WHEN ONE STOCK SHARE IS A BIOLOGICAL INDIVIDUAL: A STYLIZED SIMULATION OF THE POPULATION DYNAMICS IN AN ORDER-DRIVEN MARKET

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ABSTRACT

The demand-supply relationship plays an important role in an order-driven stock market. In this thesis, we propose a stylized model by defining demand (supply) over a stock at a certain time as how many shares are on the bid (ask) side, which includes all buy (sell) limit orders and buy (sell) market orders. We treat two types of shares as two different species with an interaction effect and construct generalized Lotka-Volterra equations based on some properties or assumptions of an order-driven market. Also, we apply the model to simulate how the population of the two types of shares evolves over time under the condition that there is no signal information influencing the decisions of investors. The model suggests that the population of bid and ask shares moves either to a fixed point in the phase space or exhibits periodical dynamics. Also, our model explains, though not perfectly, why it is that stock prices sometimes behave chaotically.

Keywords: Order-driven markets; Bid and ask shares; Generalized Lotka-Volterra equation; Demand-supply relationship; Population dynamics

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ODE	Ordinary Differential Equation
GLV	Generalized Lotka-Volterra
LV	Lotka-Volterra
LOB	Limit Order Book
ABM	Agent-based Model
CPU	Central Processing Unit
PCS	Personal Communications Services
LCD	Liquid-crystal-display
PDP	Plasma Display Panel

CHAPTER 1. INTRODUCTION

Nowadays, more than half of stock exchange markets are order-driven. Because of its trading mechanism, an order-driven market is also recognized as an auction market, where almost all transactions are executed by using either limit orders or market orders. There is no doubt that the demand-supply relationship plays an important role in such an "auction" market. For instance, the market price of the commodity (or stock) largely depends on the demand-supply relationship (Samuelson, 1951). To define demand in this thesis, we first check a general definition which describes demand as the quantity of a good that consumers are able to purchase at various prices during a given period of time (O'sullivan & Sheffrin, 2003). On the basis of this definition, a simple statement of the demand for a stock at a certain time can be the number of shares on the bid side of a stock, which includes all buy limit orders and buy market orders at that time. In other words, we evaluate the power of demand at a certain time by observing the population of shares in all buy orders at that time (both limit and market)[1]. Conversely, supply can be stated as the number of shares on the ask side, which indicates supply.

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[1] We assume that the stock is only traded in one exchange and there are only two types of orders.

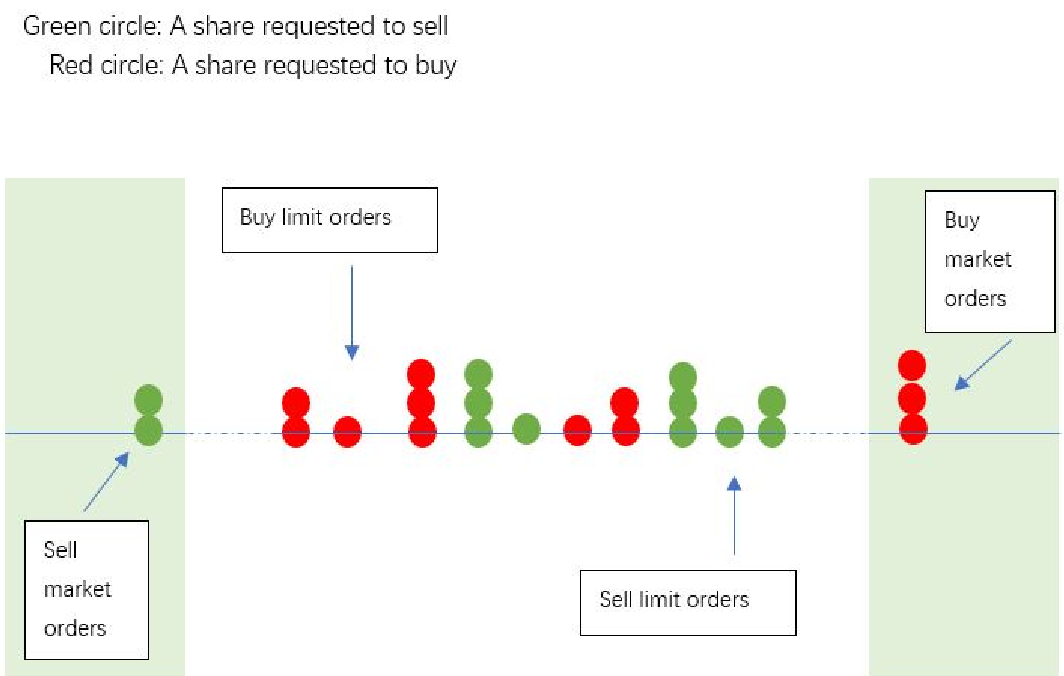

Figure 1: A "Snap" at a Single Stock in The Market: the horizontal line indicates the price grid

As we can see in fig 1. The demand at a certain time is indicated by the total number of red circles, while supply is indicated by the number of green circles. Studying such a population dynamic between two types of shares helps us to gain a better understanding of the demand-supply relationship in an order-driven stock market. By simply analyzing the trading process of an order-driven market, we determined that, for a single stock, the total number of shares on the bid side (red circles) and shares on the ask side (green circles) keeps changing with time. Moreover, the two categories of shares are similar to two animal species with an interaction effect.

First, we will briefly explain the basic trading mechanics of an order-driven market. As rules for lot size, which is the smallest amount of the asset that can be traded within it, are different in different stock exchanges, for the purposes of simplicity and clarity, we make the following assumption for this thesis. That is the lot size is always one share. Also, we use the term “bid share” to indicate a share on the bid side and “ask share” to indicate a share on the ask side. The bid price at time t is the highest price among all active bid shares, while the ask price at time t is the lowest price among all active ask shares. The mechanics here are straightforward. A newly placed bid share in a limit order, for instance, can either match an outstanding ask share and both of them “disappear” as they are matched and executed, or the bid share cannot match anything and becomes an outstanding share on the bid side of the limit order book. Moreover, in this thesis, we propose a method to distinguish the shares in market orders from those in limit orders. The trading rule is that, in fig 1, the example, bid shares in buy market orders match with ask shares in sell limit orders in the order from the lowest price to the highest price of sell limit orders until all current bid shares in buy market orders are executed. Meanwhile, ask shares in sell market orders match with bid shares in buy limit orders in the order from the highest price to the lowest price of sell limit orders until all current ask shares in buy market orders are executed. Then, if there are still some shares in limit orders which match in price, they will trigger transactions among shares in limit orders (see fig 2). Note that those three steps can happen at the same time, as long as the “snap” is determined. We place all bid shares in buy market orders on the price level of positive infinite, indicating that no matter what the prices are of those sell limit orders, transactions will happen. Similarly, all ask shares in sell markets orders are placed at the price level of 0. Moreover, if two shares have the same price, they will be executed from the bottom to the top (see fig 4)[2]. So, each share in the "snap" has a unique location on the price grid represented by the horizontal line. The "destiny" of each share is determined by its location. The "destiny" describes whether this share will be executed and, if it will, which share it will match. Overall, we distinguish shares in limit orders and market orders by their locations on the price grid, not by the "appearance" of those green and red circles (as what is shown in fig 1). In this thesis, we use X to indicate the total amount of bid shares, and Y to indicate the total amount of ask shares. It is obvious that the numbers of X and Y will change with time[3].

There are three factors affecting the population of two interactive animal species, namely newborns, interaction effect and death without interaction. There are also three factors responsible for the change in X and Y, and they correspond to those for species. The three factors for shares are newly placed orders, successful matches and cancellations. Now, we explore it further to see how such correspondences work.

The action of investors placing new orders in (which contain some bid-ask shares) is similar to individuals of a single species being born. This is the only way a

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In the real market, if two shares are at a same price, we always have a method to determine which share should be executed first by several dimensions, such as the time of placing orders, the size of the order and the grade of the investors. Then we array them from the bottom to the top based on the order of executing at that price level.

[3] Aggregating all shares is difficult in practice, we just propose an idea here and such an idea is theoretically possible.

a population can grow. Cancelation happens when investors do not want to trade anymore, or do not think their orders are at appropriate prices, therefore they want to cancel their orders. Cancelation is motivated by investors themselves, not by shares on the opposite side. Thus, it should be viewed as a natural death or as a death not impacting the population of the other species. Both cancelation and death lead to a decrease in population. If a bid share is at a price higher than or equal to a price where an ask share is, then the transaction happens, and both of them will be removed from the "auction" market. Note that a transaction happens only when two shares have an interaction.

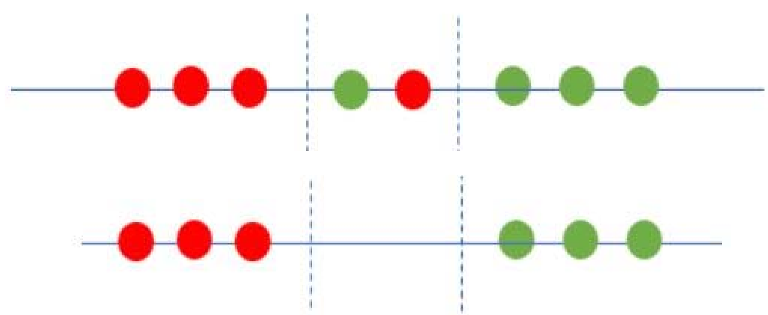

Figure 2: A Successful Match Among Shares in Limit Orders

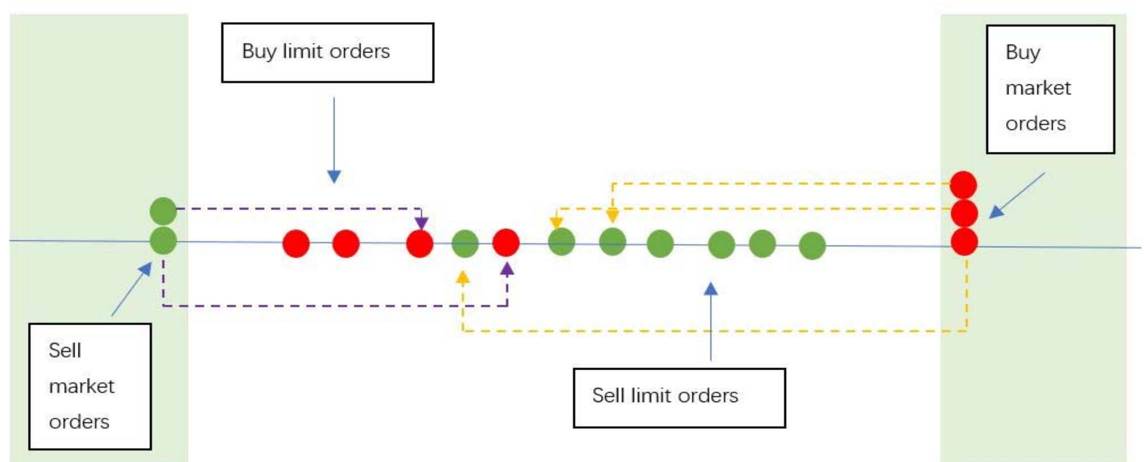

Figure 3: Matches Initiated by Market Shares

As we can see from figure 2, two shares will successfully match as the one on the bid side is at a higher price than the one on the ask side. And they will be immediately canceled out (therefore a time lag is not necessary). If we consider only interactions in fig 1, then it will evolve as follows based on the trading rule.

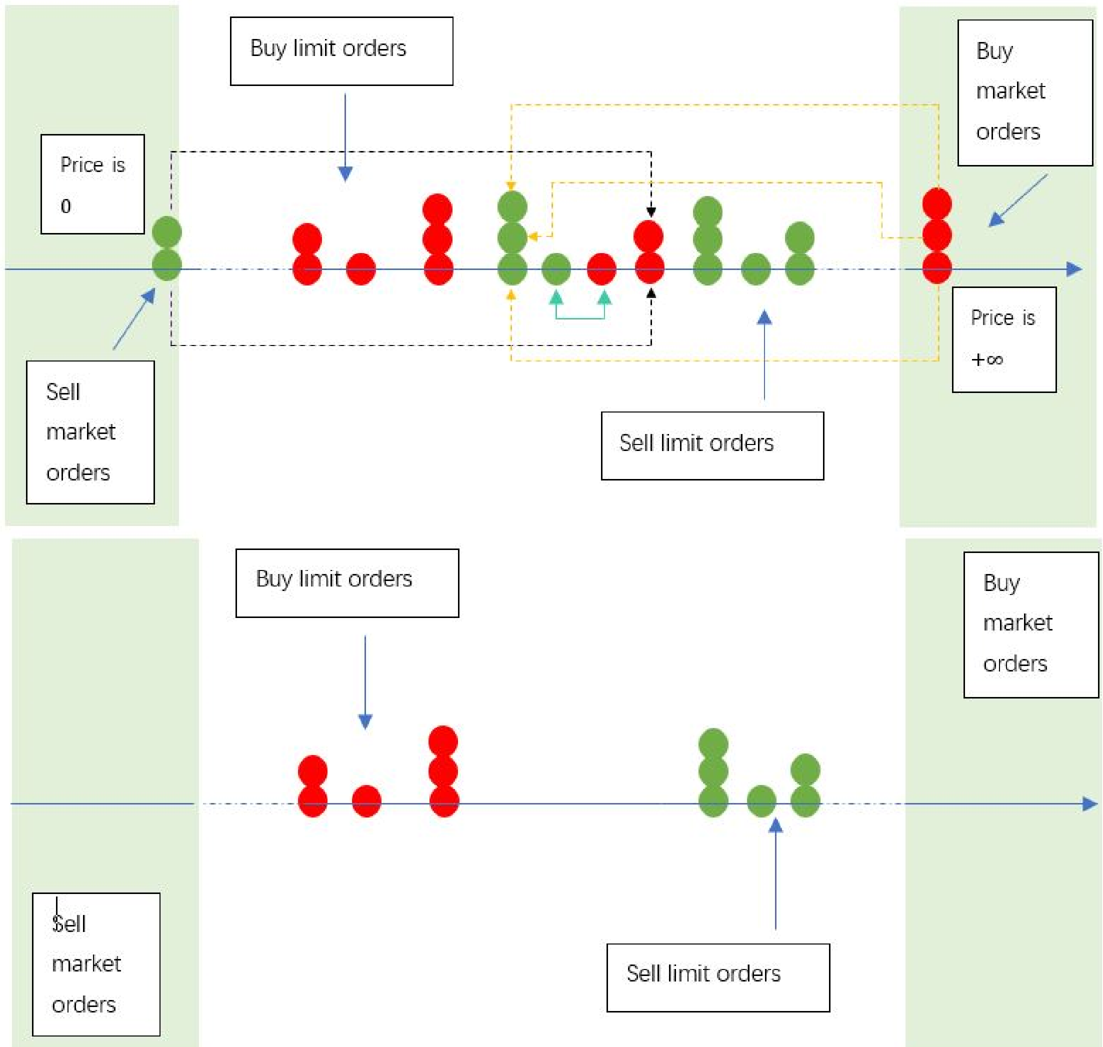

Figure 4: The “Next Moment” of Figure 1: consider matches only

Such an interaction effect between the two types of shares (or orders) did not escape the attention of previous studies. For instance, there is a category of studies of modeling on limit order books called the “zero-intelligence approach” (details are later in this thesis), where shares are modeled as positron and negatron, two different particles which will annihilate after an encounter each other. Similarly, if we model bid-ask shares as two competitive or mutually antagonistic animal species, any two individuals from different species will kill each other and perish together after an encounter. Also, in both types of studies, an encounter means two shares in different catagories get matched in price and therefore trigger a transaction.

