Topological expansion in Dynkin type isomorphisms for matrix valued fields

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TOPOLOGICAL EXPANSION IN DYNKIN TYPE ISOMORPHISMS FOR MATRIX VALUED FIELDS

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Abstract. We consider Gaussian fields of symmetric or Hermitian matrices over an electrical network, and describe how Dynkin type isomorphisms with random walks for these fields make appear topological expansions encoded by ribbon graphs. A particular case of this, in continuum, is that of a Dyson's Brownian motion for β equal to 1 or 2. We further consider matrix valued Gaussian fields twisted by an orthogonal or unitary connection. In this case the isomorphisms make appear traces of holonomies of the connection along random walk loops parametrized by cycles of ribbon graphs.

1. Introduction

There is a family of identities, usually referred to as "isomorphism theorems", that relate the square of a Gaussian free field (GFF) to the occupation times of symmetric Markov process (random walk on an electrical network, Brownian motion, etc.). One usually cites the Dynkin's isomorphism [Dyn84a, Dyn84b], who expressed it for

$$\mathbb{E}\Big[\prod_{k=1}^{2q}\phi(x_k)F(\phi^2/2)\Big],$$

ϕ being a GFF. Dynkin's identity is related to earlier works of Symanzik [Sym65, Sym66, Sym69] in Euclidean Quantum Field Theory and of Brydges, Fröhlich and Spencer [BFS82] on spin systems. Subsequently, other versions of isomorphism theorems appeared, such as Eisenbaum's isomorphism [Eis95], generalized Ray-Knight theorems [EKM$^+$00], Sznitman's isomorphism for random interlacements [Szn12a] and Le Jan's isomorphism for Markovian loop-soups [LJ11]. We refer to [MR06, Szn12b] for a survey on these.

More recently, new directions have been opened in the topic, such as relating the sign of a scalar free field to Markovian trajectories [Lup16, Zha18], connecting in dimension 2 the isomorphism theorems to the Schramm-Loewner Evolution [Lup19, QW15, ALS18] or isomorphisms relating hyperbolic or spherical fields to interacting random walks [BHS18, BHS19] (see also [DSZ10, ST15]).

A particular extension that we will use in this paper is that by Kassel and Lévy who considered vector valued GFFs twisted by an orthogonal or unitary connection [KL16]. In this setting isomorphism theorems make appear the holonomy of the connection along the random walks. Holonomies along random walk or Brownian loops have been also studied in [LJ17, LJ16].

In this paper we will consider fields of random Gaussian matrices, symmetric or Hermitian, on an electrical network. These are matrix valued GFFs. The matrix above any vertex of the network is proportional to a GOE matrix, in the symmetric case, or a GUE matrix, in the Hermitian case. One context where such fields have been studied is that of Dyson's Brownian motion (dimension one, continuum).

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Here we will write an isomorphism for

$$\Big\langle \Big(\prod_{l=1}^{m(\alpha)} \operatorname{Tr}\Big(\prod_{k=\alpha_1+\dots+\alpha_{l-1}+1}^{\alpha_1+\dots+\alpha_l} \Phi(x_k) \Big) \Big) F(\operatorname{Tr}(\Phi^2)/2) \Big\rangle_{\beta,n}, \tag{1.1}$$

where Φ is the matrix valued GFF, $\beta \in \{1,2\}$, n is the size of the matrices, $\alpha_1, \dots, \alpha_{m(\alpha)}$ are positive integers with $|\alpha| := \alpha_1 + \dots + \alpha_{m(\alpha)}$ even, and $x_1, \dots, x_{|\alpha|}$ vertices on the network. By taking the x_k-s equal inside each of the trace, we get a product of symmetric polynomials in the eigenvalues. By expanding the traces and the product above, one can write a Dynkin's isomorphism for each of the terms of the sum. However, one gets many different terms that give identical contributions, many terms with contributions that cancel out, being of opposite sign, and many terms that give zero contribution. By regrouping the terms surviving to cancellation into powers of n, one gets a combinatorial structure known as topological expansion. The terms of the expansion correspond to ribbon graphs with $m(\alpha)$ vertices, obtained by pairing and gluing $|\alpha|$ ribbon half-edges. Each gluing may be straight or twisted. The power of n is then given by the number of cycles formed by the borders of the ribbons. It can be also expressed in terms of genera of compact surfaces, orientable or not.

The topological expansion has been introduced by 't Hooft for the study of Quantum Chromodynamics [tH74], and further developed by Brézin, Itzykson, Parisi and Zuber [BIPZ78, IZ80]. Nowadays there is a broad, primarily physics literature on this topic. In particular, topological expansion of one matrix or several matrix integrals is used for the enumeration of maps on surfaces and other graphical objects [BIZ80, Zvo97, LZ04, Eyn16]. Compared to the case of one matrix integrals, where each ribbon edge comes only with a scalar weight, in our setting each ribbon edge will be associated to a measure on random walk paths between two vertices x_k and $x_{k'}$ on the network. For a gentle introduction to the topological expansion we refer to Zvonkin [Zvo97], and for a more advanced one to the lecture notes by Eynard, Kimura and Ribault [EKR18].

We will further extend our framework and consider matrix valued free fields twisted by a connection of orthogonal ($\beta = 1$) or unitary ($\beta = 2$) matrices. We rely for this on results of Kassel and Lévy for twisted vector valued GFFs [KL16]. If $\widehat{\Phi}$ is the twisted matrix valued GFF and $(\widehat{\lambda}_1, \dots, \widehat{\lambda}_n)$ its fields of eigenvalues, then the isomorphism for

$$\Big\langle \prod_{l=1}^{m(\alpha)} \Big(\sum_{i=1}^{n} \widehat{\lambda}_i(x_{\alpha_l})^{\alpha_l} \Big) F\Big(\frac{1}{2} \sum_{i=1}^{n} \widehat{\lambda}_i^2 \Big) \Big\rangle_{\beta,n}$$

involves a topological expansion where instead of n to the power the number of cycles in a ribbon graph appears a product of traces of holonomies of the connection, one per each cycle of the ribbon graph. The holonomies are taken along loops made of concatenated random walk paths.

This paper is organized as follows. In Section 2 we present our main results and the background necessary for stating them. The proofs are postponed to Section 3. In Subsection 2.1 we recall the original Dynkin's isomorphism, and in Subsection 2.2 that of Kassel and Lévy. In Subsection 2.3 we introduce the ribbon graphs and the related combinatorial objects. In Subsection 2.4 we recall how the moments of the GOE and GUE (one matrix integrals) are expressed using ribbon graphs. In Subsection 2.5 we introduce the matrix valued free fields (without connection). We state our results for these. Theorem 2.6 gives the isomorphism for 1.1. Theorem 2.7 contains a more general version where one intertwines deterministic matrices in the product of $\Phi(x_k)$-s. In Subsection 2.6 we give the isomorphism for matrix valued GFFs twisted by a connection (Theorem 2.8). Section 3 contains the proofs, those of Theorems 2.6 and 2.7 in Subsection 3.1, that of Theorem 2.8 in Subsection 3.2. Section 4 contains an extension of Theorem 2.6 to the Dyson's Brownian motion with $\beta \in \{1,2\}$ (Proposition 4.1).

2. Preliminaries and main statements

2.1. **Dynkin's isomorphism.** Let $\mathcal{G} = (V, E)$ be an undirected connected graph, with V finite or countable, and all vertices $x \in V$ of finite degree. We do not allow multiple edges or self-loops. Edges $\{x, y\} \in E$ are endowed with conductances $C(x, y) = C(y, x) > 0$. There also may be a killing measure $(\kappa(x))_{x \in V}$, with $\kappa(x) \geqslant 0$. $\mathcal{G}$ will be further referred to as electrical network. Let X_t be the Markov jump process to nearest neighbors with jump rates given by the conductances. X_t is also killed by κ. We assume that X_t is transient, which implies in case V is finite that κ is not uniformly zero. Let $\zeta \in (0, +\infty]$ be the first time X_t explodes to $+\infty$ or gets killed by κ. If this does not happen in finite time, $\zeta = +\infty$.

Let $(G(x, y))_{x,y \in V}$ be the Green's function:

$$G(x, y) = G(y, x) = \mathbb{E}_{X_0 = x}\Big[\int_0^\zeta \mathbf{1}_{X_t = y} dt\Big].$$

Let $p_t(x, y)$ be the transition probabilities of $(X_t)_{0 \leqslant t < \zeta}$. Then $p_t(x, y) = p_t(y, x)$ and

$$G(x, y) = \int_0^{+\infty} p_t(x, y) dt.$$

Let $\mathbb{P}_t^{x,y}$ be the bridge probability measure from x to y, where one conditions by $t < \zeta$. Let $\mu_{x,y}$ be the following measure on paths from x to y in finite time:

$$\mu^{x,y}(d\gamma) = \int_0^{+\infty} \mathbb{P}_t^{x,y}(d\gamma) p_t(x, y) dt. \tag{2.1}$$

$\mu^{x,y}$ has total mass $G(x, y)$. The image of $\mu^{x,y}$ by time reversal is $\mu^{y,x}$.

In general, for a path γ and $x \in V$, $L(\gamma)$ will denote the occupation field of γ,

$$L(\gamma)(x) = \int_0^{T(\gamma)} \mathbf{1}_{\gamma(t) = x} dt,$$

where $T(\gamma)$ is the life-time of the path.

The Gaussian free field (GFF) $(\phi(x))_{x \in V}$ will denote here the centered Gaussian process with covariance

$$\mathbb{E}[\phi(x)\phi(y)] = G(x, y).$$

If V is finite, the distribution of $(\phi(x))_{x \in V}$ is given by

$$\frac{1}{(2\pi \det G)^{\frac{1}{2}\operatorname{Card}(V)}} \exp\Big(-\frac{1}{2}\sum_{x \in V} \kappa(x)\varphi(x)^2 - \frac{1}{2}\sum_{\{x,y\} \in E} C(x, y)(\varphi(y) - \varphi(x))^2\Big) \prod_{x \in V} d\varphi(x). \tag{2.2}$$

The Dynkin's isomorphism [Dyn84a, Dyn84b] relates the square of the GFF $(\phi(x)^2)_{x \in V}$ and the measures on paths $\mu^{x,y}$. This is related to earlier works of Symanzik [Sym65, Sym66, Sym69] and Brydges, Fröhlich and Spencer [BFS82]. For more on isomorphism theorems, see [MR06, Szn12b].

Theorem 2.1 (Dynkin's isomorphism [Dyn84a, Dyn84b])**.** *Let $q \in \mathbb{N}\backslash\{0\}$, $x_1, x_2, \dots, x_{2q} \in V$ and F a bounded measurable function $\mathbb{R}^V \to \mathbb{R}$. Then*

$$\mathbb{E}\Big[\prod_{k=1}^{2q} \phi(x_k) F(\phi^2/2)\Big] = \sum_{\substack{\text{partitions of}\\ \{1,\dots,2q\}\\ \text{in pairs}\\ \{\{a_1,b_1\},\dots,\{a_q,b_q\}\}}} \int_{\gamma_1,\dots\gamma_q} \mathbb{E}\Big[F(\phi^2/2 + L(\gamma_1) + \dots + L(\gamma_q))\Big] \prod_{i=1}^{q} \mu^{x_{a_i}, x_{b_i}}(d\gamma_i),$$

where the sum runs over the $(2q)!/(2^q q!)$ partitions of $\{1, \dots, 2q\}$ in pairs.

2.2. **Dynkin's isomorphism for the Gaussian free field twisted by a connection.** In [KL16] Kassel and Lévy introduced the vector valued GFF twisted by an orthogonal/unitary connection, and generalized the Dynkin's isomorphism to this case. Here we will do a less abstract, more computational presentation of the same object.

For simplicity, we will assume that the graph $\mathcal{G}$ is finite and that the killing measure $(\kappa(x))_{x \in V}$ is not uniformly zero. Let $d \in \mathbb{N}$, $d \geqslant 2$. We consider that each undirected edge in E consists of two directed edges of opposite direction. We consider a family of $d \times d$ orthogonal matrices $(\mathcal{U}(x,y))_{\{x,y\} \in E}$, with

$$\mathcal{U}(y,x) = \mathcal{U}(x,y)^{\mathsf{T}} = \mathcal{U}(x,y)^{-1}, \quad \forall \{x,y\} \in E,$$

M^{T} denoting the transpose of a matrix M. $(\mathcal{U}(x,y))_{\{x,y\} \in E}$ is our *connection* on the vector bundle with base space $\mathcal{G}$ and fiber $\mathbb{R}^d$.

Given a nearest neighbor oriented discrete path $\gamma = (y_1, y_2, \dots, y_j)$, the *holonomy* of $\mathcal{U}$ along γ is the product

$$\mathsf{hol}^{\mathcal{U}}(\gamma) = \mathcal{U}(y_1,y_2)\mathcal{U}(y_2,y_3)\dots\mathcal{U}(y_{j-1},y_j).$$

If the path γ is a nearest neighbor path parametrized by continuous time, and does only a finite number of jumps, the holonomy $\mathsf{hol}^{\mathcal{U}}(\gamma)$ is defined as the holonomy along the discrete skeleton of γ. $\overleftarrow{\gamma}$ will denote the time-reversal of a path γ. We have that

$$\mathsf{hol}^{\mathcal{U}}(\overleftarrow{\gamma}) = \mathsf{hol}^{\mathcal{U}}(\gamma)^{\mathsf{T}} = \mathsf{hol}^{\mathcal{U}}(\gamma)^{-1}. \tag{2.3}$$

A connection is said *flat*, if for any closed path γ (i.e. loop, i.e. path having the same starting and endpoint), $\mathsf{hol}^{\mathcal{U}}(\gamma) = I_d$ (the $d \times d$ identity matrix).

The Green's function associated to the connection $\mathcal{U}$, $G^{\mathcal{U}}$ is a function from $V \times V$ to $\mathbb{R}^d \otimes \mathbb{R}^d$ (i.e. the $d \times d$ matrices with real entries), with the entries given by

$$G^{\mathcal{U}}_{ij}(x,y) = \int_{\gamma} \mathsf{hol}^{\mathcal{U}}_{ij}(\gamma)\mu^{x,y}(d\gamma), \quad x,y \in V,\ i,j \in \{1,\dots,d\},$$

where the measure on paths $\mu^{x,y}(d\gamma)$ is given by (2.1). Since the image of $\mu^{x,y}$ by time reversal is $\mu^{y,x}$, and because of (2.3), we have that

$$G^{\mathcal{U}}_{ij}(x,y) = G^{\mathcal{U}}_{ji}(y,x), \qquad G^{\mathcal{U}}_{ij}(x,x) = G^{\mathcal{U}}_{ji}(x,x).$$

i.e. $G^{\mathcal{U}}(x,y)^{\mathsf{T}} = G^{\mathcal{U}}(y,x)$ and $G^{\mathcal{U}}(x,x)$ is symmetric. $G^{\mathcal{U}}$ can be seen as a symmetric linear operator on $(\mathbb{R}^d)^V$. It is positive definite (see Proposition 3.14 in [KL16]). $\det G^{\mathcal{U}}$ we will denote the determinant of this operator.

The $\mathbb{R}^d$-valued Gaussian free field on $\mathcal{G}$ twisted by the connection $\mathcal{U}$ is a random Gaussian function $\widehat{\phi} : V \to \mathbb{R}^d$ ($\widehat{\phi}(x) = (\widehat{\phi}_1(x), \dots, \widehat{\phi}_d(x))$) with the distribution given by

$$\dfrac{1}{Z^{\mathcal{U}}_{GFF}} \exp\Big(-\dfrac{1}{2}\sum_{x \in V} \kappa(x)\|\widehat{\varphi}(x)\|^2 - \dfrac{1}{2}\sum_{\{x,y\} \in E} C(x,y)\|\widehat{\varphi}(x) - \mathcal{U}(x,y)\widehat{\varphi}(y)\|^2\Big) \prod_{x \in V}\prod_{i=1}^{d} d\widehat{\varphi}(x)_i,$$

where $\|\cdot\|$ is the usual $\mathbb{L}^2$ norm on $\mathbb{R}^d$ and

$$Z^{\mathcal{U}}_{GFF} = (2\pi \det G^{\mathcal{U}})^{\frac{1}{2} d\, \mathrm{Card}(V)}.$$

Note that if $\{x,y\} \in E$,

$$\|\widehat{\varphi}(x) - \mathcal{U}(x,y)\widehat{\varphi}(y)\|^2 = \|\widehat{\varphi}(y) - \mathcal{U}(y,x)\widehat{\varphi}(x)\|^2.$$

Remark 2.2. The free field $\widehat{\phi}$ above is the same object as the covariant Gaussain free field in Section 5 in [KL16]. Our construction is more concrete but less intrinsic than that of [KL16]. Indeed, we implicitly made a choice of an orthonormal basis on each fiber of the vector bundle. However, in a continuum setting, on manifolds, such a choice (called section) cannot be in general done in a continuous way.

