On Cohomological Algebras in Supersymmetric Quantum Field Theories

by

Nafiz Ishtiaque

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Physics

Waterloo, Ontario, Canada, 2019

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner:	Erich Poppitz Professor, Dept. of Physics, University of Toronto
Supervisor:	Jaume Gomis Faculty, Perimeter Institute for Theoretical Physics Adjunct Professor, Dept. of Physics and Astronomy, University of Waterloo
Internal Member:	Davide Gaiotto Faculty, Perimeter Institute for Theoretical Physics Adjunct Professor, Dept. of Physics and Astronomy, University of Waterloo
Internal Member:	Niayesh Afshordi Associate Faculty, Perimeter Institute for Theoretical Physics Associate Professor, Dept. of Physics and Astronomy, University of Waterloo
Internal-External Member:	Ruxandra Moraru Associate Professor, Dept. of Pure Mathematics, University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based on the following papers:

1. Jaume Gomis and **Nafiz Ishtiaque**. Kähler potential and ambiguities in 4d $\mathcal{N} = 2$ SCFTs. *JHEP*, 04:169, 2015, 1409.5325.
2. Efrat Gerchkovitz, Jaume Gomis, **Nafiz Ishtiaque**, Avner Karasik, Zohar Komargodski, and Silviu S. Pufu. Correlation Functions of Coulomb Branch Operators. *JHEP*, 01:103, 2017, 1602.05971.
3. **Nafiz Ishtiaque**. 2D BPS Rings from Sphere Partition Functions. *JHEP*, 04:124, 2018, 1712.02551.
4. **Nafiz Ishtiaque**, Seyed Faroogh Moosavian, and Yehao Zhou. Topological Holography: The Example of The D2-D4 Brane System. 2018, 1809.00372.

Abstract

In this thesis we compute certain supersymmetric subsectors of the algebra of observables in some Quantum Field Theories (QFTs) and demonstrate an application of such computation in checking an instance of Holographic duality. Computing the algebra of observables beyond perturbative approximation in weakly coupled field theories is far from a tractable problem. In some special, yet interesting large classes of supersymmetric theories, supersymmetry can be used to extract exact nonperturbative information about certain subsets of observables. This is an old idea which we advance in this thesis by introducing new techniques of computations, computing certain observalbes for the first time, and reproducing earlier results about some other observables. We also propose a new toy model of holographic duality involving topological/holomorphic theories, demonstrating the power of exact computations in supersymmetric subsectors. To be more specific, the subject of this thesis includes the following:

1. Computing the algebra of chiral and twisted chiral operators in 2d $\mathcal{N} = (2, 2)$ theories – while these algebras were previously known, we demonstrate how they can be computed using relatively modern techniques of supersymmetric localization.

2. Computing the chiral rings of 4d $\mathcal{N} = 2$ Superconformal Field Theories (SCFTs) – we compute this algebra for the first time. We use the same method of supersymmetric localization that we use in the 2d case.[1]

3. Computing the algebra of operators on a defect in the topological 2d BF theory, along with its holographic dual. This is a new toy model of holographic duality set in the world of 6d topological string theory. We also argue that this setup is in fact a certain supersymmetric subsector of the holographic duality involving 4d $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory and its 10d supergravity dual – both involving some defects.

In order to be able to discuss these different theories in different dimensions with different symmetries without sounding disparate and ad hoc, we employ the language of cohomological algebra. Since this is perhaps not a language most commonly used in the standard physics literature, we would like to emphasize that this is not a novel idea, it is merely a convenient thematic and linguistic umbrella that covers all the topics of this thesis. In

[1] Chronologically this is the first work demonstrating the use of localization in computing supersymmetry invariant operator algebra. Application of this method to the 2d case came later. However, pedagogically it seems more relevant to present the 2d case first and then the 4d case.

the Batalin-Vilkovisky (BV) formulation of a Quantum Field Theory (QFT), the algebra of observables is presented as the cohomology algebra of a certain complex consisting of fields and anti-fields. In this language restriction to supersymmetric subsectors correspond to modifying the BV differential by the addition of the relevant supersymmetry generator. We simply refer to this modification as reduction to cohomology (with respect to the choice of supersymmetry).

Acknowledgements

I would like to thank my supervisor Jaume Gomis for his patient guidance whenever needed and incredible support in helping me shape my own path in physics. I am also grateful to him for allowing me to focus entirely on my research without being burdened by other usual chores of academia.

I would like to thank Kevin Costello, Davide Gaiotto, and Jaume Gomis for their generosity in sharing their valuable insights that helped making significant leaps in understanding possible for me.

In addition to my supervisor, I must thank Efrat Gerchkovitz, Avner Karasik, Zohar Komargodski, Seyed Faroogh Moosavian, Silviu Pufu, and Yeaho Zhou – this thesis stands on our collaborations.

I thank Nima Doroud, Hee-Cheol Kim, Dalimil Mazac, Junya Yagi, and Yehao Zhou for being extremely kind in answering my uncountable questions on basic physics and mathematics throughout the years which I can not imagine being even remotely enjoyable for them – though they have never been anything but enthusiastically helpful, I could not have asked for better teachers.

Perimeter Institute has been an extraordinary distraction-free workplace. I especially thank Debbie Guenther for always carefully taking care of all the formal affairs allowing me to not worry about them.

Finally, I would like to express my gratitude toward Mahbub Majumdar and Arshad Momen. They were my first exposure to theoretical physics, they supported my hope for higher studies when they had no reason to, and without their encouraging and optimistic support I would never have had any chance of pursuing my studies in physics.

Table of Contents

Abbreviations

BV Batalin-Vilkovisky vi

CFT Conformal Field Theory 25

CFTs Conformal Field Theories 3

CS Chern-Simons 93

CY Calabi-Yau 22

FI Fayet-Iliopoulos 32

GLSM Gauged Linear Signa Model 22

IR infrared 26

KW Kapustin-Witten 140

LG Landau-Ginzburg 22

OPE Operator Product Expansion 3

QFT Quantum Field Theory vi

QFTs Quantum Field Theories v

QM Quantum Mechanics 2

RG Renormalization Group 14

RW Rozansky-Witten 142

SCFT Superconformal Field Theory 25

SCFTs Superconformal Field Theories v, 22

SUSY Supersymmetry 196

SYM Super Yang-Mills v

UV ultraviolet 22

Chapter 1

Introduction

We begin with an elementary discussion about operator algebras in quantum field theories and their restriction to supersymmetric subsectors. This discussion will be broadly general and perhaps lacking in rigor – the aim being simply to serve as a motivation and a common theme behind the concrete and specific investigations into different theories that we lay out in the latter chapters. Anyone already interested in the particular theories considered in the latter portions of the thesis can safely ignore the introduction without any loss of essential technical background.

1.1 Operator Algebra

Among the primary objects of interest in a quantum field theory are *observables*. For any QFT defined on a space-time[1] manifold M, observables are *probes* assigned to submanifolds of M. In a field theoretic model of some physical dynamics, the observables represent measurements made in some spatial location at some time. Observables are also referred to as *operators*[2] and we shall use these two terms interchangeably. Observables assigned to points are called *local*, those assigned to 1-dimensional submanifolds are called *line operators*, the ones assigned to 2 dimensinoal submanifolds are *surface operators*, and so on. We will mostly be concerned with local operators in this thesis.

[1] In this thesis we are only concerned with *Riemannian* space-time manifolds, i.e., there will be no distinguished temporal direction.

[2] There is a canonical way of assigning vector spaces to codimension 1 submanifolds of M and observables defined on submanifolds of codimension at least 1 can act on these vector spaces as operators [149].

The collection of observables in a QFT has a certain algebraic structure encoding the bulk of the dynamical information. This algebra is known as a *factorization algebra*. We will not delve into the exact definition of such an algebra – this is part of the standard literature on mathematical physics and we refer to [40] for the technical details. In this introduction we only sketch an intuitive picture of this algebra and in the latter chapters when we focus on specific theories we will concretely define the algebras that we will compute. In fact, the main theme of this thesis is that in favorable conditions, supersymmetry can be used to define certain reductions of these very general factorization algebras to more familiar and much simpler algebras such as associative algebras (in Quantum Mechanics (QM)) or vertex operator algebras [135] – these simpler, often computable,[3] algebras can still contain highly nontrivial dynamical information.

Following [40], let us quickly motivate the basic structures behind operator algebras. If we imagine observables as measuring devices for some kind of experiment in our dynamical system, then it makes sense to position these devices at arbitrary locations in space-time so longs as we don't put multiple devices at the same place at the same time. For any open subset $U \subseteq M$ of our space-time let us denote by $\mathrm{Obs}(U)$ the set of observables that can be inserted in the region U. Now, if $U \subseteq M$ and $U' \subseteq M$ are two disjoint open subsets of M, then combining any two observables located in U and U' we should get an observable located in $U \cup U'$ – i.e., we should have a map:

$$\mathrm{Obs}(U) \times \mathrm{Obs}(U') \to \mathrm{Obs}(U \cup U') \,. \tag{1.1}$$

If U and V are open subsets of M such that U is contained in V, i.e. $U \subseteq V \subseteq M$, then there should be a way to think of the measurements done in the region U as measurements done in the region V as well, thus we expect a map:

$$\mathrm{Obs}(U) \to \mathrm{Obs}(V) \,. \tag{1.2}$$

The above map should satisfy a compatibility condition. If we have a chain of inclusions of open subsets of the space-time $U \subseteq V \subseteq W \subseteq M$ then we can view a measurement done in the region U as a measurement done in the region W in two ways:

1. Using the inclusion $U \subseteq W$,

2. By first thinking of it as a measurement done in the region V and then using the inclusion $V \subseteq W$.

[3] As opposed to the full factorization algebras themselves.

These two ways of thinking should be the same for physical consistency and thus we expect the following diagram to commutes:

$$\begin{tikzcd}
\mathrm{Obs}(U) \arrow[rr] \arrow[dr] & & \mathrm{Obs}(W) \\
& \mathrm{Obs}(V) \arrow[ur] &
\end{tikzcd} \tag{1.3}$$

The conditions (1.1), (1.2), and (1.3) together define the structure of a *prefactorization algebra*, which is a short step away from a factorization algebra. Intuitively, a factorization algebra is a prefactorization algebra where the observables associated to an open set $U \subseteq M$ can be constructed using information about the observables associated to all the open subsets of U. This intuition can be made precise though we do not attempt to do so here as this is mostly a technicality that will play no role in this thesis – the precise characterization of a factorization algebra can be found in Definition 1.3.1 of [40].

Remark 1.1.1 (Where are correlation functions?). In the standard textbook treatment of QFTs the emphasis is often on computing *correlation functions* or *expectation values*[4] of observables, rather than the algebra of observables.[5] Formally, the ability to compute expectation values of observables require a small piece of information *in addition to* the factorization algebra – this extra information is a choice of a *vacuum*, which is a map:[6]

$$\langle - \rangle : \mathrm{Obs}(M) \to \mathbb{C}[\![\hbar]\!] \,. \tag{1.4}$$

Note that for any two disjoint open subsets U and V of M the map $\langle - \rangle$ extends to the product $\mathrm{Obs}(U) \otimes \mathrm{Obs}(V)$ by composing with (1.1) and (1.2):

$$\mathrm{Obs}(U) \times \mathrm{Obs}(V) \to \mathrm{Obs}(U \cup V) \to \mathrm{Obs}(M) \xrightarrow{\langle - \rangle} \mathbb{C}[\![\hbar]\!] \,. \tag{1.5}$$

We require this extension to be linear on both factors. In this fashion we can extend $\langle - \rangle$ to act multilinearly on arbitrary products $\prod_{i=1}^{n} \mathrm{Obs}(U_i)$ for a set $\{U_i\}_{i=1}^{n}$ of mutually disjoint open subsets of M and given an operator $O_i \in \mathrm{Obs}(U_i)$ from each open set we can compute the *correlation function*:

$$\langle O_1 \cdots O_n \rangle \in \mathbb{C}[\![\hbar]\!] \,. \tag{1.6}$$

△

[4] We use these two terms interchangeably.

[5] A potential exception is a discussion on Conformal Field Theories (CFTs) where computing the Operator Product Expansion (OPE) – a name for the product (1.1) – takes up a significant amount of attention.

[6] We use the *Planck's constant* here for the first time. In this thesis the primary role of this parameter will be to keep track of the loop order in perturbation theory. When this is not needed we will often set $\hbar$ to 1 and not worry about it.

1.2 Scalar Field Theory

Let us consider an example of a real scalar field theory on a d-dimensional space-time M with a Riemannian metric g and apply some of the general terminology we have been using to the practical case. The dynamical variables, the fields, of this theory are real valued smooth functions on M:

$$\text{Space of fields, } \mathcal{E} := C^\infty(M)\,. \tag{1.7}$$

The action is a real valued function on the space of fields which is bounded from bellow:

$$S(\phi) = \frac{1}{2}\int_M \mathrm{d}^d x\sqrt{g}\,\phi(\Delta + m^2)\phi + I(\phi)\,. \tag{1.8}$$

Here Δ is the Laplacian on M defined with respect to the metric g, m is a mass parameter of the theory, and $I(\phi)$ is a polynomial function on the space of fields. Observables or operators of any theory are generally functions on the space of fields of the theory.[7] Given any open subset $U \subseteq M$ of M, fields with support on U belong to $C^\infty(U)$ and a first approximation to the space of observables with support in U is given by a space of functions on $C^\infty(U)$:[8]

$$\mathrm{Obs}(U) := \mathscr{O}(C^\infty(U))\,. \tag{1.9}$$

A *linear* operator with support in U can be defined given a distribution γ with support in U as:

$$O(\gamma) : \phi \mapsto \int_M \mathrm{d}^d x\sqrt{g}\,\gamma\phi\,. \tag{1.10}$$

Since a distribution with support in U can be trivially extended to be considered as a distribution with support in V for any open set $V \supseteq U$ containing U, we have a natural map $\mathrm{Obs}(U) \hookrightarrow \mathrm{Obs}(V)$ given simply by inclusion.[9] If γ is a Dirac delta function with support at some point p then we write:

$$O(p) : \phi \mapsto \int_M \mathrm{d}^d x\sqrt{g}\,\delta^{(d)}(x-p)\phi = \phi(p)\,. \tag{1.11}$$

Operators with delta function support are *local*. It is also common practice to write $\phi(p)$ to refer to the operator $O(p)$.

[7] Functions on the space of fields are sometimes referred to as *functionals*.

[8] We are not being specific here about what type of functions we want. In specific situations we will choose some appropriate type of functions so that we can integrate them with respect the measure on the field space defined by the action.

[9] The diagram (1.3) is automatically commutative if the arrows are inclusions.

Remark 1.2.1 (Equations of motion). Saying that (1.9) is the space of observables is not strictly correct. Because, the functions on the field space – the operators – must satisfy the equations of motion and therefore, the true space of obsevables should be the quotient of $\mathscr{O}(C^\infty(U))$ by the equations of motion. A formal (homological) way of doing that is to replace $\mathscr{O}(C^\infty)$ by a cochain complex which is a resolution of the desired quotient. This is the core of the BV formalism, which assigns to each open set U a cochain complex. This assignment is also a factorization algebra, and it is this factorization algebra that gives us the physical operator product. △

Given an operator O, its expectation value is simply its average over the field space $\mathcal{E}$ with respect to a measure $\mathcal{D}\phi\, e^{-\frac{1}{\hbar}S(\phi)}$:[10]

$$\langle O \rangle = \int_{\mathcal{E}} \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S(\phi)} O\,. \tag{1.12}$$

The measure $\mathcal{D}\phi$ on the field space is an infinite dimensional generalization of the Euclidean measure on $\mathbb{R}^n$.

Product of operators must be compatible with computing correlation functions, i.e., if the product of two operators O_1 and O_2 is O_3, then we must have:

$$\langle O_1 O_2 \mathcal{O} \rangle = \langle O_3 \mathcal{O} \rangle\ , \tag{1.13}$$

for any arbitrary operator $\mathcal{O}$. In particular, if there is an operator O_0 such that any correlation function containing it vanishes:

$$\langle O_0 \mathcal{O} \rangle = 0\,, \tag{1.14}$$

for any operator $\mathcal{O}$, then we should identify O_0 with 0 in the operator algebra. Finding operators that look nontrivial when written in terms of fields but have vanishing correlators with all other operators mean imposing relations among the operators in (1.9). For example, there will be relations among the observables defined in (1.9) coming from the equations of motion. The description of the operator algebra that we have given so far often suffices, in fact, this is how we shall identify certain operator algebras in chapters 2 and 3 – we shall first characterize the operators of interest in terms of the fields (as in (1.9) and (1.11)) and we shall find out the relations that always holds among these operators inside correlation functions.

[10]These are the *unnormalized* expectation values. Conventionally one normalizes the path integral so that the expectation value of the identity operator is 1. This is not important for our current general discussion.

Remark 1.2.2 (An analogy with finite dimensional integrations). There is some resemblance between a QFT and a theory of integration. Computing correlation functions such as (1.12) is essentially about defining a measure of integration on the space of fields, which is generally a daunting task. On a finite dimensional manifold with a metric one can define various measures of integration which can be used to integrate functions on said manifold. Instead of worrying about all the functions one can focus attention only to those that are closed with respect to the de Rham differential and this reduces the problem of integration to the "simpler" theory of de Rham cohomology. It is simpler in the sense that integration of cohomology classes only requires topological information about the manifold and all finer structures (such as differential geometry) can be safely ignored. Supersymmetry can be used to serve a similar purpose in a QFT. This is the key motivation behind the works of this thesis. △

1.3 Supersymmetry and Cohomology

A (Lie) symmetry algebra of a QFT is an algebra $\mathfrak{g}$ that acts on the field space[11] $\mathcal{E}$ (or rather on the functions on the field space, the observables) and leaves invariant the integration measure on $\mathcal{E}$ defined by the action. A supersymmetry algebra is a $\mathbb{Z}_2$ graded symmetry algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of a QFT with a graded bracket:

$$[-,-] : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0 \,, \qquad [-,-] : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1 \,, \qquad \{-,-\} : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0 \,. \tag{1.15}$$

The subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$ is called *bosonic* and its representation $\mathfrak{g}_1$ is called *fermionic*. Suppose we have a fermionic generator, also called a *supercharge*, $Q \in \mathfrak{g}_1$ which squares to 0:

$$\{Q, Q\} = 0 \,. \tag{1.16}$$

Then Q is a nilpotent operator that acts on the functions on our field space (the operators) and leaves our integration measure on the field space invariant:[12]

$$Q : \mathrm{Obs}(M) \to \mathrm{Obs}(M) \,, \qquad Q\left(\mathcal{D}\Phi\, e^{-S(\Phi)/\hbar}\right) = 0 \,. \tag{1.17}$$

It is then natural to consider the cohomology of Q:[13]

$$H_Q^\bullet(\mathrm{Obs}(M)) \,, \tag{1.18}$$

[11] For the special case of a scalar field theory this was defined in (1.7)

[12] We assume that the QFT is defined on a space-time manifold M with an action S and we use Φ to schematically represent all the fields of the theory.

[13] Note that only the stabilizer of Q in $\mathfrak{g}$ has a sensible action on the cohomology. Symmetry of the cohomological theory will therefore generally be a subgroup of the symmetry of the original theory. And the action of the ideal of $\mathfrak{g}$ generated by Q on the observables of the theory becomes trivial.

and wonder whether:

1. integrating these cohomology classes is simpler than integrating arbitrary functions,
2. the algebra of the cohomology classes is simpler to find than the factorization algebra of all operators and if there is any interesting information left in the simpler algebra.

The answer to these questions will vary from case to case. Note that just as de Rham cohomology is invariant under smooth deformations of the underlaying manifold, the algebraic structure that survives in $H^\bullet_Q(\mathrm{Obs}(M))$ is also invariant under any deformation of the QFT that is Q-exact. In particular, any deformation of the QFT that leads to a Q-exact deformation of the integraiton measure on field space leaves the Q-cohomology invariant. One such deformation is given by a Q-exact deformation of the action:

$$S \to S + \{Q, V\}\,, \tag{1.19}$$

where V is some fermionic functional. Invariance under such deformations has been used to great effect in explicitly showing that integrating Q-cohomology classes can indeed be a much simpler problem than integrating arbitrary operators. These results go by the name of *supersymmetric localization* [140] and we shall use some of these results in the latter chapters.

Let us take a somewhat detailed look at the simplest possible example of supersymmetry where taking cohomology leads to considerable simplification while preserving some nontrivial structure.

1.3.1 A Toy Model: 0 Dimensional QFT

A 0-dimensional QFT is simply a theory of maps from a point to a target space with an action functional. Since maps from a point are completely parametrized by the the image of the point, the space of fields in this theory can be identified with the target space itself. The action functional is then an ordinary function on the target space with a lower bound and the path integral measure is easily constructed from the integration measure on the target space. In this section we set $\hbar = 1$, it does not influence the claims of this section.

Suppose X is a smooth compact connected oriented Riemannian manifold of dimension n. TX is the tangent bundle of X and let ΠTX denote the *shifted* tangent bundle of X, by which we mean the tangent bundle with the tangential directions being parametrized by grassmann variables. We will see that a generic 0-dimensional QFT with the target space being ΠTX captures the smooth geometry of differential forms on X and their integration. A 0-dimensional QFT with the additional structure of supersymmetry allows us to take cohomology, which in this case corresponds precisely to reduction to de Rham cohomology of X.

Let us choose local coordinates x^i on X and coordinates θ^i for the grassmannian/odd tangential directions.[14] The even coordinates x^i are commutative and the odd coordinates θ^i are anti-commutative:

$$x^i x^j = x^j x^i\,, \qquad \theta^i\theta^j = -\theta^j\theta^i\,, \qquad x^i\theta^j = \theta^j x^i\,. \tag{1.20}$$

We now consider a 0-dimensional QFT of maps from a point to ΠTX. The fields of this theory consist of the bosonic fields x^i and the fermionic fields θ^i. The space of fields is the target space ΠTX itself:

$$\text{Spae of fields, } \mathcal{E} = \Pi TX\,. \tag{1.21}$$

The action of the theory is a function of the fields:

$$S \in C^\infty(\Pi TX)\,, \tag{1.22}$$

which defines a measure of integration on the field space:[15]

$$\mathrm{d}^n x \mathrm{d}^n\theta\, e^{-S(x,\theta)}\,. \tag{1.23}$$

Observables of the theory are smooth functions on the field space:

$$\mathrm{Obs} = C^\infty(\Pi TX)\,. \tag{1.24}$$

For any operator $O \in \mathrm{Obs}$, its expectation value is its integral over the field space with the measure defined by the action:

$$\langle O\rangle = \int_{\Pi TX} \mathrm{d}^n x\mathrm{d}^n\theta\, e^{-S(x,\theta)} O(x,\theta)\,. \tag{1.25}$$

[14] Coordinates on a vector space are valued in the dual of the vector space. Which means, in particular, that under diffeomorphism of X, θ^i transforms as $\mathrm{d}x^i$.

[15] Note that the measure $\mathrm{d}^n x\mathrm{d}^n\theta e^{-S(x,\theta)}$ is diffeomorphism invariant. Under a coordinate transformation $x^i \mapsto \phi^i(x)$ the measure $\mathrm{d}^n x$ transforms as $\mathrm{d}^n x \mapsto \det\left(\frac{\partial\phi^i}{\partial x^j}\right)\mathrm{d}^n x$. Under the same transformation θ^i transforms as $\theta^i \mapsto \frac{\partial\phi^i}{\partial x^j}\theta^j$ (since θ^i transforms as $\mathrm{d}x^i$). Therefore the product $\theta^1\cdots\theta^n$ transforms as $\theta^1\cdots\theta^n \mapsto \det\left(\frac{\partial\phi^i}{\partial x^j}\right)\theta^1\cdots\theta^n$ and the invariance of the grassmann integration $\int \mathrm{d}\theta^1\cdots\mathrm{d}\theta^n\,\theta^1\cdots\theta^n = (-1)^{n-1}$ then implies that the measure $\mathrm{d}^n\theta$ transforms as $\mathrm{d}^n\theta \mapsto \det\left(\frac{\partial\phi^i}{\partial x^j}\right)^{-1}\mathrm{d}^n\theta$. The measure $\mathrm{d}^n x\mathrm{d}^n\theta$ and in turn $\mathrm{d}^n x\mathrm{d}^n\theta e^{-S(x,\theta)}$ (since S is a scalar) is therefore invariant.