Therefore, in this thesis, we will treat shares on the bid side and shares on the ask side as two different species which interact (a single share corresponds to an individual of one species). We then apply generalized Lotka-Volterra equations (Hofbauer & Sigmund, 1998) to simulate how the species populations evolve based on some stylized facts of an order-driven market. Since the population of the two categories of shares indicates the power of demand and supply, respectively, the trend of stock price (market price) can be forecasted to some extent.

This study focuses on the dynamic of an order-driven market. It does not simulate the dynamic of stock price, but instead it simulates the dynamic of bid orders and ask orders (they are bid and ask shares in this thesis), which are the lower dimensions of stock price. We can infer the behavior of stock price by observing the dynamic of its lower dimensions.

Compared with the previous studies about the stock dynamic, the model in this thesis has several features. First, the simulation is not about the evolution of a limit order book, which records only outstanding shares in limit orders. Instead, this model tracks those shares in either limit or market orders being executed. In other words, it describes the evolution of all shares in all orders for a single stock. Since it is a pure simulation of the population dynamics, we do not track the price element in a limit order book, such as bid-ask spread and mid-price. Second, the tool in this thesis is generalized Lotka-Volterra equations, which was not commonly used as a tool in previous studies. Moreover, our study specified the similarity between the interaction effect of two species and that of bid-ask shares. Third, the model is a deterministic dynamics system, so it is "statistics free", meaning that the model does not involve any statistics or probability.

For the academic community, the models offer insights on combining the study of stock markets and the study of mathematical biology with deterministic dynamical systems. The simulation discloses how the heterogeneity among investors generates stock market dynamics, and how external information impacts the market dynamic. Also, the simulation explains the possible reasons for some phenomena observed in a stock market, such as periodicity and chaos. The model may also help investors make decisions under uncertainty (when they do not have any information about the stock). For example, periodical or quasi-periodical movements of stock price may indicate that there is a lack of external information for this stock, resulting in chances for arbitrage by exploiting predictable external information

CHAPTER 2: CONTINUOUS DYNAMICAL SYSTEMS

As our purpose is to simulate how the population of two different items change with time together, dynamical systems are no doubt a proper tool. Generally, a dynamical system can be classified as continuous or discrete. In this thesis, we will select continuous dynamical systems, because both the matching and placing of orders can happen in continuous time.

2.1 Definition

Firstly, we give a strict mathematical definition of dynamical systems described by Zhang (1987). The concept of dynamical system originated from the study of ordinary differential equations (ODEs). Now, consider the system of ordinary differential equations

$$d\mathbf{X}/dt=G(t;\mathbf{X}) \quad (1)$$

which has the initial condition $\mathbf{X}(0)=\mathbf{X}_0$ defined on R^n, where R^n is a n-dimensional Euclidean space; $\mathbf{X}$ is a n-dimensional vector; G is a map on R^n such that $G \in C^r(R^n, R^n)$[4]. There is always solution for (1) in local area. If $G(t;\mathbf{X})$ meets a certain condition, then its solution $g(t,\mathbf{X}_0)$ is meaningful to all $t \in R$, $X_0 \in R^n$. Replace X_0 with X, the solution $g(t,\mathbf{X})$ should meet two conditions. 1) $g(0,\mathbf{X})=\mathbf{X}$, $\forall X \in R^n$. 2) $g(s+t,\mathbf{X})=g(s,g(t,\mathbf{X}))$, $\forall s, t \in R$, $\forall X \in R^n$. We call the map $g: R^n \times R^n \to R^n$, which meets conditions 1) and 2), a dynamical system or flow on R^n. In particular, if the

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That is rth order continuous differentiable function on R^n

right-hand side of (1) does not contain t, we call the system an autonomous system. An autonomous system with two dimensions is an autonomous plane system, which is our focus in this thesis.

In simple words, a dynamical system reveals how things change with time. Here, we take prey-predator equations as an example (From Wikipedia).

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \delta xy - \gamma y \end{cases}$$

In the system above, x is the number of rabbits (prey), and y is the number of foxes (predator). The two equations tell us how the numbers of prey and predators change over time. Given the certain coefficients in the equations, the graph of figure 5 is obtained.

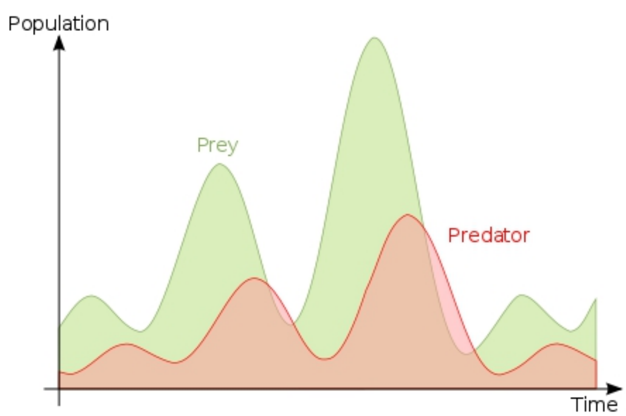

Figure 5: Population Dynamics in Prey-Predator Equations

When G(t;**X**)=**0**, the derivatives of all the variables are equal to 0, which implies the system is in a static situation. In the study of a dynamical system, such a situation is named as a stationary state, which represents a critical point in the phase space. In the research of economics and finance, we are usually concerned about the ultimate form of a system, or say, where will the system finally go. Under such a circumstance, studying the situation at critical points becomes extremely important. Because for a stable critical point, for instance, when the system is disturbed by external factors, the particle[5] will depart from the critical point. However, due to the property of the stable critical point (this property will be explained later in the thesis), the particle will be attracted back to the critical point. This means that the particle will always be trapped in a neighborhood of the critical point, and that is the final form of the system. More information about critical points and stability will be given later.

2.2 Some concepts and properties

In this section, we will explain some of the basic concepts and properties which will be used in this thesis.

Nullcline: The set of points on which $\frac{\mathrm{dx}}{\mathrm{dt}} = 0$ or $\frac{\mathrm{dy}}{\mathrm{dt}} = 0$. Usually, a nullcline divides the plane into several parts. For instance, if we have a system $\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{y} - x^2$, $\frac{\mathrm{dy}}{\mathrm{dt}} = x - 2$. The nullcline for $\frac{\mathrm{dx}}{\mathrm{dt}}$ is $\mathrm{y} - x^2 = 0$, and is $\mathrm{x} = 2$ for $\frac{\mathrm{dy}}{\mathrm{dt}}$.

Critical points: Critical points can be viewed as the intersection of nullclines. A critical point (x_0, y_0) is that point at which, $\dot{x}(x_0, y_0) = 0$ and $\dot{y}(x_0, y_0) = 0$. In other words, the system will become static at critical points.

Imagine that there is a particle, which moves along the trace of the solution with a certain initial point.

Stability at critical point: The strict definition and classification of Lyapunov stability are as follows: Consider an autonomous nonlinear dynamical system $d\mathbf{X}/dt=G(\mathbf{X})$, $\mathbf{X}(0)=\mathbf{X}_0$. Suppose the system has an equilibrium at which $G(\mathbf{X}_1) = \mathbf{0}$. Then

1. This equilibrium is Lyapunov stable if, $\forall\varepsilon>0$, $\exists\sigma>0$, and $\| \mathbf{X}(0) - \mathbf{X}_1 \| < \sigma$, then for any $t>0$, we have $\| \mathbf{X}(t) - \mathbf{X}_1 \| < \varepsilon$.

2. The equilibrium of the above system is asymptotically stable if it is Lyapunov stable, and $\exists\sigma>0$, such that, if $\|\mathbf{X}(0)-\mathbf{X}_1\|<\sigma$, then $\lim_{t\to\infty} \| \mathbf{X}(t) - \mathbf{X}_1 \| =0$

3. The equilibrium of the above system is said to be unstable if $\exists\varepsilon>0$, for $\forall\sigma>0$, there is at least one initial position $\mathbf{X}(0)$ which satisfies $\|\mathbf{X}(0)-\mathbf{X}_1\|<\sigma$, and the corresponding solution $\mathbf{X}(t)$ satisfies $\|\mathbf{X}(t)-\mathbf{X}_1\|>\varepsilon$ for at least one $t>0$.

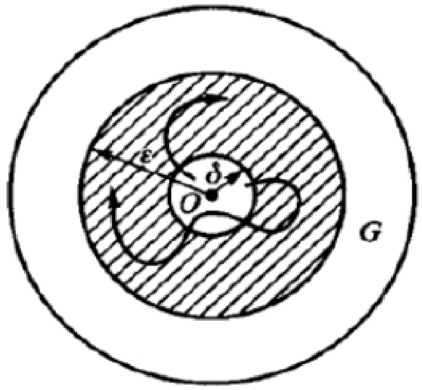

Figure 6: Stable Critical Point

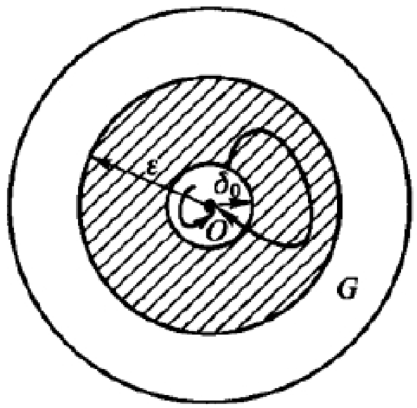

Figure 7: Asymptotically Stable Point

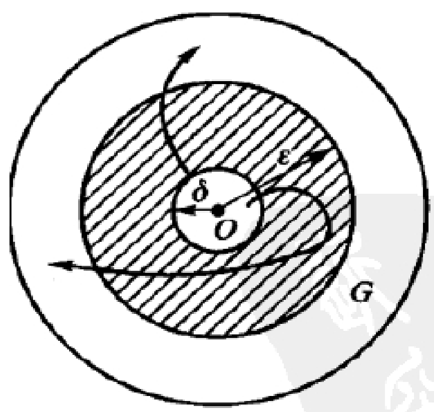

Figure 8: Unstable Critical Point

The content above is from Wang (2006). In simple terms, stability is a concept to describe whether those points within a neighborhood of a critical point will be attracted towards or be repelled away from the critical point.

The classification of critical points is based on eigenvalues: We only illustrate the cases of critical points in this thesis. Such classification is based on the eigenvalues of the Jacobian Matrix of the system at critical points (Zill & Wright, 2012).

a) If $\lambda_1 = \lambda_2 > 0$, then this is an unstable node.

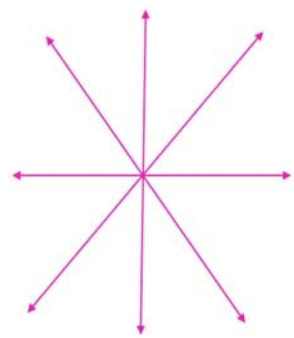

Figure 9: An Unstable Node

b) If $\lambda_1 = \lambda_2 < 0$, then this is a stable node.

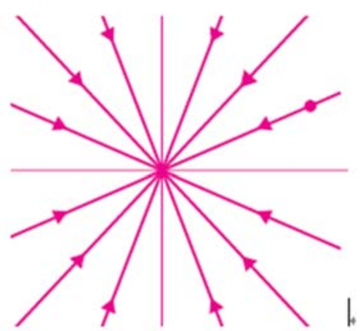

Figure 10: A Stable Node

c) If $\lambda_1 < 0, \lambda_2 < 0, \lambda_1 \neq \lambda_2$, then this is also a stable node.

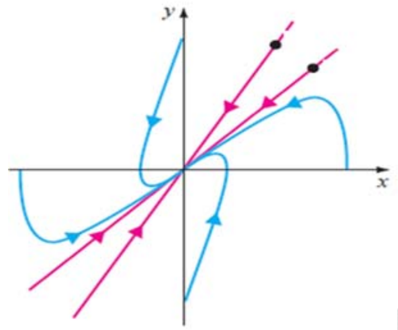

Figure 11: A Stable Node

d) If $\lambda_1 < 0, \lambda_2 > 0$, then this is a saddle point, which is unstable.

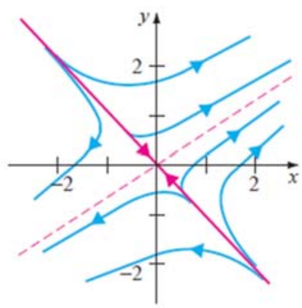

Figure 12: A Saddle Point

e) If λ_1 and λ_2 are two complex numbers with the only imaginary part. The critical point is the center of a spiral under such a circumstance.

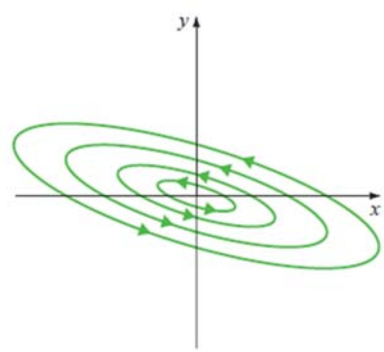

Figure 13: A Spiral Point

Unique Solution: The concept of "unique solution" is crucial for the study of differential equations and dynamical systems. When we analyze the solution of a system, we cannot be certain that the solution matching a certain condition is unique. Also, the unique solution ensures that two traces of different solutions cannot intersect, otherwise a solution with an initial point at the intersection can have two different traces. Usually, if a system matches the Lipschitz condition, then it has the local unique solution. The Lipschitz condition is not easy to check under most circumstances, but we can use another more obvious condition to take the place of Lipschitz constant. That is, if $\mathrm{G}(\mathbf{X}) \in C^1(\mathrm{U})$, where $\mathrm{G}(\mathbf{X})$ was defined in 2.1 and $\mathbf{X} = (x_1, x_2 \dots \dots x_n)$, on an open set $\mathrm{U} \subset R^n$, the system meets the Lipschitz condition (the proof is in the Appendix 2). All systems in this thesis have such partial derivatives, and this is easy to verify.

2.3 Generalized Lotka-Volterra (GLV) equations

The prey-predator model in the previous section is proposed by Alfred J. Lotka (Lotka, 1926) and Vito Volterra (Volterra, 1927) , and serves as the foundation of modeling theory for populations of species. In this thesis, we will apply a family of generalized Lotka-Volterra equations in the form:

$$\frac{\mathrm{d}x_i}{dt} = \mathrm{r}(x_i) + \sum_{j=1, j\neq i}^{\infty} \gamma(x_i, x_j) x_i x_j$$

where $\mathrm{r}(x_i)$ indicates the growth of the *i*th species， and $\gamma(x_i, x_j) x_i x_j$ indicates the interaction effect between the *i*th species and *j*th species. Specifically, if we apply the logistic equation for $\mathrm{r}(x_i)$, the equations become generalized Verhulst-Lotka-Volterra equations (Simin et al., 2018).

During the past several decades, the application of GLV equations has not been limited to biology alone, but has expanded to many other areas, such as economics, finance, management, marketing and many other social sciences. In this thesis, we will apply different types of function for both growth and interaction parts, and discuss the qualities of each model. Specifically, linear function and logistics growth will be applied to the growth term; constant and non-linear functions will be applied to the interaction term.