We have that $\mathbb{E}[\widehat{\phi}] \equiv 0$. As for the covariance structure, we have (see Proposition 5.1 in [KL16]):

$$\mathbb{E}[\widehat{\phi}_i(x)\widehat{\phi}_j(y)] = G^{\mathcal{U}}_{ij}(x,y).$$

Remark 2.3. If the connection $\mathcal{U}$ is flat, the field $\widehat{\phi}$ can be reduced to d i.i.d. scalar GFFs as follows. Fix $x_0 \in V$. For $x \in V$, choose $\gamma^{x_0,x}$ a nearest neighbor path from x_0 to x. Define

$$\mathfrak{U}^{x_0}(x) = \mathsf{hol}^{\mathcal{U}}(\gamma^{x_0,x}), \quad x \in V.$$

Since the connection $\mathcal{U}$ is flat, $\mathfrak{U}^{x_0}$ does not depend on the particular choice of $(\gamma^{x_0,x})_{x\in V}$. It is the only function satisfying $\mathfrak{U}^{x_0}(x_0) = I_d$ and

$$\forall x, y \in V \text{ such that } \{x,y\} \in E, \quad \mathfrak{U}^{x_0}(x)^{-1}\mathfrak{U}^{x_0}(y) = \mathcal{U}(x,y).$$

Then the coordinates of the field $(\mathfrak{U}^{x_0}(x)\widehat{\phi}(x))_{x\in V}$ are d i.i.d. copies of the scalar GFF with distribution (2.2).

In [KL16], Theorems 6.1 and 8.3, Kassel and Lévy gave a Dynkin-type isomorphism for GFFs twisted by connections.

Theorem 2.4 (Kassel-Lévy [KL16]). *Let $q \in \mathbb{N}\backslash\{0\}$, $x_1, x_2, \dots, x_{2q} \in V$, $J(1), J(2), \dots, J(2q) \in \{1,\dots,d\}$ and F a bounded measurable function $\mathbb{R}^V \to \mathbb{R}$. Then*

$$\mathbb{E}\Big[\prod_{k=1}^{2q}\widehat{\phi}_{J(k)}(x_k)F(\|\widehat{\phi}\|^2/2)\Big]$$
$$= \sum_{\substack{\text{partitions of}\\ \{1,\dots,2q\}\\ \text{in pairs}\\ \{\{a_1,b_1\},\dots,\{a_q,b_q\}\}}} \int_{\gamma_1,\dots\gamma_q} \mathbb{E}\Big[F(\|\widehat{\phi}\|^2/2 + L(\gamma_1) + \dots + L(\gamma_q))\Big]\prod_{i=1}^{q}\mathsf{hol}^{\mathcal{U}}_{J(a_i)J(b_i)}(\gamma_i)\mu^{x_{a_i},x_{b_i}}(d\gamma_i),$$

where the sum runs over the $(2q)!/(2^q q!)$ partitions of $\{1,\dots,2q\}$ in pairs.

2.3. **Ribbon graphs and surfaces.** Here we describe the ribbon graphs and the related two-dimensional surfaces. For more details, we refer to [Eyn16], Sections 2.2 and 2.3, [EKR18], Chapter 2, [LZ04], Sections 3.2 and 3.3, [Zvo97], and [MT01], Section 3.3.

Let be $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, where $m \geqslant 1$, and for all $l \in \{1,2,\dots,m\}$, $\alpha_l \in \mathbb{N}\backslash\{0\}$. We will denote

$$m(\alpha) = m, \qquad |\alpha| = \sum_{l=1}^{m(\alpha)} \alpha_l.$$

We will assume that $|\alpha|$ is even.

Given α as above, we consider $m(\alpha)$ vertices, where each vertex has adjacent unoriented half-edges. The first vertex has α_1 half-edges, the second α_2, etc. The half-edges are numbered from 1 to $|\alpha|$. See the next picture for an example with $\alpha = (4,3,1)$.

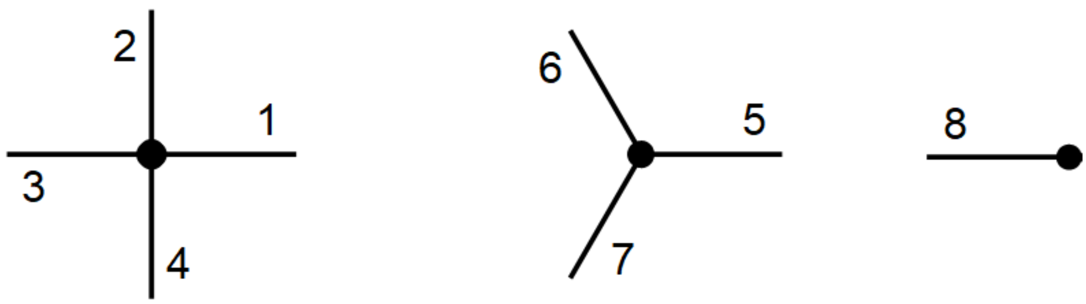

FIGURE 1. The case of $\alpha = (4,3,1)$. Three vertices with in total eight half-edges.

Since the total number of half-edges, $|\alpha|$, is even, one can pair them to obtain an unoriented graph (not necessarily connected), with $m(\alpha)$ vertices and $|\alpha|/2$ edges. See the picture below for an example. Since the half-edges are enumerated, the total number of different possible pairings is

$$\frac{|\alpha|!}{2^{|\alpha|/2}(|\alpha|/2)!}.$$

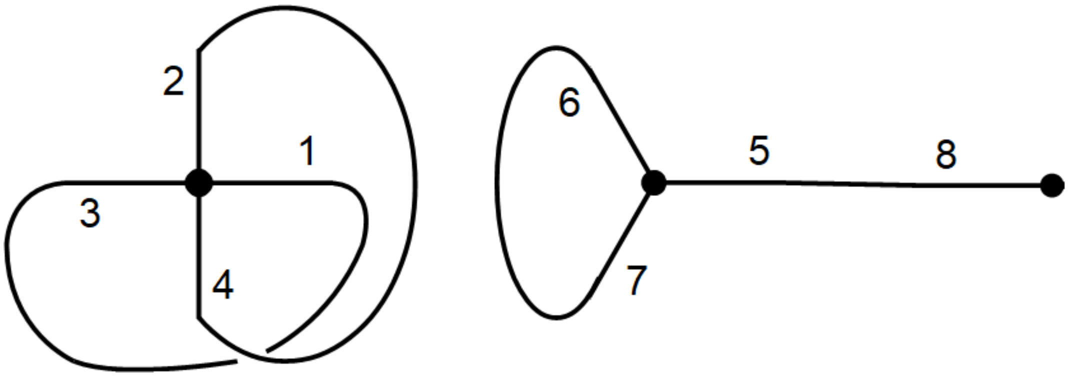

FIGURE 2. A pairing of half-edges in the case of $\alpha = (4, 3, 1)$.

Now, instead of considering the half-edges just as lines, we see them as two-dimensional ribbons, and call them *ribbon half-edges*. The ribbons are considered to be oriented. Also, the ribbon half-edges around each vertex are ordered in a cyclic way. See the picture below.

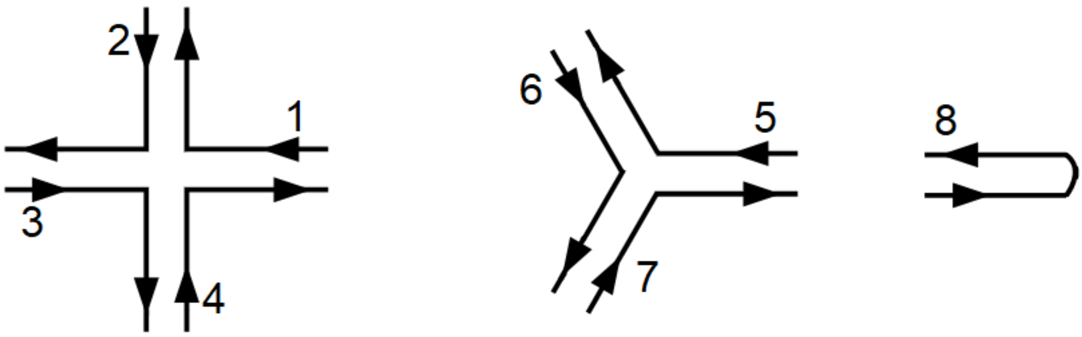

FIGURE 3. Ribbon half-edges in the case of $\alpha = (4, 3, 1)$.

Each time we pair two half-edges, we can glue the corresponding ribbons in two different ways. Either the orientations of the two ribbon half-edges match, or are opposite. In the first case we get a *straight ribbon edge*, in the second a *twisted ribbon edge*. See the illustration below.

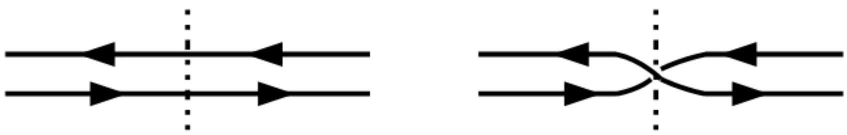

FIGURE 4. A straight ribbon edge on the left and a twisted ribbon edge on the right.

A *ribbon pairing* is a pairing of ribbon edges with a choice of straight or twisted pairing each time. The result is a *ribbon graph*. Below is an example of a ribbon pairing, with two straight edges and two twisted edges.

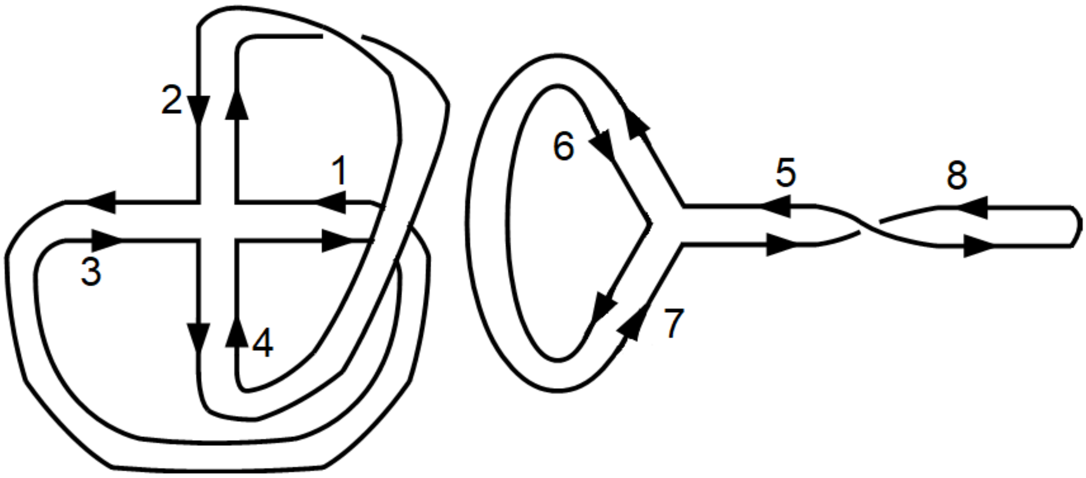

FIGURE 5. A ribbon pairing in the case of $\alpha = (4, 3, 1)$.

Let $\mathcal{R}_\alpha$ be the set of all possible ribbon pairings associated to α. The number of different ribbon pairings is

$$\mathrm{Card}(\mathcal{R}_\alpha) = \frac{|\alpha|!}{2^{|\alpha|/2}(|\alpha|/2)!} 2^{|\alpha|/2} = \frac{|\alpha|!}{(|\alpha|/2)!}.$$

Given a pairing $\rho \in \mathcal{R}_\alpha$, one can see the corresponding ribbon graph as a two-dimensional compact bordered surface (not necessarily connected). We will denote it $\check{\Sigma}_\alpha(\rho)$. On the example of Figure 5, $\check{\Sigma}_\alpha(\rho)$ has two connected components. The border of $\check{\Sigma}_\alpha(\rho)$, denoted $\partial\check{\Sigma}_\alpha(\rho)$ is given by the borders of the ribbon. Topologically it is a disjoint union of circles. $f_\alpha(\rho)$ will denote the number of connected components of $\partial\check{\Sigma}_\alpha(\rho)$, that is to say the number of distinct cycles formed by the borders of ribbons in the pairing ρ. On the example of Figure 5, $f_\alpha(\rho) = 3$.

Given $\rho \in \mathcal{R}_\alpha$, on can glue along each connected component of $\partial\check{\Sigma}_\alpha(\rho)$ a disk ($f_\alpha(\rho)$ disks in total), an obtained this way a two-dimensional compact surface (not necessarily connected) without border. We will denote it $\check{\Sigma}^+_\alpha(\rho)$, and consider it up to diffeomorphisms. On the example of Figure 5, $\check{\Sigma}^+_\alpha(\rho)$ has two connected components. On the left we get topologically a Klein bottle and on the right topologically a sphere. The connected components of $\check{\Sigma}^+_\alpha(\rho)$ can be orientable or non-orientable. The condition for orientability is that in the corresponding connected component of the ribbon graph, each closed path crosses twisted edges an even number of times (an edge can be crossed multiple times and one has to count the multiplicity). In particular, if there are no twisted edges, the connected component is orientable.

A ribbon pairing can be represented in a dual way as a paring of edges of a family of polygons (α_1-gone, α_2-gone, ..., $\alpha_{m(\alpha)}$-gone). This is shown on the next picture. This gluing of polygons also gives $\check{\Sigma}^+_\alpha(\rho)$.

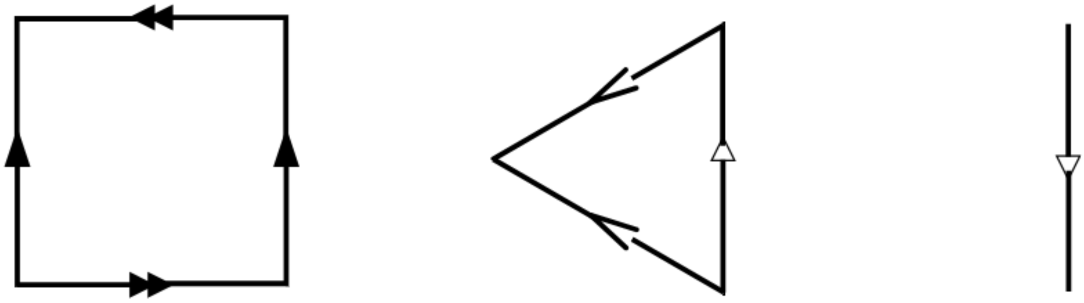

FIGURE 6. The gluing of polygons dual to the pairing of ribbon edges on Figure 5.

The connected compact surfaces without border, Σ, are classified by their orientability and their genus $g(\Sigma)$, an non-negative integer. The orientable genus zero surface is a sphere. The orientable genus one surface is a torus. Orientable surfaces of higher genera are "bretzels" with $g(\Sigma)$ handles. There are no non-orientable surfaces genus 0. A non-orientable surfaces of genus one is the real projective plane, of genus two - a Klein bottle, etc. A non-orientable surfaces of genus k is a connected sum of k real projective planes. Its orientation double cover is an orientable surface of genus $k-1$. For more on the classification of surfaces, we refer to [MT01], Chapter 3.

$\mathfrak{C}(\check{\Sigma}^+_\alpha(\rho))$ will denote the set of connected components of $\check{\Sigma}^+_\alpha(\rho)$. $f_\alpha(\rho)$ can be expressed in terms of connected components of $\check{\Sigma}^+_\alpha(\rho)$ as follows:

$$f_\alpha(\rho) = \frac{|\alpha|}{2} - m(\alpha) + \sum_{\Sigma \in \mathfrak{C}(\check{\Sigma}^+_\alpha(\rho))} (2 - (1 + \mathbf{1}_{\Sigma \text{ orientable}}) g(\Sigma)). \tag{2.4}$$

This can be shown using Euler's formula, as the Euler's characteristic of a connected compact surface without border is

$$\chi(\Sigma) = 2 - (1 + \mathbf{1}_{\Sigma \text{ orientable}}) g(\Sigma).$$

Now let be N and $(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k < k' \leqslant |\alpha|}$ be abstract formal commuting variables. We will also use the convention $\mathsf{Y}_{k'k} = \mathsf{Y}_{kk'}$ and $\widehat{\mathsf{Y}}_{k'k} = \widehat{\mathsf{Y}}_{kk'}$. Given a ribbon pairing $\rho \in \mathcal{R}_\alpha$, we will associate to it a monomial in the variables $(\mathsf{N}, (\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k < k' \leqslant |\alpha|})$:

$$P_{\alpha,\rho}(\mathsf{N}, (\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k < k' \leqslant |\alpha|}) = \mathsf{N}^{f_\alpha(\rho)} \prod_{\{a,b\} \in E_{\text{str}}(\rho)} \mathsf{Y}_{a,b} \prod_{\{a',b'\} \in E_{\text{tw}}(\rho)} \widehat{\mathsf{Y}}_{a',b'}, \tag{2.5}$$

where $E_{\text{str}}(\rho)$ denotes the set of straight edges of ρ, and $E_{\text{tw}}(\rho)$ the set of twisted edges of ρ. The monomial $P_{\alpha,\rho}$ is of degree $f_\alpha(\rho) + |\alpha|/2$, and of degree $|\alpha|/2$ in the variables $(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k < k' \leqslant |\alpha|}$. In the example of Figure 5,

$$P_{\alpha,\rho} = \mathsf{N}^3 \mathsf{Y}_{13} \mathsf{Y}_{67} \widehat{\mathsf{Y}}_{24} \widehat{\mathsf{Y}}_{58}.$$

We will associate to α the following polynomial in the variables $(\mathsf{N}, (\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k < k' \leqslant |\alpha|})$:

$$P_\alpha = \sum_{\rho \in \mathcal{R}_\alpha} P_{\alpha,\rho}. \tag{2.6}$$

The degree in N of P_α is

$$\deg_{\mathsf{N}} P_\alpha = \frac{|\alpha|}{2} + \operatorname{Card}(\{l \in \{1, \dots, m(\alpha)\} | \alpha_l \text{ even}\}).$$

It is attained by the ribbon pairings with only straight edges, which isolate the vertices with α_l even and partition in pairs the vertices with α_l odd, and that create only surfaces which are topologically spheres by respecting the cyclic order of the half-edges.

