1968

Some Thermal Plate Deflections Determined by the Complex Variable Method

James N. Maytum

SOME THERMAL PLATE DEFLECTIONS DETERMINED BY THE COMPLEX VARIABLE METHOD

BY

JAMES N. MAYTUM

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Mechanical Engineering, South
Dakota State University

1968

SOME THERMAL PLATE DEFLECTIONS

DETERMINED BY THE COMPLEX

VARIABLE METHOD

This thesis is approved as creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser   Date

Head, Mechanical Engineering Department   Date

ACKNOWLEDGMENTS

To Dr. Mansa C. Singh my thesis advisor for his patience, guidance, and for the many hours he spent assisting me in this work; to Dr. Paul L. Koepsell for his assistance with the computor programming and for his time which was so freely given; to Dr. Edward Lumsdaine for the assistance he gave me in the solutions of the heat equations in this work; to my wife Moreen who typed the rough drafts and was a ready source of encouragement.

JNM

TABLE OF CONTENTS

Chapter		Page
I	INTRODUCTION	1
II	PLATE DEFLECTION THEORY	5
III	EQUATIONS GOVERNING PLATE DEFLECTION IN COMPLEX FORM .	21
IV	DERIVATION OF PLATE DEFLECTION EQUATIONS	37
	Circular Plate Loaded over a Circular Region . . .	38
	Elliptic Plate Loaded over a Circular Region . . .	46
	Round Cornered Square Plate Loaded over a Circular Region	52
V	CALCULATION OF PLATE DEFLECTION	57
	Development of Heat Equation	57
	Circular Plate Loaded over a Circular Region . . .	60
	Elliptic Plate Loaded over a Circular Region . . .	61
	Round Cornered Square Plate Loaded over a Circular Region	67
VI	APPROXIMATE SOLUTIONS TO THE PLATE PROBLEMS	71
VII	RESUME AND RECOMMENDATIONS	87
	Resume	87
	Recommendations	90
LITERATURE CITED		92
APPENDIX I		94
APPENDIX II		111

LIST OF FIGURES

Figure		Page
2.1	Orientation of Coordinate System in a Plate	6
2.2	Cross Section of Deflected Plate	8
2.3	Plate Element in Equilibrium	11
2.4	Mid-Surface of Deflected Plate in xz Plane	15
3.1	Complex Plane	22
3.2	General Plate Plan View	34
4.1	Circular Plate Loaded over a Circular Region . . .	39
4.2	Elliptic Plate Loaded over a Circular Region . . .	47
4.3	Round Corner Square Plate Loaded over a Circular Region	53
5.1	Deflection of Round Plate	62
6.1	Orientation of Finite-Difference System in Two Coordinates	72
6.2	Orientation of Finite-Difference System in Three Coordinates	
6.3	Elliptic Plate Laid out for Finite-Difference Solution	80
6.4	Round Cornered Square Plate Laid out for Finite-Difference Solution	81
6.5	Comparison of Results for Elliptic Plate	85
6.6	Comparison of Results for Round Cornered Square Plate	86

LIST OF SYMBOLS

Symbol	Meaning
x, y, z	Cartesian coordinates, x, y in plane of plate
r, θ, z	cylinderical coordinates, r, θ in plane of plate
ρ, ϕ, w	cylinderical coordinates in plane, ρ, ϕ in plane of plate
u, v, w	displacements in x, y, z directions respectively, or Cartesian coordinates in ζ plane, u, v in plane of plate
z, ζ	complex planes
$\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$	direct and shear strain in plane z = const.
$\sigma_x, \sigma_y, \tau_{xy}$	direct and shear stresses in plane z = const.
N_x, N_y, N_{xy}	direct and shear forces per unit length in plane of plate with thermal effects
Q_x, Q_y	transverse shear forces per unit length
M_x, M_y, M_{xy}	bending and twisting moments per unit length with thermal effects
M_T, N_T	thermal bending moment and direct force per unit length in plane of plate
n_x, n_y, n_{xy}	direct and shear forces per unit length in plane of plate without thermal effects
E, G	Young's modulus and shear modulus
ν	Possion's ratio
h	plate thickness
D	flexural rigidity of the plate, $Eh^3/\left[12(1-\nu^2)\right]$
a, b, c	typical plate dimensions

w_p	particular integral
w_g	general function
t	time
T	tangent to boundary, or temperature ^oR
N	normal to boundary
q	transverse loading per unit area
F	force
∇^2	two dimensional Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
B	constant thermal loading, $\left[\nabla^2 M_T/(1-\nu)\right]$
$\mathcal{I}$	"the imaginary part of"
R	"the real part of"
r	radius
$d_1, d_2, \ldots d_n$	real or complex constants of Taylor's series
$g_o, g_1, \ldots g_n$	real or complex constants of Taylor's series
K	complex constants of integration
m	constant describing an ellipse, major plus minor axis
n	constant describing an ellipse, major minus minor axis
L	constant, 25/48b where b is width of square plate
λ	constant, -1/25
α	coefficient of thermal expansion
β	thermal diffusivity
Γ	inner plate boundary
γ	outer plate boundary

$\chi, \varphi, \Omega, \Phi$	complex potential functions
σ	variable ζ on the boundary of unit circle

CHAPTER I

INTRODUCTION

The first application of complex variables to elasticity dates with the beginning of this century. In 1902, L. N. G. Filon published a paper in which he developed the rudements of the problem of plane theory of elasticity in complex variables. In 1909 G. V. Kolosov derived the equations of plane theory of elasticity in complex form and effected solutions to some boundary value problems. The major work relating complex variables and the theory of elasticity was given by N. I. Muskhelishvili whose published works began in 1919. Muskhelishvili's [11] outstanding contribution is his brilliant monograph "Some Basic Problems in the Mathematical Theory of Elasticity", published in 1933. This outstanding work was the catalyst necessary to initiate further research in the theory of elasticity and other related fields; however, it was not until 1953 that it was translated into English by J. R. M. Radok. In 1940, A. C. Stevenson independently developed the technique given by Muskhelishvili but his work was not published until 1945.

Muskhelishvili's work is very precise, complete, and lengthy. I. S. Sokolnikoff [16] published in 1946 his book in which a portion is devoted to the abbreviation and extension of the Muskhelishvili method.

Extension of these results in the theory of elasticity to the theory of plates was effected by S. G. Lekhnitsky, I. N. Vekua, and Lur'e as outlined in [14]. The general solution of a plate loaded uniformly over its entire surface is given by L. I. Deveral [7] in a brief work which serves as a guide to solution of the plate problem using complex variables. Mansfield [9] gives a more complete study of this problem with a detailed discussion of the plate equations expressed in complex form. Both Deveral and Mansfield apply the material to specific problems.

Plates subjected to mechanical loading over a partial region have been treated using the complex variable approach by Bassali [1], Bassali and Hanna [2], and Bassali and Nassif [3]. These papers use continuity equations between the loaded and unloaded regions to support the boundary conditions in developing solutions of the two field equations defining the deflection in the two regions. The plate boundary conditions insure proper deflection and slope patterns at the outer edges while the continuity equations insure that the deflection, slope, radii of curvature, and shear match on the common boundary of the loaded and unloaded regions.

W. Nowacki [13] applied the technique of Green functions to determine the deflections of a plate loaded over a partial region by thermal effects. In this work Nowacki used an assumed solution in the form of a double Fourier sine series.

The objective of this thesis is to determine the deflections of thin, homogeneous, isothropic plate configurations thermally loaded over partial regions. These plate configurations will be of constant thickness and loaded over an inner circular region. The outer region will be unloaded with the boundaries clamped. The calculations will be confined to small deflection theory in which the stresses remain entirely elastic. The temperature function in the loaded region will be confined to a particular case in which the Laplacion of the thermal moment is a constant. This type of problem, solved by the complex variable technique, could not be found in a review of published literature. However, parallel solutions have been effected for plates which are mechanically loaded. The general outline of the solution is as follows: The differential equations defining the deflection of the two regions are established in the complex plane. The solutions of these equations are written in terms of the particular solution of the differential equations plus two complex potentials. The particular solution being known leaves the solution of these complex potentials as the solution of the plate deflection problem. These complex potentials are established by assuming a solution of the deflection in the outer region and evaluating the necessary constants using the boundary conditions. The continuity equations are then used to establish the deflection of the inner loaded regions from the deflections in the outer region. The solutions established in this thesis will be compared to Bassali's [3] work wherever possible. In the other cases the answers will be compared to approximate solutions determined by the finite-difference technique.

The Muskhelishvili method in theory of elasticity is a very powerful technique and it has found applications in other areas as well as in the theory of plates. Savin [15] has applied the complex variable method to the calculation of stress concentrations around holes. His work not only deals with simply connected plates but also with those which are multiply connected. Novozhilov [12] has applied this approach to the theory of shells. This work in the curvilinear coordinate system exemplifies the versatility of this process. Savin and Novozhilov are but two examples of the widening scope of this technique which points out its importance in the area of solid mechanics.

CHAPTER II

PLATE DEFLECTION THEORY

A plate is a body in which two dimensions are large in comparison to the third. In this thesis the plates considered will be of uniform thickness h having a "mid-plane" that occurs at an equal distance of h/2 from each face. A rectangular Cartesian coordinate system x, y, z will be chosen such that the x, y plane corresponds to the plate mid-plane and the positive z axis is downward forming a right handed system. (See figure 2.1) The deflection of a point p (x, y, z) is represented by the components of its displacement u, v, and w in the x, y, and z directions respectively.

The basic assumptions in the formulation of the plate problem are:

(1) The plate is considered thin; i.e., the h dimension be considered small as compared to the plate length or width.

(2) The plate material is homogenous, isotropic and elastic.

(3) All strain components are small as compared to unity.

(4) The stress in the z direction σ_{zz} is negligible.

(5) Kirchoff's hypothesis of linear elements holds; i.e., a line element perpendicular to the plate mid-plane before deformation remains perpendicular to the plate mid-surface after deflection and does not change in length.

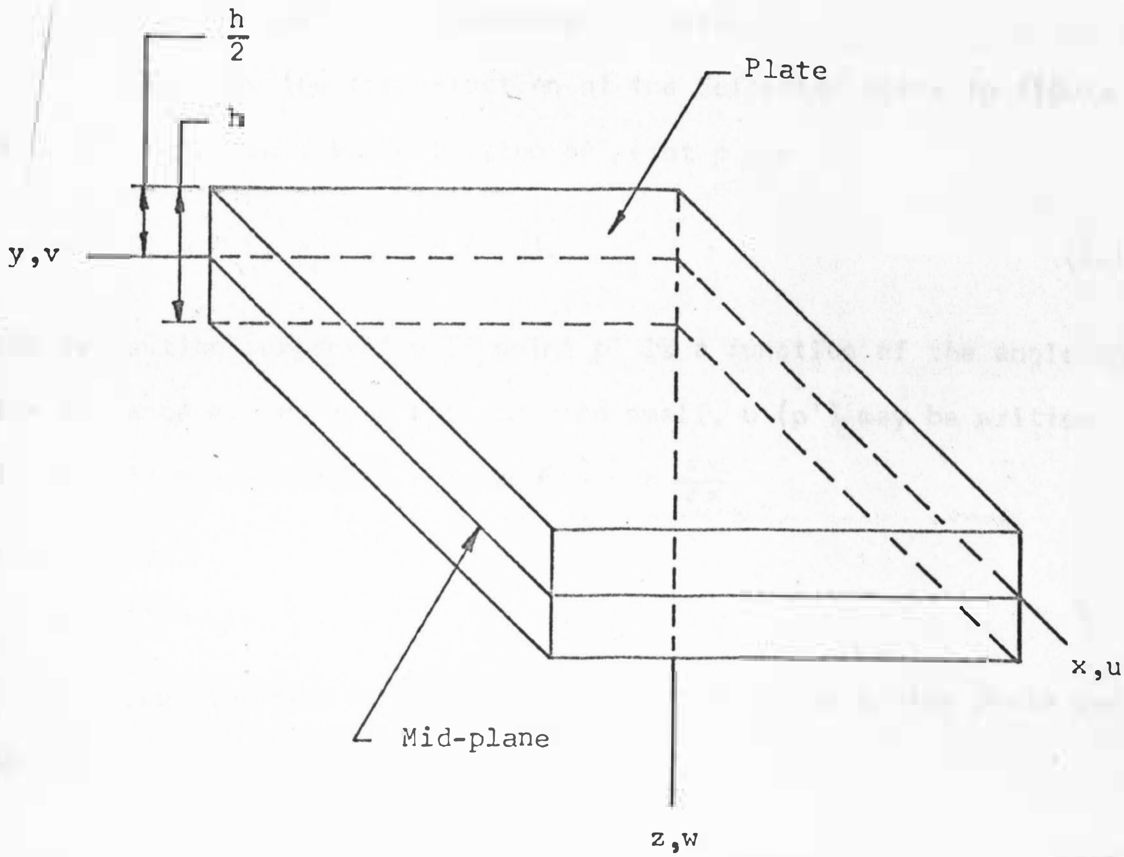

Figure 2.1. Orientation of Coordinate System in a Plate

(6) The deflection w is small as compared to the dimension h.

(7) The deflection components u and v of any point p (x, y, z = 0) are considered negligibly small.

(8) The slope of any point p (x, y, z = 0) after deflection is small when compared to unity.

Consider the cross-section of the deflected plate in figure 2.2. The components of deflection of point p are

$$u = 0, \quad v = 0, \quad \text{and } w = f(x,y). \tag{2.1}$$

The deflection component u of point p' is a function of the angle and the distance z. Because γ is assumed small, u (p') may be written

$$u(p') = -z \sin\gamma \doteq -z \tan\gamma \doteq -z \frac{\partial w}{\partial x}$$

similiarly

$$v(p') = -z \frac{\partial w}{\partial y} .$$

Thus the deflection components of any point in the plate can be given by

$$u = -z \frac{\partial w}{\partial x}, \quad v = -z \frac{\partial w}{\partial y}, \quad \text{and} \quad w = f(x,y). \tag{2.2}$$

Making use of the strain-displacement relations from Boresi [5] and the equations (2.2) one obtains the strain-displacement equations for a plate as

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2} . \tag{2.3a}$$

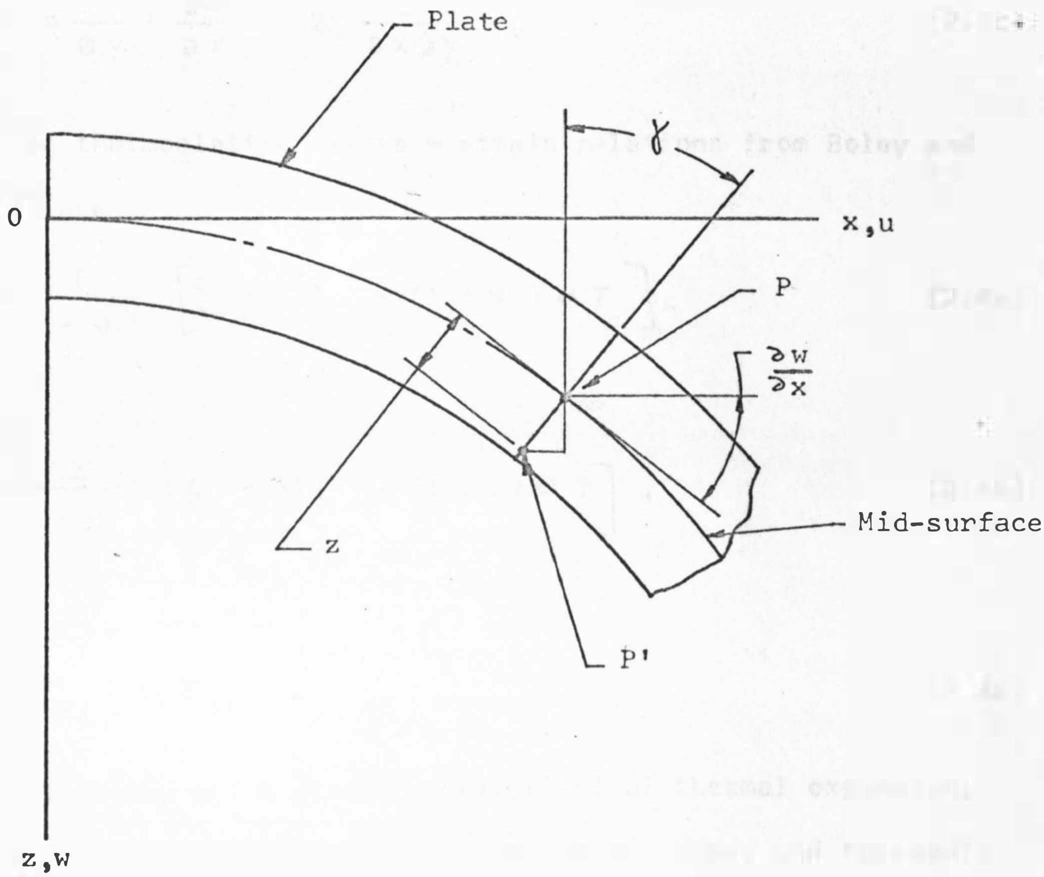

Figure 2.2. Cross Section of Deflected Plate

$$\varepsilon_y = \frac{\partial v}{\partial y} = - z \frac{\partial^2 w}{\partial y^2} , \tag{2.3b}$$

and $$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = - 2z \frac{\partial^2 w}{\partial x \partial y} . \tag{2.3c}$$

The thermoelastic stress - strain relations from Boley and Wiener [4] are

$$\sigma_x = \frac{E}{1 - \nu^2} \left[\varepsilon_x + \nu \varepsilon_y - (1 + \nu) \alpha T \right], \tag{2.4a}$$

$$\sigma_y = \frac{E}{1 - \nu^2} \left[\varepsilon_y + \nu \varepsilon_x - (1 + \nu) \alpha T \right] , \tag{2.4b}$$

and $$\tau_{xy} = G \gamma_{xy} \tag{2.4c}$$

where α, T, E, G, and ν are the coefficient of thermal expansion, temperature, moxulus of elasticity, modulus of shear, and Poisson's ratio respectively.

The substitution of equations (2.3) into equations (2.4) yields the stresses in terms of deflection w which is a function of the coordinates x and y; i.e.,

$$\sigma_x = \frac{- E}{1 - \nu^2} \left[z \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) - (1 + \nu) \alpha T \right] , \tag{2.5a}$$

$$\sigma_y = \frac{-E}{1-\nu^2}\left[z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - (1+\nu)\alpha T\right], \quad (2.5b)$$

and $$\gamma_{xy} = \frac{-E}{(1+\nu)} \frac{\partial^2 w}{\partial x \partial y}. \quad (2.5c)$$

Let us consider a plate which is in equilibrium subjected to mechanical loading and temperature distribution. Take an infinitesimal element from this plate which is dx wide, dy long, and h deep. The external load qdxdy and internal stress resultants acting upon this element are shown in figure 2.3. These internal stresses are the shears per unit length Q_x and Q_y, the normal stresses per unit length N_x, N_y and N_{xy}, and the moments per unit length M_x, M_y and M_{xy}. The moments are represented by double headed arrows which comply with the right hand screw rule. These stress resultants and stress couples are related to the stresses by the following equations

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \quad (2.6a)$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz \quad (2.6b)$$

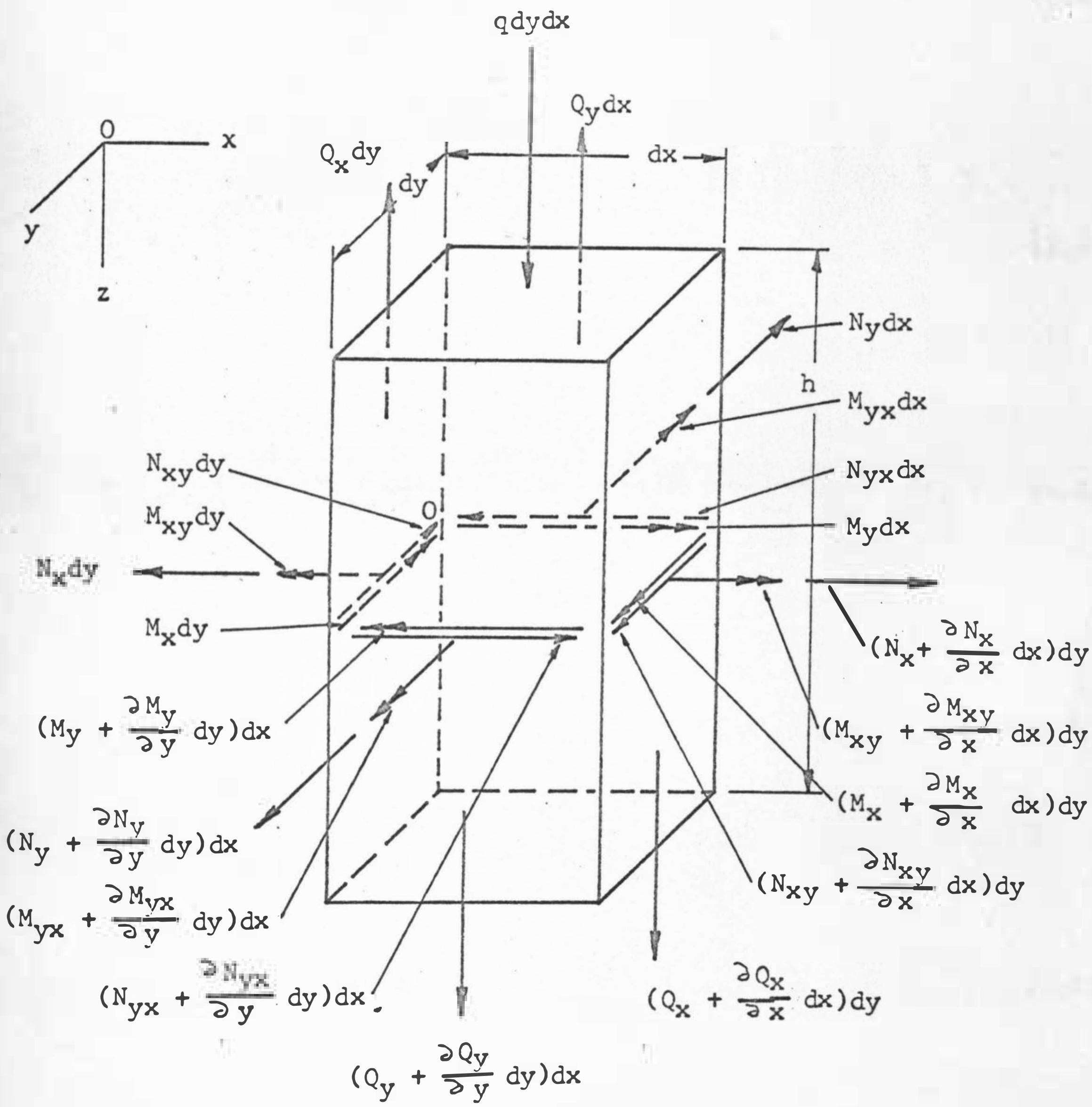

Figure 2.3. Plate Element in Equilibrium

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, dz \tag{2.6c}$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \, dz \tag{2.6d}$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, dz \tag{2.6e}$$

$$M_x = \int_{-h/2}^{h/2} z \, \sigma_x \, dz \tag{2.6f}$$

$$M_y = \int_{-h/2}^{h/2} z \, \sigma_y \, dz \tag{2.6g}$$

$$M_{xy} = \int_{-h/2}^{h/2} z \, \sigma_{xy} \, dz \tag{2.6h}$$

$$N_T = \alpha E \int_{-h/2}^{h/2} T dz \tag{2.6j}$$

$$M_T = \alpha E \int_{-h/2}^{h/2} T z dz \tag{2.6k}$$

As the infinitesimal element of figure 2.3 is taken from a plate in equilibrium, the element itself must be in equilibrium. Thus, the vector sum of all the forces and moments must vanish. The sum of the vertical components of all the forces yields

$$q + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \quad . \tag{2.7}$$

Summing moments about the x axis yields

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0 \quad . \tag{2.8}$$

Summing moments about the y axis gives

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \quad . \tag{2.9}$$

Solving equations (2.7), (2.8), and (2.9) simultaneously we get

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \quad . \tag{2.10}$$

Solving equations (2.6 f, g, h) using equations (2.5) and substituting the results into equation (2.10) yields

$$\nabla^4 w = \frac{1}{D} \left[q - \frac{\nabla^2 M_T}{1 - \nu} \right] , \qquad (2.11)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\nabla^4 = \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$,

and $D = \frac{E h^3}{12 (1 - \nu^2)}$.

From the element shown in figure 2.3 the equation

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0 \qquad (2.12)$$

is obtained by summing forces on the plate mid-plane in the x direction.

Similiarly for the y direction

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 . \qquad (2.13)$$

Consider now the section of the mid-surface of a deflected plate in the xz plane shown in figure 2.4. The force $N_x dy$ and the force $(N_x + \frac{\partial N_x}{\partial x} dx)\, dy$ have components in the z direction. Summing these components and neglecting the higher order terms yields

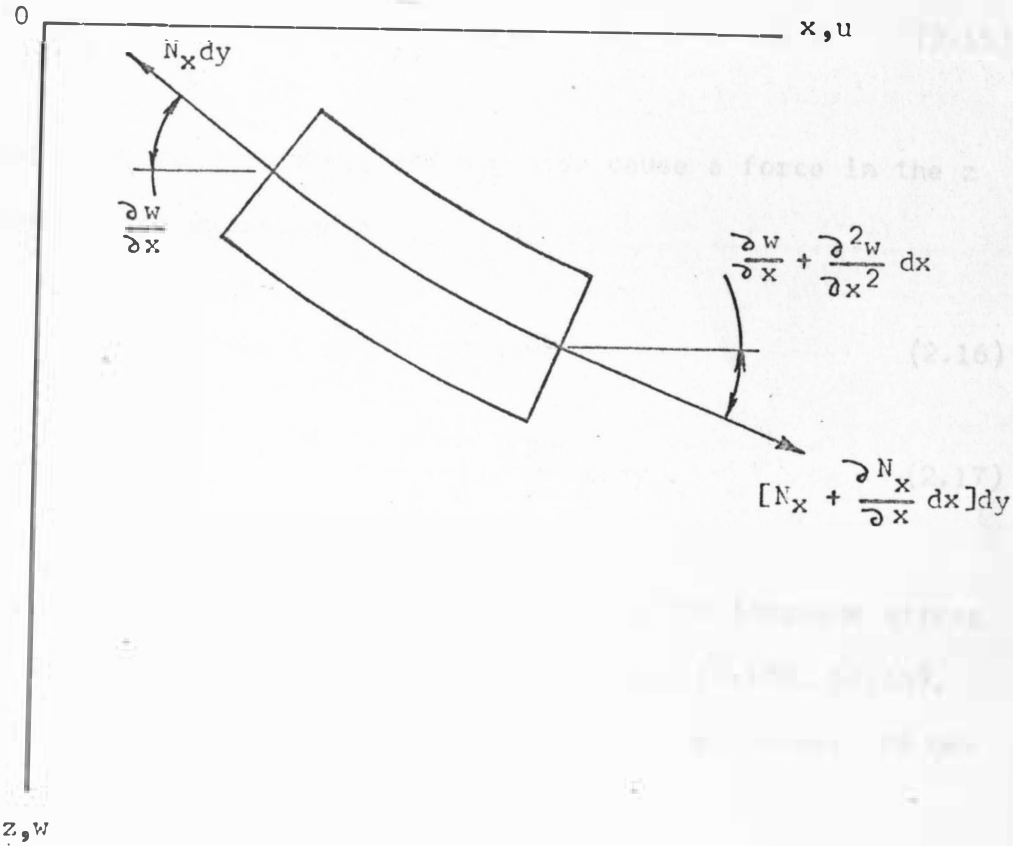

Figure 2.4. Mid-Surface of Deflected Plate in xz Plane

$$F_{zNx} = N_x \frac{\partial^2 w}{\partial x^2} dydx + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} dxdy. \tag{2.14}$$

Similiarly, $N_y dx$ and $(N_y + \frac{\partial N_y}{\partial y} dy)\, dx$ have components in the z direction. Summing these components yields

$$F_{zNy} = N_y \frac{\partial^2 w}{\partial y^2} dxdy + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} dxdy. \tag{2.15}$$

The in-plane shears N_{xy} and N_{yx} also cause a force in the z direction which are expressed as

$$F_{zNxy} = N_{xy} \frac{\partial^2 w}{\partial x \partial y} dxdy + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} dxdy \tag{2.16}$$

and

$$F_{z\,Nyx} = N_{yx} \frac{\partial^2 w}{\partial x\, \partial y} + \frac{\partial N_{yx}}{\partial y} \frac{\partial w}{\partial x} dxdy\ . \tag{2.17}$$

The total vertical force exerted on the plate by the in-plane stress resultants and shears is found by adding equations (2.14), (2.15), (2.16), and (2.17) and neglecting the terms of higher order. We get

$$F_z = N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x\, \partial y}\ . \tag{2.18}$$

This vertical force acting on the plate is of the same character as the load q. Thus by adding (2.18) to equation (2.11) and solving N_x, N_y, and N_{xy} using equations (2.5) and equations (2.6c, d, e, and j) one obtains

$$\nabla^4 w = \frac{1}{D}\left[q + n_x \frac{\partial^2 w}{\partial x^2} + n_y \frac{\partial^2 w}{\partial y^2} + 2n_{xy}\frac{\partial^2 w}{\partial x \partial y} - \frac{\nabla^2 M_T}{1-\nu} - \frac{N_T}{1-\nu}\nabla^2 w\right]. \quad (2.19)$$

Equation (2.19) along with equations (2.12) and (2.13) are the differential equations of equilibrium of a plate subjected to any combination of mechanical loads and thermal effects so long as these loads stay within the bounds of small deflection theory. The term q(x, y) is the normal pressure exerted over the surface of the plate. The terms n_x, n_y, and n_{xy} are the mechanical stresses applied to the plate mid-plane. M_T and N_T are the thermal moment and thermal stress resultants respectively. Equation (2.19) will be applied to a plate which is only thermally loaded. Thus,

$$q = n_x = n_y = n_{xy} = 0, \quad (2.20)$$

which identically satisfies the equilibrium equations (2.12) and (2.13) leaving equation (2.19) defining the deflection throughout the rest of this thesis. The thermal loading will also be restricted to cases which cause N_T to be zero. Referring to equation (2.6j), N_T would be zero if the temperature T is an odd function with respect to the z axis. That is, if

$$T(x, y, z) = - T(x, y, -z) . \quad (2.21)$$

Equation (2.19) with the restrictions (2.20) and (2.21) has the form

$$\nabla^4 w = \frac{-\nabla^2 M_T}{D(1-\nu)} . \tag{2.22}$$

This equation defines the deflection of a plate which is thermally loaded in a manner such that the in-plane stresses caused by N_T disappear. That is, the average temperature through the plate thickness approaches the ambient temperature (T_0).