CHAPTER 3. AN OVERVIEW OF THE MODEL

3.1. What should the model look like?

Based on GLV equations, we write the model directly in the form applied in this thesis,

$$\begin{cases} \dfrac{dX}{dt} = R_1(X) - \gamma(X,Y)XY \\ \dfrac{dY}{dt} = R_2(Y) - \gamma(X,Y)XY \end{cases}$$

where $R_1(X)$ and $R_2(Y)$ are the growth of bid and ask shares respectively, and $\gamma(X,Y)XY$ is the interaction term indicating the trade volume. What is special here is that the interaction term indicates the trade volume at a certain time, which is identical to both X and Y, so the interaction term should be the same in the two equations. Also, the sign of $\gamma(X,Y)$ should always be negative, since successful matches make both X and Y decrease. The interaction of the two "species" here is competition (Odum, 1971), to some extent. But we should note that such a kind of competition is different from the typical competitive LV equation, which has the form $\frac{\mathrm{dx}}{\mathrm{dt}} = rx(1 - \frac{x}{K} - \frac{ay}{K})$. In this equation, two species exist in the same space and compete for the same resource, so individuals in one species will squeeze the living space of individuals in the other species. The competitive L-V equation is widely used in the study of marketing where several agents compete for the same client group for market share. In our case, bid and ask shares do not have such a quality (we cannot say that buying orders occupy the space of selling orders).

3.2. When can we apply the model?

An obvious question is that given that behaviors of a stock market are normally erratic and diversified, can a dynamical system with ordinary differential equations cover all situations? The answer to this question is that such a system cannot cover all situations. Thus, we find a trade-off and describe only one situation.

First, we define something as "signal information", which provides investors with clear signals, according to which investors make decisions to buy or to sell. A canonical example of such signal information is good or bad news. Most investors tend to buy the stock on good news and to sell on bad news. Although such a signal is clear to investors, it can be faked or be incorrect, and can be appropriately classed as "noise".

Apparently, a single stock can behave disparately under different kinds of signal information, which makes the system unpredictable. For instance, let the growth of bid shares be rX, where r is a constant referring for the growth rate. If the current r is r_1, then the value of r in the next moment will be unpredictable under the impact of signal information. For example, if good news arrives when r=r_1, then r can rise to a number different than r_1, because most investors will be motivated to buy the stock under bull news. Such a change has nothing to do with the current values of X and Y, but totally depends on existing signal information. So, what we have to do is try to avoid the impact of such information. Thus, the model in the thesis is limited to the situation, where there is no signal information influencing the decision of investors. Under such a circumstance, investors make decisions randomly and differently due to heterogeneity. Also, the analytical solution is smooth and follows the same pattern under this situation. The condition can be viewed as a stable environment; therefore, no sudden change is allowed. For example, the growth rate of *X* cannot rise sharply within a short time interval. Therefore, we created a zero-information situation. In more sophisticated words, we use such a zero-information situation as a benchmark, we can separate out the part of change on *X* and *Y* due to the impact of signal information, rather than let the impact of signal information change the parameters of the system.

$$\begin{cases} \dfrac{dX}{dt} = R_1(X) - \gamma(X,Y)XY + \varepsilon_1 \\ \dfrac{dY}{dt} = R_2(Y) - \gamma(X,Y)XY + \varepsilon_2 \end{cases}$$

We absorb the part of change in *X* and *Y* due to the information to ε_1 and ε_2[6], which indicate the change in *X* and *Y* under current signal information. In other words, ε_1 and ε_2, are treated as the disturbance term. By doing so, we isolate the impact of signal information from the system. We still use the previous one as an example; good news comes when $r{=}r_1$. We assume r_1 jumps to a larger number r_2, and in the next moment, the growth becomes $r_2 x$. Now, we rewrite this term as $r_1X + (r_2 - r_1)X$. In this equation, r_1X remains the same, and we attribute $(r_2 - r_1)X$ to ε_1 as this is the change due to the signal information. In this thesis, we try to describe the situation when there is no impact due to signal information. Hence, both ε_1 and ε_2 should be equal

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Since the relationships between the signal information and, *X, Y* and t are uncertain, we do not pose a function form for the disturbance.

to zero under this assumption. Overall, if we compare the signal information to a road sign which gives an obvious clue to passersby about which road to choose, what we try to describe in this thesis is the situation where such a road sign is missing.

3.3. What do the system parameters suggest?

i. R: The growth rate of X and Y indicates the frequency of newly placed shares when there is no signal information. It is also equal to the frequency of investors placing their orders. We make such rates for X and Y equal, because, when there is no signal information, the motivation of selling and buying should be the same. Note that rX, for example, is determined by the difference between the frequency of newly placed orders and the frequency of canceled orders. We assume this is always positive.

ii. K: The capacity of the environment. Here, that means the maximum values of X and Y. A detailed description appears later in this thesis.

iii. σ: The parameter used to standardize the trade volume σXY. σ is positively related to the trade volume, and is a measure of the possibility of successful matches, indicating the motivation of investors to make successful transactions. In reality, σ indicates the motivation of investors to choose market orders or to place limit buy (sell) orders at prices higher (lower) than current ask (bid) price.

In this thesis, we will fix the growth rate r and capacity K, and classify different situations for the system based on σ.

CHAPTER 4. RELATED LITERATURE

4.1 Modelling of limit order books

In this section, we will review some literature which focused on the evolution of limit order books (LOBs). Although our study is not directly of a LOB, there are still some similarities. Also, we applied some ideas and methods from the study of LOBs to this thesis. Martin et al. (2003) summarized and classified the models of LOBs into three categories. The first one is the perfect-rationality approach, the aim of which is to find the best trading strategy for investors to maximize their utility. For this category of study, the model is usually static. Scholars rely heavily on the assumption of fundamental value and perfectly rational investors. Kyle's model (1985) assumes that there are both well-informed traders and noise traders in a market. In the first round, noise traders place limit orders with the price deviating from fundamental prices. Well-informed investors will place market orders to match those limit orders at a wrong price and therefore obtain profits in the second round. Gottler, Parlour and Rajan (2006) proposed a model in which traders place orders following a Poisson process, and can arbitrarily choose the type and the price level of an order. Also, they can cancel and modify orders submitted. Once their orders are executed, they are not allowed to re-enter the market. An interesting fact about this model is that all traders can pay to become and remain informed until they leave the market. Another branch of perfect-rationality study is about minimizing the market impact of executed orders. For those traders whose goal is to minimize trading cost generated by executing a big order, which has to be finished in several steps, Bertsimas and Lo (1998) obtained an optimal trading strategy. They demonstrated that if the movement of price follows a random walk process, traders should divide the order into N equal parts and submit those parts with the same time interval. Also, if the price is influenced by external information, then the optimal strategy involves properly adjusting trade quantities at every step. Almgren and Chriss (2001) derived an analogical strategy for those traders who want to maximize the utility of trading revenue for executing a large order. However, the assumption of fundamental value has been rejected by many researchers already, and as we mentioned before, such a model is usually static whereas the tool in this thesis is dynamical systems. Nevertheless, we will also make some assumptions about investors based on rationality, but such a level of rationality is far away from perfect-rationality and fundamentalism.

Another category is the "zero-intelligence approach". This approach focuses on the state and the evolution of LOBs, assuming that arrivals, cancellations and matches are purely stochastic processes. Thus, no trading strategy is involved, and stochastic processes totally govern the evolution. The framework of zero-intelligence approaches proposed by Bak, Paczuski and Shubik (1997) is similar to the one in this thesis. The price grid was described as a tube, and bid orders and ask orders were described as two kinds of particles (positrons and negatrons) injected from two ends of the tube. These particles are randomly diffused in this tube, where location indicates the price until two particles in different categories encounter each other. This process can also be viewed as an interaction effect between two particles, and is similar to our model. Because we apply a continuous model in this thesis, we will focus on some zero-intelligence approaches in continuous time intervals. The first zero-intelligence model in continuous time was proposed by Daniels *et al.* (2003). They produced an equation for $L(t)$, which is the state of a limit order book at time t, under the assumption that arrivals of limit and market orders, and cancellations are all Poisson processes. Also, they assumed that limit orders arrive at the same rate at each relative price level. Smith *et al.* (2003) solved the equation with tick size approaching to zero by using a mean-field approximate on that the depths available at neighboring prices are independent. By removing several assumptions in Smith's article (2003), Cont, Stoikov and Talreja (2010) proposed a stochastic model for order dynamics. They retained the assumption about Poisson processes and applied a two-way Laplace transform to obtain the conditional probability in the next moment of several market events, including the increase of mid-price, a limit order executed before the price moves and both buy and sell limit order executed before the price moves. The model performs well on predicting the evolution of order books in a short time. Similarly, Cont and Larrard (2013) proposed another stochastic model for describing the evolution of a limit order book, in which arrivals of market order, limit orders and order cancellations are described as a Markovian queueing system, which has the property of analytical tractability. Because of this property, they obtained analytical expressions for various quantities for next moment based on the state of the current order book, such as the distribution of the duration between price changes, the distribution and autocorrelation of price changes, and the probability of price going up. The previous two models are usually viewed as successful in predicting the dynamics of limit order books and many other sequential studies modified and optimized them. For instance, a quixotic assumption in the model of Cont's paper (2010) is that all orders are restricted to be the unit size. Huang and Kercheval (2012) relaxed this assumption in 2012. They also applied the Laplace transform to compute the conditional probability of several events which are similar to those in the article of Cont (2011). Also, based on some empirical studies, they relaxed the assumption that the arrival and the cancellation follow the Poisson process. Instead, the author applied a Hawkes process to describe the arrival and the cancellation of orders. In such a process, the arrival rate of market events is a function of the rate and the number of recent arriving orders. Toke (2011) also replaced the Poisson processes with the Hawkes processes. He observed that after the arrival of a market order, the average time for the next limit order arriving is less than the corresponding unconditional meantime. Based on this observation, Toke simulated the order flow that matches his observations more closely than those produced by a Poisson-process. As we can see, most zero-intelligence models are founded on the assumption that market events follow stochastic processes.

There is also a third category lying between those of perfect-rationality and zero-intelligence, namely, the "agent-based model" (ABM), which describes a large number of possibly heterogeneous agents interacting in a specified way (2007). The advantage of such models is that they can incorporate both the performance of individuals and their aggregate effect. But the disadvantage is also obvious: Due to the complexity of interactions among different agents, it is difficult to compute parameters in such a model. Below are some selected agent-based models. In 1988, Cont and Bouchaud (2000) proposed a model for an order-driven stock market where random interactions among agents generate a heavy tail in the distribution of stock price changes in the form of a truncated power-law, which is similar to distributions observed in some empirical studies of high-frequency market data. Two famous market phenomena are linked by the model. The one is heavy tails observed in the distribution of stock returns, and the other one is the ’herding’ behavior in financial markets. Also, the study suggests a relationship between the excess kurtosis observed in asset returns and the market order flow, and the tendency of market participants to imitate each other. They introduced an order-driven market model with heterogeneous agents who set bids and asks, and post-market or limit orders following certain exogenous rules and sharing a common valuation for the traded asset. They studied how trading strategies impact price dynamics, volatility and the trade volume. In their model, they noticed that the volatility produced by the simulation is by far lower than that in empirical data, and there is no volatility clustering. Therefore, they argued that there must be substantial heterogeneity among traders in a real order-driven market. Challet and Stinchcombe (2003) studied how allowing the parameters in a simple ABM for limit order books to change with time affects traded price series. They concluded that the mid-term over-diffusive price behavior is a result of the variability of market order and limit order rates. Also, they argued the heavy-tailed distribution of mid-price changes and volatility clustering are led by such time-dependence. They also noted that, in many LOB models, parameters are assumed to remain constant, and they inferred that some stylized facts might be caused by erratic actions on the paper of traders. Lillo (2007) was engaged in the problem of the optimal limit order price of a financial asset in the framework of the maximization of the utility function of investors. By solving a utility maximization equation, he gave insight into the origin of observed power-law distribution of limit order prices. He concluded that the emergence of such power-law distribution is probably due to the power law heterogeneity of traders' investment time horizons. Specifically, he showed that if mid-price movements are assumed to obey a Brownian motion, then each agent (assumed to be perfectly rational) will choose the relative price of her orders to be

$$\delta^{x*} = \sqrt{2T} g^{-1}(\alpha) V$$

where $g^{-1}(\alpha)$ is the degree of risk aversion of an agent, T is the time horizon of investors, and V is the volatility. Then, he studied how observed homogeneity affects the price choices of interacting agents with different risk aversions and different maximum time horizons. He also showed that heterogeneity of T is the main source of the power law distribution of δ^{x*}.

4.2 Application of dynamical systems in finance

In this section, we will review some literature applying dynamical systems to model financial markets at the microstructure level. A deficiency of classic rational expectations models or equilibrium models is that they lack the strength to explain the formation of price bubbles. Dynamical systems are usually an alternative which is applied to explain this financial phenomenon. Caginalp and Ermentrout (1990) developed a complete dynamical system for investor behavior. The model assumes a kinetic reaction among investors who rely on a fundamental value component and a trend-based component. The latter is based on a memory of price history decaying in time, and it captures the tendencies among investors to buy a recently rising stock and to sell a recently declining stock. Based on this, a dynamical model introduced by Caginalp and Balenovich in 1993 (G Caginalp & Ba1enovich, 1993) did an excellent work on predicting price patterns after calibrating a previous experimental bubble, given the initial condition for a new bubble and its controlled fundamental value. As an extension of the work of Caginalp and Balenovich , Porter and Smith (Porter & Smith, 1994) reviewed the results of more than 70 laboratory asset market experiments which incorporate experimental treatments for suppressing bubbles that are suggested by the rational expectations theory or dominant prescription. The results suggest bubbles are the result of uncertain behaviors of investors.

In 2016. Cheriyan and Kleywegt (2016) proposed a model of boundedly rational investor behavior, which involves asset price forecasts based on past price data. The model demonstrates how bubbles, crashes and asset price cycles may be led by the behavior of boundedly rational decision makers. Investor behaviors are represented by the parameters of the dynamical system. It is shown that the dynamical system can converge to a fixed point, which corresponds to the fundamental value of the asset, or may converge to a limit cycle with both positive and negative bubbles, or may exhibit uncertain chaotic behavior.

Dynamical systems are also applied to areas other than the study of financial bubbles. For instance, Slanina and Zhang (1999) introduced a model of an open economy composed of producers and speculators, and the model is investigated by numerical simulations. The economy is viewed as an open dynamical system, and the influx of capital leads to the coexistence of producers and speculators. They demonstrated that the existence of speculators can be useful to the economy by suppressing price fluctuations. They also show that the optima for producers and speculators lie close one to the other, and thus their mutual coexistence can be better described as symbiosis than parasitism. By employing random dynamical systems theory, Evstigneev, Hens and Schenk-Hoppe (2006) derived necessary and sufficient conditions for the evolutionary stability of portfolio rules. Specifically, the market is evolutionary stable if and only if stocks are evaluated by expected relative dividends. More often than not, there are links between dynamical systems, financial systems and ecology systems. Farmer (2002) proposed several dynamical system models for the financial market in 2002. The models can be concisely summarized into three words, which are "Force", "Ecology" and "Evolution". The first one indicates how a nonequilibrium price is formed by the market forces, which is mainly led by directional traders submitting market orders. The word "Ecology" implies that the interrelationships of financial agents are similar to the biological case. In this section, the authors applied several common trading strategies and studied how they work on the price dynamics. "Evolution" literally indicates the short-term and long-term performance of the system. They found the combination of value investing and trend following can result in a boom-bust circle and in an excess of volatility. For the long-term, the evolution is studied in terms of flows of money. The results indicated the reinvestment of profits leads to a capital allocation model which is equivalent to Lotka-Volterra equations.