Next we give two tractable examples of polynomials P_α. Consider $\alpha = (4)$ (one vertex with four half-edges). Consider the half-edges numbered from 1 to 4 in a cyclic order. If we pair 1 with 2 and 3 with 4, or 1 with 4 and 2 with 3, in a straight way, we get a sphere ($f_\alpha(\rho) = 3$). If one of the edge is twisted, we get a real projective plane ($f_\alpha(\rho) = 2$). If both edges are twisted, we get a Klein bottle ($f_\alpha(\rho) = 1$). If we pair 1 with 3 and 2 with 4 and use only straight edges, we get a torus ($f_\alpha(\rho) = 1$). If one of the edges is twisted, we get a Klein bottle again ($f_\alpha(\rho) = 1$). If both edges are twisted, we get again a real projective plane ($f_\alpha(\rho) = 2$). So,

$$\begin{aligned} P_{(4)} &= \mathsf{N}^3(\mathsf{Y}_{12}\mathsf{Y}_{34} + \mathsf{Y}_{14}\mathsf{Y}_{23}) \\ &+ \mathsf{N}^2(\widehat{\mathsf{Y}}_{12}\mathsf{Y}_{34} + \mathsf{Y}_{12}\widehat{\mathsf{Y}}_{34} + \widehat{\mathsf{Y}}_{14}\mathsf{Y}_{23} + \mathsf{Y}_{14}\widehat{\mathsf{Y}}_{23} + \widehat{\mathsf{Y}}_{13}\widehat{\mathsf{Y}}_{24}) \\ &+ \mathsf{N}(\widehat{\mathsf{Y}}_{12}\widehat{\mathsf{Y}}_{34} + \widehat{\mathsf{Y}}_{14}\widehat{\mathsf{Y}}_{23} + \mathsf{Y}_{13}\mathsf{Y}_{24} + \mathsf{Y}_{13}\widehat{\mathsf{Y}}_{24} + \widehat{\mathsf{Y}}_{13}\mathsf{Y}_{24}). \end{aligned} \tag{2.7}$$

Further, for $\alpha = (2,2)$, we get

$$P_{(2,2)} = \mathsf{N}^4\mathsf{Y}_{12}\mathsf{Y}_{34} + \mathsf{N}^3(\widehat{\mathsf{Y}}_{12}\mathsf{Y}_{34} + \mathsf{Y}_{12}\widehat{\mathsf{Y}}_{34}) + \mathsf{N}^2(\widehat{\mathsf{Y}}_{12}\widehat{\mathsf{Y}}_{34} + \mathsf{Y}_{13}\mathsf{Y}_{24} + \mathsf{Y}_{14}\mathsf{Y}_{23} + \widehat{\mathsf{Y}}_{13}\widehat{\mathsf{Y}}_{24} + \widehat{\mathsf{Y}}_{14}\widehat{\mathsf{Y}}_{23}) + \mathsf{N}(\widehat{\mathsf{Y}}_{13}\mathsf{Y}_{24} + \mathsf{Y}_{13}\widehat{\mathsf{Y}}_{24} + \widehat{\mathsf{Y}}_{14}\mathsf{Y}_{23} + \mathsf{Y}_{14}\widehat{\mathsf{Y}}_{23}). \tag{2.8}$$

Next we introduce more combinatorial objects related to the ribbon pairings. We will consider tuples $(k_1, \mathtt{s}_1, k_2, \mathtt{s}_2, \dots, k_j, \mathtt{s}_j)$, where $j \in \mathbb{N}\backslash\{0\}$, each of the k_i is in $\mathbb{N}\backslash\{0\}$, and each of the $\mathtt{s}_i$ is one of the three abstract symbols $\rightarrow$, $\leftarrow$ or $\doteq$. We will endow such tuples by an equivalence relation $\approx$ generated by the following rules.

- Cyclic permutation: for any $i \in \{2, \dots, j\}$, $(k_i, \mathtt{s}_i, \dots, k_j, \mathtt{s}_j, k_1, \mathtt{s}_1, \dots, k_{i-1}, \mathtt{s}_{i-1})$ is identified to $(k_1, \mathtt{s}_1, k_2, \mathtt{s}_2, \dots, k_j, \mathtt{s}_j)$.
- Reversal of the direction: $(k_j, \mathtt{r}(\mathtt{s}_j), \dots, k_2, \mathtt{r}(\mathtt{s}_2), k_1, \mathtt{r}(\mathtt{s}_1))$ is identified to $(k_1, \mathtt{s}_1, k_2, \mathtt{s}_2, \dots, k_j, \mathtt{s}_j)$, where $\mathtt{r}(\rightarrow)$ is $\leftarrow$, $\mathtt{r}(\leftarrow)$ is $\rightarrow$, and $\mathtt{r}(\doteq)$ is $\doteq$.

For the lack of a better name, we will call *trails* the equivalence classes of $\approx$.

Given a ribbon pairing $\rho \in \mathcal{R}_\alpha$, we will associate to ρ a set $\mathcal{T}_\alpha(\rho)$ made of $f_\alpha(\rho)$ trails, one per each border cycle in the ribbon pairing. One starts on such a border cycle in an arbitrary place, and travels along it in any of the two directions. Then one successively visits ribbon half-edges with labels $k_1, k_2, \dots, k_j$ and then returns to the half-edge k_1. One can go from the half-edge k_i to the half-edge k_{i+1} either by following a gluing, and we will denote this $k_i \doteq k_{i+1}$, or by going through a vertex. In the latter case, one either does a turn clockwise, and we will denote this $k_i \rightarrow k_{i+1}$, or counterclockwise, and we will denote this $k_i \leftarrow k_{i+1}$. A special rule is applied if the vertex has only one outgoing half-edge, one just makes the arrows in the trail and on the picture match, as in the example below. This is how a trial is obtained. Note that by construction, there is an alternation between on one hand $\doteq$, and on the other hand $\rightarrow$ or $\leftarrow$. In the example of Figure 5, there are three trails:

$$(1, \rightarrow, 2, \doteq, 4, \leftarrow, 3, \doteq, 1, \leftarrow, 4, \doteq, 2, \rightarrow, 3, \doteq), \quad (5, \rightarrow, 6, \doteq, 7, \rightarrow, 5, \doteq, 8, \leftarrow, 8, \doteq), \quad (6, \rightarrow, 7, \doteq).$$

Further, to each trail $\mathtt{t}$, we will associate an abstract formal variable $\mathsf{Z}_\mathtt{t}$. We will consider polynomials in the commuting variables $((\mathsf{Z}_\mathtt{t})_{\mathtt{t}\text{ trail}}, (\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k < k' \leqslant |\alpha|})$. Given a ribbon pairing $\rho \in \mathcal{R}_\alpha$, we will associate to it the monomial

$$Q_{\alpha,\rho}((\mathsf{Z}_\mathtt{t})_{\mathtt{t}\text{ trail}}, (\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k < k' \leqslant |\alpha|}) = \prod_{\mathtt{t} \in \mathcal{T}_\alpha(\rho)} \mathsf{Z}_\mathtt{t} \prod_{\{a,b\} \in E_{\mathrm{str}}(\rho)} \mathsf{Y}_{a,b} \prod_{\{a',b'\} \in E_{\mathrm{tw}}(\rho)} \widehat{\mathsf{Y}}_{a',b'}. \tag{2.9}$$

We define the polynomial Q_α as

$$Q_\alpha = \sum_{\rho \in \mathcal{R}_\alpha} Q_{\alpha,\rho}. \tag{2.10}$$

Note that if in Q_α one substitutes each variable $\mathsf{Z}_\mathtt{t}$ with the variable N, one gets the polynomial P_α.

2.4. **One matrix integrals and topological expansion.** $\mathcal{S}_n(\mathbb{R})$ will denote the set of $n \times n$ real symmetric matrices and $\mathcal{H}_n(\mathbb{C})$ the Hermitian matrices. An inner product on $\mathcal{S}_n(\mathbb{R})$ and on $\mathcal{H}_n(\mathbb{C})$ is given by

$$(M, M') \mapsto \mathrm{Tr}(MM').$$

$E_{\beta,n}$ will denote $\mathcal{S}_n(\mathbb{R})$ for $\beta = 1$ and $\mathcal{H}_n(\mathbb{C})$ for $\beta = 2$.

$$\dim E_{\beta=1,n} = \frac{(n+1)n}{2}, \qquad \dim E_{\beta=2,n} = n^2.$$

The Gaussian Orthogonal Ensemble GOE(n) and the Gaussian Unitary Ensemble GUE(n) are Gaussian probability measures on $E_{\beta,n}$, with $\beta = 1$ for the GOE(n) and $\beta = 2$ for the

GUE(n). The density with respect to the Lebesgue measure on $E_{\beta,n}$ is given by

$$\frac{1}{Z_{\beta,n}} e^{-\frac{1}{2}\operatorname{Tr}(M^2)}.$$

The distribution of the ordered family of eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ of GOE(n) and GUE(n) is given by

$$\frac{1}{Z^{\mathrm{ev}}_{\beta,n}} \mathbf{1}_{\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n} \prod_{1\leqslant i<j\leqslant n} (\lambda_i - \lambda_j)^{\beta} e^{-\frac{1}{2}(\lambda_1^2+\cdots+\lambda_n^2)} d\lambda_1 \dots d\lambda_n.$$

For more on random matrices see [Meh04].

Let be $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{m(\alpha)})$, where for all $l \in \{1, 2, \dots, m(\alpha)\}$, $\alpha_l \in \mathbb{N}\backslash\{0\}$, and

$$|\alpha| = \sum_{l=1}^{m(\alpha)} \alpha_l$$

is even. Next we recall the expressions for the matrix integrals

$$\langle \prod_{l=1}^{m(\alpha)} \operatorname{Tr}(M^{\alpha_l}) \rangle_{\beta,n} = \frac{1}{Z_{\beta,n}} \int_{E_{\beta,n}} \Big(\prod_{l=1}^{m(\alpha)} \operatorname{Tr}(M^{\alpha_l}) \Big) e^{-\frac{1}{2}\operatorname{Tr}(M^2)} dM, \qquad \beta \in \{1,2\}. \tag{2.11}$$

Note that if $|\alpha|$ is odd, the above integrals are zero. The expression for (2.11) is known since the works of 't Hooft [tH74] and Brézin, Itzykson, Parisi and Zuber [BIPZ78]. It makes appear a polynomial in n, with powers $n^{f_\alpha(\rho)}$, where $\rho \in \mathcal{R}_\alpha$ are ribbon pairings associated to α. Since $f_\alpha(\rho)$ can be expressed using genera and orientabilities of surfaces as in (2.4), the expression for (2.11) is often referred to as *topological expansion*. For details on how it is obtained, we refer to [Eyn16], Chapter 2, [EKR18], Chapter 2, [LZ04], Chapter 3, and [Zvo97]. We consider the polynomial P_α defined by (2.6) and (2.5). We will also need the following weights $(w_{\mathrm{str}}(\beta), w_{\mathrm{tw}}(\beta))_{\beta\in\{1,2\}}$:

$$w_{\mathrm{str}}(1) = w_{\mathrm{tw}}(1) = \frac{1}{2}, \qquad w_{\mathrm{str}}(2) = 1, \quad w_{\mathrm{tw}}(2) = 0.$$

Note that in both cases, $w_{\mathrm{str}}(\beta) + w_{\mathrm{tw}}(\beta) = 1$.

Theorem 2.5 ('t Hooft [tH74], Brézin-Itzykson-Parisi-Zuber [BIPZ78])**.** *For $|\alpha|$ even, the value of the matrix integral $\langle\prod_{l=1}^{m(\alpha)} \operatorname{Tr}(M^{\alpha_l})\rangle_{\beta,n}$ (2.11) is given by evaluating the polynomial P_α in $\mathsf{N} = n$, $\mathsf{Y}_{kk'} = w_{\mathrm{str}}(\beta)$ and $\widehat{\mathsf{Y}}_{kk'} = w_{\mathrm{tw}}(\beta)$:*

$$\langle \prod_{l=1}^{m(\alpha)} \operatorname{Tr}(M^{\alpha_l}) \rangle_{\beta,n} = P_\alpha(\mathsf{N} = n, (\mathsf{Y}_{kk'} = w_{\mathrm{str}}(\beta), \widehat{\mathsf{Y}}_{kk'} = w_{\mathrm{tw}}(\beta))_{1\leqslant k<k'\leqslant|\alpha|}). \tag{2.12}$$

The consistency of the identity (2.12) can be tested by computing $\langle \operatorname{Tr}(M^2)\rangle_{\beta,n}$. We have that

$$P_{(2)} = \mathsf{N}^2\mathsf{Y}_{12} + \mathsf{N}\widehat{\mathsf{Y}}_{12}.$$

So (2.12) gives us

$$\langle \operatorname{Tr}(M^2)\rangle_{\beta=1,n} = \frac{1}{2}n^2 + \frac{1}{2}n, \qquad \langle \operatorname{Tr}(M^2)\rangle_{\beta=2,n} = n^2.$$

In both cases one gets $\dim E_{\beta,n}$.

Let us continue the examples with $\alpha = (4)$ and $\alpha = (2,2)$, as we computed the corresponding polynomials P_α (2.7),(2.8). We get

$$\begin{aligned} \langle \operatorname{Tr}(M^4)\rangle_{\beta=1,n} &= \frac{1}{2}n^3 + \frac{5}{4}n^2 + \frac{5}{4}n, & \langle (\operatorname{Tr}(M^2))^2\rangle_{\beta=1,n} &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{5}{4}n^2 + n, \\ \langle \operatorname{Tr}(M^4)\rangle_{\beta=2,n} &= 2n^3 + n, & \langle (\operatorname{Tr}(M^2))^2\rangle_{\beta=2,n} &= n^4 + 2n^2. \end{aligned}$$

2.5. **Matrix valued free fields, isomorphisms and topological expansion.** Let $\mathcal{G} = (V, E)$ be an electrical network as in Section 2.1. For $\beta \in \{1, 2\}$, $\Phi^{(\beta,n)}$ will be a random Gaussian function from V to $E_{\beta,n}$. If V is finite, the distribution of $\Phi^{(\beta,n)}$ is

$$(2.13)\quad \frac{1}{Z^{\mathcal{G}}_{\beta,n}} \exp\Big(-\frac{1}{2}\sum_{x\in V}\kappa(x)\operatorname{Tr}(M(x)^2) - \frac{1}{2}\sum_{\{x,y\}\in E} C(x,y)\operatorname{Tr}((M(y)-M(x))^2)\Big)\prod_{x\in V} dM(x).$$

In general, if V is finite or not, one can take an i.i.d. family of n^2 scalar Gaussian free fields $(\phi^{\mathbf{r}}_{ij})_{1\leqslant i\leqslant j\leqslant n}$, $(\phi^{\mathbf{i}}_{ij})_{1\leqslant i<j\leqslant n}$, and define the matrix valued field $\Phi_{\beta,n}$ by its entries:

$$(2.14)\qquad \Phi^{(\beta=1,n)}_{ij} = \begin{cases} \phi^{\mathbf{r}}_{ii} & \text{if } i=j,\\ \phi^{\mathbf{r}}_{ij}/\sqrt{2} & \text{if } i<j,\\ \phi^{\mathbf{r}}_{ji}/\sqrt{2} & \text{if } i>j,\end{cases}\qquad \Phi^{(\beta=2,n)}_{ij} = \begin{cases} \phi^{\mathbf{r}}_{ii} & \text{if } i=j,\\ (\phi^{\mathbf{r}}_{ij}+\mathbf{i}\phi^{\mathbf{i}}_{ij})/\sqrt{2} & \text{if } i<j,\\ (\phi^{\mathbf{r}}_{ji}-\mathbf{i}\phi^{\mathbf{i}}_{ji})/\sqrt{2} & \text{if } i>j.\end{cases}$$

For any $x \in V$, $\Phi^{(\beta,n)}(x)/\sqrt{G(x,x)}$ is distributed as a GOE(n) matrix ($\beta = 1$) or a GUE(n) matrix ($\beta = 2$). $\langle\cdot\rangle_{\beta,n}$ will denote the expectation with respect to the law of $\Phi^{(\beta,n)}$ In the sequel we will drop the subscripts (β, n) in $\Phi^{(\beta,n)}$ and just write Φ.

