Output Feedback Stabilization by Reduced Order Finite Time Observers using a Trajectory Based Approach

Frédéric Mazenc, Saeed Ahmed, Michael Malisoff

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Abstract— **We use finite time reduced order continuous-discrete observers to solve an output feedback stabilization problem for a broad class of nonlinear systems whose output contains uncertainty. Unlike earlier works, our feedback control is discontinuous, but it does not contain any distributed terms. We use a trajectory based approach based on a contractivity condition. We illustrate our new control design using a tracking problem for nonholonomic systems in chained form.**

Index Terms— **Observer, stability, time-varying**

I. Introduction

This paper continues our development and use of finite time observers that can cope with uncertain or intermittent output observations and nonlinearities, while also reducing the dimension of the required observers. While our work [12] provided full order finite observers (whose dimensions equal the dimension of the original system) that allowed intermittent output observations, and our work [10], [11] provided reduced order observers that led to continuous output feedback controllers, the present work provides an alternative to [11] in which the controls contain a mixture of continuous and discrete time dynamics (and therefore are called continuous-discrete) but do not contain any distributed terms. This can help further reduce the computational burden by eliminating the need for distributed terms; see [2] and [14] for the relevance of distributed terms.

Our work is motivated by the importance of estimating values of solutions of systems, which produces difficult challenges from the applied and theoretical viewpoints. Much of the observers literature is based on the Luenberger observer (from [7] and [8]) or other asymptotic observers, which have been constructed for large families of nonlinear systems. On the other hand, there are significant applications that call for finite time state estimation, e.g., fault detection [18], where asymptotic observers may present the disadvantage that they only present a useful estimate after a transient period.

Finite time observers can be used to exactly construct the solutions in an arbitrarily short amount of time when there are no perturbations, and they can quantify the effects of the perturbations on the estimation error. Some finite time observers such as [6] and [17] use nonsmooth functions, but since they are based on homogeneity properties, they do not lend themselves to the design of smooth observers. Other finite time observers are computed using past values of the output or dynamic extensions; see for instance the works [3] and [19] for linear systems, and see [13], [16], and [20] for analogs for nonlinear systems. These earlier finite time observers provide estimates for all of the state variables, which can produce redundancies because oftentimes, some of the state components are already available for measurement and therefore do not need to be estimated by observers.

By adapting the main results of [13] and [20], our work [11] constructed finite time reduced order observers for a family of nonlinear time-varying systems. We will use the main result of [11] as a key building block for our control design in this work, which we believe provides the first reduced order finite time observer for nonlinear systems that does not require distributed terms. Studying time-varying systems is motivated by the fact that tracking problems can be recast into problems whose objectives are the stabilization of the zero equilibrium of a time-varying system (namely, the tracking error dynamics). As was the case for [4] and [1, Chapt. 4, Sec. 4.4.3], our work only provides estimate of the unmeasured variables, leading to an output feedback control that can be computed using the observer values and the perturbed measurements of the outputs. By reducing the order of the observer, we obtain more user friendly observer and feedback formulas, where one computes the fundamental matrix for a system whose dimension is the dimension of the unmeasured variable (instead of the higher dimension of the original system). This is valuable because of the well known difficulty of computing fundamental matrices for higher order systems.

After presenting our class of systems and our assumptions and theorem in Section II, we provide two key lemmas in Section III including a trajectory based result from [15]. We prove our theorem in Section IV, and we apply our method to a nonholonomic dynamics in Section V. We close in Section VI with our ideas for future research.

We use standard notation, in which the dimensions of our Euclidean spaces are arbitrary unless otherwise noted, and which will be simplified whenever no confusion would arise. We use $|\cdot|$ to denote the usual Euclidean norm and the induced matrix norm, $|\cdot|_J$ is the sup over any interval J, $|\cdot|_\infty$ is the usual essential supremum, and I is the identity matrix in the dimension under consideration. Given a constant $\tau > 0$ and a continuous function $\varphi : [-\tau, +\infty) \to \mathbb{R}^n$ and values $t \geq 0$, we define φ_t by $\varphi_t(m) = \varphi(t + m)$ for all $m \in [-\tau, 0]$. For each continuous function $\Omega : [-\tau, +\infty) \to$

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Malisoff was supported by NSF Grant 1711299.

$\mathbb{R}^{n\times n}$, let Φ_Ω denote the function such that

$$\tfrac{\partial \Phi_\Omega}{\partial t}(t,t_0) = -\Phi_\Omega(t,t_0)\Omega(t) \tag{1}$$

and $\Phi_\Omega(t_0,t_0) = I$ for all $t \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. Then $\mathcal{M}(t,s) = \Phi_\Omega^{-1}(t,s)$ is the fundamental solution associated to Ω for $\dot{x} = \Omega(t)x$; see [21, Lemma C.4.1]. We let $\mathcal{K}_\infty$ denote the set of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$ that are strictly increasing, unbounded, and satisfy $\gamma(0) = 0$. We say that a function $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ is proper and positive definite provided there are class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that $\underline{\alpha}(|x|) \le V(t,x) \le \overline{\alpha}(|x|)$ holds for all $(t,x) \in [0,\infty)\times\mathbb{R}^n$. We also use the standard definitions of input-to-state stability (or ISS) and $\mathcal{KL}$ functions [5]. Finally, we say that a function $\rho : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^m$ is globally Lipschitz with respect to its second variable uniformly in t provided there exists a constant $\bar{\rho} \ge 0$ such that $|\rho(t,a) - \rho(t,b)| \le \bar{\rho}|a-b|$ holds for all $t \ge 0$ and all a and b in $\mathbb{R}^n$.

