# A HIGH-ORDER, CONSERVATIVE INTEGRATOR WITH LOCAL TIME-STEPPING*

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**Abstract.** We present a family of multistep integrators based on the Adams-Bashforth methods. These schemes can be constructed for arbitrary convergence order with arbitrary step size variation. The step size can differ between different subdomains of the system. It can also change with time within a given subdomain. The methods are linearly conservative, preserving a wide class of analytically constant quantities to numerical roundoff, even when numerical truncation error is significantly higher. These methods are intended for use in solving conservative PDEs in discontinuous Galerkin formulations, but are applicable to any system of ODEs. A numerical test demonstrates these properties and shows that significant speed improvements over the standard Adams-Bashforth schemes can be obtained.

**Key words.** local time-stepping, multirate time integration, Adams methods, adaptive time stepping, conservation laws

**AMS subject classifications.** 65L06, 65L60, 65M20, 65M60, 65Z05

## 1. Introduction.

A common problem in computational fields is to find approximate solutions to partial differential equations (PDEs). For hyperbolic PDEs, where a solution typically describes an evolution of one or more fields through time, the most common approach is to apply the method of lines, where the spatial coordinates in the PDE are discretized, producing a large system of coupled ordinary differential equations (ODEs). These systems of equations can then be discretized in time and solved using standard explicit integration schemes.

To obtain correct solutions to these equations, the time discretization must be fine enough for the integration to be stable. For a method-of-lines system, this limit is primarily because of the Courant-Friedrichs-Lewy (CFL) condition, which limits the step size to approximately the information propagation time between grid points. The resulting step size can show large variation across the spatial domain because of changes in the propagation speed or, more commonly, because of changes in the spacing of the evaluation points. It is often desirable to increase the spatial resolution in some regions to resolve rapidly varying parts of the solution, but this then restricts the step size allowed for stability. Furthermore, in order to evaluate the system right-hand side, it is necessary to know the entire state of the system at the time of interest. The time step for the whole system is then set by the most restrictive of the conditions over the entire domain. If the problematic points make up a small fraction of the system, then the forced evaluations at the remaining points can dominate the computational expense.

To reduce the computational cost of finding these solutions, we would like to evaluate each point at intervals set by its own stability limit, rather than the smallest limit for all the points. A method allowing this is known as a *local time-stepping* (LTS)

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(or *multirate*) method, as opposed to a global time-stepping (GTS) method. Such a method must describe an update scheme for the frequently evaluated degrees of freedom that does not require knowing the full state of the system.

Modifying a GTS method into an LTS one can have significant drawbacks. The individual steps near locations of time step changes are typically more expensive than for a GTS method, so the benefit of fewer derivative evaluations must outweigh this overhead. Care must be taken when calculating the CFL limit near step size changes to take into account variations in the characteristic speeds of the system in the neighborhood of the element. [1] Furthermore, modifying the GTS scheme can destroy numerically desirable properties of the integrator, such as a high convergence order. LTS schemes also do not naturally provide exact conservation of linear conserved quantities [2], a property often taken for granted for GTS integrators. In a physical system, errors accumulated in these quantities (which can represent, for example, total mass) can produce an approximate solution qualitatively different from the true solution.

Early LTS schemes (for example [3]) typically used GTS integrators with different time steps and performed interpolation to obtain data at times at which it was not produced directly. Such schemes are easy to adapt to arbitrary mesh configurations and can be constructed to obtain the same convergence order as the underlying GTS method, but they do not preserve conserved quantities of the system. Corrections to more accurately treat conservation laws were developed [4], but still only resulted in approximate conservation.

More recently, many methods have been investigated as starting points for more sophisticated LTS methods, including both substep [5, 6, 7, 8, 9, 10] and multistep [2, 11, 12] integrators and also less common methods such as leapfrog [11, 13], Richardson extrapolation [14], ADER [15], and implicit methods [16]. Demirel *et al.* [17] have even explored LTS schemes constructed from multiple unrelated GTS integrators. Recently, Günther and Sandu [9] presented a very general family of multirate Runge-Kutta-like methods based on the GARK family of integrators [18] that unifies many of the previous Runge-Kutta-based LTS schemes. These methods are applicable to any problem and can be constructed to have any order of accuracy, but they are not conservative. Sandu and Constantinescu [2] presented an Adams-Bashforth-based scheme based on evaluating the right-hand side of the evolution equations using a combination of data at different times. This system is conservative and applicable to any system of equations, but the method is limited to second-order accuracy at times at which all degrees of freedom are evaluated and first-order accuracy at intermediate times.

LTS integrators for the special case of linear systems have been developed based on Adams-Bashforth [11], Runge-Kutta [7, 10], and leapfrog [11, 13] schemes. Of particular interest here, starting from the Adams-Bashforth methods, Grote and Mitkova [11] found a family of high-order, conservative methods for integral ratios between step sizes on different degrees of freedom. These methods use the linearity of the system to split the equations into a form resembling multiple copies of the standard Adams-Bashforth method.

Some authors have derived methods specialized to the discontinuous Galerkin or finite volume formalisms. The structure of elements coupled comparatively weakly in a standard way by exchange of fluxes allows for some simplifications to the problem. Winters and Kopriva [12] presented a scheme using dense output of the integrators for each element to calculate fluxes at intermediate times. This scheme is high-order and allows for arbitrary step ratios and varying time steps, but it sacrifices the con-

servative nature of its parent scheme. Gassner *et al.* [6] presented a similar method, but restored conservation by treating the element and flux terms as a predictor and corrector. Krivodonova [5] constructed a method based on a Runge-Kutta integrator which, while not naturally conservative, was made so by adding a correction to cancel any error in conservation whenever neighboring cells are aligned in time. Cavalcanti *et al.* [19] considered the addition of nonlinear operations, such as slope limiting, to the integration step.

In this paper we present a generic, high-order, conservative scheme based on the Adams-Bashforth family of explicit multistep methods. The method uses the idea of performing single right-hand side evaluations using values from different times, in a similar manner to previous work presented by Sandu and Constantinescu [2]. The scheme is conservative and has the same convergence order as the Adams-Bashforth integrator it is based on. The method allows for generic ratios of step sizes between different degrees of freedom, as well as for arbitrarily varying the individual degrees of freedom's step sizes in time. While the applications discussed here are to discontinuous Galerkin systems, the method is fully general and can be applied to any set of coupled ODEs. When applied to a linear system with integral step size ratios, this scheme reduces to the Adams-Bashforth-based scheme presented by Grote and Mitkova [11].

The remainder of this paper is structured as follows: Section 2 presents a derivation of the integration scheme. Section 3 discusses simplifications that are applicable when the method is applied to some common special cases. Section 4 applies the method to numerical test cases. An appendix lists specific formulas for methods of order 2, 3, and 4.

## 2. The method.

### 2.1. Adams-Bashforth methods.

Suppose we wish to numerically solve a set of coupled first-order ordinary differential equations

$$\frac{d\mathbf{y}}{dt} = \mathbf{D}(\mathbf{y}), \tag{2.1}$$

where $\mathbf{D}(\mathbf{y})$, the time-derivative operator, is the right-hand side evaluated when the system is in state $\mathbf{y}$. A common method is to solve for the variables at a (monotonic) sequence of times $t_0, t_1, \ldots$ using a $k$th-order Adams-Bashforth method

$$\Delta\mathbf{y}_n = \Delta t_n \sum_{j=0}^{k-1} \alpha_{nj} \mathbf{D}(\mathbf{y}_{n-j}) \tag{2.2}$$

with $\Delta\mathbf{y}_n = \mathbf{y}_{n+1} - \mathbf{y}_n$, $\Delta t_n = t_{n+1} - t_n$ and the coefficients corresponding to the step given by [12]

$$\alpha_{nj} = \frac{1}{\Delta t_n} \int_{t_n}^{t_{n+1}} dt\, \ell_j\big(t; t_n, t_{n-1}, \ldots, t_{n-(k-1)}\big)\,. \tag{2.3}$$

Here

$$\ell_n(t; t_0, \ldots, t_{k-1}) = \prod_{\substack{j=0 \\ j\neq n}}^{k-1} \frac{t - t_j}{t_n - t_j} \tag{2.4}$$

are Lagrange polynomials. We will not concern ourselves here with the process of starting the evolution, that is, evaluating $\Delta\mathbf{y}_n$ for $n < k - 1$.

If different degrees of freedom require different time steps for stability, it may be desirable to evaluate these variables at different frequencies, in order to avoid unnecessary computations for the more stable variables. Suppose we divide $\mathbf{y}$ into $S$ sets $\mathbf{y}^1, \ldots, \mathbf{y}^S$, and that we wish to evaluate $\mathbf{y}^s$ at times $t_0^s, t_1^s, \ldots$. We can then split (2.1) into an equation for each of these sets:

$$\frac{d\mathbf{y}^s}{dt} = \mathbf{D}^s\big(\mathbf{y}^1, \ldots, \mathbf{y}^S\big)\,, \tag{2.5}$$

where $\mathbf{D}^s$ is the result of $\mathbf{D}$ restricted to the set $s$. Any attempt to use this equation to perform an LTS evolution immediately encounters the problem that evaluating its right-hand side requires knowing the entire state of the system, which conflicts with the goal of independent evaluation times for different degrees of freedom.

**2.2. Conserved quantities.** A linear conserved quantity is a quantity $C$ expressible as an inner product of a vector $\mathbf{c}$ with the evolved variables (treated as a vector)

$$C = \mathbf{c} \cdot \mathbf{y}, \tag{2.6}$$

with

$$\mathbf{c} \cdot \mathbf{D}(\mathbf{y}) = 0 \tag{2.7}$$

for all values of $\mathbf{y}$. Such a quantity is constant under exact integration of the system and under integration using Euler's method. An integrator is called (linearly) *conservative* if all such quantities remain constant when integrating a system using it [20]. It is desirable for an integrator to keep such quantities precisely constant (up to roundoff error) rather than merely constant up to the truncation error of the scheme. Such quantities often have an intuitive physical meaning, and frequently even a small rate of drift can cause qualitative changes in the evolution of the system.

When solving a PDE representing a physical system, the most common linear conserved quantities are integrals over the computational domain of fields representing densities. The vector $\mathbf{c}$ in these cases is the vector of coefficients necessary to perform a numerical integral. In a discontinuous Galerkin scheme these coefficients would combine quadrature weights on the elements and factors arising from coordinate mappings of the elements relative to their canonical shapes.