Trivial Action: Differential Forms

If we take the action to be trivial $S = 0$, then this theory just captures the smooth structure of X, i.e., the theory of differential forms $\Omega^\bullet(X)$ and their integration on X.

Note that, when $S = 0$, the measure on ΠTX is simply $\mathrm{d}^n x \mathrm{d}^n \theta$. The integration over the odd variables in computing $\langle O \rangle$ will then pick out the coefficient of $\theta^1 \cdots \theta^n$ in the expansion of O, i.e., if we consider the expansion of O in the odd variables:

$$O(x,\theta) = O_{1\cdots n}(x)\theta^1 \cdots \theta^n + (\text{terms with fewer odd variables}) \tag{1.26}$$

then after performing the grassmann integration we get an integration over just X:

$$\langle O \rangle = \int_X \mathrm{d}^n x \, O_{1\cdots n}(x) \,. \tag{1.27}$$

This is an integration of a top form $O_{1\cdots n}(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \in \Omega^n(X)$ on X. We can go ahead and define an isomorphism

$$\mathrm{Obs} = C^\infty(\Pi TX) \xrightarrow{\sim} \Omega^\bullet(X) \,, \tag{1.28}$$

between operators in our theory and differential forms on X which acts as:

$$O(x,\theta) = \sum_{k=0}^{n} O_{i_1 \cdots i_k}(x)\theta^{i_1} \cdots \theta^{i_k} \xmapsto{\sim} \sum_k O_{i_1\cdots i_k}(x)\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} =: \widetilde{O} \,. \tag{1.29}$$

Then OPE in this theory is equivalent to wedge product of forms on X:[16]

$$\begin{tikzcd} C^\infty(\Pi TX) \times C^\infty(\Pi TX) \arrow[r, "\sim"] \arrow[d, "\mathrm{OPE}"'] & \Omega^\bullet(X) \times \Omega^\bullet(X) \arrow[d, "\wedge"] \\ C^\infty(\Pi TX) \arrow[r, "\sim"] & \Omega^\bullet(X) \end{tikzcd} \tag{1.30}$$

And as discussed earlier, expectation value of an operator is given by integration of forms:[17]

$$\langle O \rangle = \int_X \widetilde{O} \,. \tag{1.31}$$

[16]Note that this theory is too special in the sense that the space-time is just a point and there is only one open set – meaning that there is no way to see the structure of a factorization algebra. However, due to the simple nature of the theory we can still define operator product at the coincident point.

[17]Note that for some operator $O \in C^\infty(\Pi TX)$ we refer to its image in $\Omega^\bullet(X)$ under the isomorphism (1.28) as $\widetilde{O}$.

Generic Action: More Differential Forms

For an arbitrary action $S(x,\theta)$ we can expand the exponential weight e^{-S} in the grassmann variables as:

$$e^{-S(x,\theta)} = W^{(0)}(x) + W^{(1)}_i(x)\theta^i + W^{(2)}_{ij}(x)\theta^i\theta^j + \cdots\,. \tag{1.32}$$

The coefficients $W^{(k)}_{i_1\cdots i_k}$ can be identified as coefficients of a k-form on X, which we denote as $\widetilde{W}^{(k)}$:

$$\widetilde{W}^{(k)} := W^{(k)}_{i_1\cdots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\,. \tag{1.33}$$

An arbitrary operator $O \in \mathrm{Obs}$ can similarly be expanded:

$$O(x,\theta) = O^{(0)}(x) + O^{(1)}_i(x)\theta^i + O^{(2)}_{ij}\theta^i\theta^j + \cdots\,, \tag{1.34}$$

and as in (1.33) we define the forms corresponding to the coefficients:

$$\widetilde{O}^{(k)} := O^{(k)}_{i_1\cdots i_k}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\,. \tag{1.35}$$

The product $e^{-S}O$ has a similar expansion, for the purpose of integrating over the grassmann variables however, we only care about the terms with the maximal number of grassmann variables:

$$e^{-S(x,\theta)}O(x,\theta) = \sum_{m=0}^{n} W^{(m)}_{i_1\cdots i_m}O^{(n-m)}_{j_1\cdots j_{n-m}}\theta^{i_1}\cdots\theta^{i_m}\theta^{j_1}\cdots\theta^{j_{n-m}} + \cdots\,. \tag{1.36}$$

The path integral evaluating the expectation value of O now becomes:

$$\langle O\rangle = \int_X \mathrm{d}^n x\int \mathrm{d}\theta^1\cdots\mathrm{d}\theta^n\, e^{-S(x,\theta)}O(x,\theta) = \sum_{m=0}^{n}\int_X \widetilde{W}^{(m)}\wedge\widetilde{O}^{(n-m)}\,. \tag{1.37}$$

This is a slight generalization of (1.31), qualitatively not much different, we are still dealing with integration of arbitrary forms on X utilizing the smooth geometry.

Supersymmetry and Reduction to Cohomology

We can define a *fermion number* operator on our space of operators, which literally just counts the number of θ fields in an operator:[18]

$$[F, x^i] = 0\,,\qquad [F,\theta^i] = \theta^i\,. \tag{1.38}$$

[18] Note that x^i and θ^i are coordinates of our field space ΠTX and therefore they are also operators in the sense of being functions on the space of fields. These are referred to as *coordinate functions*, which takes as input a point of ΠTX and outputs the value of the respective coordinates at that point.

Supersymmetry is a transformation of the operator space:

$$Q : C^\infty(\Pi TX) \to C^\infty(\Pi TX)\,, \tag{1.39}$$

which is odd, i.e., has fermion number 1:

$$[F, Q] = Q\,. \tag{1.40}$$

This leads to a grading on our space of operators:

$$C^\infty(\Pi TX) = \bigoplus_{k=0}^{n} \mathrm{Obs}^k\,, \tag{1.41}$$

where Obs^k is defined by the property that F restricted to Obs^k acts as k:

$$[F, -]_{\mathrm{Obs}^k} = k\,\mathrm{id}_{\mathrm{Obs}^k}\,. \tag{1.42}$$

A theory is supersymmetric if the action and the field space measure it defines are invariant under this transformation:

$$[Q, S] = 0\,, \qquad [Q, \mathrm{d}^n x \mathrm{d}^n\theta\, e^{-S}] = 0\,. \tag{1.43}$$

Let us now specialize to a theory with the following supersymmetry transformations of operators:

$$[Q, x^i] = \theta^i\,, \qquad [Q, \theta^i] = 0\,. \tag{1.44}$$

The action of Q extends to arbitrary elements of $C^\infty(\Pi TX)$ via Leibniz's rule. Note that this operator is nilpotent:

$$Q^2 = 0\,, \tag{1.45}$$

and via the isomorphism (1.28) between operators of our theory and differential forms on X, corresponds precisely with the action of the de Rham differential on X:

$$\begin{tikzcd} C^\infty(\Pi TX) \arrow[r, "\sim"] \arrow[d, "Q"'] & \Omega^\bullet(X) \arrow[d, "\mathrm{d}_{\mathrm{dR}}"] \\ C^\infty(\Pi TX) \arrow[r, "\sim"] & \Omega^\bullet(X) \end{tikzcd} \tag{1.46}$$

Let us summarize the identifications between field theoretic quantities and geometric objects on X:

0-dimensional QFT		Geometry of X	
Bosonic operator	x^i	x^i	Coordinate
Fermionic operator	θ^i	$\mathrm{d}x^i$	Basis for cotangent space
Space of all operators	$C^\infty(\Pi TX)$	$\Omega^\bullet(X)$	Differential forms
Fermion number			Form degree
Space of operators with fermion no. k	Obs^k	$\Omega^k(X)$	Differential k-forms
Operator product			Wedge product
Supercharge	Q	d_{dR}	de Rham differential

(1.47)

We can now define a cohomology of operators with respect to the supercharge Q:

$$H^\bullet_Q(C^\infty(\Pi TX)) := \ker Q/\mathrm{im}\, Q\,. \tag{1.48}$$

The grading is given by fermion number. From the above identifications (1.47) it is clear that we have isomorphism between this operator cohomology and de Rham cohomology of X:

$$H^\bullet_Q(C^\infty(\Pi TX)) \cong H^\bullet_{\mathrm{dR}}(\Omega^\bullet(X))\,. \tag{1.49}$$

The reason why this isomorphism is more than just about tabulating supersymmetric operators is that if we restrict our attention only to operators in this cohomology, then their expectation values correspond to integrating cohomology classes of X on homology cycles in X. To see this, note that we can expand the weight e^{-S} in the fermionic variables and use the isomorphism (1.28) to identify the weight with a generic element of $\Omega^\bullet(X)$:

$$e^{-S} \overset{\sim}{\mapsto} e^{-\widetilde{S}} = \sum_{i=0}^{n} W^{(i)}(x)\,, \qquad W^{(i)} \in \Omega^i(X)\,. \tag{1.50}$$

In the equality above, the exponential of the differential form $\widetilde{S}$ is expanded with wedge product. The condition for supersymmetry then translates to the statement that each homogeneous piece $W^{(i)}$ is a representative of a cohomology class:

$$[Q,S]=0 \quad \Rightarrow \quad \mathrm{d}_{\mathrm{dR}}\widetilde{S}=0 \quad \Rightarrow \quad \mathrm{d}_{\mathrm{dR}}e^{-\widetilde{S}}=0 \quad \Rightarrow \quad \mathrm{d}_{\mathrm{dR}}W^{(i)}=0\,. \tag{1.51}$$

Using Poincaré duality we can relate $e^{-\widetilde{S}}$ with a linear combination of homology cycles of X:

$$e^{-\widetilde{S}} \xrightarrow{\text{Poincaré dual}} \sum_{i=0}^{n} \mathsf{X}_{(n-i)}\,, \tag{1.52}$$

where $[\mathsf{X}_{(n-i)}] \in H_{n-i}(X)$ is the Poincaré dual to $W^{(i)}$. Similarly, any operator $O \in C^\infty(\Pi TX)$ which is supersymmetric $[Q, f] = 0$, can be indeintified with a representative of a cohomology class:

$$f \xrightarrow{\sim} \widetilde{O} = \sum_{i=0}^{n} O^{(i)}\,, \qquad [O^{(i)}] \in H^i(X)\,. \tag{1.53}$$

Given that the action S and the operator O are supersymmetric the product $e^{-S}O$ is supersymmetric:

$$\left[Q, e^{-S}O\right] = 0\,, \tag{1.54}$$

and this product can now be identified with the wedge product:

$$e^{-S}O \overset{\sim}{\mapsto} e^{-\widetilde{S}} \wedge \widetilde{O}\,. \tag{1.55}$$

Furthermore, the expectation value of O with respect to the measure defined by the supersymmetric action now reduces to a pairing between homology cycles and cohomology of X:

$$\langle O\rangle = \int_X e^{-\widetilde{S}} \wedge \widetilde{O} = \sum_{i,j=0}^{n} \int_X \widetilde{W}^{(i)} \wedge \widetilde{O}^{(j)} = \sum_{i=0}^{n} \int_{\mathsf{X}_{(i)}} \widetilde{O}^{(i)}\,, \tag{1.56}$$

where the last equality follows from Poincaré duality. Thus in this theory the action makes a choice of cycles $\mathsf{X}_{(i)}$, operators are cocycles, OPE is the wedge product, and taking expectation values means pairing the cycles $\mathsf{X}_{(i)}$ with the respective cocycles. This is clearly a topological skeleton of the expectation value (1.37), and this cohomological 0-dimensional QFT is therefore invariant under continuous deformations of the target space.

The point here is that the reduction from the full 0-dimensional QFT to its Q-cohomology parallels closely the reduction from the smooth structure of the target space X to its theory of de Rham cohomology, including Poincaré duality and integration. This is one of the simplest examples where the expectations expressed at the end of §1.3 of achieving simplification and robustness by considering supersymmetric cohomologies is fulfilled.

1.4 Lessons from BV

Describing observables of a QFT as a cohomology of an infinite dimensional manifold (the field space) is actually quite ubiquitous – observables of all QFTs with a Lagrangian description can be written as a cohomology with respect to a certain differential operator on an extended field space. This is the outcome of the *Batalin-Vilkovisky (BV)* formalism of quantization [11, 12, 40]. The BV formalism is a a formal way of implementing equations of motion on the space of operators. In the computations of this thesis we will not use the BV formalism and so we will not go into much details of its technicalities. However, it provides a nice unified language to talk about operator algebras in QFTs and for this reason we gather some key statements of the formalism and state our point of view on cohomological operator algebras in QFTs using this language. There is no computation in this section, only definitions and notations. We discuss BV formalism for a 0d QFT in a bit more detail in Appendix A with an aim to making the definitions in this section seem at least somewhat reasonable by analogy. For a thorough treatment of the BV formalism we refer to [39, 40].

BV Data of A QFT

Consider an *effective* QFT on a space-time M at a *scale* L.[19] Associated to each open set $U \subseteq M$ there is a space of fields $\mathcal{E}(U)$ (contains both fields and anti-fields) which is a graded vector space.[20] An effective action functional $\mathcal{S}_L : \mathscr{O}(\mathcal{E}(M)) \to \mathbb{R}[\![\hbar]\!]$ defines the dynamics of the theory. The action functional can be separated into two parts, $\mathcal{S}_L = S_0 + \mathcal{I}_L$, where S_0 is a local quadratic functional defining a free theory and is scale independentand $\mathcal{I}_L$ is an effective interaction.[21] The quadratic free action S_0 and the scale L defines a nilpotent (cohomological) degree 1 differential operator on $\mathscr{O}(\mathcal{E}(M))$ called the *BV Laplacian*:[22]

$$\Delta_L^{\mathrm{BV}} : \mathscr{O}(\mathcal{E}(M)) \to \mathscr{O}(\mathcal{E}(M)) \,, \qquad \Delta_L^{\mathrm{BV}} \circ \Delta_L^{\mathrm{BV}} = 0 \,. \tag{1.57}$$

In turn, this operator defines a degree 1 Poisson bracket:

$$\begin{gathered} \{-,-\}_L : \mathscr{O}(\mathcal{E}(U)) \times \mathscr{O}(\mathcal{E}(U)) \to \mathscr{O}(\mathcal{E}(U)) \,, \\ \{A, B\}_L = \Delta_L^{\mathrm{BV}}(AB) - \Delta_L^{\mathrm{BV}}(A)B - (-1)^{|A|} A \Delta_L^{\mathrm{BV}}(B) \,. \end{gathered} \tag{1.58}$$

Renormalization Group (RG) flow relates effective actions at different scales. By RG we mean *Wilsonian* RG. Let us refer to the action of RG flow from a scale $\ell < L$ to L as

[19] We take L to be a *length* scale. $L \to 0$ is the UV limit of the theory, which may or may not exist.

[20] Generalizing the finite dimensional field space (A.14).

[21] For $L > 0$ the interactions are generally non-local, roughly speaking the interactions are spread out over a space-time region of length scale L. Which goes back to the claim that the limit $L \to 0$ is the UV limit, as interaction becomes point-like.

[22] This is an infinite dimensional generalization of the finite dimensional BV Laplacian (A.17), further deformed by the scale in such a way that $\lim_{L\to 0} \Delta_L^{\mathrm{BV}}$ is the straightforward generalization of the finite dimensional operator to infinite dimension.

W_ℓ^L, we write:

$$W_\ell^L(\mathcal{I}_\ell) = \mathcal{I}_L \,. \tag{1.59}$$

Given the interactions at scale ℓ one finds the interactions at scale L by integrating out modes of the fields that fall between the two energy scales defined by ℓ and L. Schematically,

$$(S_0 + \mathcal{I}_L)(\Phi_{<L^{-1}}) = -\hbar \log \int \mathcal{D}\Phi_{L^{-1}<-<\ell^{-1}} e^{-\frac{1}{\hbar}(S_0+\mathcal{I}_\ell)(\Phi_{\text{tot}})} \,, \tag{1.60}$$

where $\Phi_{\text{tot}} = \Phi_{L^{-1}<-<\ell^{-1}} + \Phi_{<L^{-1}}$ includes both high energy and low energy fields. The free theory S_0 is a reference theory for the *family of effective theories* parametrized by scale.

Observables

At any scale L, there is the following degree 1 derivation (c.f. (A.16))

$$\mathrm{d}_L := \Delta_L^{\mathrm{BV}} - \frac{1}{\hbar}\{\mathcal{S}_L, -\}_L : \mathscr{O}(\mathcal{E}(U))[\![\hbar]\!] \to \mathscr{O}(\mathcal{E}(U))[\![\hbar]\!] \tag{1.61}$$

which is nilpotent:

$$\mathrm{d}_L^2 = 0 \,. \tag{1.62}$$

This equation is also referred to as the *Quantum Master Equation.*[23]

The observables of the theory are given by a cosheaf of cochain complexes, for any open set $U \subseteq M$ the cosheaf is given by (c.f. (A.18) – observables in a 0d QFT):

$$U \mapsto \mathrm{Obs}_L(U) := (\mathscr{O}(\mathcal{E}(U))[\![\hbar]\!], \mathrm{d}_L) \,. \tag{1.63}$$

Note that the underlying graded vector space of Obs_L and Obs_ℓ are the same for any L and ℓ, it is the differential that varies with the scale. There is an isomorphism[24] between the cochain complexes Obs_L and Obs_ℓ for two different scales ℓ and $L > \ell$ given by RG:

$$\begin{aligned} \mathcal{W}_\ell^L &: \mathrm{Obs}_\ell \xrightarrow{\sim} \mathrm{Obs}_L \,, \\ \mathcal{W}_\ell^L &: O \mapsto \frac{\mathrm{d}}{\mathrm{d}\epsilon} W_\ell^L(\mathcal{I}_\ell + \epsilon O)\Big|_{\epsilon=0} \,. \end{aligned} \tag{1.64}$$

[23]This is a highly non-trivial equation. It constraints effective interactions at any scale L, and since the equation must hold at any scale, it is also a consistency condition for transformation of effective interactions under RG flow.

[24]A homotopy, as some might say.

Remark 1.4.1 (Operators as deformations). Note that ϵ is essentially being treatd as an infinitesimal variable and ϵO is being treated as an infinitesimal deformation of the effective theory defined by the effective interaction $\mathcal{I}_\ell$. Observables in this sense span the tangent space of an infinite dimensional space of effective QFTs on which RG acts. The above isomorphism identifies tangent vectors at different points of this space related by the RG action. △

Because of the isomorphism defined above we can write just $\mathrm{Obs}(U)$ without refering to any particular scale, as long as we keep in mind that an operator is really an orbit – for the RG flow – of operators in the vector space underlying $\mathrm{Obs}(U)$ (namely, $\mathscr{O}(\mathcal{E}(U))[\![\hbar]\!]$). Thus, when we say $O \in \mathrm{Obs}(U)$ is an operator, we are referring not to any single element of $\mathscr{O}(\mathcal{E}(U))[\![\hbar]\!]$, but to an entire orbit $\{O[L]\}$ related by the RG flow:

$$O[L] = \mathcal{W}_\ell^L(O[\ell])\,. \tag{1.65}$$

Factorization Product of Observables

We can define a product structure on the cosheaf $U \mapsto \mathrm{Obs}(U)$ as follows. For any two operators $O_1 \in \mathrm{Obs}(U)$ and $O_2 \in \mathrm{Obs}(V)$ in two disjoint open sets U and V, and a scale L, define:

$$\begin{aligned} - * - : \mathrm{Obs}(U) \times \mathrm{Obs}(V) &\to \mathrm{Obs}(U \sqcup V)\,, \\ (O_1 * O_2)[L] &= \lim_{\ell \to 0} \mathcal{W}_\ell^L\left(O_1[\ell] O_2[\ell]\right)\,, \end{aligned} \tag{1.66}$$

where the product inside the bracket on the right hand side is the ordinary product of functions. A key result of the BV formulation is the following theorem by Costello and Gwilliam [39]:

Theorem 1.4.2 (Costello, Gwilliam)**.** *The product $*$ makes the cosheaf $U \mapsto Obs(U)$ a factorization algebra.*

OPE of Local Observables and Singularities

Local operators located at a point $p \in M$ are the operators that belong to the intersection of $\mathrm{Obs}(U)$ for all open sets U containing p. Product of local operators is an important structure to study, especially in this thesis. In generic cases, such products have singularities when the two local operators approach each other.

In the language of factorization algebra, singularities in the operator product appears as follows. For a point $p \in M$ of space-time and a positive number r, Let us denote by

$D_{p,r}$ an open disc of radius r in a neighbourhood of p. Consider two discs of small radius ϵ centered at 0 and x inside a larger disc of radius r centered at zero:

$$D_{0,\epsilon} \sqcup D_{x,\epsilon} \subseteq D_{0,r} \,. \tag{1.67}$$

The above inclusion gives a product (the same product as in (1.66)):

$$\mathrm{Obs}(D_{0,\epsilon}) \times \mathrm{Obs}(D_{x,\epsilon}) \to \mathrm{Obs}(D_{0,r}) \,. \tag{1.68}$$

By considering ϵ to be arbitrarily small we can restrict ourselves to local operators located at 0 and x. Then the product above is only well defined for $x \neq 0$. Put differently, for any two operators $O_1(0) \in \mathrm{Obs}(D_{0,\epsilon})$, $O_2(x) \in \mathrm{Obs}(D_{x,\epsilon})$, by taking their product and allowing x to vary over $D_{0,r}\backslash\{0\}$ we get a smooth operator valued map:

$$\begin{gathered} O_1(0) * O_2(-) : D_{0,r}\backslash\{0\} \to \mathrm{Obs}(D_{0,r}) \,, \\ O_1(0) * O_2(-) : x \mapsto O_1(0) * O_2(x) \,. \end{gathered} \tag{1.69}$$

We call an operator valued map $\mathcal{O} : D_{0,r}\backslash\{0\} \to \mathrm{Obs}(D_{0,r})$ regular at 0 if there is another operator valued map $\overline{\mathcal{O}} : D_{0,r} \to \mathrm{Obs}(D_{0,r})$ whose restriction to $D_{0,r}\backslash\{0\}$ gives $\mathcal{O}$:

$$\exists \overline{\mathcal{O}} : D_{0,r} \to \mathrm{Obs}(D_{0,r}) \text{ such that } \overline{\mathcal{O}}\big|_{D_{0,r}\backslash\{0\}} = \mathcal{O} \quad \Leftrightarrow \quad \mathcal{O} \text{ is regular at } 0 \,. \tag{1.70}$$

An operator valued map is singular at 0 simply if it is not regular at 0.

1.4.1 Supersymmetry and Cohomological Algebra

With all these formalities introduced it is easy to define the structure we are interested in.