Equation (2.22) will be modified such that the Laplacian of the thermal moment ($\nabla^2 M_T$) will be limited to constant values. In order to accomplish this along with the restriction (2.20) and (2.21) the temperature gradient through the plate must be restricted to the form

$$T = T_0 + z\left[g(x, y) + h(t)\right] \tag{2.23}$$

where T_0, $g(x, y)$ and $h(t)$ are the ambient temperature, a function of x and y, and a function of time respectively. Substituting equation (2.23) into equation (2.6k) and integrating, one gets

$$M_T = \alpha E \left\{ \frac{T_0 z^2}{2} + \frac{z^3}{3}\left[g(x, y) + h(t)\right]\right\}\Bigg|_{-h/2}^{h/2}$$

or $$M_T = \frac{\alpha E h^3 \left[g(x, y) + h(t)\right]}{12} . \tag{2.24}$$

From equation (2.24) it is seen that in order for the Laplacian of the thermal moment to be constant, the Laplacian of the function g (x, y) must also be a constant.

The temperature distribution in a body for the unsteady state condition is given by Fourier's general law of heat conduction, which is

$$\frac{\partial T}{\partial t} = \beta \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \tag{2.25}$$

where β and t are diffusivity and time respectively. For the steady state condition equations (2.25) reduced to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{2.26}$$

Thus, in order for the temperature function (2.23) to satisfy the steady state conditions (2.26), the Laplacian of the thermal moment (M_T) must be zero, yielding a plate deflection of zero from equation (2.22). The plate stresses are established directly from equations (2.5). Also, function h (t) of equation (2.23) is therefore limited to

$$h\ (t) = t\beta \left(\frac{\partial^2 g\ (x,\ y)}{\partial x^2} + \frac{\partial^2 g\ (x,\ y)}{\partial y^2} \right) + \text{constant.} \tag{2.27}$$

The thermal gradient (2.23) established in this thesis now satisfies Fourier's general law of heat conduction and yields

$$\frac{\nabla^2 M_T}{1-\nu} = \frac{\alpha E h^3}{12(1-\nu)} \left(\frac{\partial^2 g(x,y)}{\partial x^2} + \frac{\partial^2 g(x,y)}{\partial y^2} \right) = B \qquad (2.28)$$

where B is a constant.

The final form of equation (2.19) with the restrictions imposed in this thesis is

$$\nabla^4 w = \frac{-B}{D} \, . \qquad (2.29)$$

This equation will be applied to plates which have the inner region thermally loaded and the outer region unloaded with the entire plate clamped at the outer boundary of the unloaded region. Thus, the deflection of the entire plate will be defined by two differential equations. The deflection of the loaded inner region by equation (2.29) and the deflection of the unloaded outer region by

$$\nabla^4 w = 0 \qquad (2.30)$$

which is the biharmonic equation.

These equations will be expressed in terms of complex variables and their solution attempted in the next chapter.

CHAPTER III

EQUATIONS GOVERNING PLATE DEFLECTION IN COMPLEX FORM

Let z be a complex number x + iy whose conjugate $\bar{z}$ is x - iy, where x and y are real as shown in figure 3.1. Let a complex or z plane be established such that the abscissa of a point z be x and the ordinate be iy, where

$$i^2 = -1.$$

This coordinate system shown in figure 3.1 may also be established for the variables r and θ using Eulers formula

$$re^{i\theta} = r\left[\cos\theta + i\sin\theta\right].$$

Substitute the z plane for the xy plane corresponding to the plate mid-plane as established in Chapter two. Now any point of the plate mid-plane may be located by the complex variable z or $\bar{z}$. Thus, the deflection w is a function of z and $\bar{z}$ and the total derivative dw is

$$dw = \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \bar{z}}d\bar{z}. \qquad (3.1)$$

Differentiating z and $\bar{z}$ with respect to x and y yields

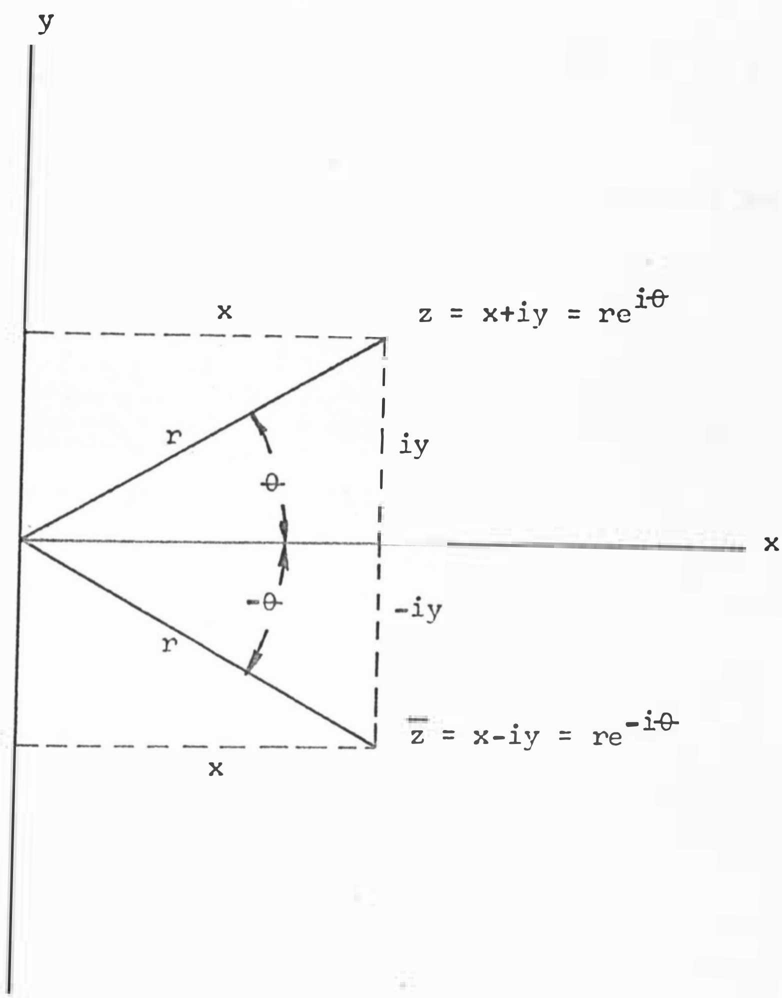

Figure 3.1. Complex Plane

$$\frac{\partial z}{\partial x} = \frac{\partial x}{\partial x} = 1 \quad , \qquad \frac{\partial \bar{z}}{\partial x} = \frac{\partial x}{\partial x} = 1 \quad ,$$

$$\frac{\partial z}{\partial y} = i \frac{\partial y}{\partial y} = i \quad , \text{ and } \qquad \frac{\partial \bar{z}}{\partial y} = - i \frac{\partial y}{\partial y} = - i \quad . \tag{3.2}$$

By taking the partial derivative of the deflection w with respect to x, one gets

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial x}$$

from which one obtains the operator

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \quad . \tag{3.3}$$

Similiarly, from

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial y}$$

one obtains the operator

$$\frac{\partial}{\partial y} = i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \quad . \tag{3.4}$$

The inverse of equations (3.3) and (3.4) is

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad . \tag{3.5a}$$

and $$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) . \tag{3.5b}$$

Combining equations (3.5a) and (3.5b) one obtains

$$\frac{4 \partial^2}{\partial z \, \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2 . \tag{3.6}$$

Equation (3.6) immediately yields the biharmonic operator in complex form; i.e.,

$$\nabla^4 = \nabla^2 \nabla^2 = \frac{16 \, \partial^4}{\partial z^2 \, \partial \bar{z}^2} . \tag{3.7}$$

Referring to equation (2.29), it is seen that the differential equation defining the deflection of a thermally loaded plate is

$$\nabla^4 w = - \frac{B}{D}$$

which, from equation (3.7), has the complex form

$$\frac{\nabla^4 w}{\partial z^2 \partial \bar{z}^2} = \frac{- B}{16 \, D} . \tag{3.8}$$

Equation (3.8) is the differential equation defining the deflection w of the region of a plate which is thermally loaded. This equation becomes

$$\frac{\partial^4 w}{\partial z^2 \, \partial \bar{z}^2} = 0 \tag{3.9}$$

for unloaded regions. The solution of equation (3.8) is given by

$$w = w_p + w_g \tag{3.10}$$

where w_g is the general solution satisfying

$$\frac{\partial^4 w_g}{\partial z^2 \partial \bar{z}^2} = 0 \tag{3.11}$$

and w_p is the particular solution satisfying

$$\frac{\partial^4 w_p}{\partial z^2 \partial \bar{z}^2} = \frac{-B}{16 D} . \tag{3.12}$$

The particular solution is found easily by integrating equation (3.12) four times as,

$$w_p = \frac{-B z^2 \bar{z}^2}{64 D} . \tag{3.13}$$

The constants of integration will be included in the general solution.

The general solution may also be found by integrating equation (3.11) four times. This was done by Goursat [11]. However, it was necessary to assume that w is real in order to reach a conclusion and the fact that the complex potentials derived in the solution must necessarily be analytic is not evident. Muskhelishvili [11] gives the more concise derivation which follows.

Consider the function U (x, y) which satisfies the bi-harmonic equation

$$\nabla^4 U = 0. \tag{3.14}$$

Also let a variable P (x, y) be defined such that

$$\nabla^2 U = P.$$

Thus, $\nabla^4 U = \nabla^2 P = 0$ which indicates that P is harmonic in the domain being considered. Let Q (x, y) be the harmonic congugate function to P. Thus, Q and P must satisy the Cauchy-Riemann conditions; i.e.,

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \text{and} \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

Now the complex function defined

$$f(z) = P(x, y) + i\, Q(x, y)$$

is necessarily analytic within the domain considered.

Furthermore define

$$\Omega(z) = p + iq = \frac{1}{4}\int f(z)\, dz. \tag{3.15}$$

Thus $\Omega'(z) = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial y} = \frac{1}{4} f(z) = \frac{P(x, y)}{4} + i\frac{Q(x, y)}{4}$

which indicates the $\Omega'(z)$ is analytic, thus requiring $\Omega(z)$ to be analytic, as per Churchill [6]. So equation (3.15) is analytic and must conform to the Cauchy-Riemann conditions from which it is seen

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{P}{4} \quad \text{and} \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} = -\frac{Q}{4} .$$

It can be shown that

$$\nabla^2 (U - px - qy) = 0. \qquad (3.16)$$

Hence, $U = px + qy + r$ where r is a harmonic function in the region considered. Let $\Phi(z)$ be the function whose real part is r. Thus (3.16) may be written

$$U = R \left[\bar{z}\, \Omega(z) + \Phi(z) \right] \qquad (3.17)$$

where R denotes " the real part of ". Equation (3.17) is the solution of the biharmonic equation (3.14) and $\Omega(z)$ and $\Phi(z)$ are called complex potentials which are harmonic, thus, necessarily analytic within the domain being considered.

From equation (3.17) it is seen that the form of the general solution is

$$w_g = \bar{z}\, \varphi(z) + z\, \overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} , \qquad (3.18)$$

where $\varphi(z)$ and $\chi(z)$ are complex potentials. Thus, the solution to equation (3.10) is obtained from equations (3.13) and (3.18); i.e.,

$$w = \bar{z}\,\varphi(z) + z\,\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} - \frac{B\,z^2\,\bar{z}^2}{64\,D} \quad . \tag{3.19}$$

If the region involved is free of thermal loading then B equals zero and equation (3.19) becomes

$$w = \bar{z}\,\varphi(z) + z\,\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} \quad . \tag{3.20}$$

The restrictions which must be placed on the complex potentials in equations (3.19) and (3.20) are easily illustrated by substituting $\varphi^*(z) + A + iB + icz$ for $\varphi(z)$ and $\chi^*(z) - A\bar{z} - iB\bar{z} + iD$ for $\chi(z)$ into either equation. For example, equation (3.20) would yield

$$w = \bar{z}\,\varphi^*(z) + z\,\overline{\varphi^*(z)} + \chi^*(z) + \overline{\chi^*(z)}$$

which is exactly the same equation as before the substitution only using different complex potentials. This condition, however, cannot be allowed to exist in that the deflections of a simply connected plate are single valued. Thus $\varphi(z)$ and $\chi(z)$ must be unique. In order to eliminate the possibility of multiple valued deflections, it is necessary that

$$\varphi(0) = 0, \tag{3.21a}$$

$$\Im\,\varphi'(0) = 0, \tag{3.21b}$$

and $\Im\,\chi(0) = 0$. (3.21c)

where $\mathcal{I}$ denotes " the imaginary part of ".

The clamped edge has two boundary conditions. First, there can not be any deflection of the edge, so

$$w = 0 \tag{3.22}$$

on the boundary. Second, the slope of plate at the boundary is zero. So

$$\frac{\partial w}{\partial N} = 0 \tag{3.23}$$

on the plate boundary. Also, because the plate is flat in the undeformed shape

$$\frac{\partial w}{\partial T} = 0 \tag{3.24}$$

where N and T are the distances measured along the normal and tengent to the boundary respectively. Squaring and adding equations (3.23) and (3.24) yields

$$\left(\frac{\partial w}{\partial N} \right)^2 + \left(\frac{\partial w}{\partial T} \right)^2 = 0$$

which is analogus to

$$\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 = 0$$

on the boundary because like N and T, x and y are perpendicular.

Substituting the operators (3.3) for $\partial/\partial x$ and (3.4) for $\partial/\partial y$ gives boundary equation (3.23) in the form

$$4 \frac{\partial w}{\partial z} \frac{\partial w}{\partial \bar{z}} = 0 \quad .$$

In that z and $\bar{z}$ are directly related, one can not be zero without the other being zero; so the final form of boundary conditions (3.23) is

$$\frac{\partial w}{\partial \bar{z}} = 0 \quad . \tag{3.25}$$

Substituting into the two boundary conditions (3.22) and (3.25) the expression for the deflection (3.20), one obtains the two boundary conditions for a clamped plate in terms of the complex potentials $\varphi(z)$ and $\chi(z)$; i.e.,

$$\bar{z}\, \varphi(z) + z\, \overline{\varphi(z)} + \chi(z) + \chi(z) = 0 \tag{3.26}$$

and $$\varphi(z) + z\, \overline{\varphi'(z)} + \overline{\chi'(z)} = 0 \quad . \tag{3.27}$$

Consider a function $f(\zeta)$ such that

$$z = f(\zeta) \qquad \text{or} \qquad \bar{z} = \overline{f(\zeta)} \tag{3.28}$$

will map all points of the plate in the z plane to points inside the unit circle in the ζ plane. If the function $f(\zeta)$ is analytic, and its derivative with respect to the stated parameter is not zero, then the transformation is conformal. The operation of conformal mapping of the outer boundary of the plate onto the unit circle greatly simplifies the determination of the complex potentials $\varphi(z)$ and $\chi(z)$. The purpose of this transformation is two fold. First, an irregular shaped plate in the z plane is mapped to a circle of radius one and all of the points within that unit circle are mapped from points on the irregular shaped plate. Thus, if the complex potentials are analytic at all points of the plate in the z plane, they are also analytic at all points within a circle of unit radius in the ζ plane. Thus, the form of $\varphi(\zeta)$ and $\chi(\zeta)$ in the ζ plane can be assumed as Taylor's series with a radius of convergence of one. Using the restrictions (3.21) the form of $\varphi(\zeta)$ and $\chi(\zeta)$ may be assumed

$$\varphi(\zeta) = \sum_{n=1}^{\infty} d_n \zeta^n \tag{3.29}$$

and $$\chi(\zeta) = \sum_{n=0}^{\infty} g_n \zeta^n \tag{3.30}$$

where g_0 and d_1 are real and the rest of the coefficients are complex. These series will converge as per Churchill [6]. Second, the boundary conditions (3.26) and (3.27) can be expressed in terms of one variable σ in the ζ plane. Take any point ζ on the outer boundary of the unit circle in the ζ plane. This can be written

$$\zeta = \rho e^{i\phi} = 1 e^{i\phi} = e^{i\phi} = \sigma \tag{3.31a}$$

and $$\bar{\zeta} = \rho e^{-i\phi} = 1 e^{-i\phi} = e^{-i\phi} = \frac{1}{\sigma} \tag{3.31b}$$

because ρ equals one on the boundary. Thus, by transforming the boundary conditions (3.26) and (3.27) to the ζ plane by the use of the conformal transformation (3.28), these boundary conditions can then be expressed in terms of one variable σ .

Using the two ideas expressed above, it is seen that the complex potentials of the boundary conditions (3.26) and (3.27) may be expressed in series form as in equations (3.29) and (3.30) and these series may be expressed in terms of the one variable σ . Then, by equating coefficients of the power series, it is possible to determine g_1, g_2, ... g_n and d_0, d_1, d_2 ..., d_n, thus determining the complex potentials $\varphi(\zeta)$ and $\chi(\zeta)$.

To transform the boundary condition (3.26) from the z plane to the ζ plane is a simple matter of substituting in the conformal transformation (3.28) which yields

$$\overline{f(\sigma)}\ \varphi(\sigma) + f(\sigma)\ \overline{\varphi(\tfrac{1}{\sigma})} + \chi(\sigma) + \overline{\chi(\tfrac{1}{\sigma})} = 0. \qquad (3.32)$$

To transform the boundary condition (3.27) it is necessary to note that

$$\varphi'(z) = \frac{\partial \varphi(z)}{\partial z} = \frac{\partial \varphi(\zeta)}{\partial \zeta}\ \frac{\partial \zeta}{\partial z} = \frac{\varphi'(\zeta)}{f'(\zeta)} \qquad (3.33)$$

Using (3.33) the boundary condition (3.27) becomes

$$\varphi(\sigma) + f(\sigma)\ \frac{\overline{\varphi'(\tfrac{1}{\sigma})}}{\overline{f'(\tfrac{1}{\sigma})}} + \frac{\overline{\chi'(\tfrac{1}{\sigma})}}{\overline{f'(\tfrac{1}{\sigma})}} = 0 . \qquad (3.34)$$

Equations (3.32) and (3.34) are the boundary conditions of a clamped plate in the ζ plane. Using these two equations it is possible to determine the complex potentials $\varphi(\zeta)$ and $\chi(\zeta)$ and thus effect a solution.

Let γ be the outer boundary of a thin elastic plate which is divided into two regions as shown in figure 3.2. The boundary between region 1 and region 2 will be designated Γ. Region 1 of the plate is thermally loaded and region 2 is unloaded.

Recall that the differential equation of a thermally loaded plate is

$$\nabla^4 w_1 = -\frac{B}{D} .$$

This equation holds in region 1. Also, the defining equation of region 2 is

$$\nabla^4 w_2 = 0 .$$

In that there are two separate differential equations defining the deflection of the plate in figure 3.2, there will be two solutions made up of the complex potentials $\varphi_1(z)$, $\chi_1(z)$, $\varphi_2(z)$ and $\chi_2(z)$. These two solutions must yield values of plate deflection such that on the boundary Γ there will be no discontinuity.

Take a point z_0 which is located on Γ. Move an infinitesimal distance away from z_0 and perpendicular to the boundary Γ into region 1. Call this point z_1. Similiarly, move into region 2 and call that

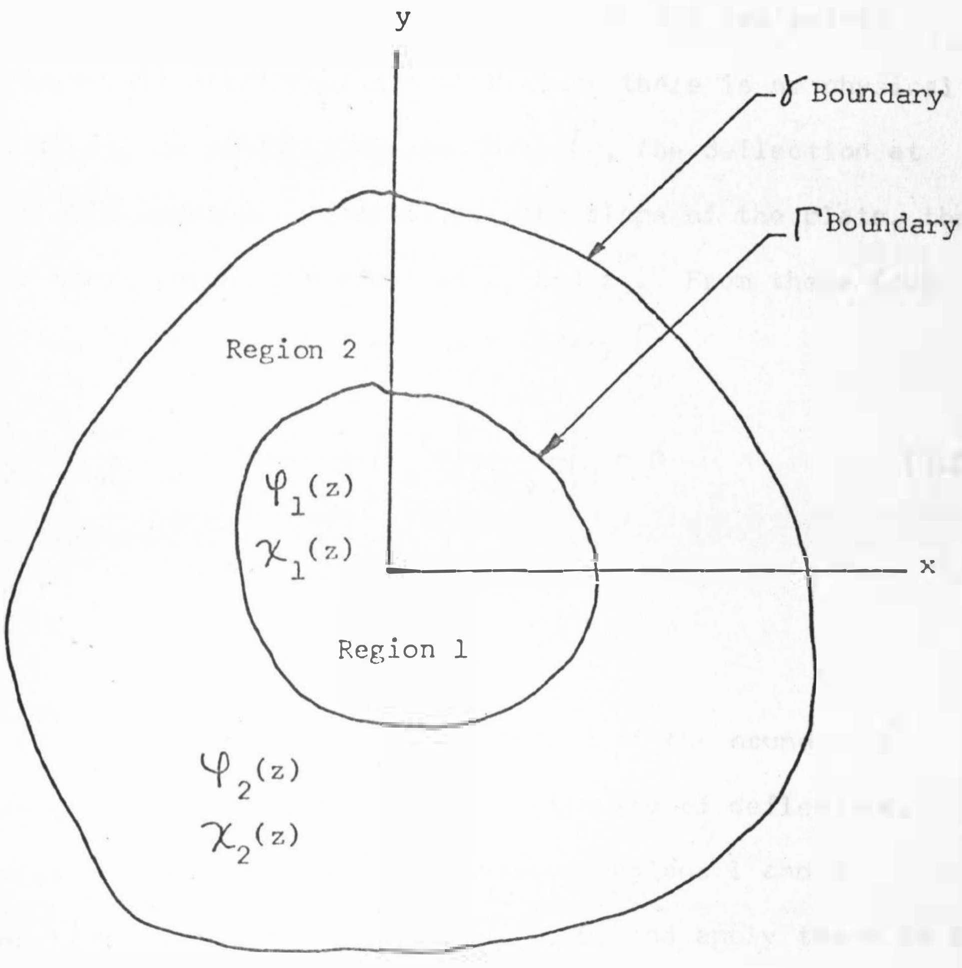

Figure 3.2. General Plate Plan View

point z_2. Now points z_1 and z_2 are separated by an infintesimal distance; however, the plate characteristics at the two points are governed by different equations. Because there is no physical discontinuity in the plate on the boundary Γ, the deflection at z_1 and z_2 should be equal. Similiarly, the slope of the plate, the moment, and shear forces are equal at z_1 and z_2. From these four considerations it is seen that on the boundary Γ

$$\left[w\right]_2^1 = \left[\frac{\partial w}{\partial z}\right]_2^1 = \left[\frac{\partial^2 w}{\partial z \partial \bar{z}}\right]_2^1 = \left[\frac{\partial^3 w}{\partial z^2 \partial \bar{z}}\right]_2^1 = 0 \tag{3.35}$$

where $\left[w\right]_2^1 = w_1 - w_2 = 0$, etc.

Equations (3.35) are the continuity equations of the boundary Γ which, when satisfied, will ensure the continuity of deflection, slope, radius of curvature, or shear between regions 1 and 2.

Recall now equations (3.19) and (3.20) and apply these to the plate of figure 3.2, which yields

$$w_1 = \bar{z} \quad \varphi_1(z) + z \quad \overline{\varphi_1(z)} + \chi_1(z) + \overline{\chi_1(z)} - \frac{Bz^2 \; \bar{z}^2}{64 \; D} \tag{3.36}$$

and $$w_2 = \bar{z} \quad \varphi_2(z) + z \quad \overline{\varphi_2(z)} + \chi_2(z) + \overline{\chi_2(z)} \quad . \tag{3.37}$$

Substituting (3.36) and (3.37) into the continuity equations (3.35) one obtains the continuity equations in terms of the complex potentials; i.e.,

$$\left[\bar{z}\ \varphi(z) + z\ \overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}\right]_2^1 = \frac{Bz^2\ \bar{z}^2}{64\ D}\ , \qquad (3.38a)$$

$$\left[\bar{z}\ \varphi'(z) + \overline{\varphi(z)} + \chi'(z)\right]_2^1 = \frac{B\ z\ \bar{z}^2}{32\ D}\ , \qquad (3.38b)$$

$$\left[\varphi'(z) + \overline{\varphi'(z)}\right]_2^1 = \frac{B\ z\ \bar{z}}{16\ D}\ , \qquad (3.38c)$$

and
$$\left[\varphi''(z)\right]_2^1 = \frac{B\ \bar{z}}{16\ D}\ . \qquad (3.38d)$$

The solution of these equations gives the difference in the complex potentials across the boundary Γ, which will have the form

$$\left[\varphi(z)\right]_2^1 = g(z)$$

and
$$\left[\chi(z)\right]_2^1 = h(z)\ .$$

When these continuity solutions have been obtained, one can find the complex potentials in region 2 knowing those in region 1 and vise versa.

The form of equations (3.38) and the solutions of these equations are dependent only upon the shape of the boundary Γ and are not affected by the outside boundary γ. For example, the solution of the continuity equations of a square plate loaded over a central circle are the same as those for an elliptic plate loaded over a central circle.

CHAPTER IV

DERIVATIONS OF PLATE DEFLECTION EQUATIONS

This chapter contains the development of the equations of deflection of three plate configurations. These will be arranged in the order of;

(A) A circular plate loaded over a circular region.

(B) An elliptic plate loaded over a circular region.

(C) A round-cornered square plate loaded over a circular region.

This is more or less the order of difficulty even though the method of solution is similar.

The solution of the circular plate loaded over the circular region is helpful in checking the solution of an elliptic plate loaded over a circular region, in that the circular plate is a special case of an elliptic plate. The round-cornered square plate loaded over a circular region is not a special case of either of the other problems.

In order to conserve reading time and present a continuous flow of material, the calculations in this chapter are given in condensed form. The complete set of mathematical calculations is given in Appendix I for those who wish to see the development in more detail.