**2.3. Second-order $2:1$ stepping.** Let us first consider as an example the case of a second-order scheme on two sets, $A$ and $B$, with $B$ being evaluated twice as often as $A$. Call their step sizes $\Delta t^A$ and $\Delta t^B = \Delta t^A/2$. This step pattern is shown in Figure 1, where for simplicity we consider the steps starting from $t = 0$ leading up to $t = \Delta t^A$. There are three types of steps to consider: the large step on set $A$, labeled $(a)$, and the first and second halves of that step on set $B$, labeled $(b)$ and $(c)$. This case is considered in Sandu and Constantinescu [2], but the method presented there only provides a second-order value when sets have stepped to the same time; intermediate values are only accurate to first order.

We will start with the small step $(b)$. For a GTS Adams-Bashforth method, this step would be given by

$$\Delta\mathbf{y}_b^B = \Delta t^B \left[\frac{3}{2}\tilde{\mathbf{D}}^B(0) - \frac{1}{2}\tilde{\mathbf{D}}^B(-\Delta t^B)\right], \tag{2.8}$$

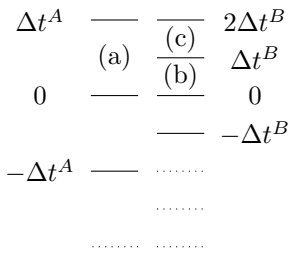



FIG. 1. *The step pattern for a* $2:1$ *method on two sets, with time steps* $\Delta t^A = 2\Delta t^B$. *There are three types of steps: the large step on element A marked (a), and the two types of small step on B marked (b) and (c). For a second order method, we use only the two most recent values of the variables when taking a step. Steps whose values are no longer needed for the indicated steps are marked with dotted lines.*

where we write $\tilde{\mathbf{D}}^B$ instead of $\mathbf{D}^B$ to remind ourselves that we cannot generally obtain these values from a straightforward application of the derivative operator. At time $t = 0$ we do have values over our entire system, so $\tilde{\mathbf{D}}^B(0)$ can, in fact, be evaluated by a simple use of the derivative operator, giving $\tilde{\mathbf{D}}^B(0) = \mathbf{D}^B[\mathbf{y}^A(0), \mathbf{y}^B(0)]$.

We cannot evaluate $\tilde{\mathbf{D}}^B(-\Delta t^B)$ in this manner, because we do not have data for $\mathbf{y}^A$ at $t = -\Delta t^B$, so we must construct it from the values at $t = 0$ and $t = -2\Delta t^B$. There are two reasonable choices of how to do this: average the known values of $\mathbf{y}^A$ to get a value at the desired time and use that to apply the derivative operator, or apply the derivative operator at both times (using the value of $\mathbf{y}^B$ at $-\Delta t^B$ both times) and average the results. We choose the latter, giving step $(b)$ as

$$\Delta \mathbf{y}_b^B = \Delta t^B \left[ \frac{3}{2} \mathbf{D}^B \big[\mathbf{y}^A(0), \mathbf{y}^B(0)\big] - \frac{1}{4} \mathbf{D}^B \big[\mathbf{y}^A(0), \mathbf{y}^B(-\Delta t^B)\big] - \frac{1}{4} \mathbf{D}^B \big[\mathbf{y}^A(-\Delta t^A), \mathbf{y}^B(-\Delta t^B)\big] \right]. \tag{2.9}$$

The error in averaging the derivatives is of order $(\Delta t^B)^2$, so it introduces an error of order $(\Delta t^B)^3$ in the value after the step, preserving the second-order quality of the base GTS method.

The second small step, $(c)$, proceeds similarly, except that we now use a derivative at $\Delta t^B$ instead of $-\Delta t^B$. Instead of averaging the derivatives at different $\mathbf{y}^A$ we must therefore perform a (linear) extrapolation to obtain our approximate derivative $\tilde{\mathbf{D}}^B(\Delta t^B)$. After doing this, we obtain the rule

$$\Delta \mathbf{y}_c^B = \Delta t^B \left[ \frac{9}{4} \mathbf{D}^B \big[\mathbf{y}^A(0), \mathbf{y}^B(\Delta t^B)\big] - \frac{3}{4} \mathbf{D}^B \big[\mathbf{y}^A(-\Delta t^A), \mathbf{y}^B(\Delta t^B)\big] - \frac{1}{2} \mathbf{D}^B \big[\mathbf{y}^A(0), \mathbf{y}^B(0)\big] \right]. \tag{2.10}$$

We could use the same procedure to evaluate the large step $(a)$, but, as this would not take into account the value $\mathbf{y}^B(\Delta t^B)$ used for taking the second small step, there is no way this procedure could be conservative. This, however, gives us a hint as to how to proceed: we treat the large step as having two internal steps, one for balancing each of the small steps. In fact, in order to remain conservative, we must take each of these internal steps using the same scheme as for the corresponding small step,

except using the part of the derivative corresponding to set $A$. This can be seen by considering a generic pair of methods for a step: $\Delta\mathbf{y}^{A,B} = \sum_i k_i^{A,B}\mathbf{D}_i^{A,B}$. The change in a linear conserved quantity during that step is

(2.11)
$$\Delta C^A + \Delta C^B = \mathbf{c}^A \cdot \sum_i k_i^A \mathbf{D}_i^A + \mathbf{c}^B \cdot \sum_i k_i^B \mathbf{D}_i^B = \sum_i \left[k_i^A \mathbf{c}\cdot\mathbf{D}_i + (k_i^B - k_i^A)\mathbf{c}^B \cdot \mathbf{D}_i^B\right].$$

The first term vanishes by (2.7), so the only way for two sets to take equal-sized steps in a conservative manner is if they use the same step rule. The procedure for the large step can therefore be found by summing (2.9) and (2.10), giving

(2.12)
$$\begin{aligned}\Delta\mathbf{y}_a^A = \\ \Delta t^A\Bigg[&\frac{9}{8}\mathbf{D}^A\left[\mathbf{y}^A(0), \mathbf{y}^B(\Delta t^B)\right] + \frac{1}{2}\mathbf{D}^A\left[\mathbf{y}^A(0), \mathbf{y}^B(0)\right] - \frac{1}{8}\mathbf{D}^A\left[\mathbf{y}^A(0), \mathbf{y}^B(-\Delta t^B)\right] \\ &- \frac{3}{8}\mathbf{D}^A\left[\mathbf{y}^A(-\Delta t^A), \mathbf{y}^B(\Delta t^B)\right] - \frac{1}{8}\mathbf{D}^A\left[\mathbf{y}^A(-\Delta t^A), \mathbf{y}^B(-\Delta t^B)\right]\Bigg].\end{aligned}$$

Note that the coefficients have changed by a factor of 2 compared to the previous equations because of the change of the leading coefficient to $\Delta t^A$. As the two small steps were accurate to second order and this is effectively their concatenation, it is also accurate to second order.

**2.4. Conservative time steppers.** Let us return now to the task of finding a general conservative, high-order LTS integrator. First, we will consider the implications of requiring an Adams-Bashforth-like LTS scheme to be conservative. For such a scheme it only makes sense to evaluate (2.6) at times at which all the degrees of freedom are evaluated. We therefore introduce a new quantity $\tilde{\mathbf{y}}_n$ that is defined for the entire set of degrees of freedom for each time $\tilde{t}_n$ at which any set is evaluated, and is equal to $\mathbf{y}$ where the latter is defined. If we provide an update rule for $\tilde{\mathbf{y}}_n$ then, as long as portions of $\tilde{\mathbf{y}}_n$ that we do not wish to evaluate are never used, we can obtain an LTS method by summing the changes in $\tilde{\mathbf{y}}$ on each of the sets between evaluations. Furthermore, if the step from $\tilde{\mathbf{y}}_n$ to $\tilde{\mathbf{y}}_{n+1}$ is conservative, then the implied full method will be as well.

The condition for this small step to be conservative is

$$0 = \mathbf{c}\cdot\Delta\tilde{\mathbf{y}}_n. \tag{2.13}$$

This is satisfied if we evaluate $\Delta\tilde{\mathbf{y}}_n$ using a standard Adams-Bashforth method, but that would require values of $\tilde{\mathbf{y}}$ that are not included in $\mathbf{y}$. Comparing (2.7) and (2.13), we see that we will obtain a conservative method if we take

$$\Delta\tilde{\mathbf{y}}_n = \sum_i \beta_{ni}\mathbf{D}(\mathbf{y}_{ni}^1, \ldots, \mathbf{y}_{ni}^S) \tag{2.14}$$

for some set of coefficients $\beta_{ni}$ and with $\mathbf{y}_{ni}^s = \mathbf{y}^s(t)$ for some time $t$ at which set $s$ is evaluated. The choices of these coefficients are not unique, but there is a natural choice. We evaluate each step using a standard order-$k$ Adams-Bashforth scheme, but instead of using the derivatives of the function that we cannot evaluate, we use approximate derivatives $\tilde{\mathbf{D}}_n$. As long as these are accurate to order $k-1$, we will lose no formal accuracy for the step. We evaluate $\tilde{\mathbf{D}}_n$ by treating $\mathbf{D}(\mathbf{y}^1(t_1), \ldots, \mathbf{y}^S(t_S))$ as a function of the times $t_1, \ldots, t_S$ independently, and then performing a multidimensional interpolation from known values. To obtain the required accuracy, we need

evaluations at at least $k$ times from each set, and it is natural to choose the most recent values. The known values of $\mathbf{D}$ then form a lattice in the multidimensional space. Multidimensional interpolation from such a lattice is not unique, but a natural choice is to perform it as a series of one-dimensional interpolations.[1] Combining all these ideas, we have

$$\Delta\tilde{\mathbf{y}}_n = \sum_{q^1=0}^{k-1} \cdots \sum_{q^S=0}^{k-1} I_{n;q^1\cdots q^S} \mathbf{D}\Big(\mathbf{y}^1_{m^1_n-q^1}, \ldots, \mathbf{y}^S_{m^S_n-q^S}\Big), \tag{2.15}$$

where $m^s_n$ is defined by $t^s_{m^s_n} \le \tilde{t}_n < t^s_{m^s_n+1}$, i.e., it is the index of the last evaluation on set $s$ that can influence $\Delta\tilde{t}_n$ (see Figure 2). The coefficients of the derivatives are

$$I_{n;q^1\cdots q^S} = \Delta\tilde{t}_n \sum_{i=0}^{k-1} \tilde{\alpha}_{ni} \prod_{s=1}^{S} \ell_{q^s}\Big(\tilde{t}_{n-i}; t^s_{m^s_n}, \ldots, t^s_{m^s_n-(k-1)}\Big), \tag{2.16}$$

with $\tilde{\alpha}_{ni}$ the Adams-Bashforth coefficients corresponding to the sequence of times $\tilde{t}_n$. For computational purposes, it is useful to rewrite these steps as

$$\Delta\tilde{\mathbf{y}}_n = \sum_{q^1=m^1_n-(k-1)}^{m^1_n} \cdots \sum_{q^S=m^S_n-(k-1)}^{m^S_n} \bar{I}_{n;q^1\cdots q^S} \mathbf{D}\big(\mathbf{y}^1_{q^1}, \ldots, \mathbf{y}^S_{q^S}\big), \tag{2.17}$$

where the coefficient is

$$\bar{I}_{n;q^1\cdots q^S} = \Delta\tilde{t}_n \sum_{i=0}^{k-1} \tilde{\alpha}_{ni} \prod_{s=1}^{S} \ell_{m^s_n-q^s}\Big(\tilde{t}_{n-i}; t^s_{m^s_n}, \ldots, t^s_{m^s_n-(k-1)}\Big). \tag{2.18}$$