II. Statement of Problem and Assumptions

We study systems of the form

$$\begin{cases} \dot{z}(t) = A_1(t)x_r(t) + B_1(t)u(t) + \rho_1(t,z(t)) + f_1(t) \\ \dot{x}_r(t) = A_2(t)x_r(t) + B_2(t)u(t) + \rho_2(t,z(t)) + f_2(t) \end{cases} \tag{2}$$

where z is valued in $\mathbb{R}^p$, x_r is valued in $\mathbb{R}^{n-p}$, the output is

$$y(t) = z(t) + \epsilon(t) \tag{3}$$

where ϵ is an unknown bounded piecewise continuous function, A_i and B_i for $i = 1,2$ are known piecewise continuous bounded matrix valued functions, $\rho = (\rho_1,\rho_2)$ is known and piecewise continuous with respect to t, and $f = (f_1,f_2)$ is an unknown locally bounded piecewise continuous function; see Remark 1 and Section V for motivation for studying systems of the form (2). We next state our three assumptions; see Remark 2 for ways to check our assumptions.

Assumption 1: There exist a known constant $\tau > 0$ and a known bounded function L of class C^1 with a bounded first derivative such that for all $t \in \mathbb{R}$ and with the choice $H(t) = A_2(t) + L(t)A_1(t)$, the matrix

$$\Lambda(t) = \Phi_{A_2}(t,t-\tau) - \Phi_H(t,t-\tau) \tag{4}$$

is invertible. Also, Λ^{-1} is a bounded function of t.

Assumption 2: There exist a function u_s that is globally Lipschitz in its second variable uniformly in t, a C^1 proper and positive definite function V, positive constants c_1 and c_2, and a function γ of class $\mathcal{K}_\infty$ such that the time derivative of V along all solutions $\chi : [0,\infty) \to \mathbb{R}^n$ of

$$\left\{\begin{array}{rcl} \dot{z}(t) & = & A_1(t)x_r(t) + B_1(t)u(t) \\ & & +\,\rho_1(t,z(t)) + h_1(t) \\ \dot{x}_r(t) & = & A_2(t)x_r(t) + B_2(t)u(t) \\ & & +\,\rho_2(t,z(t)) + h_2(t) \end{array}\right. \tag{5}$$

with the choice $u(t) = u_s(t, x_r(t) + \mu_1(t), z(t) + \mu_2(t))$ satisfies $\dot{V}(t) \le -c_1 V(t,\chi(t)) + \gamma(|(\mu,h)(t)|)$ for all choices of the locally bounded piecewise continuous functions $\mu = (\mu_1,\mu_2)$ and $h = (h_1,h_2)$ and all $t \ge 0$, and its time derivative along all trajectories $\chi : [0,\infty) \to \mathbb{R}^n$ of (2) with the choice $u(t) = 0$ satisfies $\dot{V}(t) \le c_2 V(t,\chi(t)) + \gamma(|h(t)|)$ for all $t \ge 0$.

Assumption 3: The function $\rho = (\rho_1,\rho_2)$ is globally Lipschitz in its second variable uniformly in t and there is a function $\alpha \in \mathcal{K}_\infty$ such that $|\rho(t,a)| \le \alpha(|a|)$ for all $a \in \mathbb{R}^p$ and $t \ge 0$.

In terms of the functions $\chi = (z, x_r)$ and

$$\rho_4(t,z) = -[D(t)z + \rho_3(t,z)], \text{ where} \tag{6}$$

$$\begin{array}{l} \rho_3(t,z) = L(t)\rho_1(t,z) + \rho_2(t,z) \text{ and} \\ D(t) = \dot{L}(t) - H(t)L(t) \end{array} \tag{7}$$

where H and L are from Assumptions 1-2, we will prove:

Theorem 1: Let Assumptions 1-3 hold, and let $T > 0$ be a constant such that

$$\tau < \tfrac{Tc_1}{c_1+c_2} \tag{8}$$

and set $t_i = iT$ for each integer $i \ge 0$. Then we can construct functions $\bar{\beta} \in \mathcal{KL}$ and $\bar{\gamma} \in \mathcal{K}_\infty$ such that the following ISS result is true: For all initial conditions $\chi(0) \in \mathbb{R}^n$, all solutions $\chi : [0,\infty) \to \mathbb{R}^n$ of the system (2), in closed loop with the control $u(t) = u_\star(t, \overline{x}_r(t), y(t))$ where

$$\begin{array}{l} u_\star(t,\overline{x}_r(t),y(t)) = \\ \left\{\begin{array}{ll} u_s(t,\overline{x}_r(t),y(t)) & \text{if } t \in [t_i, t_{i+1}-\tau) \text{ and } i \ge 0 \\ 0 & \text{otherwise} \end{array}\right. \end{array} \tag{9}$$

and where $\overline{x}_r$ is the state of the continuous-discrete observer

$$\begin{array}{rcl} \dot{\overline{x}}_r(t) & = & A_2(t)\overline{x}_r(t) + B_2(t)u_\star(t,\overline{x}_r(t),y(t)) \\ & & +\,\rho_2(t,y(t)) \text{ when } t \in (t_i,t_{i+1}) \text{ and } i \ge 0 \\ \overline{x}_r(t_i) & = & \Lambda(t_i)^{-1}\int_{t_i-\tau}^{t_i}[\Phi_{A_2}(m,t_i-\tau)\rho_2(m,y(m)) \\ & & +\Phi_H(m,t_i-\tau)\rho_4(m,y(m))]\,\mathrm{d}m \\ & & +\Lambda(t_i)^{-1}\left[\Phi_H(t_i,t_i-\tau)L(t_i)y(t_i)\right. \\ & & \left.-L(t_i-\tau)y(t_i-\tau)\right] \text{ for all } i \ge 1 \end{array} \tag{10}$$

with $\overline{x}_r(0) = 0$ satisfy $|\chi(t)| \le \bar{\beta}(|\chi(0)|,t) + \bar{\gamma}(|(\epsilon,f)|_{[0,t]})$ for all $t \ge 0$. □