Circular Plate Loaded over a Circular Region

Consider the circular plate of radius b thermally loaded over the region enclosed by a concentric circle of radius a as shown in figure 4.1. Let the boundary between loaded region 1 and the unloaded region 2 be Γ and the outer boundary of region 2 be γ , which is clamped.

The continuity equations of the boundary Γ expressed in terms of the complex potentials were established previously by equation (3.38) as

$$\left[\bar{z}\,\varphi(z) + z\,\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}\right]_2^1 = \frac{B\,z^2\,\bar{z}^2}{64\,D} \,,$$

$$\left[\bar{z}\,\varphi'(z) + \overline{\varphi(z)} + \chi'(z)\right]_2^1 = \frac{B\,z\,\bar{z}^2}{32\,D} \,,$$

$$\left[\varphi'(z) + \overline{\varphi'(z)}\right]_2^1 = \frac{B\,z\,\bar{z}}{16\,D} \,,$$

and

$$\left[\varphi''(z)\right]_2^1 = \frac{B\,\bar{z}}{16\,D} \,.$$

In order to solve these continuity equations the boundary Γ must be transformed to the outer boundary Γ^* of the unit circle in the ζ plane. The transformation necessary to conformally map Γ to Γ^* is

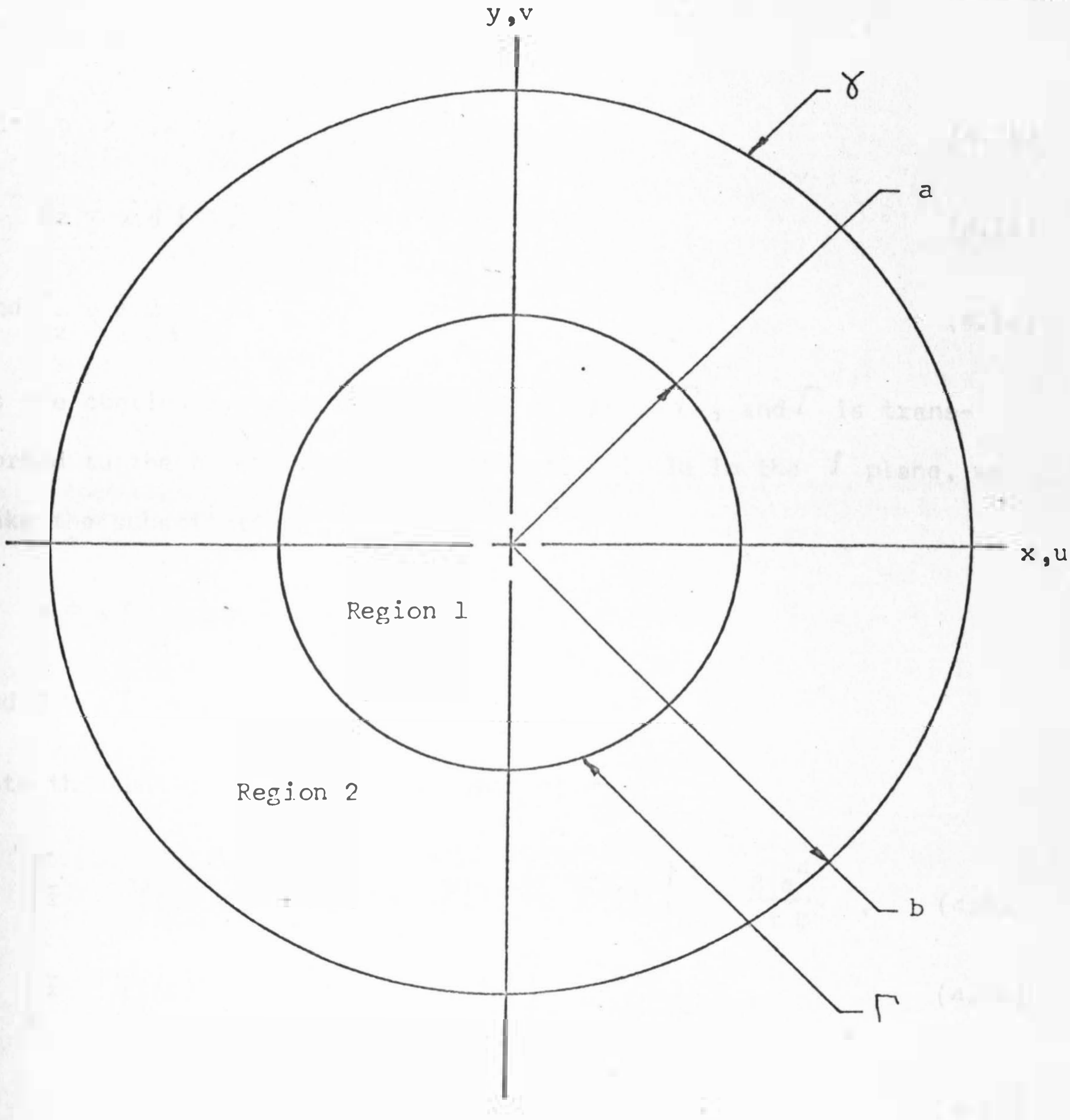

Figure 4.1. Circular Plate Loaded over a Circular Region

$$z = a\zeta \qquad (4.1a)$$

where a is the radius of the inner region.

Giving $\bar{z} = a\bar{\zeta}$, (4.1b)

$$dz = a\, d\zeta \qquad (4.1c)$$

and $\dfrac{d}{dz} = \dfrac{1}{a}\dfrac{d}{d\zeta}$. (4.1d)

As the continuity equations are valid only on Γ, and Γ is transformed to the outer boundary of the unit circle in the ζ plane, we make the substitution

$$z = a\zeta = a\sigma$$

and $\bar{z} = a\bar{\zeta} = a\sigma^{-1}$

into the continuity equations. This gives

$$\left[\bar{z}\,\varphi(z) + z\,\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}\right]_2^1 = \frac{B\,a^4}{64\,D} , \qquad (4.2a)$$

$$\left[\bar{z}\,\varphi'(z) + \overline{\varphi(z)} + \chi'(z)\right]_2^1 = \frac{B\,a^3}{32\,D\sigma} , \qquad (4.2b)$$

$$\left[\varphi'(z) + \overline{\varphi'(z)}\right]_2^1 = \frac{B\,a^2}{16\,D} , \qquad (4.2c)$$

and $$\left[\varphi''(z)\right]_2^1 = \frac{B\,a}{16\,D\sigma} . \qquad (4.2d)$$

From equation (4.2d) we see using (4.1d) that

$$\left[\varphi''(z)\right]_2^1 = \left[\frac{d}{dz}\,\varphi'(z)\right]_2^1 = \left[\frac{1}{a}\,\frac{d}{d\sigma}\,\varphi'(z)\right]_2^1 = \frac{B\,a}{16\,D\sigma}\,.$$

which may be rewritten

$$\left[\frac{d}{d\sigma}\,\varphi'(z)\right]_2^1 = \frac{B\,a^2}{16\,D\sigma}\,.$$

Integrating with respect to σ yields

$$\left[\varphi'(z)\right]_2^1 = \frac{B\,a^2}{16\,D}\log\sigma + K_1 \qquad (4.3)$$

where K_1 is the complex constant of integration of which the real part may be determined by substituting (4.3) into (4.2c) yielding

$$K_1 = \frac{B\,a^2}{32\,D} + iK_2\,.$$

We now have a solution for $\left[\varphi'(z)\right]_2^1$ which may be again integrated giving

$$\left[\varphi(z)\right]_2^1 = \frac{B\,a^3}{16\,D}\,\sigma\log\sigma - \frac{B\,a^3}{32\,D}\,\sigma + K_3 + iaK_2\sigma \qquad (4.4)$$

where K_3 again is the complex constant of integration.

Substituting (4.4) and (4.3) into the second continuity equation (4.2b) and again carrying out the integration process

$$\left[\chi(z)\right]_2^1 = \frac{B\ a^4}{32\ D} \log\sigma \ - a\,\sigma\ K_3 + i\,\frac{a^2\sigma^2}{2} K_2 + K_4 \qquad (4.5)$$

is developed with another complex constant of integration K_4. We have now established the forms of $\left[\varphi(z)\right]_2^1$ and $\left[\chi(z)\right]_2^1$. Using the final equation at our disposal (4.2a) it is shown that

$$R\left[K_4\right] = \frac{5\ Ba^4}{128\ D}$$

where R denotes "the real part of." The imaginary portions of K_2, K_3, and K_4 do not contribute to the deflection. They can not be determined and are not necessary, so they are disregarded. This leaves the solution of the continuity equations in the ζ plane as

$$\left[\varphi(z)\right]_2^1 = \frac{Ba^3}{16\ D}\,\sigma \log\sigma \ - \frac{Ba^3}{32\ D}$$

and $$\left[\chi(z)\right]_2^1 = \frac{Ba^4}{32\ D} \log\sigma \ + \frac{5Ba^4}{128\ D}\ .$$

Transforming these back to the z plane using the inverse transformation to (4.1a)

$$\zeta = z/a$$

one has

$$\left[\varphi(z)\right]_2^1 = \frac{Ba^2}{16\ D}\left[z \log\frac{z}{a} - \frac{z}{2}\right] \qquad (4.6a)$$

$$\text{and} \quad \left[\chi(z)\right]_2^1 = \frac{Ba^4}{32\,D}\left[\log\frac{z}{a} + \frac{5}{4}\right]. \tag{4.6b}$$

The equations (4.6) give the change in the complex potentials from region 1 to region 2. Thus, they somewhat indicate the form of the complex potentials in region 2. In Chapter III it was seen that the complex potentials must be analytic throughout the region being considered and that

$$\varphi(0) = 0 \;,$$

$$\Im\,\varphi'(0) = 0 \;,$$

$$\text{and} \quad \Im\,\chi'(0) = 0 \;.$$

Using the transformation

$$z = b\,\zeta$$

which conformally maps the outer boundary of the plate in the z plane onto the unit circle in the ζ plane and the considerations stated above the form of the complex potentials for region 2 in the ζ plane may be assumed as

$$\varphi_2(\zeta) = \frac{Ba^2}{16\,D}\left[d_1\zeta - b\,\zeta\log\zeta\right] \tag{4.7a}$$

$$\text{and} \quad \chi_2(\zeta) = \frac{Ba^4}{32\,D}\left[g_0 - \log\zeta\right] \tag{4.7b}$$

where d_1 and g_0 are constants.

These two complex potentials must satisfy the clamped boundary conditions of region 2 which are

$$\bar{z}\,\varphi_2(z) + z\,\overline{\varphi_2(z)} + \chi_2(z) + \overline{\chi_2(z)} = 0 \qquad (4.8a)$$

and $$\varphi_2(z) + z\,\overline{\varphi_2'(z)} + \overline{\chi_2'(z)} = 0 \quad . \qquad (4.8b)$$

Substituting (4.7a) and (4.7b) into boundary equation (4.8a) and setting ζ equal to σ, one finds that the logarithic terms cancel and the $Ba^2/16D$ terms drop out, leaving

$$2bd_1 + a^2 g_0 = 0. \qquad (4.9)$$

Substituting the assumed values of $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ into boundary condition (4.8b) we find that again the logarithmic terms cancel, leaving the coefficients of the powers of σ ; i.e.,

$$2d_1 - b - \frac{a^2}{2b} = 0 \quad . \qquad (4.10)$$

To substantiate the assumed form of $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ it is necessary that we develop two independent equations with which to determine the two assumed constants d_1 and g_0. This condition has been fulfilled by equation (4.9) and (4.10).

Using (4.9) and (4.10), the solutions of d_1 and g_0 will establish the complex potentials $\varphi(\zeta)$ and $\chi(\zeta)$ in the ζ plane. Transforming these back to the z plane through the inverse transformation

$$\zeta = \frac{z}{b}$$

gives

$$\varphi_2(z) = \frac{Ba^2}{16\ D}\left[\left(\frac{1}{2} + \frac{a^2}{4b^2}\right) z - z \log \frac{z}{b}\right] \qquad (4.11a)$$

and $$\chi_2(z) = \frac{Ba^4}{32\ D}\left[- \frac{b^2}{a^2} - \frac{1}{2} - \log \frac{z}{b}\right]. \qquad (4.11b)$$

Recalling that the solutions to the continuity equations are

$$\left[\varphi(z)\right]_2^1 = \varphi_1(z) - \varphi_2(z)$$

and $$\left[\chi(z)\right]_2^1 = \chi_1(z) - \chi_2(z) .$$

Thus, (4.11) and (4.6) give

$$\varphi_1(z) = \frac{Ba^2}{16\ D}\left[z \log \frac{b}{a} + \frac{a^2 z}{4b^2}\right] \qquad (4.12a)$$

and $$\chi_1(z) = \frac{Ba^4}{32\ D}\left[\log \frac{b}{a} + \frac{3}{4} - \frac{b^2}{a^2}\right]. \qquad (4.12b)$$

With the solution of $\varphi_1(z)$, $\varphi_2(z)$, $\chi_1(z)$ and $\chi_2(z)$ we have established the deflection w_1 and w_2.

Referring to equations (3.36) and (3.37) and recalling that $z\bar{z} = r^2$ one sees that

$$w_2 = \frac{Ba^2}{32\ D}\left[2r^2 - 2b^2 - a^2 + \frac{a^2 r^2}{b^2} - 2\,(a^2 + 2r^2)\log\frac{r}{b}\right] \qquad (4.13a)$$

and $$w_1 = \frac{Ba^2}{16\ D}\left[\frac{3a^2}{4} - b^2 + \frac{a^2 r^2}{2b^2} - \frac{r^4}{4a^2} + (a^2 + 2r^2)\log\frac{b}{a}\right]. \qquad (4.13b)$$

These are the equations of deflection of a circular plate loaded over a circular region.

Elliptic Plate Loaded over a Circular Region

Consider the thin elliptic plate shown is figure 4.2 whose outer boundary γ is clamped and whose major axis is a and minor axis is b. This plate is loaded over a central circle of radius c which will be called region 1 and unloaded in region 2 with the boundary between region 1 and region 2 designated Γ. The equations defining the deflection in terms of the complex potentials are given by (3.36) and (3.37).

The continuity equations (4.6) hold for the plate shown in figure 4.2. These equations are valid for any simply connected plate thermally loaded over a circular region whose center corresponds with the origin of the coordinate system. These continuity equations are

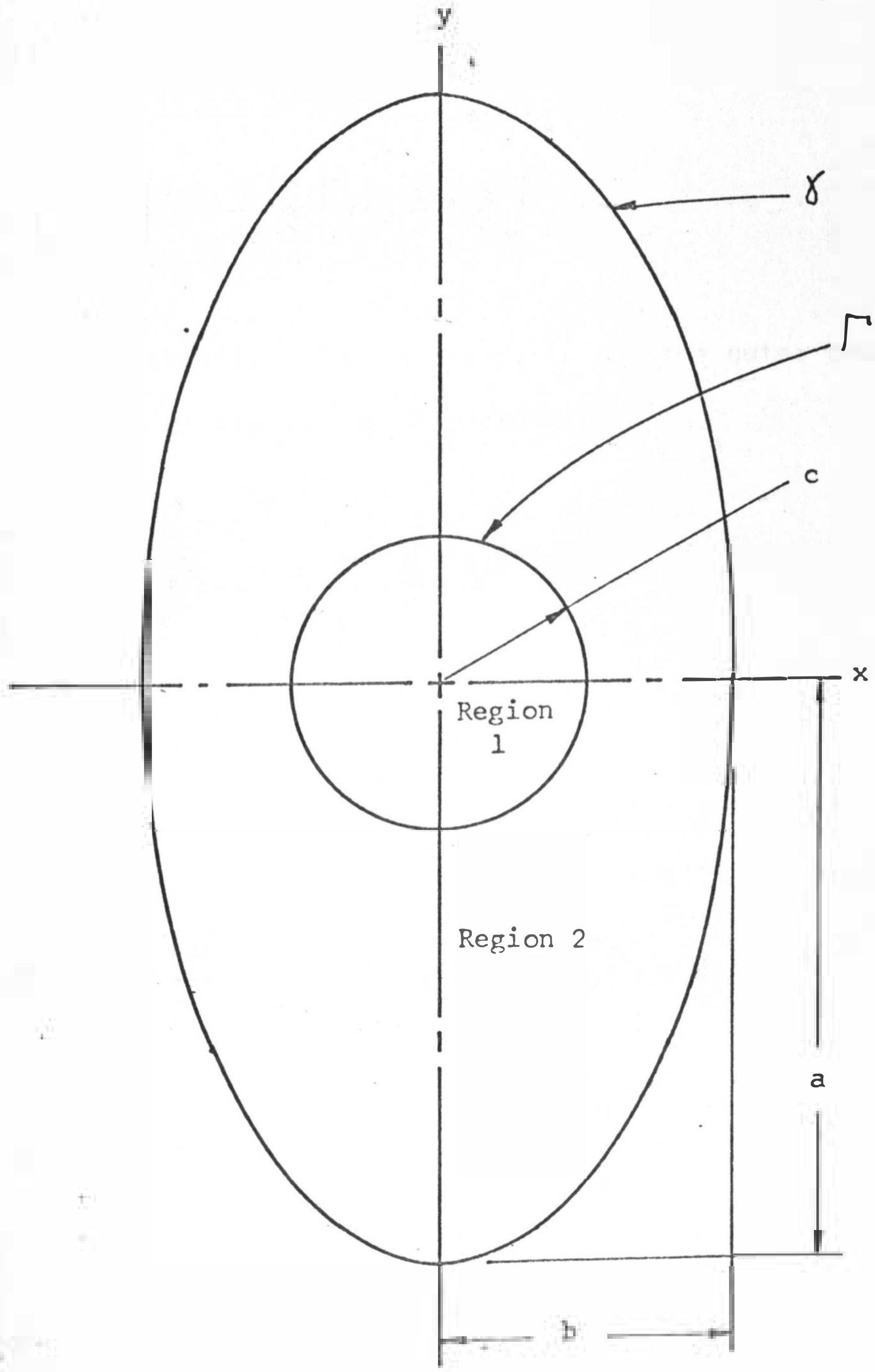

Figure 4.2. Elliptic Plate Loaded over a Circular Region

$$\Big[\varphi(z)\Big]_2^1 = \frac{Bc^2}{16\ D}\left[z \log\frac{z}{c} - \frac{z}{2}\right] \tag{4.14a}$$

and
$$\Big[\chi(z)\Big]_2^1 = \frac{Bc^4}{32\ D}\left[\log\frac{z}{c} + \frac{5}{4}\right]. \tag{4.14b}$$

The transformation necessary to map the outer boundary γ onto the unit circle in the ζ plane is

$$z = \frac{1}{2}(m\zeta + n\zeta^{-1}) \tag{4.15a}$$

or
$$\bar{z} = \frac{1}{2}(m\bar{\zeta} + n\bar{\zeta}^{-1}). \tag{4.15b}$$

where $m = a + b$ and $n = a - b$.

We now assume the form of the complex potentials $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ based on equations (4.14), (4.15) and (3.21). These are

$$\varphi_2(\zeta) = \frac{Bc^2}{32\ D}\left[-(m\zeta + n\zeta^{-1})\log\zeta + d_1\zeta + d_2\zeta^{-1}\right] \tag{4.16a}$$

and
$$\chi_2(\zeta) = \frac{Bc^4}{32\ D}\left[-\log\zeta + g_0 + g_1\zeta^{-2}\right]. \tag{4.16b}$$

The constants d_1, d_2, g_0, and g_1 will be determined from the boundary conditions of region 2 which are given by equations (3.32) and (3.34). Substituting the assumed solutions of the complex potentials $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ into equation (3.32) one sees that the constant $Bc^2/16D$ cancels and the logarithmic terms drop out, leaving

$$\frac{md_1}{2} + \frac{nd_1}{2}\sigma^2 + \frac{md_2}{2}\sigma^{-2} + \frac{nd_2}{2} + \frac{m}{2}d_1 + \frac{n}{2}d_1\sigma^{-2} + \frac{m}{2}d_2\sigma^2 + \frac{n}{2}d_2$$

$$+ 2c^2 g_0 + c^2 g_1 \sigma^{-2} + c^2 g_1 \sigma^2 = 0 \;.$$

Equating coefficients of like powers of σ gives

$$nd_1 + md_2 + 2c^2 g_1 = 0 \;, \tag{4.17a}$$

$$md_1 + nd_2 + 2c^2 g_0 = 0 \;, \tag{4.17b}$$

and $md_2 + nd_1 + 2c^2 g_1 = 0 \;,$

yielding two independent equations for determining the constants.

Substituting the assumed complex potentials (4.16) into the second boundary condition (3.34) one again sees that the constants cancel and logarithmic terms drop out, leaving

$$\left(d_1 + m + \frac{d_2 m}{n} + \frac{4c^2}{n} g_1 \right. \quad + \left(2d_2 - \frac{m}{n} d_1 + \frac{2m^2}{n} + m + \frac{m^2}{n^2} d_2\right.$$

$$\left. + \frac{2c^2}{n} + \frac{4c^2 m}{n^2} g_1\right)\sigma^{-1} + \left(m - d_1 - \frac{m^2}{n^2}d_1 + \frac{m^3}{n^2} + m + \frac{m^3}{n^2}\right.$$

$$\left. + \frac{m}{n} d_2 + \frac{m^3}{n^3} d_2 + \frac{2mc^2}{n^2} + \frac{4c^2 n^2}{m^3} g_1\right)\sigma^{-3} = 0.$$

Equating coefficients of like powers of σ one obtains three equations:

$$d_1 + m + \frac{d_2 m}{n} + \frac{4c^2}{n} g_1 = 0 \quad , \tag{4.18a}$$

$$(2 + m^2/n^2)\, d_2 - \frac{m}{n} d_1 + \frac{4c^2 m}{n^2} g_1 + \frac{2m^2}{n} + \frac{2c^2}{n} + m = 0 \quad , \tag{4.18b}$$

and $$\left(\frac{m}{n} + \frac{m^3}{n^3}\right) d_2 - \left(1 + \frac{m^2}{n^2}\right) d_1 + \frac{4c^2 n^2}{m^3} g_1 + 2m + \frac{2m}{n^3} + \frac{2mc^2}{n^2} = 0 \quad , \tag{4.18c}$$

which, along with the two equations (4.17) make five equations from which four unknown constants d_1, d_2, g_1, and g_0 are desired. Thus, our system of equations appears to be in over determinate. However, equation (4.18c) is not independent. If we multiply equation (4.18b) by m/B and add equation (4.17a) knowing the solution of g_1 is $-mn/2c^2$ from (4.17a) and (4.18a) we can develop equation (4.18c) which proves it is not independent. Thus, there are four independent equations with which to determine the four unknown constants, yielding

$$g_1 = \frac{-mn}{2c^2} \quad , \tag{4.19a}$$

$$d_2 = \frac{mn^2 - n^3 - 2nc^2}{2\,(m^2 + n^2)} \quad , \tag{4.19b}$$

$$d_1 = \frac{3mn^2 + m^3 + 2mc^2}{2\,(m^2 + n^2)} \quad , \tag{4.19c}$$

and $$g_0 = \frac{4m^2n^2 + m^4 + n^4 + (2m^2 + 2n^2)\, c^2}{-4c^2\,(m^2 + n^2)} \tag{4.19d}$$

The solution of these constants gives $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ which, coupled with the inverse transformation of (4.15)

$$\zeta = \frac{z + \sqrt{z^2 - mn}}{m}$$

or $\zeta^{-1} = \dfrac{z - \sqrt{z^2 - mn}}{n}$

and the solutions to the continuity equations (4.14) give the solutions to $\varphi_2(z)$, $\chi_2(z)$ and thus $\varphi_1(z)$ and $\chi_1(z)$ which follow

$$\varphi_2(z) = \frac{Bc^2}{32\ D}\left[-2z \log \frac{z + \sqrt{z^2 - mn}}{m} + \left(\frac{d_1}{m} + \frac{d_2}{n}\right) z + \frac{d_1}{m}\sqrt{z^2 - mn} - \frac{d_2}{n}\sqrt{z^2 - mn}\right] \tag{4.20a}$$

$$\chi_2(z) = \frac{Bc^4}{32\ D}\left[-\log \frac{z + \sqrt{z^2 - mn}}{m} + g_0 + \frac{g_1}{n^2}\left(2z^2 - 2z\sqrt{z^2 - mn} - mn\right)\right] \tag{4.20b}$$

$$\varphi_1(z) = \frac{Bc^2}{32\ D}\left[-z \log \frac{cz + c\sqrt{z^2 - mn}}{mz} + \left(\frac{d_2}{n} + \frac{d_1}{m} - 1\right) z + \frac{d_1}{m}\sqrt{z^2 - mn} - \frac{d_2}{n}\sqrt{z^2 - mn}\right] \tag{4.20c}$$

$$\chi_1(z) = \frac{Bc^4}{32\ D}\left[-\log \frac{cz + c\sqrt{z^2 - mn}}{zm} + g_0 + \frac{5}{4} + \frac{g_1}{n^2}\left(2z^2 - 2z\sqrt{z^2 - mn} - mn\right)\right] \tag{4.20d}$$

These equations coupled with the equations of deflection (3.36) and (3.37) describe the deflection of the plate in figure (4.2).

Round Cornered Square Plate Loaded over a Circular Region

This section deals with a round cornered square plate whose outer boundary γ is clamped and whose inner region 1 bounded by Γ is thermally loaded. The radius of the inner boundary Γ is c and the plate itself is b square as in figure 4.3.

The solution of the continuity equation between regions 1 and 2 are as in the previous sections (4.6); i.e.,

$$\left[\varphi(z)\right]_2^1 = \frac{Bc^2}{16\ D}\left[z \log \frac{z}{c} - \frac{z}{2}\right]$$

and $$\left[\chi(z)\right]_2^1 = \frac{Bc^4}{32\ D}\left[\log \frac{z}{c} + \frac{5}{4}\right].$$

The transformation necessary to map the boundary γ onto the unit circle in the ζ plane is

$$z = L\ (\zeta + \lambda \zeta^5)$$

or $$\bar{z} = L\ (\bar{\zeta} + \lambda \bar{\zeta}^5)$$

where $L = \frac{25}{48}$ b and $\lambda = -1/25$.

Based on this transformation, the solutions to the continuity equations, the restrictions (3.21), and the fact that $\varphi(z)$ and $\chi(z)$

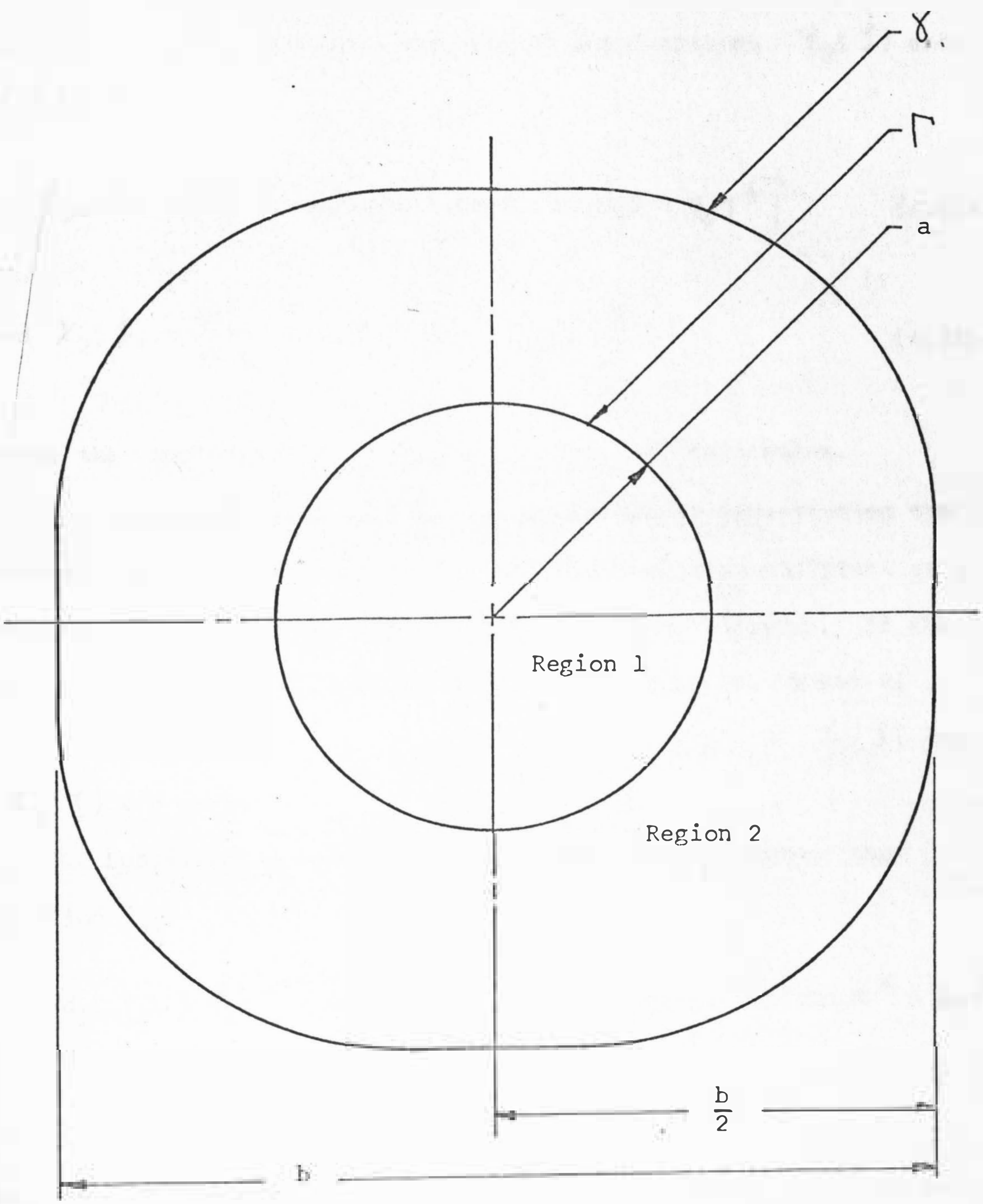

Figure 4.3. Round Cornered Square Plate Loaded over a Circular Region

must be analytic, we assume the form of the functions $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ as

$$\varphi_2(\zeta) = \frac{Bc^2}{16\ D}\left[L\ (\zeta+\lambda\zeta^5)\ \log\zeta^{-1} + d_1\zeta + d_5\zeta^5\right] \tag{4.21a}$$

and $$\chi_2(\zeta) = \frac{Bc^4}{32\ D}\left[\log\zeta^{-1} + g_o + g_4\zeta^4\right], \tag{4.21b}$$

where the constants d_1, d_5, g_0, and g_4 are real or complex.