The full change in the value of a given set of degrees of freedom over an entire step can then be obtained by summing the contributions of all these small steps. This will give for each set of degrees of freedom an equation of the form

$$\Delta\mathbf{y}^s_m = \Delta t^s_m \sum_{q^1} \cdots \sum_{q^S} a^s_{m;q^1\cdots q^S} \mathbf{D}\big(\mathbf{y}^1_{q^1}, \ldots, \mathbf{y}^S_{q^S}\big) \tag{2.19}$$

for some coefficients $a^s_{m;q^1\cdots q^S}$.

## 3. Special cases.

[1] This freedom arises from the fact that the system of equations defining this interpolation is underdetermined for $S$ and $k$ greater than 1: we must find $k^S$ fitting coefficients but there are only $\binom{k+S-1}{S}$ monomial terms of degree less than $k$ (which are the ones relevant for an order $k$ fit). A general choice of interpolation coefficients will result in an interpolating polynomial containing all terms of degree less than $k$ in each of the $t^s$ individually. We therefore have the freedom to modify the interpolation coefficients as long as the modification alters only terms in the interpolating polynomial of total degree at least $k$. This freedom could be used, for example, to set certain coefficients to zero to reduce the number of computations required or to decrease the effect of terms where the times on different sets have large mismatches.

In the case where the step size on each set is constant, the alternative sets of interpolation coefficients can be obtained by adding high-order products of discrete Chebyshev polynomials [21] to the coefficients in (2.16) or (2.18). In the general case we know of no simple method to calculate alternative coefficients. We have not investigated the use of such alternative coefficients in either of these cases.

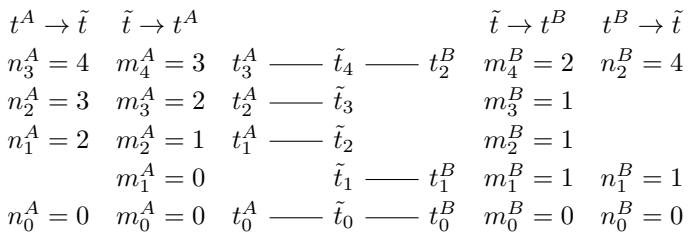



FIG. 2. *Example of the values of $m_n^s$ and $n_m^s$ for an arbitrarily chosen step pattern on two sets. These quantities give a mapping between the indices of the sequences of times $t_m^s$ and $\tilde{t}_n$, with $n_m^s$ mapping indices of $t_m^s$ to the corresponding indices of $\tilde{t}_n$ and $m_n^s$ performing the reverse map. In cases where there is no $t_m^s$ corresponding to a given $\tilde{t}_n$ the index given by $m_n^s$ is for the most recent step.*

**3.1. Element splitting.** These equations involve many more evaluations of the derivative than the standard GTS Adams-Bashforth method, so in the form (2.19) the LTS method is unlikely to be more efficient. However, if the couplings between the sets of degrees of freedom are inexpensive to calculate compared to the interactions within each set, then the required number of evaluations can be reduced.. Let us suppose that the derivative on set $s$ is split into a "volume" portion only depending on set $s$ itself and a "boundary" portion encoding the coupling to other sets:

$$\mathbf{D}^s\big(\mathbf{y}^1_{q^1}, \ldots, \mathbf{y}^S_{q^S}\big) = \mathbf{V}^s(\mathbf{y}^s_{q^s}) + \mathbf{B}^s\big(\mathbf{y}^1_{q^1}, \ldots, \mathbf{y}^S_{q^S}\big). \tag{3.1}$$

These names are motivated by finite volume and discontinuous Galerkin methods, where the terms from the interior and boundaries of elements split in this manner. Substituting this into (2.17) and summing over the small steps, the volume contribution to the full step on set $s$ is

$$(\Delta \mathbf{y}^s_m)_{\text{vol}} = \sum_{q^s=m-(k-1)}^{m} \left[ \sum_{n=n^s_m}^{n^s_{m+1}-1} \sum_{q^1=m^1_n-(k-1)}^{m^1_n} \cdots \sum\!\!\!\!\!\!\diagup_{q^s} \cdots \sum_{q^S=m^S_n-(k-1)}^{m^S_n} \bar{I}_{n;q^1\cdots q^S} \right] \mathbf{V}^s(\mathbf{y}^s_{q^s}), \tag{3.2}$$

where $n^s_m$ is defined by $\tilde{t}_{n^s_m} = t^s_m$ (see Figure 2). This is the same form as the GTS Adams-Bashforth method (2.2) using the bracketed expression as coefficients (absorbing the $\Delta t$ factor). The bracketed expression does not depend on the form of the derivative, so to evaluate it we can take the boundary coupling $\mathbf{B}^s$ to be zero, in which case this is the only contribution to the step. As this is then a $k$th-order GTS method and the Adams-Bashforth method is the unique $k$th-order method of this form, the bracketed quantity must be the standard Adams-Bashforth coefficient. Returning to the general case with a coupling, this shows that a set of degrees of freedom can be evolved using the standard Adams-Bashforth method for the volume portion with only the coupling terms evaluated using (2.17).

This simplification applies in intermediate cases as well: if the full derivative can be split into portions each of which depends on only some of the degree-of-freedom sets, each of those contributions to the step can be calculated independently using (2.17) ignoring non-contributing sets. In calculations where the sets are only coupled pairwise, this implies that only the $S = 2$ case need be considered.

**3.2. Two-set case.** In the common case where the sets of degrees of freedom are only coupled pairwise the update method reduces to a collection of standard Adams-Bashforth methods and LTS methods with $S = 2$. For the two-set case, we call the

sets $A$ and $B$ and define the selection functions $\Theta_n^A$, $\Theta_n^B$, and $\Theta_n^{AB}$ to be one if $\tilde{t}_n$ is an evaluation time for only set $A$, only set $B$, or both sets, respectively. By construction, a time evaluated on neither set can never occur. These selection functions sum to one, so we can write $\bar{I}_{n;q^Aq^B} = \bar{I}^A_{n;q^Aq^B} + \bar{I}^B_{n;q^Aq^B} + \bar{I}^{AB}_{n;q^Aq^B}$ with

$$\bar{I}^{A,B,AB}_{n;q^Aq^B} = \Delta\tilde{t}_n \sum_{i=0}^{k-1} \tilde{\alpha}_{ni} \Theta^{A,B,AB}_{n-i} \ell_{m_n^A - q^A}\Big(\tilde{t}_{n-i}; t^A_{m_n^A}, \ldots, t^A_{m_n^A-(k-1)}\Big)\, \ell_{m_n^B - q^B}\Big(\tilde{t}_{n-i}; t^B_{m_n^B}, \ldots, t^B_{m_n^B-(k-1)}\Big)\,. \tag{3.3}$$

By the definition of $m_n$, $\tilde{t}_n$ is not older than $t^{A,B}_{m_n^{A,B}}$, so, from the construction of the $\tilde{t}_n$ we see that $\tilde{t}_{n-i} \geq t^{A,B}_{m_n^{A,B}-(k-1)}$. This implies that if the $\tilde{t}_{n-i}$ is an evaluation time for either set, it is one of the control points in the corresponding Lagrange polynomial. We can therefore collapse those polynomials to obtain

$$\bar{I}^A_{n;q^Aq^B} = \Theta^A_{n^A_{q^A}}\, \Delta\tilde{t}_n \tilde{\alpha}_{n,n-n^A_{q^A}}\, \ell_{m_n^B - q^B}\Big(t^A_{q^A}; t^B_{m_n^B}, \ldots, t^B_{m_n^B-(k-1)}\Big) \tag{3.4}$$

$$\bar{I}^{AB}_{n;q^Aq^B} = \delta_{t^A_{q^A}, t^B_{q^B}}\, \Delta\tilde{t}_n \tilde{\alpha}_{n,n-n^A_{q^A}}\,, \tag{3.5}$$

and $\bar{I}^B_{n;q^Aq^B} = \bar{I}^A_{n;q^Bq^A}$. Some example values are shown in Tables 1 and 2. The meaning of, for example, the first entry for (a) in Table 1 is that in (2.19) the coefficient $a^A_{0;0,1} = 115/64$ (where we have chosen to number the steps starting from $t = 0$ so the step on set $B$ at $\Delta t^B$ is step 1), so the equation for this step begins

$$\mathbf{y}^A(\Delta t^A) - \mathbf{y}^A(0) = \Delta t^A \left[\frac{115}{64}\mathbf{D}^A\big[\mathbf{y}^A(0), \mathbf{y}^B(\Delta t^B)\big] + \cdots\right]. \tag{3.6}$$

Similarly, the lower-left entry for (c) in Table 2 indicates that one term in the second small step is

$$\mathbf{y}^B(2\Delta t^B) - \mathbf{y}^B(\Delta t^B) = \Delta t^B \left[\frac{2}{3}\mathbf{D}^A\big[\mathbf{y}^A(-2\Delta t^A), \mathbf{y}^B(\Delta t^B)\big] + \cdots\right]. \tag{3.7}$$

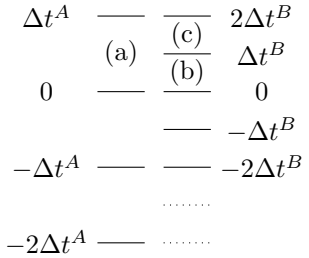
Additional tables of coefficients can be found in Appendix B.