4.3 Studies and simulations of GLV equations

Based on the model of Lotka (1926) and Volterra (1927), Odum (1971) divided the interaction between two species among the classifications of "mutualism", "competition" and "prey-predator". All these relationships are indicated by the sign of the interaction term. The interaction is mutualism if signs of interaction terms, which indicate how A impacts B and B impacts A, are both positive; competition if the two signs are negative; prey-predator if two signs are different. In 1977, Hannan and Freeman (1977) introduced ideas such as these, originally from the field of ecology to the study of organizations, and mentioned the usage of Lotka-Volterra equations in competitive markets. Also, due to the complex nature among the peer industry, many studies regarding this area applied GLV equations with different interaction terms. The most common method is to fit parameters to check the sign of the interaction term and therefore determine the relationship among different markets or firms. For instance, Kim and Lee (2006) studied the Korean mobile phone market by applying LV equations as the diffusion function. The article concluded that the relationship between two mobile phone markets (the PCS market and the cellular market) is commensalism. That means the PCS market has benefited from the cellular market while the former hardly influenced the latter. In other words, the interaction of the cellular market to the PCS market is positive, but is zero when the relationship is inversed. Similarly, Kreng and Wang (2009) studied the relationship between LCD TV and PDP TV in Taiwan. It was shown that actors on the LCD side took the role of prey and actors on the PDP side took the role of predator. These two studies described markets in which the products are different, so the relationship is not necessarily competitive. Below is some literature about the market with the same products manufactured by different firms. For instance, Michalakelis, Sphicopoulos and Varoutas (2011) applied LV equations to model the competition among three Greek mobile telephone providers, Vodafone, Cosmote and Wind. They forecasted that market shares of the three providers will finally reach an equilibrium. Samarajiva (2000) did similar research on the telecommunication providers in Sri Lanka. Moreover, some studies add a constraint to LV equations. That means they assume the total market share is 1. For instance, Huang ,Tsai and Wu (2014) modeled the market shares of different types retail industries in Taiwan. They made x(t)+y(t)=1, and substituted the y(t) with 1-x(t). Thus, the system becomes $\begin{cases} \frac{dx}{dt} = [a + bx + c(1-x)]x \\ \frac{dy}{dt} = 1 - x \end{cases}$. The general method here applies past data to fit the parameters in the LV model and forecast the evolution of the whole system. However, such a method gets constants for parameters, but in real markets, usually, one agent will change its strategy when its market share is low, and then the parameter will also change. Some studies noticed this and tried to optimize the model. For example, Tang and Zhang (2005) modeled the competition between two CPU vendors AMD and Intel. In the article, they replaced the interaction constant with a time-depended function. The system is

$$\begin{cases} \frac{dx}{dt} = r_1 x\{1 - x - [1 + (b_{21} - b_{22})x]\} \\ \frac{dy}{dt} = r_2 y\{1 - y - [1 + (b_{12} - b_{11})y]\} \end{cases}.$$

Here, b_{21} and b_{12} are creation functions, which indicate the innovation of new technology of two companies. And b_{11} and b_{22} are functions used to evaluate the pressure from other companies which benefited from the innovation of AMD and Intel, and released technology to compete with them. Note that all these four functions were time dependent without explicit functions. Similarly, in 2016, Marasco, Picucci and Romano (2016) applied a family of integrable nonautonomous LV models on modeling the utility function of different firms in a market. Their model has the formalism of

$$\frac{\mathrm{d}x_i(t)}{dt} = x_i(t)[g_i(t) - \sum_{j=1}^{N} g_j(t)\, x_j(t)].$$

In this equation, the constant in the original LV model was replaced by an integrable function. Also, such type of function made the differential equation solvable. In the previous two cases, although we had the idea of changing constants with functions, no expression was provided. Now, we look at an example with a concrete function expression proposed by Caram, Caiafa, Proto and Ausloos (2010). They did a pure simulation by GLV equations on the competition of agents with different sizes. The model is,

$$\frac{\mathrm{d}s_i}{dt} = a_i s_i (b_i - s_i) - \sum_{j=1, j \neq i}^{\infty} \gamma(s_i, s_j) s_i s_j,$$

where s_i is the size of agent I and $a_i s_i (b_i - s_i)$ is the logistic growth, which will also be applied in this thesis. The interaction function $\gamma(s_i, s_j)$ is defined as $\exp[-\left(\frac{s_i - s_j}{\sigma}\right)^2]$, which is a Gaussian function with some good mathematical properties. The author explained that the competition is higher if two agents have similar sizes ($s_i - s_j$ is small), and is lower when they are largely different in size. This is a typical example, in which the interaction function is a function of x and y, and is determined by the characteristics of what they want to model. Recently, Simin *et al.* (2018) applied this model again to simulate the size of love (the strength of love) among different people.

CHAPTER 5. PROPOSED MODELS

In this section, we will propose several GLV models from simple (herding and radical treading strategy) to the most complex one (rational and conservative treading strategy). Our purpose is to observe how the populations of bid and ask shares evolve with time and therefore estimate the current situation of demand and supply. Based on this, we are concerned with the trajectories of a solution, which is expressed as how a particle on the plane[7] moves from certain initial conditions, and following the system. Also, the ultimate situation is expressed as where the particle will finally go.

5.1 Linear growth and interaction (System 1)

5.1.1. Constructing the model

Firstly, we consider the simplest model, which is purely the equations of the population of the prey in the prey-predator model. That is rX and rY indicate the growth of bid and ask share, respectively, while σxy indicates the trade volume.

$$\begin{cases} \dfrac{dX}{dt} = rX - \sigma XY \\ \dfrac{dY}{dt} = rY - \sigma XY \end{cases} \qquad (1)$$

There is no capacity in the two equations. Both X and Y follow a linear increase. The trade volume is also a linear function of X and Y, respectively.

For consistency with the two variables, X and Y, we consider the coordinate plane composed by X and Y axes.

5.1.2. Dynamical Analysis

We will demonstrate the critical points and their types in table 1. Since the process of determining the type of each critical point is easy but tedious, we will not show it in detail for most cases.

Table 1
System 1: Critical Points and Their Types

Critical points	Type of critical points
(0, 0)	Unstable node
$(\frac{r}{\sigma},\frac{r}{\sigma})$	Saddle point

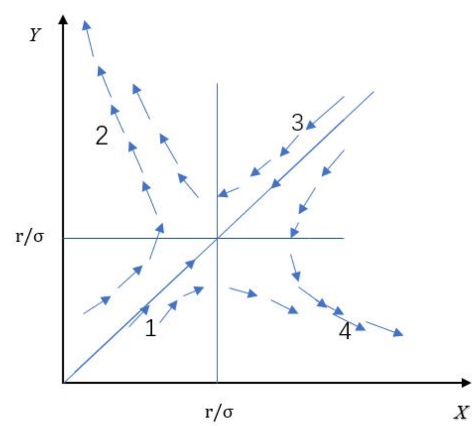

Figure 14: A Rough Stream Plot of System1

We can know the sign of dx/dt and dy/dt in different areas (in figure 14), which are divided by nullclines. For instance, in area 1, we have X>0 and $r-\sigma Y>0$, so $\frac{dX}{dt}$>0. Similarly, we can have $\frac{dY}{dt}$>0 in area 1. That means the particle in area 1 should move along a trajectory, on which, both X and Y will increase. But we should note what we can obtain from this method is just an approximated direction. In the rest of the thesis, we will show graphs of approximated moving traces based on this analytical method.

In addition to the properties of critical points, we can draw a rough stream plot. It is also important to note that the line X=Y is also a solution curve of the system. Moreover, because of the unique solution, any two of trajectories should not intersect. This property implies that any trajectory with initial points in the region of X>Y should always stay in this region and finally moves to Y=0 and X=+∞; so does a trajectory with an initial point in the region of Y>X.

In this case, we can note that the population of bid and ask shares will go to the area where either X=0, Y=+∞ or Y=0, X=+∞. This indicates two situations in the end. The first is that all people want to buy this stock. The second is that all people want to sell this stock. However, at least the value of Y cannot go infinite. This is because the number of stock shares issued by one company is limited. Hence, the maximum value of Y should be equal with or less than the number of total shares issued. Besides, this model can be viewed as when investors apply herding strategy. The truth that the growth parts rX and rY are larger when X and Y are larger reflects investors' inclination to follow the strategies of others and a generate feedback effect. Overall, herding strategy makes the stock go to the extremes.

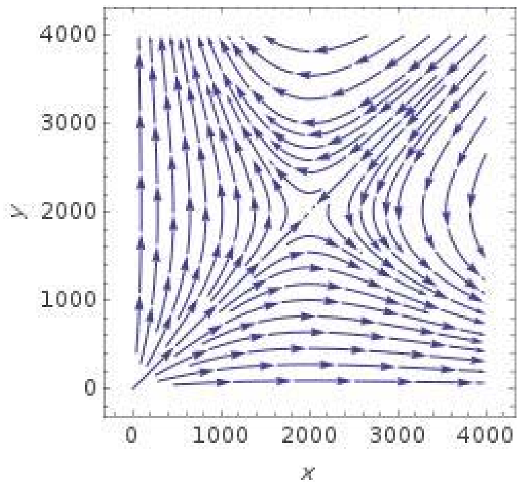

Figure 15: Simulation Data for System 1: when r=0.1, σ=5×10^{-5}, K=4000.

5.2 Logistic growth for the ask side and linear interaction (System 2)

5.2.1. Constructing the model.

First, let us think about ask shares. As the number of total shares issued by a listed company is fixed, the value of *Y* cannot go infinite. Moreover, in reality, it is nearly impossible that all shares issued are requested to sell. For example, there are some extremely rational investors and fundamentalists who know that the stock will be under-evaluated if too many shares are on the ask side, especially when there is no signal information. Thus, they will not sell if there is already a large number of ask shares. Also, some stockholders will never sell for some particular reasons. Consequently, the maximum value of *Y* should be less than the number of total shares. Now, we assume the maximum value is a fixed number of “K” and *Y* will never go beyond K even with extremely bad news.

Such a relationship can also be viewed mathematically. Let the set $\boldsymbol{Y}$={Y: Y is the possible number of ask shares}, and we have that the total number of shares issued is an upper bound of $\boldsymbol{Y}$, while K is the supremum. But there is nothing wrong with treating K directly as the number of shares issued in terms of analyzing the system. So, we have to consider the growth part with a limit. The logistic growth equation should be an ideal choice. It indicates the growth rate will slow down with the increase of population, as more individuals bring more competition for a limited resource. The general form of such growth is $\left[\mathrm{r}\left(1-\frac{\mathrm{x}}{\mathrm{K}}\right)\right]x$ where r is the logistic growth rate and K is the environment capacity indicating the maximum number of x. Notice that the growth rate part is $\mathrm{r}\left(1-\frac{\mathrm{x}}{\mathrm{K}}\right)$, which is a decreasing function of x. It is worth mentioning that most zero-intelligence models assume the placement of new orders follows the Poisson process. However, the number of new orders within a fixed time interval depends on the base number (how many orders are already there), as investors have strategies predicated on such a base number. Thus, this process is not one with an independent increment. Actually, some literature has already suggested that almost all social phenomena follow logistics growth (Solomon & Richmond, 2001). Also, if rY indicates the effect of those investors applying herding strategy, the term $\left(1-\frac{Y}{\mathrm{K}}\right)$ refers to the effect of another group of investors who know the stock will be undervalued if irrationally many investors want to sell the stock. If so, their motivation for selling will decrease with the increase in Y and no more people will sell the stock when Y reaches K (similar to the bid side). Thus, $\left[\mathrm{r}\left(1-\frac{Y}{\mathrm{K}}\right)\right]Y$ covers two groups of investors.

Now we apply the logistic growth to Y and we call,

$$\mathrm{r}x_i(1-\frac{x_i}{K})+\sum_{j=1,j\neq i}^{\infty}\gamma(x_i,x_j)x_ix_j$$

the generalized Verhulst-Lotka-Volterra equations. Then the system becomes,

$$\begin{cases}\dfrac{dX}{dt}=rX-\sigma XY\\ \dfrac{dY}{dt}=rY\left(1-\dfrac{Y}{K}\right)-\sigma XY\end{cases}\tag{2}$$

5.2.2. Dynamical Analysis: $\sigma \leq \mathrm{r}/\mathrm{K}$

We first check the case when $\mathrm{K} \leq \mathrm{r}/\sigma$ or $\sigma \leq \mathrm{r}/\mathrm{K}$. This indicates that the motivation for transactions are relatively low.

Table 2

System 2: Critical Points and Their Types ($\sigma \leq r/K$)

Critical points	Type of critical points
(0, 0)	Unstable node
$(\frac{r}{\sigma},\frac{r}{\sigma})$	Saddle point

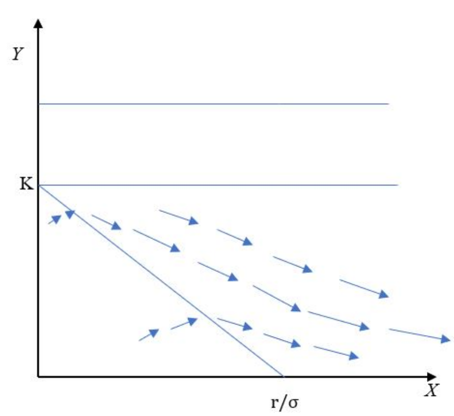

Figure 16: A Rough Stream Plot of System 2 (σ ≤r/K)

In this case, all points will move to *Y*=0 and *X* to infinite. That means this stock will finally go to the situation in which infinitely many people want to buy it. However, based on the reality of stock markets, or economic phenomena, it is impossible that infinite people want to buy an item in the long run. Because an overage of demand will finally lead to price bubbles and then to crashes.

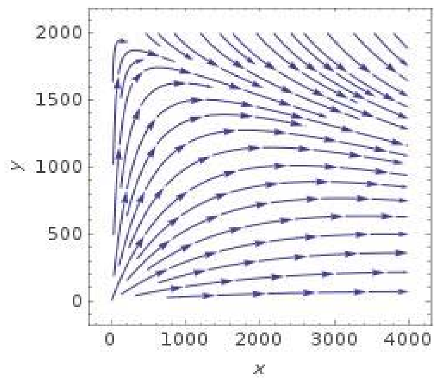

Figure 17: Simulation Data for System 2 (σ ≤r/K): when r=0.1, σ=2×10^{-5}, K=4000.

5.2.3. Dynamical Analysis： $\sigma > \mathrm{r}/\mathrm{K}$.

The other case of system (2) is when $\sigma > \mathrm{r}/K$. This indicates that the motivation for making transactions are relatively low.

Table 3

System 2: Critical Points and Their Types ($\sigma > r/K$)

Critical points	Type of critical points
(0， 0)	Unstable node
(0, K)	Stable node
$(\frac{\mathrm{r}}{\sigma}(1-\frac{\mathrm{r}}{\mathrm{k}\sigma}),\frac{\mathrm{r}}{\sigma})$	Saddle point

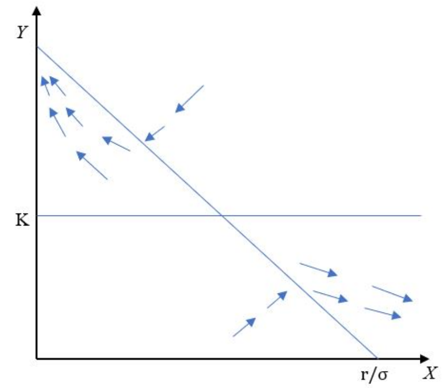

Figure 18: A Rough Stream Plot of System 2 $(\sigma > r/K)$

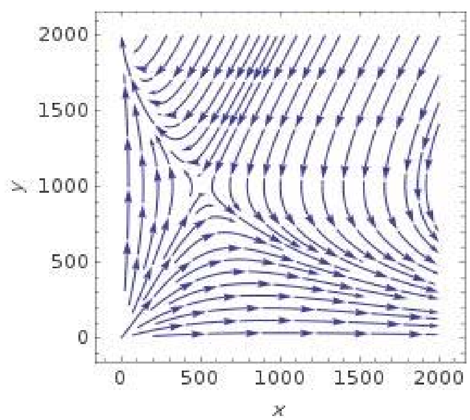

Figure 19: Simulation Data for System 2 ($\sigma > r/K$): when r=0.1, σ=1×10^{-5}, K=4000.