Remark 1: To appreciate the importance of our class of systems (2), consider the class of systems $\dot{x}(t) = Ax(t) + \delta(t)$ where A is a constant matrix and $\delta : [0,+\infty) \to \mathbb{R}^n$ is a piecewise continuous locally bounded function, with an output $y(t) = Cx(t)$ that is valued in $\mathbb{R}^p$ with $p \le n$ where C is of full rank and the pair (A,C) is observable (but analogous considerations apply for more general systems). Since C is of full rank, there exist constant matrices C_T and A_1 and A_2, a linear change of coordinates $x_T = C_T x = [y^\top, x_r^\top]^\top$, and functions δ_i for $i = 1,2$ that are piecewise continuous with respect to their first argument and linear with respect to y such that the x_T system can be written as

$$\left\{\begin{array}{rcl} \dot{y}(t) & = & A_1 x_r(t) + \delta_1(t,y(t)) \\ \dot{x}_r(t) & = & A_2 x_r(t) + \delta_2(t,y(t)) \,. \end{array}\right. \tag{11}$$

Since (A,C) is observable, the pair (A_2,A_1) is observable; see [8, pp. 304-306]. Since (A_2,A_1) is observable, one can use [13, Lemma 1] to find a constant matrix $L \in \mathbb{R}^{(n-p)\times p}$ and a constant $\tau > 0$ such that $M_\tau = e^{-A_2\tau} - e^{-H\tau}$ with the choice $H = A_2 + LA_1$ is invertible. Hence, we obtain a class of systems that satisfy Assumption 1. □

Remark 2: Assumption 1 agrees with the main assumption from [12], which led to output feedbacks that have distributed terms. As noted in [12], if there exist a function L and two constants $\tau > 0$ and $\varpi \in (0,1)$ such that $|\Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)| \leq \varpi$ for all $t \geq 0$, then Assumption 1 is satisfied and Λ^{-1} is bounded. This follows because $I - \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)$ will be invertible for all $t \geq 0$ and

$$\begin{array}{l}\left[I - \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)\right]^{-1} = \\ \sum\limits_{k=0}^{\infty}\left[\Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)\right]^k\end{array} \tag{12}$$

for all $t \geq 0$, so the geometric sum formula gives

$$\begin{array}{l}\left|\Lambda(t)^{-1}\right| = \\ \left|\left(I - \Phi_{A_2}(t,t-\tau)^{-1}\Phi_H(t,t-\tau)\right)^{-1}\Phi_{A_2}(t,t-\tau)^{-1}\right| \\ \leq \frac{\left|\Phi_{A_2}(t,t-\tau)^{-1}\right|}{1-\varpi},\end{array}$$

which is bounded by a constant because A_2 and so also $\Phi_{A_2}(t,t-\tau)^{-1}$ are bounded. □

III. Key Lemmas

In this section, we provide two lemmas that we need to prove our theorem. The first lemma is from [12], and we summarize its proof in Appendix A below. The second is a contractivity lemma from [15]. Our first lemma is:

Lemma 1: Consider the system

$$\left\{\begin{array}{lcl}\dot{z}(t) & = & A_1(t)x_r(t) + \delta_1(t,z(t)) \\ \dot{x}_r(t) & = & A_2(t)x_r(t) + \delta_2(t,z(t))\end{array}\right. \tag{13}$$

where z is valued in $\mathbb{R}^p$, x_r is valued in $\mathbb{R}^{n-p}$, the output is (3) where $\epsilon(t)$ is a piecewise continuous bounded function, the functions A_i for $i = 1$ and 2 are piecewise continuous and bounded, and δ_1 and δ_2 are functions that are piecewise continuous with respect to t and that satisfy the requirements from Assumption 3 with the choice $\rho = (\delta_1, \delta_2)$. Let Assumption 1 hold, and choose $\delta_3(t,z) = L(t)\delta_1(t,z) + \delta_2(t,z)$ and $\delta_4(t,z) = -[D(t)z + \delta_3(t,z)]$, where $D(t) = \dot{L}(t) - H(t)L(t)$. Then for each solution of (13), we have

$$\begin{array}{l}x_r(t) = \Lambda(t)^{-1}\int_{t-\tau}^{t}\left[\Phi_{A_2}(m,t-\tau)\delta_2(m,y(m)-\epsilon(m))\right. \\ \left.+\Phi_H(m,t-\tau)\delta_4(m,y(m)-\epsilon(m))\right]\mathrm{d}m \\ +\Lambda(t)^{-1}\left[\Phi_H(t,t-\tau)L(t)(y(t)-\epsilon(t))\right. \\ \left.-L(t-\tau)(y(t-\tau)-\epsilon(t-\tau))\right]\end{array}$$

for all $t \geq \tau$. □

Lemma 2: Let $T_* > 0$ be a constant. Let $w : [-T_*, \infty) \to [0,\infty)$ be a piecewise continuous locally bounded function and $d : [0,\infty) \to [0,\infty)$ be piecewise continuous. Assume that there exists a constant $\lambda \in (0,1)$ such that

$$w(t) \leq \lambda |w|_{[t-T_*,t]} + d(t) \tag{14}$$

holds for all $t \geq 0$. Then the inequality

$$w(t) \;\leq\; |w|_{[-T_*,0]}e^{\frac{\ln(\lambda)}{T_*}t} + \tfrac{1}{1-\lambda}|d|_{[0,t]} \tag{15}$$

holds for all $t \geq 0$. □

IV. Proof of Theorem 1

Let us introduce the error variable $\tilde{x}_r(t) = x_r(t) - \overline{x}_r(t)$. Then for all integers $i \geq 0$, our formulas (2) and (10) give

$$\begin{array}{rcl}\dot{\tilde{x}}_r(t) & = & A_2(t)\tilde{x}_r(t) + \rho_2(t,z(t)) + f_2(t) \\ & & -\,\rho_2(t,z(t)+\epsilon(t)) \text{ for all } t \in (t_i, t_{i+1}) \\ \tilde{x}_r(t_i) & = & x_r(t_i) - \overline{x}_r(t_i).\end{array} \tag{16}$$