**4. Numerical results.** We tested this scheme on a set of field equations evaluated using discontinuous Galerkin methods. In a DG formulation, the domain of evolution is divided into elements, with each element containing a collection of nodes. The evolution equations are evaluated locally within each element and this collection of partial solutions is coupled by adding additional terms at the element boundaries obtained from comparison with neighboring elements. The application of the LTS scheme to this type of problem is natural: the elements themselves can each be evolved uniformly using a standard GTS method, with only the couplings to the neighbors using the more complicated LTS equations.

For our test problem, we evolved the scalar wave equation:

$$\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi. \tag{4.1}$$

For numerical purposes, this is usually written in a form that contains only first order temporal and spatial derivatives. This is done by introducing the additional quantities




| (a) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{115}{64}$ | $\frac{7}{24}$ | $-\frac{11}{64}$ | $0$ |
| $-\Delta t^A$ | $-\frac{115}{96}$ | $0$ | $-\frac{11}{32}$ | $\frac{5}{24}$ |
| $-2\Delta t^A$ | $\frac{23}{64}$ | $0$ | $\frac{11}{192}$ | $0$ |

| (b) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $0$ | | $\frac{23}{12}$ | $-\frac{1}{2}$ | $0$ |
| $-\Delta t^A$ | | $0$ | $-1$ | $\frac{5}{12}$ |
| $-2\Delta t^A$ | | $0$ | $\frac{1}{6}$ | $0$ |

| (c) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{115}{32}$ | $-\frac{4}{3}$ | $\frac{5}{32}$ | |
| $-\Delta t^A$ | $-\frac{115}{48}$ | $0$ | $\frac{5}{16}$ | |
| $-2\Delta t^A$ | $\frac{23}{32}$ | $0$ | $-\frac{5}{96}$ | |

TABLE 1

*A third-order method for two sets $A$ and $B$ with $B$ evaluated twice as often as $A$. Coefficients for the derivatives in (2.19) evaluated using data from $A$ and $B$ at the times indicated for (a) a step of set $A$ from 0 to $\Delta t^A$, and steps of set $B$ (b) from 0 to $\Delta t^B$ and (c) from $\Delta t^B$ to $2\Delta t^B = \Delta t^A$.*

$\pi$, the conjugate momentum of $\psi$, and $\vec{\Phi}$, the gradient of $\psi$, given by

$$\pi = -\frac{\partial \psi}{\partial t} \tag{4.2}$$

$$\vec{\Phi} = \nabla \psi. \tag{4.3}$$

Eliminating the second derivatives in (4.1) by substituting these back in provides the system of first-order evolution equations

$$\frac{\partial \psi}{\partial t} = -\pi \tag{4.4}$$

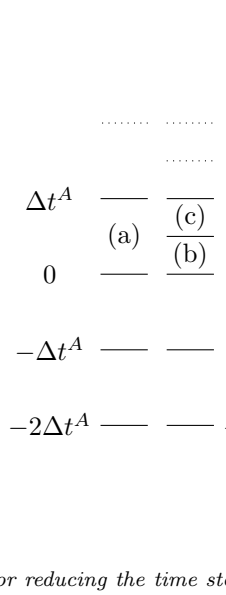
$$\frac{\partial \vec{\Phi}}{\partial t} = -\nabla \pi \tag{4.5}$$

$$\frac{\partial \pi}{\partial t} = -\nabla \cdot \vec{\Phi} \tag{4.6}$$

where we have also taken a time derivative of (4.3) to cast it in the form of an evolution equation. The DG elements were coupled using an upwind flux (see section 6 of [22]). For this system of PDEs, the integrals of $\pi$ and $\vec{\Phi}$ are linear conserved quantities. This carries over to the discretized system as long as the discretization procedure preserves the standard vector calculus identities, which the DG scheme does. The integral of the energy density of the field, $E = (\pi^2 + \vec{\Phi}^2)/2$ is also an analytically conserved quantity, but it is not linear in the evolved variables.

We used for a domain a periodic two-dimensional square divided nonuniformly into rectangular elements, as shown in Figure 3. The largest elements are 16 times as large (linearly) as the smallest elements. The nodes in each element are distributed as Legendre-Gauss-Lobatto points in each dimension.

For our test solution we used a sinusoidal plane wave propagating diagonally across the square, with wavelength equal to half the length of the diagonal. Step




| (a) | $\Delta t^B$ | 0 | $-\Delta t^A$ | $-2\Delta t^A$ |
|---|---|---|---|---|
| 0 | $\frac{5}{3}$ | $\frac{1}{4}$ | 0 | 0 |
| $-\Delta t^A$ | $-\frac{10}{9}$ | 0 | $-\frac{2}{9}$ | 0 |
| $-2\Delta t^A$ | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{12}$ |

| (b) | $\Delta t^B$ | 0 | $-\Delta t^A$ | $-2\Delta t^A$ |
|---|---|---|---|---|
| 0 | | $\frac{17}{12}$ | 0 | 0 |
| $-\Delta t^A$ | | 0 | $-\frac{7}{12}$ | 0 |
| $-2\Delta t^A$ | | 0 | 0 | $\frac{1}{6}$ |

| (c) | $\Delta t^B$ | 0 | $-\Delta t^A$ | $-2\Delta t^A$ |
|---|---|---|---|---|
| 0 | $\frac{10}{3}$ | $-\frac{11}{12}$ | 0 | |
| $-\Delta t^A$ | $-\frac{20}{9}$ | 0 | $\frac{5}{36}$ | |
| $-2\Delta t^A$ | $\frac{2}{3}$ | 0 | 0 | |

TABLE 2

*Rules for reducing the time step size in one set to start the algorithm in Table 1 from a GTS state. For $t \leq 0$ both sets step together at interval $\Delta t^A$, after which set $B$ changes to a step of $\Delta t^B = \Delta t^A/2$. For steps before $t = 0$ the standard GTS rules can be used, and for steps beyond $t = \Delta t^A$ the rules in Table 1 apply.*

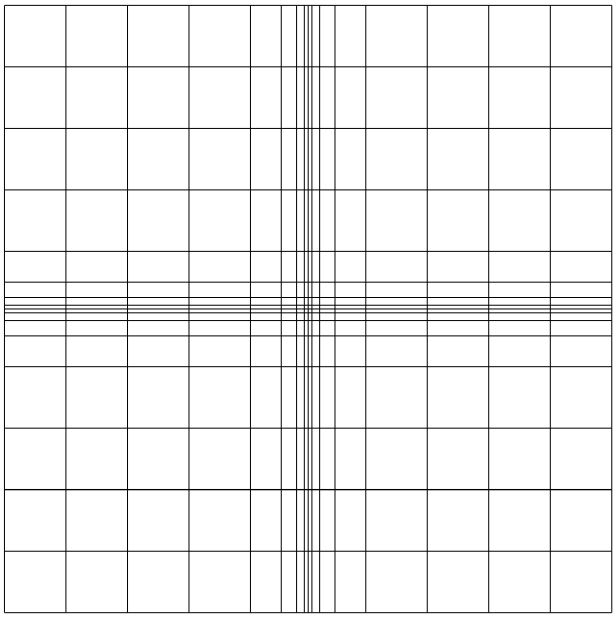


FIG. 3. *The domain used in the numerical tests: a square with periodic boundary conditions. The element pattern is symmetrical, with each half of each axis divided into four equal-sized segments and four smaller segments, each of which is half the size of its larger neighbor.*

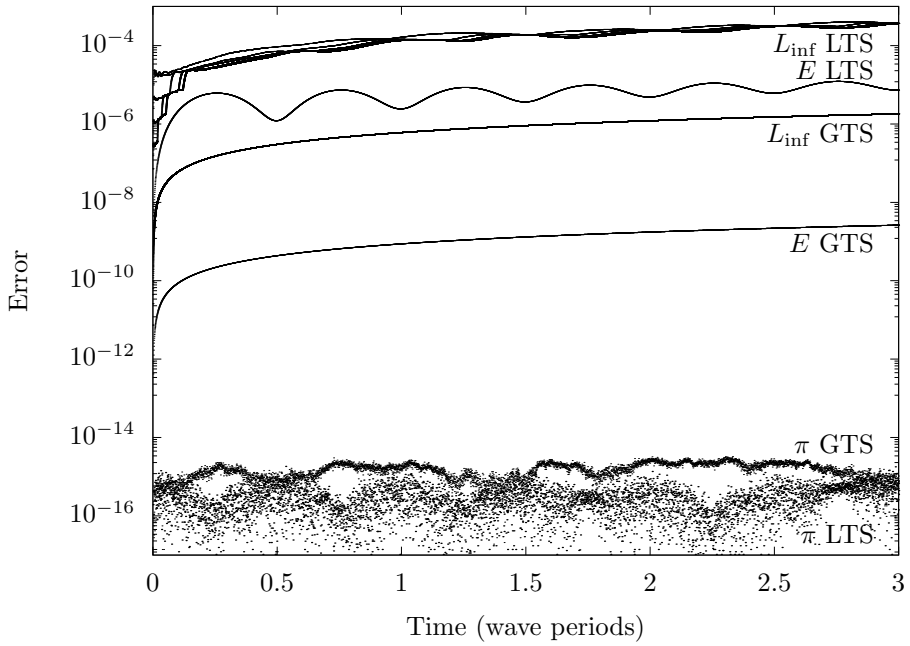



FIG. 4. *The $L_{inf}$ norm of the error in $\psi$, $\pi$, and $\vec{\Phi}$, as well as the error in the integral over the domain of the conjugate momentum and energy density, for GTS and LTS evolutions. Both runs used a second-order stepper with $9^2$ points per element. The integral of the conjugate momentum, being a linear conserved quantity, is constant to numerical roundoff for both methods, while the other errors reflect the integrator truncation error. The LTS truncation errors are larger because the error is dominated by the regions where the step size is large. The appearance of the LTS error as multiple lines is because the measured error is smaller at times when larger elements are not evaluated. The integrated quantities are only evaluated at times when all elements have data.*

sizes were restricted to be binary fractions of the wave period. When step sizes were allowed to vary, they were chosen according to the CFL condition with the restriction that they must be binary fractions of the wave period. The step size in each element is determined by its smallest dimension, so all elements along the center of the refined cross take steps of the same size, 16 times as small as the steps on the largest elements. The step sizes then increased by factors of two moving outward to each next layer of elements.