Given a supersymmetric field theory on M with a sheaf of fields $\mathcal{E}$, and a nilpotent supercharge Q (as described in §1.3), which anticommutes with the BV differential:

$$\mathrm{d}_L Q + Q \mathrm{d}_L = 0 \tag{1.71}$$

at any scale, the corresponding *cohomological field theory* at scale L is the cosheaf of cochain complexes:

$$U \mapsto \mathrm{Obs}_L^Q(U) := (\mathscr{O}(\mathcal{E}(U)), \mathrm{d}_L + Q) \,, \tag{1.72}$$

which is a factorization algebra with a product which is the descent of the BV factorization product (1.66). More spcecifically, the algebra of observalbes that we are interested in is the factorization algebra of the 0-th cohomology:

$$H^0(\mathrm{Obs}_L^Q(U)) \times H^0(\mathrm{Obs}_L^Q(V)) \to H^0(\mathrm{Obs}_L^Q(U \sqcup V)) \,. \tag{1.73}$$

In certain specific cases (such as the theories we consider in chapter 2 and 3) this product on the cohomology gives a *non-singular* product for local operators and depending on the choice of Q the non-singular algebra is traditionally referred to as the chiral ring, twisted-chiral ring, BPS ring etc.

Remark 1.4.3 (Aside on terminology: "Cohomological" vs. "Twisted"). What we are referring to as a "cohomological" algebra would perhaps more commonly be referred to as a "twisted" algebra. We chose not to use the term "twisted" as, in our experience, this term is too widely and broadly used, and not always in contexts that are qualitatively similar enough. On the other hand "cohomological" always refers to essentially the same concept of "closed modulo exact" – seeming less ambiguous and more expressive. △

Remark 1.4.4 (Ω-deformation). A physically important deformation in some supersymmetric QFTs is the so called Ω-deformation [134]. This is the situation where we have a supercharge Q which is not nilpotent, instead, it squares to a $\mathrm{U}(1)$ symmetry including space-time rotation:

$$Q^2 = \mathcal{L}_J \,, \tag{1.74}$$

where $\mathcal{L}_J$ refers to the action of the generator J of some $\mathrm{U}(1)$ symmetry of the theory including rotation in some particular $\mathbb{R}^2$ plane[25] in the space-time M. Physically, we want to consider operators that are located in the space-time region fixed by the $\mathrm{U}(1)$ action and we want to perform path integral over field configurations that are invariant under the $\mathrm{U}(1)$ action as well. Adding such a Q to the BV operator d_L does not lead to a cochain complex immediately, however by restricting to the invariant subspace $\mathscr{O}(\mathcal{E}(U))^{\mathrm{U}(1)}$ we get a good cochain complex. Since the $\mathrm{U}(1)$ acts on space-time, open sets $U \subseteq M$ are not invariant under this action and we can not get a factorization algebra on M this way. However, we should get a factorization algebra restricted to the fixed points of the $\mathrm{U}(1)$. Physically, this shall mean that we essentially have a field theory in two dimensions less than the dimension of the original space-time M. This construction is certainly physically motivated[26] and the cohomology seems similar to the mathematical construction of *equivariant cohomology*, however, we were unable to find a description of the BV formalism incorporating equivariant cohomology in the literature. We expect that there is indeed an equivariant version of the BV formulation which generalizes the cohomological description of QFTs to incorporate Ω-deformation, but we leave this for future investigations. △

[25] Or more generally, leaves of a foliation of M with $\mathrm{U}(1)$ symmetry.

[26] In fact we shall use omega deformation in addition to topological twists to relate a model of topological holography with the holography of $\mathcal{N} = 4$ SYM in chapter 4.

1.5 Cohomological Algebras and Dualities

Cohomological algebras can be useful tools in checking dualities between different QFTs. In principal, duality should imply an isomorphism between the full factorization algebras associated to the dual QFTs. However, establishing such isomorphism is exceedingly hard in general. Computations of suitably chosen cohomological algebras can be significantly simpler. A cartoon analogy that inspires us is that of the cohomology of finite dimensional manifolds. While it may be hard to establish diffeomorphism between the smooth structures of two manifolds,[27] it is often much simpler to establish homeomorphism, or at least an isomorphism of cohomology. In this vain, in chapter 4 we establish a holographic duality between two topological/holomorphic theories and we conjecture that this is in fact an isomorphism at the level of cohomology between the standard $\mathcal{N} = 4$ super Yang-Mills with defect and its cojectured supergravity dual.

1.6 Organization of The Thesis

This thesis is dedicated to explicit computations of operator algebras/correlation functions in specific theories. Computationally, we employ two different approaches to this end: 1) Supersymmetric localization and 2) Direct computation of Feynman and Witten diagrams.

Chapter 2 and 3 are closely related – in these chapters we use the technique of supersymmetric localization to compute correlation functions of operators belonging to certain cohomological algebras. Chapter 2 is based on the paper [104], in this chapter we study 2d theories with $(2, 2)$ supersymmetry. There are two much studied cohomological algebras in these theories, called the chiral ringa and the twisted-chiral ring. These operators are observables in two different topological twists called the B-model and the A-model. These theories are of significant interest in the context of mirror symmetry. We demonstrate that these operator algebras can be computed using localization. Some background regarding relevant supersymmetry in 2d, curved space supersymmetry, and some technical details of computations are presented in Appendix B. Chapter 3 is based on [75] where we study a similar algebra, namely the chiral ring, in 4d $\mathcal{N} = 2$ superconformal theories using localization. In Appendix D we gather some background regarding integrability and supersymmetry relevant for this chapter. The algebras we study in 2d and 4d have some curious connection with the integrable Toda equations. In appendix C we clarify some subtlety

[27] Think of the problem of classifying *exocit structures* on topological S^4, which remains an open problem to this day – it is not even known whether there are more than one smooth structures or not [150].

regarding defining the sphere partition function and certain supersymmetric correlation functions in the 4d supersymmetric theories – this is based on the paper [81].

In chapter 4 we present a holographic duality between two topological/holomorphic theories. We demonstrate this duality by computing operator algebras in the boundary theory and scattering algebras in the bulk theory and showing that they are isomorphic. We use the standard approach of computing Feynman diagrams and Witten diagrams to compute these algebras. Most of this chapter is based on the paper [105]. However, §4.6 contains results from ongoing projects that have not yet been published otherwise. In this section we show that the model of topological holography that we have constructed in this chapter is in fact a supersymmetric subsector of the more familiar model of holographic duality involving $\mathcal{N}=4$ super Yang-Mills theory with defects. We identify the supersymmetric twists and Ω-deformation that reduce the $\mathcal{N}=4$ duality setup to the topological setup presented in the earlier sections.

A few general remarks about the theme of the thesis as a whole and its future are mentioned in chapter 5.

The chapters of this thesis can be read independently of each other. Chapters 2 and 3 are applications of the same computational strategy in different dimensions. There is no technical link between these two chapters and chapter 4.

Chapter 2

Chiral Rings in 2 Dimensions

2.1 Prologue

The cohomological algebras we are focusing on in this chapter are the *chiral rings* and the *twisted chiral rings* of 2d $\mathcal{N} = (2,2)$ supersymmetric quantum field theories (QFTs). We refer to them as 2d *BPS rings* throughout this chapter. On a general ground, assignment of such algebras to the respective QFTs lets us distinguish between different theories and identify various types of equivalence classes of QFTs and dualities. More specifically, the BPS rings are renormalization group (RG) invariants that can be used to distinguish between different universality classes of 2d $\mathcal{N} = (2,2)$ theories.[1] These rings are interesting objects from a mathematical point of view as well, as chiral and the twisted chiral rings of a given theory belong to two different topological sectors of the theory and their structures encode complex structure invariants and Kähler structure invariants of some geometric spaces associated to the theory [151, 153, 154].

In this chapter we compute the BPS ring structure of 2d $\mathcal{N} = (2,2)$ theories using the exactly known results regarding the 2d sphere partition functions [15, 60, 61, 83]. In order for us to use the sphere partition function and still be able to infer results for the theory on flat space, we require that we must be able to canonically place the flat space theory on a sphere. This forces us to restrict to 2d $\mathcal{N} = (2,2)$ theories that flow to some conformal theories. One interesting feature of these rings in 2d is that they are not freely generated,

[1] These invariants consist of only local operators and they can not distinguish between theories with different non-local defects for example.

and our procedure will generate the ring relations.[2] This process does not rely on Mirror symmetry and therefore results obtained in this way can be used for independent checks of such symmetry.

The plan for the rest of the chapter is as follows. In §2.2 we establish the notations and conventions we use to characterize the BPS ring structures. In §2.3 we review, tailoring to the 2d case, the procedure put forward in [75] for computing chiral rings and finally in §2.4 we apply this general procedure to compute the twisted chiral ring (consisting of the Coulomb branch operators) of the Quintic Calabi-Yau (CY) Gauged Linear Signa Model (GLSM) and the chiral ring of the Landau-Ginzburg (LG) minimal models. In the appendices we present details about the superconformal algebra (§B.1), supersymmetric backgrounds on the sphere (§B.2), proof of a supersymmetric Ward identity we use (§B.3), and some explicit computations (§B.4).

2.2 The BPS Rings

We first give the definition of the BPS ring in a superconformal theory, and then explain its definition for an ultraviolet (UV) theory with a conformal fixed point. We have included some details about the relevant $(2,2)$ superconformal algebra $\mathfrak{su}(2|2)$ in appendix B.1.

In a superconformal theory

A *superconformal primary* operator is one that is annihilated by all the S-supersymmetries:

$$\mathcal{O} \text{ is a primary} \quad \Leftrightarrow \quad [S_\pm, \mathcal{O}] = [\overline{S}_\pm, \mathcal{O}] = 0\,. \tag{2.1}$$

The anti-commutation relations of the $(2,2)$ superconformal algebra (B.5) allow to consistently define the following types of primary operators with additional supersymmetry:

[2]Very analogously, chiral rings can be defined in 4d $\mathcal{N}=2$ SCFTs as well. In fact, the method of computing the structure constants of these rings using the sphere partition function was initially developed for the the 4d case in [75] (the topic of chapter 3 of this thesis). One important novelty in the 2d case compared to its 4d analogue is that the 2d chiral/anti-chiral rings are *not* freely generated, unlike their 4d analogues.

$$
\begin{aligned}
\text{Chiral:} \qquad & [\overline{Q}_\pm, \mathcal{O}] = 0\,, && (2.2a)\\
\text{Anti-chiral:} \qquad & [Q_\pm, \mathcal{O}] = 0\,, && (2.2b)\\
\text{Twisted chiral:} \qquad & [\overline{Q}_+, \mathcal{O}] = [Q_-, \mathcal{O}] = 0\,, && (2.2c)\\
\text{Twisted anti-chiral:} \qquad & [Q_+, \mathcal{O}] = [\overline{Q}_-, \mathcal{O}] = 0\,. && (2.2d)
\end{aligned}
$$

Note that we use the name *BPS* (*anti-BPS*) to refer to both chiral and twisted chiral (anti-chiral and twisted anti-chiral). The above definitions apply to local and nonlocal operators alike but for this chapter we are only concerned with local operators. Charges of chiral primaries under some of the generators of $\mathfrak{su}(2|2)$ are constrained: for example, using the Q-S anti-commutators from (B.5) it follows that the dimension and vector R-charge of a chiral primary $\mathcal{O}$ are related, so are the dimension and the vector R-charge of an anti-chiral primary $\overline{\mathcal{O}}$:

$$
2\Delta(\mathcal{O}) = J_V(\mathcal{O})\,, \qquad 2\Delta(\overline{\mathcal{O}}) = -J_V(\overline{\mathcal{O}})\,. \tag{2.3}
$$

Such constraints lead to non-singular operator product expansion (OPE) between chiral primaries [123]:

$$
(\mathcal{O}_1\mathcal{O}_2)(x) := \lim_{y\to x} \mathcal{O}_1(x)\mathcal{O}_2(y) =: \mathcal{O}_3(x)\,, \tag{2.4}
$$

where $\mathcal{O}_3$ is either zero or a chiral primary with dimension:

$$
\Delta(\mathcal{O}_3) = \Delta(\mathcal{O}_1) + \Delta(\mathcal{O}_2)\,. \tag{2.5}
$$

With this product the set of all chiral primaries becomes a ring called the *chiral ring*, which we will denote as $\mathcal{R}^{\mathrm{c}}$. The twisted chiral ring, denoted $\mathcal{R}^{\mathrm{tc}}$, is analogously defined as the ring of twisted chiral primaries.

Theories with conformal fixed points

The definition of primary opearators (2.1) does not apply in a non-conformal theory since the S-supersymmetries are not part of the symmetry in such case, so the BPS rings can not be defined in such a theory as the ring generated by the primaries. There is however an alternative definition of these rings which applies in this case. For that definition we need the following two nilpotent supercharges:

$$
\begin{aligned}
Q_A := \overline{Q}_+ + Q_-\,, \qquad & Q_B := \overline{Q}_+ + \overline{Q}_-\,,\\
Q_A^2 = 0\,, \qquad & Q_B^2 = 0\,.
\end{aligned} \tag{2.6}
$$

Now the chiral (twisted chiral) ring can be defined as the Q_B-cohomology (Q_A-cohomology) of operators:

$$\mathcal{R}^{\mathrm{c}} := H^{\bullet}_{Q_B}\,, \qquad \mathcal{R}^{\mathrm{tc}} := H^{\bullet}_{Q_A}\,, \tag{2.7}$$

where the grading refers to the $\mathrm{U}(1)_V$ R-charge for the chiral ring and the $\mathrm{U}(1)_A$ R-charge for the twisted chiral ring.[3]

To see that these cohomologies define the same ring as the ring of chiral/twisted chiral primaries in a superconformal theory we need the following two observations:

1. Suppose $\mathcal{O}$ is a Q_B-closed operator:

   $$\left[\overline{Q}_+ + \overline{Q}_-, \mathcal{O}\right] = 0\,. \tag{2.8}$$

   If $\mathcal{O}$ has spin[4] α then rotating the above equation by an angle $\pi/2$ we get:

   $$\left[-i\overline{Q}_+ + i\overline{Q}_-, e^{i\pi\alpha/2}\mathcal{O}\right] = 0 \tag{2.9}$$

   Together, (2.8) and (2.9) imply:

   $$\left[\overline{Q}_+, \mathcal{O}\right] = \left[\overline{Q}_-, \mathcal{O}\right] = 0\,. \tag{2.10}$$

   Thus we recover the chirality condition (2.2a). Similarly it can be shown that being Q_A-closed is equivalent to being twisted chiral (2.2c).

2. Any chiral (twisted chiral) operator is Q_B-cohomologous (Q_A-cohomologous) to a chiral (twisted chiral) primary [123]. Furthermore, a $Q_{A/B}$-exact operator is not a primary, since a primary is defined as the operator in a superconformal multiplet with the lowest Weyl weight, whereas an operator $[Q_{A/B}, \mathcal{O}]$ is in the same multiplet as $\mathcal{O}$ while having a higher Weyl weight than $\mathcal{O}$.

The cohomological definition of the BPS rings (2.7) is perfectly sensible in the absence of conformal symmetry and coincides with the definition in terms of superconformal primaries at a conformal fixed point.

[3] Q_A and Q_B have charge 1 under $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_V$ respectively.

[4] By spin we are referring to the charge for the generator $2J_L$ (where J_L is the generator of rotation on $\mathbb{R}^2$), note that $\overline{Q}_\pm$ has charge ∓ 1 for this generator. (Details about the symmetry algebra are provided in §B.1.)

Extremal correlators

Let us first define extremal correlators in a Superconformal Field Theory (SCFT), and then explain why we can compute them in a UV theory with Conformal Field Theory (CFT) fixed point.

Given two BPS operators, $\mathcal{O}_1$ and $\mathcal{O}_2$ of conformal dimensions Δ_1 and Δ_2 respectively, the field theory defines a Hermitian inner product:

$$\langle \mathcal{O}_1, \mathcal{O}_2 \rangle := \lim_{x\to\infty} |x|^{\Delta_1+\Delta_2} \left\langle \mathcal{O}_1(0)\overline{\mathcal{O}}_{\overline{2}}(x) \right\rangle_{\mathbb{R}^2} = \delta_{\Delta_1,\Delta_2} \lim_{x\to\infty} |x|^{2\Delta_2} \left\langle \mathcal{O}_1(0)\overline{\mathcal{O}}_{\overline{2}}(x) \right\rangle_{\mathbb{R}^2} , \quad (2.11)$$

where $\overline{\mathcal{O}}_{\overline{2}}$ is the anti-BPS primary operator conjugate to $\mathcal{O}_2$. The second equality follows from $\mathrm{U}(1)_R$ selection rule[5] and the constraint (2.3) (and its analogue for the twisted case). In order to shorten the notation of (2.11) we define:

$$\overline{\mathcal{O}}(\infty) := \lim_{x\to\infty} |x|^{2\Delta(\overline{\mathcal{O}})}\overline{\mathcal{O}}(x) \,. \quad (2.12)$$

The inner product (2.11) now becomes:

$$\langle \mathcal{O}_1, \mathcal{O}_2 \rangle = \left\langle \mathcal{O}_1(0)\overline{\mathcal{O}}_{\overline{2}}(\infty) \right\rangle_{\mathbb{R}^2} \,. \quad (2.13)$$

The correlation functions of this form, i.e., with a BPS operator at 0 and an anti-BPS operator at ∞, are called *extremal correlators* on $\mathbb{R}^2$.

There's a little more to the extremal correlators. Generally they are defined with an arbitrary number of BPS primaries $\mathcal{O}_1, \cdots, \mathcal{O}_m$ located at $x_1, \cdots, x_m$ respectively and one anti-BPS primary at infinity:

$$\left\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_m(x_m)\overline{\mathcal{O}}(\infty) \right\rangle_{\mathbb{R}^2} , \quad (2.14)$$

and this correlator is independent of the positions $x_1, \cdots, x_m$. We can see this by translating any of the BPS operators and using (B.5a), for example, the infinitesimally translated correlator $\left\langle [L_{-1}, \mathcal{O}_1](x_1) \cdots \mathcal{O}_m(x_m)\overline{\mathcal{O}}(\infty) \right\rangle_{\mathbb{R}^2}$ is proportional to:

$$\lim_{y\to\infty} |y|^{2\Delta(\mathcal{O})} \left\langle [\{Q_+, \overline{Q}_+\}, \mathcal{O}_1](x_1) \cdots \mathcal{O}_m(x_m)\overline{\mathcal{O}}(y) \right\rangle_{\mathbb{R}^2} \,. \quad (2.15)$$

Supersymmetric Ward identity allows us to pull $\overline{Q}_+$ out of $\mathcal{O}_1$ and distribute it over the rest of the operators, all the BPS operators are annihilated by $\overline{Q}_+$ and when it acts on

[5] $\mathrm{U}(1)_R = \mathrm{U}(1)_V$ if the operators are chiral and $\mathrm{U}(1)_R = \mathrm{U}(1)_A$ if the operators are twisted chiral.

$\overline{\mathcal{O}}$, the correlator behaves as $|y|^{-2\Delta(\mathcal{O})-1}$ and the limit makes the contribution zero. This position independence of the extremal correlators allows us to bring all the chiral operators to one point (say at the origin).

A similar argument shows that exact operators are zero inside extremal correlators and therefore the extremal correlators really define an inner product in the cohomology. Furthermore, the energy-momentum tensor couples to the linearized space-time metric via a D-term action [31]. Variation of a correlation function with respect to the metric then inserts an operator inside the correlator which is an integral over the entire superspace:

$$\delta_{g_{\mu\nu}} \langle \cdots \rangle_{\mathbb{R}^2} \sim \left\langle \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}(\cdots)\cdots \right\rangle_{\mathbb{R}^2} \tag{2.16}$$

Such an integrated operator can be written as an exact operator [100] which implies that as long as $\langle \cdots \rangle_{\mathbb{R}^2}$ is an extremal correlator such variations vanish. This in particular implies that the extremal correlators are scale invariant, in other words, they are RG invariant and can be computed in a UV theory even when we are interested in an infrared (IR) CFT fixed point.

Basis, structure constants, norms and relations

In a finitely and freely generated ring[6] $\mathcal{R}$ with a non-degenerate Hermitian inner product $\langle -, - \rangle$, we can choose a minimal set of generators $\{\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_N\}$ and define a metric in their basis:

$$g_{i\bar{j}} := \langle \mathcal{O}_i, \mathcal{O}_j \rangle \ . \tag{2.17}$$

The inverse metric $g^{\bar{i}j}$ is defined by imposing:

$$g^{\bar{i}j} g_{j\bar{k}} = \delta^{\bar{i}}_{\bar{k}} , \quad g_{i\bar{j}} g^{\bar{j}k} = \delta^k_i \ . \tag{2.18}$$

We define the ring structure by the structure constants in such a basis:

$$\mathcal{O}_i \mathcal{O}_j = C_{ij}{}^k \mathcal{O}_k \quad \Leftrightarrow \quad C_{ij}{}^k = C_{ij\bar{l}} g^{\bar{l}k} \quad \text{where,} \quad C_{ij\bar{l}} := \langle \mathcal{O}_i \mathcal{O}_j, \mathcal{O}_l \rangle \tag{2.19}$$

Furthermore, we can choose the basis in such way that the structure constants become trivial/diagonal in the following sense:[7]

$$C_{ij}{}^k = \delta^k_{i+j} \ . \tag{2.20}$$

[6]The 2d BPS rings are finitely but *not* freely generated, we will discuss truncation by relations momentarily.

[7]For two indices i and j referring to two operators $\mathcal{O}_i$ and $\mathcal{O}_j$, we use the index $i+j$ to refer to the operator with dimension equal to the sum of the dimensions of $\mathcal{O}_i$ and $\mathcal{O}_j$. For simplicity we are assuming that there is only one such operator, having more does not make any qualitative difference.

Now all the nontrivial information about the ring structure is encoded in the norms of the basis vectors:

$$\|\mathcal{O}_i\| := \sqrt{\langle \mathcal{O}_i, \mathcal{O}_i \rangle}\,. \tag{2.21}$$

The constraint (2.20) fixes the norms of all the basis vectors *relative to each other*. To fix this arbitrariness in case of the BPS rings, we will fix the norm of the identity operator $\mathbb{1}$ to be 1:[8]

$$\langle \mathbb{1}, \mathbb{1} \rangle := 1\,. \tag{2.22}$$

Given a complete set of generators $\{\mathcal{O}_1, \cdots, \mathcal{O}_N\}$, a freely generated ring is simply the polynomial ring:

$$\mathcal{R} = \mathbb{C}[\mathcal{O}_1, \cdots, \mathcal{O}_N]\,. \tag{2.23}$$

The only new addition to this discussion in the case of a ring *with relations*, is that there will be some polynomials $p_a \in \mathbb{C}[\mathcal{O}_1, \cdots, \mathcal{O}_N]$ for $a \in \{1, \cdots, M\}$ which will be identified with zero, i.e., we must impose the relations $p_a = 0$ for all $a \in \{1, \cdots, M\}$ and the ring will be given by:

$$\mathcal{R} = \mathbb{C}[\mathcal{O}_1, \cdots, \mathcal{O}_N] / \langle p_1, \cdots, p_M \rangle\,. \tag{2.24}$$

where $\langle p_1, \cdots, p_M \rangle$ is the ideal generated by the polynomials $\{p_1, \cdots, p_M\}$.

In the context of the 2d $\mathcal{N} = (2,2)$ BPS rings, the zero polynomials will appear as BPS operators with zero norm.[9] We will always choose a basis of the BPS operators with trivialized (as in (2.20)) structure constants and the identity operator will be defined to have unit norm, therefore, according to the above discussion all the information of the BPS rings will be encoded in the extremal correlators $\left\langle \mathcal{O}_i(0)\overline{\mathcal{O}}_{\bar{i}}(\infty)\right\rangle_{\mathbb{R}^2}$, in particular, finding the relations will amount to finding BPS operators $\mathcal{O}$ such that $\left\langle \mathcal{O}(0)\overline{\mathcal{O}}(\infty)\right\rangle_{\mathbb{R}^2} = 0$.