In this case, the particle will go either Y=0, X=+∞ or (0, K). This means that in the end, all investors want to buy this stock. Otherwise, all holders want to sell it. The problem here is similar to that of the previous case. Moreover, theoretically, it is possible that all the holders want to sell. But what we describe is when there is no signal information in the market. Under this assumption, it is unlikely that a stock will go to such an extreme situation.

5.3 Logistics growth for both sides and linear interaction (System 3)

5.3.1. Constructing the model

Now, we consider adding a limit to the buy side as well. As we mentioned in the previous paragraph, demand cannot go infinite. Also, let us consider that, in a rational market, and to avoid an excessive demand or excessive supply, the limit of ask shares should be equal with that of bid shares.

$$\begin{cases} \frac{dX}{dt} = rX(1-\frac{X}{K}) - \sigma XY \\ \frac{dY}{dt} = rY(1-\frac{Y}{K}) - \sigma XY \end{cases} \tag{3}$$

5.3.2. Dynamical Analysis： $\sigma > r/$K.

Similar to system (2), there are two cases for system (3). The first case is when $\sigma > r/K$, which also indicates that the motivation for transactions is relatively high.

Table 4

System 3: Critical Points and Their Types (σ>r/K)

Critical points	Type of critical points
(0, 0)	Unstable node
(0,K)	Stable node
(K,0)	Stable node
$(\frac{kr}{k\sigma+r}, \frac{kr}{k\sigma+r})$	Saddle point

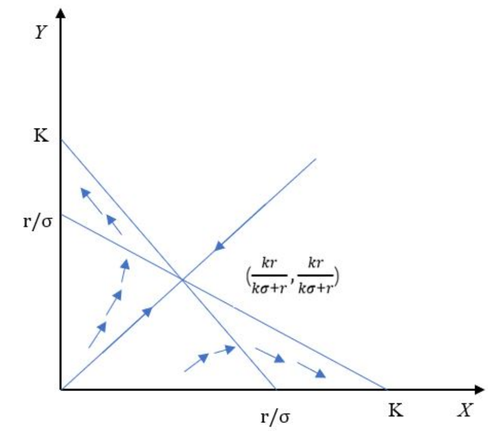

Figure 20: A Rough Stream Plot of System 3 ($\sigma > r/K$)

In this case, most of the solutions will finally go either to the critical point (K,0) or to (0, K). It is also important to note that X=Y is a solution for the system. Because for every point on X=Y, we have $\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{u-Y}{u-X}$ where u=$\frac{kr}{k\sigma+r}$, it follows that the particle on X=Y will move along this line to (u, u). In addition to the truth that any two traces of a solution cannot intersect, we can determine that the attract basin of (K,0) is the area X>Y; the attract basin of (0, K) is the area Y<X. The attract basin indicates that if there are more bid shares than ask shares at t_0, the stock will finally go to a bubble build area, where everyone wants to buy it. The opposite situation happens when there are more ask shares at the initial time. However, even if we add a logistics part to the equations, the particle will still go to the extremity. We will explain the reason in a later paragraph.

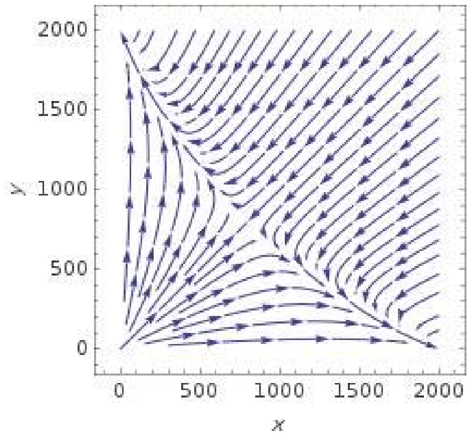

Figure 21: Simulation Data for System 3 ($\sigma > r/K$): when r=0.1, σ=0.75×10^{-4}, K=2000.

5.3.3. Dynamically Analysis: $\sigma \leq r/\mathrm{K}$

The other case is when $\sigma \leq r/K$, which indicates that the motivation for transactions is relatively low.

Table 5

System 3: Critical Points and Their Types ($\sigma \leq r/K$).

Critical points	**Type of critical points**
(0, 0)	Unstable node
(0, K)	Unstable node
(K,0)	Unstable node
$(\frac{kr}{k\sigma+r}, \frac{kr}{k\sigma+r})$	Stable node

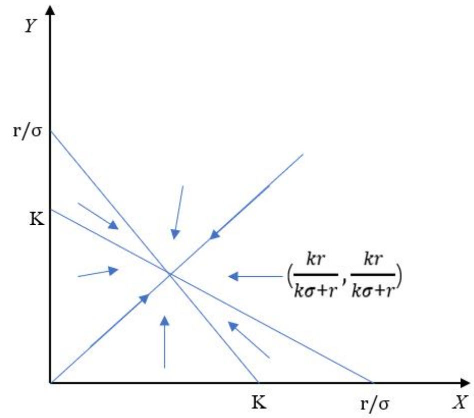

Figure 22: A Rough Stream Plot of System 3 ($\sigma \leq r/K$)

It is obvious that all the traces will finally move to $(\frac{kr}{k\sigma+r}, \frac{kr}{k\sigma+r})$. This means that the stock will reach an equilibrium in the end. The number of both bid and ask shares will become unchanged since the particle has reached the point. It is reasonable that a market reaches an equilibrium when there is neither good nor bad news. But the drawback is that this equilibrium situation is an extremely static state, in which the total number of bid and ask shares, and the trade volume, which is relatively high when the system reaches the fixed point, will all remain unchanged in the long run. Although such hyper-equilibrium seems a castle in the air, it brings us to a situation other than being mired in extremities.

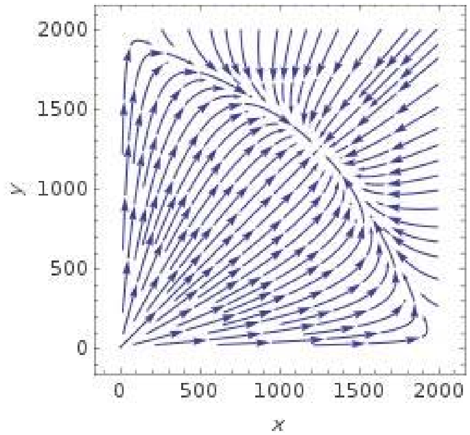

Figure 23: Simulation Data for System 3 ($\sigma \leq r/K$): when r=0.1, σ=0.3×10^{-4}, K=2000.

5.4 Logistics growth and non-linear interaction (System 4)

5.4.1. Constructing the model

The previous models are more or less flawed in the final situation. In other words, the stock is unlikely to exhibit extreme variation between supply and demand when there is no signal information. We therefore go back to check our model, which is composed of two parts. The first part indicates growth, and this part seems unflawed as both X and Y should have a limit. So, we turn our consideration to the second part—σxy. Actually, in the first place, Lotka admitted this term should be $\gamma(\mathrm{x},\mathrm{y})$xy, where $\gamma(\mathrm{x},\mathrm{y})$ is an interaction function for x and y. He made assumption that $\gamma(\mathrm{x},\mathrm{y})$ can be expanded in Taylor series with only constant term remained.

We now consider replacing the constant σ with an appropriate function. We make two simple assumptions about investors. First, if there is no signal information, the motivation of most buyers to make successful transactions (place limit orders with a price higher than ask price, or place market orders) will decline while the exercise price rises. For sellers, such motivation will decline while the exercise price falls[8]. Second, for investors who chose market orders, they will know the trend of change on the exercise price with the change on X and Y. For instance, sellers who chose sell market orders will realize that the exercise price will decrease if X remains unchanged while Y increases constantly. This is because the power of supply is rising, but demand does not change.

For our model, σXY indicates the linear relation between the trade volume and bid-ask shares. This means that when X remains unchanged, the increase in the trade volume will increase proportionally with the increase in Y. Also, such a linear relation indicates the population of either ask or bid shares is purely and positively related to the trade volume. In the following paragraphs, we will demonstrate that the population of bid shares and ask shares can result in a blocking effect acting on the trade volume. It follows because of this that an increase in X and Y is not only able to increase the probability of successful matches (just as an increase in foxes and rabbits results in more mutual interactions), but also has the tendency to make such a probability decrease, which may not be obvious in the case of two biological species with interactions.

Now we introduce a simple example, in which both X and Y are 50 at t_0, and the

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This assumption is based on the property of demand curves and supply curves.

number of successful matches is 5. These shares are distributed on the price grid represented by the horizontal line.

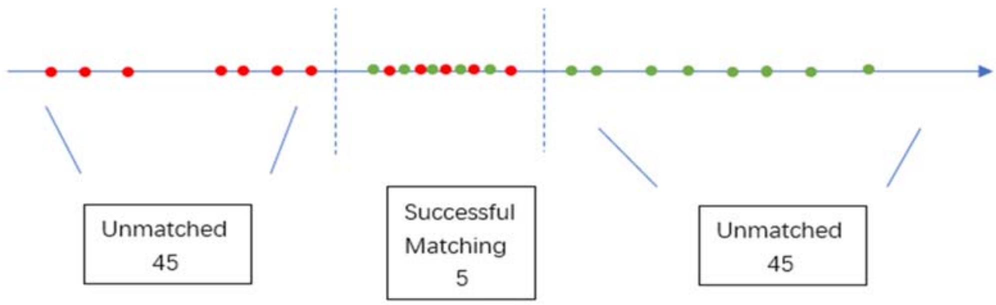

Figure 24: An Example at T_0

If there are 10 newly placed ask shares at t_1, then the number of successful matches should be 6 according to proportional increase. Here, we assume that locations of original shares at t_0 do not change at t_1.

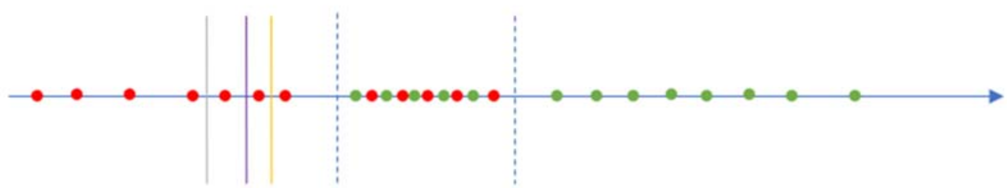

Figure 25: An Example at t_1

Note that among the ten newly placed ask shares, there should be at least one placed on the left-hand side of that yellow line, or overlapping with it, to ensure the proportional increase. That means a newly placed ask share has to be executed at a lower exercise price to make another transaction happen. If we repeat this process, to ensure another successful match, at least one newly placed order should be placed to the left of the purple line at t_2, and to the left of the grey line at t_3. That means the excise price for sellers will go down with the increase in ask shares. It is likely that, for the purple line, as the exercise price becomes lower, we need 11 ask shares to ensure there is at least one investor willing to make a transaction at that price level. And such a chance or a proportion will become less and less as *Y* keeps increasing. So, the real increase should be less than the proportional increase. Now, suppose that $\mathrm{U}(X, Y)$ is the real value of the trade volume, and $\mathrm{U}(X, Y)= \sigma XY$ when $X=X_0$ and $Y=Y_0$. Then $\forall\ \Delta X, \Delta Y > 0$, we have $\mathrm{U}(X_0 + \Delta X, Y_0 + \Delta Y) < \sigma(X_0 + \Delta X)(Y_0 + \Delta Y)$. Otherwise, the trade volume has a proportional or even a larger increase at (X_0, Y_0). Now we have to consider when $\mathrm{U}(X, Y)$ will be equal or almost equal to σXY. This is a signal of a blocking effect of bid-ask shares on the trade volume, or say, on σXY in this system. In other words, the increase in *X* or *Y* cannot be fully (linearly) reflected by the increase in the trade volume. And such a blocking effect will increase with the increase in *X* or *Y*. Apparently, only when both *X* and *Y* are extremely small (and the blocking effect is also small), we have $\mathrm{U}(X, Y) \approx \sigma XY$. For almost all points in region $[K, 0] \times [0, K]$, we have $\mathrm{U}(X, Y) < \sigma XY$.

Besides, such a blocking effect can also be generated from another source-heterogeneity, which means different investors can apply different trading strategies under the same *X* and *Y*. Such a heterogeneity can even result in a negative relation between *Y* (*X*) and the trade volume. Let us return to the previous case, where there were 50 ask shares at t_0, and consider it from the perspective of buyers. The increase in *Y*, which implies an increase in the power on the ask side, is likely to decrease the motivation of some buyers to make transactions. Because buyers do not have any information, they specifically do not know whether there is information held by others. From the perspective of buyers, the increase in *Y* may indicate the existence of negative information which they are unable to access. Thus, for the buyers' part, the motivation to place market orders to have instant transactions will be weakened. If these buyers are dominant, such a phenomenon can exacerbate the blocking effect on trade volume.

As with the first source, the more *Y* increases, the lower the number of market bid shares will be. In the previous case, at t_1, the worst situation without considering the second source is that no seller wants to trade at the price represented by the yellow line, and the trade volume will remain unchanged from that at t_0. Now we remove one bid market share at t_1 due to the second source. As we assumed that the value of *X* is fixed, we have to place a bid limit share to keep *X* unchanged because of the removed market share. What is subtle here is that the loss of market share will absolutely result in a decrease in the trade volume, while the newly placed limit share cannot guarantee a successful match. Then, the worst situation at t_1 results in that no seller wants to trade at the price represented by the yellow line, and the limit bid share applied to replace the lost market bid share is placed at a price lower than the one represented by the yellow line. If so, the trade volume at t_1 will decrease from the trade volume at t_0.

Now, we have already recognized the existence of the blocking effect, which is positively related to the value of *X* and *Y*. Another critical issue is that we set a limit K[9] for both *X* and *Y*. In this case, we should keep our system in the region of $[K, 0] \times [0, K]$. In other words, the region $[K, 0] \times [0, K]$ should be an invariant set for our system. Because of the blocking effect and the invariant set, we propose that the interaction term be,

$$\sigma XY(1-\frac{X}{\mathrm{K}})(1-\frac{Y}{K}).$$

The items within this term are explained below

1. *X* indicates the positive impact of bid shares on trade volume.
2. *Y* indicates the positive impact of ask shares on trade volume.
3. $(1-\frac{X}{\mathrm{K}})$ indicates the blocking effect of bid shares on trade volume.
4. $(1-\frac{Y}{K})$ indicates the blocking effect of ask shares on trade volume.