As shown in Figure 4, the overall error in the evolution is larger when using an LTS integrator than when using a GTS integrator of the same smallest step size. This is because the GTS integration is taking unnecessarily small steps in the large elements, while the LTS integration is providing the largest time steps consistent with stability. If desired, the LTS error could be reduced by choosing steps using a criterion other than just stability. As the time steps throughout the domain are decreased, the numerical results converge to the analytic value at the expected rate, as shown for several integrator orders in Figure 5. Even for the largest possible time steps the errors in the linear conserved quantities are at roundoff level.

Switching from a standard third-order GTS scheme to the LTS scheme, while still using uniform step sizes throughout the domain, incurs a performance penalty of a factor of approximately 0.8 because of the increased computational cost of the

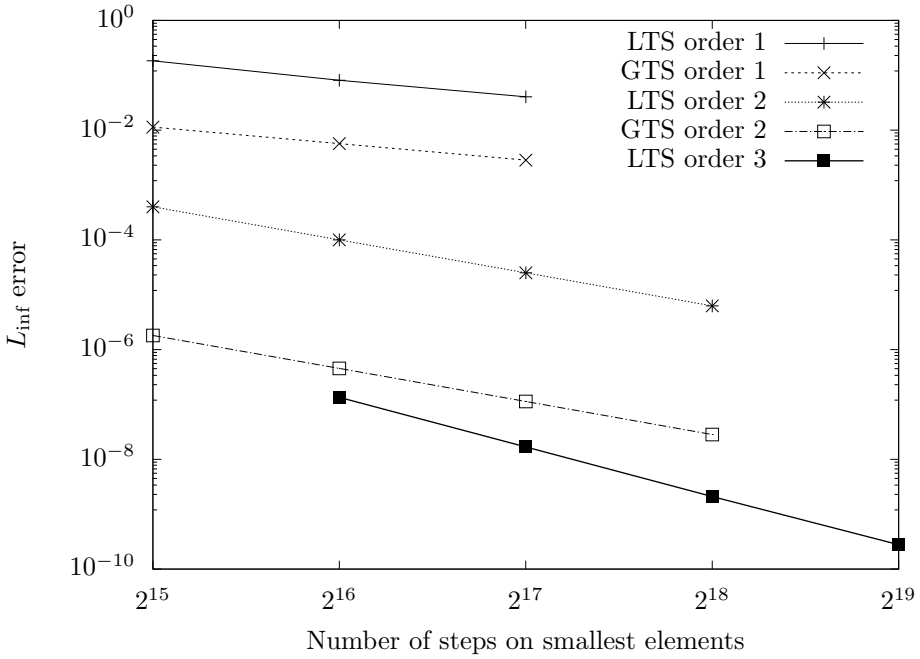



FIG. 5. *Maximum difference from the analytic solution over all grid points over the first three oscillations of the wave when artificially decreasing the step size below the CFL limit, showing the expected convergence rates. All simulations used* $9^2$ *points per element. The third-order GTS errors (not shown) are dominated by the spatial discretization error of approximately* $10^{-10}$.

boundary computations. However, once the step sizes are permitted to vary across the domain the total number of steps can be reduced by the ratio of the total number of steps across all elements during GTS and LTS, which in this case is 512/211, providing a theoretical improvement by a factor of 2.43. This provides an upper bound on the speed increase obtainable for this problem by any LTS scheme. In practice, we observe an approximate speedup of 2.1 relative to the uniform-step LTS integration, leading to a speedup of approximately 1.6 relative to the GTS scheme. The effects on integration speed from the LTS method are not strongly dependent on resolution, as shown in Figure 6, but integration is slower at higher time-stepper order, as shown in Figure 7, which is consistent with the need to evaluate more points for interpolating the couplings. The performance penalty of using an LTS scheme with a fixed step size does not vary significantly across the tested cases, which is expected because in the equal-step-size case no interpolation is necessary to compute couplings between elements.

All of these performance measurements are system-dependent, and should be improved with more complex systems than the scalar wave. The overhead from switching to LTS is independent of the system, and so should have a much smaller relative effect for systems with expensive right-hand sides. The overhead from increasing the integration order is proportional to the cost of the calculations on the element boundaries and so should decrease when these are much smaller than the right-hand-side cost.

**5. Conclusions.** When integrating systems of coupled ODEs, particularly those arising from discretizations of PDE systems, it is often the case that time-step-related

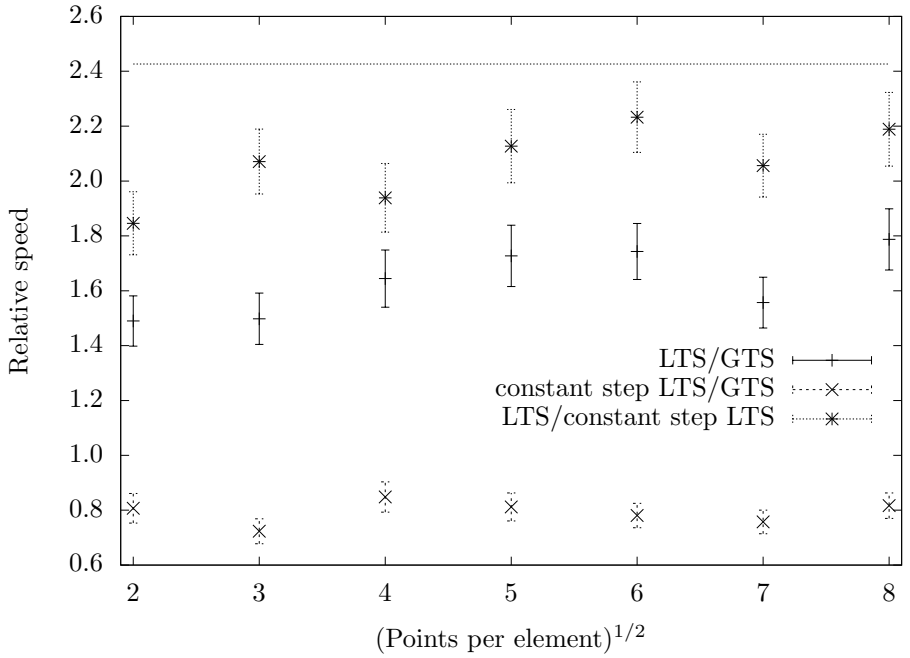



FIG. 6. *Comparisons of run speed using different third-order integrators at various element resolutions. The LTS and GTS algorithms are compared, and also compared to the LTS algorithm running with a constant global step size. The horizontal line shows the ratio of the number of steps on all elements taken when stepping globally and locally.*

instabilities arise primarily in a small subset of the variables being integrated. Using standard evolution schemes, this forces all degrees of freedom to be evolved with the most restrictive stable time step, potentially causing significant waste of computational resources. A local time-stepping integrator removes this requirement, allowing different degrees of freedom to be updated at different frequencies.

This paper has presented a local time-stepping scheme based on the Adams-Bashforth family of multistep integrators. This method allows arbitrary step choices, with a completely independent choice of time step for each variable. Unlike some previous schemes, it retains the full convergence order of the Adams-Bashforth integrator it is based on. This method is also conservative in that all linear conserved quantities of the system are constant to numerical roundoff under evolution.

The use of this method was demonstrated on a scalar wave equation evolved using the DG framework. The roundoff-level conservation of a conserved quantity was demonstrated, and the expected convergence rate for other quantities was observed for multiple integrator orders. For this problem, we observe an evolution speed improvement by a factor of approximately 1.6 from switching from the global to the local scheme, although this number is strongly dependent on the integration order. We also expect a bigger speedup if the right-hand side of the equations is more complicated than the simple wave equation we used as a test case.

This method will be used for DG evolutions of general relativity and magnetohydrodynamics in upcoming work using the SpECTRE code [23].

**Appendix A. Element splitting for general methods.** When comparing

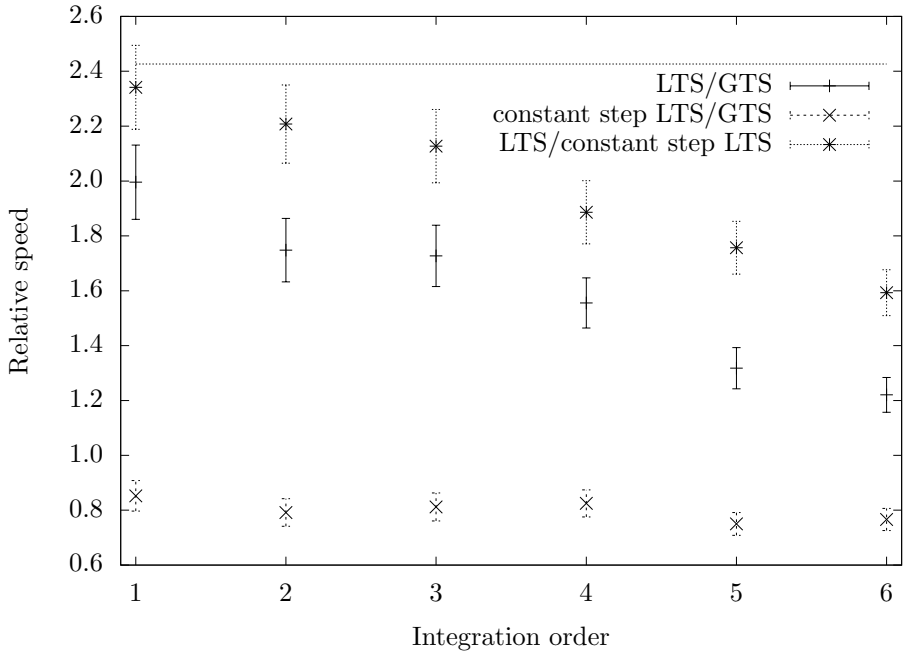



FIG. 7. *Comparisons of run speed using different integration orders on a domain with $5^2$ points per element. The LTS and GTS algorithms are compared, and also compared to the LTS algorithm running with a constant global step size. The horizontal line shows the ratio of the number of steps on all elements taken when stepping globally and locally.*

integrators, one may wish to use a GTS integrator that is not usually expressed in terms of volume terms and boundary couplings (for example, a Runge-Kutta method) in a framework designed for an LTS integrator that is so expressed. This is easiest if the GTS integrator can be cast into the element splitting form (Section 3.1).