2.3 Computing the Ring Structures

As explained in §2.2, a BPS ring structure is essentially defined by flat space extremal correlators $\langle \mathcal{O}_1(0)\overline{\mathcal{O}}_{\bar{2}}(\infty)\rangle_{\mathbb{R}^2}$ of BPS primaries once a suitable basis has been chosen. A straightforward application of Weyl Ward identity tells us that if we put our theory on a

[8]Note that $\|\mathbb{1}\|^2 = \left\langle \mathbb{1}(x)\overline{\mathbb{1}}(\infty)\right\rangle_{\mathbb{R}^2} = Z$ where Z is the partition function, therefore, in terms of Feynman diagrams, defining this norm to be one is equivalent to subtracting bubble diagrams from all our correlation functions.

[9]The identification of zero normed operators with identically zero operators is provided by the Reeh-Schlieder theorem [147].

sphere of radius r, then the extremal correlators on the sphere are related to the flat space correlators in the following way:

$$\left\langle \mathcal{O}_1(0)\overline{\mathcal{O}}_{\overline{2}}(\infty)\right\rangle_{\mathbb{R}^2} = (2r)^{2\Delta(\mathcal{O}_2)} \left\langle \mathcal{O}_1(N)\overline{\mathcal{O}}_{\overline{2}}(S)\right\rangle_{S^2} \,, \tag{2.25}$$

where N and S on the sphere are images of 0 and ∞ on $\mathbb{R}^2$ respectively, under an inverse stereographic projection. The S^2 correlators that appear in the above formula can be readily computed using localization. The main complication then, in using the above formula to compute the BPS ring structure constants, is that the identification between the flat space operators and the operators on the sphere is nontrivial due to operator mixing on the sphere. Mixing among operators of different dimensions can take place on the sphere because the sphere does not preserve scaling symmetry. Our task is therefore to "unmix" the operators on the sphere and then use the Weyl Ward identity (2.25) to compute the BPS ring structures. In this section we elaborate on this general procedure. We note that this process is essentially identical to the process of computing chiral rings in 4d $\mathcal{N} = 2$ SCFTs [75].

2.3.1 Extremal Correlators on S^2

Choice of a localizing supercharge

The first step in extracting the flat space extremal correlators from the sphere partition function is to compute their analogue on the sphere, such as $\langle \mathcal{O}_i(N)\overline{\mathcal{O}}_{\bar{j}}(S)\rangle_{S^2}$,[10] using supersymmetric localization. We begin in this section by defining our choice of localizing supercharges for the two-sphere backgrounds described in §B.2 and some of their important properties:

- **Background-A:** In accordance with the notation of §B.2, we define our choice of localizing supercharge by imposing the following chirality constraints on the constant Dirac spinors that parametrize the solutions of the Killing spinor equations (see (B.11)):
$$\chi_{0-} = 0\,, \qquad \widetilde{\chi}_{0+} = 0\,. \tag{2.26}$$
With these constraints the Killing spinors become chiral at the poles:
$$P_-\epsilon^A_{\chi_0,\widetilde{\chi}_0}(N) = P_+\widetilde{\epsilon}^A_{\chi_0,\widetilde{\chi}_0}(N) = 0\,, \quad P_+\epsilon^A_{\chi_0,\widetilde{\chi}_0}(S) = P_-\widetilde{\epsilon}^A_{\chi_0,\widetilde{\chi}_0}(S) = 0\,, \tag{2.27}$$

[10] North (N) and South (S) poles refer to two antipodal points on the sphere. We will take them to be $x = 0$ and $x = \infty$ (in stereographic coordinate) for convenience.

where $P_\pm := \frac{1}{2}(1+\gamma^3)$ are the chiral projectors. We will refer to this choice of supercharge as $\mathcal{Q}_A$.

We recall that under a generic supercharge corresponding to a generic solution ϵ and $\widetilde{\epsilon}$ of the Killing spinor equations, a twisted chiral primary Y and a twisted anti-chiral primary $\overline{Y}$, which are the bottom components of a twisted chiral mulitplet (Y,ζ,G) and a twisted anti-chiral multiplet $(\overline{Y},\overline{\zeta},\overline{G})$ respectively, transform as (B.17):

$$\delta_{\epsilon,\widetilde{\epsilon}}Y(x) = \widetilde{\epsilon}_+(x)\zeta_-(x) - \epsilon_-(x)\zeta_+(x)\,, \qquad \delta_{\epsilon,\widetilde{\epsilon}}\overline{Y}(x) = \widetilde{\epsilon}_-(x)\zeta_+(x) - \epsilon_+(x)\zeta_-(x)\,. \tag{2.28}$$

Therefore, for the supercharge $\mathcal{Q}_A$ corresponding to (2.27) we get:

$$\delta_{\mathcal{Q}_A}Y(N) = \delta_{\mathcal{Q}_A}\overline{Y}(S) = 0\,. \tag{2.29}$$

This implies that insertions of twisted chiral and twisted anti-chiral primaries at the North and the South pole respectively are invariant under $\mathcal{Q}_A$ and the corresponding correlators can be computed by supersymmetric localization using $\mathcal{Q}_A$.[11]

- **Background-B:** We impose the same chirality constraints (2.26) on the constant spinors but this leads to different constraints for the Killing spinors of this background (B.16):

$$\widetilde{\epsilon}^{B}_{\chi_0,\widetilde{\chi}_0}(N) = 0\,, \qquad \epsilon^{B}_{\chi_0,\widetilde{\chi}_0}(S) = 0\,. \tag{2.30}$$

  We will refer to this choice of supercharge by $\mathcal{Q}_B$.

  We recall the transformations of a chiral primary ϕ and an anti-chiral primary $\overline{\phi}$, which are the bottom components of a chiral multiplet (ϕ,ψ,F) and an anti-chiral multiplet $(\overline{\phi},\overline{\psi},\overline{F})$ respectively, under a generic supercharge [61]:

$$\delta_{\epsilon,\widetilde{\epsilon}}\phi(x) = \widetilde{\epsilon}(x)\psi(x)\,, \qquad \delta_{\epsilon,\widetilde{\epsilon}}\overline{\phi}(x) = \epsilon(x)\overline{\psi}(x)\,. \tag{2.31}$$

  Therefore, according to (2.30) we have:

$$\delta_{\mathcal{Q}_B}\phi(N) = \delta_{\mathcal{Q}_B}\overline{\phi}(S) = 0\,, \tag{2.32}$$

  implying that we can compute correlators with insertions of chiral and anti-chiral primaries at the North and South pole respectively by supersymmetric localization using the supercharge $\mathcal{Q}_B$.[12]

[11] Such a supercharge was used in [61] to compute the $\mathfrak{su}(2|1)_A$-invariant partition function using localization.

[12] Such a supercharge was used in [60] to compute $\mathfrak{su}(2|1)_B$-invariant partition function and correlation functions of 2d gauge theories.

A Ward identity and extremal correlators

A particularly convenient way to insert BPS (anti-BPS) primary operators at the North (South) pole of the sphere is to use a supersymmetric Ward identity. Before stating the identity, let us define for an arbitrary twisted chiral multiplet $\Psi = (Y, \zeta, G)$ with a scalar bottom component:

$$\mathcal{G}(\Psi) := G + \frac{\mathit{\Delta}(Y) - 1}{r} Y \,, \tag{2.33}$$

where $\mathit{\Delta}(Y)$ denotes the Weyl weight (equal to the dimension for a scalar operator) of Y. Now we state the Ward identity:

> Suppose we are given the following data in backgrounad-A: A supercharge $Q_A \in \mathfrak{su}(2|1)_A$, a Q_A-invariant operator[13] $\mathcal{O}$ and a twisted chiral multiplet $\Psi = (Y, \zeta, G)$ of arbitrary Weyl weight. Then, inside a correlator with $\mathcal{O}$, the $\mathfrak{su}(2|1)_A$-invariant twisted F-term action for Ψ localizes to the insertion of the bottom component Y at the fixed point of Q_A on the sphere (which we call the North pole N), in other words:
>
> $$\left\langle \left(\int_{S^2} \mathrm{d}^2 x \sqrt{g(x)}\, \mathcal{G}(\Psi) \right) \mathcal{O} \right\rangle_{S^2} = -4\pi r \left\langle Y(N) \mathcal{O} \right\rangle \,, \tag{2.34}$$
>
> where g is the determinant of the *covariant* metric on the sphere. Similarly, the conjugate twisted F-term action of the twisted anti-chiral multiplet $\overline{\Psi} = (\overline{Y}, \overline{\zeta}, \overline{G})$ localizes to the insertion of the bottom component at the South pole (fixed point of $\overline{Q}_A$):
>
> $$\left\langle \left(\int_{S^2} \mathrm{d}^2 x \sqrt{g(x)}\, \mathcal{G}(\overline{\Psi}) \right) \mathcal{O} \right\rangle_{S^2} = 4\pi r \left\langle \overline{Y}(S) \mathcal{O} \right\rangle \,. \tag{2.35}$$

There is a parallel Ward identity for background-B the statement of which simply replaces Q_A with Q_B and "twisted chiral" with "chiral". In [76] this was proven for twisted chiral multiplets in background-A and chiral multiplets in background-B of Weyl weight 1.[14] The proof for arbitrary Weyl weight requires only a trivial modification, we reproduce the modified proof in §B.3 for reference.

---

[13] The operator $\mathcal{O}$ does not have to be twisted chiral, it suffices that $\langle \mathcal{O} \rangle_{S^2}$ be an extremal correlator.

[14] Which results in the twisted F-term or the F-term action being a marginal deformation.

The twisted F-terms or the F-terms can be used to deform the theory[15] in background-A or B respectively by introducing coupling constants of appropriate Weyl weights. For example, in background-A we can have the following deformation:[16]

$$S_A[X] \to S'_A[X;\tau,\overline{\tau}] := S_A[X] + \left[-\frac{i\tau}{4\pi} \int_{S^2} \mathrm{d}^2 x \sqrt{g(x)}\, \mathcal{G}(\Psi) + \text{c.c.} \right] , \tag{2.36}$$

where the bottom component Y of the twisted chiral multiplet $\Psi = (Y, \zeta, G)$ and the coupling constant τ have Weyl weights that satisfy:

$$\Delta(Y) + \Delta(\tau) = 1 \, , \tag{2.37}$$

and X is merely a place-holder for all the dynamical fields. Using the Ward identities (2.34) and (2.35) we can now relate τ-derivatives of the partition function to extremal correlators:

$$\frac{1}{Z^A_{S^2}} \frac{1}{r^{m+n}} \partial^m_\tau \partial^n_{\overline{\tau}} Z^A_{S^2}(\tau,\overline{\tau})\big|_{\tau,\overline{\tau}=0} = \left\langle (iY)^m(N)(i\overline{Y})^n(S) \right\rangle_{S^2} \, , \tag{2.38}$$

where $Z^A_{S^2}(\tau,\overline{\tau})$ is the deformed partition function:[17]

$$Z^A_{S^2}(\tau,\overline{\tau}) = \int \mathcal{D}X \, e^{-S'_A[X;\tau,\overline{\tau}]} \, . \tag{2.39}$$

We encode the equation (2.38) in the following correspondence between derivative with respect to a coupling, and the operator it inserts at a pole after localization:

$$\frac{1}{r}\partial_\tau \longleftrightarrow iY(N) \, , \qquad \frac{1}{r}\partial_{\overline{\tau}} \longleftrightarrow i\overline{Y}(S) \, . \tag{2.40}$$

We can compute extremal correlators of chiral operators on the sphere similarly in background-B.

Remark: If the undeformed action already contains a superpotential or twisted superpotential coupling then we can compute extremal correlators of the corresponding chiral

[15] We are assuming these deformation terms to be scalar so as not to break (Euclidean) Lorentz invariance. From now on we assume that all the non-trivial operators in the BPS rings are scalars, this will be true in the examples that we will consider.

[16] The normalization of the deformation term was chosen simply to cancel some numerical factors in (2.34) and (2.35).

[17] This deformed partition function does not need to be convergent, it is just a generating function with indeterminate variables τ and $\overline{\tau}$ for correlators with integrated operators, which, due to the Ward identities, become correlators with unintegrated twisted chiral and twisted anti-chiral primaries. We need only to be able to compute these correlation functions using localization.

or twisted chiral fields without any further deformation, just by taking derivatives with respect to the corresponding coupling constant. An example of this, which will be studied in detail later, is an abelian gauge theory in background-A where the action contains a complexified Fayet-Iliopoulos (FI) coupling $t \int_{S^2} \mathrm{d}^2x \sqrt{g(x)}\, G_\Sigma$ where G_Σ is the top component of a twisted chiral multiplet Σ of Weyl weight[18] 1 known as the field strength multiplet. The bottom component of this multiplet is a complex scalar σ and we can therefore compute such extremal correlators as $\langle \sigma^m(N)\overline{\sigma}^n(S)\rangle_{S^2}$ by evaluating derivatives of the partition function with respect to the FI parameters t and $\bar{t}$ at arbitrary values of t and $\bar{t}$. We will do this in §2.4.1.

2.3.2 Chiral Ring Coefficients from Extremal Correlators on S^2

Knowing the extremal correlators on S^2, the next step is to extract from them the flat space extremal correlators.

Operator mixing

As was pointed out in [75] for the case of 4d $\mathcal{N} = 2$ SCFTs, when put on a sphere, operators of different Weyl weights can mix due to the presence of scheme dependent Weyl symmetry breaking counterterms. This is true in two dimensions as well. The important difference between the two and four dimensional story is that, in four dimensions the $\mathcal{N} = 2$ supergravity background multiplet that goes into the counterterms causing the operator mixing had Weyl weight 2, whereas the $\mathcal{N} = (2, 2)$ supergravity background multiplet in two dimensions responsible for operator mixing has Weyl weight 1. This leads to the fact that in four dimensions two operators can mix on the sphere only if their Weyl weights differ by an even integer, on the other hand in two dimensions two operators with Weyl weights differing by any integer amount can mix. More specifically, on S^2, a chiral (twisted chiral) operator $\mathcal{O}_w$ of Weyl weight w can mix with all chiral (twisted chiral) operators of

[18] By the Weyl weight of a BPS multiplet we refer to the Weyl weight of its bottom component. In particular, by a twisted chiral multiplet $\Psi = (Y, \zeta, G)$ of Weyl weight w we mean that Y has Weyl weight w. The Weyl weights of ζ and G are $\left(w + \frac{1}{2}\right)$ and $(w + 1)$ respectively.

lower weights:[19]

$$\mathcal{O}_w \to \mathcal{O}_w + \sum_{\substack{n\in\mathbb{N} \\ 0<n\leq w}} \alpha_n(\tau_{\text{mar}}) r^{-n} \mathcal{O}_{w-n}\,, \tag{2.41}$$

where the mixing coefficients α_n are arbitrary holomorphic functions of all the exactly marginal couplings, schematically written as τ_{mar}. We now construct the $\mathcal{N} = (2,2)$ supergravity counterterms giving rise to such mixings.

There are two minimal versions of $\mathcal{N} = (2,2)$ supergravity that differ in the choice of U(1) R-symmetry that is gauged [5, 31, 74, 93, 94, 102]. After choosing appropriate background values for the fields, these two versions reduce to background-A and background-B on S^2 preserving the vector and the axial R-symmetry respectively. Let us focus on the supergravity leading to background-A.

We discuss the mixing of the bottom component of a twisted chiral multiplet $\widehat{\mathcal{O}}_w = (\mathcal{O}_w, \zeta_{\mathcal{O}_w}, G_{\mathcal{O}_w})$ of Weyl weight w. In order to compute correlation functions of the operator $\mathcal{O}_w$ using the Ward identity (2.34) we need to deform the action, as in (2.36), by introducing a coupling. The manifestly supersymmetric way of doing this is to use superspace integrals to write the deformation terms. To that end we need to promote the coupling, which we denote as τ_{1-w} (making the Weyl weight explicit), to the bottom component of a background twisted chiral multiplet $\widehat{\tau}_{1-w} = (\tau_{1-w}, \zeta_{\tau_{1-w}}, G_{\tau_{1-w}})$. For this background multiplet to be supersymmetric, the $\mathfrak{su}(2|1)_A$ variations of the component fields must vanish. Consulting (B.17) we find the following background values for the fermion and the top component (given the constant value of the bottom component):

$$\zeta_{\tau_{1-w}} = 0\,, \qquad G_{\tau_{1-w}} = \frac{w-1}{r}\tau_{1-w}\,. \tag{2.42}$$

[19] In the sum we are restricting to lower weights to avoid repeated counting, since mixing with an operator of higher weight is already considered as a mixing of the higher weighted operator with the lower weighted operator. Also, we are assuming that there is at most one operator with a given Weyl weight for simplicity. If there are more than one operators of a given Weyl weight then we only need to choose an ordering of these operators and all the computations follow without any qualitative modification.

Now the superspace integral representation of the deformation (2.36) becomes:[20]

$$-\frac{i\tau_{1-w}}{4\pi}I_{w,0} := -\frac{i}{4\pi}\int_{S^2}\mathrm{d}^2x\int \mathrm{d}^2\widetilde{\theta}\,\mathcal{E}_{\mathrm{tc}}\,\widehat{\tau}_{1-w}\widehat{\mathcal{O}}_w = -\frac{i\tau_{1-w}}{4\pi}\int_{S^2}\mathrm{d}^2x\sqrt{g(x)}\,\mathcal{G}(\widehat{\mathcal{O}}_w)\,. \tag{2.43}$$

where, as in the definition (2.33), $\mathcal{G}(\Psi) = G + \frac{w-1}{r}Y$. The supergravity counterterm that leads to the mixing of the operator $\mathcal{O}_w$ with another twisted chiral operator $\mathcal{O}_{w-n}$ of lower weight ($n \in \mathbb{N}_{>0}$) necessarily involves the background coupling multiplet $\widehat{\tau}_{1-w}$, the twisted chiral multiplet $\widehat{\mathcal{O}}_{w-n} := (\mathcal{O}_{w-n}, \zeta_{\mathcal{O}_{w-n}}, G_{\mathcal{O}_{w-n}})$ of weight $(w-n)$ and a background twisted chiral multiplet $\widehat{M} = (M, \zeta_R, -\mathcal{R}/2)$ whose bottom component is a complex scalar of Weyl weight 1 coming from the supergravity multiplet and whose top component is proportional to the scalar curvature of the space-time (this multiplet appeared in [112] in the context of 2d supergravity and in [76] in constructing supergravity counterterms responsible for Kähler ambiguity in two-sphere partition function). On the sphere background, the scalar curvature is $\mathcal{R} = 2/r^2$. As we did for the background coupling multiplet $\widehat{\tau}_{1-w}$, we now find the supersymmetric background values for the component fields of $\widehat{M}$ (this time given the constant value of the top component):

$$M = \frac{1}{r}\,, \qquad \zeta_R = 0\,, \qquad -\frac{\mathcal{R}}{2} = -\frac{1}{r^2}\,. \tag{2.44}$$

Apart from the multiplets just mentioned, we have the freedom to include an arbitrary holomorphic function α of the exactly marginal couplings τ_{mar}, including this we can now write down the mixing counterterm:

$$-\frac{i\tau_{1-w}}{4\pi}I_{w,n} := -\frac{i}{4\pi}\int_{S_2}\mathrm{d}^2x\int \mathrm{d}^2\widetilde{\theta}\,\mathcal{E}_{\mathrm{tc}}\,\widehat{\tau}_{1-w}\alpha(\widehat{\tau}_{\mathrm{mar}})\widehat{M}^n\widehat{\mathcal{O}}_{w-n}\,, \tag{2.45}$$

where we have promoted the exactly marginal couplings to background twisted chiral multiplets of Weyl weight 0. Just to avoid cluttering the notation too much, let us introduce a symbol for the product multiplet:

$$\widehat{\mathcal{O}}^{\alpha}_{w,n} := \alpha(\widehat{\tau}_{\mathrm{mar}})\widehat{M}^n\widehat{\mathcal{O}}_{w-n}\,. \tag{2.46}$$

[20]The equality in (2.43) can be proven as follows. For a twisted chiral multiplet $\Psi_1 = (Y, \zeta, G)$ of Weyl weight $w = 1$, the relevant superspace integral just picks up the top component, i.e., $\int_{S^2}\mathrm{d}^2x\int \mathrm{d}^2\widetilde{\theta}\,\mathcal{E}_{\mathrm{tc}}\,\Psi_1 = \int_{S^2}\mathrm{d}^2x\sqrt{g(x)}\,G$. That this is supersymmetric can also be checked by noting that the $\mathfrak{su}(2|1)_A$ variation of G, namely $\delta G = \nabla_m(-i\widetilde{\epsilon}_-\gamma^m\zeta_- + i\epsilon_+\gamma^m\zeta_+)$ (see (B.17)), is a total derivative. Now assume $\Psi_w = (Y, \zeta, G)$ is a twisted chiral multiplet of some arbitrary Weyl weight w and $\widehat{\tau}_{1-w} = \left(\tau, 0, \frac{w-1}{r}\tau\right)$ is a *supersymmetric background* twisted chiral multiplet of Weyl weight $(1-w)$. Then $\widehat{\tau}_{1-w}\Psi_w = \left(\tau Y, \tau\zeta, \tau\left(G + \frac{w-1}{r}Y\right)\right)$ [31] is a twisted chiral multiplet of Weyl weight 1 and therefore $\int_{S^2}\mathrm{d}^2x\int \mathrm{d}^2\widetilde{\theta}\,\mathcal{E}_{\mathrm{tc}}\,\widehat{\tau}_{1-w}\Psi_w = \int_{S^2}\mathrm{d}^2x\sqrt{g(x)}\left(\tau\left(G + \frac{w-1}{r}Y\right)\right)$, and it is supersymmetric.

Since this is a multiplet of Weyl weight w, we can use (2.43) to evaluate the superspace integral in (2.45) which leads to:

$$I_{w,n} = \int_{S^2} \mathrm{d}^2x \sqrt{g(x)}\, \mathcal{G}(\widehat{\mathcal{O}}^{\alpha}_{w,n})\,. \tag{2.47}$$

The Ward identity (2.34) tells us that, inside an extremal correlator, the integrated operator $I_{w,n}$ will localize to the insertion of the bottom component of $\widehat{\mathcal{O}}^{\alpha}_{w,n}$ at the North pole. The bottom component of a product multiplet is simply the product of the bottom components of the individual multiplets in the product [31]. Therefore, in presence of the counterterm (2.47), the correspondence between coupling derivatives and operators (2.40) is modified:

$$\frac{1}{r}\partial_{\tau_{1-w}} \longleftrightarrow i\mathcal{O}_w(N) + i\alpha(\tau_{\text{mar}})r^{-n}\mathcal{O}_{w-n}(N)\,. \tag{2.48}$$

In general, we must consider all possible counterterms, $\tau_{1-w}I_{w,n}$ for all $n \in \mathbb{N}$ with $0 < n \leq w$ and this leads to the general form of the mixing (2.41).

"Un-mixing" the operators

Let us define $\mathfrak{O}_w$ to be the mixed operator in (2.41):

$$\mathfrak{O}_w := \mathcal{O}_w + \sum_{\substack{n\in\mathbb{N}\\ 0<n\leq w}} \alpha_n(\tau_{\text{mar}})r^{-n}\mathcal{O}_{w-n} \quad \Rightarrow \quad \frac{1}{r}\partial_{\tau_{1-w}} \longleftrightarrow i\mathfrak{O}_w\,. \tag{2.49}$$

Note that the mixing coefficients are scheme dependent,[21] so these operators are not physical. But due to the mixing counterterms, such as (2.47), taking derivatives of the deformed sphere partition function with respect to the coupling constants computes extremal correlation functions of these operators:

$$\frac{1}{Z^A_{S^2}}\frac{1}{r^2}\partial_{\tau_{1-w}}\partial_{\overline{\tau}_{1-w'}} Z^A_{S^2}\big|_{\tau_{1-w}=\tau_{1-w'}=0} = \left\langle i\mathfrak{O}_w(N) i\overline{\mathfrak{O}}_{w'}(S)\right\rangle_{S^2}\,. \tag{2.50}$$

We are of course interested in the flat space correlation functions of the physical operators, such as $\left\langle \mathcal{O}_w(0)\overline{\mathcal{O}}_{w'}(\infty)\right\rangle_{\mathbb{R}^2}$. Once we properly identify the flat space operators with their counterparts on the sphere, we can relate the correlators on $\mathbb{R}^2$ with the correlatros on S^2 by the Weyl Ward identity (2.25).

