varphi - T^{\mathcal{P}}\varphi \|^2_{L^2(\Omega)} +$$

$$+ \| T^{\mathcal{P}}\varphi - \Pi^{\mathcal{P}}\overline{\varphi}_h \|^2_{L^2(\Omega)}$$

Thanks to Lemma 4.7 and Lemma 4.8 we have

$$\frac{1}{3}\int_\Omega | \varphi - \Pi^{\mathcal{P}}\overline{\varphi}_h |^2 \ dx \leq C\, h^2 + \| \widetilde{E}^{\mathcal{P}}_h \|^2_{L^2(\Omega)} \tag{4.24}$$

and

$$\frac{1}{3}\int_\Omega | \varphi - \Pi^{\mathcal{D}}\overline{\varphi}_h |^2 \ dx \leq C\, h^2 + \| \widetilde{E}^{\mathcal{D}}_h \|^2_{L^2(\Omega)} \tag{4.25}$$

where $\widetilde{E}^{\mathcal{P}}_h$ and $\widetilde{E}^{\mathcal{D}}_h$ are two cellwise constant functions identified with the two subvectors $\widetilde{E}^{\mathcal{P}}$ and $\widetilde{E}^{\mathcal{D}}$ of the vector $\widetilde{E}^{\mathcal{A}}$ whose components are defined as:

$$(\widetilde{E}^{\mathcal{A}})_P = \varphi_P - \overline{\varphi}_P - C^{\mathcal{P}} \qquad \forall\, P \in \mathcal{P}$$
$$(\widetilde{E}^{\mathcal{A}})_{D^\star} = \varphi_{D^\star} - \overline{\varphi}_{D^\star} - C^{\mathcal{D}} \qquad \forall\, D^\star \in \mathcal{D}$$

where $C^{\mathcal{P}}$ and $C^{\mathcal{D}}$ are chosen such that the vector $\widetilde{E}^{\mathcal{A}}$ lies in the space $\mathbb{R}^{\mathcal{A}}_{Perio,0}$. From summation side by side of inequalities (4.24) and (4.25), we see that

$$\frac{1}{3} \parallel \varphi - \overline{\varphi} \parallel^2_{L^2} \leq C\, h^2 + \parallel \widetilde{E}^{\mathcal{A}} \parallel^2_{L^2(\Omega)}$$

Remarking that $| \widetilde{E}^{\mathcal{A}} |_{\mathcal{A}} = | E^{\mathcal{A}} |_{\mathcal{A}}$ and thanks to Propositions 4.3 and 4.6, the proof is ended.

■

# 5 Numerical results

In this section, we confirm with some numerical tests the theoretical results we have proven in the previous section. For each test-case, a uniform rectangular mesh is used with different levels of refinement materialized by successive decreasing values assigned to the mesh size $h$. In the following test-cases we have taken: $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and $\frac{1}{64}$.

## 5.1 Notations

In the following tables $ndu$ denotes the number of discrete unknowns. Recall that $| \,.\, |_{\mathcal{A}}$ stands for the discrete $H^1_0-$norm (see Proposition 4.3) and let $\parallel . \parallel_{1,\mathcal{A}}$ denote the discrete $H^1-$norm defined as:

$$\parallel v \parallel_{1,\mathcal{A}} = \left[ | v |^2_{\mathcal{A}} + \parallel v \parallel^2_{L^2(\Omega)} \right]^{\frac{1}{2}} \tag{5.1}$$

When the exact solution is available, the relative discrete $L^2-$norm of the error for the exact potential is defined as:

$$erL^2 = \left( \frac{\sum\limits_{P\in\mathcal{P}} meas\,(C_P)\,[\varphi\,(x^p) - \bar{\varphi}_P]^2}{\sum\limits_{P\in\mathcal{P}} meas\,(C_P)\,[\varphi\,(x^p)]^2} \right)^{\frac{1}{2}}$$