These constants will now be determined by substituting the assumed complex potentials (4.21) into the boundary conditions of a clamped plate (3.33) and (3.34) and equating coefficients. If the number of equations derived by this method equals the number of constants assumed, then in all likelyhood, the forms of $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ are correct.

Substituting equations (4.21) into (3.33) develops the equation

$$2d_1L + L\lambda d_1\sigma^{4} + Ld_5\sigma^{-4} + 2L\lambda d_5 + L\lambda d_1\sigma^{-4} + Ld_5\sigma^{4} + g_0a^2 + \frac{g_4a^2}{2}\sigma^{-4} + \frac{g_4a^2}{2}\sigma^{4} = 0.$$

Equating coefficients of like powers of σ gives three equations of which

$$2Ld_1 + 2L\lambda d_5 + a^2g_0 = 0 \tag{4.22a}$$

and $$2L\lambda d_1 + 2Ld_5 + a^2g_4 = 0 \tag{4.22b}$$

are independent. The substitution of equations (4.18) into boundary conditions (3.34) yields the infinite series

$$(d_5 + d_1\lambda - L\lambda)\sigma^5 + \left[5\lambda d_5 + (2 - 5\lambda^2) d_1 + 4L\lambda^2 - L - a^2/2L\right]\sigma$$

$$+ \left[(5 - 25\lambda^2) d_5 + (25\lambda^3 - 5\lambda)d_1 + 4L\lambda - 20 L\lambda^3 + 5a^2\lambda/2L\right.$$

$$\left. + 2a^2 g_4/L\right]\sigma^{-3} + (125\lambda^3 d_5 - 10a^2\lambda g_4 - 25L\lambda^4 + \ldots)\sigma^{-7}$$

$$+ (\ldots)\sigma^{-11} + \ldots = 0.$$

Equating coefficients of like powers of σ develops an infinite number of equations. A closer examination, however, reveals that by multiplying the coefficients of σ^{-3} by -5λ one obtains the coefficients of σ^{-7}. Similiarly, by multiplying (4.22b) by 4/L, and adding the coefficients of σ times -5λ plus the coefficients of σ^5 one can develop the equation for the coefficients of σ^{-3}. Thus, two more independent equations

$$d_5 + d_1\lambda - L\lambda = 0 \qquad (4.23a)$$

and $$5\lambda d_5 + (2 - 5\lambda^2) d_1 + 4L\lambda^2 - L - a^2/2L = 0 \qquad (4.23b)$$

are developed, giving us four equations with which to develop the four unknowns; i.e.,

$$d_1 = \frac{2L^2 - 18L^2\lambda^2 + a^2}{4L - 2L\lambda^2} \qquad (4.24a)$$

$$d_5 = \frac{2\lambda L^2 - 2L^2\lambda^3 - a^2\lambda}{4L - 20L\lambda^2} , \tag{4.24b}$$

$$g_0 = \frac{-2L^2 + 16L^2\lambda^2 + 2L^2\lambda^4 + (\lambda^2 - 1)a^2}{a^2(2 - 10\lambda^2)} , \tag{4.24c}$$

and
$$g_4 = \frac{-L^2\lambda + 9L^2\lambda^3 - \lambda^2 L^2 + L^2\lambda^4 + (\lambda - 1)\ \frac{1}{2}\lambda a^2}{1 - 5\ ^2} \tag{4.24d}$$

These equations (4.24), when substituted into the assumed forms of $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ (4.18), yield the complex potentials in region 2 which, with the solutions to the continuity equations (4.6), yield the complex potentials of region 1. These equations will, from necessity, be written in terms of ζ, where $z = L\zeta + L\lambda\zeta^5$ because the inverse of this transformation cannot be written. Thus, the complex potentials defining the deflection of the plate shown in figure 4.3 are (4.21a and b) and

$$\varphi_1(\zeta) = \frac{Bc^2}{16\ D}\Big[L(\zeta + \lambda\zeta^5)\log\frac{(L + L\lambda\zeta^4)}{c} + (c_1 - \tfrac{L}{2})\zeta + (d_5 - \tfrac{L\lambda}{2})\zeta^5\Big] \tag{4.25a}$$

and
$$\chi_1(\zeta) = \frac{Bc^4}{32\ D}\Big[\log\frac{L + L\lambda\zeta^4}{c} + g_0 + \frac{5}{4} + g_4\zeta^4\Big] . \tag{4.25b}$$

CHAPTER V

CALCULATION OF THE PLATE DEFLECTION

Development of Heat Equation

The temperature field within the inner circular region of the plate configurations must satisfy Fourier's general law of heat conduction and be so constructed that the Laplacian of the thermal moment M_T is a constant over the entire heated region. The temperature function must yield an average temperature equal to the ambient temperature so that the inner region will not expand on the average. This is necessary in order for the inner boundary between the heated and unheated region to be free from in-plane stresses N_T.

In order for the Laplacian of the thermal moment to be a constant it is seen from equations (2.25) and (2.28) that the Laplacian of the temperature function (2.23) must also be a constant. When this is the case, Fourier's general law of heat conductions (2.25) takes the form of Poisson's equation. i.e.,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\beta}\frac{\partial T}{\partial t} = \text{constant}, \tag{5.1}$$

or in cylinderical coordinates,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\beta}\frac{\partial T}{\partial t} = \text{constant}. \tag{5.2}$$

Equation (5.1) is the particular case in which the temperature varies linearly with time or when there is a heat source or sink within the plate itself.

The temperature function in the outer region, that is, region 2 of the plates, is a constant equal to the ambient temperature. From the assumed form of the temperature function (2.23) it is seen that the mid-plane of the heated region also remains at the ambient temperature. In order for equation (5.1) along with the conditions (2.21) to be satisfied, it is necessary for the temperature to vary linearly with z as in (2.23). The temperature field will also be considered independent of θ and vary only as a function of r, thus, the heat equation (2.23) has the form

$$T = T_0 + z \left[g(r) + h\ (t) \right] \qquad (5.3)$$

and equation (5.2) has the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\beta}\frac{\partial T}{\partial t} \qquad (5.4)$$

Substituting (5.3) into (5.4) gives

$$\frac{d^2 g\ (r)}{dr^2} + \frac{1}{r}\frac{d^2 g\ (r)}{dr^2} = C \qquad (5.5)$$

where C is a constant equal to $h'(t)/\beta$.

Equation (5.5) is an ordinary differential equation whose solution is

$$g\ (r) = \frac{Cr^2}{4} + C_1 \ln r + C_2.$$

The constants C_1 and C_2 may be determined from the boundary conditions, which are

when $r = a$, $\quad g(r) = 0$

when $r = a$, $\quad \frac{\partial}{\partial r} g(r) = 0$.

These boundary conditions correspond to the temperature being ambient on the inner boundary and the slope zero at that boundary. It is seen that

$$g(r) = \frac{C}{4} \left[r^2 - a^2 \right] + \frac{Cra}{2} \ln \frac{a}{r}$$

The temperature function (5.3) becomes

$$T = T_o + z \left[\frac{C}{4} (r^2 - a^2) + h(t) \right] . \tag{5.6}$$

The function h(t) (2.27) is linear with respect to time. Thus, the heating function (5.6) will continue to increase in magnitude indefinitely, which of course, is impossible. Therefore, a limit must be placed upon the time interval such that the temperature build up does not cause the material to exceed its elastic properties. It is also apparent that the deflection is dependent upon the rate of heating; thus, a heating rate is picked such that the deflections do not exceed the small deflection theory presented here. The rate of heating h'(t) is 22.443 R^o/hr which when applied to a steel plate .25 inches thick gives

$$\frac{\nabla^2 M_T}{(1-\nu)} = \frac{\alpha E h^3}{12(1-\nu)} \left(\frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} \right) = 8.271 \times 10^2 = B \tag{5.7}$$

from equation (2.28) where

$h = .25$ inches,

$E = 30 \times 10^6$ psi,

$\alpha = .06$ in/in,

and $\nu = .3$.

Circular Plate Loaded over a Circular Region

The equations of deflection of a circular plate loaded over a circular region are given by equations (4.13). i.e.,

$$w_2 = \frac{Ba^2}{32\ D}\left[2r^2 - 2b^2 - a^2 + \frac{a^2 r^2}{2b^2} - 2(a^2 + 2r^2)\log\frac{r}{b}\right]$$

and $w_1 = \frac{Ba^2}{16\ D}\left[\frac{3a^2}{4} - b^2 + \frac{a^2 r^2}{2b^2} - \frac{r^4}{4a^2} + (a^2 + 2r^2)\log\frac{b}{a}\right]$.

These equations of deflection were solved using the digital computor to determine the deflections at various points on the plate. These positions were selected in a manner such that the boundary and continuity conditions could be checked along with the general deflection pattern.

The plate considered was made of steel and of the dimensions,

$a = 2.5$ inches,

$b = 5.0$ inches,

$B = 8.274 \times 10^2$,

and $D = 4.285 \times 10^4$.

This plate was heated on the inner region by the temperature function (5.6) with a rate of heating expressed by equation (5.7). The deflections derived from the above equations are shown in figure 5.1. The computer program used to develop these deflections is given in Appendix II.

The equations of deflection (4.13) given above were obtained from the complex potentials (4.11) and (4.12). These complex potentials are analytic in the region involved and conform to the restriction (3.21) and thus fulfill all of the necessary conditions established previously.

Elliptic Plate Loaded over a Circular Region

The equations of deflection of an elliptic plate loaded over a circular region are given by the substitution of the complex potentials (4.20) into the deflection formulas (3.36) and (3.37). These equations are

$$w_2 = \frac{Bc^2}{16\,D}\Bigg[-(c^2 + 2x^2 + 2y^2)\log\frac{\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 - mn}}{m} + \left(\frac{d_1}{m} + \frac{d_2}{n}\right)$$

$$(x^2 + y^2) + \left(\frac{d_1}{m} + \frac{d_2}{n}\right)\sqrt{x^4 + 2x^2y^2 + y^4 - mn(x^2 - y^2)} + c^2 g_0 + \frac{c^2}{n^2}\, g_1$$

$$(2x^2 - 2y^2 - mn - 2\sqrt{x^4 - 6x^2y^2 + y^4 - mn(x^2 - y^2)}\Bigg] \qquad (5.8a)$$

and $$w_1 = \frac{Bc^2}{16\,D}\Bigg[-(c^2 + 2x^2 + 2y^2)\log\frac{c}{M}\left(1 + \sqrt{1 - \frac{mn(x^2 + y^2)}{x^4 + 2x^2y^2 + y^4}}\right)$$

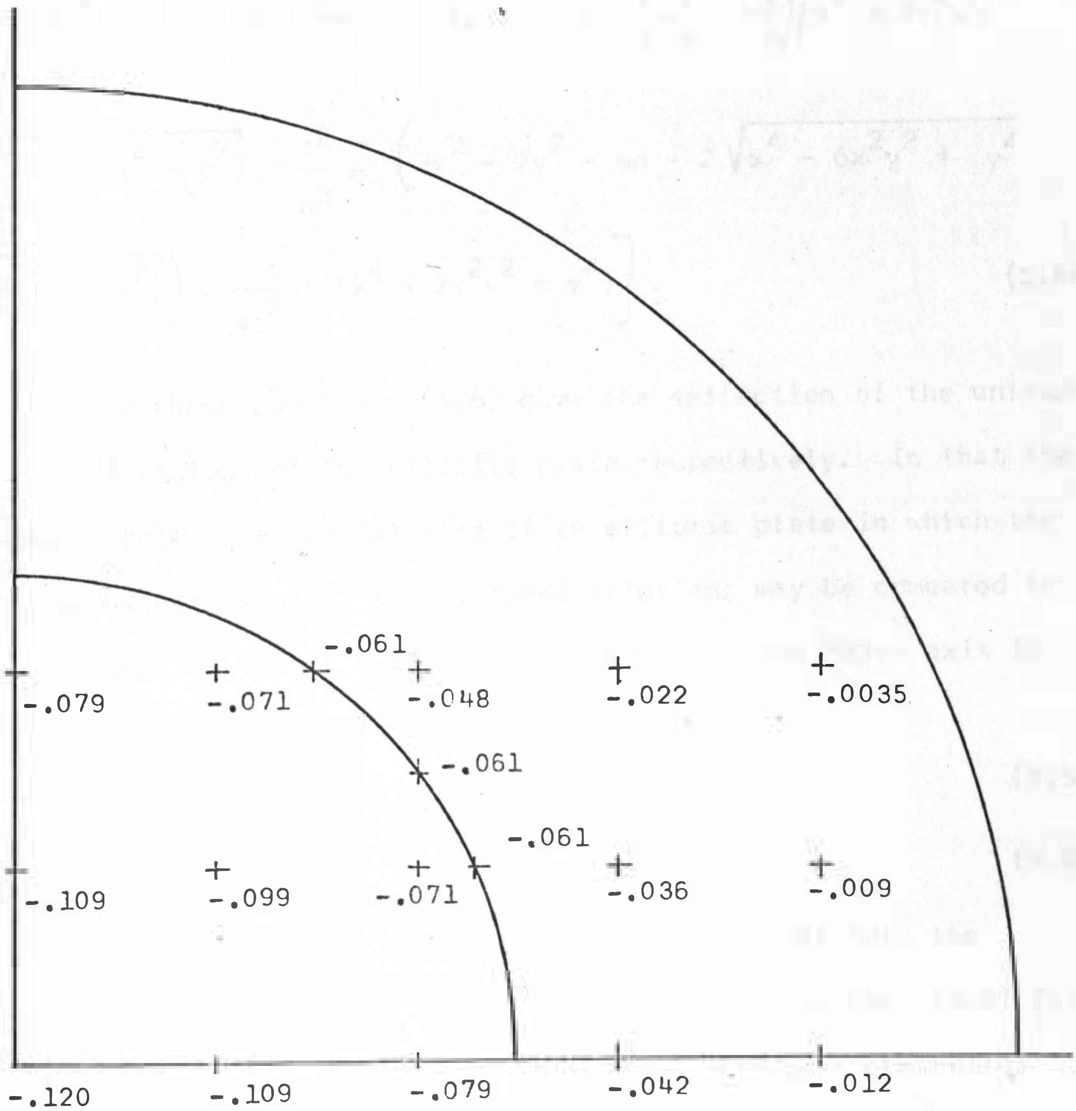

Figure 5.1. Deflection of a Round Plate

$$+\left(g_o + \frac{5}{4}\right) c^2 + \left(\frac{d_1}{m} + \frac{d_2}{n} - 1\right) (x^2 + y^2) + \left(\frac{d_1}{m} - \frac{d_2}{n}\right)\sqrt{x^4 + 2x^2y^2 + y^4 - mn(x^2 - y^2)} + \frac{c^2}{n^2} g_1 \left(2x^2 - 2y^2 - mn - 2\sqrt{x^4 - 6x^2y^2 + y^4 - mn(x^2 - y^2)}\right) - \frac{1}{4c^2} (x^4 + 2x^2y^2 + y^4)\Bigg] . \quad (5.8b)$$

The above equations (5.8) give the deflection of the unloaded and loaded regions of the elliptic plate respectively. In that the circular plate is a special case of an elliptic plate in which the major and minor axis are equal, these solutions may be compared to the equations (4.13) of a circular plate. When the major axis is equal to the minor axis, then a equals b, and

$$m = 2a \quad (5.9a)$$

$$n = 0 \quad (5.9b)$$

from equations (4.15). Substituting equations (5.9) into the equations (4.19) one developes the constants of equations (5.8) for the particular case of a circular plate loaded over a concentric circle; i.e.,

$$g_1 = 0, \quad (5.10a)$$

$$d_2 = 0, \quad (5.10b)$$

$$d_1 = a + \frac{c^2}{2a} \quad (5.10c)$$

and $g_o = -\frac{a^2}{c^2} - \frac{1}{2}$. (5.10d)

The substitution of (5.9) and (5.10) into the deflection equations (5.8) gives

$$w_2 = \frac{Bc^2}{32\,D}\left[-\,2(c^2 + 2r^2)\log\frac{r}{a} + 2r^2 + \frac{c^2 r^2}{2a^2} - 2a^2 - c^2\right] \qquad (5.11a)$$

and $$w_1 = \frac{Bc^2}{16\,D}\left[-\,(c^2 + 2r^2)\log\frac{c}{a} - a^2 + \frac{3}{4}c^2 + \frac{c^2 r^2}{2a^2} - \frac{r^4}{4c^2}\right]. \qquad (5.11b)$$

where $r^2 = x^2 + y^2$.

The equations (5.11) compare exactly with the equations (4.13) which were derived in Chapter IV. Thus, the particular case of the elliptic plate deflections described above satisfies identically the deflections of a circular plate similiarly loaded. This check indicates that the deflection formulas (5.8) are correct when the outer ellipse is reduced to a circle.

The equations (5.8), because of the large number of radicals, are difficult to solve on the digital computor so these equations are transformed back to the ζ plane where only a few radicals appear in the formula. These equations are written in terms of u and v where $\zeta = u + iv = \quad {}^{i0}$; i.e.,

$$w_2 = \frac{Bc^2}{32\,D}\left\{-\left[m^2(u^2 + v^2) + 2mn\left(\frac{u^2 - v^2}{u^2 + v^2}\right) + \frac{n^2}{u^2 + v^2} + 2c^2\right]\log\sqrt{u^2 + v^2}\right.$$

$$+ md_1\,(u^2 + v^2) + (md_2 + nd_1)\left(\frac{u^2 - v^2}{u^2 + v^2}\right) + \frac{nd_2}{u^2 + v^2} + 2g_0c^2$$

$$\left. + 2g_1c^2\left(\frac{u^2 - v^2}{u^4 + 2u^2v^2 + v^4}\right)\right\} \qquad (5.12a)$$

and
$$w_1 = \frac{Bc^2}{32\ D}\left\{\left[m^2(u^2 + v^2) + 2mn\left(\frac{u^2 - v^2}{u^2 + v^2}\right) + \frac{n^2}{u^2 + v^2} + 2c^2\right]\right.$$

$$\log\sqrt{\left(\frac{m}{2c} \quad \frac{n}{2c^2} \operatorname{Cos} 2\phi\right)^2 + \frac{n^2}{4c^2\rho^4}\sin^2 2\phi} + \left(d_1 - \frac{m}{2}\right)\left[m(u^2 + v^2)\right.$$

$$\left. + n\left(\frac{u^2 - v^2}{u^2 + v^2}\right)\right] + \left(d_2 - \frac{n}{2}\right)\left[m\left(\frac{u^2 - v^2}{u^2 + v^2}\right) + \frac{n}{(u^2 + v^2)}\right] + 2.5c^2$$

$$+ 2g_oc^2 + 2g_oc^2\left(\frac{u^2 - v^2}{u^4 + 2u^2v^2 + v^4}\right) - \left(\frac{1}{2c^2}\right)\left[\frac{m^4}{16}(u^4 + 2u^2v^2 + v^4)\right.$$

$$+ \frac{m^3n}{4}(u^2 - v^2) + \frac{m^2n^2}{8}\left(\frac{u^4 - 6u^2v^2 + v^4}{u^4 + 2u^2v^2 + v^4}\right) + \frac{m^2n^2}{4} + \left(\frac{mn^3}{4}\right)$$

$$\left.\left.\left(\frac{u^2 - v^2}{u^4 + 2u^2v^2 + v^4}\right) + \frac{n^4}{16\ (u^4 + 2u^2v^2 + v^4)}\right]\right\} \qquad (5.12b)$$

Examining the logarithm term of equation (5.12b) one sees that at the point $\rho = 0$ the deflection becomes infinite. This singularity in the deflection is due to the singularity in the complex potentials (4.20c and d). This singularity is also evident in the complex potentials (4.20a and b) but does not affect them in that they are only valid in the outer region, thus analytic over the region involved. The complex potentials $\varphi_1(z)$ and $\chi_1(z)$ are, however, not analytic at the origin thus invalidating their solution. Upon examination it is seen that this singularity is introduced in the transformation $z = m\zeta + n\zeta^{-1}$ which is not conformal at the origin.

The solution presented here is an attempt at using a non-conformal transformation to solve this type of problem. This procedure is not invalid as shown by Bassali and Hanna [3]; however, it does present difficulties. The theory presented in this work makes no mention of the requirement that the transformation must be conformal. It does, however, state that the complex potentials must be analytic which is difficult but not impossible to obtain using nonconformal transformations. The procedure used is to select the proper form of the complex potentials in the outer region so that combining them with the solutions to the continuity equations presents analytic complex potentials in the inner region. This type of analysis was attempted; however, an exact function could not be formulated. The solution presented here is thus only valid for one particular case of the elliptic plate and that is when it is circular.

Equations (5.12) were solved using the temperature function (5.7) applied to a steel plate whose major axis is six inches and whose minor axis is three inches. Thus,

$h = .25$ inches,

$a = 6.$ inches,

$b = 3.$ inches,

$c = 1.5$ inches,

$B = 8.274 \times 10^2$,

and $D = 4.285 \times 10^4$.

The deflections were obtained by use of the digital computer. The results are given in figure 6.5 of Chapter VI where the pairs of numbers indicate the plate deflection at that point with the top number being the approximate solution and the bottom number the results of equations (5.12). The computer program used is given in the Appendix II.

Round Cornered Square Plate Loaded over a Circular Region

The equations of deflection of a round cornered square plate are derived by the substitution of the complex potentials (4.25) and the constants (4.24) into the deflection formulas (3.36) and (3.37); i.e.,

$$w_2 = \frac{Bc^2}{16\ D}\Big\{\Big[c^2 - 2L^2(u^2+v^2) - 4\lambda L^2(u^6 - 5u^4v^2 - 5u^2v^4 + v^6)$$

$$- 2L^2\lambda^2(u^{10} + 5u^8v^2 + 10u^6v^4 + 10u^4v^6 + 5u^2v^8 + v^{10})\Big] \log\sqrt{u^2+v^2}$$

$$+ c^2g_0 + 2Ld_1\ (u^2+v^2) + (2Ld_5 + 2L\lambda d_1)(u^6 - 5u^4v^2 - 5u^2v^4 + v^6)$$

$$+ 2L\lambda d_5(u^{10} + 5u^8v^2 + 10u^6v^4 + 10u^4v^6 + 5u^2v^8 + v^{10})$$

$$+ c^2g_4(u^4 - 6u^2v^2 + v^4)\Big\}\ , \tag{5.13a}$$

and

$$w_1 = \frac{Bc^2}{16\ D}\Big\{\Big[c^2 + 2L^2\ (u^2+v^2) + 4L^2\lambda(u^6 - 5u^4v^2 - 5u^2v^4 + v^6)$$

$$+ 2L^2\lambda^2\ (u^{10} + 5u^8v^2 + 10u^6v^4 + 10u^4v^6 + 5u^2v^8 + v^{10})\Big] \log$$

$$\left(\frac{L+L\lambda\ (u^4 + 2u^2v^2 + v^4)}{c}\right) + \left(\frac{5}{4} + g_0\right)c^2 + (2d_1\ L - L^2)(u^2+v^2)$$

$$+ (2L\lambda d_1 + 2d_5 - 2L^2\lambda)(u^6 - 5u^4v^2 - 5u^2v^4 + v^6) + (L^2\lambda^2 + 2L\lambda d_5)$$

$$(u^{10} + 5u^8v^2 + 10u^6v^4 + 10u^4\ v^6 + 5u^2v^8 + v^{10}) - \frac{L^4}{4c^2}\ (u^4 + 2u^2v^2 + v^4)$$

$$- \frac{L^4\lambda}{c^2}\ (u^8 - 4u^6v^2 - 10u^4v^4 - 4u^2v^6 + v^8) - \frac{L^4\lambda^2}{2c^2}\ (u^{12} - 26u^{10}v^2$$

$$+ 15u^8v^4 + 84u^6v^6 + 15u^4v^8 - 26u^2v^{10} + v^{12}) - \frac{L^4\lambda^2}{c^2}(u^{12} + 6u^{10}v^2$$

$$+ 15u^8v^4 + 20u^6v^6 + 15u^4v^8 + 6u^2v^{10} + v^{12}) - \frac{L^4\lambda^3}{c^2}(u^{16} - 20u^{12}v^4$$

$$- 64u^{10}v^6 - 90u^8v^8 - 64u^6v^{10} - 20u^4v^{12} + v^{16}) - \frac{L^4\lambda^4}{4c^2}(u^{20} + 10u^{18}v^2$$

$$+ 45u^{16}v^4 + 120u^{14}v^6 + 210u^{12}v^8 + 252u^{10}v^{10} + 210u^6v^{14} + 45u^4v^{16}$$

$$+ 10u^2v^{18} + v^{20}) + c^2g_4(u^4 - 6u^2v^2 + v^4)\Big\} . \qquad (5.13b)$$

These equations are from necessity written in terms of ζ where $\zeta = u + iv = \rho e^{i\phi}$. This is due to the fact that the transformation $z = L\zeta + L\lambda\zeta^5$ which maps the outer boundary of the plate in the z plane onto the unit circle in the ζ plane does not have an inverse with which to transform the equations back to the z plane. Thus, the deflection equations will be written in terms of u and v and deflections of various points in the ζ plane will be established. These particular points will be determined as follows:

From the transformation

$$z = L\zeta + L\lambda\zeta^5$$

it is seen that

$$x + iy = L(u + iv) + \lambda L(u^5 + i5u^4v - i10u^2v^3 - 10u^3v^2 + 5uv^4 + iv^5).$$

Equating the real and imaginary parts gives

$$x = Lu + \lambda Lu^5 - 10\lambda Lu^3v^2 + 5\lambda Luv^4 \qquad (5.14a)$$

and $y = Lv + 5\lambda Lu^4 v - 10\lambda Lu^2 v^3 + \lambda Lv^5$. (5.14b)

In that u and v must be equal to or less than one and the term λ is small with respect to L equations (5.14) may be approximated by

$$u_1 \doteq \frac{x}{L} , \tag{5.15a}$$

and $$v_1 \doteq \frac{y}{L} . \tag{5.15b}$$

The second approximation is developed by solving equations (5.14) for u and v as follows:

$$u_2 \doteq \frac{x}{L} - \lambda u_1^5 + 10\lambda u_1^3 v_1^2 - 5\lambda u_1 v_1^4 , \tag{5.16a}$$

and $$v_2 \doteq \frac{y}{L} - 5\lambda u_1^4 v_1 + 10\lambda u_1^2 v_1^3 - xv_1^5 , \tag{5.16b}$$

and substituting into these equations the solutions (5.15). Now u_2 and v_2 may be again substituted into equations (5.16) for a further approximation.