Since $u_\star(t,\overline{x}_r(t),y(t)) = 0$ for all $t \in [t_i-\tau, t_i)$ and integers $i \geq 1$, we also have

$$\begin{array}{l}x_r(t_i) = \Lambda(t_i)^{-1}\int_{t_i-\tau}^{t_i}\left[\Phi_{A_2}(m,t_i-\tau)\rho_2(m,z(m))\right. \\ \left.+\Phi_H(m,t_i-\tau)\rho_4(m,z(m))\right]\mathrm{d}m + \Delta(t_i) \\ +\Lambda(t_i)^{-1}[\Phi_H(t_i,t_i-\tau)L(t_i)z(t_i) - L(t_i-\tau)z(t_i-\tau)]\end{array} \tag{17}$$

for all $i \geq 1$, where

$$\begin{array}{l}\Delta(t) = \Lambda(t)^{-1}\int_{t-\tau}^{t}\left[\Phi_{A_2}(m,t-\tau)f_2(m)\right. \\ \left.-\,\Phi_H(m,t-\tau)\left(L(m)f_1(m) + f_2(m)\right)\right]\mathrm{d}m\end{array} \tag{18}$$

for all $t \geq 0$; this follows by applying Lemma 1 with the choices $\delta_i(t,z) = B_i(t)u(t) + \rho_i(t,z) + f_i(t)$ for $i = 1, 2$ and then noting that $u(t) = 0$ on each of the intervals $[t_i - \tau, t_i]$ where (17) is being computed (by our formula (9)).

Consequently, our formulas (10) give

$$\begin{array}{l}\tilde{x}_r(t_i) = \\ \Lambda(t_i)^{-1}\int_{t_i-\tau}^{t_i}\left[\Phi_{A_2}(m,t_i-\tau)(\rho_2(m,z(m))\right. \\ \left.-\rho_2(m,y(m)))\right]\mathrm{d}m \\ +\Lambda(t_i)^{-1}\int_{t_i-\tau}^{t_i}\left[\Phi_H(m,t_i-\tau)(\rho_4(m,z(m))\right. \\ \left.-\rho_4(m,y(m)))\right]\mathrm{d}m \\ +\Lambda(t_i)^{-1}\left[\Phi_H(t_i,t_i-\tau)L(t_i)(z(t_i)-y(t_i))\right. \\ \left.-L(t_i-\tau)(z(t_i-\tau)-y(t_i-\tau))\right] + \Delta(t_i)\end{array} \tag{19}$$

for all $i \geq 1$. From Assumptions 1 and 3 and the bound $\sup_{t_i-\tau<m<t_i}|\Phi_{A_2}(m,t_i-\tau)| \leq e^{\tau|A_2|_\infty}$ (and an analogous bound for H), we can find a constant $c_4 > 0$ such that

$$|\tilde{x}_r(t_i)| \;\leq\; c_4 \sup_{s\in[t_i-\tau,t_i]}|(\epsilon,f)(s)| \tag{20}$$

for all $i \geq 1$, namely,

$$\begin{array}{l}c_4 = |\Lambda^{-1}|_\infty\left(e^{\tau|A_2|_\infty}(\bar{\rho}+1)\tau +\right. \\ \left.\tau e^{\tau|H|_\infty}(|D|_\infty + (|L|_\infty+1)(\bar{\rho}+1)) + \left(1+e^{\tau|H|_\infty}\right)|L|_\infty\right)\end{array}$$

where $\bar{\rho}$ is a global Lipschitz constant satisfying the Lipschitzness requirement from Assumption 3.

Moreover, by integrating the first equality in (16), we deduce there is a constant $c_5 > 0$ such that

$$\begin{array}{l}|\tilde{x}_r(t)| \leq c_5|\tilde{x}_r(t_i)| \\ +c_5\int_{t_i}^{t}|\rho_2(\ell,z(\ell)) - \rho_2(\ell,z(\ell)+\epsilon(\ell))|\mathrm{d}\ell \\ \text{for all } t \in (t_i, t_{i+1}) \text{ and } i \geq 0,\end{array} \tag{21}$$

namely, $c_5 = e^{|A_2|_\infty T}$. It follows from (20) and Assumption 3 that for all $t \in [t_i, t_{i+1}]$ and $i \geq 1$, we have

$$\begin{array}{rcl}|\tilde{x}_r(t)| & \leq & c_5c_4\sup\limits_{s\in[t_i-\tau,t_i]}|(\epsilon,f)(s)| + c_5\bar{\rho}\int_{t_i}^{t}|\epsilon(\ell)|\mathrm{d}\ell \\ & \leq & c_6\sup\limits_{s\in[t_i-\tau,t]}|(\epsilon,f)(s)|\;.\end{array}$$

where $c_6 = c_5(c_4 + \bar{\rho}T)$. Next observe that the closed-loop system admits the representation

$$\left\{\begin{array}{rcl} \dot{z}(t) & = & A_1(t)x_r(t)+B_1(t)u_a(t) \\ & & +\rho_1(t,z(t))+f_1(t) \\ \dot{x}_r(t) & = & A_2(t)x_r(t)+B_2(t)u_a(t) \\ & & +\rho_2(t,z(t))+f_2(t) \end{array}\right. \tag{22}$$

where $u_a(t)=u_\star(t,x_r(t)-\tilde{x}_r(t),z(t)+\epsilon(t))$. We deduce from Assumption 2 that, for all $t\in[t_{i+1}-\tau,t_{i+1})$ and $i\geq 1$,

$$\dot{V}(t)\leq c_2V(t,\chi(t))+\gamma(|f|_{[0,t]}) \tag{23}$$

and, when $t\in[t_i,t_{i+1}-\tau)$ and $i\geq 1$, we have $\dot{V}(t)\leq -c_1V(t,\chi(t))+\gamma\left(|(\tilde{x},\epsilon,f)(t)|\right)$.