Of the properties we analyzed earlier, this term has the following: 1)The value of the term is less than σxy 2) The self-block term $(1-\frac{X}{\mathrm{K}})(1-\frac{Y}{K})$ is a decreasing function for *X* and *Y*. 3) The increase in the trade volume is less than the proportional increase. 4) The region of $[K, 0] \times [0, K]$ is an invariant set for our system (an explanation appears later in this thesis). However, a remarkable feature about this interaction term is that the trade volume is zero when either *X* or *Y* reaches K. In reality, it is hard to say what the trade volume should be when *X* or *Y* reaches its maximum

We assumed that *X* and *Y* can never go beyond K even with the impact of signal information.

possible value since there is a lack of empirical study to describe such a situation, especially since we prescribed there should be no signal information. Fortunately, we do not have to worry about such an "undefined" situation (such as $\infty - \infty$ in some cases of mathematics). That is because the trace of a solution is unable to reach the boundary of the $[K, 0] \times [0, K]$ plane if the particle moves purely under the impact of our system, unless the initial point is on the boundary. In other words, a single stock cannot go to the extremity of almost all investors wanting to buy or to sell when there is no signal information. The only possibility for K is that the particle moves to $[K, K]$. We will clarify this later. The model follows:

$$\begin{cases} \dfrac{dX}{dt} = rX(1 - \dfrac{X}{K}) - \sigma XY(1 - \dfrac{X}{K})(1 - \dfrac{Y}{K}) \\ \dfrac{dY}{dt} = rY(1 - \dfrac{Y}{K}) - \sigma XY(1 - \dfrac{X}{K})(1 - \dfrac{Y}{K}) \end{cases} \qquad (4)$$

We do a small transformation firstly.

$$\begin{cases} \dfrac{dX}{dt} = X\left(1 - \dfrac{X}{K}\right)[r - \sigma Y + \dfrac{\sigma Y^2}{k}] \\ \dfrac{dY}{dt} = Y\left(1 - \dfrac{Y}{K}\right)[r - \sigma Y + \dfrac{\sigma Y^2}{k}] \end{cases}$$

The nullclines of the system are:

$$\text{X:}\ X = 0, X = \text{K},\ r - \sigma Y + \frac{\sigma Y^2}{k} = 0$$

$$\text{Y:}\ Y = 0, Y = \text{K},\ r - \sigma X + \frac{\sigma X^2}{k} = 0$$

As both X and Y are from 0 to K, $X\left(1 - \frac{X}{K}\right)$ and $Y\left(1 - \frac{Y}{K}\right)$ are positive. So, if $r - \sigma X + \frac{\sigma X^2}{k}$ and $r - \sigma Y + \frac{\sigma Y^2}{k}$ are also positive when X and Y are from 0 to K, both $\frac{dX}{dt}$ and $\frac{dY}{dt}$ are positive. Under such a circumstance, the stream plot is very simple.

All the traces will finally flow to (K, K), which is a stable node. Another thing we have to notice is that, as *X*=K and *Y*=K are nullclines for *Y* and *X* respectively, all traces with initial points in the square region of $[0, \mathrm{K}] \times [0, \mathrm{K}]$ will always be trapped in this region. In other words, the region is an invariant set for the system. This fact is important, as it ensures the unique solution for all $\mathrm{t} \geq t_0$. Proof can be found in page 61 of Khalil's book (1996). This global condition is stronger than the local unique solution.

Now let $\mathrm{f}(X) = r - \sigma X + \frac{\sigma X^2}{k}$ as an example, for which $\Delta = \sigma^2 - \frac{4\sigma r}{k} = \sigma(\sigma - \frac{4r}{k})$. This indicates that when $\sigma \leq \frac{4r}{k}$, the function will be always positive. Besides, note that if f(*X*)=0 has two roots, the smaller one (say, L_1) must be between zero and K/2 and the larger one (say, L_2) must be between K/2 and K. If the situation above happens, that means σ is very small. A small σ indicates that the motivation of investors to place market orders is extremely low, therefore few transactions happen. But there are still buy orders and sell orders placed as limit buy or sell orders at relatively low or high prices respectively. In other words, there is both demand and supply in the market, but buyers and sellers do not reach a consensus for the trading price. Finally, although there are lots of bid shares and ask shares, there are very few successful transactions. Such "K vs K" stagnation is not very common in the real market, but it is theoretically feasible, as the motivation for trading can be very low when there is no signal information.

Now we attach more importance on the other case when $\sigma > \frac{4r}{k}$. In this case, the motivation level for trading is high and successful transactions happen more frequently.

5.4.2. Dynamical Analysis

Unlike previous cases, this case has eight critical points.

Table 6
System 4: Critical Points and Their Types (σ>4r/K)

Critical points	**Type of critical points**
(0, 0)	Unstable node
$(0, \mathrm{K})$	Saddle point
(K,0)	Saddle point
(K, K)	Stable node
$(\mathrm{L}_1, \mathrm{L}_1)$	Saddle point
$(\mathrm{L}_2, \mathrm{L}_2)$	Saddle point
$(\mathrm{L}_1, \mathrm{L}_2)$	Spiral point
$(\mathrm{L}_2, \mathrm{L}_1)$	Spiral point

We will give a detailed description about the situations at (L_1, L_2) and (L_2, L_1), since determining the type of them is not as easy as others. Also, these two points are important for the rest of thesis. Since the system is symmetric, we take (L_1, L_2) as an example. The Jacobian Matrix at this point is

$$\begin{matrix} 0 & -\sigma L_1(1-\frac{L_1}{K})(1-\frac{2L_2}{K}) \\ -\sigma L_2(1-\frac{L_2}{K})(1-\frac{2L_1}{K}) & 0 \end{matrix}$$

And, we have

$$\lambda^2=\sigma^2 L_2 L_1(1-\frac{L_2}{K})(1-\frac{2L_1}{K})(1-\frac{L_1}{K})(1-\frac{2L_2}{K}).$$

Notice that

$$L_1 < \frac{2}{K}, L_2 > \frac{2}{K},$$

thus

$$\sigma^2 L_2 L_1 \left(1-\frac{L_2}{K}\right)\left(1-\frac{2L_1}{K}\right)\left(1-\frac{L_1}{K}\right)\left(1-\frac{2L_2}{K}\right) < 0.$$

Let

$$\sigma^2 L_2 L_1 \left(1-\frac{L_2}{K}\right)\left(1-\frac{2L_1}{K}\right)\left(1-\frac{L_1}{K}\right)\left(1-\frac{2L_2}{K}\right) = -a,$$

and we have

$$\lambda = \pm\sqrt{a}i.$$

So, the eigenvalues at $(L_2, \ L_1)$ and $(L_1, \ L_2)$ are two pure imaginary numbers, which implies the existence of spirals around the two points. Also, the presence of spirals suggests the existence of periodical solutions. However, the sizes of such spirals are unknown, so further analysis is needed.

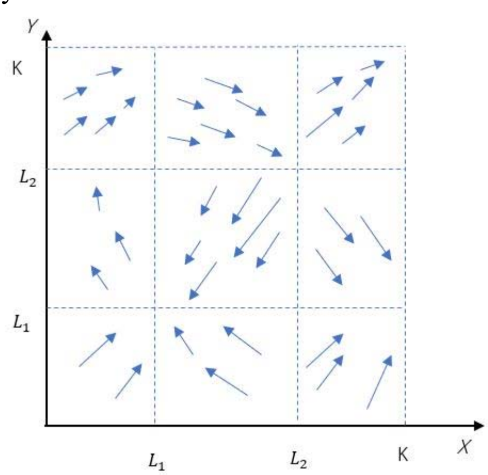

Figure 26: A Rough Stream Plot for System 4 (σ>4r/k)

However, this stream plot above gives us little information. So, we have to explore the phase plot further. Based on the system, we have that

$$\frac{\mathrm{d}Y}{\mathrm{dx}} = \frac{rY(1-\frac{Y}{K})-\sigma XY(1-\frac{X}{K})(1-\frac{Y}{K})}{rX(1-\frac{X}{K})-\sigma XY(1-\frac{X}{K})(1-\frac{Y}{K})}.$$

Transform this equation into

$$\left\{\mathrm{Kr} - \left[\frac{1}{Y(K-r)}\right] - \sigma\right\} dy = \left\{\mathrm{Kr} - \left[\frac{1}{X(K-r)}\right] - \sigma\right\} dx.$$

As *X* and *Y* are independent, we can integrate both sides for *Y* and *X* separately.

$$\int \left\{\mathrm{Kr} - \left[\frac{1}{Y(K-r)}\right] - \sigma\right\} dY = \left\{\mathrm{Kr} - \left[\frac{1}{X(K-r)}\right] - \sigma\right\} dX.$$

Thus, we have

$$\mathrm{r}\ln Y - \mathrm{r}\ln(\mathrm{K} - Y) - \sigma Y = \mathrm{r}\ln X - \mathrm{r}\ln(\mathrm{K} - X) - \sigma X + \mathrm{C}.$$

Constant C is determined by the initial point. As the initial point and C_0 are determined, all the points satisfying the equation

$$\mathrm{r}\ln Y - \mathrm{r}\ln(\mathrm{K} - Y) - \sigma Y = \mathrm{r}\ln X - \mathrm{r}\ln(\mathrm{K} - X) - \sigma X + C_0$$

should be on the same curve, which is the trace of a solutions meeting the initial condition C=C_0. Note that this is an implicit function and we cannot get an explicit form, but we can still figure out a way to draw the graph of the function and do the dynamical analysis for the system. Let

$$\mathrm{F}(X) = \mathrm{r}\ln X - \mathrm{r}\ln(\mathrm{K} - X) - \sigma X.$$

Then $\mathrm{F}(Y)$ can be inferred by vertically moving the graph of $\mathrm{F}(X)$ by C units. We will apply the following method to get the graph for the solution of $\mathrm{F}(Y)$ =$\mathrm{F}(X)$+C. As the system is totally symmetric, we can assume without loss of generality that C>0.

First, draw the graph of F(x), which is an elementary function, and

$$\lim_{X\to 0} \mathrm{F}(X) = -\infty, \lim_{X\to K} \mathrm{F}(X) = +\infty.$$

The derivative is

$$\frac{\mathrm{dF(x)}}{\mathrm{dx}} = \frac{r}{\mathrm{x}} + \frac{r}{K-\mathrm{x}} - \quad \sigma = \frac{\sigma \mathrm{x}^2 - \sigma k \mathrm{x} + rk}{\mathrm{x}(k-\mathrm{x})}$$

and

$$\Delta = (\sigma k)^2 - 4\sigma r k.$$

In this case, the discriminant Δ is positive. And the two roots are:

$$L_1 = \frac{K}{2}\left(1 - \sqrt{1 - \frac{4r}{\sigma K}}\right) < \frac{K}{2}, \qquad L_2 = \frac{K}{2}\left(1 + \sqrt{1 - \frac{4r}{\sigma K}}\right) > \frac{K}{2}.$$

They are the values of *X* at the maximum and the minimum points. Based on the sign of $\frac{dF(X)}{dX}$, we have F(*X*) is monotonously increasing when $X \in [0, L_1) \cup (L_{2,} K]$, and monotonously decreasing when $X \in [L_1, L_{2,}]$. Thus, we have a graph.

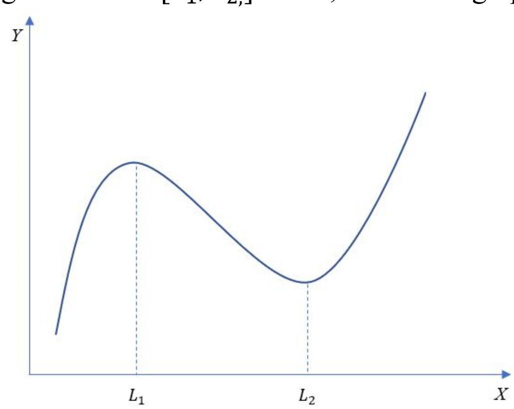

Figure 27: Approximated Graph of F(x).

We first consider the simplest case when C=0. Under such a circumstance, we have $\mathrm{rln}Y - \mathrm{rln}(\mathrm{K} - Y) - \sigma Y = \mathrm{rln}X - \mathrm{rln}(\mathrm{K} - X) - \sigma X$

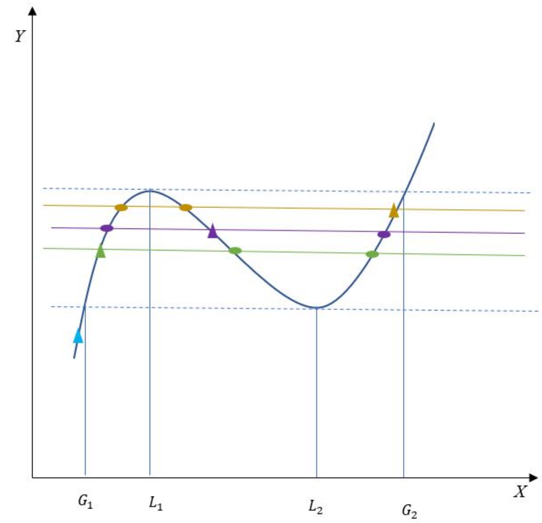

Figure 28: The Graph (Horizontal) (C=0)[10].

The solutions of X and Y for the equation above are those points sharing the same height. For instance, in figure 28, we have $F(G_1)$=$F(L_2)$. That means there should be a point with the coordinates (G_1, L_2) on the trace of the solution. Note that X=Y is always a solution for $rlnY - rln(K - Y) - \sigma Y = rlnX - rln(K - X) - \sigma X$, as all the points on X=Y satisfy this equation. Now, we try to find all the Y corresponding to every X on the bottom line. When 0<X< G_1, each X corresponds to only one Y which is the value of X itself. As we can see in figure 25, there is no other point sharing the same height with the blue triangle. If X=G_1, then Y=L_2 and Y=G_1 correspond to it. That means there should be two points on the curve, (G_1, G_1) and $(G_{1,}L_2)$. When $G_1 <$

[1] For this type of graph, we use the word “horizontal” since the graph demonstrates the relationship between *X* and *Y* on the horizontal direction. In other words, how triangles correspond to ellipses on the horizontal direction.

$X < L_1$, there are two other Y's corresponding to it (except X itself). As in the graph, two green ellipses correspond to the green triangle. As the triangle moves from G_1 to L_1, the right ellipse will move from L_2 to L_1, while the left one moves from L_2 to G_2. When $L_1 < X < L_2$, there are still two other ellipses. The left one will move from L_1 to G_1; the right one will move from G_2 to L_2. A similar situation occurs when X is between L_2 and G_2. Again, when X is larger than G_2, only X itself (on X=Y) corresponds to it. So, when C=0, it is obvious that the curve of the solution is determined.

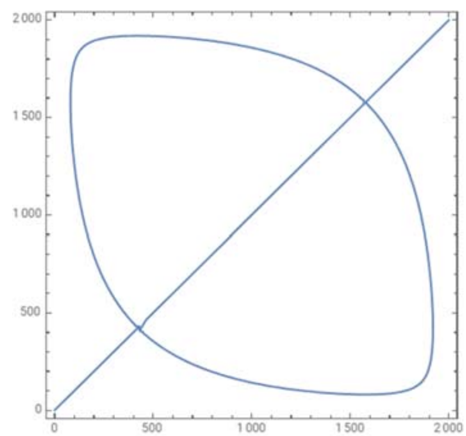

Figure 29: The Graph of $F(Y) = F(X) + C$ (C=0): plotted by WolframAlpha when r=0.1, K=2000, C=0, σ=0.0003.

Notice that, the curve $r\ln y - r\ln(K - y) - \sigma y = r\ln x - r\ln(K - x) - \sigma x$ (ignore x=y here) divide the plane into two parts. That is the one inside the circle, and the one outside the circle. This divide is useful in later analysis.

Another possible case is that $\mathrm{C} \geq \mathrm{F}(L_1) - F(L_2)$.