Defined by analogy, $er- \parallel . \parallel_{1,\mathcal{A}}$ is the relative discrete $H^1-$norm of the error for the exact potential. For a given mesh, since different levels $i$ of refinement may be considered, we denote by $erL^2(i)$ and $er- \parallel . \parallel_{1,\mathcal{A}}\,(i)$ the corresponding relative discrete $L^2-$norm and relative discrete $H^1-$norm of the exact potential. Let us set for any integer $i$ (with $i \geq 2$):

$$raL^2(i) = -2\frac{ln\left[erL^2(i)\right] - ln\left[erL^2(i-1)\right]}{ln\left[ndu(i)\right] - ln\left[ndu(i-1)\right]}$$

We define $ra- \parallel . \parallel_{1,\mathcal{A}} (i)$, for any integer $i \geq 2$ , with the same relation as for $raL^2(i)$, except that in this relation $erL^2(i)$ is replaced with $er- \parallel . \parallel_{1,\mathcal{A}} (i)$. On the other hand, $erflm$ stands for $L^\infty - norm$ of the error on the exact mean value of the flux across the mesh edges. So we have:

$$erflm = \max_{\sigma\in\mathcal{E}} \frac{1}{meas\,(\sigma)} \left|Q_\sigma - \bar{Q}_\sigma\right|$$

where $Q_\sigma$ and $\bar{Q}_\sigma$ are respectively the exact and the approximate flux across $\sigma$ which is either a primal edge or a dual edge. The symbol $erL^\infty$ denotes the pressure error for $L^\infty-$norm.

Let $ocv[\times\times\times]$ denote the error order of convergence to zero for the norm $[\times\times\times]$ which may be taken to be one of the following norms $\parallel . \parallel_{L^2}$, $\parallel . \parallel_{1,\mathcal{A}}$ and $\parallel . \parallel_{L^\infty}$. The first two norms are used for pressure error estimates while the last one serves for the flux error estimate. The quantity $ocv[\times\times\times]$ is defined as:

$$ocv[\times\times\times] = \frac{ln(er[\times\times\times](imax)) - ln(er[\times\times\times](imax-1))}{ln(h_{imax}) - ln(h_{imax-1})}$$

were $imax$ is the maximum level of refinement of a given primal mesh.

At last, we denote by $slope[\times\times\times]$ the error order of convergence to zero for the norm $[\times\times\times]$ when computed with the least-square method. The quantity $slope[\times\times\times]$ obeys the relation:

$$ln\left[er[\times\times\times]\,(i)\right] = slope[\times\times\times]\; ln\left[h(i)\right] \;+\; \nu,$$

where $\nu$ is a real number.

## 5.2 Test problems

We consider two groups of test problems. In the first group, the medium is homogeneous and so is spatially periodic. In the second group, the medium is taken to be heterogeneous and spatially periodic.

### 5.2.1 Group I

We consider in this group two cases: a homogeneous isotropic medium and a homogeneous anisotropic medium.

**Data from Test problem 1:** The medium $\Omega = ]0;1[\times]0;1[$ is associated with the following diffusion coefficient matrix.

$$K(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.2}$$

The exact solution to equations (1.7)-(1.8) such that:

$$\int_\Omega u(x)\,dx \;=\; 0 \tag{5.3}$$

is

$$u(x,y) \;=\; sin(2\pi x)sin(2\pi y).$$

Note that it is easy to determine the corresponding (source term) function $f$ and check that this function satisfies the compatibility condition (1.11). The same remark remains valid for all the test problems analyzed in this section.