Therefore, when $t\in[t_i-\tau,t_{i+1})$ and $i\geq 1$, we have (23), while when $t\in[t_i,t_{i+1}-\tau)$ and $i\geq 1$, we have

$$\dot{V}(t)\leq -c_1V(t,\chi(t))+\gamma\left((c_6+1)|(\epsilon,f)|_{[0,t]}\right). \tag{24}$$

Combining the previous two cases gives

$$\begin{array}{rcl} V(t,\chi(t)) & \leq & e^{c_2\tau-(T-\tau)c_1}V(t-T,\chi(t-T)) \\ & & +T^\sharp\gamma\left((c_6+1)|(\epsilon,f)|_{[0,t]}\right) \end{array} \tag{25}$$

for all $t\geq T$, where

$$T^\sharp=\left(\tfrac{1}{c_1}+\tfrac{1}{c_2}\right)\left(1+e^{(c_1+c_2)\tau}\right), \tag{26}$$

and where (25) was obtained by separately considering the cases where $t\in[t_i,t_{i+1}-\tau)$ or $t\in[t_{i+1}-\tau,t_{i+1})$ for some $i\geq 1$. Next note that (8) gives $c_2\tau-(T-\tau)c_1<0$. We can then complete the proof using Lemma 2, as follows.

We set

$$w(t)=V(t+T,\chi(t+T))$$

and $d(t)=T^\sharp\gamma((c_6+1)|(\epsilon,f)|_{[0,t+T]})$ along any solution of the closed loop system from the conclusion of the theorem, and $\lambda=e^{c_2\tau-(T-\tau)c_1}$. Then $\lambda\in(0,1)$, so the conclusion (14) from Lemma 2 is satisfied with $T_*=T$. Choosing class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that $\underline{\alpha}(|\chi|)\leq V(t,\chi)\leq\overline{\alpha}(|\chi|)$ for all t and χ, it follows from the conclusion (15) of Lemma 2 that

$$\begin{array}{rcl} \underline{\alpha}(|\chi(t)|) & \leq & e^{\ln(\rho)(t-T)/T}\overline{\alpha}(|\chi|_{[0,T]}) \\ & & +\frac{T^\sharp}{1-\rho}\gamma\left((c_6+1)|(\epsilon,f)|_{[0,t]}\right) \end{array} \tag{27}$$

for all $t\geq T$. Hence, we can use the fact that $\underline{\alpha}^{-1}(a+b)\leq\underline{\alpha}^{-1}(2a)+\underline{\alpha}^{-1}(2b)$ for all nonnegative values a and b to get

$$\begin{array}{rcl} |\chi(t)| & \leq & \underline{\alpha}^{-1}\left(2e^{\ln(\rho)(t-T)/T}\overline{\alpha}(|\chi|_{[0,T]})\right) \\ & & +\underline{\alpha}^{-1}\left(\frac{2T^\sharp}{1-\rho}\gamma\left((c_6+1)|(\epsilon,f)|_{[0,t]}\right)\right) \end{array} \tag{28}$$

for all $t\geq T$. We can also use our global Lipschitzness assumptions to find a constant $\bar{G}>0$ such that

$$|\chi(t)|\leq e^{\max\{0,T-t\}}\bar{G}|\chi(0)|+\bar{G}|(\epsilon,f)|_{[t_0,t]} \tag{29}$$

holds along all solutions of the closed loop system from the statement of the theorem for all $t\in[0,T]$. The final ISS estimate now follows by using (29) to upper bound the $|\chi|_{[0,T]}$ in (28), then using the property $\overline{\alpha}(a+b)\leq\overline{\alpha}(2a)+\overline{\alpha}(2b)$ for suitable nonnegative a and b and the same property for $\underline{\alpha}^{-1}$, and then taking the maximum of the resulting right side of (28) and (29), to get an ISS estimate for all $t\geq 0$.

V. Application to Nonholonomic System

A. Tracking problem

We illustrate Theorem 1 using this system from [9, p. 137]:

$$\left\{\begin{array}{rcl} \dot{\xi}_4 & = & \xi_3v_1 \\ \dot{\xi}_3 & = & \xi_2v_1 \\ \dot{\xi}_2 & = & v_2 \\ \dot{\xi}_1 & = & v_1 \end{array}\right. \tag{30}$$

with $(\xi_1,\xi_2,\xi_3,\xi_4)$ valued in $\mathbb{R}^4$ and the input $(v_1,v_2)\in\mathbb{R}^2$, which is a nonholonomic system in chained form. We assume that ξ_4, ξ_2 and ξ_1 are measured, but that ξ_3 is not measured. Therefore, we cannot integrate $\dot{\xi}_3=\xi_2v_1$ to obtain $\xi_3(t)$ because $\xi_3(0)$ is not known. Instead, we design a dynamic output feedback making the system (30) track the trajectory

$$(\xi_{1r}(t),\xi_{2r}(t),\xi_{3r}(t),\xi_{4r}(t))=\left(t+\tfrac{1}{2}\sin(t),0,0,0\right) \tag{31}$$

by combining Theorem 1 with a backstepping approach. While the preceding problem was solved in [10], this earlier work produced a feedback control that contained distributed terms (meaning, the observer and so also the feedback control were expressed in terms of an integral equation whose integrand contained past values of the estimate, and this integral equation did not admit an explicit solution). Here, we use our new method from Theorem 1 to produce a feedback control that is free of distributed terms.