Equations (5.13) were programmed and solved for a ten inch square steel plate thermally loaded over circle of radius 2.5 inches. Thus, the terms of equations (5.13) are

$B = 8.274 \times 10^2$,

$h = .25$ inches,

$$D = \frac{E\,h^3}{12\,(1-\nu^2)} = 4.284 \times 10^4 ,$$

$c = 2.5$ inches,

and $b = 10$ inches.

These, when substituted into equations (5.13), yield the solutions shown in figure 6.6. The computor program used is given in Appendix II.

CHAPTER VI

APPROXIMATE SOLUTIONS TO THE PLATE PROBLEMS

This chapter will present the solutions of the elliptic plate and round cornered square plate by the finite-difference technique. These solutions will then be compared to the solutions given in Chapter V.

Consider the two dimensional curve shown in figure 6.1 in which the absissa and ordinates of points on the curve are given by x_i, y_i; x_{i+1}, y_{i+1}; etc. Let a constant λ be defined such that

$$\lambda = x_i - x_{i-1} = x_{i+1} - x_i = \text{etc.} \tag{6.1}$$

As the function $y = f\,(x)$ is assumed continuous one may expand it at the point (x_i, y_i) by the Taylor's series

$$y_{i-1} = y_i + y'_i \frac{(x-x_i)}{1!} + y''_i \frac{(x-x_i)^2}{2!} + \ldots, \tag{6.2}$$

where the prime indicates differentiation. Equation (6.2) can be written

$$y_{i-1} = y_i + \frac{y'_i(-\lambda)}{1!} + \frac{y''_i(-\lambda)^2}{2!} + \frac{y'''_i(-\lambda)^3}{3!} + \ldots \tag{6.3}$$

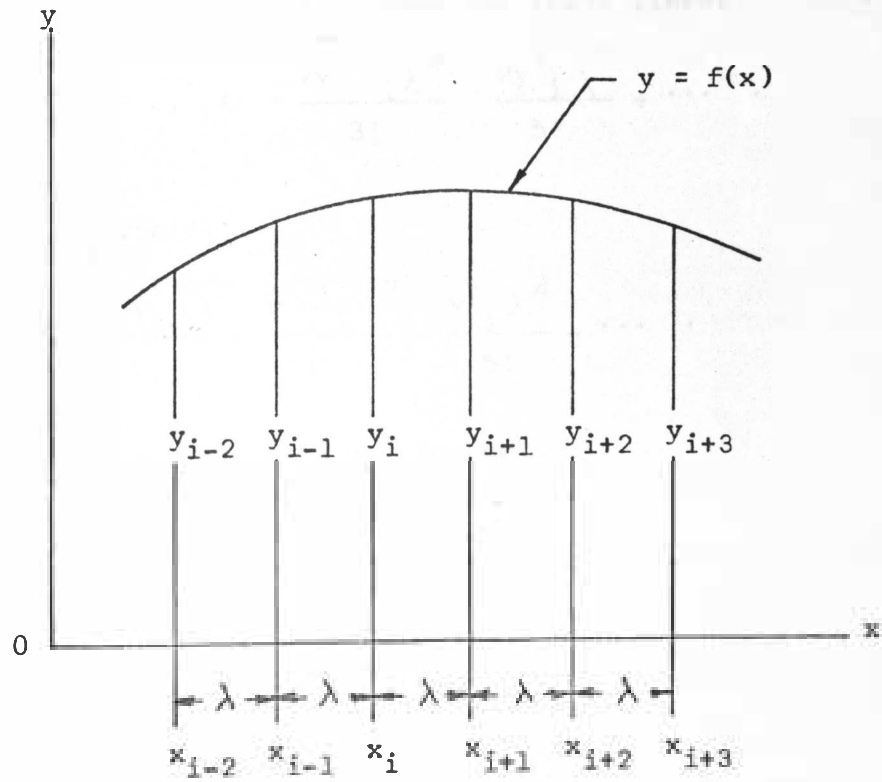

Figure 6.1. Orientation of Finite-Difference System in Two Coordinates

Similiarly one gets

$$y_{i+1} = y_i + \frac{y'_i}{1!} + \frac{y''_i\lambda^2}{2!} + \frac{y'''_i\lambda^3}{3!} + \cdots \qquad (6.4)$$

Subtracting equation (6.3) from equation (6.4) yields

$$y_{i+1} - y_{i-1} = 2y'_i\lambda + \frac{2y'''_i\lambda^3}{3!} \quad \frac{2y^V_i\lambda^5}{5!} + \cdots \; .$$

Solving for y'_i yields

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2\lambda} - \frac{y'''_i{}^2}{3!} - \frac{y^V_i\lambda^4}{5!} - \cdots \; .$$

Neglecting the higher order terms one gets

$$y'_i \doteq \frac{y_{i+1} - y_{i-1}}{2\lambda} \; . \qquad (6.5)$$

Similiarly adding equations (6.3) and (6.4) gives

$$y'_i \doteq \frac{y_{i+1} - 2y_i + y_{i-1}}{\lambda^2} \; . \qquad (6.6)$$

For derivatives of higher order (differentiate) equation (6.5) on both sides, which gives

$$y''_i \doteq \frac{y'_{i+1} - y'_{i-1}}{2\lambda}$$

with the help of equation (6.5) this can be expressed as

$$y''_i \doteq \frac{(y_{i+2}-y\ )-(y_i-y_{i-2})}{(2\lambda)\qquad\qquad(2\lambda)}$$

which yields $y''_i \doteq \dfrac{y_{i+2}-2y_i+y_{i-2}}{4\lambda^2}$. (6.7)

Similiarly, $y'''_i \doteq \dfrac{y_{i+2}-2y_{i+1}+2y_{i-1}+y_{i-2}}{2\lambda^3}$

and $y^{IV}_i \doteq \dfrac{y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}}{\lambda^4}$. (6.8)

The differential equation defining the deflection of a plate in the region which is thermally loaded is

$$\nabla^4 w = \frac{-B}{D}$$

or $\dfrac{\partial^4 w}{\partial x^4} + 2\dfrac{\partial^4 w}{\partial x^2\,\partial y^2} + \dfrac{\partial^4 w}{\partial y^4} = \dfrac{-B}{D}$. (6.9)

This differential equation is two dimensional so a portion of the x y plane is subdivided into a grid-work composed of λ by λ squares as shown in figure 6.2. Select an intersection point (x_i, y_j) of the grid. Using this as a reference point one may write first derivatives on the basis of equation (6.5) where $w_{i+1,j}$ is the deflection of the point (x_{i+1}, y_j). Similar meaning is given to the other symbols as is clear from figure 6.2. Thus

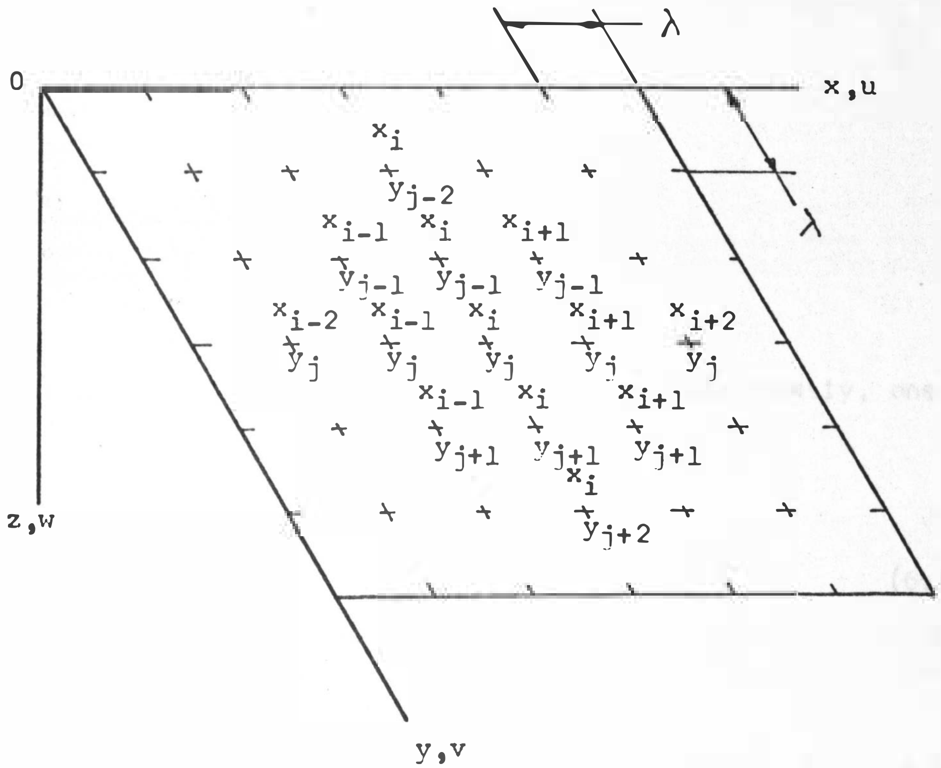

Figure 6.2. Orientation of Finite-Difference System in Three Coordinates

$$\left(\frac{\partial w}{\partial x}\right)_{i,j} \doteq \frac{w_{i+1,j} - w_{i-1,j}}{2\lambda}$$

at the point (x_i, y_j)

Similiarly, $\partial w/\partial y$ at the point (x_i, y_j) is

$$\left(\frac{\partial w}{\partial y}\right)_{i,j} \doteq \frac{w_{i,j+1} - w_{i,j-1}}{2\lambda}$$

Determining the derivatives of higher order as shown previously, one obtains

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{i,j} \doteq \frac{w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}}{\lambda^4} \tag{6.10}$$

and $$\left(\frac{\partial^4 w}{\partial y^4}\right)_{i,j} \doteq \frac{w_{i,j+2} - 4w_{i,j+1} + 6w_{i,j} - 4w_{i,j-1} + w_{i,j-2}}{\lambda^4} . \tag{6.11}$$

In order to evaluate the $\partial^2 w/\partial x\,\partial y$ at (x_i, y_j) it is necessary to write the equation in the form

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{i,j} = \frac{\partial\left(\frac{\partial w}{\partial x}\right)_{i,j}}{\partial y} \doteq \frac{\left(\frac{\partial w}{\partial x}\right)_{i,j+1} - \left(\frac{\partial w}{\partial x}\right)_{i,j-1}}{2\lambda}$$

which yields

$$\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_{i,j} \doteq \frac{(w_{i+1,j+1} - w_{i-1,j+1}) - (w_{i+1,j-1} - w_{i-1,j-1})}{(2\lambda)\ (2\lambda)}$$

or $$\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_{i,j} \doteq \frac{w_{i+1,j+1} - w_{i-1,j+1} - w_{i+1,j-1} + w_{i-1,j-1}}{4\lambda^2}$$

Similiarly, $$\left(\frac{\partial^4 w}{\partial x^2\,\partial y^2}\right)_{i,j} \doteq \frac{\partial^2 \left(\frac{\partial^2 w}{\partial x^2}\right)_{i,j}}{\partial y^2}$$

which yields

$$\left(\frac{\partial^4 w}{\partial x^2\,\partial y^2}\right)_{i,j} = \frac{w_{i+1,j+1} - 2w_{i+1,j} + w_{i+1,j-1} - 2(w_{i,j-1} - 2w_{i,j}}{\lambda^4}$$

$$\frac{+ w_{i,j+1})}{} + \frac{w_{i-1,j-1} - 2w_{i-1,j} + w_{i-1,j+1}}{\lambda^4} \qquad (6.12)$$

By substituting equations (6.10), (6.11) and (6.12) into equation (6.9) the differential equation defining the deflection of point (x_i, y_j) may be expressed approximately in terms of the deflections of the adjacent intersections and the constant distance λ. If the deflection of each grid point on the plate is similiarly expressed, one would have n equations with which to determine the deflection at the n intersections, thus effecting a solution. The finite-difference equation at a point (x_i, y_j) is expressed as a molecule which is a physical representation of multiplication factors with which the deflection at the intersections neighboring this point influence the deflection equation written for this point. This molecule is written as

$$w_{i,j} = \frac{\lambda^4 q_{i,j}}{D_{i,j}} \quad . \tag{6.13}$$

The terms $q_{i,j}$ and $D_{i,j}$ are the transverse pressure and flexural rigidity at the point (x_i, y_j). $w_{i,j}$ is the deflection at (x_i, y_j) whose multiplication factor is 20 from the molecule (6.13). The grid spacing will be established such that the constant λ is unity.

When applying this molecule to an intersection near the outer boundary of the plate it becomes necessary to establish points outside the physical boundaries of the plate. In order to develop this mathematical model we must apply the boundary conditions of a clamped plate. We know that $\partial w / \partial n = 0$ on the boundary. Thus, assuming a vertical edge as the boundary, this becomes

$$\left(\frac{\partial w}{\partial x}\right)_{i,j} = 0 = \frac{w_{i+1,j} - w_{i-1,j}}{2\lambda} . \tag{6.14}$$

Solving for the deflections yields

$$w_{i+1,j} = w_{i-1,j} \tag{6.15}$$

where the point (x_i, y_j) is on the boundary. Thus, the boundary conditions are satisfied if the deflection of the intersection outside the plate boundary is equal to that of the intersection inside the plate boundary. That is, a mirror image exists at a clamped plate boundary.

Consider the plates shown in figures 6.3 and 6.4. The deflection of the elliptic plate in figure 6.3 is symetrical about the x and y axes. Using this symmetry, the deflections at intersections in three quadrants can be given in terms of the deflections established in the first quadrant. The round cornered square has four axes of symmetry. This means that the deflection throughout the plate can be given in terms of the deflection of a half quadrant.

When the molecule (6.13) is applied to each of the fifteen grid intersections shown in figure 6.3, fifteen simultaneous independent linear algebraic equations are developed in terms of the fifteen unknown deflections. These equations are:

1 $\quad 21w_1 - 8w_2 + w_3 + 4w_8 = 0$ (6.16a)

2 $\quad -8w_1 + 20w_2 - 8w_3 + w_4 - 16w_8 + 4w_9 = 0$ (6.16b)

3 $\quad w_1 - 8w_2 + 20w_3 - 8w_4 + w_5 + 4w_8 - 16w_9 + 4w_{10} + 2w_7 = 0$ (6.16c)

4 $\quad w_2 - 8w_3 + 20w_4 - 8w_5 + w_6 + 4w_9 - 16w_{10} + 4w_{11} + 2w_{13} = 0$ (6.16d)

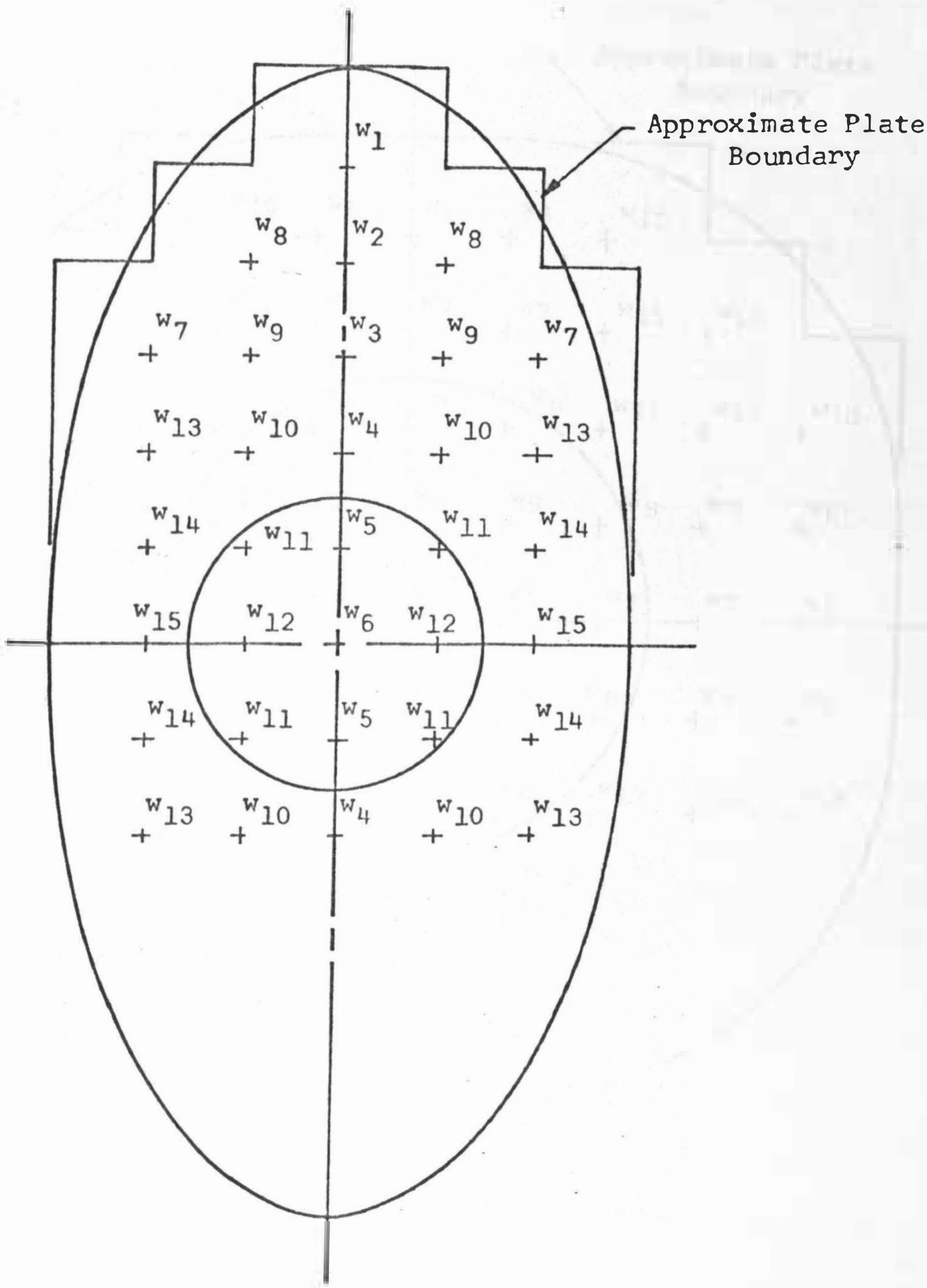

Figure 6.3. Elliptic Plate Laid out for Finite-Difference Solution

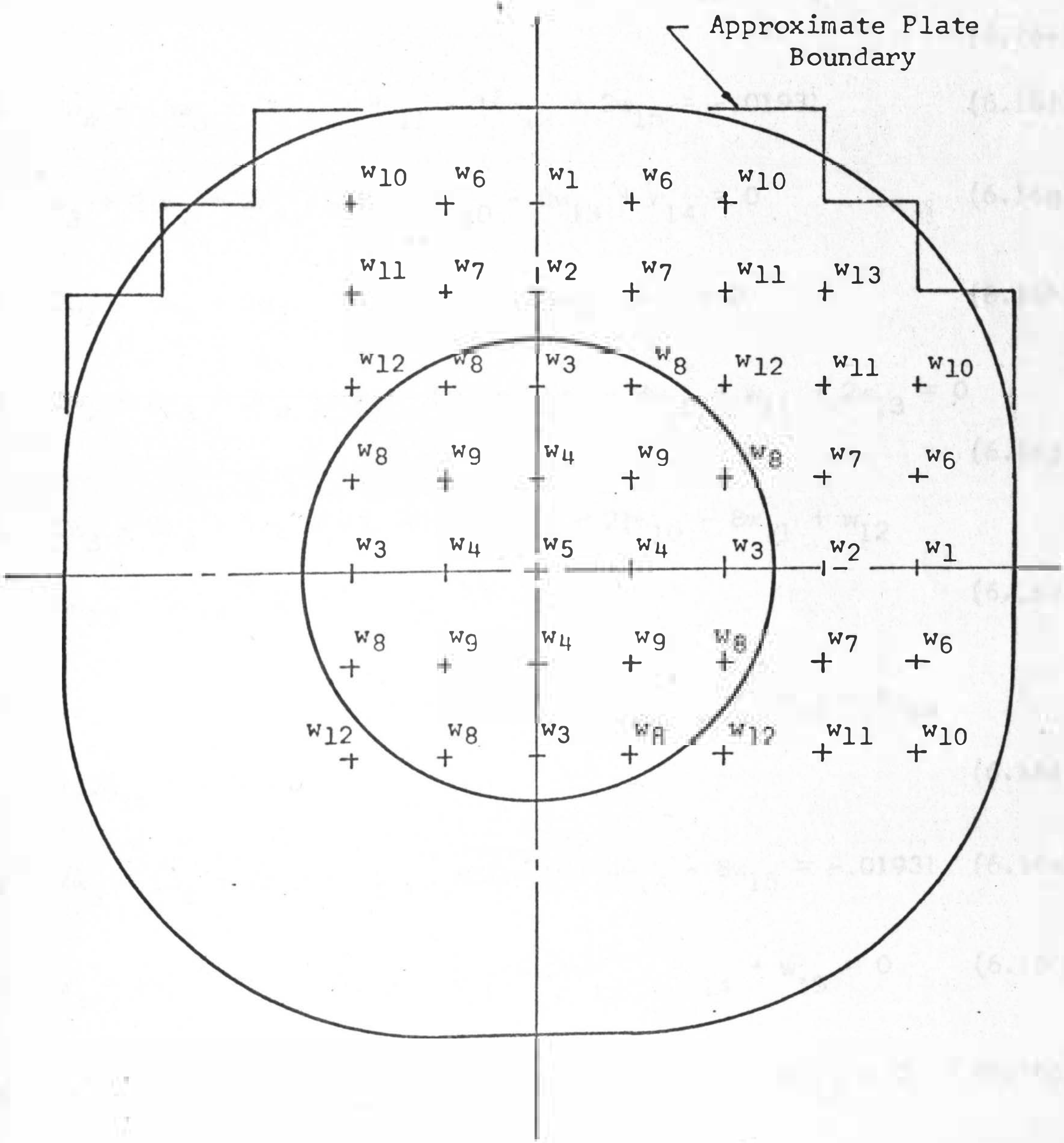

Figure 6.4. Round Cornered Square Plate Laid out for Finite-Difference Solution

5 $w_3 - 8w_4 + 21w_5 - 8w_6 + 4w_{10} - 16w_{11} + 4w_{12} + 2w_{14} = -.01931$ (6.16e)

6 $2w_4 - 16w_5 + 20w_6 + 8w_{11} - 16w_{12} + 2w_{15} = -.01931$ (6.16f)

7 $w_3 + 21w_7 + 2w_8 - 8w_9 + 2w_{10} - 8w_{13} + w_{14} = 0$ (6.16g)

8 $2w_1 - 8w_2 + 2w_3 - 2w_7 + 21w_8 - 8w_9 + w_{10} = 0$ (6.16h)

9 $2w_2 - 8w_3 + 2w_4 - 8w_7 - 8w_8 + 21w_9 - 8w_{10} + w_{11} + 2w_{13} = 0$ (6.16j)

10 $2w_3 - 8w_4 + 2w_5 + 2w_7 + w_8 - 8w_9 + 21w_{10} - 8w_{11} + w_{12}$

$8w_{13} + 2w_{14} = 0$ (6.16k)

11 $2w_4 - 8w_5 + 2w_6 + w_9 - 8w_{10} + 22w_{11} - 8w_{12} + 2w_{13} - 8w_{14}$

$+ 2w_{15} = -.01931$ (6.16l)

12 $4w_5 - 8w_6 + 2w_{10} - 16w_{11} + 21w_{12} + 4w_{14} - 8w_{15} = -.01931$ (6.16m)

13 $w_4 - 8w_7 + 2w_9 - 8w_{10} + 2w_{11} + 21w_{13} - 8w_{14} + w_{15} = 0$ (6.16n)

14 $w_5 + w_7 + 2w_{10} - 8w_{11} + 2w_{12} - 8w_{13} + 22w_{14} - 8w_{15} = 0$ (6.16p)

15 $w_6 + 4w_{11} - 8w_{12} + 2w_{13} - 16w_{14} + 21w_{15} = 0$ (6.16q)

Applying the molecule (6.13) to the thirteen intersections of the round cornered square plate of figure 6.4 yields the following thirteen equations:

1 $$21w_1 - 8w_2 + w_3 - 16w_6 + 4w_7 + 2w_{10} = 0 \qquad (6.17a)$$

2 $$-8w_1 + 20w_2 - 8w_3 + w_4 + 4w_6 - 16w_7 + 4w_8 + 2w_{11} = 0 \qquad (6.17b)$$

3 $$w - 8w + 25w - 8w + w + 4w - 16w + 4w + 2w = -.01931 \qquad (6.17c)$$

4 $$w_2 - 8w_3 + 25w_4 - 8w_5 + 6w_8 - 16w_9 = -.01931 \qquad (6.17d)$$

5 $$4w_3 - 32w_4 + 20w_5 + 8w_9 = -.01931 \qquad (6.17e)$$

6 $$-8w_1 + 2w_2 + 22w_6 - 8w_7 + w_8 - 8w_{10} + 2w_{11} = 0 \qquad (6.17f)$$

7 $$2w_1 - 8w_2 + 2w_3 - 8w_6 + 21w_7 - 8w_8 + w_9 + 2w_{10} - 8w_{11} + 2w_{12} + w_{13} = 0 \qquad (6.17g)$$

8 $$2w_2 - 8w_3 + 3w_4 + w_6 - 8w_7 + 23w_8 - 8w_9 + 3w_{11} - 8w_{12} = .01931 \qquad (6.17h)$$

9 $$4w_3 - 16w_4 + 2w_5 + 2w_7 - 16w_8 + 22w_9 + 2w_{12} = -.01931 \qquad (6.17j)$$

10 $$w_1 - 8w_6 + 2w_7 + 21w_{10} - 8w_{11} + w_{12} + 2w_{13} = 0 \qquad (6.17k)$$

11 $$w_2 + 2w_6 - 8w_7 + 3w_8 - 8w_{10} + 22w_{11} - 8w_{12} - 8w_{13} = 0 \qquad (6.17l)$$

12 $2w_3 + 4w_7 - 16w_8 + 2w_9 + 2w_{10} - 16w_{11} + 20w_{12} + 2w_{13} = 0$ (6.17m)

13 $2w_7 + 4w_{10} - 16w_{11} + 2w_{12} + 20w_{13} = 0$

The equations (6.16) and (6.17) were solved by matrix inversion. This process requires a large core space in the digital computor which limits the number of equations. The solutions of these equations are given in figures 6.5 and 6.6 where the upper number is the finite-difference solution and the lower number is the solution given by the complex variable method.

The solution of deflection of the elliptic plate given by the complex variables method in figure 6.5 shows the effect of the singularity in the inner region. The continuity conditions between the two regions are not satisfied and the deflection of the inner region is unrealistic. This singularity is discussed more fully in Chapter V. The plate deflection in the outer region is well behaved and satisfies the boundary conditions nicely.

The solution of deflection of the round cornered square plate matches well at the inner boundary and conforms to the outer boundary conditions. The deflection obtained from the complex variables method also compares favorably with the approximate solution given by the finite-difference method.