Let be $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m(\alpha)})$, where for all $l \in \{1, 2, \ldots, m(\alpha)\}$, $\alpha_l \in \mathbb{N}\backslash\{0\}$, and $|\alpha|$ even. Let $x_1, x_2, \ldots, x_{|\alpha|}$ be vertices in V, not necessarily distinct and F a bounded measurable function $\mathbb{R}^V \to \mathbb{R}$. By applying Theorem 2.1, one can *a priori* write an isomorphism for

$$\Big\langle\Big(\prod_{l=1}^{m(\alpha)}\operatorname{Tr}\Big(\prod_{k=\alpha_1+\cdots+\alpha_{l-1}+1}^{\alpha_1+\cdots+\alpha_l}\Phi(x_k)\Big)\Big)F(\operatorname{Tr}(\Phi^2)/2)\Big\rangle_{\beta,n}$$

However, if one expands the traces and the product, one gets many terms that give identical contributions, many terms with contributions that compensate, and many terms that do not contribute at all. For instance, if $\alpha = (1, 1)$,

$$\begin{aligned}\big\langle\phi^{\mathbf{r}}_{11}(x_1)\phi^{\mathbf{r}}_{11}(x_2)F(\operatorname{Tr}(\Phi^2)/2)\big\rangle_{\beta,n} &= \big\langle\phi^{\mathbf{r}}_{22}(x_1)\phi^{\mathbf{r}}_{22}(x_2)F(\operatorname{Tr}(\Phi^2)/2)\big\rangle_{\beta,n},\\ \big\langle\phi^{\mathbf{r}}_{11}(x_1)\phi^{\mathbf{r}}_{22}(x_2)F(\operatorname{Tr}(\Phi^2)/2)\big\rangle_{\beta,n} &= 0.\end{aligned}$$

Here we will be interested in the exact combinatorics that appear. What emerges is a topological expansion, generalizing that of Theorem 2.5.

Let be $\mu^{x_1,x_2,\ldots,x_{|\alpha|}}_{\alpha,\beta,n}$, the positive measure on families of $|\alpha|/2$ nearest neighbor paths on $\mathcal{G}$ obtained by substituting in the polynomial P_α (2.5),(2.6) the variable N by n, the variables $\mathsf{Y}_{kk'}$ by the measures $w_{\text{str}}(\beta)\mu^{x_k,x_{k'}}$, and the variables $\widehat{\mathsf{Y}}_{kk'}$ by $w_{\text{tw}}(\beta)\mu^{x_k,x_{k'}}$. The product on the measures is the usual tensor product $\otimes$. We will not need to distinguish between $\mu^{x_k,x_{k'}}$ and $\mu^{x_{k'},x_k}$, and between $\mu^{x_a,x_b}\otimes\mu^{x_{a'},x_{b'}}$ and $\mu^{x_{a'},x_{b'}}\otimes\mu^{x_a,x_b}$. The total mass of $\mu^{x_1,x_2,\ldots,x_{|\alpha|}}_{\alpha,\beta,n}$ equals

$$P_\alpha(\mathsf{N} = n, (\mathsf{Y}_{kk'} = w_{\text{str}}(\beta)G(x_k, x_{k'}), \widehat{\mathsf{Y}}_{kk'} = w_{\text{tw}}(\beta)G(x_k, x_{k'}))_{1\leqslant k<k'\leqslant|\alpha|}).$$

Next we give examples.

$$\mu^{x_1,x_2}_{\alpha=(2),\beta=1,n} = \Big(\frac{1}{2}n^2 + \frac{1}{2}n\Big)\mu^{x_1,x_2},\qquad \mu^{x_1,x_2}_{\alpha=(1,1),\beta=1,n} = n\mu^{x_1,x_2},$$

$$\begin{aligned}\mu^{x_1,x_2,x_3,x_4}_{\alpha=(4),\beta=1,n} &= \Big(\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n\Big)\mu^{x_1,x_2}\otimes\mu^{x_3,x_4} + \Big(\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n\Big)\mu^{x_1,x_4}\otimes\mu^{x_2,x_3}\\ &\quad + \Big(\frac{1}{4}n^2 + \frac{3}{4}n\Big)\mu^{x_1,x_3}\otimes\mu^{x_2,x_4},\end{aligned}$$

$$\begin{aligned}\mu^{x_1,x_2,x_3,x_4}_{\alpha=(2,2),\beta=1,n} &= \Big(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\Big)\mu^{x_1,x_2}\otimes\mu^{x_3,x_4} + \Big(\frac{1}{2}n^2 + \frac{1}{2}n\Big)\mu^{x_1,x_4}\otimes\mu^{x_2,x_3}\\ &\quad + \Big(\frac{1}{2}n^2 + \frac{1}{2}n\Big)\mu^{x_1,x_3}\otimes\mu^{x_2,x_4},\end{aligned}$$

$$\mu^{x_1,x_2}_{\alpha=(2),\beta=2,n} = n^2\mu^{x_1,x_2},\qquad \mu^{x_1,x_2}_{\alpha=(1,1),\beta=2,n} = n\mu^{x_1,x_2},$$

$$\mu^{x_1,x_2,x_3,x_4}_{\alpha=(4),\beta=2,n} = n^3\mu^{x_1,x_2}\otimes\mu^{x_3,x_4} + n^3\mu^{x_1,x_4}\otimes\mu^{x_2,x_3} + n\mu^{x_1,x_3}\otimes\mu^{x_2,x_4},$$

$$\mu^{x_1,x_2,x_3,x_4}_{\alpha=(2,2),\beta=2,n} = n^4\mu^{x_1,x_2}\otimes\mu^{x_3,x_4} + n^2\mu^{x_1,x_4}\otimes\mu^{x_2,x_3} + n^2\mu^{x_1,x_3}\otimes\mu^{x_2,x_4}.$$

In the example of Section 2.3, the term in $\mu^{x_1,x_2,\dots,x_8}_{\alpha=(4,3,1),\beta,n}$ corresponding to the pairing displayed on Figure 5 is 0 for $\beta=2$, because of the twisted edges, and for $\beta=1$ it is

$$\frac{1}{16}n^3\mu^{x_1,x_3}\otimes\mu^{x_2,x_4}\otimes\mu^{x_5,x_8}\otimes\mu^{x_6,x_7}.$$

The topological expansion is as follows.

Theorem 2.6. *For $\beta\in\{1,2\}$ and F a bounded measurable function $\mathbb{R}^V\to\mathbb{R}$, one has the following equality:*

$$\Big\langle\Big(\prod_{l=1}^{m(\alpha)}\mathrm{Tr}\Big(\prod_{k=\alpha_1+\dots+\alpha_{l-1}+1}^{\alpha_1+\dots+\alpha_l}\Phi(x_k)\Big)\Big)F(\mathrm{Tr}(\Phi^2)/2)\Big\rangle_{\beta,n}$$
$$= \int_{\gamma_1,\dots\gamma_{|\alpha|/2}}\Big\langle F\Big(\mathrm{Tr}(\Phi^2)/2 + L(\gamma_1)+\dots+L(\gamma_{|\alpha|/2})\Big)\Big\rangle_{\beta,n}\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n}(d\gamma_1,\dots,d\gamma_{|\alpha|/2}),$$

where $\langle\cdot\rangle_{\beta,n}\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n}(\cdot)$ is a product measure. As for the moments, we have

$$\Big\langle\Big(\prod_{l=1}^{m(\alpha)}\mathrm{Tr}\Big(\prod_{k=\alpha_1+\dots+\alpha_{l-1}+1}^{\alpha_1+\dots+\alpha_l}\Phi(x_k)\Big)\Big)\Big\rangle_{\beta,n}$$
$$= P_\alpha(\mathsf{N}=n,(\mathsf{Y}_{kk'}=w_{\mathrm{str}}(\beta)G(x_k,x_{k'}),\widehat{\mathsf{Y}}_{kk'}=w_{\mathrm{tw}}(\beta)G(x_k,x_{k'}))_{1\leqslant k<k'\leqslant|\alpha|}).$$

In particular, if $\lambda_1(x)\geqslant\lambda_2(x)\geqslant\dots\lambda_n(x)$ is the family of eigenvalues of $\Phi(x)$, $x\in V$, and

$$x_1=\dots=x_{\alpha_1},\qquad x_{\alpha_1+1}=\dots=x_{\alpha_1+\alpha_2},\qquad\dots,\qquad x_{|\alpha|-\alpha_{m(\alpha)}+1}=\dots=x_{|\alpha|},$$

then

$$\Big\langle\prod_{l=1}^{m(\alpha)}\Big(\sum_{i=1}^n\lambda_i(x_{\alpha_l})^{\alpha_l}\Big)F\Big(\frac12\sum_{i=1}^n\lambda_i^2\Big)\Big\rangle_{\beta,n}$$
$$= \int_{\gamma_1,\dots\gamma_{|\alpha|/2}}\Big\langle F\Big(\frac12\sum_{i=1}^n\lambda_i^2 + L(\gamma_1)+\dots+L(\gamma_{|\alpha|/2})\Big)\Big\rangle_{\beta,n}\mu^{(x_{\alpha_1},\alpha_1),\dots,(x_{|\alpha|},\alpha_{m(\alpha)})}_{\alpha,\beta,n}(d\gamma_1,\dots,d\gamma_{|\alpha|/2}),$$

where the notation $(x_{\alpha_1+\dots+\alpha_l},\alpha_l)$ means that $x_{\alpha_1+\dots+\alpha_l}$ is repeated α_l times.

Now, let be a family of $|\alpha|$ square matrices with complex entries of size $n\times n$:

$$A(1,2),\dots,A(\alpha_1-1,\alpha_1),A(\alpha_1,1),$$
$$A(\alpha_1+1,\alpha_1+2),\dots,A(\alpha_1+\alpha_2-1,\alpha_1+\alpha_2),A(\alpha_1+\alpha_2,\alpha_1+1),$$
$$\dots,A(|\alpha|-\alpha_{m(\alpha)}+1,|\alpha|-\alpha_{m(\alpha)}+2),\dots,A(|\alpha|-1,|\alpha|),A(|\alpha|,|\alpha|-\alpha_{m(\alpha)}+1).$$

Note that by convention, for each $l\in\{1,\dots,m(\alpha)\}$ such that $\alpha_l=1$, we have a single matrix $A(\alpha_1+\dots+\alpha_l,\alpha_1+\dots+\alpha_l)$. For $l\in\{1,\dots,m(\alpha)\}$, $\Pi_{\alpha,l}(\Phi,A)$ will denote the product

(2.15)
$$\Pi_{\alpha,l}(\Phi,A) = \Phi(x_{\alpha_1+\dots+\alpha_{l-1}+1})A(\alpha_1+\dots+\alpha_{l-1}+1,\alpha_1+\dots+\alpha_{l-1}+2)\Phi(x_{\alpha_1+\dots+\alpha_{l-1}+2})$$
$$\dots A(\alpha_1+\dots+\alpha_l-1,\alpha_1+\dots+\alpha_l)\Phi(x_{\alpha_1+\dots+\alpha_l})A(\alpha_1+\dots+\alpha_l,\alpha_1+\dots+\alpha_{l-1}+1).$$

In case $\alpha_l=1$, $\Pi_{\alpha,l}(\Phi,A)=\Phi(x_{\alpha_1+\dots+\alpha_l})A(\alpha_1+\dots+\alpha_l,\alpha_1+\dots+\alpha_l)$.

Next we will write an isomorphism for

$$\Big\langle\Big(\prod_{l=1}^{m(\alpha)}\mathrm{Tr}\Big(\Pi_{\alpha,l}(\Phi,A)\Big)\Big)F(\mathrm{Tr}(\Phi^2)/2)\Big\rangle_{\beta,n}.$$

For this we introduce the (complex valued) measure $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n,A}$ on $|\alpha|/2$ nearest neighbor paths in $\mathcal{G}$, constructed as follows. In the polynomial Q_α (2.9), (2.10) we substitute the variables $\mathsf{Y}_{kk'}$ by the measures $w_{\text{str}}(\beta)\mu^{x_k,x_{k'}}$, and the variables $\widehat{\mathsf{Y}}_{kk'}$ by $w_{\text{tw}}(\beta)\mu^{x_k,x_{k'}}$. This part is similar to the construction of $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n}$. The difference comes from what is substituted for $\mathsf{Z}_{\mathtt{t}}$, $\mathtt{t}$ trail. We replace $\mathsf{Z}_{\mathtt{t}}$ by the trace of a product of matrices of form $A(k,k')$ or $A(k',k)^{\mathsf{T}}$. For each sequence $k \to k'$ in the trail $\mathtt{t}$ we add the factor $A(k,k')$ to the product, and for each sequence $k \leftarrow k'$, we add the factor $A(k',k)^{\mathsf{T}}$, all by respecting cyclic order of the trail. We will denote this product $\Pi_{\mathtt{t}}(A)$, and its trace $\mathrm{Tr}(\Pi_{\mathtt{t}}(A))$ is substituted for $\mathsf{Z}_{\mathtt{t}}$. For instance, if

$$\mathtt{t} = (5, \to, 6, \doteq, 7, \to, 5, \doteq, 8, \leftarrow, 8, \doteq),$$

which is one of the trails on Figure (5), then

$$\Pi_{\mathtt{t}}(A) = A(5,6)A(7,5)A(8,8)^{\mathsf{T}}.$$

Note that while the product $\Pi_{\mathtt{t}}(A)$ depends on the particular representative of the equivalence class $\mathtt{t}$, its trace does not. Indeed, the trace is invariant by a cyclic permutation of the factors. Moreover, reversing the direction of a representative of $\mathtt{t}$ amounts to taking the transpose of the product, which has the same trace.

Next we give examples of measures $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n,A}$:

$$\begin{aligned}
\mu^{x_1,x_2}_{\alpha=(2),\beta=1,n,A} &= \Big(\frac{1}{2}\,\mathrm{Tr}(A(1,2))\,\mathrm{Tr}(A(2,1)) + \frac{1}{2}\,\mathrm{Tr}(A(1,2)A(2,1)^{\mathsf{T}})\Big)\mu^{x_1,x_2},\\
\mu^{x_1,x_2}_{\alpha=(1,1),\beta=1,n,A} &= \Big(\frac{1}{2}\,\mathrm{Tr}(A(1,1)A(2,2)) + \frac{1}{2}\,\mathrm{Tr}(A(1,1)A(2,2)^{\mathsf{T}})\Big)\mu^{x_1,x_2},\\
\mu^{x_1,x_2}_{\alpha=(2),\beta=2,n,A} &= \mathrm{Tr}(A(1,2))\,\mathrm{Tr}(A(2,1))\mu^{x_1,x_2},\\
\mu^{x_1,x_2}_{\alpha=(1,1),\beta=2,n,A} &= \mathrm{Tr}(A(1,1)A(2,2))\mu^{x_1,x_2}.
\end{aligned}$$

In the example of Section 2.3, the term in $\mu^{x_1,x_2,\dots,x_8}_{\alpha=(4,3,1),\beta,n,A}$ corresponding to the pairing displayed on Figure 5 is 0 for $\beta = 2$, because of the twisted edges, and for $\beta = 1$ it is

$$\frac{1}{16}\,\mathrm{Tr}(A(1,2)A(3,4)^{\mathsf{T}}A(4,1)^{\mathsf{T}}A(2,3))\,\mathrm{Tr}(A(5,6)A(7,5)A(8,8)^{\mathsf{T}})\,\mathrm{Tr}(A(6,7))\\
\mu^{x_1,x_3}\otimes\mu^{x_2,x_4}\otimes\mu^{x_5,x_8}\otimes\mu^{x_6,x_7}.$$

Note that if all of the matrices $A(k,k')$ are equal to I_n, the $n\times n$ identity matrix, then $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n,A}$ is just $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n}$, because all of the traces $\mathrm{Tr}(\Pi_{\mathtt{t}}(A))$ equal then n.

Theorem 2.7. *For $\beta \in \{1,2\}$ and F a bounded measurable function $\mathbb{R}^V \to \mathbb{R}$, one has the following equality:*

$$\Big\langle\Big(\prod_{l=1}^{m(\alpha)}\mathrm{Tr}\,\Big(\Pi_{\alpha,l}(\Phi,A)\Big)\Big)F(\mathrm{Tr}(\Phi^2)/2)\Big\rangle_{\beta,n}\\
= \int\limits_{\gamma_1,\dots\gamma_{|\alpha|/2}}\Big\langle F\big(\mathrm{Tr}(\Phi^2)/2 + L(\gamma_1)+\dots+L(\gamma_{|\alpha|/2})\big)\Big\rangle_{\beta,n}\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n,A}(d\gamma_1,\dots,d\gamma_{|\alpha|/2}),$$

where $\langle\cdot\rangle_{\beta,n}\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n,A}(\cdot)$ is a product measure.

2.6. **Isomorphisms and topological expansion for matrix valued fields twisted by a connection.** In this section we assume that the electrical network $\mathcal{G}$ is finite. $\mathbb{U}_{\beta,n}$ will denote the Lie group of $n\times n$ orthogonal, if $\beta = 1$, or unitary, if $\beta = 2$, matrices. Given a square matrix M, M^* will denote its adjoint (i.e. conjugate transpose), which in the case of matrices with real entries is the transpose.