[21] A choice of scheme is a choice of the holomorphic functions α_n of the the exactly marginal couplings.

On flat space, operators of different Weyl weights are orthogonal, this changes on the sphere.[22] It is a standard procedure to compute the inner products in an orthogonal basis (the $\mathcal{O}_w$'s) given the inner products in the mixed basis (the $\mathfrak{O}_w$'s), called the *Gram-Schmidt* procedure. In order to state the result, it is convenient to define some matrices. Given a complete set of operators $\{\mathfrak{O}_w\}$ indexed by their Weyl weights, define the following matrices:

$$M_{(w)} := \begin{pmatrix} M_{0,0} & \cdots & M_{0,w} \\ \vdots & \ddots & \vdots \\ M_{w,0} & \cdots & M_{w,w} \end{pmatrix}, \qquad M_{i,j} := \left\langle i\mathfrak{O}_i(N) i\overline{\mathfrak{O}}_j(S) \right\rangle_{S^2} . \tag{2.51}$$

Now, we can express the flat space correlators of interest as follows (for $w \geq w'$):

$$\left\langle i\mathcal{O}_w(0) i\overline{\mathcal{O}}_{w'}(\infty) \right\rangle_{\mathbb{R}^2} = \delta_{w,w'}(2r)^{2w'} \frac{\det M_{(w)}}{\det M_{(w-1)}} . \tag{2.52}$$

We will use this formula in examples to compute chiral and twisted chiral ring relations in the following section.

2.4 Some Examples

In this section we illustrate the general points made so far by applying them to a couple of well known $\mathcal{N} = (2,2)$ theories, namely the Quintic GLSM and Landau-Ginzburg minimal models.

2.4.1 Twisted Chiral Ring of the Quintic GLSM

This is a U(1) gauge theory with $\mathcal{N} = (2,2)$ supersymmetry and the bosonic global symmetry is $\mathrm{U}(1)_L \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$, where $\mathrm{U}(1)_L$ is the spacetime rotation and $\mathrm{U}(1)_V, \mathrm{U}(1)_A$ are the vector and axial R-symmetries respectively. It is a theory of six chiral multiplets Φ_i with $i \in \{1, \cdots, 6\}$ interacting via a superpotential:

$$W(\Phi_1, \cdots, \Phi_6) = \Phi_6 P(\Phi_1, \cdots, \Phi_5) , \tag{2.53}$$

[22] For example, if τ is an exactly marginal coupling then the partition function $Z_{S^2}(\tau, \overline{\tau})$ depends on it and has a nonzero derivative: $\frac{1}{Z_{S^2}(\tau,\overline{\tau})}\partial_\tau Z_{S^2}(\tau,\overline{\tau}) = \langle \mathcal{O}_\tau \rangle_{S^2}$. Here $\mathcal{O}_\tau$ is the bottom component of a BPS multiplet whose top component is an exactly marginal operator of Weyl weight 2. The bottom component $\mathcal{O}_\tau$ has Weyl weight 1 and the fact that it has a nonzero one-point function indicates that it has mixed with the identity operator (of Weyl weight 0).

where P is a homogeneous polynomial of degree five. We will denote the vector multiplet by V and the associated *twisted chiral* "field strength" multiplet is defined as (in superfield notation):

$$\Sigma := \overline{D}_+ D_- V \,, \tag{2.54}$$

where $\overline{D}_+$ and D_- are two of the four superspace derivatives that commute with the supercharges. Σ has Weyl weight[23] 1 and its top component is $\mathrm{D} - iF_{12}$, where D is the real scalar in V and F_{12} is the field strength of the gauge field in V. This field strength multiplet defines the twisted superpotential action:[24]

$$-\frac{t}{4\pi}\int_{\mathbb{R}^2} \mathrm{d}^2x \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^- \, \Sigma - \text{c.c.} = -\frac{i}{2\pi}\int_{S^2} \mathrm{d}^2x \left(\xi \mathrm{D} - \frac{\theta}{2\pi} F_{12}\right) , \tag{2.55}$$

where ξ and θ are the real FI parameter and the topological theta angle respectively:

$$t = i\xi + \frac{\theta}{2\pi} \,. \tag{2.56}$$

Charges of the fields of this theory under the gauge and the global symmetries are as follows (we have also included the charges of the superspace coordinates for quick reference):[25]

	$\Phi_1, \cdots, \Phi_5$	Φ_6	Σ	$\theta^\pm$	$\overline{\theta}^\pm$
$\mathrm{U}(1)_L$	0	0	0	$\pm$	$\pm$
$\mathrm{U}(1)_{\text{gauge}}$	1	-5	0	0	0
$\mathrm{U}(1)_V$	$2q_V$	$2-10q_V$	0	$+$	$-$
$\mathrm{U}(1)_A$	q_A	$-5q_A$	2	$\pm$	$\mp$

(2.57)

This model gets its name from the fact that for the value of the real FI parameter in a certain range (namely $\xi \gg 0$), the IR CFT fixed point of this theory is described by a non-linear sigma model with target [154]:

$$\Sigma = \Phi_6 = 0 \,, \qquad \left\{\left(\sum_{i=1}^{5} |\Phi_i|^2 = \Im t\right)\right\} \Big/ \mathrm{U}(1)_{\text{gague}} \cap \{P(\Phi_1, \cdots, \Phi_5) = 0\} \,, \tag{2.58}$$

[23]The kinetic term for the vector multiplet is normalized as $\frac{1}{e^2}\int \mathrm{d}^2x \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\, \Sigma\overline{\Sigma}$ where e is the gauge coupling of dimension 1.

[24]Our normalization of this term has an extra factor of $\frac{1}{2\pi}$ compared to that of [15, 61].

[25]The charges of the fields follow from the following arguments. The chiral fields appear in the action in the term $\int \mathrm{d}^2x \int \mathrm{d}\theta^+ \mathrm{d}\theta^- \Phi_6 P(\Phi_1, \cdots, \Phi_5)$ and the twisted chiral field appears in $\int \mathrm{d}^2x \int \mathrm{d}\theta^+ \overline{\theta}^- \Sigma$ and these terms have to be neutral. The bosonic measure is neutral under all the symmetry groups. Noting that θ and $\mathrm{d}\theta$ have opposite charges we see that the fermionic measure $\mathrm{d}\theta^+\mathrm{d}\theta^-$ has charge $(0, 0, -2, 0)$ under $\mathrm{U}(1)_L \times \mathrm{U}(1)_{\text{gauge}} \times \mathrm{U}(1)_V \times \mathrm{U}(1)_{\mathcal{A}}$, and $\mathrm{d}\theta^+\mathrm{d}\overline{\theta}^-$ has charge $(0, 0, 0, -2)$ under the same group.

which is known as the Quintic Calabi-Yau (CY) threefold.[26]

Twisted chiral ring

The twisted chiral ring $\mathcal{R}^{\text{tc}}$ of this theory is generated by the complex scalar operator σ, which is the bottom compotent of the field strength multiplet Σ.[27] An orthogonal spanning set for $\mathcal{R}^{\text{tc}}$ is given by $\{\sigma^m\}_{m=0}^{\infty}$. This set also satisfies the triviality constraint for the structure constants (2.20) since we have:

$$\sigma^m \sigma^n = \sigma^{m+n} \,. \tag{2.59}$$

Therefore, after fixing the norm of $\sigma^0 = \mathbb{1}$ to be 1, all we have left to compute to determine $\mathcal{R}^{\text{tc}}$ are the following extremal correlators:

$$\|\sigma^m\|^2 = \langle \sigma^m(0)\overline{\sigma}^m(\infty)\rangle_{\mathbb{R}^2} \quad \forall m \geq 0 \,. \tag{2.60}$$

In the following we will compute these correlation functions and we will find that $\mathcal{R}^{\text{tc}}$ is not freely generated, we will find the null operators as well.

From sphere to flat space

Extremal correlators of twisted chiral operators on the sphere can be computed by putting the theory in background-A (see §2.3.1). The partition function of a generic $\mathcal{N} = (2,2)$ gauge theory in background-A has been computed explicitly in [15, 61] and this result has been applied to the specific case of the Quintic GLSM (among several others) in [107].[28]

In the computation of the partition function in background-A, we can ignore the superpotential and only the twisted superpotential is important. The twisted superpotential action (2.55) of the theory gets modified in the sphere background by the appearance of nontrivial integration measures:

$$-\frac{1}{4\pi}\int_{S^2} \mathrm{d}^2x \int \mathrm{d}^2\widetilde{\theta}\, \mathcal{E}_{\text{tc}}\, \widehat{t}\,\Sigma - \text{c.c.} = -\frac{t}{4\pi}\int_{S^2} \mathrm{d}^2x \sqrt{g(x)}\, \mathcal{G}(\Sigma) - \text{c.c.} \,, \tag{2.61}$$

[26] At the conformal fixed point the target space metric (i.e., the metric that appears in the kinetic term of the non-linear sigma model) is necessarily the Ricci flat metric.

[27] It should be noted that the non linear sigma model description in the IR Calabi-Yau phase of this GLSM contains more twisted chiral operators which are fermionic. These are the operators that survive the A-twist [153]. However, the description of these operators in the GLSM is unclear. We thank Cyril Closset for pointing out this subtlety.

[28] These localization results were further extended to extremal correlators on S^2 and used to find evidence for Seiberg-like dualities for $(2,2)$ gauge theories [15, 16, 61].

This being the only twisted superpotential of the theory, the $\mathfrak{su}(2|1)_A$ preserving partition function $Z^A_{S^2}(t,\bar{t})$ for the Quintic is a function only of the complexified FI parameter. Due to the presence of mixing counterterms (see (2.47)), derivatives of the partition function $Z^A_{S^2}(t,\bar{t})$ compute correlation functions of mixed operators. We denote these mixed operators as $\mathfrak{s}_m := \sigma^m + \mathcal{O}(r^{-1})$ (c.f. (2.49)) so that t and $\bar{t}$-derivatives of $Z^A_{S^2}$ can be equated readily with correlation functions of $\mathfrak{s}_m$ and $\bar{\mathfrak{s}}_n$ (c.f. (2.50)):

$$\frac{1}{r^{m+n} Z^A_{S^2}} \partial^m_t \partial^n_{\bar{t}} Z^A_{S^2}(t,\bar{t}) = \langle \mathfrak{s}_m(N) \bar{\mathfrak{s}}_n(S) \rangle_{S^2} \ . \tag{2.62}$$

According to (2.52), in terms of these correlators, the extremal correlators on the flat space are given by:

$$\langle \sigma^m(0) \bar{\sigma}^n(\infty) \rangle_{\mathbb{R}^2} = \delta_{m,n} \frac{(2r)^{2n}}{Z^A_{S^2}} \frac{\det_{i,j\in\{0,\cdots,m\}} \partial^i_t \partial^j_{\bar{t}} Z^A_{S^2}}{\det_{i,j\in\{0,\cdots,m-1\}} \partial^i_t \partial^j_{\bar{t}} Z^A_{S^2}} \, . \tag{2.63}$$

From now on we will set the radius of the sphere to 1 for simplicity.

The partition function $Z^A_{S^2}(t,\bar{t})$ does not depend on the axial R-charges, so instead of writing q_V all the time we will simply write q. For $\xi \gg 0$, the partition function can be written as [107]:

$$Z^A_{S^2}(t,\bar{t}) = (w\bar{w})^q \oint \frac{\mathrm{d}\epsilon}{2\pi i} (w\bar{w})^{-\epsilon} \frac{\pi^4 \sin(5\pi\epsilon)}{\sin^5(\pi\epsilon)} \left| \sum_{k=0}^{\infty} (-w)^k \frac{\Gamma(1+5k-5\epsilon)}{\Gamma(1+k-\epsilon)^5} \right|^2 , \tag{2.64}$$

where we have defined:[29]

$$w := e^{it} \, . \tag{2.65}$$

In (2.64), the contour surrounds only the pole at $\epsilon = 0$ and in computing the absolute value of the infinite sum complex conjugation does not act on ϵ. The infinite sum appearing in (2.64) converges at $\epsilon = 0$ for large enough ξ, as can be seen from ratio test:

$$\lambda_k := (-w)^k \frac{\Gamma(1+5k)}{\Gamma(1+k)^5} \, , \qquad \frac{\lambda_{k+1}}{\lambda_k} \xrightarrow{k\to\infty} -e^{\frac{i\theta}{2\pi} - \xi} 5^5 \xrightarrow{\xi\to\infty} 0 \, . \tag{2.66}$$

Let us denote the series at $\epsilon = 0$ as:

$$X(t) := \sum_{k=0}^{\infty} (-e^{it})^k \frac{\Gamma(1+5k)}{\Gamma(1+k)^5} \, . \tag{2.67}$$

[29] There is again a factor of 2π offset with respect to the convention of [107], where they use $e^{2\pi i t}$. This offset compensates for our choice of normalization in (2.55).

The residue of the integrand at the pole is:

$$\operatorname*{Res}_{\epsilon=0} \frac{(w\overline{w})^{-\epsilon}}{2\pi i} \frac{\pi^4 \sin(5\pi\epsilon)}{\sin^5(\pi\epsilon)} \left| \sum_{k=0}^{\infty} (-w)^k \frac{\Gamma(1+5k-5\epsilon)}{\Gamma(1+k-\epsilon)^5} \right|^2 = -\frac{10}{3\pi} i\xi(-5\pi^2+\xi^2) X(t)\overline{X}(\bar{t})\,. \quad (2.68)$$

Therefore the partition function as a function of the FI parameter looks like:

$$Z^A_{S^2}(t,\bar{t}) = \frac{20}{3} e^{-2q\xi} \xi(-5\pi^2+\xi^2) X(t)\overline{X}(\bar{t})\,. \tag{2.69}$$

One interpretation of this partition function is that it computes the Kähler potential of the moduli space of CFTs that can be reached from the GLSM under RG flow by varying the FI parameter [76]:

$$Z^A_{S^2}(t,\bar{t}) = e^{-K(t,\bar{t})}\,. \tag{2.70}$$

From this perspective, the partition function is defined only upto a Kähler transformation of the Kähler potential:

$$K(t,\bar{t}) \to K(t,\bar{t}) + F(t) + \overline{F}(\bar{t})\,, \tag{2.71}$$

where F and $\overline{F}$ are arbitrary holomorphic and anti-holomorphic functions (in particular, they can be taken as $F = \log X$, $\overline{F} = \log \overline{X}$), and a Kähler transformation can be interpreted as a change of the UV regularization scheme [76], which does not affect any physical observables.[30] We now go to a simpler scheme by doing a Kähler transformation:

$$Z^A_{S^2}(t,\bar{t}) \to \widetilde{Z}^A_{S^2}(t,\bar{t}) := \frac{Z^A_{S^2}(t,\bar{t})}{X(t)\overline{X}(\bar{t})} = \frac{20}{3} e^{-2q\xi} \xi(-5\pi^2+\xi^2)\,. \tag{2.72}$$

Using $\partial_t = -\frac{i}{2}\partial_\xi + \pi\partial_\theta$ and $\partial_{\bar{t}} = \frac{i}{2}\partial_\xi + \pi\partial_\theta$ we can now compute flat space correlation functions using $\widetilde{Z}^A_{S^2}$ in (2.63). Next we find the ring relations.

Relations and the ring

The correlation functions (2.62) are the matrix components from (2.51):

$$M_{m,n} = \langle \mathfrak{s}_m(N)\bar{\mathfrak{s}}_n(S)\rangle_{S^2} = \frac{\partial_t^m \partial_{\bar{t}}^n \widetilde{Z}^A_{S^2}}{\widetilde{Z}^A_{S^2}} = (-1)^m (i/2)^{m+n} \frac{\partial_\xi^{m+n} \widetilde{Z}^A_{S^2}}{\widetilde{Z}^A_{S^2}}\,, \tag{2.73}$$

[30] A careful proof of the fact that a Kähler transformation does not affect the extremal correlators on flat space, in the analogous situation of 4d $\mathcal{N}=2$ SCFTs, can be found in the appendix of [75], the proof applies without any significant change to the 2d case as well.

where we could replace all the t and $\bar{t}$-derivatives with ξ-derivatives because the partition function does not depend on the theta angle. Now the flat space correlators are given by:

$$\langle \sigma^m(0)\overline{\sigma}^n(\infty)\rangle_{\mathbb{R}^2} = \delta_{m,n} 2^{2n} \frac{\det M_{(m)}}{\det M_{(m-1)}} \,. \tag{2.74}$$

We can get a recursion relation for $m > 3$ and $n \geq 0$:

$$M_{m,n} = \frac{20}{3\widetilde{Z}^A_{S^2}}(-1)^m (i/2)^{m+n} \sum_{k=0}^{3} (-2q)^{m+n-k} e^{-2q\xi} \partial^k_\xi (-5\pi^2\xi + \xi^3) = iqM_{m-1,n} \,. \tag{2.75}$$

This shows that for $m > 3$ the last $(m-2)$ rows of the matrix $M_{(m)}$ are multiples of each other and therefore $M_{(m)}$ has $(m-3)$ zero eigenvalues, i.e., $\det M_{(m)} \sim 0^{m-3}$. For the correlation functions the implication is:

$$\frac{0^{m-3}}{0^{m-4}} \sim \langle \sigma^m(0)\overline{\sigma}^m(\infty)\rangle_{\mathbb{R}^2} = 0 \,, \qquad \forall m > 3 \,. \tag{2.76}$$

Since the above correlation function is equivalent to an operator norm in a unitary theory, the operators σ^m for $m > 3$ themselves are identically zero. Thus we have fully determined the ring:

$$\text{Twisted chiral ring for the Quintic, } \mathcal{R}^{\text{tc}} = \mathbb{C}[\sigma]/\langle \sigma^4 \rangle \,. \tag{2.77}$$

This result was previously obtained in [32, 130] using the topologically A-twisted version of the GLSM which doesn't require the supersymmetric localization and the counterterm analysis that we did. The upshot of going through the more elaborate method that we have presented is that this allows us to see the change in the ring structure as we move along the CFT moduli space; which we now discuss.

Toda and tt^*-geometry of the bundle of BPS primaries

The nontrivial coupling dependence of the extremal correlators[31] can be given a geometric interpretation [46], where we view the twisted chiral primaries as forming a holomorphic bundle with nontrivial connection over the conformal manifold parametrized by the exactly marginal coupling constants, the FI parameter in the present case. More generally, this picture applies to both chiral and twisted chiral rings in CFTs with arbitrary dimensional

[31] the coupling dependence of the correlators (2.74) derives from the coupling dependence of the partition function (2.72) via (2.73).

conformal manifolds. The CFT dynamics imposes the following curvature constraints (a.k.a. the tt^* equations) on the geometry of these bundles:[32]

$$[\nabla_\mu, \nabla_\nu]_i^j = [\overline{\nabla}_{\overline{\mu}}, \overline{\nabla}_{\overline{\nu}}]_i^j = 0\,, \tag{2.78a}$$

$$[\nabla_\mu, \overline{\nabla}_{\overline{\nu}}]_i^j = -[C_\mu, \overline{C}_{\overline{\nu}}]_i^j + g_{\mu\overline{\nu}}\delta_i^j \left(1 + \frac{R}{4c}\right)\,. \tag{2.78b}$$

Let us explain the notations: μ, ν refer to tangential directions on the conformal manifold[33] and i, j refer to all BPS primaries. A bar over an index corresponds to an anti-BPS primary. In the second line, R is the $\mathrm{U}(1)_R$ charge of the bundle and c is the central charge of the CFT. The matrices C_μ, or more generally C_i, with indices expressed as $C_{ij}{}^k$ are the structure constants of our ring as defined in (2.19), and finally, the metric $g_{\mu\overline{\nu}}$, or more generally $g_{i\overline{j}}$, is defined in terms of extremal correlators as in (2.17).

With a suitable choice for the basis of the BPS primaries over the conformal manifold, the tt^* equation (2.78b) can be put into a more explicit form:[34]

$$\frac{\partial}{\partial\overline{\tau}^{\overline{\nu}}}\left(g^{\overline{k}j}\frac{\partial}{\partial\tau^\mu}g_{i\overline{k}}\right) = [C_\mu, \overline{C}_{\overline{\nu}}]_i^j - g_{\mu\overline{\nu}}\delta_i^j\,. \tag{2.79}$$

Specializing to the case of a one dimensional conformal manifold, such as the Quintic (where the coordinate parametrizing the conformal manifold is t), and choosing basis of operators with diagonal structure constants and orthgonal metric $g_{i\overline{j}} = g_i\delta_{ij}$,[35] the above equation further simplifies to:

$$\partial_{\overline{t}}\partial_t \log g_k = \frac{g_{k+1}}{g_k} - \frac{g_k}{g_{k-1}} - g_1\,, \qquad k \in \{1, \cdots, \dim\mathcal{R} - 1\}, \quad g_{\dim\mathcal{R}} = 0\,, \tag{2.80}$$

where $\mathcal{R}$ is the BPS ring of interest, for example, $\dim\mathcal{R}_{\mathrm{tc}} = 4$ for the twisted chiral ring of the Quintic. The above equation is known as the Toda equation in the literature, which is usually written (after defining $q_k := \log g_k + Z_{S^2}$) in the more familiar form:

$$\partial_{\overline{t}}\partial_t q_k = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}}\,, \qquad k \in \{1, \cdots, \dim\mathcal{R} - 1\}, \quad q_{\dim\mathcal{R}} = -\infty\,, \tag{2.81}$$

[32]The commutator of structure constants in (2.78b) looks like: $[C_\mu, \overline{C}_{\overline{\nu}}]_i^j = C_{\mu i}{}^k g_{k\overline{l}}\overline{C}_{\overline{\nu}\overline{m}}{}^{\overline{l}} g^{\overline{m}j} - g_{i\overline{k}}\overline{C}_{\overline{\nu}\overline{l}}{}^{\overline{k}} g^{\overline{l}m} C_{\mu m}{}^j$.

[33]i.e., they are indices for the exactly marginal couplings.

[34]For details see [9] where the underlying theory was four dimensional but this portion of the computation applies to 2d as well.

[35]Such as the basis $\{\sigma^i\}_{i=0}^3$ for the twisted chiral ring of the Quintic.

The condition $q_{\dim \mathcal{R}} = -\infty$ signifies that the above equations are the equations of motion of a finite non-periodic Toda chain consisting of $\dim \mathcal{R}$ sites located at $q_0, \cdots, q_{\dim \mathcal{R}-1}$ where q_k and is bound to q_{k+1} by the potential $e^{q_{k+1}-q_k}$.

The tt^* equations (2.78) are known to be integrable. In particular, the Toda equation (2.81) can be solved explicitly given g_1 which we can compute using localization. Note that the Toda equation for the norms of the operators in an orthogonal basis with diagonal structure constant, in a one parameter theory, can be derived simply from the expression of these norms as a ratio of determinants (such as (2.74)). An explicit derivation was presented in [75] in the context of 4d $\mathcal{N} = 2$ SCFT with SU(2) gauge group, the proof remains unchanged for one parameter 2d BPS rings.