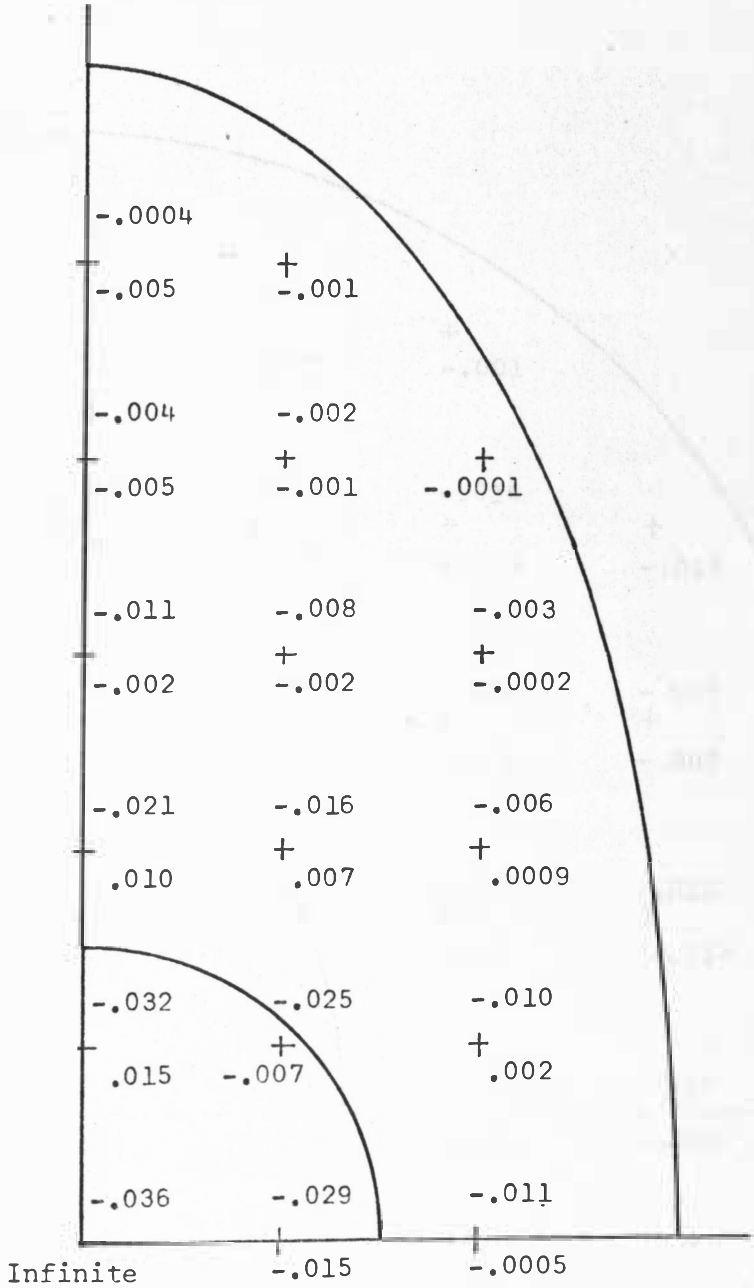

Figure 6.5. Comparison of Results for Elliptic Plate.

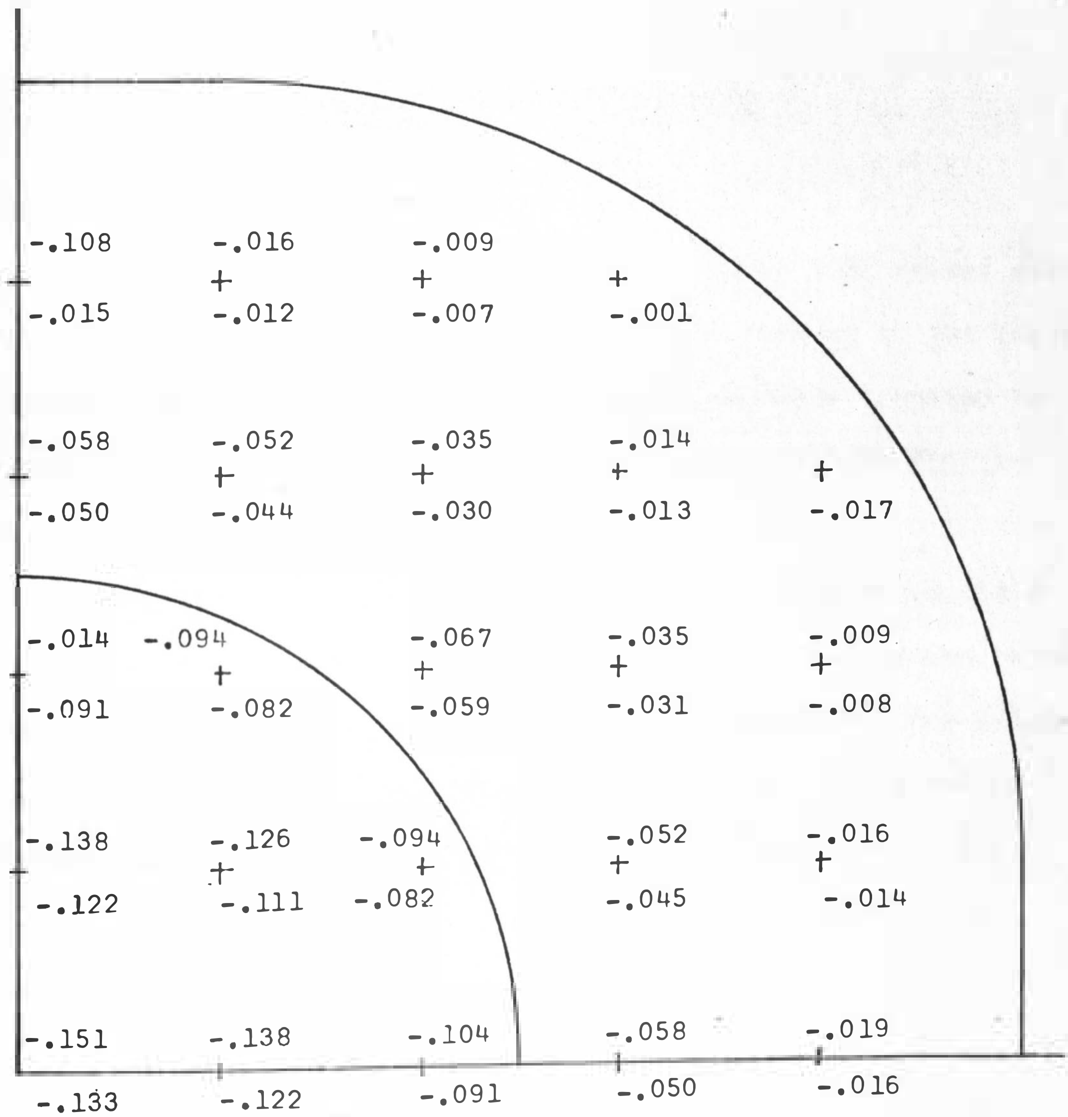

Figure 6.6. Comparison of Results for Round Cornered Square Plate

CHAPTER VII

RESUME AND RECOMMENDATIONS

Resume

The plate problems presented in this thesis were solved using the Muskhelishvili method by applying complex variables to the theory of plates. The solutions obtained in this manner were compared to existing data or to approximate solutions obtained by the finite-difference technique.

The plates presented were deflected by a temperature field which existed over a partial region of the plate. This heated region was in all cases circular with its center coincident with the origin of the coordinate system. The temperature function of the heated region was defined in a manner such that only the effects of the thermal moment contributed to the deflection. By comparing the temperature function of this region to Fouriers general law of heat transfer it was discovered that a steady state temperature field would not produce plate deflection. It was also found that the temperature function should vary linearly with time.

The general outline of the solution was based on the use of continuity equations between the loaded and unloaded regions of the plate to supplement the boundary conditions in satisfying the two differential equations which govern the deflection in the two regions. The plate deflection in the complex plane was expressed in terms of four complex potentials, two in each region, and the boundary conditions were expressed in terms of two of these complex potentials. The solutions of the continuity equations were developed by transforming the inner boundary between the loaded and the unloaded region on to a unit circle in another complex plane. There the change in the complex potentials necessary to insure the continuity of the plate deflection, slope, radii of curvature and shear across the boundary were developed. These changes in the complex potentials from one region to another were then transformed back to the original coordinate system where they were used to assume the forms of the complex potentials in the outer region. Note that the transformation used for the inner boundary must necessarily have an inverse in order to develop the change of the complex potentials in the original coordinate system. This change in the complex potentials across the inner boundary gives an indication of the form of the complex potentials in the outer region. Using this knowledge coupled with the fact that the complex potentials must be analytic in the regions considered and must satisfy the restrictions (3.21) discussed in Chapter three, one assumes a general solution to the complex potentials in the outer region. The assumed solutions consisted of an algebraic polynominal and a trancendental function. These assumed functions were then conformally transformed to a complex plane in which the outer boundary of the plate corresponds to the unit circle. The complex constants in the assumed forms of the complex potentials are then determined using the boundary conditions and equating coefficients of like powers of the single variable σ. The transformation used in this procedure need not have an inverse in that the deflection can be determined in that plane. Once the solution of the complex potentials in the outer region is developed, the change in the complex potentials developed from the continuity equations can be used to develop the complex potentials in the inner region.

The particular type of temperature function used in this thesis corresponds directly with the case of a plate loaded by a uniform transverse pressure. Thus, the problems solved here can be compared to work previously done by researchers working with transverse loads. The solution (4.13a) and (4.13b) developed in this thesis for the round plate loaded over a circular region compare identically with those given by Bassali and Hanna [3].

The solution of the elliptic plate loaded over a circular region, was checked by two methods. The first was to compare the particular case of the problem, that is, when the outer boundary of the ellipse becomes circular, to the solution previously derived. This was done and the comparison proved to be correct. The second check was to compare the deflections developed by the finite-difference technique with those developed by the complex variable methods. Upon examination it was found that the complex variable solution to the inner plate region contained a singularity. This caused the deflection at the origin to become infinite, thus invalid. This singularity was introduced in the mapping function and should be removable as discussed in Chapter V. The solution given by this thesis for the outer, unloaded, region does compare favorably with the finite-difference solution and the complex potentials in that region are analytic.

It should be noted that it was not proven that the transformations used must be conformal. It was seen, however, that conformal transformations do facilitate the procedure.

The round cornered square plate loaded over a circular region was not found in the literature review. The solutions given here were thus compared to the finite-difference solution. This comparison was favorable. The solutions developed matched nicely at the inner boundary and corresponded well to the outer boundary conditions. It was therefore concluded that the solutions developed were acceptable.

Recommendations

On the basis of the preceding analysis the following recommendations are made for further work in this field.

(1) Investigate ways of removing the singularity from the solution of the inner region of the elliptic plate.

(2) Develop a technique which does not require the inverse of the conformal transformation of the inner boundary.

(3) Investigate the possibility of using the assumed forms of $\varphi_2(\zeta)$ and $\chi_2(\zeta)$ as Taylor's series without the use of transendental functions.

(4) Investigate an approximation technique which would combine the Swartz-Christoffel transformation and the technique present in this thesis to solve the deflections of complicated regions for which no conformal transformation exists.

(5) Extend this technique to plates thermally loaded over the outer region and unloaded over the inner region.

(6) Extend this technique to plates in which the thermal loading is more general than in this thesis.

LITERATURE CITED

[1] Bassali, W. A., "Transverse Bending of Thin Circular Plates Loaded Normally Over Eccentric Circle," Proceedings of the Cambridge Philosophical Society, Vol. 52, Pt. 4, pp. 742-749, October, 1956.

[2] Bassali, W. A. and Hanna, N. C. M.; "Bending of Curvilinear and Rectilinear Polygonal Plates Symetrically Loaded Over a Concentric Circle," Proceedings of the Cambridge Philosophical Society, Vol. 57, pp. 166-179, January, 1961.

[3] Bassali, W. A. and Nassif, M., "A Thin Circular Plate Normally and Uniformly Loaded Over a Concentric Elliptic Patch," Proceedings of the Cambridge Philosophical Society, Vol. 55, pp. 101-109, 1959.

[4] Boley, B. A. and Weiner, J. H., Theory of Thermal Stresses, John Wiley and Sons, Inc., New York, N. Y., 1960.

[5] Boresi, A. P., Elasticity in Engineering Mechanics, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965.

[6] Churchill, R. V., Complex Variables and Applications, 2nd Ed., Mc Graw-Hill Book Company, Inc., New York, N. Y., 1960.

[7] Deverall, L. I., "Solution of Some Problems in Bending of Thin Clamped Plates by Means of the Method of Muskhelishvili," Journal of Applied Mechanics, Vol. 24, pp. 295-298, 1957.

[8] Flugge, W., Handbook of Engineering Mechanics, Mc Graw-Hill Book Company, Inc., New York, N. Y., 1962.

[9] Mansfield, E. H., The Bending and Stretching of Plates, Macmillan Co., New York, N. Y., 1964.

[10] Milne-Thomson, L. M., Plane Elastic Systems, Springer - Verlag, Berlin, Gottingen Heidelberg, 1960.

[11] Muskhelishvili, N. I., Some Basic Problems of the Mathematic Theory of Elasticity, 3rd Ed., P. Noordhoff Ltd., Groningen, Holland, 1953.

[12] Novozhilov, V. V., Thin Shell Theory, 2nd Ed., P. Noorhoff Ltd., Groningen, Netherlands, 1964.

[13] Nowacki, W., Thermoelasticity, Pergamon Press, New York, N. Y., 1962.

[14] Rabinovich, I. M., Structural Mechanics in the U.S.S.R., 1917-1957; Pergamon Press, New York, N. Y., 1960.

[15] Savin, G., Stress Concentration Around Holes, Pergamon Press, New York, N. Y., 1961.

[16] Sokolnikoff, I. S., Mathematical Theory of Elasticity, 2nd Ed., Mc Graw-Hill Book Company, Inc., New York, N. Y., 1956.

APPENDIX I

APPENDIX I

CONTINUITY EQUATIONS OF A CIRCULAR REGION

The continuity equations of a circular region are

$$\left[\bar{z}\,\varphi(z) + z\,\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}\right]_2^1 = \frac{Ba^4}{64\ D} \quad , \tag{1}$$

$$\left[\bar{z}\,\varphi'(z) + \overline{\varphi(z)} + \chi'(z)\right]_2^1 = \frac{Ba^2}{32\ D\sigma} \quad , \tag{2}$$

$$\left[\varphi'(z) + \overline{\varphi'(z)}\right]_2^1 = \frac{Ba^2}{16\ D} \quad , \tag{3}$$

and

$$\left[\varphi''(z)\right]_2^1 = \frac{Ba}{16\ D\sigma} \tag{4}$$

where $\zeta = \sigma$ and $z = a\zeta$. From equation (4)

$$\left[\frac{d}{d\sigma}\ \varphi'(z)\right]_2^1 = \frac{Ba^2}{16\ D\sigma} \ .$$ Integrating this equation with respect to σ gives

$$\left[\varphi'(z)\right]_2^1 = \frac{Ba^2}{16\ D}\log\sigma + K_1 \ . \tag{5}$$

From equation (3).

$$\frac{Ba^2}{16\ D}\log\sigma + K_1 + \frac{Ba^2}{16\ D}\log\sigma^{-1} + K_2 = \frac{Ba^2}{16\ D} \ .$$

Thus,

$$K_1 = \frac{Ba^2}{32\ D} + iK_2 \ .$$

From equation (5)

$$\left[\frac{d}{d\sigma}\ \varphi(z)\right]_2^1 = \frac{Ba^3}{16\ D}\log\sigma + \frac{Ba^3}{32\ D} + iaK_2 \ .$$ Integrating gives

$$\left[\varphi(z)\right]_2^1 = \frac{Ba^3}{16\ D}\sigma\log\sigma - \frac{Ba^3}{16\ D}\sigma + \frac{Ba^3}{32\ D}\sigma + K_3 + ia\sigma K_2 \ . \qquad (6)$$

Substituting (5) and (6) into (2) yields

$$\frac{Ba^3}{16\ D}\frac{\log}{\sigma} + \frac{Ba^3}{32D\sigma} + \frac{Ba^3}{16D\sigma}\log\sigma^{-1} - \frac{Ba^3}{32D\sigma} + \bar{K}_3 - ia\sigma K_2 + \frac{iaK_2}{\sigma}$$

$$+ \left[\chi'(z)\right]_2^1 = \frac{Ba^3}{32D\sigma}\ ,\ \text{or}$$

$$\left[\frac{d}{d\sigma}\chi(z)\right]_2^1 = \frac{Ba^4}{32D\sigma} - \frac{Ba^4}{16D\sigma}\log\sigma - \frac{Ba^4}{16D\sigma}\log\sigma^{-1} - aK_3 - ia^2\sigma K_2$$

$$+ \frac{ia^2}{\sigma}K_2 \ .$$

Which when integrated gives

$$\left[\chi(z)\right]_2^1 = \frac{Ba}{32\ D}\log\sigma - a\sigma K_3 - \frac{ia^2\sigma^2}{2}K_2 + ia^2 K_2\log\sigma + K_4 \ . \qquad (7)$$

Substituting (6) and (7) into (1) one finds

$$\frac{Ba^4}{16\ D}\log\sigma - \frac{Ba^4}{16\ D} + \frac{Ba^4}{32\ D} + a\sigma^{-1}K_3 + ia^2K_2 + \frac{Ba^4}{16\ D}\log\sigma^{-1} - \frac{Ba^4}{16\ D} + \frac{Ba^4}{32\ D}$$

$$+ a\sigma\bar{K}_3 - ia^2K_2 + \frac{Ba^4}{32\ D}\log\sigma - a\sigma\bar{K}_3 - \frac{ia^2\sigma^2K_2}{2}$$

$$+ ia^2K_2 \log\sigma + K_4 + \frac{Ba^4}{32\ D} \log\sigma^{-1} - a\sigma^{-1}K_3 + \frac{ia^2\sigma^{2}K_2}{2} - ia^2K_2 \log\sigma$$

$$+ K_4 = \frac{Ba^4}{64\ D} ,$$ or

$$\frac{Ba^4}{8\ D}\left[\log\sigma + \log\sigma^{-1}\right] - \frac{.5Ba^4}{64\ D} + 2\ K_4 = 0.$$

So $K_4 = \frac{5Ba^4}{128}$, and K_2 and K_3 are neglected.

Thus

$$\left[\chi(z)\right]_{\frac{1}{2}} = \frac{Ba^4}{32\ D}\left[\log\frac{z}{a} + \frac{5}{4}\right] \quad (8)$$

and $\left[\varphi(z)\right]_{\frac{1}{2}} = \frac{Ba^2}{16\ D} z \left[\log\frac{z}{a} - \frac{1}{2}\right]$ because $\zeta = \frac{z}{a} = \sigma$. (9)

SOLUTION OF CIRCULAR PLATE CIRCULAR REGION

Assume $$\varphi_2(\zeta) = \frac{Ba^2}{16\ D}\left[d_1\zeta - b\zeta\log\zeta\right] \quad (10)$$

and $$\chi_2(\zeta) = \frac{Ba^4}{32\ D}\left[g_o - \log\zeta\right] . \quad (11)$$

The first boundary condition of a clamped plate is

$$\bar{z}\ \varphi(z) + z\ \overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} = 0 .$$

Substituting into this equations (10) and (11) gives

$$b\sigma^{-1}\left(\frac{Ba^2}{16\ D}\right)(d_1\sigma - b\sigma\ \log\sigma) + b\sigma\left(\frac{Ba^2}{16\ D}\right)(d_1\sigma^{-1} - b\sigma^{-1}$$

$$\log\sigma^{-1}) + \frac{Ba^4}{32\ D}(g_0 - \log\sigma) + \frac{Ba^4}{32\ D}(g_0 - \log\sigma^{-1}) = 0.$$

Which reduces to

$$2bd_1 + a^2 g_0 = 0\ . \qquad (12)$$

The second boundary condition of a clamped plate is

$$\varphi(z) + z\ \overline{\varphi'(z)} + \overline{\chi'(z)} = 0\ .$$

Substituting into this (10) and (11) one finds

$$\left(\frac{Ba^2}{16\ D}\right)(d_1\sigma - b\sigma\log\sigma) + \left(\frac{b\sigma}{b}\right)\left(\frac{Ba^3}{16\ D}\right)(d_1 - b\ \log\sigma^{-1} - b)$$

$$- \frac{Ba^4}{32\ D}\left(\frac{1}{b}\right)\sigma = 0\ ,$$

or $2d_1\sigma - b\sigma - \frac{a^2}{2b}\sigma = 0$.

Equating coefficients gives $2d_1 - b - \frac{a^2}{2b} = 0$. (13)

So (12) and (13) give $d_1 = \frac{b}{2} + \frac{a^2}{4b}$ and $g_0 = -\frac{b^2}{a^2} - \frac{1}{2}$.

Thus (10) and (11) yield

$$\varphi_2(z) = \frac{Ba^2}{16\ D} \left[\frac{z}{2} + \frac{a^2 z}{4b^2} - z \log \frac{z}{b} \right] \tag{14}$$

and $$\chi_2(z) = \frac{Ba^4}{32\ D} \left[-\tfrac{1}{2} - \frac{b^2}{a^2} - \log \frac{z}{b} \right]. \tag{15}$$

Because $$\left[\chi(z)\right]_2^1 = \chi_1(z) - \chi_2(z)$$

and $$\left[\varphi(z)\right]_2^1 = \varphi_1(z) - \varphi_2(z)$$

$$\varphi_1(z) = \frac{Ba^2}{16\ D} \left[z \log \frac{b}{a} + \frac{a^2 z}{4b^2} \right]$$

and $$\chi_1(z) = \frac{Ba^4}{32\ D} \left[\log \frac{b}{a} + \frac{3}{4} - \frac{b^2}{a^2} \right]$$

from (14), (15), (8), and (9).

SOLUTION OF THE ELLIPTIC PLATE CIRCULAR REGION

Assume

$$\varphi_2(\zeta) = \frac{Bc^2}{32D} \left[-(m\zeta + n\zeta^{-1}) \log \zeta + d_1 \zeta + d_2 \zeta^{-1} \right] \tag{16}$$

and $$\chi_2(\zeta) = \frac{Bc^4}{32D} \left[-\log \zeta + g_0 + g_1 \zeta^{-2} \right] \tag{17}$$

because $2z = m\zeta + n\zeta^{-1}$.

$\bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} = 0$ is a boundary condition which with (16) and (17) gives

$$\left(\frac{m}{2}\sigma^{-1} + \frac{n}{2}\sigma\right)\left(\frac{Bc^2}{32D}\right)(-m\sigma\log\sigma - n\sigma^{-1}\log\sigma + d_1\sigma + d_2\sigma^{-1})$$

$$+\left(\frac{m}{2}\sigma + \frac{n}{2}\sigma^{-1}\right)\left(\frac{Bc^2}{32D}\right)(-m\sigma^{-1}\log\sigma^{-1} + n\sigma\log\sigma^{-1} + d_1\sigma^{-1} + d_2\sigma)$$

$$+ \frac{Bc^4}{32\ D}(-\log\sigma + g_o + g_1\sigma^{-2}) + \frac{Bc^4}{32\ D}(-\log\sigma^{-1} + g_o + g_1\sigma^2) = 0\ .$$

Multiplying gives

$$-\frac{m^2}{2}\log\sigma - \frac{mn}{2}\sigma^{-2}\log\sigma - \frac{mn}{2}\sigma^2\log\sigma - \frac{n^2}{2}\log\sigma + \frac{md_1}{2} + \frac{nd_1}{2}\sigma^2$$

$$+\frac{md_2}{2}\sigma^{-2} + \frac{n}{2}d_2 - \frac{m^2}{2}\log\sigma^{-1} - \frac{mn}{2}\sigma^2\log\sigma^{-1} - \frac{mn}{2}\sigma^{-2}\log\sigma^{-1} - \frac{n^2}{2}$$

$$\log\sigma^{-1} + \frac{m}{2}d_1 + \frac{n}{2}d_1\sigma^{-1} + \frac{m}{2}d_2\sigma^{-2} + \frac{m}{2}d_2 - c^2\log\sigma + c^2g_o + c^2g_1\sigma^{-2}$$

$$- c^2\log\sigma^{-1} + c^2g_o + c^2g_1\sigma^2 = 0\ .$$

Equating coefficients of like powers of σ

$$(\sigma^2) \qquad \frac{nd_1}{2} + \frac{md_2}{2} + c^2g_1 = 0 \tag{18}$$

$$(\sigma^0) \qquad md_1 + nd_2 + 2c^2g_o = 0 \tag{19}$$

$$(\sigma^{-2}) \qquad \frac{md_2}{2} + \frac{n}{2}d_1 + c^2g_1 = 0$$

$$\varphi(z) + z\,\overline{\varphi'(z)} + \chi'(z) = 0$$

is a boundary condition which combined with (16) and (17) gives

$$\varphi(\zeta) + \frac{\left(\frac{m}{2}\zeta + \frac{n}{2}\zeta^{-1}\right)\overline{\varphi'(\zeta)}}{\frac{m}{2} - \frac{n}{2}\bar{\zeta}^{-2}} + \frac{\overline{\chi'(\zeta)}}{\frac{m}{2} - \frac{n}{2}\zeta^{-2}} = 0\,. \qquad (20)$$

By dividing

$$\frac{\frac{m}{2}\sigma + \frac{n}{2}\sigma^{-1}}{\frac{m}{2} - \frac{n}{2}\sigma^{2}} = -\frac{m}{n}\sigma^{-1} - \left(1 + \frac{m^2}{n^2}\right)\sigma^{-3} - \left(\frac{m}{n} + \frac{m^3}{n^3}\right)\sigma^{-5} - \cdots\,.$$

Similiarly

$$\frac{2}{m - n\sigma^{2}} = -\frac{2}{n}\sigma^{-2} - \frac{2m}{n^2}\sigma^{-4} - \frac{2m^2}{n^3}\sigma^{-6} - \cdots\,.$$

From (16)

$$\varphi'_2(\zeta) = \frac{Bc^2}{32D}\left[-m\log\sigma - m + n\sigma^{-2}\log\sigma - n\sigma^{-2} + d_1 - d_2\sigma^{-2}\right]$$

so $$\overline{\varphi'_2(\zeta)} = \frac{Bc^2}{32D}\left[(n\sigma^{-2} - m)\log\sigma^{-1} + d_1 - m + (n + d_2)\sigma^{2}\right]\,.$$

Similiarly

$$\chi'_2(\zeta) = \frac{Bc^4}{32D}\left[-\sigma^{-1} - 2g_1\sigma^{-3}\right]$$

and $\overline{\chi'_2(\zeta)} = \frac{Bc^4}{32D}\left[-\sigma - 2g_1\sigma^3\right]$.

Now (20) gives

$$\frac{Bc^2}{32\ D}(-m\sigma\log\sigma - n\sigma^{-1}\log\sigma + d_1\sigma + d_2\sigma^{-1}) + \frac{Bc^2}{32\ D}$$

$$\Big[-\frac{m}{n}(n\sigma^2 - m)\sigma^{-1}\log\sigma^{-1}$$

$$-\frac{m}{n}(d_1 - m)\sigma^{-1} + (n+d_2)\frac{m}{n}\sigma - (n\sigma^2 - m)\sigma^{-3}\log\sigma^{-1} - (d_1 - m)\sigma^{-3}$$

$$+ (n+d_2)\sigma^{-1} - \frac{m^2}{n^2}(n\sigma^2 - m)\sigma^{-3}\log\sigma^{-1} - \frac{m^2}{n^2}(d_1 - m)\sigma^{-3}$$

$$+ (n+d_2)\frac{m^2}{n^2}\sigma^{-1} - \left(\frac{m}{n} + \frac{m^3}{n^3}\right)(n\sigma^2 - m)\sigma^{-5}\log\sigma^{-1} - (d_1 - m)\left(\frac{m}{n} + \frac{m^3}{n^3}\right)\sigma^{-5}$$

$$+\left(\frac{m}{n} + \frac{m^3}{n^3}\right)(n+d_2)\sigma^{-3} + \ldots\Big] + \frac{Bc^4}{32\ D}\left(\frac{2}{n}\sigma^{-1} + \frac{2m}{n^2}\sigma^{-3} + \frac{2m^2}{n^3}\sigma^{-5}\right.$$

$$\left. + \ldots - \frac{4}{n}g_1\sigma - \frac{4m}{n^2}g_1\sigma^{-1} - \frac{4m^2}{n^3}g_1\sigma^{-3} - \ldots\right) = 0 .$$

Which in reduced form is

$$d_1\sigma + d_2\sigma^{-1} - \frac{m}{n}d_1\sigma^{-1} + \frac{m^2}{n}\sigma^{-1} + \left(m + \frac{d_2 m}{n}\right)\sigma - d_1\sigma^{-3} + m\sigma^{-3}$$

$$+ (n+d_2)\sigma^{-1} - \frac{m^2}{n^2}d_1\sigma^{-3} + \frac{m^3}{n^2}\sigma^{-3} + \left(\frac{m^2}{n} + \frac{m^2}{n^2}d_2\right)\sigma^{-1}$$

$$+ \left(-d_1 \frac{m}{n} - d_1 \frac{m^3}{n^3} + \frac{m^2}{n} + \frac{m^4}{n^3}\right)\sigma^{-5} + \left(m + \frac{m^3}{n^2} + \frac{m}{n} d_2 + \frac{m^3}{n^3} d_2\right)\sigma^{-3}$$

$$+ \ldots + \frac{2c^2}{n}\sigma^{-1} + \frac{2mc^2}{n^2}\sigma^{-3} + \frac{2m^2c^2}{n^3}\sigma^{-5} + \ldots + \frac{4c^2}{n} g_1 \sigma +$$

$$\frac{4c^2m}{n^2} g_1 \sigma^{-1} + \frac{4c^2m^2}{n^3}\sigma^{-3} + \ldots = 0 .$$

Equating coefficients of like powers of σ gives

$$(\sigma) \qquad d_1 + m + \frac{m}{n} d_2 + \frac{4c^2}{n} g_1 = 0 \qquad (21)$$

$$(\sigma^{-1}) \qquad 2d_2 - \frac{m}{n} d_1 + \frac{m^2}{n} + n + \frac{m^2}{n} + \frac{m^2}{n^2} d_2 + \frac{2c^2}{n} + \frac{4c^2m}{n^2} g_1 = 0 \qquad (22)$$

$$(\sigma^{-3}) \qquad -d_1 + m - \frac{m^2}{n^2} d_1 + \frac{m^3}{n^2} + m + \frac{m^3}{n^2} + \frac{m}{n} d_2 + \frac{m^3}{n^3} d_2 + \frac{2mc^2}{n^2}$$

$$+ \frac{4c^2m^2}{n^3} g_1 = 0$$

$$(\sigma^{-5}) \qquad - d_1 \frac{m}{n} - d_1 \frac{m^3}{n^3} + \frac{m^2}{n} + \frac{m^4}{n^3} + \ldots = 0$$

$$(\sigma^{-7}) \qquad \ldots = 0$$

The independent equations are (18), (19), (21), and (22).