| i | h(i) | ndu | er$L^\infty(i)$ | er$L^2(i)$ | er-$\|\,.\,\|_{1,\mathcal{A}}$ | erflm | ra$L^2(i)$ | ra-$\|\,.\,\|_{1,\mathcal{A}}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1/4 | 41 | 0.2711 | 0.3197 | 1.9039 | 0.9348 | - | - |
| 2 | 1/8 | 145 | 0.0530 | 0.0532 | 0.4636 | 0.3000 | 2.349 | 2.401 |
| 3 | 1/16 | 245 | 0.0130 | 0.0130 | 0.1153 | 0.0793 | 2.122 | 2.100 |
| 4 | 1/32 | 2113 | 0.0032 | 0.0032 | 0.02878 | 0.0201 | 2.091 | 2.048 |
| 5 | 1/64 | 8321 | 8.03E-04 | 8.03E-04 | 0.007145 | 0.005 | 2.017 | 2.033 |
| 6 | 1/128 | 33025 | 2.0E-04 | 2.0E-04 | - | - | 2.010 | - |
| $slope[\times\times\times]$ | | | 2.08 | 2.13 | 2.01 | 1.89 | | |
| $ocv[\times\times\times]$ | | | 2.0031 | 2.0031 | 2.01 | 2.007 | | |

Table 1: Convergence rate of flux and pressure errors for $L^\infty - norm$, $L^2 - norm$ and discrete $H^1 - norm$ in Test-problem number 1.

According to Table 1 above, DDFV computations of the approximate solution to Test problem number one display a quadratic convergence for $L^2-$norm and discrete $H^1 - norm$ concerning the pressure. The same rate of convergence is observed for $\|\,.\,\|_\infty$ $-$norm concerning the interface fluxes. The quadratic convergence for $L^2-$norm and discrete $H^1 - norm$ numerically obtained in the homogeneous diffusion analyzed here is not in contradiction with our theoretical results (see Proposition 4.6). Indeed, the order one of convergence proven in Proposition 4.6 is based upon much weaker assumptions on the diffusion coefficient which is supposed to be piecewise constant. Note that similar results have been obtained for Dirichlet and Neumann boundary conditions by diverse authors in FVCA5 Benchmark (see [HH 08]).

**Data from Test-problem 2:** Let $\Omega = ]0;1[\times]0;1[$ be a square with the following diffusion coefficient:

$$K(x,y) = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \tag{5.4}$$

The exact solution to (1.7)-(1.8) such that

$$\int_\Omega u(x,y)dx = 0 \tag{5.5}$$

is

$$u(x,y) \;=\; sin(2\pi x)\,cos(2\pi y).$$

Results from Table 2 confirm the comment we have developed about the homogeneous flow in $\Omega$ exposed in Table 1. The result trends do not change even if one considers homogeneous media with contrasting diffusion coefficients like

| i | h(i) | ndu | er-$L^\infty$ | er$L^2$ | er-$\|.\|_{1,\mathcal{A}}$ | erflm | ra$L^2$ | ra$\|.\|_{1,\mathcal{A}}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1/4 | 41 | 1 | 0.3996 | 1.9160 | 0.9348 | - | - |
| 2 | 1/8 | 145 | 0.7071 | 0.1347 | 0.4826 | 0.300 | 1.703 | 2.28 |
| 3 | 1/16 | 245 | 0.3827 | 0.0362 | 0.1208 | 0.0793 | 1.984 | 2.12 |
| 4 | 1/32 | 2113 | 0.1951 | 9.2E-03 | 3.0139E-02 | 0.0201 | 2.021 | 2.062 |
| 5 | 1/64 | 8321 | 0.0980 | 2.3E-03 | 0.7558E-02 | 0.0050 | 2.013 | 2.028 |
| $slope[\times\times\times]$ | | | 0.955 | 1.87 | 2.01 | 1.89 | | |
| $ocv[\times\times\times]$ | | | 1.005 | 2.009 | 1.995 | 2.007 | | |

Table 2: Convergence rate of flux and pressure errors for $L^\infty - norm$, $L^2 - norm$ and discrete $H^1 - norm$ in Test-problem number 2.

$$K(x) = \begin{bmatrix} 1 & 10 \\ 10 & 1000 \end{bmatrix} \tag{5.6}$$

#### 5.2.2 Group II

Now we consider a nonhomogeneous isotropic and anisotropic porous domain $\Omega$. Computation results are presented in Table 3 below.