All common explicit GTS integrators (both multistep and substep) can be written in the form

$$u_{n+1} - u_n = \sum_i A_n^i (u_n - u_{n-i}) + \Delta t_n \sum_i B_n^i D(u_{n-i}). \tag{A.1}$$

Adams-Bashforth integrators are usually written in this form with the $A_n^i = 0$. Runge-Kutta methods take some manipulation. For example, the second-order midpoint method

$$u_{n+1} - u_n = \Delta t D\left[u_n + \frac{1}{2}\Delta t D(u_n)\right] \tag{A.2}$$

can be written as

$$u_{2n+1} - u_{2n} = \Delta t_{2n} D(u_{2n}) \tag{A.3}$$

$$u_{2n+2} - u_{2n+1} = -(u_{2n+1} - u_{2n}) + 2\Delta t_{2n+1} D(u_{2n+1}), \tag{A.4}$$

where we have renumbered the steps so that the even numbered ones are the results of complete RK steps and $\Delta t_n = \Delta t/2$.

For local time-stepping, the derivative values can depend on an additional set of values $v_j$ (which have their own, similar, update equation), but where we still expect the update rule to have the form of a linear combination:

$$u_{n+1} - u_n = \sum_i A_n^i (u_n - u_{n-i}) + \Delta t_n \sum_{i,j} B_n^{ij} D(u_{n-i}, v_{m_n - j}). \tag{A.5}$$

We now perform an element splitting as in Section 3.1 by writing $D(u,v) = V(u) + B(u,v)$. Substituting this in gives

$$\begin{aligned} u_{n+1} - u_n = &\Big[\sum_i A_n^i (u_n - u_{n-i}) + \Delta t_n \sum_i \Big(\sum_j B_n^{ij}\Big) V(u_{n-i})\Big] \\ &+ \Delta t_n \sum_{i,j} B_n^{ij} B(u_{n-i}, v_{m_n - j}). \end{aligned} \tag{A.6}$$

Since a general method must be independent of the details of the $V$ and $B$ functions, the bracketed terms in (A.6) must be the standard GTS method operating with only the "volume" portion of the equations, and the last term is a coupling correction. Notably, the coupling term does not require the function values directly, but only the value of the coupling evaluated at those values.

When using a GTS method in an LTS framework, the $u$ and $v$ will be evaluated at the same sequence of times and the coefficients $B_n^{ij}$ will be diagonal in $i,j$. Comparing (A.1) to the bracketed term in (A.6), we see that $\sum_j B_n^{ij} = B_n^i$, so for a GTS method $B_n^{ij} = \delta_{ij} B_n^i$. Combining all this, we find that

$$u_{n+1} - u_n = \Big[\sum_i A_n^i (u_n - u_{n-i}) + \Delta t_n \sum_i B_n^i V(u_{n-i})\Big] + \Delta t_n \sum_i B_n^i B(u_{n-i}, v_{n-i}), \tag{A.7}$$

that is, when using an arbitrary GTS integrator in a framework designed for LTS, one can evaluate the volume term using the standard GTS rule and the coupling contribution by using the usual update formula but with all the non-derivative terms set to zero. For the midpoint Runge-Kutta scheme above, this gives the split rule

$$u_{2n+1} - u_{2n} = \Delta t_{2n} V(u_{2n}) + \Delta t_{2n} B(u_{2n}, v_{2n}) \tag{A.8}$$

$$u_{2n+2} - u_{2n+1} = \Big[(u_{2n+1} - u_{2n+1-1}) + \Delta t_{2n+1} V(u_{2n+1})\Big] + \Delta t_{2n+1}\, B(u_{2n+1}, v_{2n+1}). \tag{A.9}$$

## Appendix B. Tables of coefficients for $2:1$ LTS rules.

Below are tables of coefficients for order 2, 3, and 4 LTS rules with $2:1$ stepping, as well as the coefficients for transitioning between LTS and GTS stepping in these cases. The step patterns corresponding to these tables are shown in Figure 8.

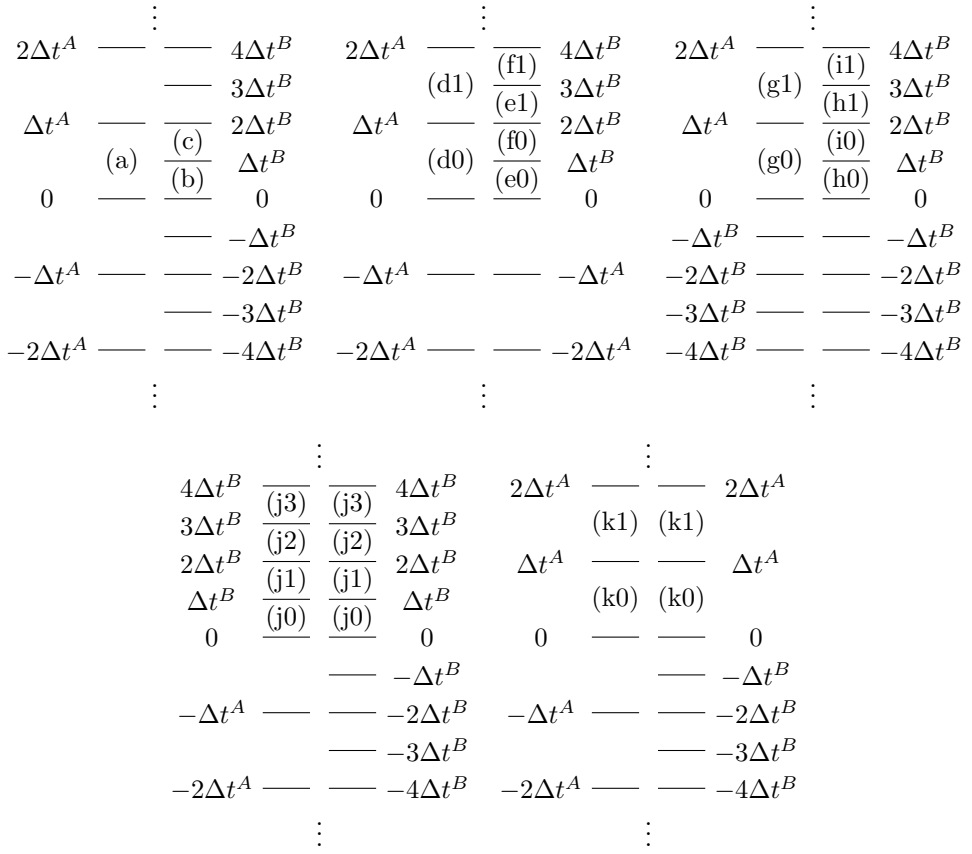



FIG. 8. *Step patterns during (a–c) steady state* 2 : 1 *evolution, (d–f) transition to LTS by decreasing a step size, (g–i) transition to LTS by increasing a step size, (j) transition back to GTS by decreasing a step size, and (k) transition back to GTS by increasing a step size. The labels correspond to the tables given in Appendix B. The coefficients for transitioning back to GTS are the same for both elements. The numbered labels are extended upwards as necessary until the steady-state values are reached.*

For the transition rules, the number of steps requiring special coefficients depends on the order of the integrator. Only tables for steps affected by the transition are shown below, after which either the 2 : 1 rule or the GTS rule should be used, as appropriate.

### B.1. Order 2.

**LTS** 2 : 1 **rule**

| (a) | $\Delta t^B$ | 0 | $-\Delta t^B$ |
|---|---|---|---|
| 0 | $\frac{9}{8}$ | $\frac{1}{2}$ | $-\frac{1}{8}$ |
| $-\Delta t^A$ | $-\frac{3}{8}$ | 0 | $-\frac{1}{8}$ |

| (b) | 0 | $-\Delta t^B$ |
|---|---|---|
| 0 | $\frac{3}{2}$ | $-\frac{1}{4}$ |
| $-\Delta t^A$ | 0 | $-\frac{1}{4}$ |

| (c) | $\Delta t^B$ | 0 |
|---|---|---|
| 0 | $\frac{9}{4}$ | $-\frac{1}{2}$ |
| $-\Delta t^A$ | $-\frac{3}{4}$ | 0 |

**Transition to LTS by decreasing a step size**

| (d0) | $\Delta t^B$ | $0$ | $-\Delta t^A$ |
|---|---|---|---|
| $0$ | $\frac{9}{8}$ | $\frac{3}{8}$ | $0$ |
| $-\Delta t^A$ | $-\frac{3}{8}$ | $0$ | $-\frac{1}{8}$ |

| (e0) | $0$ | $-\Delta t^A$ |
|---|---|---|
| $0$ | $\frac{5}{4}$ | $0$ |
| $-\Delta t^A$ | $0$ | $-\frac{1}{4}$ |

| (f0) | $\Delta t^B$ | $0$ |
|---|---|---|
| $0$ | $\frac{9}{4}$ | $-\frac{1}{2}$ |
| $-\Delta t^A$ | $-\frac{3}{4}$ | $0$ |

**Transition to LTS by increasing a step size**

| (g0) | $\Delta t^B$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|
| $0$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $0$ |
| $-\Delta t^B$ | $-\frac{3}{4}$ | $0$ | $-\frac{1}{4}$ |

| (h0) | $0$ | $-\Delta t^B$ |
|---|---|---|
| $0$ | $\frac{3}{2}$ | $0$ |
| $-\Delta t^B$ | $0$ | $-\frac{1}{2}$ |

| (i0) | $\Delta t^B$ | $0$ |
|---|---|---|
| $0$ | $3$ | $-\frac{1}{2}$ |
| $-\Delta t^B$ | $-\frac{3}{2}$ | $0$ |

**Transitioning to GTS**

| (j0) | $0$ | $-\Delta t^B$ |
|---|---|---|
| $0$ | $\frac{3}{2}$ | $-\frac{1}{4}$ |
| $-\Delta t^A$ | $0$ | $-\frac{1}{4}$ |

| (k0) | $0$ | $-\Delta t^B$ |
|---|---|---|
| $0$ | $2$ | $-\frac{1}{2}$ |
| $-\Delta t^A$ | $0$ | $-\frac{1}{2}$ |