Subtracting (21) from (18) one finds

$$nd_1 + md_2 + 2c^2g_1 - nd_1 - mn - md_2 - 4\,c^2g_1 = 0$$

$$2c^2g_1 - mn\ 4mc^2g_1 = 0$$

$$g_1 = \frac{-mn}{2c^2} \ . \qquad (23)$$

From (18) $d_1 = m - \frac{m}{n} d_2$.

So (22) gives

$$\left(2 + \frac{m^2}{n^2}\right) d_2 - \frac{m^2}{n} + \frac{m^2}{n^2} d_2 + \frac{2m^2}{n} + m + \frac{2c^2}{n} - \frac{2m^2}{n} = 0$$

$$\left(n + \frac{m^2}{n}\right) 2d_2 + m^2 + n^2 + 2c^2 - 2m^2 = 0$$

$$d_2 = \frac{nm^2 - n^3 - 2nc^2}{2(m^2 + n^2)} \qquad (24)$$

and

$$d_1 = m - \frac{nm^3 - n^3m - 2nmc^2}{2n(m^2 + n^2)}$$

$$d_1 = \frac{3mn^2 + m^3 + 2mc^2}{2(m^2 + n^2)} \ . \qquad (25)$$

From (19) $$g_o = \frac{- md_1 - nd_2}{2c^2}$$

or $$g_o = \frac{3m^2n^3 + m^4n + 2m^2nc^2 + n^3m^2 + n^5 + 2n^3c^2}{- 4nc^2 (m^2 + n^2)}$$

$$g_o = \frac{4m^2n^2 + m^4 + n^4 + 2 (m^2 + n^2) c^2}{- 4c^2 (m^2 + n^2)} . \qquad (26)$$

$2z = m\zeta + n\zeta^{-1}$ as shown previously whose inverse is

$$\zeta = \frac{z + \sqrt{z^2 - mn}}{m} \quad \text{and} \quad \zeta^{-1} = \frac{z - \sqrt{z^2 - mn}}{n} .$$

So $$\varphi_2(z) = \frac{Bc^2}{32D} \left[- 2z \log \frac{z + \sqrt{z^2 - mn}}{m} + \left(\frac{d_1}{n} + \frac{d_2}{n} \right) z + \left(\frac{d_1}{m} - \frac{d_2}{n} \right) \sqrt{z^2 - mn} \right]$$

and $$\chi_2(z) = \frac{Bc^4}{32D} \left[- \log \frac{z + \sqrt{z^2 - mn}}{m} + g_o - \frac{g_1 m}{n} + \frac{2g_1 z^2}{n^2} - \frac{2g_1}{n^2} z \sqrt{z^2 - mn} \right]$$

from equations (16) and (17) .

From the continuity equations of a round region (8) and (9) one sees,

$$\varphi_1(z) = \frac{Bc^2}{32D} \left[- z \log \frac{cz + c\sqrt{z^2 - mn}}{zm} + \left(\frac{d_1}{m} + \frac{d_2}{n} + 1 \right) z + \left(\frac{d_1}{m} - \frac{d_2}{n} \right) \sqrt{z^2 - mn} \right]$$

and $\chi_1(z) = \frac{Bc^4}{32D}\left[-\log\frac{cz+c\sqrt{z^2-mn}}{m} + g_0 - \frac{g_1 m}{n} + \frac{5}{4} + \frac{2g_1 z^2}{n^2} - \frac{2g_1 z}{n^2}\right.$

$\left.\sqrt{z^2-mn}\right].$

SOLUTION OF THE ROUND CORNERED SQUARE PLATE CIRCULAR REGION

Assume

$$\varphi_2(\zeta) = \frac{Ba^2}{16D}\left[-L(\zeta+\lambda\zeta^5)\log\zeta + d_1\zeta + d_5\zeta^5\right] \tag{27}$$

and $$\chi_2(\zeta) = \frac{Ba^4}{32D}\left[-\log\zeta + g_0 + g_4\zeta^4\right] \tag{28}$$

where $z = L\zeta + L\lambda\zeta^5$.

$z\,\varphi_2(z) + z\,\overline{\varphi_2(z)} + \chi_2(z) + \overline{\chi_2(z)} = 0$ is a boundary condition which

with (27) and (28) gives

$$(L\sigma + L\lambda\sigma^5)\left(\frac{Ba^2}{16D}\right)\left[-(L\sigma^{-1} + L\lambda\sigma^{-5})\log\sigma^{-1} + d_1\sigma^{-1} + d_5\sigma^{-5}\right]$$

$$+ (L\sigma^{-1} + L\lambda\sigma^{-5})\left(\frac{Ba^2}{16D}\right)\left[-(L\sigma + L\lambda\sigma^5)\log\sigma + d_1\sigma + d_5\sigma^5\right]$$

$$+ \frac{Ba^4}{32D}\left[-\log\sigma + g_0 + g_4\sigma^4\right]$$

$$+ \frac{Ba^4}{32D}\left[-\log\sigma^{-1} + g_0 + g_4\sigma^{-4}\right] = 0 .$$

Multiplying through one finds

$$(-L^2-L^2\lambda\sigma^{-4} -L^2\lambda\sigma^4 -L^2\lambda^2)\log\sigma^{-1} + d_1L + L\lambda d_1\sigma^4 + d_5L\sigma^{-4} + d_5L\lambda$$

$$(-L^2-L^2\lambda\sigma^{-4} -L^2\lambda\sigma^{-4} -L^2\lambda^2)\log\sigma + d_1L + d_1L\lambda\sigma^{-4} + d_5L\sigma^4 + d_5L\lambda$$

$$-\frac{a^2}{2}\log\sigma + \frac{g_0a^2}{2} + \frac{g_4a^2}{2}\sigma^4 - \frac{a^2}{2}\log\sigma^{-1} + \frac{g_0a^2}{2} + \frac{g_4a^2}{2}\sigma^{-4} = 0 \; .$$

Equating coefficients of like powers of σ

$$(\sigma^4) \qquad L\lambda d_1 + d_5L + \frac{g_4a^2}{2} = 0 \tag{29}$$

$$(\sigma^0) \qquad 2Ld_1 + 2L\lambda d_5 + a^2g_0 = 0 \tag{30}$$

$$(\sigma^{-4}) \qquad d_5L + L\lambda d_1 + \frac{g_4a^2}{2} = 0$$

$\varphi(z) + z\,\overline{\varphi'(z)} + \overline{\chi'(z)} = 0$ is the second boundary condition

or $\varphi(\zeta) + \dfrac{(L\sigma + L\lambda\sigma^5)}{(L+5L\lambda\sigma^{-4})}\,\overline{\varphi'(\zeta)} + \dfrac{\overline{\chi'(\zeta)}}{(L+5L\lambda\sigma^{-4})} = 0$ in the ζ plane.

By dividing one obtains

$$\frac{\sigma+\lambda\sigma^5}{1+5\lambda\sigma^{-4}} = \lambda\sigma^5 + (1-5\lambda^2)\sigma + (25\lambda^3-5\lambda)\sigma^{-3} + (25\lambda^2-125\lambda^4)\sigma^{-7} + \dots$$

and

$$\frac{1}{L(1+5\lambda\sigma^{-4})} = \frac{1}{L} - \frac{5\lambda}{L}\sigma^{-4} + \frac{25\lambda^2}{L}\sigma^{-8} - \frac{125\lambda^3}{L}\sigma^{-12} + \frac{625\lambda^4}{L}\sigma^{-16} + \cdots$$

From (27)

$$\varphi'_2(\zeta) = \frac{Ba^2}{16\ D}\left[- L \log\sigma - L - 5L\lambda\sigma^4 \log\sigma - L\lambda\sigma^4 + d_1 + 5d_5\sigma^4\right]$$

so

$$\overline{\varphi'_2(\zeta)} = \frac{Ba^2}{16\ D}\left[-(L + 5L\lambda\sigma^{-4}) \log\sigma^{-1} + d_1 - L + (5d_5 - L\lambda)\sigma^{-4}\right].$$

Similiarly

$$\chi'_1(\zeta) = \frac{Ba^4}{32D}\left[-\sigma^{-1} + 4g_4\sigma^3\right]$$

$$\overline{\chi'_1(\zeta)} = \frac{Ba^4}{32D}\left[-\sigma + 4g_4\sigma^3\right].$$

Now the boundary condition gives

$$\frac{Ba^2}{16D}\left[- L(\sigma - \lambda\sigma^5) \log\sigma + d_1\sigma + d_5\sigma^5\right] + \frac{Ba^2}{16D}\left[- (L\lambda\sigma^5\right.$$

$$+ 5L\lambda^2\sigma) \log\sigma^{-1}$$

$$+(d_1\lambda - L\lambda)\sigma^5 + (5d_5 - L\lambda)\lambda\sigma - (L\sigma + 5L\lambda\sigma^{-3}) \log\sigma^{-1}$$

$$+ (d_1 - L)\sigma + (5d_5 - L\lambda)\sigma^{-3} + (5L\lambda^2\sigma + 25L\lambda^3\sigma^{-3}) \log\sigma^{-1}$$

$$- (5\lambda^2)(d_1 - L)\sigma - (5\lambda^2)(5d_5 - L\lambda)\sigma^{-3} - (25\lambda^3)(L\sigma^{-3} + 5L\lambda\sigma^{-7}) \log\sigma^{-1}$$

$$+ (25\lambda^3)(d_1 - L)\sigma^{-3} + (25\lambda^3)(5d_5 - L\lambda)\sigma^{-7} + (5\lambda L\sigma^{-3} + 25L\lambda^2\sigma^{-7})\log\sigma^{-1}$$

$$- (5\lambda)(d_1 - L)\sigma^{-3} - (5\lambda)(5d_5 - L\lambda)\sigma^{-7} - (25\lambda^2 L\sigma^{-7} + 125L\lambda^3\sigma^{-11})\log\sigma^{-1}$$

$$+ (25\lambda^2)(d_1 - L)\sigma^{-7} + 25\lambda^2(5d_5 - L\lambda)\sigma^{-11} + \ldots\Big]$$

$$+ \frac{Ba^4}{32DL}\left[-\sigma + 4g_4\sigma^{-3} + 5\lambda\sigma^{-3} - 20g_4\lambda\sigma^{-7} - 25\lambda^2\sigma^{-7} + 100\lambda^2 g_4\sigma^{-1} + \ldots\right]$$

$$= 0 .$$

Rewriting gives

$$d_1\sigma + d_5\sigma^5 + (d_1\lambda - L\lambda)\sigma^5 + (5\lambda d_5 - L\lambda^2)\sigma + (d_1 - L)\sigma + (d_5 - \lambda)\sigma^{-3}$$

$$-(5d_1\lambda^2 - 5L\lambda^2)\sigma - (25\lambda^2 d_5 - 5L\lambda^3)\sigma^{-3} + (25\lambda^3 d_1 - 25L\lambda^3)\sigma^{-3}$$

$$+ (125\lambda^3 d_5 - 25L\lambda^4)\sigma^{-7} - (5\lambda d_1 - 5\lambda L)\sigma^{-3} - (25\lambda d_5 - 5L\lambda^2)\sigma^{-7}$$

$$+ (25\lambda^2 d_1 - 25L\lambda^2)\sigma^{-7} + (125\lambda^2 d_5 - 25L\lambda^3)\sigma^{-11} + \ldots - \frac{a^2}{2L}\sigma$$

$$+ \frac{2a}{L}g_4\sigma^{-3} + \frac{5a^2\lambda}{2L}\sigma^{-3} - \frac{10a^2 g_4\lambda}{L}\sigma^{-7} - \frac{25a^2\lambda^2}{2L}\sigma^{-7} + \frac{50a^2\lambda^2 g_4}{L}\sigma^{-11}$$

$$+ \ldots = 0.$$

Equating coefficients of like powers of σ

(σ^{5}) $\quad d_5 + d_1 - L\lambda = 0$ (31)

(σ) $\quad 5\lambda d_5 + (2-5\lambda^2)\, d_1 + 4L\lambda^2 - L - \frac{a^2}{2L} = 0$ (32)

(σ^{-3}) $\quad (5-25\lambda^2)\, d_5 + (25\lambda^3 - 5\lambda)\, d_1 + \frac{2a^2 g_4}{L} + 4\lambda L - 20L\lambda^3$

$$+ \frac{5a^2}{2L}\lambda = 0$$

(σ^{-7}) $\quad 125\lambda^3 d_5 - 25L\lambda^4 - 25\lambda d_5 - 5L\lambda^2 + \ldots = 0$

Equations (29), (30), (31), and (32) are independent.

$d_5 = L\lambda - \lambda d_1$ from (31).

(32) then becomes

$$5L\lambda^2 + (-10\lambda^2 + 2)\, d_1 = L - 4L\lambda^2 + a^2/2L$$

$$(4L - 20L\lambda^2)\, d_1 = 2L^2 - 18L^2\lambda^2 + a^2$$

$$d_1 = \frac{2L^2 - 18L^2\lambda^2 + a^2}{4L - 20L\lambda^2} \quad (33)$$

So

$$d_5 = L\lambda - \frac{2\lambda L^2 - 18L\lambda^3 + a^2\lambda}{4L - 20L\lambda^2}$$

$$d_5 = \frac{2\lambda L^2 - 2L^2\lambda^3 - a^2\lambda}{4L - 20L\lambda^2} \quad (34)$$

From (30)

$$g_o = -\frac{2L}{a^2} d_1 - \frac{2L\lambda}{a^2} d_5$$

$$g_o = \frac{-4L^3\lambda + 36L^3\lambda^2 - 21a^2 - 4\lambda^2 L^3 + 4L^3\lambda^4 + 2L\lambda^2 a^2}{a^2(4L - 20L\lambda^2)}$$

$$g_o = \frac{-2L^2 + 16L^2\lambda^2 + 2L^2\lambda^4 + (\lambda^2 - 1)a^2}{a^2(2 - 10\lambda^2)} \quad (35)$$

From (29)

$$g_4 = -\frac{2L\lambda}{a^2} d_1 - \frac{2L\lambda}{a^2} d_5$$

$$g_4 = -\frac{4L^3\lambda + 36L^3\lambda^3 - 2L\lambda a^2 - 4\lambda^2 L^3 + 4L^3\lambda^4 + 2L\lambda^2 a^2}{a^2(4L - 20L\lambda^2)}$$

$$g_4 = \frac{-L^2\lambda + 9L^2\lambda^3 - \lambda^2 L^2 + L^2\lambda^4 + (\lambda - 1).50\lambda a^2}{1 - 5\lambda^2} \quad (36)$$

Using the continuity equations of a circular region gives,

$$\varphi_1(\zeta) = \frac{Ba^2}{16D}\left[(L\zeta + L\lambda\zeta^5)\log\frac{L + L\lambda\zeta^4}{a} + (d_1 - \frac{L}{2})\zeta + (d_5 + \frac{L\lambda}{2})\zeta^5\right]$$

and $\chi_1(\zeta) = \frac{Ba^4}{32D}\left[\log\frac{L + L\lambda\zeta^5}{a} + \frac{5}{4} + g_o + g_4\zeta^4\right]$.

APPENDIX II

APPENDIX II

CIRCULAR PLATE LOADED OVER A CIRCULAR REGION
DEFLECTION OF THE INNER REGION

```
ZZJOB 5
ZZFORX5
    5 FORMAT(F10.3,3X ,F10.3,3X ,E14.8)
    6 FORMAT(4F14.11)
    7 FORMAT(2F10.3)
      READ 6,B,A,D,BB
   30 READ 7,X,Y
      R=SQRTF(X*X+Y*Y)
      S=2.*R*R+A*A
      WONE=BB*A*A/(16.*D)*(.75*A*A-B*B+.5*A*A*R*R/(B*B)-.25*R**4/(A*A)+S*
     1*LOGF(B/A))
      PUNCH 5,X,Y,WONE
      GO TO 30
      END
   5.00           2.500            .004292         .000082895
     0.0            0.000    -.12056310E+00
     1.0            0.000    -.10946296E+00
     2.0            0.000    -.79783909E-01
     2.5            0.000    -.61089380E-01
     0.0            1.000    -.10946296E+00
     1.0            1.000    -.98966400E-01
     2.0            1.000    -.71098015E-01
     2.3            1.000    -.60784682E-01
     2.0            1.500    -.61089380E-01
     0.0            2.000    -.79783909E-01
     0.0            2.500    -.61089380E-01
     1.0            2.000    -.71098015E-01
     1.4            2.000    -.63327371E-01
```

CIRCULAR PLATE LOADED OVER A CIRCULAR REGION

DEFLECTION OF THE OUTER REGION

```
ZZJOB 5
ZZFORX5
    5 FORMAT(F10.3,3X ,F10.3,3X ,E14.8)
    6 FORMAT(4F14.11)
    7 FORMAT(2F10.3)
      READ 6,B,A,D,BB
   30 READ 7,X,Y
      R=SQRTF(X*X+Y*Y)
      S=2.*R*R+A*A
      WTWO=BB*A*A/(16.*D)*(R*R-B*B+.5*R*R*A*A/(B*B)-.5*A*A-S*LOGF(R/B)) *
      PUNCH 5,X,Y,WTWO
      GO TO 30
      END
```

5.00	2.500	.004292	.000082895
2.5	0.000	-.61089380E-01	
3.0	0.000	-.42343171E-01	
4.0	0.000	-.11993894E-01	
5.0	0.000	.00000000E-99	
2.3	1.000	-.60784697E-01	
3.0	1.000	-.36580766E-01	
4.0	1.000	-.93441563E-02	
4.9	1.000	.10169943E-07	
2.0	1.500	-.61089380E-01	
1.5	2.000	-.61089380E-01	
2.0	2.000	-.48652738E-01	
3.0	2.000	-.22297209E-01	
4.0	2.000	-.35067180E-02	
4.6	2.000	-.33432558E-05	
0.0	2.500	-.61089380E-01	

ELLIPTIC PLATE LOADED OVER A CIRCULAR REGION
DEFLECTION OF THE INNER REGION

```
ZZJOB 5
ZZFORX5
    5 FORMAT(F10.3,3X,F10.3,3X,F10.3,3X,F10.3,3X,E14.8)
    6 FORMAT(5F14.11)
    7 FORMAT(2F10.3)
      READ 6,A,B,C,D,BB
   30 READ 7,U,V
      E=A+B
      F=A-B
      Z=E*E+F*F
      G=U*U+V*V
      R=U*U-V*V
      S=R/(U**4+2.*U*U*V*V+V**4)
      RO=SQRTF(G)
      PI=ATANF(V/U)
      T=.5*E/C+.5*F*COSF(2.*PI)/(C*RO*RO)
      W=.25*F*F*SINF(2.*PI)*SINF(2.*PI)/(C*C*RO**4)
      X=.5*E*U+.5*F*U/G
      Y=.5*E*V-.5*F*V/G
      D1=(3.*F*F+2.*C*C+E*E)/(2.*E+2.*F*F/E)
      D2=F-F*D1/E
      G0=(-E*D1-F*D2)/(2.*C*C)
      G1=(-F*D1-E*D2)/(2.*C*C)
      H=U**4-6.*U*U*V*V+V**4
      GG=U**8+4.*U**6*V*V+6.*U**4*V**4+4.*U*U*V**6+V**8
      HH=H/GG
      WONE=BB*C*C/(32.*D)*((E*E*G+2.*E*F*R/G+F*F/G+2.*C*C)*LOGF(SQRTF(T**
     1T+W))+(D1-.5*E)*(E*G+F*R/G)+(D2-.5*F)*(E*R/G+F/G)+2.5*C*C+2.*G0*C**
     1C+2.*G1*S-(1./(2.*C*C))*(E**4*G*G/(16.)+.25*E**3*F*R+.125*E*E*F*F**
     1H/(G*G)+.25*E*E*F*F+.25*E*F**3*R/(G*G)+F**4/(16.*G*G)))
      PUNCH 5,X,Y,U,V,WONE
      GO TO 30
      END
```

6.00	3.00	1.50	.004292	.000082895	*
1500.004	0.0	.001	0.000	-.15276220E+10	*
750.009	0.0	.002	0.000	-.95455734E+08	*
500.013	0.0	.003	0.000	-.18849409E+08	*
375.018	0.0	.004	0.000	-.59616156E+07	*
300.022	0.0	.005	0.000	-.24406728E+07	*
250.027	0.0	.006	0.000	-.11763530E+07	*
214.317	0.0	.007	0.000	-.63456129E+06	*
187.536	0.0	.008	0.000	-.37170716E+06	*
166.707	0.0	.009	0.000	-.23187816E+06	*
150.045	0.0	.010	0.000	-.15201073E+06	*
136.413	0.0	.011	0.000	-.10373484E+06	*
125.054	0.0	.012	0.000	-.73176160E+05	*
115.443	0.0	.013	0.000	-.53076166E+05	*
107.205	0.0	.014	0.000	-.39420245E+05	*
100.067	0.0	.015	0.000	-.29881754E+05	*
93.822	0.0	.016	0.000	-.23057535E+05	*
88.311	0.0	.017	0.000	-.18071806E+05	*
83.414	0.0	.018	0.000	-.14361350E+05	*
79.032	0.0	.019	0.000	-.11554287E+05	*
75.090	0.0	.020	0.000	-.93992847E+04	*
71.523	0.0	.021	0.000	-.77228990E+04	*
68.280	0.0	.022	0.000	-.64031802E+04	*
65.320	0.0	.023	0.000	-.53529192E+04	*
62.608	0.0	.024	0.000	-.45087888E+04	*
60.112	0.0	.025	0.000	-.38241539E+04	*
30.225	0.0	.050	0.000	-.22924617E+03	*
20.337	0.0	.075	0.000	-.43152764E+02	*
15.450	0.0	.100	0.000	-.13005072E+02	*
12.562	0.0	.125	0.000	-.50920931E+01	*
10.675	0.0	.150	0.000	-.23637291E+01	*
9.358	0.0	.175	0.000	-.12402715E+01	*
8.4	0.0	.200	0.000	-.71566745E+00	*
7.679	0.0	.225	0.000	-.44647070E+00	*
7.125	0.0	.250	0.000	-.29779239E+00	*

6.692	0.0	.275	0.000	-.21070567E+00	*
6.350	0.0	.300	0.000	-.15721503E+00	*
5.550	0.0	.400	0.000	-.75400051E-01	*
5.250	0.0	.500	0.000	-.59576830E-01	*
5.2	0.0	.600	0.000	-.62550873E-01	*
5.292	0.0	.700	0.000	-.76130671E-01	*
0.0	-14.550	0.000	.100	-.67557003E+01	*
7.950	-7.050	.100	.100	-.15644718E+01	*
6.9	-2.550	.200	.100	-.29168835E+00	*
5.850	-1.050	.300	.100	-.10905302E+00	*
5.329	-.432	.400	.100	-.66551282E-01	*
5.134	-.126	.500	.100	-.57373658E-01	*
5.132	.044	.600	.100	-.61766693E-01	*
5.250	.150	.700	.100	-.75732070E-01	*
5.446	.219	.800	.100	-.99285704E-01	*
0.0	-6.6	0.000	.200	.54027174E+00	*
3.450	-5.1	.100	.200	.29032050E+00	*
4.650	-2.850	.200	.200	.77531996E-01	*
4.811	-1.407	.300	.200	-.14054000E-01	*
4.8	-.6	.400	.200	-.40613861E-01	*
4.836	-.134	.500	.200	-.49167482E-01	*
4.950	.150	.600	.200	-.58450501E-01	*
5.131	.333	.700	.200	-.74061942E-01	*
5.364	.458	.800	.200	-.98514317E-01	*
0.0	-3.650	0.000	.300	.25380496E+00	*
1.950	-3.150	.100	.300	.18845663E+00	*
3.207	-2.111	.200	.300	.88532903E-01	*
3.850	-1.150	.300	.300	.22501137E-01	*
4.2	-.450	.400	.300	-.15394271E-01	*
4.455	.026	.500	.300	-.36143647E-01	*
4.7	.350	.600	.300	-.51742163E-01	*
4.960	.574	.700	.300	-.70450628E-01	*
5.243	.733	.800	.300	-.96859167E-01	*
0.0	-1.950	0.000	.400	.83989806E-01	*
1.332	-1.729	.100	.400	.72202564E-01	*

2.4	-1.2	.200	.400	.45520802E-01	*
3.150	-.6	.300	.400	.18593915E-01	*
3.675	-.075	.400	.400	-.46041818E-02	*
4.079	.336	.500	.400	-.24494471E-01	*
4.430	.646	.600	.400	-.43322103E-01	*
4.765	.876	.700	.400	-.65293620E-01	*
5.1	1.050	.800	.400	-.94522123E-01	*
0.0	-.750	0.000	.500	.14558935E-01	*
1.026	-.634	.100	.500	.15366268E-01	*
1.934	-.336	.200	.500	.14673931E-01	*
2.673	.044	.300	.500	.90185853E-02	*
3.263	.420	.400	.500	-.17106576E-02	*
3.750	.750	.500	.500	-.16700531E-01	*
4.175	1.020	.600	.500	-.35697698E-01	*
4.568	1.236	.700	.500	-.60081641E-01	*
0.0	.2	0.000	.600	-.86210112E-02	*
.855	.267	.100	.600	-.54318426E-02	*
1.650	.450	.200	.600	.68154175E-03	*
2.350	.7	.300	.600	.35138890E-02	*
2.953	.969	.400	.600	-.40716092E-03	*
3.479	1.224	.500	.600	-.11698777E-01	*
3.950	1.450	.600	.600	-.30210078E-01	*
4.385	1.641	.700	.600	-.56487153E-01	*
0.0	1.007	0.000	.700	-.10329552E-01	*
.750	1.050	.100	.700	-.74129378E-02	*
1.466	1.168	.200	.700	-.11517975E-02	*
2.125	1.339	.300	.700	.32022026E-02	*
2.723	1.534	.400	.700	.13757167E-02	*
3.263	1.731	.500	.700	-.86307162E-02	*
3.758	1.914	.600	.700	-.27516776E-01	*
4.221	2.078	.700	.700	-.55954272E-01	*

ELLIPTIC PLATE LOADED OVER A CIRCULAR REGION
DEFLECTION OF THE OUTER REGION

```
ZZJOB 5
ZZFORX5
    5 FORMAT(F10.3,3X,F10.3,3X,F10.3,3X,F10.3,3X,E14.8)
    6 FORMAT(5F14.11)
    7 FORMAT(2F10.3)
      READ 6,A,B,C,D,BB
   30 READ 7,U,V
      E=A+B
      F=A-B
      Z=E*E+F*F
      G=U*U+V*V
      R=U*U-V*V
      S=R/(U**4+2.*U*U*V*V+V**4)
      X=.5*E*U+.5*F*U/G
      Y=.5*E*V-.5*F*V/G
      D1=(3.*F*F+2.*C*C+E*E)/(2.*E+2.*F*F/E)
      D2=F-F*D1/E
      G0=(-E*D1-F*D2)/(2.*C*C)
      G1=(-F*D1-E*D2)/(2.*C*C)
      WTWO=BB*C*C/(32.*D)*(-(E*E*G+2.*E*F*R/G+F*F/G+2.*C*C)*LOGF(SQRTF(G*
     1))+E*D1*G+(E*D2+F*D1)*R/G+F*D2/G+2.*G0*C*C+2.*G1*C*C*S)
      PUNCH 5,X,Y,U,V,WTWO
      GO TO 30
      END
```