2.4.2 Chiral Rings of the LG Minimal Models

These are theories of chiral multiplets X_i, and the theories are characterized by a superpotential $W(X_i)$ and a Kähler potential $K(X_i, \overline{X}_i)$. If the superpotential has a quasi-homogenous singularity

$$W(\lambda^{m_i} X_i) = \lambda^2 W(X_i) \tag{2.82}$$

the Landau-Ginzburg model flows in the IR to a $(2, 2)$ SCFT. The universality class of the SCFT is insensitive to the choice of Kähler potential, which henceforth we take to be canonical: $K(X, \overline{X}) = \frac{1}{2}\delta^{ij} X_i \overline{X}_j$.

The equations of motion of the Landau-Ginzburg model gives:

$$\partial_i W \propto \overline{D}^2 \overline{X}_i \,. \tag{2.83}$$

Thus the bottom component of $\partial_i W$ is not only $\overline{Q}_B$-closed, but also $\overline{Q}_B$-exact. Therefore the chiral operator (the bottom component of) $\partial_i W$ is represented by 0 in correlation functions with other chiral operators ($\partial_i W$ is not represented by 0 in an arbitrary correlator). Therefore the chiral ring of the SCFT is the quotient

$$\mathcal{R}^{\mathrm{c}} = C[X_1, \ldots, X_n]/\langle \mathrm{d}W \rangle \,. \tag{2.84}$$

The chiral ring is spanned by polynomials in the fields subject to the relations $dW = 0$. Unorbifolded Landau-Ginzburg models have a trivial twisted chiral ring. We will recover the result (2.84) from the sphere partition function in what follows.

The $(2, 2)$ unitary minimal models admit a Landau-Ginzburg description. The minimal model modular invariants pertain to an ADE classification. This is mirrored by the following ADE family of Landau-Ginzburg models:

$$\begin{aligned}
A_k: &\qquad W = X^{k+1} \\
D_k: &\qquad W = X^{k-1} + XY^2 \\
E_6: &\qquad W = X^3 + Y^4 \\
E_7: &\qquad W = X^3 + XY^3 \\
E_8: &\qquad W = X^3 + Y^5
\end{aligned} \tag{2.85}$$

The associated minimal models have the following properties:

- Central charge $c = 3 - \frac{6}{h}$, where h is the Coxeter number of the corresponding Lie group G.
- Dimension of the chiral ring, $\dim(\mathcal{R}^{\mathrm{c}}_G) = \mathrm{rank}(G)$.
- The chiral operators $\mathcal{O}_i \in \mathcal{R}^{\mathrm{c}}_G$ have dimension $\Delta_i = \frac{d_i - 2}{h}$, where d_i is the order of the i-th Casimir of G.

Since $d_i \leq h$, all operators in the minimal models are relevant.

Extremal correlators of chiral and anti-chiral operators in a Landau-Ginzburg model on S^2 with a superpotential $W(X_i)$ are given by [83] (we set the radius of the sphere to 1)

$$\left\langle \mathcal{O}_i(N) \overline{\mathcal{O}}_{\bar{j}}(S) \right\rangle_{S^2} = \int \prod_i dX_k d\overline{X}_{\bar{k}}\, \mathcal{O}_i(X)\, \overline{\mathcal{O}}_{\bar{j}}(\overline{X}) e^{-4\pi i (W(X) + \overline{W}(\overline{X}))}. \tag{2.86}$$

Recall from the general discussion of §2.3.1 that these are precisely the correlators that can be computed in background-B by localizing the path integral with respect to the supercharge $\mathcal{Q}_B$.

A_{k+1} minimal model

We start first by analyzing the Landau-Ginzburg representation of the A_{k+1} minimal model. The $\mathfrak{su}(2|1)_B$-invariant S^2 partition function for this Landau-Ginzburg minimal model is given by

$$
\begin{aligned}
Z_{S^2}^{A_{k+1}} &= \int_{\mathbb{C}} \mathrm{d}X \mathrm{d}\overline{X}\, e^{-4\pi i W(X) - 4\pi i \overline{W}(\overline{X})} \\
&= \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, e^{-4\pi i (x+iy)^{k+2} - 4\pi i (x-iy)^{k+2}} \\
&= \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, r e^{-8\pi i r^{k+2}\cos((k+2)\theta)} = \frac{\pi}{k+2}(4\pi)^{-\frac{2}{k+2}} \frac{\Gamma\left(\frac{1}{k+2}\right)}{\Gamma\left(\frac{k+1}{k+2}\right)}\,.
\end{aligned} \tag{2.87}
$$

We would like to observe that this matches exactly with the $\mathfrak{su}(2|1)_A$-invariant S^2 partition function of the same Landau-Ginzburg model. In this theory, the $\mathrm{U}(1)_V$-charge of X is fixed by the superpotential to be $\frac{2}{k+2}$, and it follows from the formulae in [15, 61] that the $\mathfrak{su}(2|1)_A$-invariant partition function is indeed (2.87). The physical interpretation of this equality of partition functions is mirror symmetry for the k-th minimal model MM_k, which exchanges

$$
\mathrm{MM}_k \Longleftrightarrow \frac{\mathrm{MM}_k}{Z_k}\,. \tag{2.88}
$$

The ring relation $X^{k+1} = 0$ can be derived from the S^2 partition function. Indeed, using the identity

$$
\int dX d\overline{X} \frac{d}{dX} e^{-4\pi i (X^{k+2} + \overline{X}^{k+2} + \bar{t}\overline{X})} = 0\,, \tag{2.89}
$$

from which it follows that

$$
\left\langle X^{k+1}(N) \overline{X}^{\ell}(S) \right\rangle = 0 \qquad \forall \ell\,, \tag{2.90}
$$

implying that

$$
X^{k+1} = 0\,. \tag{2.91}
$$

This can be obtained from a differential equation. First we deform the supertotential by adding a source for the generator of the ring $W = X^{k+2} + tX$. The two-sphere partition function then obeys

$$
\partial_t^{k+1} Z_{S^2}^{A_{k+1}}|_{t=0} = 0 \qquad\qquad \partial_{\bar{t}}^{k+1} Z_{S^2}^{A_{k+1}}|_{\bar{t}=0} = 0\,. \tag{2.92}
$$

The two-point functions of chiral operators are given by (we do the computation in §B.4 using Riemann bilinear identity):

$$\begin{aligned} M^{A_{k+1}}_{m,n} &:= \left\langle X^m(N)\overline{X}^n(S)\right\rangle_{S^2} \\ &= \frac{1}{Z_{S^2}}\int_{\mathbb{C}} \mathrm{d}X\mathrm{d}\overline{X}\, X^m\overline{X}^n e^{-4\pi i W(X)-4\pi i\overline{W}(\overline{X})} \qquad (2.93a)\\ &= \begin{cases} \frac{\pi(-i)^q}{(k+2)Z_{S^2}}(4\pi)^{-\frac{2(1+m)}{k+2}-q}\frac{\Gamma\left(\frac{m+1}{k+2}+q\right)}{\Gamma\left(\frac{k-m+1}{k+2}\right)} & \text{if } q:=\frac{n-m}{k+2}\in\mathbb{Z} \\ 0 & \text{otherwise}\end{cases} \qquad (2.93b)\end{aligned}$$

We see that any two operators with integer dimensions can mix.

Recall that, $\Gamma(-s)$ has poles at $s\in\mathbb{N}_{\geq 0}$. Therefore (2.93b) implies:

$$M^{A_{k+1}}_{m,n}=0 \quad \text{for} \quad \frac{k-m+1}{k+2}=-s \;\Rightarrow\; m=k+1+s(k+2) \quad \text{where} \quad s\in\mathbb{N}_{\geq 0}\,. \qquad (2.94)$$

This in particular shows that, for $q=s=0$, $M^{A_{k+1}}_{k+1,k+1}=0$. Now, let us denote by $M^{A_{k+1}}_{(n)}$ the matrix $M^{A_{k+1}}_{i,j}$ with $i,j=0,\cdots,n$. Then $M^{A_{k+1}}_{(k+1)}$ is a diagonal matrix with exactly one zero row (the $(k+2)$'th row). This implies, using the Gram-Schmidt procedure (c.f. (2.52)),[36]

$$\left\langle X^{k+1}(0)\overline{X}^{k+1}(\infty)\right\rangle_{\mathbb{R}^2}=4^{\frac{k+1}{k+2}}\frac{\det M^{A_{k+1}}_{(k+1)}}{\det M^{A_{k+1}}_{(k)}}=0 \qquad (2.95)$$

By the Reeh-Schlieder theorem, we arrive at the ring relation

$$X^{k+1}=0 \qquad (2.96)$$

This implies that the chiral ring is given by:

$$\mathcal{R}^{\mathrm{c}}_{A_{k+1}}=\mathbb{C}[X]/\langle X^{k+1}\rangle\,. \qquad (2.97)$$

It can also be explicitly checked that $\left\langle X^m(0)\overline{X}^m(\infty)\right\rangle_{\mathbb{R}^2}=0$ for all $m>k$ implying that

$$X^m=0 \qquad m>k\,. \qquad (2.98)$$

[36]In this particular case it is actually unnecessary to use the Gram-Schmidt procedure, because the operator X^{k+1} has dimension $\frac{k+1}{k+2}<1$ and the operators $\mathbb{1},X,\cdots,X^{k+1}$ do not mix with each other (mixing can occur only at integer gaps in dimensions).

This can be shown as follows. (2.93b) allows to write a recursion relation between $M^{A_{k+1}}_{m,n}$ and $M^{A_{k+1}}_{m+k+2,n}$. Note that if we shift $m \to m+k+2$ then $m-n+q(k+2)=0$ can be maintained by simultaneously shifting $q \to q-1$. Therefore,

$$\begin{aligned} M^{A_{k+1}}_{m+k+2,n} &= \frac{\pi(-i)^{q-1}}{(k+2)Z_{S^2}}(4\pi)^{-\frac{2(1+m)}{k+2}-q+1}\frac{\Gamma\left(\frac{m+1}{k+2}+q\right)}{\Gamma\left(\frac{k-m+1}{k+2}-1\right)} \\ \Rightarrow \frac{M^{A_{k+1}}_{m+k+2,n}}{M^{A_{k+1}}_{m,n}} &= -4\pi i\left(\frac{m+1}{k+2}\right) \end{aligned} \tag{2.99}$$

This implies that for any $m > k+1$ the $(m+1)$'th row of $M^{A_{k+1}}_{(m)}$ is a multiple of the $(m-k-1)$'th row of $M^{A_{k+1}}_{(m)}$. Therefore, for $m > k$, the number of 0 eigenvalues of $M^{A_{k+1}}_{(m)}$ is $m-k$ (note that $M^{A_{k+1}}_{(k+1)}$ already has one zero eigenvalue). Hence:

$$\left\langle X^m(0)\overline{X}^m(\infty)\right\rangle_{\mathbb{R}^2} = 2^{\frac{2m}{k+2}}\frac{\det M^{A_{k+1}}_{(m)}}{\det M^{A_{k+1}}_{(m-1)}} \sim \frac{0^{m-k}}{0^{m-k-1}} = 0\,, \qquad \text{for} \quad m > k\,. \tag{2.100}$$

Finally, we can make contact with previously known results about the OPE structure of these chiral primaries obtained from CFT methods [27, 131]. First, let us normalize the chiral ring operators by defining $\widehat{X^n} := X^n/\|X^n\|$ where $\|X^n\|^2 = \left\langle X^m(0)\overline{X}^m(\infty)\right\rangle_{\mathbb{R}^2}$. Then using the relation $X^mX^n = X^{m+n}$ we can compute the OPE coefficients for the normalized operators:

$$\widehat{X^m}\,\widehat{X^n} = \mathcal{F}^{A_{k+1}}_{m,n}\widehat{X^{m+n}}\,, \qquad \mathcal{F}^{A_{k+1}}_{m,n} = \frac{\|X^{m+n}\|}{\|X^m\|\|X^n\|}\,. \tag{2.101}$$

These OPE coefficients depend only on the dimensions of the operators and the central charge c [27, 131]. For the ADE models, the central charge $c = 3 - \frac{6}{h}$ where h is the Coxeter number of the corresponding Lie group. For example, the Coxeter number of A_{k+1} is $(k+2)$, so the central charge of the A_{k+1} model is:

$$c_{A_{k+1}} = \frac{3k}{k+2}\,. \tag{2.102}$$

The central charge dependence of the OPE coefficients is implicit in (2.101) and can be seen through the dependence of the correlation functions on k (see (2.93b)). These OPE coefficients can be computed using (2.101), (2.93b), and (2.52), the result is:

$$\mathcal{F}^{A_{k+1}}_{m,n} = \sqrt{\frac{\Gamma\left(\frac{1}{k+2}\right)\Gamma\left(\frac{k-m+1}{k+2}\right)\Gamma\left(\frac{k-n+1}{k+2}\right)\Gamma\left(\frac{m+n+1}{k+2}\right)}{\Gamma\left(\frac{k+1}{k+2}\right)\Gamma\left(\frac{m+1}{k+2}\right)\Gamma\left(\frac{n+1}{k+2}\right)\Gamma\left(\frac{k-m-n+1}{k+2}\right)}}\,. \tag{2.103}$$

These coefficients can also be read off from the results of [27,131] and they match with our expression.

D_{k+1} minimal model

The same analysis can be carried out for the D_{k+1} minimal model in a completely analogous way. The superpotential for the LG theory describing this minimal model is:

$$W(X,Y) = X^k + XY^2\,, \tag{2.104}$$

with two generators X and Y corresponding to conformal primaries of dimensions $\frac{1}{k}$ and $\frac{k-1}{2k}$ respectively. With the canonical kinetic Lagrangian $\int \mathrm{d}^4\theta(X\overline{X} + Y\overline{Y})$ the equations of motion tells us:

$$\overline{D}^2\overline{X} \propto kX^{k-1} + Y^2\,, \qquad \overline{D}^2\overline{Y} \propto XY\,. \tag{2.105}$$

Therefore, the chiral ring can be described as:

$$\mathcal{R}^{\mathrm{c}}_{D_{k+1}} = \mathbb{C}[X,Y]/\langle kX^{k-1} + Y^2, XY\rangle\,. \tag{2.106}$$

We will derive these ring relations by computing correlation functions of the generators and showing that the operators $kX^{k-1} + Y^2$ and XY are zero in the chiral ring. Note that according to the relations, a minimal (dimension-wise) basis for the ring is given by:

$$\mathbb{1}, X, \cdots, X^{k-1}, \quad \text{and} \quad Y\,. \tag{2.107}$$

Here X^{k-1} has the highest dimension, $\frac{k-1}{k} < 1$. Based on our supergravity analysis we can expect that operators can only mix at integer gaps in dimensions,[37] in fact we will establish this by explicit computation. This implies that there is no mixing among the operators in the minimal basis, simplifying our computations.

When put on a two-sphere, a general extremal correlation function in this LG model is given by:

$$\begin{aligned} M^{D_{k+1}}_{m,n,p,q} &:= \left\langle X^m(N)Y^n(N)\overline{X}^p(S)\overline{Y}^q(S)\right\rangle_{S^2} \\ &= \frac{1}{Z^{D_{k+1}}_{S^2}} \int_{\mathbb{C}^2} \mathrm{d}X\mathrm{d}\overline{X}\mathrm{d}Y\mathrm{d}\overline{Y}\; X^m Y^n \overline{X}^p \overline{Y}^q e^{-4\pi i\left(X^k + XY^2 + \overline{X}^k + \overline{XY}^2\right)} \end{aligned} \tag{2.108}$$

[37] As we found explicitly for the A_{k+1} model following (2.93b).

We can carry out the $Y, \overline{Y}$ integrals first:

$$\begin{aligned}\int_{\mathbb{C}} \mathrm{d}Y \mathrm{d}\overline{Y}\, Y^n \overline{Y}^q e^{-4\pi i (XY^2 + \overline{X}\overline{Y}^2)} &= X^{-\frac{n+1}{2}} \overline{X}^{-\frac{q+1}{2}} \int_{\mathbb{C}} \mathrm{d}Y \mathrm{d}\overline{Y}\, Y^n \overline{Y}^q e^{-4\pi i (Y^2 + \overline{Y}^2)}\,, \\ &= X^{-\frac{n+1}{2}} \overline{X}^{-\frac{q+1}{2}} Z^{A_1}_{S^2} M^{A_1}_{n,q}\,. \qquad \text{[c.f. (2.93a)]}\end{aligned} \tag{2.109}$$

Then we can perform the integrals over $X, \overline{X}$, which leads to an expression for the D-type extremal correlators in terms of the A-type extremal correlators:

$$M^{D_{k+1}}_{m,n,p,q} = \frac{Z^{A_1}_{S^2} Z^{A_{k-1}}_{S^2}}{Z^{D_{k+1}}_{S^2}} M^{A_1}_{n,q} M^{A_{k-1}}_{m-\frac{n+1}{2}, p-\frac{q+1}{2}}\,. \tag{2.110}$$

Let us define some symbols:

$$\widetilde{m} := m - \frac{n+1}{2}\,, \quad \widetilde{p} := p - \frac{q+1}{2}\,, \quad \upsilon := \frac{q-n}{2}\,, \quad \chi := \frac{\widetilde{p} - \widetilde{m}}{k}\,, \tag{2.111}$$

then we can write (2.110) more explicitly as (c.f. (2.93b)):

$$M^{D_{k+1}}_{m,n,p,q} = \begin{cases} \frac{1}{Z^{D_{k+1}}_{S^2}} \frac{(-i)^{\upsilon+\chi}}{32k} (4\pi)^{1-n-\upsilon-\chi-\frac{2}{k}(\widetilde{m}+1)} \frac{\Gamma\left(\frac{1+n}{2}+\upsilon\right)\Gamma\left(\frac{1+\widetilde{m}}{k}+\chi\right)}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{k-\widetilde{m}-1}{k}\right)} & \upsilon, \chi \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{2.112}$$

Noting the difference in dimension:

$$\Delta(X^m Y^n) - \Delta(X^p Y^q) = \upsilon + \chi\,, \tag{2.113}$$

we observe that mixing between different operators can occur on the sphere only if their dimensions differ by an integer amount. Since all the operators in the minimal basis (2.107) have dimensions less than 1, there is no mixing among them. Therefore, their correlation functions on the sphere of unit radius, namely the ones given by (2.112), are simply proportional to the corresponding flat space correlators.

In order to check the ring relations, we define the following two operators:

$$\mathcal{O}_1 := kX^{k-1} + Y^2, \quad \text{and} \quad \mathcal{O}_2 := XY\,. \tag{2.114}$$

Using the explicit form of the correlation functions (2.112) we can easily check that these two operators have zero norms:

$$\begin{aligned}\|\mathcal{O}_1\|^2 = \left\langle \mathcal{O}_1(0)\overline{\mathcal{O}}_1(\infty)\right\rangle_{\mathbb{R}^2} &= 4^{\frac{k-1}{k}} \left\langle \mathcal{O}_1(N)\overline{\mathcal{O}}_1(S)\right\rangle_{S^2} \\ &= 4^{\frac{k-1}{k}} \left(k^2 M^{D_{k+1}}_{k-1,0,k-1,0} + kM^{D_{k+1}}_{k-1,0,0,2} + kM^{D_{k+1}}_{0,2,k-1,0} + M^{D_{k+1}}_{0,2,0,2}\right) = 0\,,\end{aligned} \tag{2.115}$$

and similarly,

$$\|\mathcal{O}_2\|^2 = \left\langle \mathcal{O}_2(0)\overline{\mathcal{O}}_2(\infty)\right\rangle_{\mathbb{R}^2} = 2^{\frac{k+1}{k}} \left\langle \mathcal{O}_2(N)\overline{\mathcal{O}}_2(S)\right\rangle_{S^2} = 2^{\frac{k+1}{k}} M^{D_{k+1}}_{1,1,1,1} = 0\,. \tag{2.116}$$

By the Reeh-Schlieder theorem, we arrive at the ring relations:

$$kX^{k-1} + Y^2 = XY = 0\,, \tag{2.117}$$

giving us the chiral ring (2.106).

Once we normalize the chiral primaries, $\widehat{X^mY^n} := \frac{X^mY^n}{\|X^mY^n\|}$, we can compute the OPE coefficients for the product of two arbitrary chiral primaries in terms of the extremal correlators (as we did for the A series in (2.101) and (2.103)).

Let us make a few remarks about the computation. We define the OPE coefficients of the normalized operators by the following equation:

$$\widehat{X^mY^n}\,\widehat{X^pY^q} = \mathcal{F}^{D_{k_2}}_{(m,n),(p,q)}\widehat{X^{m+n}Y^{p+q}}\,, \qquad \mathcal{F}^{D_{k_2}}_{(m,n),(p,q)} = \frac{\|X^{m+n}Y^{p+q}\|}{\|X^mY^n\|\,\|X^pY^q\|}\,. \tag{2.118}$$

As we mentioned after (2.101), these coefficients depend only the dimensions of the operators and the central charge, in particular, they should be computable without making a choice a modular invariant. Therefore we should expect the OPE coefficients defined in (2.101) and the ones in (2.118) to be the same:

$$\mathcal{F}^{A_{k+1}}_{m,n} = \mathcal{F}^{D_{k'+1}}_{(p,q),(r,s)}\,, \tag{2.119}$$

whenever the central charges of the two theories and the dimensions of the involved operators coincide, i.e.:[38]

$$c_{A_{k+1}} = c_{D_{k'+1}} \quad \Rightarrow \quad k' = \frac{k+2}{2}\,, \tag{2.120a}$$

$$\Delta_{A_{k+1}}(X^m) = \Delta_{D_{k'+1}}(X^pY^q) \quad \Rightarrow \quad m = \frac{1}{2}(4p+kq)\,, \tag{2.120b}$$

$$\Delta_{A_{k+1}}(X^n) = \Delta_{D_{k'+1}}(X^rY^s)\,, \quad \Rightarrow \quad n = \frac{1}{2}(4r+ks)\,. \tag{2.120c}$$

Explicit computation shows that the equality (2.119) indeed holds given that the above conditions are satisfied.

[38]The Coxeter numbers of A_{k+1} and D_{k+1} are $(k+2)$ and $2k$ respectively. Two ADE models have the same central charge as long as the corresponding Lie groups have the same Coxeter number.

Exceptional minimal models

Our discussion so far for the A and the D-series of minimal models readily extends to the E_6, E_7 and E_8 models. Like the D-series, the chiral rings of these models have two generators. The E_6 and the E_8 minimal model superpotentials, namely X^3+Y^4 and X^3+Y^5 (2.85), are decoupled sums of two polynomials in these two generators and therefore the chiral rings of these two models are simply Cartesian products of two A-type chiral rings:

$$\mathcal{R}^{\mathrm{c}}_{E_6} = \mathcal{R}^{\mathrm{c}}_{A_2} \times \mathcal{R}^{\mathrm{c}}_{A_3}\,, \qquad \mathcal{R}^{\mathrm{c}}_{E_8} = \mathcal{R}^{\mathrm{c}}_{A_2} \times \mathcal{R}^{\mathrm{c}}_{A_4}\,. \tag{2.121}$$

The E_7 model involves nontrivial coupling between the two generators analogous to the D-series. Recall that the E_7 superpotential is $X^3 + XY^3$ and therefore the chiral ring is:

$$\mathcal{R}^{c}_{E_7} = \mathbb{C}[X,Y]/\langle 3X^2+Y^3, XY^2\rangle\,. \tag{2.122}$$

There are two chiral primaries X and Y of dimensions $\frac{1}{3}$ and $\frac{2}{9}$ respectively. A basis for the chiral ring is given by $\{\mathbb{1}, Y, X, Y^2, XY, Y^3, X^2Y\}$ in increasing order of dimension. The highest dimensional chiral primary X^2Y has dimension $\frac{8}{9} < 1$ and therefore there is no mixing among these basis operators when we put the corresponding LG theory on a sphere. Thus once again we have proportionality between extremal correlators on a sphere of unit radius and extremal correlators on $\mathbb{R}^2$. Just as we did in (2.110), we can write down the extremal correlators in this model in terms of A-series extremal correlators:

$$M^{E_7}_{m,n,p,q} := \left\langle X^m(N) Y^n(N) \overline{X}^p(S) \overline{Y}^q(S)\right\rangle_{S^2} = \frac{Z^{A_2}_{S^2} Z^{A_2}_{S^2}}{Z^{E_7}_{S^2}} M^{A_2}_{n,q} M^{A_2}_{m-\frac{n+1}{3},p-\frac{q+1}{3}}\,. \tag{2.123}$$

Using the explicit form of the A-type extremal correlators (2.93b), we can verify that in the E_7 model the operators $\mathcal{O}_1 := 3X^2+Y^3$, $\mathcal{O}_2 := XY^2$ have zero norms:

$$\begin{aligned} \|\mathcal{O}_1\|^2 &= 2^{\frac{4}{3}}\left(9M^{E_7}_{2,0,2,0} + 3M^{E_7}_{2,0,0,3} + 3M^{E_7}_{0,3,2,0} + M^{E_7}_{0,3,0,3}\right) = 0\,, \\ \|\mathcal{O}_2\|^2 &= 2^{\frac{14}{9}} M^{E_7}_{1,2,1,2} = 0\,. \end{aligned} \tag{2.124}$$

This establishes the ring relations in (2.122) by identifying $\mathcal{O}_1$ and $\mathcal{O}_2$ with the null operator.