Following [10], we use the time-varying change of variables and the feedback

$$x_1=\xi_1-\xi_{1r}(t)\text{ and }v_1(t,x_1)=-x_1+1+\frac{1}{2}\cos(t) \tag{32}$$

which produces the x_1 subsystem $\dot{x}_1=-x_1$ and which therefore prompts us to consider the problem of globally asymptotically stabilizing the tracking dynamics

$$\left\{\begin{array}{rcl} \dot{\xi}_4 & = & \xi_3\left[1+\frac{1}{2}\cos(t)\right] \\ \dot{\xi}_3 & = & \xi_2\left[1+\frac{1}{2}\cos(t)\right] \\ \dot{\xi}_2 & = & v_2 \end{array}\right. \tag{33}$$

to 0, by replacing x_1 by 0 in the dynamics. In the first step of our analysis, we apply Theorem 1 to

$$\left\{\begin{array}{rcl} \dot{z} & = & x_r\left[1+\frac{1}{2}\cos(t)\right] \\ \dot{x}_r & = & u\left[1+\frac{1}{2}\cos(t)\right]+f_2(t) \end{array}\right. \tag{34}$$

where u is the input, and then in a second step, we complete the stabilization design of (33) using backstepping.

B. Applying Theorem 1

The system (34) has the form (2) with $A_1(t)=B_2(t)=1+\frac{1}{2}\cos(t)$ and with ϵ, ρ_1, ρ_2, A_2, B_1, and f_1 all being the zero function. We next show how to satisfy the assumptions of Theorem 1 for the system 34. We choose $L(t)=-\frac{1}{3}$ for all $t\in\mathbb{R}$, which gives $H(t)=A_2(t)+L(t)A_1(t)=-\frac{1}{3}\left(1+\frac{1}{2}\cos(t)\right)$ for all $t\in\mathbb{R}$. Then for any constant $\tau>0$, we have $\Phi_{A_2}(t,t_0)=1$,

$$\begin{array}{l} \Phi_H(t,t_0)=e^{\frac{1}{3}(t-t_0)+\frac{1}{6}(\sin(t)-\sin(t_0))}, \\ \text{and }\Lambda(t)=1-e^{\frac{\tau}{3}+\frac{1}{6}[\sin(t)-\sin(t-\tau)]}, \end{array} \tag{35}$$

where Λ is defined in (4). Since $\frac{1}{3}\tau+\frac{1}{6}[\sin(t)-\sin(t-\tau)]\geq\frac{\tau}{6}>0$ for all $t\in\mathbb{R}$, the matrix $\Lambda(t)$ satisfies Assumption 1. Assumption 3 is satisfied with $\rho=0$.

We now check that Assumption 2 is satisfied with $u_s(t,\chi)=-2(x_r+z)$ and $V(\chi)=z^2+\frac{1}{2}x_r^2+zx_r$, where $\chi=(z,x_r)$. The system which corresponds to (5) is

$$\left\{\begin{array}{rcl}\dot z(t) & = & \left[1+\frac{1}{2}\cos(t)\right]x_r+h_1(t)\\ \dot x_r(t) & = & \left[1+\frac{1}{2}\cos(t)\right][-2(x_r+z)+\bar\mu(t)]\\ & & +h_2(t)\ ,\end{array}\right. \tag{36}$$

where $\bar\mu=-2(\mu_1+\mu_2)$. Along all solutions of (36),

$$\begin{array}{l}\dot V(t)=-2\left[1+\frac{1}{2}\cos(t)\right]V(\chi)+(2z+x_r)h_1(t)\\ +(x_r+z)\left[\bar\mu(t)\left(1+\frac{1}{2}\cos(t)\right)+h_2(t)\right]\\ \leq -V(\chi)+\{(2z+x_r)h_1(t)\\ +(x_r+z)\left[\bar\mu(t)\left(1+\frac{1}{2}\cos(t)\right)+h_2(t)\right]\}\end{array} \tag{37}$$

holds for all $t\geq 0$. We can also use the inequality $zx_r\geq-\frac{1}{3}x_r^2-\frac{3}{4}z^2$ to get the lower bound $V(\chi)\geq\frac{1}{6}x_r^2+\frac{1}{4}z^2$. We easily deduce that $\dot V(t)\leq-\frac{1}{2}V(\chi)+\gamma(|(\mu,h)|(t)|)$ holds along all solutions of the dynamics (36) for all $t\geq 0$, where $\gamma(s)=240s^2$ by using the bounds

$$ab\leq\tfrac{1}{96}a^2+24b^2\ \text{ and }\ ab\leq\tfrac{1}{48}a^2+12b^2$$

for suitable real numbers a and b to get

$$\begin{array}{l}(2z+x_r)h_1\leq\frac{1}{96}(2z+x_r)^2+24h_1^2\\ \leq\frac{1}{24}(z^2+x_r^2)+24h_1^2\ \leq\ \frac{1}{4}V(z,x_r)+24h_1^2\end{array}$$

and

$$\begin{array}{l}(x_r+z)\left[\bar\mu\left(1+\frac{1}{2}\cos(t)\right)+h_2(t)\right]\\ \leq\frac{1}{48}(x_r+z)^2+12\left[\bar\mu\left(1+\frac{1}{2}\cos(t)\right)+h_2(t)\right]^2\\ \leq\frac{1}{48}(x_r+z)^2+12(3|\mu_1+\mu_2|+|h_2(t)|)^2\\ \leq\frac{1}{4}V(z,x_r)+12(18)|(\mu,h)(t)|^2\end{array}$$

respectively. Therefore, we can choose $c_1=\frac{1}{2}$. To get a value for c_2, notice that along all solutions of