### B.2. Order 3.

**LTS $2:1$ rule**

| (a) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{115}{64}$ | $\frac{7}{24}$ | $-\frac{11}{64}$ | $0$ |
| $-\Delta t^A$ | $-\frac{115}{96}$ | $0$ | $-\frac{11}{32}$ | $\frac{5}{24}$ |
| $-2\Delta t^A$ | $\frac{23}{64}$ | $0$ | $\frac{11}{192}$ | $0$ |

| (b) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|
| $0$ | $\frac{23}{12}$ | $-\frac{1}{2}$ | $0$ |
| $-\Delta t^A$ | $0$ | $-1$ | $\frac{5}{12}$ |
| $-2\Delta t^A$ | $0$ | $\frac{1}{6}$ | $0$ |

| (c) | $\Delta t^B$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|
| $0$ | $\frac{115}{32}$ | $-\frac{4}{3}$ | $\frac{5}{32}$ |
| $-\Delta t^A$ | $-\frac{115}{48}$ | $0$ | $\frac{5}{16}$ |
| $-2\Delta t^A$ | $\frac{23}{32}$ | $0$ | $-\frac{5}{96}$ |

**Transition to LTS by decreasing a step size**

| (d0) | $\Delta t^B$ | $0$ | $-\Delta t^A$ | $-2\Delta t^A$ |
|---|---|---|---|---|
| $0$ | $\frac{5}{3}$ | $\frac{1}{4}$ | $0$ | $0$ |
| $-\Delta t^A$ | $-\frac{10}{9}$ | $0$ | $-\frac{2}{9}$ | $0$ |
| $-2\Delta t^A$ | $\frac{1}{3}$ | $0$ | $0$ | $\frac{1}{12}$ |

| (e0) | $0$ | $-\Delta t^A$ | $-2\Delta t^A$ |
|---|---|---|---|
| $0$ | $\frac{17}{12}$ | $0$ | $0$ |
| $-\Delta t^A$ | $0$ | $-\frac{7}{12}$ | $0$ |
| $-2\Delta t^A$ | $0$ | $0$ | $\frac{1}{6}$ |

| (f0) | $\Delta t^B$ | $0$ | $-\Delta t^A$ |
|---|---|---|---|
| $0$ | $\frac{10}{3}$ | $-\frac{11}{12}$ | $0$ |
| $-\Delta t^A$ | $-\frac{20}{9}$ | $0$ | $\frac{5}{36}$ |
| $-2\Delta t^A$ | $\frac{2}{3}$ | $0$ | $0$ |

**Transition to LTS by increasing a step size**

| (g0) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{23}{8}$ | $\frac{7}{24}$ | $0$ | $0$ |
| $-\Delta t^B$ | $-\frac{23}{8}$ | $0$ | $-\frac{11}{24}$ | $0$ |
| $-2\Delta t^B$ | $\frac{23}{24}$ | $0$ | $0$ | $\frac{5}{24}$ |

| (h0) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|
| $0$ | $\frac{23}{12}$ | $0$ | $0$ |
| $-\Delta t^B$ | $0$ | $-\frac{4}{3}$ | $0$ |
| $-2\Delta t^B$ | $0$ | $0$ | $\frac{5}{12}$ |

| (i0) | $\Delta t^B$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|
| $0$ | $\frac{23}{4}$ | $-\frac{4}{3}$ | $0$ |
| $-\Delta t^B$ | $-\frac{23}{4}$ | $0$ | $\frac{5}{12}$ |
| $-2\Delta t^B$ | $\frac{23}{12}$ | $0$ | $0$ |

| (g1) | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ | $0$ |
|---|---|---|---|---|
| $\Delta t^A$ | $\frac{23}{12}$ | $\frac{7}{24}$ | $-\frac{11}{72}$ | $0$ |
| $0$ | $-\frac{23}{12}$ | $0$ | $-\frac{11}{24}$ | $\frac{5}{24}$ |
| $-\Delta t^B$ | $\frac{23}{24}$ | $0$ | $\frac{11}{72}$ | $0$ |

| (h1) | $2\Delta t^B$ | $\Delta t^B$ | $0$ |
|---|---|---|---|
| $\Delta t^A$ | $\frac{23}{12}$ | $-\frac{4}{9}$ | $0$ |
| $0$ | $0$ | $-\frac{4}{3}$ | $\frac{5}{12}$ |
| $-\Delta t^B$ | $0$ | $\frac{4}{9}$ | $0$ |

| (i1) | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ |
|---|---|---|---|
| $\Delta t^A$ | $\frac{23}{6}$ | $-\frac{4}{3}$ | $\frac{5}{36}$ |
| $0$ | $-\frac{23}{6}$ | $0$ | $\frac{5}{12}$ |
| $-\Delta t^B$ | $\frac{23}{12}$ | $0$ | $-\frac{5}{36}$ |

**Transitioning to GTS by decreasing a step size**

| (j0) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|
| $0$ | $\frac{23}{12}$ | $-\frac{1}{2}$ | $0$ |
| $-\Delta t^A$ | $0$ | $-1$ | $\frac{5}{12}$ |
| $-2\Delta t^A$ | $0$ | $\frac{1}{6}$ | $0$ |

| (j1) | $\Delta t^B$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|
| $\Delta t^B$ | $\frac{23}{12}$ | $0$ | $-\frac{5}{36}$ |
| $0$ | $0$ | $-\frac{4}{3}$ | $\frac{5}{12}$ |
| $-\Delta t^A$ | $0$ | $0$ | $\frac{5}{36}$ |

**Transitioning to GTS by increasing a step size**

| (k0) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|
| $0$ | $\frac{19}{6}$ | $-\frac{5}{4}$ | $0$ |
| $-\Delta t^A$ | $0$ | $-\frac{5}{2}$ | $\frac{7}{6}$ |
| $-2\Delta t^A$ | $0$ | $\frac{5}{12}$ | $0$ |

| (k1) | $\Delta t^A$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|
| $\Delta t^A$ | $\frac{37}{18}$ | $0$ | $-\frac{5}{36}$ |
| $0$ | $0$ | $-\frac{13}{6}$ | $\frac{5}{6}$ |
| $-\Delta t^A$ | $0$ | $0$ | $\frac{5}{12}$ |

### B.3. Order 4.

**LTS $2:1$ rule**

| (a) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ | $-3\Delta t^B$ |
|---|---|---|---|---|---|
| $0$ | $\frac{1925}{768}$ | $-\frac{1}{12}$ | $-\frac{55}{384}$ | $0$ | $\frac{3}{256}$ |
| $-\Delta t^A$ | $-\frac{1925}{768}$ | $0$ | $-\frac{55}{128}$ | $\frac{7}{12}$ | $-\frac{27}{256}$ |
| $-2\Delta t^A$ | $\frac{385}{256}$ | $0$ | $\frac{55}{384}$ | $0$ | $-\frac{27}{256}$ |
| $-3\Delta t^A$ | $-\frac{275}{768}$ | $0$ | $-\frac{11}{384}$ | $0$ | $\frac{3}{256}$ |

| (b) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ | $-3\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{55}{24}$ | $-\frac{295}{384}$ | $0$ | $\frac{3}{128}$ |
| $-\Delta t^A$ | $0$ | $-\frac{295}{128}$ | $\frac{37}{24}$ | $-\frac{27}{128}$ |
| $-2\Delta t^A$ | $0$ | $\frac{295}{384}$ | $0$ | $-\frac{27}{128}$ |
| $-3\Delta t^A$ | $0$ | $-\frac{59}{384}$ | $0$ | $\frac{3}{128}$ |

| (c) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{1925}{384}$ | $-\frac{59}{24}$ | $\frac{185}{384}$ | $0$ |
| $-\Delta t^A$ | $-\frac{1925}{384}$ | $0$ | $\frac{185}{128}$ | $-\frac{3}{8}$ |
| $-2\Delta t^A$ | $\frac{385}{128}$ | $0$ | $-\frac{185}{384}$ | $0$ |
| $-3\Delta t^A$ | $-\frac{275}{384}$ | $0$ | $\frac{37}{384}$ | $0$ |

**Transition to LTS by decreasing a step size**

| (d0) | $\Delta t^B$ | $0$ | $-\Delta t^A$ | $-2\Delta t^A$ | $-3\Delta t^A$ |
|---|---|---|---|---|---|
| $0$ | $\frac{833}{384}$ | $\frac{47}{384}$ | $0$ | $0$ | $0$ |
| $-\Delta t^A$ | $-\frac{833}{384}$ | $0$ | $-\frac{37}{128}$ | $0$ | $0$ |
| $-2\Delta t^A$ | $\frac{833}{640}$ | $0$ | $0$ | $\frac{461}{1920}$ | $0$ |
| $-3\Delta t^A$ | $-\frac{119}{384}$ | $0$ | $0$ | $0$ | $-\frac{25}{384}$ |

| (e0) | $0$ | $-\Delta t^A$ | $-2\Delta t^A$ | $-3\Delta t^A$ |
|---|---|---|---|---|
| $0$ | $\frac{99}{64}$ | $0$ | $0$ | $0$ |
| $-\Delta t^A$ | $0$ | $-\frac{187}{192}$ | $0$ | $0$ |
| $-2\Delta t^A$ | $0$ | $0$ | $\frac{107}{192}$ | $0$ |
| $-3\Delta t^A$ | $0$ | $0$ | $0$ | $-\frac{25}{192}$ |

| (f0) | $\Delta t^B$ | $0$ | $-\Delta t^A$ | $-2\Delta t^A$ |
|---|---|---|---|---|
| $0$ | $\frac{833}{192}$ | $-\frac{125}{96}$ | $0$ | $0$ |
| $-\Delta t^A$ | $-\frac{833}{192}$ | $0$ | $\frac{19}{48}$ | $0$ |
| $-2\Delta t^A$ | $\frac{833}{320}$ | $0$ | $0$ | $-\frac{37}{480}$ |
| $-3\Delta t^A$ | $-\frac{119}{192}$ | $0$ | $0$ | $0$ |

| (d1) | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ | $0$ | $-\Delta t^A$ |
|---|---|---|---|---|---|
| $\Delta t^A$ | $\frac{1925}{768}$ | $-\frac{25}{192}$ | $-\frac{65}{768}$ | $0$ | $0$ |
| $0$ | $-\frac{1925}{768}$ | $0$ | $-\frac{65}{256}$ | $\frac{29}{96}$ | $0$ |
| $-\Delta t^A$ | $\frac{385}{256}$ | $0$ | $\frac{65}{768}$ | $0$ | $-\frac{3}{64}$ |
| $-2\Delta t^A$ | $-\frac{275}{768}$ | $0$ | $-\frac{13}{768}$ | $0$ | $0$ |

| (e1) | $2\Delta t^B$ | $\Delta t^B$ | $0$ | $-\Delta t^A$ |
|---|---|---|---|---|
| $\Delta t^A$ | $\frac{211}{96}$ | $-\frac{125}{192}$ | $0$ | $0$ |
| $0$ | $0$ | $-\frac{125}{64}$ | $\frac{47}{48}$ | $0$ |
| $-\Delta t^A$ | $0$ | $\frac{125}{192}$ | $0$ | $-\frac{3}{32}$ |
| $-2\Delta t^A$ | $0$ | $-\frac{25}{192}$ | $0$ | $0$ |

| (f1) | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ | $0$ |
|---|---|---|---|---|
| $\Delta t^A$ | $\frac{1925}{384}$ | $-\frac{59}{24}$ | $\frac{185}{384}$ | $0$ |
| $0$ | $-\frac{1925}{384}$ | $0$ | $\frac{185}{128}$ | $-\frac{3}{8}$ |
| $-\Delta t^A$ | $\frac{385}{128}$ | $0$ | $-\frac{185}{384}$ | $0$ |
| $-2\Delta t^A$ | $-\frac{275}{384}$ | $0$ | $\frac{37}{384}$ | $0$ |