We consider a connection on $\mathcal{G}$, $(U(x,y))_{\{x,y\}\in E}$, with $U(x,y)\in\mathbb{U}_{\beta,n}$ and

$$U(y,x) = U(x,y)^* = U(x,y)^{-1}.$$

If γ is a nearest neighbor path with finitely many jumps, $(y_1, y_2, \dots, y_j)$ the sequence of successively visited vertices by γ, the holonomy of U along γ is

$$\mathsf{hol}^U(\gamma) = U(y_1, y_2)U(y_2, y_3)\dots U(y_{j-1}, y_j).$$

For $\overleftarrow{\gamma}$ the time reversal of γ,

$$\mathsf{hol}^U(\overleftarrow{\gamma}) = \mathsf{hol}^U(\gamma)^* = \mathsf{hol}^U(\gamma)^{-1}.$$

Let be $\langle\cdot\rangle^U_{\beta,n}$ the following probability measure on $(E_{\beta,n})^V$, defined by the density

(2.16)
$$\frac{1}{Z^{\mathcal{G},U}_{\beta,n}} \exp\Big(-\frac{1}{2}\sum_{x\in V}\kappa(x)\operatorname{Tr}(M(x)^2) - \frac{1}{2}\sum_{\{x,y\}\in E} C(x,y)\operatorname{Tr}((M(y)-U(y,x)M(x)U(x,y))^2)\Big).$$

Note that if $\{x, y\} \in E$, then

$$\operatorname{Tr}((M(x)-U(x,y)M(y)U(y,x))^2) = \operatorname{Tr}((M(y)-U(y,x)M(x)U(x,y))^2).$$

$\widehat{\Phi}$ will denote the field under the measure $\langle\cdot\rangle^U_{\beta,n}$. It is the matrix valued (symmetric if $\beta = 1$, Hermitian if $\beta = 2$) Gaussian free field twisted by the connection U.

As in Section 2.5, we take $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{m(\alpha)})$, where for all $l \in \{1, 2, \dots, m(\alpha)\}$, $\alpha_l \in \mathbb{N}\backslash\{0\}$, and $|\alpha|$ even. Let $x_1, x_2, \dots, x_{|\alpha|}$ be vertices in V, not necessarily distinct. We also consider a family of $|\alpha|$ square matrices with complex entries of size $n \times n$:

$$A(1,2), \dots, A(\alpha_1 - 1, \alpha_1), A(\alpha_1, 1),$$
$$A(\alpha_1 + 1, \alpha_1 + 2), \dots, A(\alpha_1 + \alpha_2 - 1, \alpha_1 + \alpha_2), A(\alpha_1 + \alpha_2, \alpha_1 + 1),$$
$$\dots, A(|\alpha| - \alpha_{m(\alpha)} + 1, |\alpha| - \alpha_{m(\alpha)} + 2), \dots, A(|\alpha| - 1, |\alpha|), A(|\alpha|, |\alpha| - \alpha_{m(\alpha)} + 1).$$

By convention, for each $l \in \{1, \dots, m(\alpha)\}$ such that $\alpha_l = 1$, we have a single matrix $A(\alpha_1 + \dots + \alpha_l, \alpha_1 + \dots + \alpha_l)$. $\Pi_{\alpha,l}(\widehat{\Phi}, A)$ will denote the product (2.15) defined similarly to $\Pi_{\alpha,l}(\widehat{\Phi}, A)$, with $\widehat{\Phi}$ instead of Φ.

Next we will write an isomorphism for

$$\Big\langle \Big(\prod_{l=1}^{m(\alpha)} \operatorname{Tr}\Big(\Pi_{\alpha,l}(\widehat{\Phi}, A)\Big)\Big) F(\operatorname{Tr}(\widehat{\Phi}^2)/2) \Big\rangle^U_{\beta,n}.$$

We introduce the (complex valued) measure $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n,A,U}$ on $|\alpha|/2$ nearest neighbor paths in $\mathcal{G}$, constructed as follows. In the polynomial Q_α (2.9), (2.10), we substitute the variables $\mathsf{Y}_{kk'}$, $k < k'$ by the measures $w_{\mathrm{str}}(\beta)\mu^{x_k,x_{k'}}$, and the variables $\widehat{\mathsf{Y}}_{kk'}$, $k < k'$, by $w_{\mathrm{tw}}(\beta)\mu^{x_k,x_{k'}}$. This part is similar to the construction of $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n}$. Next we explain what is substituted for $\mathsf{Z}_{\mathtt{t}}$, $\mathtt{t}$ trail. We replace $\mathsf{Z}_{\mathtt{t}}$ by a density function depending on the nearest neighbor paths, which is the trace of a product of matrices of form $A(k, k')$ or $A(k', k)^{\mathsf{T}}$, and $\mathsf{hol}^U(\gamma)$ or $\mathsf{hol}^U(\gamma)^*$, γ being one of the paths. For each sequence $k \to k'$ in the trail $\mathtt{t}$ we add the factor $A(k, k')$ to the product, and for each sequence $k \leftarrow k'$, we add the factor $A(k', k)^{\mathsf{T}}$, as in the construction of $\mu^{x_1,x_2,\dots,x_{|\alpha|}}_{\alpha,\beta,n,A}$. Moreover, for each sequence $k \doteq k'$ with $k < k'$, a measure $\mu^{x_k,x_{k'}}(d\gamma_i)$ is present, and we add to the product the factor $\mathsf{hol}^U(\gamma_i)$. For each sequence $k \doteq k'$ with $k > k'$, a measure $\mu^{x_{k'},x_k}(d\gamma_i)$ is present, and we add to the product the factor $\mathsf{hol}^U(\gamma_i)^*$. In the product the factors respect the cyclic order on the trail. We will denote this product $\Pi_{\mathtt{t}}(U, A)(\gamma_1, \dots, \gamma_{|\alpha|/2})$, and its trace $\operatorname{Tr}(\Pi_{\mathtt{t}}(U, A)(\gamma_1, \dots, \gamma_{|\alpha|/2}))$ is substituted for $\mathsf{Z}_{\mathtt{t}}$.

In the example of Section 2.3, the term in $\mu^{x_1,x_2,\dots,x_8}_{\alpha=(4,3,1),\beta,n,A}$ corresponding to the pairing displayed on Figure 5 is 0 for $\beta = 2$, because of the twisted edges, and for $\beta = 1$ it is

$$\frac{1}{16}\operatorname{Tr}(A(1,2)\mathsf{hol}^U(\gamma_2)A(3,4)^{\mathsf{T}}\mathsf{hol}^U(\gamma_1)^*A(4,1)^{\mathsf{T}}\mathsf{hol}^U(\gamma_2)^*A(2,3)\mathsf{hol}^U(\gamma_1)^*)$$
$$\times \operatorname{Tr}(A(5,6)\mathsf{hol}^U(\gamma_4)A(7,5)\mathsf{hol}^U(\gamma_3)A(8,8)^{\mathsf{T}}\mathsf{hol}^U(\gamma_3)^*) \times \operatorname{Tr}(A(6,7)\mathsf{hol}^U(\gamma_4)^*)$$
$$\mu^{x_1,x_3}(d\gamma_1)\mu^{x_2,x_4}(d\gamma_2)\mu^{x_5,x_8}(d\gamma_3)\mu^{x_6,x_7}(d\gamma_4).$$

Next are more examples of measures $\mu_{\alpha,\beta,n,A,U}^{x_1,x_2,\dots,x_{|\alpha|}}$:

$$\begin{aligned}
\mu_{\alpha=(2),\beta=1,n,A,U}^{x_1,x_2} &= \Big(\frac{1}{2}\operatorname{Tr}(A(1,2)\mathsf{hol}^U(\gamma)^*)\times\operatorname{Tr}(A(2,1)\mathsf{hol}^U(\gamma))\\
&\quad+\frac{1}{2}\operatorname{Tr}(A(1,2)\mathsf{hol}^U(\gamma)^*A(2,1)^{\mathsf{T}}\mathsf{hol}^U(\gamma))\Big)\mu^{x_1,x_2}(d\gamma),\\
\mu_{\alpha=(1,1),\beta=1,n,A,U}^{x_1,x_2} &= \Big(\frac{1}{2}\operatorname{Tr}(A(1,1)\mathsf{hol}^U(\gamma)A(2,2)\mathsf{hol}^U(\gamma)^*)\\
&\quad+\frac{1}{2}\operatorname{Tr}(A(1,1)\mathsf{hol}^U(\gamma)A(2,2)^{\mathsf{T}}\mathsf{hol}^U(\gamma)^*)\Big)\mu^{x_1,x_2}(d\gamma),\\
\mu_{\alpha=(2),\beta=2,n,A,U}^{x_1,x_2} &= \operatorname{Tr}(A(1,2)\mathsf{hol}^U(\gamma)^*)\times\operatorname{Tr}(A(2,1)\mathsf{hol}^U(\gamma))\mu^{x_1,x_2}(d\gamma),\\
\mu_{\alpha=(1,1),\beta=2,n,A,U}^{x_1,x_2} &= \operatorname{Tr}(A(1,1)\mathsf{hol}^U(\gamma)A(2,2)\mathsf{hol}^U(\gamma)^*)\mu^{x_1,x_2}(d\gamma).
\end{aligned}$$

Theorem 2.8. *For $\beta\in\{1,2\}$ and F a bounded measurable function $\mathbb{R}^V\to\mathbb{R}$, one has the following equality:*

$$\begin{aligned}
(2.17)\quad &\Big\langle\Big(\prod_{l=1}^{m(\alpha)}\operatorname{Tr}\Big(\Pi_{\alpha,l}(\widehat{\Phi},A)\Big)\Big)F(\operatorname{Tr}(\widehat{\Phi}^2)/2)\Big\rangle_{\beta,n}^{U}\\
&=\int_{\gamma_1,\dots\gamma_{|\alpha|/2}}\Big\langle F\Big(\operatorname{Tr}(\widehat{\Phi}^2)/2+L(\gamma_1)+\dots+L(\gamma_{|\alpha|/2})\Big)\Big\rangle_{\beta,n}^{U}\mu_{\alpha,\beta,n,A,U}^{x_1,x_2,\dots,x_{|\alpha|}}(d\gamma_1,\dots,d\gamma_{|\alpha|/2}),
\end{aligned}$$

where $\langle\cdot\rangle_{\beta,n}^{U}\mu_{\alpha,\beta,n,A,U}^{x_1,x_2,\dots,x_{|\alpha|}}(\cdot)$ is a product measure.

Remark 2.9. Note that in the measure $\mu_{\alpha,\beta,n,A,U}^{x_1,x_2,\dots,x_{|\alpha|}}(d\gamma_1,\dots,d\gamma_{|\alpha|/2}$, the holonomy along each path γ_i appears twice.

Remark 2.10. Consider the particular case when the connection U is flat, i.e. for any close path (loop) γ, $\mathsf{hol}^U(\gamma)=I_n$. Fix $x_0\in V$ and let $\mathfrak{U}^{x_0}:V\to\mathbb{U}_{\beta,n}$ be the unique function satisfying $\mathfrak{U}^{x_0}(x_0)=I_n$, and

$$\forall x,y\in V\text{ such that }\{x,y\}\in E,\quad \mathfrak{U}^{x_0}(x)^{-1}\mathfrak{U}^{x_0}(y)=U(x,y).$$

See also Remark 2.3. Then for any $x,y,\in V$ and γ nearest neighbor path from x to y,

$$\mathsf{hol}^U(\gamma)=\mathfrak{U}^{x_0}(x)^{-1}\mathfrak{U}^{x_0}(y).$$

The field $\widehat{\Phi}$ twisted by the connection U has same law as $(\mathfrak{U}^{x_0}(x)^{-1}\Phi(x)\mathfrak{U}^{x_0}(x))_{x\in V}$, where Φ is the field with density (2.13). So, in the particular case of a flat connection, Theorem 2.8 follows from Theorem 2.7, since

$$\operatorname{Tr}(\Pi_{\alpha,l}(\widehat{\Phi},A))=\operatorname{Tr}(\Pi_{\alpha,l}(\Phi,\widehat{A})),$$

with $\widehat{A}(k,k')=\mathfrak{U}^{x_0}(x_k)A(k,k')\mathfrak{U}^{x_0}(x_{k'})^{-1}$.

Remark 2.11. A special case of particular interest in Theorem 2.8 is when all the matrices $A(k,k')$ equal I_n and

$$x_1=\dots=x_{\alpha_1},\qquad x_{\alpha_1+1}=\dots=x_{\alpha_1+\alpha_2},\qquad \dots,\qquad x_{|\alpha|-\alpha_{m(\alpha)}+1}=\dots=x_{|\alpha|}.$$

Then, on the left-hand side of (2.17), we have

$$\forall l\in\{1,\dots,m(\alpha)\},\operatorname{Tr}(\Pi_{\alpha,l}(\widehat{\Phi},A))=\sum_{i=1}^{n}\widehat{\lambda}_i(x_{\alpha_l})^{\alpha_l},$$

where $\widehat{\lambda}_1(x)\geqslant\widehat{\lambda}_2(x)\geqslant\dots\widehat{\lambda}_n(x)$ is the family of eigenvalues of $\widehat{\Phi}$. On the right-hand side of (2.17) appears a product of traces of holonomies along loops (i.e. closed paths). Indeed, each $\Pi_{\mathtt{t}}(U,A)(\gamma_1,\dots,\gamma_{|\alpha|/2})$ is then a holonomy along a loop formed by concatenating some of the paths γ_k. Note that if γ is a loop, the trace of the holonomy along γ does not depend on where the loop is rooted. If $(y_1, \dots, y_j, y_1)$ is the sequence of vertices visited by γ, and $\tilde{\gamma}$ is the loop visiting $(y_i, \dots, y_j, y_1, \dots, y_{i-1}, y_i)$ $(i \in \{2, \dots, j\})$, and if γ' is the path visiting $(y_1, \dots, y_i)$ then

$$\mathsf{hol}^U(\tilde{\gamma}) = \mathsf{hol}^U(\gamma')^{-1}\mathsf{hol}^U(\gamma)\mathsf{hol}^U(\gamma') \qquad \text{and} \qquad \mathrm{Tr}(\mathsf{hol}^U(\tilde{\gamma})) = \mathrm{Tr}(\mathsf{hol}^U(\gamma)).$$

3. Proofs

3.1. **Proof of Theorems 2.6 and 2.7.** Our proof of Theorem 2.6 will use an elementary approach, similar to that of [Zvo97] for one matrix integrals. From now on we will consider fixed the integer $n \in \mathbb{N}\backslash\{0\}$ and the tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{m(\alpha)})$, where for all $l \in \{1, 2, \dots, m(\alpha)\}$, $\alpha_l \in \mathbb{N}\backslash\{0\}$, and $|\alpha|$ even. We also fix $x_1, \dots, x_{|\alpha|}$ vertices in V, not necessarily all distinct. We will use the notation $(\Phi_{ij}(x))_{1 \leqslant i,j \leqslant n}$ for the entries of the matrix $\Phi(x)$, $x \in V$.

We consider a family of abstract formal commuting variables $(\mathsf{M}_{ij}(k))_{\substack{1 \leqslant i,j \leqslant n, \\ 1 \leqslant k \leqslant |\alpha|}}$, and the space of multivariate polynomials in these variables, with complex coefficients, $\mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1 \leqslant i,j \leqslant n, \\ 1 \leqslant k \leqslant |\alpha|}}\Big]$. For a given $k \in \{1, \dots, |\alpha|\}$, we will see the variables $(\mathsf{M}_{ij}(k))_{1 \leqslant i,j \leqslant n}$ as entries of a matrix $\mathsf{M}(k)$. In this way, for any sequence $(k_1, \dots, k_q)$ of integers in $\{1, \dots, |\alpha|\}$,

$$\mathrm{Tr}\Big(\prod_{l=1}^{q} \mathsf{M}(k_l)\Big)$$

is a polynomial in $\mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1 \leqslant i,j \leqslant n, \\ 1 \leqslant k \leqslant |\alpha|}}\Big]$. We would like emphasize the difference between the abstract variables $\mathsf{M}_{ij}(k)$ and the complex entries M_{ij} of a generic matrix M. $\mathsf{M}_{ij}(k)$ are not complex numbers, but can be evaluated into complex numbers.

We consider another family of abstract formal commuting variables $(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k \leqslant k' \leqslant |\alpha|}$. This variables have been already introduced in Section 2.3, except for $(\mathsf{Y}_{kk}, \widehat{\mathsf{Y}}_{kk})_{1 \leqslant k \leqslant |\alpha|}$. For $k > k'$, we will identify $\mathsf{Y}_{kk'}$ to $\mathsf{Y}_{k'k}$ and $\widehat{\mathsf{Y}}_{kk'}$ to $\widehat{\mathsf{Y}}_{k'k}$. We also consider the space of multivariate polynomials $\mathbb{C}\Big[(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k \leqslant k' \leqslant |\alpha|}\Big]$. We also consider the $\mathbb{C}$-linear map

$$\mathcal{E} : \mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1 \leqslant i,j \leqslant n, \\ 1 \leqslant k \leqslant |\alpha|}}\Big] \to \mathbb{C}\Big[(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k \leqslant k' \leqslant |\alpha|}\Big]$$

defined by the following rules.