2.5 Epilogue

In this chapter we have approached the computation of 2d BPS ring structure constants from the conceptually straightforward way of computing the relevant correlation functions.

The main tool at our disposal was supersymmetric localization allowing exact computation of these correlation functions on the sphere. The main obstacle in using these results to compute the ring structure is the presence of conformal anomalies on the sphere leading to operator mixing. We have explored the roots of this mixing in the supergravitational descriptions of the sphere, and we have outlined a way of computing the flat space correlators from the sphere partition function in the presence of such mixing. We have demonstrated our method in some familiar theories of interest and reproduced known results about the structure constants that were previously obtained via CFT methods and we have also verified all the ring relations. This provides a simple elementary perspective on the cohomological algebras of 2d $\mathcal{N} = (2, 2)$ theories.

Chapter 3

Chiral Rings in 4 Dimensions

3.1 Introduction and Conclusions

The correlation functions of local operators are amongst the most well-studied observables in Quantum Field Theory (QFT). In Conformal Field Theories (CFTs), the two- and three-point functions of the local operators in the theory completely determine all the n-point functions.

In this chapter we find a formula that computes exactly the correlation functions[1]

$$\left\langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)...\mathcal{O}_{I_n}(x_n)\overline{\mathcal{O}}_{\bar{J}}(y)\right\rangle_{\mathbb{R}^4} \tag{3.1}$$

of any number of chiral primary operators $\mathcal{O}_{I_i}$ and one anti-chiral primary operator $\overline{\mathcal{O}}_{\bar{J}}$ in four-dimensional $\mathcal{N}=2$ superconformal field theories (SCFTs). Such correlation functions are henceforth referred to as extremal correlators. We determine these correlators as functions of the exactly marginal couplings of the SCFT, which span the so-called conformal manifold of the SCFT.[2] Our results apply to any SCFT with exactly marginal couplings that admit a Lagrangian description somewhere on the conformal manifold.[3] Near a weakly coupled point on the conformal manifold we find that the correlators (3.1) are given by an infinite series of perturbative corrections dressed by an infinite sequence of nonperturbative instanton corrections. Special cases of (3.1) are the two- and three-point functions, which

[1]Chiral primary operators sit in short representations of the four dimensional $\mathcal{N}=2$ superconformal algebra. See section 3.1.2.

[2]See section 3.1.1 for more details.

[3]The structure we find also applies to SCFTs that are inherently non-Lagrangian.

we refer to as the "chiral ring data" of the SCFT. As we will review below, once the chiral ring data is known, all the extremal correlators can be reconstructed.

Roughly speaking, our strategy is to express the flat space ($\mathbb{R}^4$) correlators given in (3.1) in terms of the four-sphere (S^4) partition function of a suitable *deformation* of the SCFT[4]

$$Z_{\text{deformed}}[S^4] \Longrightarrow \left\langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)\ldots\mathcal{O}_{I_n}(x_n)\overline{\mathcal{O}}_{\bar{J}}(y)\right\rangle_{\mathbb{R}^4}. \tag{3.2}$$

This partition function can be in turn evaluated exactly by supersymmetric localization, expanding upon Pestun's computation of the undeformed $\mathcal{N}=2$ partition function [140]. As was the case with the undeformed sphere partition function studied by Pestun, $Z_{\text{deformed}}[S^4]$ is also expressed as integral over the norm of the deformed partition function in the Ω-background [134] (see section 3.2.3).

An important subtlety in the relation (3.2) between $Z_{\text{deformed}}[S^4]$ and the extremal correlators on $\mathbb{R}^4$ is due to conformal anomalies, which cause operator mixing on S^4. Diagonalizing the operator mixing matrix on S^4 à la Gram-Schmidt leads to a representation of the extremal correlators on $\mathbb{R}^4$ in terms of determinants of derivatives of the deformed sphere partition function $Z_{\text{deformed}}[S^4]$. This induces the action of a system of integrable differential equations on the extremal correlators of $\mathcal{N}=2$ SCFTs.

As an illustrative example, we can consider SU(2) SQCD with 4 fundamental hypermultiplets, which contains precisely one chiral primary operator of dimension $2n$ for every integer $n \geq 1$. This case is special in that one does not have to consider any deformations of the S^4 partition function in order to calculate extremal correlators. The two-point functions of the dimension $2n$ chiral primary operators $\mathcal{O}_n$ can be expressed succinctly as the ratio of determinants

$$\left\langle \mathcal{O}_n(0)\overline{\mathcal{O}}_m(\infty)\right\rangle_{\mathbb{R}^4} = \frac{16^n\delta_{nm}}{Z[S^4]}\frac{\det_{(k,l)=0,\ldots,n}\left(\partial_\tau^k\partial_{\bar{\tau}}^l Z[S^4]\right)}{\det_{(k,l)=0,\ldots,n-1}\left(\partial_\tau^k\partial_{\bar{\tau}}^l Z[S^4]\right)}, \tag{3.3}$$

where τ is the complexified coupling constant of the theory. This formula neatly encodes all the two-point functions of chiral primary operators in terms of the sphere partition function, which can be computed exactly by supersymmetric localization.

The partition function $Z_{\text{deformed}}[S^4]$, and therefore the extremal correlators, can be explicitly calculated to all orders in perturbation theory. The instanton corrections to

[4]Our essential ideas and techniques can be also applied to $(2,2)$ theories in $d=2$. However, we do not pursue this direction here and concentrate on $\mathcal{N}=2$ theories in $d=4$. In fact, technically the case of $d=2$ is simpler since no new instanton contributions need to be computed in the Coulomb branch representation [15, 61].

$Z_{\text{deformed}}[S^4]$ can be computed in some theories using results already available in the literature, while for other theories it requires first writing down the instanton partition function of the deformed SCFT in the Ω-background, which is an interesting open problem (see section 3.2.3).

Our identification, summarized by the schematic equation (3.2), provides a broad extension of the formula derived in [76, 81, 82] relating the undeformed S^4 partition function of the SCFT to the Kähler potential K on the conformal manifold[5]

$$Z[S^4] = r^{-4a} e^{\frac{1}{12} K(\tau^i, \bar{\tau}^{\bar{i}})} \,, \tag{3.4}$$

where $\tau^i, \bar{\tau}^{\bar{i}}$ are the exactly marginal couplings of the SCFT, a is the Euler conformal anomaly and r the radius of S^4. The two-point functions of the dimension-two chiral primary operators, denoted by $\mathcal{O}_i$, are determined in terms of the S^4 partition function of the SCFT through

$$\left\langle \mathcal{O}_i(0) \overline{\mathcal{O}}_{\bar{i}}(\infty) \right\rangle_{\mathbb{R}^4} = 16 \frac{\partial}{\partial \tau^i} \frac{\partial}{\partial \bar{\tau}^{\bar{i}}} \ln Z[S^4] \,. \tag{3.5}$$

Our results extend this formula to arbitrary chiral primary operators $\mathcal{O}_I$. See, for example, equation (3.3).

As mentioned above, the chiral ring data obtained from the deformed partition function $Z_{\text{deformed}}[S^4]$ obeys a system of differential equations with respect to the exactly marginal couplings $\tau^i, \bar{\tau}^{\bar{i}}$. For SCFTs with one exactly marginal coupling and a one-dimensional Coulomb branch, namely for $\mathcal{N} = 2$ SU(2) SQCD with four fundamental hypermultiplets and $\mathcal{N} = 4$ SU(2) super-Yang-Mills, we show that the equations obeyed by the chiral ring data obtained from $Z_{\text{deformed}}[S^4]$ are those of a semi-infinite Toda chain, which are integrable.

The fact that the chiral ring data of these theories obeys the semi-infinite Toda chain system was exhibited in [7–9] starting from the the tt^* equations of the four-dimensional SCFT [139]. In Appendix D.1 we show that the tt^* equations of any four-dimensional $\mathcal{N} = 2$ SCFT are integrable and governed by a Hitchin system, in parallel with the tt^* equations of two-dimensional $(2, 2)$ QFTs [28]. In Appendix D.4 we show that the chiral ring data of SU(N) SQCD with $2N$ fundamental hypermultiplets computed through our correspondence (3.2) indeed obeys the corresponding tt^* equations. For the special case of $\mathcal{N} = 4$ super-Yang-Mills with an arbitrary gauge group $G \neq$ SU(2), the chiral ring can be organized in terms of decoupled semi-infinite Toda chains. However, this is not the case in SU(N) SQCD with $2N$ fundamental hypermultiplets.

[5] Extending the earlier result in two-dimensional $\mathcal{N} = (2, 2)$ SCFTs [76, 82, 83, 107].

The tt^* equations themselves are not sufficient to determine the chiral ring data of the SCFT since these equations have several solutions. Rather, the chiral ring data is found through the partition function of the deformed SCFT on S^4 via (3.2). One can view (3.2) as a particular solution to the tt^* equations. This allows us to obtain new results in four-dimensional $\mathcal{N} = 2$ SCFTs.

The computation of the correlation function of local operators (3.1) in a four-dimensional QFT contributes to the recent progress in the exact determination of certain observables in supersymmetric QFTs. Particularly striking are those observables that depend non-holomorphically on the coupling constants of the theory. These include the computation of Wilson loops [140], 't Hooft loops [88], domain walls [62, 101] and cusp anomalous dimensions at small angles [66] in four dimensional $\mathcal{N} = 2$ QFTs. For some previous work on the partition function of SCFTs on spheres consult [15, 60, 61, 98, 106, 109, 140].

The extremal correlators (3.1) should transform under the action of dualities. Indeed, a chiral ring operator is expected to transform as a modular form under S-duality, with the modular weight determined by the dimension of the operator (c.f. [87, 96]). It would be interesting to study in detail the action of duality on these correlation functions. The exact computation of the extremal correlators in this chapter can be generalized by adding supersymmetric circular Wilson loops, 't Hooft loops and/or domain walls supported on S^3 in $\mathbb{R}^4$, to yield, for example, the correlators[6]

$$\left\langle \mathcal{O}_I(0) D\, \overline{\mathcal{O}}_{\bar{J}}(\infty) \right\rangle_{\mathbb{R}^4}, \tag{3.6}$$

where D denotes a judiciously chosen supersymmetric spherical defect operator in the SCFT.

The results of our work are complementary to those coming from the superconformal bootstrap of 4d $\mathcal{N} = 2$ theories [14, 121, 122, 124]. In particular, [14, 122] considered four-point correlation functions of two chiral and two anti-chiral operators and obtained bounds on various OPE coefficients including some that can be computed from extremal correlators. Their results pertaining to dimension-2 chiral primaries can be interpreted as bounds on the curvature of the conformal manifold. It would be interesting to extract bounds on the curvature of the bundles of higher-dimension conformal operators in a similar way.

The plan of the rest of the chapter is as follows. In the remaining of the present section we provide some relevant preparatory material: a brief discussion of conformal manifolds

[6] One could also compute correlators in the presence of a surface operator by figuring out the interplay between vortices and instantons with the higher dimensional chiral primary operators.

in CFTs, a review of the chiral ring of four-dimensional $\mathcal{N}=2$ SCFTs, and a discussion of some subtleties that arise in defining CFTs on S^4. In section 3.2 we show that the chiral ring data of a SCFT can be extracted from the partition function of a deformation of the SCFT on S^4, and we provide an algorithm to determine the Hermitian metric on the chiral ring. In section 3 we study in detail SU(N) SQCD and $\mathcal{N}=4$ super-Yang-Mills and discuss the relation with the four-dimensional tt^* equations. We also consider some of the asymptotic properties of the perturbative expansion in SU(N) SQCD. Many technical results are collected in five appendices.

3.1.1 Conformal Manifolds

Let us review very briefly the notion of a conformal manifold. Given a CFT in d dimensions, we suppose that there exists a (Hermitian) scalar marginal operator, O. If we deform the theory by $\delta S = \lambda \int d^d x\, O$ with some coefficient λ, then in general there would be a nontrivial beta function for λ computable in conformal perturbation theory

$$\frac{d\lambda}{d\ln\mu} = \beta_1\lambda^2 + \beta_2\lambda^3 + \cdots \ . \tag{3.7}$$

However, under some circumstances, all the coefficients vanish $\beta_a = 0$. We then say that O is an exactly marginal operator; adding it to the action does not break the conformal symmetry. The coupling λ in this case defines a line of CFTs along which the critical exponents can vary continuously. More generally, imagine that there is a set of such exactly marginal operators O_i. We can define the Zamolodchikov metric [159] in the space of theories, that is, in the conformal manifold, via

$$\langle O_i(x) O_j(0)\rangle_{\{\lambda^i\}} = \frac{g_{ij}(\lambda^i)}{x^{2d}} \ , \tag{3.8}$$

where we evaluate the two-point function in the CFT with couplings λ^i. While the metric itself is as usual ambiguous (by choosing appropriate contact terms for our operators, we can choose the metric and the Christoffel symbols to be trivial at any given point [116]), there are various invariants such as the Ricci scalar that can be constructed out of it, and which are interesting observables of the CFT.

The vanishing of all the coefficients $\beta_a = 0$ in (3.7) is common in $c=1$ models in $d=2$ but otherwise requires new symmetries in addition to the conformal symmetry [26]. One such extra symmetry is current algebra symmetry, in which case the spectrum of exactly marginal operators can be determined [67]. Another additional symmetry is supersymmetry. Indeed, exactly marginal operators are common in supersymmetric theories in $2 \leq d \leq 4$. Let us consider first $\mathcal{N}=1$ theories in $d=4$. In these theories the conformal manifold is a Kähler manifold with local complex coordinates τ^i, $\bar{\tau}^{\bar{i}}$ associated to the descendants of $\mathcal{N}=1$ chiral primaries and anti-chiral primaries of dimension 3. Not every marginal operator is necessarily exactly marginal, but there are nevertheless many examples with exactly marginal operators [92, 120]. $\mathcal{N}=2$ theories, being a special case of $\mathcal{N}=1$ theories, also admit a Kähler conformal manifold and the complex coordinates τ^i,$\bar{\tau}^{\bar{i}}$ correspond to descendants of $\mathcal{N}=2$ chiral primaries of dimension 2 (see section 3.1.2).[7] In an $\mathcal{N}=2$ theory every marginal operator is necessarily exactly marginal.[8] One can further argue that in $\mathcal{N}=2$ theories the Kähler class is trivial, in other words, there are no two-cycles in the conformal manifold through which the Kähler two-form has flux. This global restriction implies, for example, that the $\mathcal{N}=2$ conformal manifold cannot be compact [82].

In four-dimensional $\mathcal{N}=2$ SCFTs the Kähler potential (and hence the Zamolodchikov metric) on the conformal manifold can be determined exactly from the partition function of the SCFT on S^4 via (3.4).

In some theories, different points in the conformal manifold may be mapped into each other by a duality transformation, possibly relating the theory in a regime where perturbation theory is valid to a strongly coupled regime. This picture can give rise to an intricate pattern of dualities, where the conformal manifold can acquire an elegant geometrical and mathematical interpretation, as in [72].

The extremal correlators (3.1) provide novel QFT data that transforms naturally under dualities. It would be interesting to study in detail the action of strong-weak coupling dualities on these extremal correlation functions.

3.1.2 The Chiral Ring of $\mathcal{N}=2$ SCFTs

Local operators in $\mathbb{R}^4$ or equivalently states on the cylinder in an $\mathcal{N}=2$ SCFT fit into unitary highest weight representations of the superconformal algebra $\mathfrak{su}(2,2|2)$. The algebra $\mathfrak{su}(2,2|2)$ contains the following generators (in Euclidean signature):

- The conformal algebra $\mathfrak{so}(5,1)$

[7] This is for exactly marginal operators that preserve $\mathcal{N}=2$ supersymmetry.

[8] Here is an argument along the lines of [92]. There is a scheme in which the superpotential is not renormalized. Then if the beta function is nonzero it has to be reflected by a D-term in the action $\int d^4x d^8\theta\, \mathcal{U}$ with $\mathcal{U}$ some real primary operator. But since the τ^i are classically dimensionless, $\Delta(\mathcal{U})=0$ in the original fixed point. Therefore, $\mathcal{U}$ has to be the unit operator and the deformation $\int d^4x d^8\theta\, \mathcal{U}$ is therefore trivial. This proves that $\beta_a=0$.

- The Poincaré supercharges $Q^a_\alpha, \overline{Q}^a_{\dot\alpha}$ and the conformal supercharges $S^a_\alpha, \overline{S}^a_{\dot\alpha}$ $\quad (a = 1, 2)$
- The $\mathfrak{su}(2)_R \times \mathfrak{u}(1)_R$ R-symmetry

The (anti)-commutation relations can be found, for example, in [69].

A highest weight representation is labeled by the quantum numbers $(\Delta; j_l, j_r; s; R)$ of its highest weight state under dilatations, Lorentz, and $\mathfrak{su}(2)_R \times \mathfrak{u}(1)_R$. This state is created by a superconformal primary operator $\mathcal{O}$, defined by $[S^a_\alpha, \mathcal{O}(0)] = [\overline{S}^a_{\dot\alpha}, \mathcal{O}(0)] = 0$.

An interesting class of superconformal primaries are the so-called chiral primary operators $\mathcal{O}_I$, annihilated by

$$[\overline{Q}^a_{\dot\alpha}, \mathcal{O}_I] = 0\,, \tag{3.9}$$

together with the conjugate anti-chiral primaries $\overline{\mathcal{O}}_{\bar I}$

$$[Q^a_\alpha, \overline{\mathcal{O}}_{\bar I}] = 0\,. \tag{3.10}$$

Unitarity of the SCFT and the anticommutators of the $\mathcal{N} = 2$ superconformal algebra

$$\{Q^a_\alpha, S^b_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab}\left(\Delta + \frac{R}{2}\right) + \epsilon^{ab}M_{\alpha\beta} + \epsilon_{\alpha\beta}J^{ab} \tag{3.11}$$

$$\{\overline{Q}^a_{\dot\alpha}, \overline{S}^b_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\epsilon^{ab}\left(\Delta - \frac{R}{2}\right) + \epsilon^{ab}M_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}J^{ab} \tag{3.12}$$

imply that[9]

$$\mathcal{O}_I : \;\; \Delta = \frac{R}{2}\,, \qquad\quad j_r = s = 0\;, \tag{3.13}$$

$$\overline{\mathcal{O}}_{\bar I} : \;\; \Delta = -\frac{R}{2}\,, \qquad j_l = s = 0\,. \tag{3.14}$$

Therefore, a chiral primary must transform as a scalar under $\mathfrak{su}(2)_R$ and its dimension is completely determined by its $\mathfrak{u}(1)_R$ charge R. A priori, a chiral primary can carry Lorentz spin $(j_l, 0)$. However, for SCFTs that admit a Lagrangian description somewhere in their conformal manifold, one can easily show that all chiral primaries must be Lorentz scalars, so that $j_r = j_l = 0$. Furthermore, no example of a chiral primary with spin has been found to date in non-Lagrangian theories.[10] See [20] for a further discussion about spinning chiral

[9] In our conventions $[R, Q^a_\alpha] = -Q^a_\alpha$.

[10] If spinning chiral primaries existed, they would be visible in the superconformal index [113]. We are grateful to Leonardo Rastelli for a discussion.

primaries. Henceforth, we only discuss chiral primary operators that are Lorentz scalars. These chiral primary operators parametrize the Coulomb branch of vacua of the SCFT, where $\mathfrak{su}(2)_R$ is preserved and $\mathfrak{u}(1)_R$ is spontaneously broken.

A chiral primary operator of dimension Δ can be realized as the bottom component of an $\mathcal{N}=2$ chiral superfield $\mathcal{O}$ of Weyl weight Δ (we denote the superfield by the same symbol as the bottom component). This superfield is annihilated by the four-dimensional $\mathcal{N}=2$ right-handed superspace derivatives

$$\overline{D}^a_{\dot\alpha}\mathcal{O}=0\,. \tag{3.15}$$

The spacetime integral of the top component of a chiral superfield with $\Delta=2$, denoted by C, defines an $\mathcal{N}=2$ superconformal invariant, constructed by integrating the chiral superfield over the chiral half of the $\mathcal{N}=2$ superspace[11]

$$\int d^4x\, d^4\theta\, \mathcal{O}=\int d^4x\, C\,. \tag{3.16}$$

Therefore, chiral primary operators with $\Delta=2$, which we denote by $\mathcal{O}_i$, give rise to exactly marginal operators, C_i. Geometrically, the C_i can be viewed as tangent vectors to the conformal manifold.

An important property of chiral primary operators in $\mathcal{N}=2$ SCFTs is that they cannot disappear from the spectrum as we explore the conformal manifold. This is because the short representation of the $\mathcal{N}=2$ superconformal algebra built out of a chiral primary highest weight cannot combine (at a generic point) with any other multiplet of the $\mathcal{N}=2$ superconformal algebra to become a long multiplet (see [59] for the list of possible multiplet recombinations).

While chiral primary operators cannot disappear, they can mix when transported around the conformal manifold. Thus, chiral primary operators can be described as sections of a holomorphic vector bundle over the conformal manifold [139]. The connection captures the operator mixing [116, 145].[12]

The operator product expansion (OPE) of chiral primary operators is non-singular since singular terms in the OPE would necessarily violate the unitarity bound $\Delta\geq R/2$.

[11] See Appendix D.2 for some details about the component structure of a chiral multiplet.

[12] Operator mixing is nontrivial when the curvature of the connection is non-vanishing. In $\mathcal{N}=4$ super-Yang-Mills and for the Higgs branch operators in $\mathcal{N}=2$ SCFTs the situation is rather simple due to the fact that the corresponding curvatures vanish. This is however not the case for chiral primaries (which we study in this chapter) in $\mathcal{N}=2$ SCFTs.

Therefore, chiral primary operators furnish a ring, the chiral ring

$$\mathcal{O}_I(x)\mathcal{O}_J(0) = \sum_K C_{IJ}^K \mathcal{O}_K(0) + \dots\,, \tag{3.17}$$

where ... denote $\overline{Q}$-exact terms. The multiplicative operation in this commutative ring is the CFT OPE. It is believed that for $\mathcal{N}=2$ SCFTs the chiral ring is *freely generated*, that is, there exists a finite-dimensional basis of chiral operators such that any element of the chiral ring has a unique representation as a polynomial in the basis elements. For Lagrangian $\mathcal{N}=2$ theories, it is easy to show that indeed the chiral ring is freely generated. The number of generators of the chiral ring is the dimension of the Coulomb branch of the SCFT.