**Transition to LTS by increasing a step size**

| (g0) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ | $-3\Delta t^B$ |
|---|---|---|---|---|---|
| $0$ | $\frac{55}{12}$ | $-\frac{1}{12}$ | $0$ | $0$ | $0$ |
| $-\Delta t^B$ | $-\frac{55}{8}$ | $0$ | $-\frac{11}{24}$ | $0$ | $0$ |
| $-2\Delta t^B$ | $\frac{55}{12}$ | $0$ | $0$ | $\frac{7}{12}$ | $0$ |
| $-3\Delta t^B$ | $-\frac{55}{48}$ | $0$ | $0$ | $0$ | $-\frac{3}{16}$ |

| (h0) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ | $-3\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{55}{24}$ | $0$ | $0$ | $0$ |
| $-\Delta t^B$ | $0$ | $-\frac{59}{24}$ | $0$ | $0$ |
| $-2\Delta t^B$ | $0$ | $0$ | $\frac{37}{24}$ | $0$ |
| $-3\Delta t^B$ | $0$ | $0$ | $0$ | $-\frac{3}{8}$ |

| (i0) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{55}{6}$ | $-\frac{59}{24}$ | $0$ | $0$ |
| $-\Delta t^B$ | $-\frac{55}{4}$ | $0$ | $\frac{37}{24}$ | $0$ |
| $-2\Delta t^B$ | $\frac{55}{6}$ | $0$ | $0$ | $-\frac{3}{8}$ |
| $-3\Delta t^B$ | $-\frac{55}{24}$ | $0$ | $0$ | $0$ |

| (g1) | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|---|---|
| $\Delta t^A$ | $\frac{275}{96}$ | $-\frac{1}{12}$ | $-\frac{11}{96}$ | $0$ | $0$ |
| $0$ | $-\frac{275}{48}$ | $0$ | $-\frac{11}{16}$ | $\frac{7}{12}$ | $0$ |
| $-\Delta t^B$ | $\frac{275}{48}$ | $0$ | $\frac{11}{24}$ | $0$ | $-\frac{3}{16}$ |
| $-2\Delta t^B$ | $-\frac{55}{32}$ | $0$ | $-\frac{11}{96}$ | $0$ | $0$ |

| (h1) | $2\Delta t^B$ | $\Delta t^B$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|---|
| $\Delta t^A$ | $\frac{55}{24}$ | $-\frac{59}{96}$ | $0$ | $0$ |
| $0$ | $0$ | $-\frac{59}{16}$ | $\frac{37}{24}$ | $0$ |
| $-\Delta t^B$ | $0$ | $\frac{59}{24}$ | $0$ | $-\frac{3}{8}$ |
| $-2\Delta t^B$ | $0$ | $-\frac{59}{96}$ | $0$ | $0$ |

| (i1) | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ | $0$ |
|---|---|---|---|---|
| $\Delta t^A$ | $\frac{275}{48}$ | $-\frac{59}{24}$ | $\frac{37}{96}$ | $0$ |
| $0$ | $-\frac{275}{24}$ | $0$ | $\frac{37}{16}$ | $-\frac{3}{8}$ |
| $-\Delta t^B$ | $\frac{275}{24}$ | $0$ | $-\frac{37}{24}$ | $0$ |
| $-2\Delta t^B$ | $-\frac{55}{16}$ | $0$ | $\frac{37}{96}$ | $0$ |

| (g2) | $5\Delta t^B$ | $4\Delta t^B$ | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ |
|---|---|---|---|---|---|
| $2\Delta t^A$ | $\frac{165}{64}$ | $-\frac{1}{12}$ | $-\frac{11}{80}$ | $0$ | $\frac{3}{320}$ |
| $\Delta t^A$ | $-\frac{275}{96}$ | $0$ | $-\frac{11}{24}$ | $\frac{7}{12}$ | $-\frac{3}{32}$ |
| $0$ | $\frac{165}{64}$ | $0$ | $\frac{11}{48}$ | $0$ | $-\frac{9}{64}$ |
| $-\Delta t^B$ | $-\frac{55}{48}$ | $0$ | $-\frac{11}{120}$ | $0$ | $\frac{3}{80}$ |

| (h2) | $4\Delta t^B$ | $3\Delta t^B$ | $2\Delta t^B$ | $\Delta t^B$ |
|---|---|---|---|---|
| $2\Delta t^A$ | $\frac{55}{24}$ | $-\frac{59}{80}$ | $0$ | $\frac{3}{160}$ |
| $\Delta t^A$ | $0$ | $-\frac{59}{24}$ | $\frac{37}{24}$ | $-\frac{3}{16}$ |
| $0$ | $0$ | $\frac{59}{48}$ | $0$ | $-\frac{9}{32}$ |
| $-\Delta t^B$ | $0$ | $-\frac{59}{120}$ | $0$ | $\frac{3}{40}$ |

| (i2) | $5\Delta t^B$ | $4\Delta t^B$ | $3\Delta t^B$ | $2\Delta t^B$ |
|---|---|---|---|---|
| $2\Delta t^A$ | $\frac{165}{32}$ | $-\frac{59}{24}$ | $\frac{37}{80}$ | $0$ |
| $\Delta t^A$ | $-\frac{275}{48}$ | $0$ | $\frac{37}{24}$ | $-\frac{3}{8}$ |
| $0$ | $\frac{165}{32}$ | $0$ | $-\frac{37}{48}$ | $0$ |
| $-\Delta t^B$ | $-\frac{55}{24}$ | $0$ | $\frac{37}{120}$ | $0$ |

**Transitioning to GTS by decreasing a step size**

| (j0) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ | $-3\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{55}{24}$ | $-\frac{295}{384}$ | $0$ | $\frac{3}{128}$ |
| $-\Delta t^A$ | $0$ | $-\frac{295}{128}$ | $\frac{37}{24}$ | $-\frac{27}{128}$ |
| $-2\Delta t^A$ | $0$ | $\frac{295}{384}$ | $0$ | $-\frac{27}{128}$ |
| $-3\Delta t^A$ | $0$ | $-\frac{59}{384}$ | $0$ | $\frac{3}{128}$ |

| (j1) | $\Delta t^B$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $\Delta t^B$ | $\frac{55}{24}$ | $0$ | $-\frac{37}{120}$ | $0$ |
| $0$ | $0$ | $-\frac{59}{24}$ | $\frac{37}{32}$ | $0$ |
| $-\Delta t^A$ | $0$ | $0$ | $\frac{37}{48}$ | $-\frac{3}{8}$ |
| $-2\Delta t^A$ | $0$ | $0$ | $-\frac{37}{480}$ | $0$ |

| (j2) | $2\Delta t^B$ | $\Delta t^B$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|---|
| $2\Delta t^B$ | $\frac{55}{24}$ | $0$ | $0$ | $-\frac{3}{32}$ |
| $\Delta t^B$ | $0$ | $-\frac{59}{24}$ | $0$ | $\frac{3}{8}$ |
| $0$ | $0$ | $0$ | $\frac{37}{24}$ | $-\frac{9}{16}$ |
| $-\Delta t^A$ | $0$ | $0$ | $0$ | $-\frac{3}{32}$ |

**Transitioning to GTS by increasing a step size**

| (k0) | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ | $-3\Delta t^B$ |
|---|---|---|---|---|
| $0$ | $\frac{9}{2}$ | $-\frac{55}{24}$ | $0$ | $\frac{1}{12}$ |
| $-\Delta t^A$ | $0$ | $-\frac{55}{8}$ | $\frac{31}{6}$ | $-\frac{3}{4}$ |
| $-2\Delta t^A$ | $0$ | $\frac{55}{24}$ | $0$ | $-\frac{3}{4}$ |
| $-3\Delta t^A$ | $0$ | $-\frac{11}{24}$ | $0$ | $\frac{1}{12}$ |

| (k1) | $\Delta t^A$ | $0$ | $-\Delta t^B$ | $-2\Delta t^B$ |
|---|---|---|---|---|
| $\Delta t^A$ | $\frac{8}{3}$ | $0$ | $-\frac{3}{8}$ | $0$ |
| $0$ | $0$ | $-\frac{35}{6}$ | $\frac{27}{8}$ | $0$ |
| $-\Delta t^A$ | $0$ | $0$ | $\frac{27}{8}$ | $-\frac{11}{6}$ |
| $-2\Delta t^A$ | $0$ | $0$ | $-\frac{3}{8}$ | $0$ |

| (k2) | $2\Delta t^A$ | $\Delta t^A$ | $0$ | $-\Delta t^B$ |
|---|---|---|---|---|
| $2\Delta t^A$ | $\frac{71}{30}$ | $0$ | $0$ | $-\frac{3}{40}$ |
| $\Delta t^A$ | $0$ | $-\frac{17}{6}$ | $0$ | $\frac{3}{8}$ |
| $0$ | $0$ | $0$ | $\frac{8}{3}$ | $-\frac{9}{8}$ |
| $-\Delta t^A$ | $0$ | $0$ | $0$ | $-\frac{3}{8}$ |

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