(1) $\mathcal{E}(1) = 1$.
(2) For any monomial R of odd total degree, $\mathcal{E}(R) = 0$.
(3) For any monomial of form $\mathsf{M}_{ij}(k)\mathsf{M}_{i'j'}(k')$,

$$\mathcal{E}(\mathsf{M}_{ij}(k)\mathsf{M}_{i'j'}(k')) = \begin{cases} 0 & \text{if } (i,j) \neq (i',j') \text{ and } (i,j) \neq (j',i'), \\ \mathsf{Y}_{kk'} & \text{if } (i,j) = (j',i') \text{ and } i \neq j, \\ \widehat{\mathsf{Y}}_{kk'} & \text{if } (i,j) = (i',j') \text{ and } i \neq j, \\ \mathsf{Y}_{kk'} + \widehat{\mathsf{Y}}_{kk'} & \text{if } i = j = i' = j'. \end{cases}$$

(4) For any monomial of even total degree of form $M_{i_1j_1}(k_1)M_{i_2j_2}(k_2)\dots M_{i_{2q}j_{2q}}(k_{2q})$,

$$\mathcal{E}\big(M_{i_1j_1}(k_1)M_{i_2j_2}(k_2)\dots M_{i_{2q}j_{2q}}(k_{2q})\big) = \sum_{\substack{\text{partitions of} \\ \{1,\dots,2q\} \\ \text{in pairs} \\ \{\{a_1,b_1\},\dots,\{a_q,b_q\}\}}} \prod_{l=1}^{q} \mathcal{E}(\mathsf{M}_{i_{a_l}j_{a_l}}(k_{a_l})\mathsf{M}_{i_{b_l}j_{b_l}}(k_{b_l})),$$

where the sum runs over the $(2q)!/(2^q q!)$ partitions of $\{1, \dots, 2q\}$ in pairs.

One can recognize in (4) the Wick's rule for Gaussian random variables. Next we give an example:

$$\mathcal{E}\big(7M_{2,3}(1)M_{4,1}(3)^2M_{3,2}(5)+M_{1,2}(2)M_{2,2}(5)M_{2,1}(6)+3M_{1,1}(3)M_{1,1}(5)+2\mathbf{i}M_{4,4}(3)M_{4,4}(5)+9\big)$$
$$= 7\mathsf{Y}_{1,5}\widehat{\mathsf{Y}}_{3,3} + (3+2\mathbf{i})(\mathsf{Y}_{3,5} + \widehat{\mathsf{Y}}_{3,5}) + 9$$

We need one more piece of formalism. Ω_1 will denote the space of nearest neighbor cadlag paths on the graph $\mathcal{G}$, parametrized by continuous time, of finite total duration, that do a finite number of jumps. $\mathcal{F}_1$ will denote the natural σ-algebra on Ω_1 (we do not detail the construction). For $q \in \mathbb{N}$, $q \geqslant 2$, Ω_q will denote the space of collection of q paths in Ω_1, not necessarily distinct, considered up to permutation. In other words, Ω_q is the quotient of $(\Omega_1)^q$ by the action of the symmetric group $\mathfrak{S}_q$. We endow Ω_q with the σ-algebra $\mathcal{F}_q = (\mathcal{F}_1)^{\odot q}$. A function $F(\gamma_1, \dots, \gamma_q)$ is measurable with respect $\mathcal{F}_q$ if and only if it is measurable with respect the product σ-algebra $(\mathcal{F}_1)^{\otimes q}$ and is symmetric, i.e. for any $\sigma \in \mathfrak{S}_q$, $F(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(q)}) = F(\gamma_1, \dots, \gamma_q)$. Ω_0 will denote the space of no-paths, endowed with the trivial σ-algebra $\mathcal{F}_0 = \{\varnothing, \Omega_0\}$. As a set, one may take for Ω_0 just a singleton. Finally, Ω will denote the disjoint union

$$\Omega = \coprod_{q\geqslant 0} \Omega_0. \tag{3.1}$$

We endow Ω with the σ-algebra $\mathcal{F}$ generated by $(\mathcal{F}_q)_{q\geqslant 0}$. A generic element of Ω will be denoted ω. It is a finite unordered collection of paths on the graph $\mathcal{G}$, not all necessary distinct.

Given a polynomial $P \in \mathbb{C}\Big[(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1\leqslant k\leqslant k'\leqslant|\alpha|}\Big]$, one can construct out of it two complex-valued measures with finite total variation on $(\Omega, \mathcal{F})$, one for $\beta = 1$ and one for $\beta = 2$, by replacing in P each variable $\mathsf{Y}_{kk'}$ by the measure $w_{\mathrm{str}}(\beta)\mu^{x_k,x_{k'}}$ (2.1) and each variable $\widehat{\mathsf{Y}}_{kk'}$ by the measure $w_{\mathrm{tw}}(\beta)\mu^{x_k,x_{k'}}$:

$$P\big((\mathsf{Y}_{kk'} = w_{\mathrm{str}}(\beta)\mu^{x_k,x_{k'}}, \widehat{\mathsf{Y}}_{kk'} = w_{\mathrm{tw}}(\beta)\mu^{x_k,x_{k'}})_{1\leqslant k\leqslant k'\leqslant|\alpha|}\big)(d\omega).$$

This substitution is a $\mathbb{C}$-linear operation, at the monomials of degree q in P are sent to measures supported on Ω_q.

Lemma 3.1. *Let be a polynomial $\mathcal{P} \in \mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant|\alpha|}}\Big]$. For $\beta \in \{1,2\}$ and F a bounded measurable function $\mathbb{R}^V \to \mathbb{R}$, one has the following equality:*

$$\Big\langle \mathcal{P}\Big(\big(\mathsf{M}_{ij}(k) = \Phi_{ij}(x_k)\big)_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant|\alpha|}}\Big) F(\mathrm{Tr}(\Phi^2)/2)\Big\rangle_{\beta,n} = \tag{3.2}$$
$$\int_\Omega \Big\langle F\big(\mathrm{Tr}(\Phi^2)/2 + \sum_{\gamma\in\omega} L(\gamma)\big)\Big\rangle_{\beta,n} \mathcal{E}(\mathcal{P})\big((\mathsf{Y}_{kk'} = w_{\mathrm{str}}(\beta)\mu^{x_k,x_{k'}}, \widehat{\mathsf{Y}}_{kk'} = w_{\mathrm{tw}}(\beta)\mu^{x_k,x_{k'}})_{\substack{1\leqslant k\leqslant\\ k'\leqslant|\alpha|}}\big)(d\omega),$$

where $\langle\cdot\rangle_{\beta,n}\mathcal{E}(\mathcal{P})\big((\mathsf{Y}_{kk'} = w_{\mathrm{str}}(\beta)\mu^{x_k,x_{k'}}, \widehat{\mathsf{Y}}_{kk'} = w_{\mathrm{tw}}(\beta)\mu^{x_k,x_{k'}})_{\substack{1\leqslant k\leqslant\\ k'\leqslant|\alpha|}}\big)(d\omega)$ is a product measure.

Proof. By linearity of the identity (3.2), it is enough to check it on the four rules, (1) to (4), defining $\mathcal{E}$.

(1): This is trivial. If $\mathcal{P} = 1$, we have on both sides of (3.2) the quantity $\langle F(\mathrm{Tr}(\Phi^2)/2)\rangle_{\beta,n}$.

(2): Assume $\mathcal{P}$ is a monomial of odd total degree. $-\Phi$ has same law as Φ. Moreover, $\mathcal{P}(-\Phi) = -\mathcal{P}(\Phi)$. Thus, $\langle \mathcal{P}(\Phi)F(\mathrm{Tr}(\Phi^2)/2)\rangle_{\beta,n} = -\langle \mathcal{P}(\Phi)F(\mathrm{Tr}(\Phi^2)/2)\rangle_{\beta,n} = 0$.

(3): Assume $\mathcal{P} = \mathsf{M}_{ij}(k)\mathsf{M}_{i'j'}(k')$. Without loss of generality, $i \leqslant j$. If $(i,j) \neq (i',j')$ and $(i,j) \neq (j',i')$, then Φ_{ij} and $\Phi_{i'j'}$ are independent fields. By replacing Φ_{ij} by $-\Phi_{ij}$, we neither change the law of Φ nor change $\mathrm{Tr}(\Phi^2)$. Thus

$$\big\langle \Phi_{ij}(x_k)\Phi_{i'j'}(x_{k'})F(\mathrm{Tr}(\Phi^2)/2)\big\rangle_{\beta,n} = -\big\langle \Phi_{ij}(x_k)\Phi_{i'j'}(x_{k'})F(\mathrm{Tr}(\Phi^2)/2)\big\rangle_{\beta,n} = 0.$$

In other cases, $\Phi_{ij}(x_k)\Phi_{i'j'}(x_{k'})$ equals:

- $\dfrac{1}{2}\phi^{\mathbf{r}}_{ij}(x_k)\phi^{\mathbf{r}}_{ij}(x_{k'})$ if $\beta = 1$, $i \neq j$, and $(i,j) = (j',i')$ or $(i,j) = (i',j')$;

- $\phi^{\mathbf{r}}_{ii}(x_k)\phi^{\mathbf{r}}_{ii}(x_{k'})$ if $i = j = i' = j'$, and $\beta = 1$ or $\beta = 2$;
- $\dfrac{1}{2}\phi^{\mathbf{r}}_{ij}(x_k)\phi^{\mathbf{r}}_{ij}(x_{k'}) + \dfrac{1}{2}\phi^{\mathbf{i}}_{ij}(x_k)\phi^{\mathbf{i}}_{ij}(x_{k'})$ if $\beta = 2$, $i \neq j$, and $(i, j) = (j', i')$;
- $\dfrac{1}{2}\phi^{\mathbf{r}}_{ij}(x_k)\phi^{\mathbf{r}}_{ij}(x_{k'}) - \dfrac{1}{2}\phi^{\mathbf{i}}_{ij}(x_k)\phi^{\mathbf{i}}_{ij}(x_{k'})$ if $\beta = 2$, $i \neq j$, and $(i, j) = (i', j')$.

Then one applies the Dynkin's isomorphism (Theorem 2.1) to the GFFs $\phi^{\mathbf{r}}_{ij}$ and $\phi^{\mathbf{i}}_{ij}$, and sees that the coefficients obtained match with $w_{\mathrm{str}}(\beta)$, $w_{\mathrm{tw}}(\beta)$ or $w_{\mathrm{str}}(\beta) + w_{\mathrm{tw}}(\beta)$.

(4): This is a simple application of the Wick's rule. □

We move to the proof of Theorem 2.6.

Proof of Theorem 2.6. Let $\mathcal{P}_\alpha$ be the polynomial

$$\mathcal{P}_\alpha = \Big(\prod_{l=1}^{m(\alpha)} \mathrm{Tr}\Big(\prod_{k=\alpha_1+\cdots+\alpha_{l-1}+1}^{\alpha_1+\cdots+\alpha_l} \mathsf{M}(k)\Big)\Big) \in \mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant|\alpha|}}\Big].$$

Let $\widetilde{P}_\alpha$ be the polynomial in $\mathbb{C}\Big[(\mathsf{Y}_{kk'}, \hat{\mathsf{Y}}_{kk'})_{1\leqslant k\leqslant k'\leqslant|\alpha|}\Big]$ obtained by partially evaluating P_α in $\mathsf{N} = n$. In view of Lemma 3.1, it is enough to show that $\mathcal{E}(\mathcal{P}_\alpha) = \widetilde{P}_\alpha$.

For $\rho \in \mathcal{R}_\alpha$ a ribbon pairing, we will denote by $\mathcal{Y}_\rho$ the polynomial

$$\mathcal{Y}_\rho = P_{\alpha,\rho}(\mathsf{N} = 1) = \prod_{\{a,b\}\in E_{\mathrm{str}}(\rho)} \mathsf{Y}_{a,b} \prod_{\{a',b'\}\in E_{\mathrm{tw}}(\rho)} \hat{\mathsf{Y}}_{a',b'} \in \mathbb{C}\Big[(\mathsf{Y}_{kk'}, \hat{\mathsf{Y}}_{kk'})_{1\leqslant k\leqslant k'\leqslant|\alpha|}\Big].$$

Note that unlike $P_{\alpha,\rho}$, the polynomial $\mathcal{Y}_\rho$ depends on α only through $|\alpha|$. Let $\mathcal{I}$ denote the set of multi-indices

$$\mathcal{I} = \{(i_1, j_1, i_2, j_2, \ldots, i_{|\alpha|}, j_{|\alpha|}) \in \{1, \ldots, n\}^{2|\alpha|}\}.$$

We will use $\overrightarrow{\imath\jmath}$ as a short notation for a generic element of $\mathcal{I}$, and $R_{\overrightarrow{\imath\jmath}}$ will denote the monomial

$$R_{\overrightarrow{\imath\jmath}} = \mathsf{M}_{i_1j_1}(1)\mathsf{M}_{i_2j_2}(2)\ldots\mathsf{M}_{i_{|\alpha|}j_{|\alpha|}}(|\alpha|) \text{ for } (i_1, j_1, i_2, j_2, \ldots, i_{|\alpha|}, j_{|\alpha|}) = \overrightarrow{\imath\jmath}.$$

Given $\overrightarrow{\imath\jmath} \in \mathcal{I}$, it is easy to see with the rules (3) and (4) defining $\mathcal{E}$ that

$$\mathcal{E}(R_{\overrightarrow{\imath\jmath}}) = \sum_{\rho\in\mathcal{R}_\alpha} c_{\overrightarrow{\imath\jmath}}(\rho)\mathcal{Y}_\rho,$$

with

$$\forall \rho \in \mathcal{R}_\alpha, c_{\overrightarrow{\imath\jmath}}(\rho) \in \{0, 1\}.$$

For $\rho \in \mathcal{R}_\alpha$, denote

$$\mathcal{I}(\rightharpoondown \rho) = \{\overrightarrow{\imath\jmath} \in \mathcal{I} | c_{\overrightarrow{\imath\jmath}}(\rho) = 1\}.$$

A multi-index $\overrightarrow{\imath\jmath} = (i_1, j_1, i_2, j_2, \ldots, i_{|\alpha|}, j_{|\alpha|})$ is in $\mathcal{I}(\rightharpoondown \rho)$ if and only if the following conditions are satisfied:

$$(3.3) \qquad \forall \{k, k'\} \in E_{\mathrm{str}}(\rho), i_k = j_{k'}, j_k = i_{k'}, \qquad \forall \{k, k'\} \in E_{\mathrm{tw}}(\rho), i_k = i_{k'}, j_k = j_{k'}.$$

Now, denote $\mathcal{I}(\circlearrowleft \alpha)$ the subset of $\mathcal{I}$ defined by the following cyclic conditions:

$$(3.4) \quad j_1 = i_2, \ldots, j_{\alpha_1-1} = i_{\alpha_1}, j_{\alpha_1} = i_1, \ldots$$
$$\ldots, j_{|\alpha|-\alpha_{m(\alpha)}+1} = i_{|\alpha|-\alpha_{m(\alpha)}+2}, \ldots, j_{|\alpha|-1} = i_{|\alpha|}, j_{|\alpha|} = i_{|\alpha|-\alpha_{m(\alpha)}+1}.$$

For possible $\alpha_l = 1$, the condition is simply $j_{\alpha_1+\ldots\alpha_l} = i_{\alpha_1+\ldots\alpha_l}$. We have that

$$\mathcal{P}_\alpha = \sum_{\overrightarrow{\imath\jmath}\in\mathcal{I}(\circlearrowleft\alpha)} R_{\overrightarrow{\imath\jmath}}.$$

Thus,

$$\mathcal{E}(\mathcal{P}_\alpha) = \sum_{\rho\in\mathcal{R}_\alpha} \mathrm{Card}(\mathcal{I}(\circlearrowleft \alpha) \cap \mathcal{I}(\rightharpoondown \rho))\mathcal{Y}_\rho.$$

It only remains to count $\mathrm{Card}(\mathcal{I}(\circlearrowleft \alpha) \cap \mathcal{I}(\rightharpoondown \rho))$, which the number of multi-indexes satisfying both the condition (3.4) and (3.3). This conditions can be represented in a graphical way as follows. Given a ribbon pairing $\rho \in \mathcal{R}_\alpha$, for each ribbon half-edge of ρ, with a label $k \in \{1, \ldots, |\alpha|\}$, write on each side of the half-edge the index i_k, respectively j_k, with i_k on the side of the out-going arrow and j_k on the side of the in-going arrow. This is displayed on the picture below.