| Parameters | | | Test problem 3 | | | Test problem 4 | | |
|---|---|---|---|---|---|---|---|---|
| i | h(i) | ndu | er$L^\infty$ | er$L^2$ | er-$\|.\|_{1,\mathcal{A}}$ | er$L^\infty$ | er$L^2$ | er-$\|.\|_{1,\mathcal{A}}$ |
| 1 | 1/4 | 41 | 2.00 | 0.7551 | 3.4897 | 4.9834 | 0.5182 | 2.716 |
| 2 | 1/8 | 145 | 1.311 | 0.193 | 1.2036 | 0.2748 | 0.2134 | 1.379 |
| 3 | 1/16 | 245 | 0.6619 | 0.060 | 0.5813 | 0.1212 | 0.0835 | 0.549 |
| 4 | 1/32 | 2113 | 0.316 | 0.0235 | 0.3223 | 0.101 | 0.0439 | 0.3209 |
| 5 | 1/64 | 8321 | 0.1502 | 0.0085 | 0.1616 | 0.0694 | 0.0209 | 0.1595 |
| $slope[\times\times\times]$ | | | 0.90 | 1.48 | 1.007 | 0.496 | 0.97 | 0.98 |
| $ocv[\times\times\times]$ | | | 1.07 | 1.52 | 0.94 | 0.54 | 1.08 | 1.02 |

Table 3: Convergence rate of pressure error for $L^\infty-$norm, $L^2 - norm$ and for discrete $H^1-$norm in Test-problems 3 and 4.

**Data for Test-problem 3:** We have taken $\Omega = ]0;1[\times]0;1[$ associated with the following diffusion coefficient:

$$K(x,y) = \begin{bmatrix} 2 & sin(\pi x)sin(\pi y) \\ sin(\pi x)sin(\pi y) & 1 \end{bmatrix}$$

The exact solution to the system (1-1)-(1-2) such that

$$\int_\Omega u(x,y)\,dx = 0 \tag{5.7}$$

is what follows:

$$u(x,y) \;=\; 2\,sin[\pi(x+y)]\,cos[\pi(x+y)].$$

**Data for Test-problem 4:** Let $\Omega$ be the square $]0;1[\times]0;1[$ associated with the following diffusion coefficient:

$$K(x) = \begin{cases} \begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix} & if \quad x \in [\frac{1}{4};\frac{3}{4}]\times[\frac{1}{4};\frac{3}{4}] \\ \\ \begin{bmatrix} 750 & 0 \\ 0 & 2000 \end{bmatrix} & otherwise. \end{cases} \tag{5.8}$$

The exact solution to the system (1-1)-(1-2) such that

$$\int_\Omega u(x,y)dx = 0 \tag{5.9}$$

is what follows:

$$u(x,y) \;=\; 2\,sin[\pi(x+y)]cos[\pi(x+y)]. \quad \square$$

### 5.2.3 Concluding remarks

The numerical experiments were performed on uniform square meshes and have shown that:
(i) For isotropic homogeneous media one gets a quadratic convergence of the approximate pressure for $L^\infty-$norm, $L^2-$norm and discrete $H^1-$norm as well. The same convergence rate is observed concerning the flux for the vector max-norm. These results are in accordance with those published in the Finite Volume literature (see test-problem number 1).
(ii) For anisotropic homogeneous media one can see that the rate of convergence remains globally the same, except for the $L^\infty-$norm that displays a linear convergence (see test-problem number 2).
(iii) For spatially varying diffusion coefficients $D$, the cell mean value of $D$ is taken to be the cell-center diffusion coefficients. Approximations of pressure are performed with the order one of convergence for $L^\infty-$norm and discrete $H^1-$norm while a 1.50 order of convergence is got for $L^2-$norm (see test-problem number 3).
(iv) For piecewise constant diffusion coefficients $D$, the same results as for spatially varying diffusion coefficients are obtained except for the $L^\infty-$norm that led to a 0.5 order of convergence (see test-problem number 4).

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