$$\left\{\begin{array}{rcl}\dot z(t) & = & \left[1+\frac{1}{2}\cos(t)\right]x_r+h_1(t)\\ \dot x_r(t) & = & h_2(t)\end{array}\right. \tag{38}$$

for all $t\geq 0$, we can use the triangle inequality to get

$$\begin{array}{rcl}\dot V & = & (2z+x_r)\left(1+\frac{1}{2}\cos(t)\right)x_r+(2z+x_r)h_1\\ & & +(x_r+z)h_2\\ & \leq & \frac{3}{2}(x_r^2+2|x_rz|)+\frac{1}{2}|h|^2+\frac{1}{2}(2z+x_r)^2\\ & & +\frac{1}{2}(x_r+z)^2\\ & \leq & \frac{3}{2}(2x_r^2+z^2)+5(x_r^2+z^2)+\frac{1}{2}|h|^2\\ & \leq & 48\left(\frac{1}{6}(z^2+x_r^2)\right)\frac{1}{2}|h|^2\ \leq\ 48V(z,x_r)+\frac{1}{2}|h|^2\end{array}$$

which allows us to choose $c_2=48$. Therefore, we can apply Theorem 1 using $D(t)=\dot L(t)-H(t)L(t)=-\frac{1}{9}\left(1+\frac{1}{2}\cos(t)\right)$.

By Theorem 1, it follows that the (ξ_4,ξ_3)-subsystem of

$$\left\{\begin{array}{rcl}\dot\xi_4 & = & \left(1+\frac{1}{2}\cos(t)\right)\xi_3\\ \dot\xi_3 & = & \left(1+\frac{1}{2}\cos(t)\right)\left(u_\star(t,\bar\xi_3,\xi_4)+\omega\right)\\ \dot\omega & = & v_2+2\xi_3(t)\left(1+\frac{\cos(t)}{2}\right)\end{array}\right. \tag{39}$$

satisfies ISS with respect to ω, where $\omega=\xi_2+2\xi_4$ and

$$\begin{array}{l}u_\star(t,\bar\xi_3(t),\xi_4(t))=\\ \left\{\begin{array}{ll}-2(\bar\xi_3(t)+\xi_4) & \text{if } t\in[t_i,t_{i+1}-\tau) \text{ and } i\geq 0\\ 0 & \text{otherwise}\end{array}\right.\end{array} \tag{40}$$

and $\bar\xi_3$ is the state of the continuous-discrete observer

$$\begin{array}{l}\dot{\bar\xi}_3(t)=\left(1+\frac{\cos(t)}{2}\right)u_\star(t,\bar\xi_3(t),\xi_4(t))\\ \text{for all } t\in(t_i,t_{i+1}) \text{ and } i\geq 0, \text{ and}\\ \bar\xi_3(t_i)=\\ -\Lambda(t_i)^{-1}\int_{t_i-\tau}^{t_i}\Phi_H(m,t_i-\tau)D(m)\xi_4(m)\mathrm{d}m\\ -\frac{1}{3}\Lambda(t_i)^{-1}\left[\Phi_H(t_i,t_i-\tau)\xi_4(t_i)-\xi_4(t_i-\tau)\right]\\ \text{for all } i\geq 1\end{array} \tag{41}$$

with $\bar\xi_3(0)=0$. We now set $v_2=-\omega(t)-2\bar\xi_3(t)(1+0.5\cos(t))$ to obtain

$$\begin{array}{l}\dot\xi_4(t)=\left(1+\frac{1}{2}\cos(t)\right)\xi_3\\ \dot\xi_3(t)=\left(1+\frac{1}{2}\cos(t)\right)\left(u_\star(t,\bar\xi_3,\xi_4)+\omega(t)\right)\\ \dot\omega(t)=-\omega(t)+(2+\cos(t))(\xi_3(t)-\bar\xi_3(t)).\end{array} \tag{42}$$

Hence, the desired UGAS property for (42) follows under an appropriate bound on T, because the ω subsystem of (42) is globally exponentially stable to 0; see Appendix B below.

VI. Conclusions

We advanced the state of the art for the design of observers for nonlinear systems, through the construction of reduced order finite time observers and corresponding output feedbacks that are free of distributed terms. Since our observers only required computing fundamental matrices for subsystems that have the dimension of the unknown states, and since we eliminated the need for distributed terms, our method can reduce the computational burden relative to existing methods. We hope to combine Theorem 1 with the result of [12] to cover delays and disturbances in the input and intermittent output observations.

Appendix A: Proof of Lemma 1

We summarize the proof of Lemma 1; see [12] for the proof. The variable $x_v(t)=\Phi_{A_2}(t,0)x_r(t)$ satisfies

$$\begin{array}{rcl}\dot x_v(t) & = & -\Phi_{A_2}(t,0)A_2x_r(t)\\ & & +\Phi_{A_2}(t,0)[A_2(t)x_r(t)+\delta_2(t,z(t))]\\ & = & \Phi_{A_2}(t,0)\delta_2(t,z(t)).\end{array} \tag{A.1}$$

Integrating (A.1) on $[t-\tau,t]$ for any $t\geq\tau$ gives

$$x_v(t)=x_v(t-\tau)+\int_{t-\tau}^t\Phi_{A_2}(m,0)\delta_2(m,z(m))\mathrm{d}m\ .$$

Then we can use the definition of x_v to get

$$\begin{array}{l}\Phi_{A_2}(t-\tau,0)^{-1}\Phi_{A_2}(t,0)x_r(t)=x_r(t-\tau)\\ +\int_{t-\tau}^t\Phi_{A_2}(t-\tau,0)^{-1}\Phi_{A_2}(m,0)\delta_2(m,z(m))\mathrm{d}m.\end{array} \tag{A.2}$$