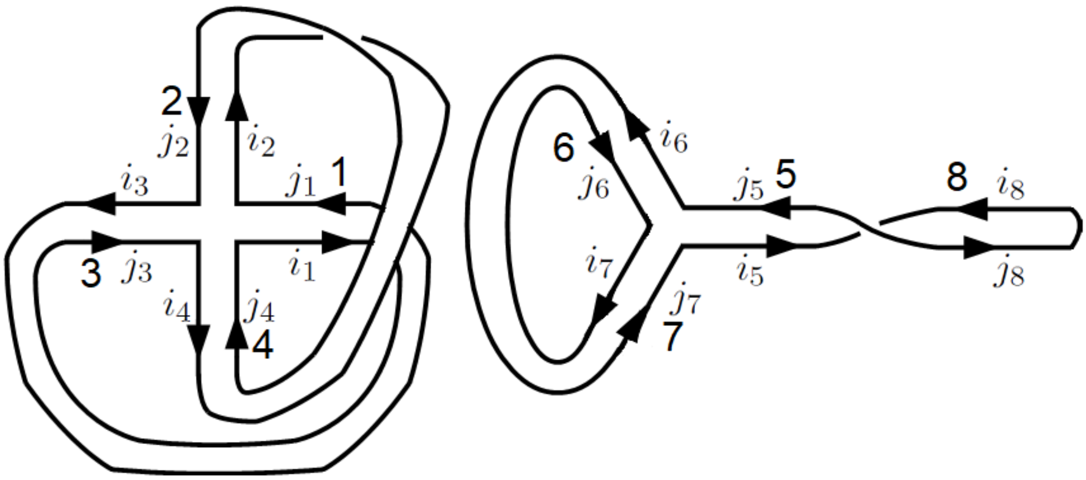

FIGURE 7. A ribbon pairing in the case of $\alpha = (4,3,1)$, with indices i_k and j_k displayed on both sides of the half-edges.

The conditions (3.4) and (3.3) simply says that on each cycle formed by the ribbon graph ρ, i.e. on each connected component of $\partial\check{\Sigma}_\alpha(\rho)$, all the indices are equal. For each cycle, there are n possible common values for the indices, and since there are $f_\alpha(\rho)$ cycles,

$$\mathrm{Card}(\mathcal{I}(\circlearrowleft \alpha) \cap \mathcal{I}(\rightharpoonup \rho)) = n^{f_\alpha(\rho)}.$$

So, $\mathcal{E}(\mathcal{P}_\alpha) = \widetilde{P}_\alpha$. $\square$

We continue with the proof of Theorem 2.7. As in Section 2.5, we consider the matrices

$$A(1,2), \ldots, A(\alpha_1 - 1, \alpha_1), A(\alpha_1, 1),$$
$$A(\alpha_1 + 1, \alpha_1 + 2), \ldots, A(\alpha_1 + \alpha_2 - 1, \alpha_1 + \alpha_2), A(\alpha_1 + \alpha_2, \alpha_1 + 1),$$
$$\ldots, A(|\alpha| - \alpha_{m(\alpha)} + 1, |\alpha| - \alpha_{m(\alpha)} + 2), \ldots, A(|\alpha| - 1, |\alpha|), A(|\alpha|, |\alpha| - \alpha_{m(\alpha)} + 1).$$

For $l \in \{1, \ldots, m(\alpha)\}$, $\Pi_{\alpha,l}(\mathsf{M}, A)$ will denote the following product of matrices:

$$\Pi_{\alpha,l}(\mathsf{M}, A) = \mathsf{M}(\alpha_1 + \cdots + \alpha_{l-1} + 1) A(\alpha_1 + \cdots + \alpha_{l-1} + 1, \alpha_1 + \cdots + \alpha_{l-1} + 2) \mathsf{M}(\alpha_1 + \cdots + \alpha_{l-1} + 2)$$
$$\ldots A(\alpha_1 + \cdots + \alpha_l - 1, \alpha_1 + \cdots + \alpha_l) \mathsf{M}(\alpha_1 + \cdots + \alpha_l) A(\alpha_1 + \cdots + \alpha_l, \alpha_1 + \cdots + \alpha_{l-1} + 1).$$

$\mathrm{Tr}\left(\Pi_{\alpha,l}(\mathsf{M}, A)\right)$ is a polynomial of $\mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1 \leqslant i,j \leqslant n, \\ 1 \leqslant k \leqslant |\alpha|}}\Big]$. Let $\mathcal{P}_{\alpha,A}$ be the polynomial

$$\mathcal{P}_{\alpha,A} = \prod_{l=1}^{m(\alpha)} \mathrm{Tr}\left(\Pi_{\alpha,l}(\mathsf{M}, A)\right). \tag{3.5}$$

For $\rho \in \mathcal{R}_\alpha$ a ribbon pairing, let be

$$\widetilde{Q}_{\alpha,\rho,A} = \prod_{\mathtt{t} \in \mathcal{T}_\alpha(\rho)} \mathrm{Tr}\left(\Pi_{\mathtt{t}}(A)\right) \prod_{\{a,b\} \in E_{\mathrm{str}}(\rho)} \mathsf{Y}_{a,b} \prod_{\{a',b'\} \in E_{\mathrm{tw}}(\rho)} \widehat{\mathsf{Y}}_{a',b'} = \Big(\prod_{\mathtt{t} \in \mathcal{T}_\alpha(\rho)} \mathrm{Tr}\left(\Pi_{\mathtt{t}}(A)\right)\Big) \mathcal{Y}_\rho.$$

Denote

$$\widetilde{Q}_{\alpha,A} = \sum_{\rho \in \mathcal{R}_\alpha} \widetilde{Q}_{\alpha,\rho,A} \in \mathbb{C}\Big[(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1 \leqslant k \leqslant k' \leqslant |\alpha|}\Big].$$

Proof of Theorem 2.7. We will use the notations of the proof of Theorem 2.6. We only need to check that

$$\mathcal{E}\big(\mathcal{P}_{\alpha,A}\big) = \widetilde{Q}_{\alpha,A}.$$

We have that

$$\mathcal{E}\big(\mathcal{P}_{\alpha,A}\big) = \sum_{\rho\in\mathcal{R}_\alpha}\Big(\sum_{\overrightarrow{\imath\jmath}\in\mathcal{I}(\rightharpoonup\rho)}\Pi_{\alpha,\overrightarrow{\imath\jmath}}(A)\Big)\mathcal{Y}_\rho,$$

where $\Pi_{\alpha,\overrightarrow{\imath\jmath}}(A)$ denotes

$$\prod_{l=1}^{m(\alpha)}\Big(\Big(\prod_{k=\alpha_1+\dots+\alpha_{l-1}+1}^{\alpha_1+\dots+\alpha_l-1}A_{j_k i_{k+1}}(k,k+1)\Big)A_{j_{\alpha_1+\dots+\alpha_l}i_{\alpha_1+\dots+\alpha_{l-1}+1}}(\alpha_1+\dots+\alpha_l,\alpha_1+\dots+\alpha_{l-1}+1)\Big)$$

By construction, if $A_{j_k i_{k'}}(k,k')$ appears in the product above, then the indices j_k and $i_{k'}$ are on the same trail $\mathtt{t}\in\mathcal{T}_\alpha(\rho)$ and adjacent (see Figure 7). Thus, instead of writing $\Pi_{\alpha,\overrightarrow{\imath\jmath}}(A)$ as a product of $m(\alpha)$ factors, one per each ribbon vertex, one can refactorize it as a product of $f_\alpha(\rho)$ factors, one for each trail $\mathtt{t}\in\mathcal{T}_\alpha(\rho)$,

$$\Pi_{\alpha,\overrightarrow{\imath\jmath}}(A) = \prod_{\mathtt{t}\in\mathcal{T}_\alpha(\rho)}\Pi_{\mathtt{t},\overrightarrow{\imath\jmath}}(A),$$

where in $\Pi_{\mathtt{t},\overrightarrow{\imath\jmath}}(A)$ are gathered the $A_{j_k i_{k'}}(k,k')$ with j_k and $i_{k'}$ on the trail $\mathtt{t}$. The condition (3.3) ensures that for $\overrightarrow{\imath\jmath}\in\mathcal{I}(\rightharpoonup\rho)$, $\Pi_{\mathtt{t},\overrightarrow{\imath\jmath}}(A)$ is a term in the expansion of the trace $\mathrm{Tr}\big(\Pi_{\mathtt{t}}(A)\big)$. Thus

$$\sum_{\overrightarrow{\imath\jmath}\in\mathcal{I}(\rightharpoonup\rho)}\Pi_{\alpha,\overrightarrow{\imath\jmath}}(A) = \prod_{\mathtt{t}\in\mathcal{T}_\alpha(\rho)}\mathrm{Tr}\big(\Pi_{\mathtt{t}}(A)\big).$$

It follows that $\mathcal{E}\big(\mathcal{P}_{\alpha,A}\big) = \widetilde{Q}_{\alpha,A}$. ☐

3.2. **Proof of Theorem 2.8.** We will prove Theorem 2.8. As in Section 2.6, we assume that the electrical network $\mathcal{G}$ is finite. $(U(x,y))_{\{x,y\}\in E}$ will be a connection with values in $\mathbb{U}_{\beta,n}$, $\beta\in\{1,2\}$. $\widehat{\Phi}$ will denote the matrix valued GFF twisted by the connection U, with density (2.16). We start by computing the two-point function of $\widehat{\Phi}$.

Lemma 3.2. *Let $x,y\in V$, $i,j,i',j'\in\{1,\dots,n\}$ and F a bounded measurable function $\mathbb{R}^V\to\mathbb{R}$. Then*

$$\Big\langle\widehat{\Phi}_{ij}(x)\widehat{\Phi}_{i'j'}(y)F(\mathrm{Tr}(\widehat{\Phi}^2)/2)\Big\rangle^U_{\beta,n} =$$
$$\begin{cases}\int_\gamma\Big\langle F\big(\mathrm{Tr}(\widehat{\Phi}^2)/2+L(\gamma)\big)\Big\rangle^U_{\beta,n}\big(\mathsf{hol}^U_{ij'}(\gamma)\mathsf{hol}^U_{ji'}(\gamma)/2+\mathsf{hol}^U_{ii'}(\gamma)\mathsf{hol}^U_{jj'}(\gamma)/2\big)\mu^{x,y}(d\gamma) & \text{if } \beta=1;\\ \int_\gamma\Big\langle F\big(\mathrm{Tr}(\widehat{\Phi}^2)/2+L(\gamma)\big)\Big\rangle^U_{\beta,n}\mathsf{hol}^U_{ij'}(\gamma)\overline{\mathsf{hol}^U_{ji'}(\gamma)}\mu^{x,y}(d\gamma) & \text{if } \beta=2.\end{cases}$$

Proof. For x and y neighbor vertices in $\mathcal{G}$, let $\mathcal{U}(x,y)$ be the linear endomorphism of $E_{\beta,n}$ defined by

$$\mathcal{U}(x,y)(M) = U(x,y)MU(y,x) = U(x,y)MU(x,y)^*.$$

$\mathcal{U}(x,y)$ is orthogonal for the inner product $(M,M')\mapsto\mathrm{Tr}(MM')$, and

$$\mathcal{U}(y,x) = \mathcal{U}(x,y)^{-1}.$$

So, we see $(\mathcal{U}(x,y))_{\{x,y\}\in E}$ as an orthogonal connection over a vector bundle with fiber $E_{\beta,n}$. For a path γ on $\mathcal{G}$ and $M\in E_{\beta,n}$,

$$\big(\mathsf{hol}^{\mathcal{U}}(\gamma)\big)(M) = \mathsf{hol}^U(\gamma)M\mathsf{hol}^U(\gamma)^*.$$

According to Theorem 2.4, for any $M, M' \in E_{\beta,n}$, $x, y \in V$ and F a bounded measurable function $\mathbb{R}^V \to \mathbb{R}$,

$$\Big\langle \mathrm{Tr}\,\big(\widehat{\Phi}(x)M\big)\,\mathrm{Tr}\,\big(\widehat{\Phi}(y)M'\big)F(\mathrm{Tr}(\widehat{\Phi}^2)/2)\Big\rangle^{U}_{\beta,n}$$
$$= \int_{\gamma}\Big\langle F\big(\,\mathrm{Tr}(\widehat{\Phi}^2)/2 + L(\gamma)\big)\Big\rangle^{U}_{\beta,n}\mathrm{Tr}(M\big(\mathsf{hol}^{\mathcal{U}}(\gamma)\big)(M'))\mu^{x,y}(d\gamma)$$
$$= \int_{\gamma}\Big\langle F\big(\,\mathrm{Tr}(\widehat{\Phi}^2)/2 + L(\gamma)\big)\Big\rangle^{U}_{\beta,n}\mathrm{Tr}(M\mathsf{hol}^{U}(\gamma)M'\mathsf{hol}^{U}(\gamma)^*)\mu^{x,y}(d\gamma).$$

For $i,j \in \{1,\dots,n\}$, let D^{ij} be the matrix with all entries zero, except the entry ij, which equals 1. Then the following holds.

- In the case $\beta = 1$, $\widehat{\Phi}_{ij}(x) = \mathrm{Tr}\,\big(\widehat{\Phi}(x)(D^{ij}/2 + D^{ji}/2)\big)$, $\widehat{\Phi}_{i'j'}(y) = \mathrm{Tr}\,\big(\widehat{\Phi}(y)(D^{i'j'}/2 + D^{j'i'}/2)\big)$, and

$$\mathrm{Tr}((D^{ij}/2+D^{ji}/2)\mathsf{hol}^{U}(\gamma)(D^{i'j'}/2+D^{j'i'}/2)\mathsf{hol}^{U}(\gamma)^*) = \mathsf{hol}^{U}_{ij'}(\gamma)\mathsf{hol}^{U}_{ji'}(\gamma)/2+\mathsf{hol}^{U}_{ii'}(\gamma)\mathsf{hol}^{U}_{jj'}(\gamma)/2.$$

  This includes the case $i = j$ or $i' = j'$.
- In the case $\beta = 2$,

$$\widehat{\Phi}_{ij}(x) = \mathrm{Tr}\,\big(\widehat{\Phi}(x)(D^{ij}/2 + D^{ji}/2)\big) + \mathbf{i}\,\mathrm{Tr}\,\big(\mathbf{i}\widehat{\Phi}(x)(D^{ij}/2 - \mathbf{i}D^{ji}/2)\big) = \mathrm{Tr}\,\big(\widehat{\Phi}(x)D^{ji}\big),$$
$$\widehat{\Phi}_{i'j'}(y) = \mathrm{Tr}\,\big(\widehat{\Phi}(y)(D^{i'j'}/2 + D^{j'i'}/2)\big) + \mathbf{i}\,\mathrm{Tr}\,\big(\mathbf{i}\widehat{\Phi}(y)(D^{i'j'}/2 - \mathbf{i}D^{j'i'}/2)\big) = \mathrm{Tr}\,\big(\widehat{\Phi}(y)D^{j'i'}\big),$$

  and

$$\mathrm{Tr}(D^{ji}\mathsf{hol}^{U}(\gamma)D^{j'i'}\mathsf{hol}^{U}(\gamma)^*) = \mathsf{hol}^{U}_{ij'}(\gamma)\overline{\mathsf{hol}^{U}_{ji'}(\gamma)}.$$

  This includes the case $i = j$ or $i' = j'$.

□

In what follows, we will give a detailed proof of Theorem 2.8 only in the case $\beta = 1$. The case $\beta = 2$ is similar, and actually simpler, since no twisted ribbon edges are involved.

As in Section 3.1, we will consider the space of formal polynomials $\mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant|\alpha|}}\Big]$. We will also use the formal variables $(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1\leqslant k\leqslant k'\leqslant|\alpha|}$. In addition to the latter, we will also introduce formal commuting variables $(\mathcal{X}_{ij}(k,k'))_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant k'\leqslant|\alpha|}}$, also commuting with $(\mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{1\leqslant k\leqslant k'\leqslant|\alpha|}$. For $k > k'$, we will identify $\mathcal{X}_{ij}(k,k')$ to $\mathcal{X}_{ji}(k',k)$. We will see $(\mathcal{X}_{ij}(k,k'))_{1\leqslant i,j\leqslant n}$ as a formal matrix $\mathcal{X}(k,k')$. We will use the space of formal polynomials $\mathbb{C}\Big[(\mathcal{X}_{ij}(k,k'), \mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant k'\leqslant|\alpha|}}\Big]$. We also consider the $\mathbb{C}$-linear map

$$\widehat{\mathcal{E}} : \mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant|\alpha|}}\Big] \to \mathbb{C}\Big[(\mathcal{X}_{ij}(k,k'), \mathsf{Y}_{kk'}, \widehat{\mathsf{Y}}_{kk'})_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant k'\leqslant|\alpha|}}\Big]$$

defined by the following rules.

(1) $\widehat{\mathcal{E}}(1) = 1$.
(2) For any monomial R of odd total degree, $\widehat{\mathcal{E}}(R) = 0$.
(3) For any monomial of form $\mathsf{M}_{ij}(k)\mathsf{M}_{i'j'}(k')$, with $k \leqslant k'$,

$$\widehat{\mathcal{E}}(\mathsf{M}_{ij}(k)\mathsf{M}_{i'j'}(k')) = \mathcal{X}_{ij'}\mathcal{X}_{ji'}(k,k')\mathsf{Y}_{kk'} + \mathcal{X}_{ii'}\mathcal{X}_{jj'}(k,k')\widehat{\mathsf{Y}}_{kk'}.$$

(4) For any monomial of even total degree of form $M_{i_1j_1}(k_1)M_{i_2j_2}(k_2)\dots M_{i_{2q}j_{2q}}(k_{2q})$,

$$\widehat{\mathcal{E}}\big(M_{i_1j_1}(k_1)M_{i_2j_2}(k_2)\dots M_{i_{2q}j_{2q}}(k_{2q})\big) = \sum_{\substack{\text{partitions of}\\ \{1,\dots,2q\}\\ \text{in pairs}\\ \{\{a_1,b_1\},\dots,\{a_q,b_q\}\}}} \prod_{l=1}^{q}\widehat{\mathcal{E}}(\mathsf{M}_{i_{a_l}j_{a_l}}(k_{a_l})\mathsf{M}_{i_{b_l}j_{b_l}}(k_{b_l})),$$

where the sum runs over the $(2q)!/(2^q q!)$ partitions of $\{1,\ldots,2q\}$ in pairs.