6.00	3.00	1.50	.004292	.000082895	*
6.0	0.0	1.000	0.000	.00000000E-99	*
5.399	1.307	.900	.435	.16296051E-07	*
4.8	1.8	.800	.600	.00000000E-99	*
4.199	2.142	.700	.714	.11950437E-07	*
3.6	2.4	.600	.800	.00000000E-99	*
2.999	2.598	.500	.866	.95060301E-08	*
2.399	2.749	.400	.916	.27160086E-08	*

1.799	2.861	.300	.953	.95060301E-08	*
1.199	2.939	.200	.979	.67900215E-08	*
.599	2.985	.100	.995	.54320172E-08	*
5.969	.3	.995	.100	.17654055E-07	*
5.878	.6	.979	.200	.20370064E-07	*
5.723	.9	.953	.300	.13580043E-07	*
5.499	1.2	.916	.400	.67900215E-08	*
5.196	1.5	.866	.500	.14938047E-07	*
4.285	2.1	.714	.700	.12493639E-07	*
2.615	2.7	.435	.900	.81480258E-08	*
0.0	3.0	0.000	1.000	.00000000E-99	*
5.970	0.0	.990	0.000	-.12234260E-04	*
5.973	0.0	.991	0.000	-.99079993E-05	*
5.976	0.0	.992	0.000	-.78288947E-05	*
5.979	0.0	.993	0.000	-.59928729E-05	*
5.982	0.0	.994	0.000	-.44012919E-05	*
5.985	0.0	.995	0.000	-.30595836E-05	*
5.988	0.0	.996	0.000	-.19555261E-05	*
5.991	0.0	.997	0.000	-.11013414E-05	*
5.994	0.0	.998	0.000	-.49023955E-06	*
5.997	0.0	.999	0.000	-.12357839E-06	*
6.0	0.0	1.000	0.000	.00000000E-99	*
6.003	0.0	1.001	0.000	-.12086238E-06	*
6.006	0.0	1.002	0.000	-.48888154E-06	*
6.009	0.0	1.003	0.000	-.11013414E-05	*
6.012	0.0	1.004	0.000	-.19555261E-05	*
6.015	0.0	1.005	0.000	-.30582256E-05	*
6.018	0.0	1.006	0.000	-.43999339E-05	*
6.021	0.0	1.007	0.000	-.59874409E-05	*
6.024	0.0	1.008	0.000	-.78166727E-05	*
6.027	0.0	1.009	0.000	-.98917033E-05	*
6.030	0.0	1.010	0.000	-.12209816E-04	*
6.350	0.0	.300	0.000	-.11588813E+00	*
5.550	0.0	.400	0.000	-.67271146E-01	*
5.250	0.0	.500	0.000	-.39796478E-01	*

5.2	0.0	.600	0.000	-.22882898E-01	*
5.292	0.0	.700	0.000	-.11997913E-01	*
5.475	0.0	.800	0.000	-.50985565E-02	*
5.716	0.0	.900	0.000	-.12404717E-02	*
5.850	-1.050	.300	.100	-.56668242E-01	*
5.329	-.432	.400	.100	-.49634880E-01	*
5.134	-.126	.500	.100	-.33526554E-01	*
5.132	.044	.600	.100	-.20387916E-01	*
5.250	.150	.700	.100	-.10941425E-01	*
5.446	.219	.800	.100	-.46543492E-02	*
5.696	.267	.900	.100	-.10868108E-02	*
4.650	-2.850	.200	.200	.20122361E+00	*
4.811	-1.407	.300	.200	.21983259E-01	*
4.8	-.6	.400	.200	-.17459969E-01	*
4.836	-.134	.500	.200	-.20117227E-01	*
4.950	.150	.600	.200	-.14539862E-01	*
5.131	.333	.700	.200	-.83235152E-02	*
5.364	.458	.800	.200	-.35225735E-02	*
5.638	.547	.900	.200	-.70429769E-03	*
3.207	-2.111	.200	.300	.15319402E+00	*
3.850	-1.150	.300	.300	.44938586E-01	*
4.2	-.450	.400	.300	.28047202E-02	*
4.455	.026	.500	.300	-.84973954E-02	*
4.7	.350	.600	.300	-.84853581E-02	*
4.960	.574	.700	.300	-.53120769E-02	*
5.243	.733	.800	.300	-.21590598E-02	*
5.550	.850	.900	.300	-.28172749E-03	*
0.0	-1.950	0.000	.400	.18333579E+00	*
1.332	-1.729	.100	.400	.15162304E+00	*
2.4	-1.2	.200	.400	.87723195E-01	*
3.150	-.6	.300	.400	.35938979E-01	*
3.675	-.075	.400	.400	.82851367E-02	*
4.079	.336	.500	.400	-.22537846E-02	*
4.430	.646	.600	.400	-.41794415E-02	*
4.765	.876	.700	.400	-.28321233E-02	*

5.1	1.050	.800	.400	-.99028253E-03	*
5.441	1.181	.900	.400	-.22298430E-04	*
0.0	-.750	0.000	.500	.78540178E-01	*
1.026	-.634	.100	.500	.67948147E-01	*
1.934	-.336	.200	.500	.44158310E-01	*
2.673	.044	.300	.500	.21262265E-01	*
3.263	.420	.400	.500	.65583093E-02	*
3.750	.750	.500	.500	-.14036875E-03	*
4.175	1.020	.600	.500	-.17952492E-02	*
4.568	1.236	.700	.500	-.12005219E-02	*
4.948	1.407	.800	.500	-.24890589E-03	*
0.0	.2	0.000	.600	.32565546E-01	*
.855	.267	.100	.600	.28756109E-01	*
1.650	.450	.200	.600	.19703900E-01	*
2.350	.7	.300	.600	.10157798E-01	*
2.953	.969	.400	.600	.34118676E-02	*
3.479	1.224	.500	.600	.12284384E-03	*
3.950	1.450	.600	.600	-.66683035E-03	*
4.385	1.641	.700	.600	-.31912870E-03	*
0.0	1.007	0.000	.700	.12015941E-01	*
.750	1.050	.100	.700	.10660688E-01	*
1.466	1.168	.200	.700	.73614683E-02	*
2.125	1.339	.300	.700	.37614342E-02	*
2.723	1.534	.400	.700	.11767745E-02	*
3.263	1.731	.500	.700	-.54944853E-05	*
3.758	1.914	.600	.700	-.16265319E-03	*
4.221	2.078	.700	.700	-.48263472E-05	*
0.0	1.725	0.000	.800	.34235953E-02	*
.680	1.753	.100	.800	.29989990E-02	*
1.341	1.835	.200	.800	.19725392E-02	*
1.966	1.956	.300	.800	.88811851E-03	*
2.550	2.1	.400	.800	.19138354E-03	*
3.092	2.251	.500	.800	-.21983373E-04	*
3.6	2.4	.600	.800	.00000000E-99	*
0.0	2.383	0.000	.900	.50827249E-03	*

.632	2.403	.100	.900	.41998319E-03	*
1.252	2.461	.200	.900	.22251443E-03	*
1.850	2.550	.300	.900	.55632004E-04	*
2.418	2.658	.400	.900	.73060631E-06	*

ROUND CORNERED SQUARE PLATE LOADED OVER A CIRCULAR REGION
DEFLECTION OF THE INNER REGION

```
ZZJOB 5
ZZFORX5
    5 FORMAT(F10.3,3X,F10.3,3X,F10.3,3X,F10.3,3X,E14.8)
    6 FORMAT(4F14.11)
    7 FORMAT(2F10.3)
      READ 6,B,C,D,BB
   30 READ 7,X,Y
      Q=25./48.*B
      R=-.04
      U1=X/Q
      V1=Y/Q
      U2=X/Q-R*V1**5+10.*R*U1**3*V1*V1-5.*R*V1**4*U1
      V2=Y/Q-R*V1**5+10.*R*U1*U1*V1**3-5.*R*U1**4*V1
      U3=X/Q-R*V2**5+10.*R*U2**3*V2*V2-5.*R*V2**4*U2
      V3=Y/Q-R*V2**5+10.*R*U2*U2*V2**3-5.*R*U2**4*V2
      U4=X/Q-R*V3**5+10.*R*U3**3*V3*V3-5.*R*V3**4*U3
      V4=Y/Q-R*V3**5+10.*R*U3*U3*V3**3-5.*R*U3**4*V3
      U5=X/Q-R*V4**5+10.*R*U4**3*V4*V4-5.*R*V4**4*U4
      V5=Y/Q-R*V4**5+10.*R*U4*U4*V4**3-5.*R*U4**4*V4
      U6=X/Q-R*V5**5+10.*R*U5**3*V5*V5-5.*R*V5**4*U5
      V6=Y/Q-R*V5**5+10.*R*U5*U5*V5**3-5.*R*U5**4*V5
      U7=X/Q-R*V6**5+10.*R*U6**3*V6*V6-5.*R*V6**4*U6
      V7=Y/Q-R*V6**5+10.*R*U6*U6*V6**3-5.*R*U6**4*V6
      U8=X/Q-R*V7**5+10.*R*U7**3*V7*V7-5.*R*V7**4*U7
      V8=Y/Q-R*V7**5+10.*R*U7*U7*V7**3-5.*R*U7**4*V7
      U9=X/Q-R*V8**5+10.*R*U8**3*V8*V8-5.*R*V8**4*U8
      V9=Y/Q-R*V8**5+10.*R*U8*U8*V8**3-5.*R*U8**4*V8
      U=U9
      V=V9
      E=U*U+V*V
      F=U**6-5.*U**4*V*V-5.*U*U*V**4+V**6
      FF=U**4+2.*U*U*V*V+V**4
```

```
      G=FF
      H=U**10+5.*U**8*V**2+10.*U**6*V**4+10.*U**4*V**6+5.*U*U*V**8+V**10*
      P=U**4-6.*U*U*V*V+V**4
      D1=(2.*Q*Q-18.*Q*Q*R*R+C*C)/(4.*Q-20.*Q*R*R)
      D5=Q*R-R*D1
      G0=(-2.*Q*D1-2.*Q*R*D5)/(C*C)
      G4=(-2.*Q*R*D1-2.*Q*D5)/(C*C)
      S=U**8-4.*U**6*V*V-10.*U**4*V**4-4.*U*U*V**6+V**8
      T=U**12-26.*U**10*V*V+15.*U**8*V**4+84.*U**6*V**6+15.*U**4*V**8-26*
     1.*U*U*V**10+V**12
      W=U**16-20.*U**12*V**4-64.*U**10*V**6-90.*U**8*V**8-64.*U**6*V**10*
     1-20.*U**4*V**12+V**16
      A=U**20+10.*U**18*V*V+45.*U**16*V**4+120.*U**14*V**6+210.*U**12*V**
     1*8+252.*U**10*V**10+210.*U**8*V**12+120.*U**6*V**14+45.*U**4*V**16*
     1+10.*U*U*V**18+V**20
      REST=-Q**4*R**4/(4.*C*C)*A+C*C*G4*P
      EE=U**12+6.*U**10*V*V+15.*U**8*V**4+20.*U**6*V**6+15.*U**4*V**8+6.*
     1*U*U*V**10+V**12
      WONE=BB*C*C/(16.*D)*((C*C+2.*Q*Q*E+4.*Q*Q*R*F+2.*Q*Q*R*R*H)*LOGF((*
     1Q+Q*R*G)/C)+(1.25+G0)*C*C+(2.*D1*Q-Q*Q)*E+(2.*Q*R*D1+2.*D5-2.*Q*Q**
     1R)*F+(Q*Q*R*R+2.*Q*R*D5)*H-Q**4/(4.*C*C)*FF-Q**4*R/(C*C)*S-Q**4*R**
     1R/(2.*C*C)*T-Q**4*R*R/(C*C)*EE-Q**4*R**3/(C*C)*W+REST)
       PUNCH 5,X,Y,U,V,WONE
      GO TO 30
      END
10.00           2.50             .004292          .000082895
     0.0           0.0            0.000            0.000     -.13384457E+00       *
     1.0           0.0             .192            0.000     -.12222606E+00       *
     2.0           0.0             .384            0.000     -.90842735E-01       *
     2.5           0.0             .480            0.000     -.70795736E-01       *
     0.0           1.0            0.000             .192     -.12222484E+00       *
     1.0           1.0             .191             .191     -.11135594E+00       *
     2.0           1.0             .383             .192     -.82274957E-01       *
     2.291         1.0             .438             .192     -.71740190E-01       *
     2.0           1.5             .382             .287     -.72390958E-01       *
```

2.0	1.5	.382	.287	-.72390958E-01	*
1.5	2.0	.288	.382	-.72320764E-01	*
1.0	2.291	.193	.439	-.71567413E-01	*
0.0	2.5	.001	.481	-.70574505E-01	*
0.0	2.0	0.000	.384	-.90775762E-01	*
1.0	2.0	.192	.383	-.82188294E-01	*

ROUND CORNERED SQUARE PLATE LOADED OVER A CIRCULAR REGION
DEFLECTION OF THE OUTER REGION

```
ZZJOB 5
ZZFORX5
    5 FORMAT(F10.3,3X,F10.3,3X,F10.3,3X,F10.3,3X,E14.8)
    6 FORMAT(4F14.11)
    7 FORMAT(2F10.3)
      READ 6,B,C,D,BB
   30 READ 7,X,Y
      Q=25./48.*B
      R=-.04
      U1=X/Q
      V1=Y/Q
      U2=X/Q-R*V1**5+10.*R*U1**3*V1*V1-5.*R*V1**4*U1
      V2=Y/Q-R*V1**5+10.*R*U1*U1*V1**3-5.*R*U1**4*V1
      U3=X/Q-R*V2**5+10.*R*U2**3*V2*V2-5.*R*V2**4*U2
      V3=Y/Q-R*V2**5+10.*R*U2*U2*V2**3-5.*R*U2**4*V2
      U4=X/Q-R*V3**5+10.*R*U3**3*V3*V3-5.*R*V3**4*U3
      V4=Y/Q-R*V3**5+10.*R*U3*U3*V3**3-5.*R*U3**4*V3
      U5=X/Q-R*V4**5+10.*R*U4**3*V4*V4-5.*R*V4**4*U4
      V5=Y/Q-R*V4**5+10.*R*U4*U4*V4**3-5.*R*U4**4*V4
      U6=X/Q-R*V5**5+10.*R*U5**3*V5*V5-5.*R*V5**4*U5
      V6=Y/Q-R*V5**5+10.*R*U5*U5*V5**3-5.*R*U5**4*V5
      U7=X/Q-R*V6**5+10.*R*U6**3*V6*V6-5.*R*V6**4*U6
      V7=Y/Q-R*V6**5+10.*R*U6*U6*V6**3-5.*R*U6**4*V6
      U8=X/Q-R*V7**5+10.*R*U7**3*V7*V7-5.*R*V7**4*U7
      V8=Y/Q-R*V7**5+10.*R*U7*U7*V7**3-5.*R*U7**4*V7
      U9=X/Q-R*V8**5+10.*R*U8**3*V8*V8-5.*R*V8**4*U8
      V9=Y/Q-R*V8**5+10.*R*U8*U8*V8**3-5.*R*U8**4*V8
      U=U9
      V=V9
      E=U*U+V*V
      F=U**6-5.*U**4*V*V-5.*U*U*V**4+V**6
      H=U**10+5.*U**8*V**2+10.*U**6*V**4+10.*U**4*V**6+5.*U*U*V**8+V**10*
```

```
      P=U**4-6.*U*U*V*V+V**4
      D1=(2.*Q*Q-18.*Q*Q*R*R+C*C)/(4.*Q-20.*Q*R*R)
      D5=Q*R-R*D1
      G0=(-2.*Q*D1-2.*Q*R*D5)/(C*C)
      G4=(-2.*Q*R*D1-2.*Q*D5)/(C*C)
      S=U**8-4.*U**6*V*V-10.*U**4*V**4-4.*U*U*V**6+V**8
      T=U**12-26.*U**10*V*V+15.*U**8*V**4+84.*U**6*V**6+15.*U**4*V**8-26*
     1.*U*U*V**10+V**12
      W=U**16-20.*U**12*V**4-64.*U**10*V**6-90.*U**8*V**8-64.*U**6*V**10*
     1-20.*U**4*V**12+V**16
      A=U**20+10.*U**18*V*V+45.*U**16*V**4+120.*U**14*V**6+210.*U**12*V**
     1*8+252.*U**10*V**10+210.*U**8*V**12+120.*U**6*V**14+45.*U**4*V**16*
     1+10.*U*U*V**18+V**20
      WTWO=BB*C*C/(16.*D)*(-(C*C+2.*Q*Q*F+4.*R*Q*Q*F+2.*Q*Q*R*R*H)*LOGF(*
     1SQRTF(E))+C*C*G0+2.*Q*D1*E+(2.*Q*D5+2.*Q*R*D1)*F+2.*Q*R*D5*H+C*C*G*
     14*P)
        PUNCH 5,X,Y,U,V,WTWO
       GO TO 30
       END
10.00          2.50           .004292          .000082895
     2.5          0.0           .480        0.000    -.70868197E-01     *
     3.0          0.0           .576        0.000    -.50574226E-01     *
     4.0          0.0           .768        0.000    -.16673796E-01     *
     2.291        1.0           .438         .192    -.71410022E-01     *
     3.0          1.0           .573         .195    -.45005403E-01     *
     4.0          1.0           .760         .203    -.14313015E-01     *
     2.0          1.5           .382         .287    -.71781642E-01     *
     1.5          2.0           .288         .382    -.71708976E-01     *
     2.0          2.0           .382         .382    -.58806170E-01     *
     3.0          2.0           .567         .384    -.30575678E-01     *
     4.0          2.0           .746         .395    -.82135196E-02     *
     1.0          2.291         .193         .439    -.71233068E-01     *
     0.0          2.5           .001         .481    -.70647887E-01     *
     0.0          3.0           .002         .578    -.50040426E-01     *
     1.0          3.0           .197         .575    -.44393056E-01     *
```

2.0	3.0	.387	.569	-.30105085E-01	*
3.0	3.0	.566	.566	-.13351988E-01	*
4.0	3.0	.734	.571	-.17741255E-02	*
0.0	4.0	.012	.779	-.15113513E-01	*
1.0	4.0	.215	.770	-.12757199E-01	*
2.0	4.0	.404	.753	-.71197419E-02	*
3.0	4.0	.577	.739	-.14171507E-02	*

ROUND CORNERED SQUARE PLATE LOADED OVER A CIRCULAR REGION
DEFLECTION OF THE OUTER REGION

```
ZZJOB 5
ZZFORX5
    5 FORMAT(F10.3,3X,F10.3,3X,F10.3,3X,F10.3,3X,E14.8)
    6 FORMAT(4F14.11)
    7 FORMAT(2F10.3)
      READ 6,B,C,D,BB
   30 READ 7,U,V
      Q=25./48.*B
      R=-.04
      X=Q*U+Q*R*U**5-10.*Q*R*U**3*V**2+5.*Q*R*U*V**4
      Y=Q*V+5.*Q*R*U**4*V-10.*R*Q*U**2*V**3+Q*R*V**5
      E=U*U+V*V
      F=U**6-5.*U**4*V*V-5.*U*U*V**4+V**6
      H=U**10+5.*U**8*V**2+10.*U**6*V**4+10.*U**4*V**6+5.*U*U*V**8+V**10*
      P=U**4-6.*U*U*V*V+V**4
      D1=(2.*Q*Q-18.*Q*Q*R*R+C*C)/(4.*Q-20.*Q*R*R)
      D5=Q*R-R*D1
      G0=(-2.*Q*D1-2.*Q*R*D5)/(C*C)
      G4=(-2.*Q*R*D1-2.*Q*D5)/(C*C)
      S=U**8-4.*U**6*V*V-10.*U**4*V**4-4.*U*U*V**6+V**8
      T=U**12-26.*U**10*V*V+15.*U**8*V**4+84.*U**6*V**6+15.*U**4*V**8-26*
     1.*U*U*V**10+V**12
      W=U**16-20.*U**12*V**4-64.*U**10*V**6-90.*U**8*V**8-64.*U**6*V**10*
     1-20.*U**4*V**12+V**16
      A=U**20+10.*U**18*V*V+45.*U**16*V**4+120.*U**14*V**6+210.*U**12*V**
     1*8+252.*U**10*V**10+210.*U**8*V**12+120.*U**6*V**14+45.*U**4*V**16*
     1+10.*U*U*V**18+V**20
      WTWO=BB*C*C/(16.*D)*(-(C*C+2.*Q*Q*E+4.*R*Q*Q*F+2.*Q*Q*R*R*H)*LOGF(*
     1SQRTF(E))+C*C*G0+2.*Q*D1*E+(2.*Q*D5+2.*Q*R*D1)*F+2.*Q*R*D5*H+C*C*G*
     14*P)
      PUNCH 5,X,Y,U,V,WTWO
      GO TO 30
      END
```

10.00	2.50	.004292	.000082895		
5.0	0.0	1.000	0.000	.98078091E-08	*
4.819	2.108	.900	.435	.56885293E-07	*
4.374	3.140	.800	.600	.11316702E-07	*
3.785	3.874	.700	.714	.61864642E-07	*
3.140	4.374	.600	.800	.11316702E-07	*
2.499	4.691	.500	.866	.59601301E-07	*
1.899	4.871	.400	.916	.31988546E-07	*
1.354	4.958	.300	.953	.60204859E-07	*
.865	4.991	.200	.979	.52056833E-07	*
.420	4.999	.100	.995	.54320173E-07	*
4.999	.420	.995	.100	.54320173E-07	*
4.991	.865	.979	.200	.52056833E-07	*
4.958	1.354	.953	.300	.60355748E-07	*
4.871	1.899	.916	.400	.32290325E-07	*
4.691	2.499	.866	.500	.59601301E-07	*
3.874	3.785	.714	.700	.61864642E-07	*
2.108	4.819	.435	.900	.58167852E-07	*
0.0	5.0	0.000	1.000	.98078091E-08	*
4.958	0.0	.990	0.000	-.32361998E-04	*
4.962	0.0	.991	0.000	-.26212501E-04	*
4.966	0.0	.992	0.000	-.20711075E-04	*
4.970	0.0	.993	0.000	-.15853191E-04	*
4.974	0.0	.994	0.000	-.11645641E-04	*
4.979	0.0	.995	0.000	-.80838981E-05	*
4.983	0.0	.996	0.000	-.51755054E-05	*
4.987	0.0	.997	0.000	-.29083926E-05	*
4.991	0.0	.998	0.000	-.12968941E-05	*
4.995	0.0	.999	0.000	-.32139436E-06	*
5.0	0.0	1.000	0.000	.98078091E-08	*
5.004	0.0	1.001	0.000	-.31913102E-06	*
5.008	0.0	1.002	0.000	-.12802963E-05	*
5.012	0.0	1.003	0.000	-.29053748E-05	*
5.016	0.0	1.004	0.000	-.51656976E-05	*
5.020	0.0	1.005	0.000	-.80838981E-05	*

5.024	0.0	1.006	0.000	-.11639606E-04	*
5.029	0.0	1.007	0.000	-.15849419E-04	*
5.033	0.0	1.008	0.000	-.20699004E-04	*
5.037	0.0	1.009	0.000	-.26205711E-04	*
5.041	0.0	1.010	0.000	-.32355208E-04	*
1.561	0.0	.300	0.000	-.10574026E+00	*
2.081	0.0	.400	0.000	-.87614954E-01	*
2.597	0.0	.500	0.000	-.66592678E-01	*
3.108	0.0	.600	0.000	-.45697535E-01	*
3.610	0.0	.700	0.000	-.27114170E-01	*
4.098	0.0	.800	0.000	-.12526125E-01	*
4.564	0.0	.900	0.000	-.32082022E-02	*
1.562	.520	.300	.100	-.10324275E+00	*
2.082	.518	.400	.100	-.85221736E-01	*
2.6	.514	.500	.100	-.64605081E-01	*
3.113	.508	.600	.100	-.44169485E-01	*
3.617	.496	.700	.100	-.26026703E-01	*
4.108	.479	.800	.100	-.11839844E-01	*
4.579	.454	.900	.100	-.28835637E-02	*
1.041	1.041	.200	.200	-.10840118E+00	*
1.563	1.041	.300	.200	-.95606908E-01	*
2.085	1.038	.400	.200	-.78240017E-01	*
2.607	1.032	.500	.200	-.58856553E-01	*
3.125	1.020	.600	.200	-.39763084E-01	*
3.638	.999	.700	.200	-.22905559E-01	*
4.139	.966	.800	.200	-.98962891E-02	*
4.623	.918	.900	.200	-.20109283E-02	*
1.041	1.563	.200	.300	-.95606908E-01	*
1.564	1.564	.300	.300	-.83188087E-01	*
2.089	1.562	.400	.300	-.67353598E-01	*
2.616	1.556	.500	.300	-.49984081E-01	*
3.144	1.541	.600	.300	-.32999994E-01	*
3.669	1.514	.700	.300	-.18169529E-01	*
4.187	1.469	.800	.300	-.70426346E-02	*
4.693	1.402	.900	.300	-.89959707E-03	*

0.0	2.081	0.000	.400	-.87614954E-01	*
.518	2.082	.100	.400	-.85221736E-01	*
1.038	2.085	.200	.400	-.78240017E-01	*
1.562	2.089	.300	.400	-.67353598E-01	*
2.091	2.091	.400	.400	-.53762519E-01	*
2.625	2.088	.500	.400	-.38999647E-01	*
3.164	2.075	.600	.400	-.24708912E-01	*
3.706	2.046	.700	.400	-.12499901E-01	*
4.247	1.995	.800	.400	-.38604924E-02	*
4.783	1.915	.900	.400	-.81622700E-04	*
0.0	2.597	0.000	.500	-.66592678E-01	*
.514	2.6	.100	.500	-.64605081E-01	*
1.032	2.607	.200	.500	-.58856553E-01	*
1.556	2.616	.300	.500	-.49984081E-01	*
2.088	2.625	.400	.500	-.38999647E-01	*
2.630	2.630	.500	.500	-.27184227E-01	*
3.182	2.623	.600	.500	-.15965362E-01	*
3.743	2.6	.700	.500	-.68148116E-02	*
4.312	2.550	.800	.500	-.11655072E-02	*
0.0	3.108	0.000	.600	-.45697535E-01	*
.508	3.113	.100	.600	-.44169485E-01	*
1.020	3.125	.200	.600	-.39763084E-01	*
1.541	3.144	.300	.600	-.32999994E-01	*
2.075	3.164	.400	.600	-.24708912E-01	*
2.623	3.182	.500	.600	-.15965362E-01	*
3.189	3.189	.600	.600	-.80211801E-02	*
3.773	3.179	.700	.600	-.22345984E-02	*
0.0	3.610	0.000	.700	-.27114170E-01	*
.496	3.617	.100	.700	-.26026703E-01	*
.999	3.638	.200	.700	-.22905559E-01	*
1.514	3.669	.300	.700	-.18169529E-01	*
2.046	3.706	.400	.700	-.12499901E-01	*
2.6	3.743	.500	.700	-.68148116E-02	*
3.179	3.773	.600	.700	-.22345984E-02	*
3.785	3.785	.700	.700	-.39014710E-04	*

0.0	4.098	0.000	.800	-.12526125E-01	*
.479	4.108	.100	.800	-.11839844E-01	*
.966	4.139	.200	.800	-.98962891E-02	*
1.469	4.187	.300	.800	-.70426346E-02	*
1.995	4.247	.400	.800	-.38604924E-02	*
2.550	4.312	.500	.800	-.11655072E-02	*
3.140	4.374	.600	.800	.11316702E-07	*
0.0	4.564	0.000	.900	-.32082022E-02	*
.454	4.579	.100	.900	-.28835637E-02	*
.918	4.623	.200	.900	-.20109283E-02	*
1.402	4.693	.300	.900	-.89959707E-03	*
1.915	4.783	.400	.900	-.81622700E-04	*