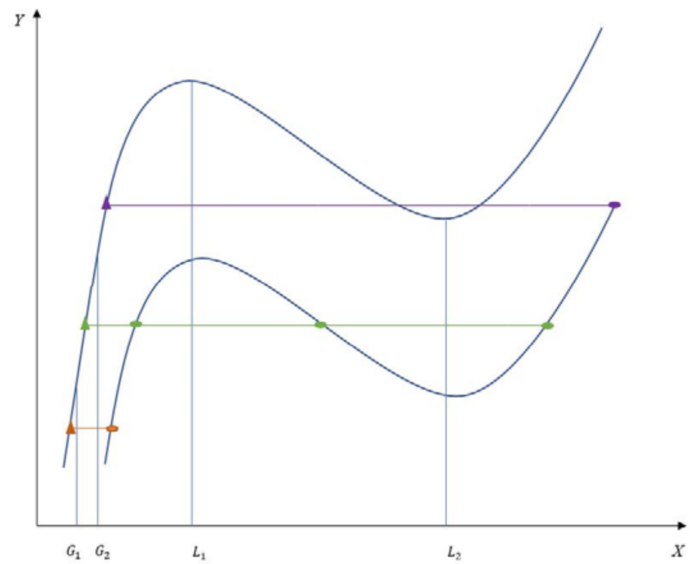

Figure 30: The Graph (Horizontal) $(C \geq F(L_1) - F(L_2))$

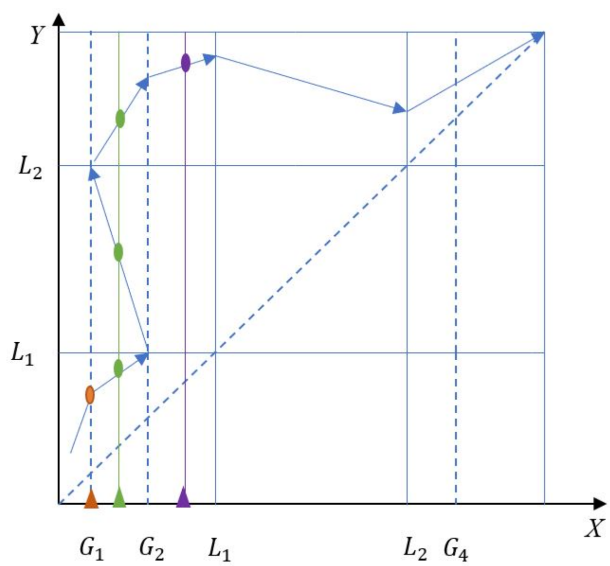

Figure 31: The Trace of a Solution $(C \geq F(L_1) - F(L_2))$[11]

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[11] This demonstrates a solution for the system 4 on *X-Y* plane when $C \geq F(L_1) - F(L_2)$.

The lower curve in figure 31 is F(Y), while the upper curve is F(X)+C, which should be equal to F(Y) to get our solution. In this case, C is larger than $\mathrm{F}(L_1) - F(L_2)$, which is also the difference between the maximum value and the minimum value. Still, we use the same method to analyze it. When 0<X< G_1, each X corresponds to a Y, and Y>X. In the graph, the brown triangle corresponds to only one brown ellipse. When X=G_1, there are two X's corresponding to it. When $G_1 < X < G_2$, each green triangle corresponds to three different green ellipses. And, the left ellipse will move from G_3 to L_1; the middle one will move from L_2 to L_1; the right one will move from L_2 to K. The left and middle ellipses will overlap at L_1 when X reaches G_2. Once X goes far beyond L_1, there will be only one Y corresponding to it (purple triangle and ellipse). It is obvious that all the traces with initial condition $\mathrm{C}_0 \geq \mathrm{F}(L_1) - F(L_2)$ will finally converge on (K, K). Below is the graph plotted by WolframAlpha.

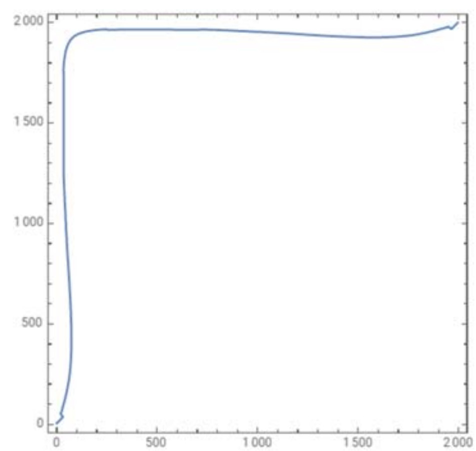

Figure 32: The graph of $F(Y)$ =$F(X)$+C ($C \geq F(L_1) - F(L_2)$): plotted by WolframAlpha when r=0.1, K=2000, σ=0.0003 and C=0.09.

The final case is when $\mathrm{C} < \mathrm{F}(L_1) - F(L_2)$

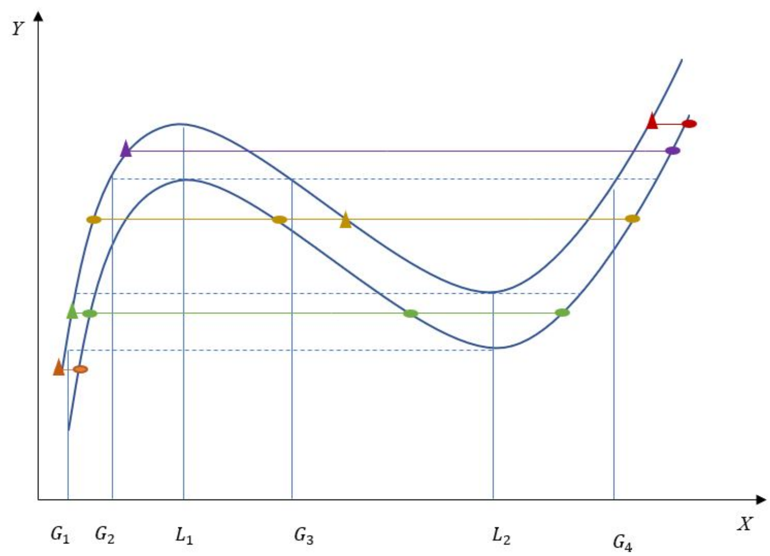

Figure 33: The Graph (Horizontal) $(C < F(L_1) - F(L_2))$

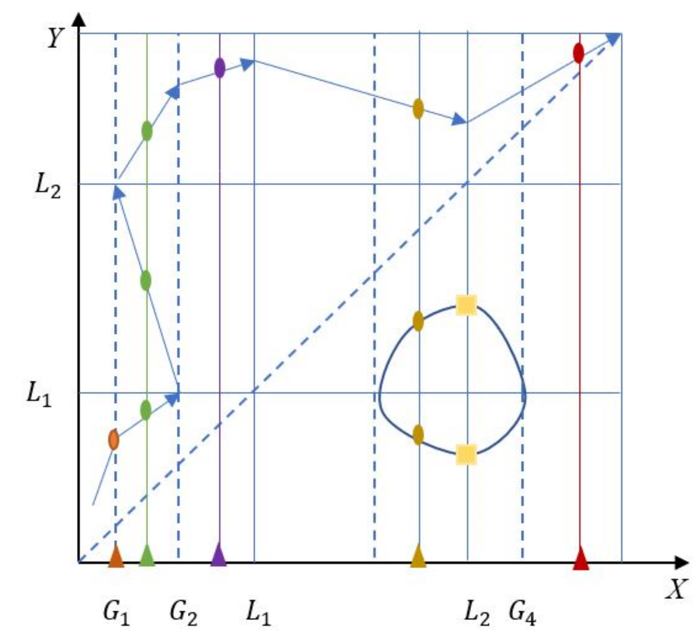

Figure 34: The Trace of a Solution $(C < F(L_1) - F(L_2))$

In this case, the curve of the solution when X is less than G_2 is almost the same as the curve in the previous case, but the location is different, as any two curves cannot intersect. The difference happens when *X* reaches G_3. As we can see from figure 33, the yellow triangle corresponds to three yellow ellipses, while it corresponds to only one in the previous case. Also, among the three ellipses, the one on the right-hand side also appears in the previous case. That means this is a regular solution no matter how C changes. In the graph, no matter how the vertical lines move, there will always be at least one intersection with the upper curve. Now, we concentrate on the left and middle ellipses. When the yellow triangle moves from G_3 to L_2, the left ellipse moves from L_1 to the location of the golden square on the left; the middle one moves from L_1 to the location of the golden square on the right. When the yellow triangle moves from L_2 to G_4, the two yellow triangles move back and finally converge to the position of L_1. Therefore, this part of the trace is a closed orbit from the analysis above. And this orbit is consistent with eigenvalues at this point, which are two pure imaginary numbers. We will show a strict proof in the Appendix 1 that each solution in this area forms a closed orbit. Also, it is not difficult to see that there is a necessary and sufficient condition for a periodical solution. That is, when *X*=L_2, there should two values of *Y* corresponding to it; the smaller one should be less than L_1; the larger one should be between L_1 and L_2. And the situation is the same when *Y*=L_2 (that is, the graph should be symmetric). This can easily be observed from figure 31. Above all, for both cases, the trace of the solution should either go (K, K) or be periodical.

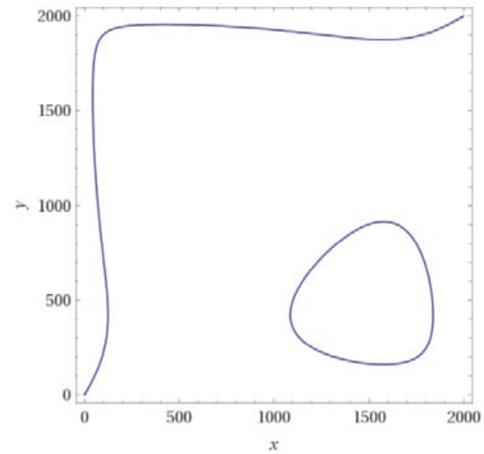

Figure 35: The Graph of $F(Y)$ =$F(X)$+C ($C < F(L_1) - F(L_2)$): plotted by WolframAlpha when r=0.1, K=2000, σ=0.0003, C=0.05.

Now, we move a little further to classify which curve will go to (K, K) and which will be periodical, based on different initial points. Also, there is another critical fact: In this system, none of the curves can go across X=Y. Because any point on this line will move along this line ($\frac{dY}{dX} = 1$). Note that, when C=0, the closed curve meeting

$$\mathrm{rln}Y - \mathrm{rln}(\mathrm{K} - Y) - \sigma Y = \mathrm{rln}X - \mathrm{rln}(\mathrm{K} - X) - \sigma X$$

divides the plane into two parts. For simplicity, we call this curve "V".

Proposition: All the traces with initial points inside V will periodically move within this area, and the rest will move to (K, K).

Proof: Firstly, we prove the first half of this proposition. If a trace of a solution with the initial point inside this curve is not periodical, it must go (K, K) which is located outside the curve V. However, V is a closed curve and the solution is continuous. The trace must go across V to reach K. This contradicts with unique solution of the system.

In other words, any two traces cannot intersect. Thus, any periodical solution with the initial point inside V cannot move out of V, otherwise it intersects V.

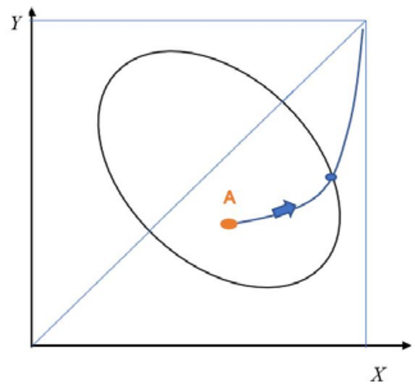

Figure 36: A Trace with the Initial Point Inside V: if a trace with the initial point A inside V moves to (K, K), then it must have an intersection with V

Now we prove the second part. As we mentioned above, there is a sufficient and necessary condition for a periodical solution. That is, when X=L_2, there should be two values of Y corresponding to it; the smaller one should be less than L_1; the larger one should be between L_1 and L_2. Also, the line segment between (L_2, L_2) and (L_2, L_1) is inside V. So, if a trace with an initial point outside V moves across any point on the segment, given the fact that V is a closed curve and that the traces of the solutions are continuous, then the trace must have an intersection with V. This also contradicts with the unique solution property.

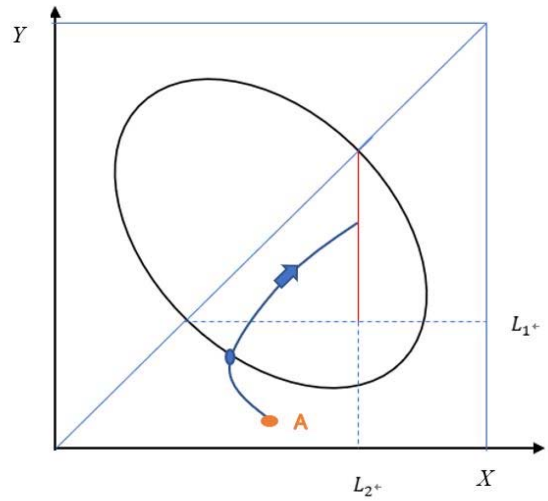

Figure 37: A Trace with the Initial Point Outside *V*: if a trace with the initial point A outside *V* moves across the red line then it must have an intersection with *V*

Now, we can conclude that any trace with an initial point satisfying $[F(y) - F(x)] \times (y - x) < 0$ is periodical, otherwise it will finally converge to (K, K).

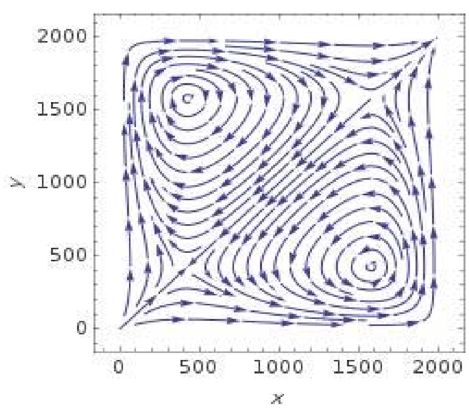

Figure 38: The Simulation Date for The System 4 (σ>4r/K): plot by simulated data, where r=0.1, K=2000, σ=0.0003

5.4.3. Remarks

The model, as we can see, has several interesting qualities. The size of the periodical region, which is encircled by *V*, depends on the value of σ. The larger σ is, the larger the area is. In other words, the system is more likely to be periodical if investors are more willing to choose market orders. In the rest of the paper, we will assume σ is large enough to cover the majority of the region of $[\mathrm{K}, 0] \times [\mathrm{K}, 0]$.

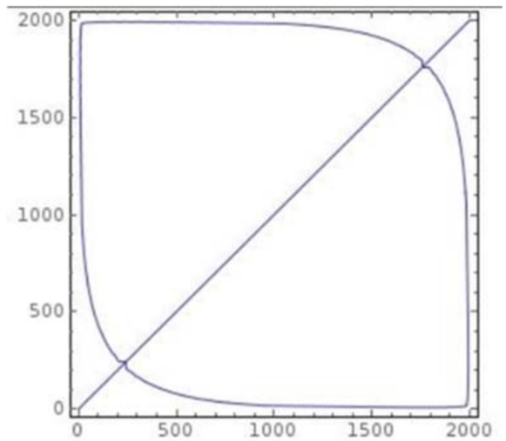

Figure 39: The Size of The Periodical Region (σ=0.0005): when r=0.1, K=2000, σ=0.0005

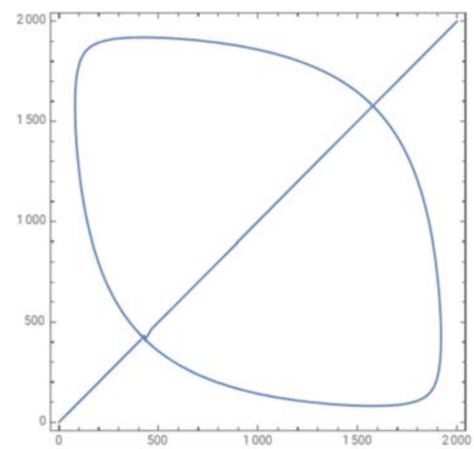

Figure 40: The Size of The Periodical Region (σ=0.0003): when r=0.1, K=2000, σ=0.0003

The single stock cannot switch from the buyer's market to the seller's market or vice versa without being impacted by signal information. As we can see, the particle cannot go across *X*=*Y* if the movement of the particle follows the system. For example, if there are more bid shares than ask shares at t_0, then *X* will always be larger than *Y*.