Using the semigroup properties of flow maps and (A.2), we deduce that

$$\begin{array}{l}\Phi_{A_2}(t,t-\tau)x_r(t)=x_r(t-\tau)\\ +\int_{t-\tau}^t\Phi_{A_2}(m,t-\tau)\delta_2(m,z(m))\mathrm{d}m\ .\end{array} \tag{A.3}$$

Moreover, the choice $x_s(t) = x_r(t) + L(t)z(t)$ gives

$$\begin{array}{rcl} \dot{x}_s(t) & = & A_2(t)x_r(t) + \delta_2(t, z(t)) + \dot{L}(t)z(t) \\ & & +L(t)[A_1(t)x_r(t) + \delta_1(t, z(t))] \\ & = & H(t)x_r(t) + \dot{L}(t)z(t) + \delta_3(t, z(t)) \\ & = & H(t)x_s(t)+[\dot{L}(t)-H(t)L(t)]z(t)+\delta_3(t, z(t)). \end{array}$$

Using variation of parameters and viewing Φ_H as the inverse of the fundamental matrix for $\dot{q} = H(t)q$ now gives

$$\begin{array}{l} \Phi_H(t, t-\tau)x_s(t) = x_s(t-\tau)+ \\ \int_{t-\tau}^{t} \Phi_H(m, t-\tau)[D(m)z(m) + \delta_3(m, z(m))]\mathrm{d}m \end{array}$$

for all $t \geq \tau$. Then the definition of x_s gives

$$\begin{array}{l} \Phi_H(t, t-\tau)x_r(t) = x_r(t-\tau) \\ -\Phi_H(t, t-\tau)L(t)z(t) + L(t-\tau)z(t-\tau) \\ + \int_{t-\tau}^{t} \Phi_H(m, t-\tau)[D(m)z(m)+\delta_3(m, z(m))]\mathrm{d}m. \end{array} \quad \text{(A.4)}$$

The lemma now follows by subtracting (A.4) from (A.3), and then left multiplying the result by $\Lambda^{-1}(t)$.

APPENDIX B: ANALYSIS OF ω SUBSYSTEM OF (42)

To show the required exponential stability of the ω subsystem of (42), we will introduce a suitable condition on the sample rate T. To this end, first note that in the special case where A_2, ϵ, f_1, and ρ_2 are all the zero function, our formulas (16), (18), and (19) give $\dot{\hat{x}}_r(t) = f_2(t)$ and

$$\Delta(t_i) = \Lambda^{-1}(t_i) \int_{t_i-\tau}^{t_i} (1 - \Phi_H(m, t_i - \tau))\, f_2(m)\mathrm{d}m$$

and $\tilde{x}_r(t_i) = \Delta(t_i)$ for all $i \geq 1$. The preceding equalities imply that for all $i \geq 1$ and $t \in [t_i, t_{i+1})$, we have

$$\begin{array}{l} |\tilde{x}_r(t)| = \Big|\Lambda^{-1}(t_i) \int_{t_i-\tau}^{t_i} (1 - \Phi_H(m, t_i - \tau))\, f_2(m)\mathrm{d}m \\ + \int_{t_i}^{t} f_2(m)\mathrm{d}m\Big| \\ \leq\ 2 \int_{(t-2T)^+}^{t} |f_2(m)|\mathrm{d}m, \end{array}$$

where $a^+ = \max\{a, 0\}$ for all $a \in \mathbb{R}$, and where we used the bound $\max_{t_i-\tau\leq m\leq t_i} |\Lambda^{-1}(t_i)(1 - \Phi_H(m, t_i - \tau))| \leq 1$ (which follows from the monotonicity of $\Phi_H(m, t_i - \tau)$ as a function of m). Specializing the preceding analysis to our case where $x_r = \xi_3$ and $f_2 = (1 + 0.5\cos(t))\omega$, we obtain

$$\big|(1+0.5\cos(t))(\xi_3(t)-\overline{\xi}_3(t))\big| \leq 3 \int_{(t-2T)^+}^{t}|\omega(m)|\mathrm{d}m.$$

Hence, along all solutions of the ω subsystem of (42), we can use Jensen's inequality and then the triangle inequality to check that the time derivative of $V_0(\omega) = \frac{1}{2}\omega^2$ satisfies

$$\begin{array}{l} \dot{V}_0(\omega) \leq -\omega^2 + 3|\omega| \int_{(t-2T)^+}^{t} |\omega(m)|\mathrm{d}m \\ \leq -\frac{1}{2}\omega^2 + 18T \int_{(t-2T)^+}^{t} \omega^2(m)\mathrm{d}m \end{array} \quad \text{(B.1)}$$

for all $t \geq 0$. We now assume that

$$T < \frac{1}{6\sqrt{2}} \quad \text{(B.2)}$$

and we pick a constant $\epsilon_0 > 0$ close enough to zero such that $18T^2(2 + \epsilon_0) < \frac{1}{2}$. Then the time derivative of

$$\begin{array}{l} V_0^\sharp(\omega_t) = V_0(\omega(t)) \\ + (2 + \epsilon_0)9T \int_{(t-2T)^+}^{t} \int_{\ell}^{t} \omega^2(m)\mathrm{d}m\mathrm{d}\ell \end{array} \quad \text{(B.3)}$$

along all solutions of the ω subsystem of (42) satisfies

$$\begin{array}{l} \dot{V}_0^\sharp(\omega_t) \leq -\left(\frac{1}{2} - 2(2 + \epsilon_0)(3T)^2\right)\omega^2(t) \\ -9\epsilon_0 T \int_{(t-2T)^+}^{t} \omega^2(m)\mathrm{d}m \end{array}$$

for all $t \geq 0$, which implies that $V_0^\sharp$ and so also ω converge to zero exponentially, as desired.

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