Let $\mathcal{V}_{\mathcal{X},\mathsf{Y}}$ be the linear subspace of $\mathbb{C}\Big[(\mathcal{X}_{ij}(k,k'),\mathsf{Y}_{kk'},\widehat{\mathsf{Y}}_{kk'})_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant k'\leqslant|\alpha|}}\Big]$ spanned by the monomials R satisfying the following conditions:

- $\forall\ \ 1\leqslant k\leqslant k'\leqslant|\alpha|$, $\deg_{\mathsf{Y}_{kk'}}R+deg_{\widehat{\mathsf{Y}}_{kk'}}R\leqslant 1$;
- $\forall\ \ 1\leqslant k\leqslant k'\leqslant|\alpha|$, if $\sum_{1\leqslant i,j\leqslant n}\deg_{\mathcal{X}_{ij}(k,k')}R\leqslant 1$, then $\deg_{\mathsf{Y}_{kk'}}R+deg_{\widehat{\mathsf{Y}}_{kk'}}R=1$,

where the notation deg with a variable in subscript means the partial degree in this variable. If $P\in\mathcal{V}_{\mathcal{X},\mathsf{Y}}$, one can construct out of P a real valued measure on the space of multi-path Ω (3.1) by substituting each variable $\mathsf{Y}_{kk'}$ and $\widehat{\mathsf{Y}}_{kk'}$, with $k\leqslant k'$ by the measure $\frac{1}{2}\mu^{x_k,x_{k'}}(d\gamma_{kk'})$, and each variable $\mathcal{X}_{ij}(k,k')$ by the density $\mathsf{hol}^U_{ij}(\gamma_{kk'})$. Indeed, on $\mathcal{V}_{\mathcal{X},\mathsf{Y}}$ there is no ambiguity to which path the holonomy refers, so the measure actually makes sense. We will denote by $\mathcal{V}_{\mathsf{M}}$ the linear subspace of $\mathbb{C}\Big[(\mathsf{M}_{ij}(k))_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant|\alpha|}}\Big]$ spanned by the monomials R such that $\forall k\in\{1,\ldots,|\alpha|\}$, $\sum_{1\leqslant i,j\leqslant n}\deg_{\mathsf{M}_{ij}(k)}\leqslant 1$. It is easy to see that $\widehat{\mathcal{E}}(\mathcal{V}_{\mathsf{M}})\subset\mathcal{V}_{\mathcal{X},\mathsf{Y}}$. For polynomials $\mathcal{P}\in\mathcal{V}_{\mathsf{M}}$, one has the analogue of Lemma 3.1. We state it without proof, as it is a straightforward consequence of Lemma 3.2 and of the definition of $\widehat{\mathcal{E}}$.

Lemma 3.3. *Let be a polynomial $\mathcal{P}\in\mathcal{V}_{\mathsf{M}}$. For F a bounded measurable function $\mathbb{R}^V\to\mathbb{R}$, one has the following equality:*

$$\Big\langle\mathcal{P}\Big((\mathsf{M}_{ij}(k)=\widehat{\Phi}_{ij}(x_k))_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant|\alpha|}}\Big)F(\mathrm{Tr}(\widehat{\Phi}^2)/2)\Big\rangle_{\beta=1,n}=$$

$$\int_{\Omega}\Big\langle F\big(\,\mathrm{Tr}(\widehat{\Phi}^2)/2+\sum_{\gamma\in\omega}L(\gamma)\big)\Big\rangle_{\beta=1,n}\times$$

$$\times\,\widehat{\mathcal{E}}(\mathcal{P})\big((\mathcal{X}_{ij}(k,k')=\mathsf{hol}^U_{ij}(\gamma_{kk'}),\mathsf{Y}_{kk'}=\widehat{\mathsf{Y}}_{kk'}=\frac{1}{2}\mu^{x_k,x_{k'}}(d\gamma_{kk'}))_{\substack{1\leqslant i,j\leqslant n\\ 1\leqslant k\leqslant k'\leqslant|\alpha|}}\big)(d\omega).$$

We can now prove Theorem 2.8.

Proof of Theorem 2.8, case $\beta=1$. We will use the notations of the proofs of Theorems 2.6 and 2.7.

For $\rho\in\mathcal{R}_\alpha$ a ribbon pairing and $\mathtt{t}\in\mathcal{T}_\alpha(\rho)$ a trail, we will associated to $\mathtt{t}$ a product of matrices $A(k,k')$ or $A(k',k)^{\mathsf{T}}$, and $\mathcal{X}(k,k')$. For each sequence $k\to k'$ in the trail $\mathtt{t}$ we add the factor $A(k,k')$ to the product, and for each sequence $k\leftarrow k'$, we add the factor $A(k',k)^{\mathsf{T}}$. Moreover, for each sequence $k\doteq k'$, we add to the product the factor $\mathcal{X}(k,k')$. In the product the factors respect the cyclic order on the trail. We will denote this product $\Pi_{\mathtt{t}}(\mathcal{X},A)$. Although the product itself depends not only on the trail $\mathtt{t}$, but on the specific representative of the equivalence class of $\mathtt{t}$, the trace $\mathrm{Tr}\left(\Pi_{\mathtt{t}}(\mathcal{X},A)\right)$ does not depend on the specific representative chosen.

Let $\widehat{Q}_{\alpha,A}$ be the polynomial of $\mathbb{C}\Big[(\mathcal{X}_{ij}(k,k'),\mathsf{Y}_{kk'},\widehat{\mathsf{Y}}_{kk'})_{\substack{1\leqslant i,j\leqslant n,\\ 1\leqslant k\leqslant k'\leqslant|\alpha|}}\Big]$ obtained by the partial substitution

$$\begin{aligned}\widehat{Q}_{\alpha,A}&=Q_\alpha\big((\mathsf{Z}_{\mathtt{t}}=\mathrm{Tr}\left(\Pi_{\mathtt{t}}(\mathcal{X},A)\right))_{\mathtt{t}\text{ trail}},(\mathsf{Y}_{kk'},\widehat{\mathsf{Y}}_{kk'})_{1\leqslant k<k'\leqslant|\alpha|}\big)\\&=\sum_{\rho\in\mathcal{R}_\alpha}\Big(\prod_{\mathtt{t}\in\mathcal{T}_\alpha(\rho)}\mathrm{Tr}\left(\Pi_{\mathtt{t}}(\mathcal{X},A)\right)\Big)\mathcal{Y}_\rho.\end{aligned}$$

The polynomial $\mathcal{P}_{\alpha,A}$ (3.5) is in $\mathcal{V}_{\mathsf{M}}$. So, Lemma 3.3 ensures that we only need to show that $\widehat{\mathcal{E}}(\mathcal{P}_{\alpha,A})=\widehat{Q}_{\alpha,A}$.

We have that

$$\widehat{\mathcal{E}}(\mathcal{P}_{\alpha,A}) =$$
$$\sum_{\rho\in\mathcal{R}_\alpha}\Big(\sum_{\overrightarrow{\imath\jmath}\in\mathcal{I}}\Pi_{\alpha,\overrightarrow{\imath\jmath}}(A)\prod_{\{a,b\}\in E_{\rm str}(\rho)}\mathcal{X}_{i_aj_b}(a,b)\mathcal{X}_{j_ai_b}(a,b)\prod_{\{a',b'\}\in E_{\rm tw}(\rho)}\mathcal{X}_{i_{a'}i_{b'}}(a',b')\mathcal{X}_{j_{a'}j_{b'}}(a,b)\Big)\mathcal{Y}_\rho.$$

Each time a factor $A_{j_ki_{k'}}(k,k')$, $\mathcal{X}_{i_aj_b}(a,b)$, $\mathcal{X}_{j_ai_b}(a,b)$, $\mathcal{X}_{i_{a'}i_{b'}}(a',b')$, respectively $\mathcal{X}_{j_{a'}j_{b'}}(a,b)$ appears in the product above, the corresponding pair of indices $j_ki_{k'}$, i_aj_b, j_ai_b, $i_{a'}i_{b'}$, respectively $j_{a'}j_{b'}$ lies on the same trail of ρ (see Figure 7). Thus, one can refactorize the product

$$\Pi_{\alpha,\overrightarrow{\imath\jmath}}(A)\prod_{\{a,b\}\in E_{\rm str}(\rho)}\mathcal{X}_{i_aj_b}(a,b)\mathcal{X}_{j_ai_b}(a,b)\prod_{\{a',b'\}\in E_{\rm tw}(\rho)}\mathcal{X}_{i_{a'}i_{b'}}(a',b')\mathcal{X}_{j_{a'}j_{b'}}(a,b)$$

trail by trail:

$$\prod_{\mathtt{t}\in\mathcal{T}_\alpha(\rho)}\Pi_{\mathtt{t},\overrightarrow{\imath\jmath}}(\mathcal{X},A).$$

Each factor $\Pi_{\mathtt{t},\overrightarrow{\imath\jmath}}(\mathcal{X},A)$ is a term in the expansion of the trace $\mathrm{Tr}\big(\Pi_{\mathtt{t}}(\mathcal{X},A)\big)$. ☐

4. The case of Dyson's Brownian motion

In this section we explain how Theorem 2.6 extends to the continuous one-dimensional case and the so-called Dyson's Brownian motion.

Our space will be $\mathbb{R}$. Let $(B_t)_{t\geqslant 0}$ denote a standard Brownian motion on $\mathbb{R}$, and T_0^B the first hitting time of 0. $p_t^B(x,y)$ will denote the heat kernel

$$p_t^B(x,y)=\frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{(y-x)^2}{2t}\Big).$$

$\mathbb{P}_t^{x,y,B}$ will denote the standard Brownian bridge probability measure from x to y in time t. Fix $\kappa>0$ a constant. $\mu_\kappa^{x,y}$ will be the following measure on Brownian paths from x to y:

$$\mu_\kappa^{x,y}(d\gamma)=\int_0^{+\infty}\mathbb{P}_t^{x,y,B}(d\gamma)p_t^B(x,y)e^{-\kappa t}dt.$$

We see κdx as a uniform killing measure on $\mathbb{R}$, and $\mu_\kappa^{x,y}(d\gamma)$ as a measure on massive Brownian paths, using the Quantum Field Theory terminology, the mass being given by κ. Given a Brownian path γ, one can define its family of Brownian local times (see [RY99], Chapter VI):

$$\ell_t^B(\gamma)(x)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t\mathbf{1}_{|\gamma(s)-x|<\varepsilon}ds.$$

If γ is a Brownian path of total duration $T(\gamma)$, we will denote

$$L^B(\gamma)(x)=\ell^B_{T(\gamma)}(\gamma)(x).$$

The Green's function of $-\frac{1}{2}\frac{d^2}{dx^2}+\kappa$ on $\mathbb{R}$ is

$$G_\kappa(x,y)=\frac{1}{\sqrt{2\kappa}}e^{-\sqrt{2\kappa}|y-x|}=\mathbb{E}_x\Big[\kappa\int_0^{+\infty}\ell_t^B(B)(y)e^{-\kappa t}dt\Big].$$

$G_\kappa(x,y)$ is the total mass of the measure $\mu_\kappa^{x,y}$.

Consider now that $(\phi_{ij}^{\mathbf{r}}(x))_{x\in\mathbb{R},1\leqslant i\leqslant j\leqslant n}$, $(\phi_{ij}^{\mathbf{i}}(x))_{x\in\mathbb{R},1\leqslant i<j\leqslant n}$ are n^2 i.i.d. stochastic processes, with $(\phi_{ij}^{\mathbf{r}}(x))_{x\in\mathbb{R}}$ a stationary Ornstein-Uhlenbeck process, solution to the stochastic differential equation

$$d\phi_{ij}^{\mathbf{r}}(x)=\sqrt{2}dW(x)-\sqrt{2\kappa}\phi_{ij}^{\mathbf{r}}(x)dx,$$

$dW(x)$ being the white noise. The covariances of $(\phi_{ij}^{\mathbf{r}}(x))_{x\in\mathbb{R}}$ are given by the massive Green's function G_κ. For $\beta\in\{1,2\}$, let $(\Phi^{(\beta,n)}(x))_{x\in\mathbb{R}}$ be the matrix valued stochastic process, with values in $E_{\beta,n}$ defined as in (2.14). Its family of eigenvalues $\lambda_1^{(\beta,n)}(x)\geqslant\lambda_2^{(\beta,n)}(x)\geqslant\cdots\geqslant$

$\lambda_n^{(\beta,n)}(x)$ is the Dyson's Brownian motion (see [Meh04], Chapter 9). It is stationary and satisfies the SDE

$$d\lambda_i^{(\beta,n)}(x) = \sqrt{2}dW_i(x) - \sqrt{2\kappa}\lambda_i^{(\beta,n)}(x)dx + \beta\sqrt{2\kappa}\sum_{\substack{1\leqslant j\leqslant n\\ j\neq i}}\frac{1}{\lambda_i^{(\beta,n)}(x)-\lambda_j^{(\beta,n)}(x)}dx,$$

where $dW_i(x)$ are independent white noises. The invariant distribution of $(\lambda_1^{(\beta,n)},\lambda_2^{(\beta,n)},\dots,\lambda_n^{(\beta,n)})$ is proportional to

$$\mathbf{1}_{\lambda_1\geqslant\lambda_2\geqslant\dots\geqslant\lambda_n}\prod_{1\leqslant i<j\leqslant n}(\lambda_i-\lambda_j)^\beta e^{-\sqrt{\frac{\kappa}{2}}(\lambda_1^2+\dots+\lambda_n^2)}d\lambda_1\dots d\lambda_n.$$

Let be $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_{m(\alpha)})$, where for all $l\in\{1,2,\dots,m(\alpha)\}$, $\alpha_l\in\mathbb{N}\backslash\{0\}$, and $|\alpha|$ even. Let be $x_1,x_2,\dots,x_{|\alpha|}\in\mathbb{R}$ with

$$x_1=\dots=x_{\alpha_1},\qquad x_{\alpha_1+1}=\dots=x_{\alpha_1+\alpha_2},\qquad \dots,\qquad x_{|\alpha|-\alpha_{m(\alpha)}+1}=\dots=x_{|\alpha|}.$$

So there are at most $m(\alpha)$ distinct points, and we use multiple indices for the same point for commodity of notations. Let $\mu_{\alpha,\beta,n,\kappa}^{(x_{\alpha_1},\alpha_1),\dots,(x_{|\alpha|},\alpha_{m(\alpha)})}$ be the measure on families of $|\alpha|/2$ Brownian paths obtained by replacing in the polynomial P_α the variable N by n, the variables $\mathsf{Y}_{kk'}$ by the measures $w_{\mathrm{str}}(\beta)\mu_\kappa^{x_k,x_{k'}}$, and the variables $\widehat{\mathsf{Y}}_{kk'}$ by the measures $w_{\mathrm{tw}}(\beta)\mu_\kappa^{x_k,x_{k'}}$. Theorem 2.6 generalizes in the continuous one-dimensional setting in a straightforward way as follows.

Proposition 4.1. *For $\beta\in\{1,2\}$ and F a bounded measurable function $\mathbb{R}^V\to\mathbb{R}$, one has the following equality:*

$$\mathbb{E}\Big[\prod_{l=1}^{m(\alpha)}\Big(\sum_{i=1}^n\lambda_i^{(\beta,n)}(x_{\alpha_l})^{\alpha_l}\Big)F\Big(\frac{1}{2}\sum_{i=1}^n(\lambda_i^{(\beta,n)})^2\Big)\Big] =$$

$$\int_{\gamma_1,\dots\gamma_{|\alpha|/2}}\mathbb{E}\Big[F\Big(\frac{1}{2}\sum_{i=1}^n(\lambda_i^{(\beta,n)})^2+L^B(\gamma_1)+\dots+L^B(\gamma_{|\alpha|/2})\Big)\Big]\mu_{\alpha,\beta,n,\kappa}^{(x_{\alpha_1},\alpha_1),\dots,(x_{|\alpha|},\alpha_{m(\alpha)})}(d\gamma_1,\dots,d\gamma_{|\alpha|/2}),$$

where $\mathbb{E}[\ \cdot\ ]\mu_{\alpha,\beta,n,\kappa}^{x_1,x_2,\dots,x_{|\alpha|}}(\cdot)$ is a product measure. In particular, the moments are given by

$$\mathbb{E}\Big[\prod_{l=1}^{m(\alpha)}\Big(\sum_{i=1}^n\lambda_i^{(\beta,n)}(x_{\alpha_l})^{\alpha_l}\Big)\Big]$$
$$= P_\alpha(\mathsf{N}=n,(\mathsf{Y}_{kk'}=w_{\mathrm{str}}(\beta)G_\kappa(x_k,x_{k'}),\widehat{\mathsf{Y}}_{kk'}=w_{\mathrm{tw}}(\beta)G_\kappa(x_k,x_{k'}))_{1\leqslant k<k'\leqslant|\alpha|}).$$

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