However, this does not necessarily mean that the stock price will rise forever. There may be an uptrend in the long run, but not at every moment. Such an inability of switching also matches reality, as switching from one type of market to the other usually needs a stimulation, or an obvious change in the relationship between demand and supply. The switch cannot be done under a mild and stable environment which is assumed in this thesis.

Periodicity or quasi-periodicity is a typical phenomenon in the stock market. When X and Y, which represent the power of demand and supply respectively, change periodically, it is likely that the stock price will oscillate within an interval. Such oscillation of stock prices indicates that most investors have a “wait and see” attitude, which can also be interpreted to indicate that current information is not powerful enough to motivate investors.

In this case, there will also be some traces converging to (K, K). But in reality, as mentioned above, such a “K vs K” situation is not common. This is because, for most initial points, it takes a long time for a trace to reach (K, K). So, during this process, and with a large probability, some piece of signal information will arrive to disturb the system. This fact will be made clearer after an explanation in section 6.2.

CHAPTER 6. EXPLORATORY DISCUSSIONS

6.1 Moving along a single circle

Now, we discuss the situation when the particle moves along a single circle. As discussed above, X and Y change periodically. The price of a stock, as with a commodity, is largely dependent on the relationship between demand and supply. If the powers of demand and supply change periodically, it is likely that stock price will also change in a periodic trend. However, a single circle is either in the area of $X>Y$ (where the power of demand surpasses the power of supply), or in the area of $Y>X$ (where the power of supply surpasses the power of demand). For this reason, if the circle is in the area of $X>Y$, the stock price may move in a quasi-periodicity with an up-trend in the long run. Such quasi-periodicity was noted in some studies. Ramsey and Zhang (1996) used waveform dictionaries to study Standard and Poor's 500 stock market index and they found tentative quasi-periodical activities across all frequencies. However, they did not firmly claim the existence of quasi-periodicity, as such a phenomenon was not that obvious (writing that the phenomenon "may not be so clear visually" in the paper). In terms of our model, such a "tentative quasi-periodicity" may result from the switch between information and non-information situations. Quasi-periodicity appears only when the power of external news is slight. Any coming news will disturb the quasi-periodicity. Thus, it is not clearly observed if external news arrives often. Also, having different frequencies implies having circles of different sizes.

6.2 Dynamic between information and non-information

Our model tries to describe the movement of X and Y when there is no signal information on this stock, and we name the set containing all the movements under such circumstance, "M_1". However, once an essential news comes, no matter whether it is good or bad, the system is going to be broken by it, and the movement will no longer follow the original trace. Then, the movement totally depends on the information itself, and we name the set containing all the movements under such circumstance, "M_2". Once the impact of such information cease (we assume such impact will cease), the movement will follow our system again from a new initial point. So, it is a switch under two different situations. Below is an example:

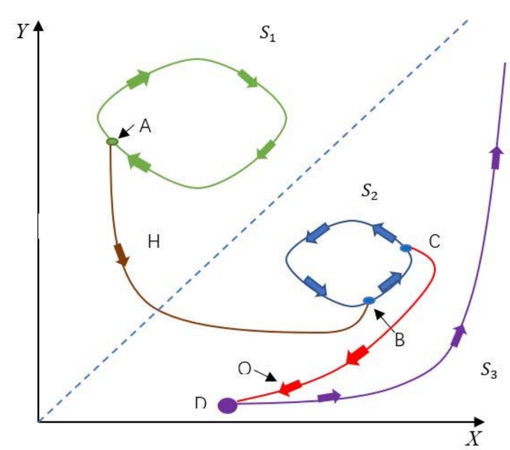

Figure 41: How a Stock Switches between Two Situations

We assume that, in the first phase, the particle moves along S_1, which is the green circle. Signal information arrives when the particle is at A. Then the particle moves along H (the brown curve) to B be under the effect of the information. In the second phase, the impact of the signal information vanishes at B, where the movement follows the system again. Point B is the initial point of the new phase and it will determine the trace of a new trace of movement. Later, another piece of signal information arrives when the particle is at C; then the particle will be "dragged" again by the information to D where the third phase begins. As we can see in fig 38, $S_{i,}\ i = 1, 2, 3$ are traces of movement without information and they follow our system. Curves O and H are traces when there is signal information and they do not follow the system. Overall, the dynamic of X and Y keeps oscillating between the sets M_1 and M_2. Moreover, the impacts of the signal information represented by H and O are strong, since they are able to drag the particle a long distance. But such strong signal information is not common in the real world. If the impact of the signal information is slight, it may just "shock" the particle from one place to another nearby. Since two close trajectories are very similar to each other, such slight change is not easily observable.

6.3 Self-similarity and chaos

Obviously, the particle can move along different traces at different times. Each circle in the phase diagram represents a function (with a different constant, C_0),

$$\mathrm{r}\ln Y - \mathrm{r}\ln(\mathrm{K} - Y) - \sigma Y = \mathrm{r}\ln X - \mathrm{r}\ln(\mathrm{K} - X) - \sigma X + C_0.$$

The graphs of this family of functions are similar in shape (although not totally homothetic), but different in size.

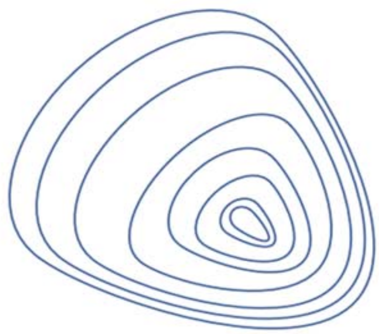

Figure 41: Similar Trajectories

Also, the size of each circle represents the length and range of different periodicities, while the shape represents the configuration of each periodicity. In other words, they are scale-free to some extent and thus may generate some self-similar periodicities for the movement of stock price[12]. Self-similarity is a canonical expression of chaos and has been confirmed by many previous studies since first noticed by Mandelbrot (1963). For instance, the famous Elliot wave principle (Frost & Prechter, 2005) has already suggested that the market price consists of periodicities with different

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1 For this case, we need to imagine that the particle is carried by the signal information from one circle to another constantly.

size waves. In addition to some other studies in chaos theory, our model explains, though not perfectly, the reason that stock prices can behave chaotically. This principle conforms to our model for both periodicity and self-similarity. Another interesting finding is that the sizes of circles are positively related to the lengths of periodicities. When the length is shorter, such as a couple of minutes or hours, it is more likely that the system holds. For instance, the probability of neither good nor bad news arriving during several hours is apparently larger than that of arriving during several months. This may be the reason why intraday periodicity is widely recognized (Andersen & Bollerslev, 1997) , while there are fewer studies for longer periods.

CHAPTER 7: CONCLUSION

We determined that bid and ask shares in limit and market orders are similar to two animal species with an interaction effect. Thus, we are motivated to simulate how the population of bid-ask shares evolve over time by using GLV equations. Both growth functions and interaction functions should match some properties of bid-ask shares in the real stock market. Besides this, the model works when there is a lack of signal information which influences the trading strategy of agents. The model in 5.1 is an expression of a pure herding strategy market where investors copy the strategy of others and are craving to make transactions happen. Thus, the system will finally go to a situation in which either everyone buys or everyone sells. Most of the models in 5.1-5.3 have a similar extremity situation. They are all led by the linear interaction effect, which indicates a large trade volume, regardless of what items are in the growth part. The only exception is the case 2 in 5.3. The model gives a clue that if the trade volume is small (σ is very small there), something different than an extremity can happen. After detecting a blocking effect acting on the trade volume resulted by the X and Y, we proposed our final model. The main result of the simulation is that, given a certain initial point, the population of bid and ask shares will move periodically along a specified trace. This result indicates the stock price is likely to have a quasi-periodical movement until a piece of signal information arriving to disturb the system. Also, the movement of the particle keeps switching between a situation with signal information and a situation without signal information. In the latter situation, traces for the movement are different circles, which are similar in the configuration but different in size. Since such circles may correspond to self-similarity and chaos theory, our model indicates that the reason that chaotic behavior in the stock price results from heterogeneity and signal information, which can "drag" the particle from one trace to another. The limit is that since X and Y are two abstract concepts, and the impact of signal information is difficult to measure, it is difficult to statistically test the model. Besides, the interaction function proposed is just one possible form.

A future research opportunity is to work on the impact of signal information to make the model more "realistic". Specifically, if we assume the impact of signal information follows a stochastic process, then the ordinary dynamical system may be modified to a random dynamical system. Based on the classification made by Saaty (2012), the impact of signal information can result in three types of randomness to the system. 1) Random initial conditions. 2) Random forcing functions. 3) Random coefficients. Besides, we can modify the model directly to a discrete dynamical system, which may show us chaos even without the impact of signal information.

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APPENDIX 1: THE PROOF OF THE PERIODICAL SOLUTION

As there are two critical points on the border of the target area, the Poincare-Bendixson theorem fails here. And this proof is almost the same as the proof of periodical solution for prey-predator model. We largely refer to the method from Hirsch and Smale et al (2012) and more information can be found there. Let

$$L_{t_n}^{(x_0,y_o)}(x,y) = \mathrm{rlny} - \mathrm{rln(K-y)} - \sigma\mathrm{y} - [\mathrm{rlnx} - \mathrm{rln(K-x)} - \sigma\mathrm{x}].$$

The solution with initial point (x_0, y_0) which is different than (L_2, L_1) . There are two qualities about $L_{t_n}^{(x_0,y_o)}(x,y)$: 1) The trace of this solution is not a limit circle, as $L_{t_n}^{(x_0,y_o)}(x,y)$ is not constant under any open set. 2) It is easy to check $L_{t_n}^{(x_0,y_o)}(L_1, L_2)$ is a strict local maximum point. We proved that the trace inside V will be always be trapped in this area already. That is to say, the trace moves spirally around critical point (L_2, L_1). It indicates the trace will move across nullcline X=L_2 countless times. And, apparently, X=L_2 is also a local section of this system. So, there is a doubly infinite sequence $\dots. < t_{-1} < t_0 < t_1 \dots \dots$ and $L_{t_n}^{(x_0,y_o)}(x,y)$ will move across X=L_2 countless times. If points on $L_{t_n}^{(x_0,y_o)}(x,y)$ are not on a closed orbit, then they are monotone on X=L_2. Since there is no limit circle, $L_{t_n}^{(x_0,y_o)}(x,y)$ should be approach (L_2, L_1), when either $\mathrm{n} \to +\infty$, or $\mathrm{n} \to -\infty$. Also,we have that

$$L_{t_n}^{(x_0,y_o)}(x,y)$$

is a constant alone a certain solution. This indicates

$$L_{t_n}^{(x_0,y_o)}(x_0,y_0) = \mathrm{L}(L_1,L_2)$$

However, this contradicts with the fact that $\mathrm{L}(L_1,L_2)$ is a local maximum point

APPENDIX 2: THE PROOF ABOUT THE LIPSCHITZ CONDITION

Here, we want to prove that continuous first order partial derivatives indicate the Lipschitz condition in a plane autonomous system, which is applied in the thesis. Let the system be as follow:

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

Both $\mathrm{f(x,y)}$ and $\mathrm{g(x,y)}$ have continuous first order partial derivatives for x and y. Then we have,

$$\frac{\|G(t_2;X_2) - G(t_1;X_1)\|}{\|X_2 - X_1\|} = \sqrt{\frac{[f(x_2,y_2)-f(x_1,y_1)]^2+[g(x_2,y_2)-g(x_1,y_1)]^2}{(x_2-x_1)^2+(y_2-y_1)^2}}$$

Suppose that, when t_1 and t_2 are very close, we can linearize the part

$$[f(x_2,y_2)-f(x_1,y_1)]^2+[g(x_2,y_2)-g(x_1,y_1)]^2$$

as follows:

$$[\frac{\partial \mathrm{f}}{\partial \mathrm{x}}|_{x=x_1,y=y_1}(x_2-x_1)-\frac{\partial \mathrm{g}}{\partial \mathrm{x}}|_{x=x_1,y=y_1}(x_2-x_1)]^2 + [\frac{\partial \mathrm{f}}{\partial \mathrm{y}}|_{x=x_1,y=y_1}(y_2-y_1)-\frac{\partial \mathrm{g}}{\partial \mathrm{y}}|_{x=x_1,y=y_1}(y_2-y_1)]^2,$$

which is equal to

$$(\frac{\partial f}{\partial x}|_{x=x_1,y=y_1}-\frac{\partial g}{\partial x}|_{x=x_1,y=y_1})^2 \times (x_2-x_1)^2+(\frac{\partial f}{\partial y}|_{x=x_1,y=y_1}-\frac{\partial g}{\partial y}|_{x=x_1,y=y_1})^2 \times (y_2-y_1)^2.$$

Based on our assumption of continuous first order partial derivatives for x and y, $\frac{\partial f}{\partial x}|_{x=x_1,y=y_1}$, $\frac{\partial g}{\partial x}|_{x=x_1,y=y_1}$, $\frac{\partial f}{\partial y}|_{x=x_1,y=y_1}$, $\frac{\partial g}{\partial y}|_{x=x_1,y=y_1}$

should be all bounded in the closed zone $[K,0] \times [K,0]$. Thus, we have

$$(\frac{\partial f}{\partial x}|_{x=x_1,y=y_1}-\frac{\partial g}{\partial x}|_{x=x_1,y=y_1})^2$$

and

$$(\frac{\partial f}{\partial y}|_{x=x_1,y=y_1}-\frac{\partial g}{\partial y}|_{x=x_1,y=y_1})^2$$

are bounded. Let both of them $\leq$ L. Then,

$$[\frac{\partial f}{\partial x}|_{x=x_1,y=y_1}(x_2-x_1)-\frac{\partial g}{\partial x}|_{x=x_1,y=y_1}(x_2-x_1)]^2 + [\frac{\partial f}{\partial y}|_{x=x_1,y=y_1}(y_2-y_1)-\frac{\partial g}{\partial y}|_{x=x_1,y=y_1}(y_2-y_1)]^2$$

$$= (\frac{\partial f}{\partial x}|_{x=x_1,y=y_1}-\frac{\partial g}{\partial x}|_{x=x_1,y=y_1})^2 \times (x_2-x_1)^2+(\frac{\partial f}{\partial y}|_{x=x_1,y=y_1}-\frac{\partial g}{\partial y}|_{x=x_1,y=y_1})^2 \times (y_2-y_1)^2 \leq \mathrm{L}[(x_2-x_1)^2+(y_2-y_1)^2].$$

Therefore, we have

$$\frac{\|G(t_2;X_2)-G(t_1;X_1)\|}{\|X_2-X_1\|} = \sqrt{\frac{[f(x_2,y_2)-f(x_1,y_1)]^2+[g(x_2,y_2)-g(x_1,y_1)]^2}{(x_2-x_1)^2+(y_2-y_1)^2}} \leq \mathrm{L},$$

where L can be treated as Lipschitz constant.