For a freely generated ring, we can always "diagonalize" the product structure in the ring such that

$$\mathcal{O}_I(x)\mathcal{O}_J(0) = \mathcal{O}_I\mathcal{O}_J(0) + \dots\,, \tag{3.18}$$

so that the matrix $(C_I)_J^K$ in (3.17) has a single nonzero entry for each row. While in this basis the ring structure constants are trivialized, the two-point functions of chiral primaries with anti-chiral primaries are nontrivial functions of the coupling constants

$$\left\langle \mathcal{O}_I(x)\overline{\mathcal{O}}_{\bar{J}}(0)\right\rangle_{\{\tau^i,\bar{\tau}^{\bar{i}}\}} = \frac{G_{I\bar{J}}(\tau^i,\bar{\tau}^{\bar{i}})}{|x|^{2\Delta_I}}\delta_{\Delta_I\Delta_{\bar{J}}}\,. \tag{3.19}$$

The metric $G_{I\bar{J}}$ defined by the two-point functions (3.19) is a Hermitian metric on the vector bundle. In this basis, the chiral ring data is captured by the Hermitian metric $G_{I\bar{J}}$.

For completeness we would like to remind that $\mathcal{N}=2$ SCFTs contain another class of half-supersymmetric superconformal primary operators, $\mathcal{H}_I$. These are annihilated by supercharges of both chiralities[13]

$$[Q_\alpha^1, \mathcal{H}_I] = [\overline{Q}_{\dot{\alpha}}^1, \mathcal{H}_I] = 0\,. \tag{3.20}$$

Unitarity and the anticommutation relations (3.11)(3.12) imply that $\mathcal{H}_I$ obey

$$\Delta = 2s\,, \qquad j_l = j_r = R = 0\,. \tag{3.21}$$

Thus, these operators are Lorentz scalars, have vanishing $\mathfrak{u}(1)_R$ charge and the conformal dimension is completely determined in terms of the $\mathfrak{su}(2)_R$ isospin s. Furthermore, they are highest weight of $\mathfrak{su}(2)_R$. The operators $\mathcal{H}_I$ form a ring under the OPE, but unlike

[13]The two conditions are compatible since $\{Q_\alpha^1, \overline{Q}_{\dot{\alpha}}^1\} = 0$.

the chiral ring, this one is not freely generated. The operators in this ring parametrize the Higgs branch of vacua of the SCFT, where $\mathfrak{u}(1)_R$ is unbroken and $\mathfrak{su}(2)_R$ is spontaneously broken.

The representations of the $\mathcal{N} = 2$ superconformal algebra with highest weight $\mathcal{H}_I$ with $s > 3/2$ can recombine with other short multiplets of the $\mathcal{N} = 2$ superconformal algebra to become a long representation.[14] The operators which do not recombine can be described as sections of a vector bundle over the conformal manifold. The curvature of this connection is vanishing. The ring data associated to these operators is independent of the exactly marginal couplings [6, 13]. This is unlike the chiral ring data which we study in this chapter, where there is a nontrivial dependence on the exactly marginal couplings.

In this chapter we relate the chiral ring data, $G_{I\bar{J}}$, of arbitrary $\mathcal{N} = 2$ SCFTs admitting a Lagrangian description somewhere in the conformal manifold to a certain partition function of the SCFT on S^4. This partition function, in turn, can be computed exactly by supersymmetric localization. More precisely, one can determine the S^4 partition function to all orders in perturbation theory and in some, but not all, cases also the instanton corrections. We discuss this in detail in chapter 3.

From the chiral ring of $\mathcal{N} = 2$ SCFTs we can obtain all of the so-called extremal correlators

$$\left\langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)...\mathcal{O}_{I_n}(x_n)\overline{\mathcal{O}}_{\bar{J}}(y)\right\rangle \tag{3.22}$$

everywhere on the conformal manifold, where by the $\mathfrak{u}(1)_R$ selection rule

$$\Delta_{I_1} + \Delta_{I_2} + ... + \Delta_{I_n} = \Delta_{\bar{J}}\,. \tag{3.23}$$

These correlators are, in general, non-holomorphic functions of $\tau^i, \bar{\tau}^{\bar{i}}$. Since there is only one anti-chiral operator in (3.22), these correlators are, in some sense, the simplest non-holomorphic local observables in the theory.

Let us now demonstrate that the extremal correlators (3.22) can be obtained from the chiral ring data. Without loss of generality we can put the operator $\overline{\mathcal{O}}_{\bar{J}}$ at infinity by writing as usual $\overline{\mathcal{O}}_{\bar{J}}(\infty) \equiv \lim_{y\to\infty} y^{2\Delta_J}\overline{\mathcal{O}}_{\bar{J}}(y)$. The next step is to observe that $\left\langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)...\mathcal{O}_{I_n}(x_n)\overline{\mathcal{O}}_{\bar{J}}(\infty)\right\rangle$ is independent of the coordinates x_i. One proves this by differentiating the correlator with respect to the position of the k-th chiral primary and noting that

$$\frac{\partial}{\partial x_k^{\alpha\dot{\alpha}}}\mathcal{O}_{I_k}(x_k) \propto \epsilon_{ab}\{\overline{Q}^a_{\dot{\alpha}}, [Q^b_\alpha, \mathcal{O}_{I_k}]\}\,. \tag{3.24}$$

[14] We would like to thank Leonardo Rastelli for discussions about multiplet recombination.

By the supersymmetry Ward identity, we can let $\overline{Q}^a_{\dot\alpha}$ act on the rest of the operators. Using that $[\overline{Q}^a_{\dot\alpha}, \mathcal{O}_I] = 0$ and that $\overline{Q}^a_{\dot\alpha}$ acting on $\overline{\mathcal{O}}_{\bar J}(y)$ yields a correlator that decays as $y^{-2\Delta_J-1}$ completes the proof. Therefore, since $\langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)...\mathcal{O}_{I_n}(x_n)\overline{\mathcal{O}}_{\bar J}(\infty)\rangle$ is independent of the coordinates x_i we can bring all the chiral primaries on top of each other and repeatedly use the OPE (3.18) to reduce any extremal correlation function to a two-point function in the chiral ring. Then, if we know $G_{I\bar J}(\tau^i, \bar\tau^{\bar i})$ for all the $I, \bar J$, we are done.

In the special case of maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$), extremal correlators have played an important role in the context of the AdS/CFT correspondence. Indeed, it was conjectured in [56, 103, 119] that extremal correlators can be computed exactly just from their tree-level diagrams, which allowed a comparison with supergravity. See [6] for a field theory proof of these nonrenormalization theorems in $\mathcal{N} = 4$ using Ward identities.

We will see that in general $\mathcal{N} = 2$ theories there are both perturbative and non-perturbative corrections to extremal correlators.

3.1.3 Subtle Aspects of Conformal Field Theories on S^4

In this subsection our discussion pertains to general CFTs (i.e. not necessarily supersymmetric ones) in four dimensions. We can start from the CFT in flat space deformed by sources $\lambda^I(x)$ that couple to all the scalar primary operators $O_I(x)$

$$\int d^4x \sum_I \lambda^I(x) O_I(x) \ .$$

From the partition function

$$Z[\mathbb{R}^4](\lambda^I(x))$$

one can compute all the n-point functions of the scalar primary operators. For example, it follows trivially that the one-point functions of all the operators other than the unit operator vanish.

In order to define the theory on S^4, one needs to specify various additional contact terms. This is in spite of the fact that S^4 is conformally flat. The simplest example of the sort of subtleties that arise is the following: if there is an operator O_0 with $\Delta_0 \in 2\mathbb{N}$, then we can add to the action the local counterterm[15]

$$\alpha \int d^4x \sqrt{g}\, \lambda_0\, R^{\Delta_0/2}\, \mathbb{1} \ , \tag{3.25}$$

[15] From now on R will denote the Ricci scalar of the background metric and should not be confused with the $\mathfrak{u}(1)_R$ charge.

with R being the Ricci scalar (more generally, it could be a combination of Riemann tensors). Unlike separated-points correlation functions in flat space, this term depends on the scheme. As a result, the one-point function of O_0 on S^4 is scheme dependent $\langle O_0 \rangle \sim \alpha r^{-\Delta_0}$, with r being the radius of the sphere. $\alpha = 0$ is obviously a preferred scheme, but it is not guaranteed that a given definition of the theory (say, by some RG flow) corresponds to this scheme.

Importantly for our analysis later, we can interpret $\alpha \int d^4x \sqrt{g}\, \lambda_0 R^{\Delta_0/2}\, \mathbb{1}$ as a scheme-dependent operator mixing between O_0 and the unit operator $\mathbb{1}$. This mixing can arise only in curved space, such as S^4. More generally, in curved space, the source for an operator O_{Δ_0} can have scheme-dependent non-minimal couplings to lower-dimensional operators due to nontrivial background fields, such as the curvature of space. This is only possible if the operators' dimensions differ by an even integer. These give rise to scheme-dependent operator mixing with all the operators of lower dimension in jumps by two units

$$O_{\Delta_0} \to O_{\Delta_0} + \alpha_1 R\, O_{\Delta_0 - 2} + \alpha_2 R^2\, O_{\Delta_0 - 4} + \cdots + \alpha_{\Delta_0/2} R^{\Delta_0/2}\, \mathbb{1}\ . \tag{3.26}$$

If the CFT has exactly marginal couplings λ^i, then the coefficients α_k can depend on them. From the point of view of the CFT in $\mathbb{R}^4$, the terms in (3.26) induce contact terms between O_{Δ_0} and the energy-momentum tensor. These contact terms can be chosen at will according to the renormalization scheme. But once the theory is put on S^4, these contact terms translate to operator mixing.

The conclusion from this discussion is that even for primary operators in a CFT, the transition from $\mathbb{R}^4$ to S^4 is nontrivial. One has to handle the possible operator mixing that is induced by various contact terms.

3.2 The Chiral Ring in 4d $\mathcal{N} = 2$ SCFTs and S^4

In this section we explain how the chiral ring and the extremal correlators (3.1) of an $\mathcal{N} = 2$ SCFT can be computed everywhere on the conformal manifold. Near a weakly coupled point on the conformal manifold, the answer can be in principle expanded into a perturbative series in the exactly marginal couplings $\tau^i, \bar{\tau}^{\bar{i}}$ dressed by an infinite sequence of instanton corrections. The key ingredient in obtaining the exact chiral ring data is the relation we establish below with a partition function on S^4. The S^4 partition function is of a suitable deformation of the $\mathcal{N} = 2$ SCFT. For some theories, the partition function can be explicitly evaluated by supersymmetric localization using formulae already available in the literature.

3.2.1 Placing the Deformed Theory on S^4

We are interested in studying the Lagrangian of an $\mathcal{N}=2$ SCFT deformed by the top component of a chiral multiplet corresponding to an arbitrary chiral primary operator $\mathcal{O}$, which we denote by C. This is done by adding to the Lagrangian in $\mathbb{R}^4$ the following term[16]

$$-\frac{1}{32\pi^2}\tau_{\mathcal{O}}\int d^4\theta\ \mathcal{O}+\text{c.c.}=-\frac{1}{32\pi^2}\tau_{\mathcal{O}}\,C+\text{c.c.} \tag{3.27}$$

If $\Delta(\mathcal{O})\neq 2$, this deformation breaks the conformal symmetry as well as the $\mathfrak{u}(1)_R$ symmetry, while it preserves $\mathfrak{su}(2)_R$ and the $\mathcal{N}=2$ super-Poincaré symmetry. If $\Delta(\mathcal{O})=2$ then the full $\mathfrak{su}(2,2|2)$ superconformal symmetry is preserved.

We will show that the deformed SCFT can be placed on S^4 while preserving $\mathfrak{osp}(2|4)$, the supersymmetry algebra of the most general massive $\mathcal{N}=2$ theory on S^4. The $\mathfrak{so}(2)_R\subset\mathfrak{osp}(2|4)$ is the Cartan generator of $\mathfrak{su}(2)_R$, and $sp(4)$ is the isometry of S^4.

We now explicitly construct the deformed SCFT on S^4. Placing the theory on S^4 requires deforming the flat space expression (3.27) by specific $1/r$ and $1/r^2$ terms, where r is the radius of S^4, as in [65]. The deformed Lagrangian on S^4 can be derived by promoting the coupling $\tau_{\mathcal{O}}$ in (3.27) to a supersymmetric background chiral multiplet of Weyl weight $2-\Delta(\mathcal{O})$. The $\mathfrak{osp}(2|4)$ invariant Lagrangian on S^4 is constructed by deforming the SCFT with the modified top component[17] (see Appendix D.2)

$$\mathcal{C}(x)\equiv C(x)+2\frac{(\Delta(\mathcal{O})-2)(\Delta(\mathcal{O})-3)}{r^2}\mathcal{O}(x)-i\frac{(\Delta(\mathcal{O})-2)}{r}\tau_1^{ij}B_{ij}(x)\,, \tag{3.28}$$

where B_{ij} is a middle component of the chiral multiplet $\mathcal{O}$ (see Appendix D.2 for details of chiral multiplet components). Indeed, if we add to the action of the SCFT on S^4 the deformation $-\frac{\tau_{\mathcal{O}}}{32\pi^2}\int d^4x\sqrt{g}\,\mathcal{C}(x)+\text{c.c.}$, the $\mathfrak{osp}(2|4)$ supersymmetry on S^4 is preserved. In superspace formalism, the sphere deformation (3.28) is given by the following F-term

$$-\frac{1}{32\pi^2}\int d^4x\int d^4\theta\,\mathcal{E}\,\tau_{\mathcal{O}}\,\mathcal{O}\,, \tag{3.29}$$

where $\mathcal{E}$ is the $\mathcal{N}=2$ chiral density.

Note that for an exactly marginal deformation, which descends from a chiral primary with $\Delta(\mathcal{O})=2$, there are no $1/r$ and $1/r^2$ corrections in (3.28).

[16] We change the normalization of the deformation by a factor of $1/32$ with respect to [76,81] in order to make formulae below simpler. In this normalization, the coefficient multiplying K in equation (3.4) should be $1/(2^{12}\times 3)$.

[17] Here $\tau^{ij}_{1,2,3}$ are the charge conjugated Pauli matrices defined as $\tau_p^{ij}\equiv\{i\sigma_3,-\mathbb{1}_{2\times2},-i\sigma_1\}=:\tau^*_{pij}$.

3.2.2 Chiral Primary Correlators from the Deformed Partition Function

We denote the partition function on S^4 of the deformed $\mathcal{N}=2$ SCFT by

$$Z[S^4](\tau^i,\bar{\tau}^{\bar{i}};\tau^A,\bar{\tau}^{\bar{A}})\,. \tag{3.30}$$

τ^A are the couplings associated to chiral ring generators $\mathcal{O}_A$ with $\Delta\neq 2$. We recall that τ^i are the couplings associated to the chiral primary operators with $\Delta=2$, which are also chiral ring generators, from which the exactly marginal operators are constructed.

We can now study derivatives of the S^4 partition function with respect to the sources τ^I and $\bar{\tau}^{\bar{J}}$ where $\{\tau^I\}=\{\tau^i\}\cup\{\tau^A\}$. We consider first the normalized second derivative

$$\begin{aligned}&\frac{1}{Z[S^4](\tau^i,\bar{\tau}^{\bar{i}})}\partial_{\tau^I}\partial_{\bar{\tau}^{\bar{I}}}Z[S^4](\tau^i,\bar{\tau}^{\bar{i}};\tau^A,\bar{\tau}^{\bar{A}})\Big|_{\tau^A=\bar{\tau}^{\bar{A}}=0}\\&=\left(\frac{1}{32\pi^2}\right)^2\int d^4x\sqrt{g(x)}\int d^4y\sqrt{g(y)}\,\langle\mathcal{C}_I(x)\overline{\mathcal{C}}_{\bar{I}}(y)\rangle_{S^4}\,.\end{aligned} \tag{3.31}$$

This yields the integrated two-point function of the operator $\mathcal{C}_I$ and $\overline{\mathcal{C}}_{\bar{I}}$ in (3.28) on S^4. The integrated correlator is ultraviolet divergent, for example, due to the appearance of the unit operator in the OPE of $\mathcal{C}_I$ and $\overline{\mathcal{C}}_{\bar{I}}$, and must be regularized and renormalized.

If we were to ignore supersymmetry for a moment, and if the sum of the dimensions of $\mathcal{C}_I$ and $\overline{\mathcal{C}}_{\bar{I}}$ were an even integer, the integrated correlation function (3.31) would be ambiguous due to the local counterterm

$$\int d^4x\sqrt{g}\,\tau^I\,\bar{\tau}^{\bar{I}}\,\mathcal{F}(\tau^i,\bar{\tau}^{\bar{i}})R^{\left(\Delta(\mathcal{O}_I)+\Delta(\overline{\mathcal{O}}_{\bar{I}})\right)/2}\quad, \tag{3.32}$$

which shifts the result (3.31) by an arbitrary function $\mathcal{F}(\tau^i,\bar{\tau}^{\bar{i}})$.

Interestingly, in $\mathcal{N}=2$ supersymmetric theories there is a unique way to regularize the divergences as $x\to y$ in (3.31). In other words, there is a unique way to regularize the singularity $x\to y$ in a way consistent with $\mathcal{N}=2$ supersymmetry. There are two equivalent ways to understand this fact:

1. Using a supersymmetry Ward identity on S^4 one can prove, extending the analysis in §C, that (see Appendix D.3 for the extended proof):

$$\int d^4x\sqrt{g(x)}\int d^4y\sqrt{g(y)}\,\langle\mathcal{C}_I(x)\overline{\mathcal{C}}_{\bar{I}}(y)\rangle_{S^4}=(32\pi^2r^2)^2\langle\mathcal{O}_I(N)\overline{\mathcal{O}}_{\bar{I}}(S)\rangle_{S^4}\,. \tag{3.33}$$

Therefore, a supersymmetric Ward identity shows that the supersymmetrically renormalized integrated correlation function of $\mathcal{C}_I$ and $\overline{\mathcal{C}}_{\bar{I}}$ in (3.28) equals the two-point function of the associated chiral primary $\mathcal{O}_I$ at the North Pole of S^4 and of the anti-chiral primary $\overline{\mathcal{O}}_{\bar{I}}$ at the South Pole.

2. In a supersymmetric regularization, the counterterms (3.32) should be $\mathcal{N}=2$ supergravity invariants. This restricts the allowed counterterms. Since τ^I and $\bar{\tau}^{\bar{I}}$ are embedded in a background $\mathcal{N}=2$ chiral and anti-chiral multiplet respectively, the counterterms that can lead to ambiguities in (3.31) must be D-term counterterms. Therefore, potential ambiguities can at best arise from superspace integrals over all superspace ($\int d^4\theta\, d^4\bar{\theta}\,\cdot$). But all the D-terms vanish on supersymmetric backgrounds [22].[18] Therefore, the singularity $x \to y$ in (3.31) is regularized in a universal fashion.

In summary, the two-point function of two arbitrary operators in the chiral ring on S^4 can be obtained from the partition function of the deformed SCFT on S^4. The relation between S^4 and $\mathbb{R}^4$ correlation functions is not entirely straightforward, though. We will discuss this soon, after we review some properties of these four-sphere partition functions.

3.2.3 The Deformed Partition Function on S^4

In the previous section we showed that an $\mathcal{N}=2$ SCFT on S^4 can be deformed with operators that are descendants of operators in the chiral ring while preserving the $\mathfrak{osp}(2|4)$ symmetry of S^4. By adapting Pestun's localization computation of the partition function

[18] This result can be derived by expressing the D-term invariants as F-term invariants, constructed from a chiral integral over half of the superspace ($\int d^4\theta\,\cdot$) using the chiral projector operator $\bar{\Delta}$ (see [118] for details)

$$\int d^4x \int d^4\theta\, d^4\bar{\theta}\, E\,\cdot = \int d^4x \int d^4\theta\, \mathcal{E}\, \bar{\Delta}\,\cdot\,, \tag{3.34}$$

where E is the Berezinian and $\mathcal{E}$ the chiral density of $\mathcal{N}=2$ supergravity. Since all terms in $\bar{\Delta}$ for $\mathcal{N}=2$ supergravity are built out of the superspace derivatives $\overline{D}^a_{\dot{\alpha}}$ and D^a_α and supersymmetric configurations are annihilated by $\overline{D}^a_{\dot{\alpha}}$ and D^a_α, it follows that all D-terms vanish on supersymmetric backgrounds. Since S^4 is a supersymmetric background of a certain off-shell $\mathcal{N}=2$ Poincaré supergravity theory [81] and the coupling constants, $(\tau^I, \bar{\tau}^{\bar{I}})$, are supersymmetric backgrounds of a chiral multiplet with the appropriate Weyl weight, all D-term counterterms automatically vanish. This is to be contrasted with the chiral projector in e.g. 4d $\mathcal{N}=1$ old minimal supergravity, where $\bar{\Delta}=\bar{D}^2-8R$, and R is a chiral superfield whose bottom component is the auxiliary field of old minimal supergravity. However, the situation in new minimal $\mathcal{N}=1$ supergravity is rather similar to our present case [4]. We would like to thank Daniel Butter for helpful discussions.

of undeformed $\mathcal{N}=2$ theories [140], we can find the exact matrix integral representation for the partition function of the deformed SCFT on S^4.

We can localize the deformed partition function using the same supercharge $\mathcal{Q}$ in $\mathfrak{osp}(2|4)$ and $\mathcal{Q}$-exact deformation term used in [140]. This supercharge obeys

$$\mathcal{Q}^2 = J_3^L + R\,, \tag{3.35}$$

where J_3^L is the Cartan generator of the $\mathfrak{su}(2)_L \subset sp(4)$ selfdual rotations on S^4 and R is the Cartan generator of the $\mathfrak{su}(2)_R$ R-symmetry. This implies that the partition function localizes to the two fixed points of J_3^L on S^4, that define the North and South Poles of S^4. Near the poles, the action of the deformed $\mathcal{N}=2$ SCFT on S^4 approaches the action of the deformed $\mathcal{N}=2$ SCFT in the Ω-background [134].

The deformed partition function on S^4 therefore localizes to the following matrix integral[19]

$$Z[S^4](\tau^i,\bar\tau^{\bar i},\tau^A,\bar\tau^{\bar A}) = \int_{\mathfrak{t}} da\,\Delta(a)\,\left|Z_\Omega(a,\tau^i,\tau^A)\right|^2\,. \tag{3.36}$$

As above, τ^A refers to the couplings associated to the chiral ring generators with $\Delta\neq 2$. In Lagrangian theories, the τ^A correspond to the higher Casimirs of the gauge group while τ^i to the quadratic Casimirs. The matrix integral is over the Cartan subalgebra $\mathfrak{t}$ of the gauge group G of the SCFT and $\Delta(a)$ is the associated Vandermonde determinant. $Z_\Omega(a,\tau^i,\tau^A)$ is the partition function of the deformed SCFT in the Ω-background evaluated with equivariant rotation parameters $\varepsilon_1=\varepsilon_2=1/r$ and real equivariant parameters a for the action of G. From now on we set $r=1$. $Z_\Omega(a,\tau^i,\tau^A)$ can, in turn, be computed by supersymmetric localization, and takes the following form

$$Z_\Omega(a,\tau^i,\tau^A) = Z_{\Omega,\text{cl}}(a,\tau^i,\tau^A)\cdot Z_{\Omega,\text{loop}}(a)\cdot Z_{\Omega,\text{inst}}(a,\tau^i,\tau^A)\,. \tag{3.37}$$

The classical contribution for gauge group[20] $G=\mathrm{SU}(N)$ is

$$Z_{\Omega,\text{cl}}(a,\tau,\tau^A) = \exp\left[i\pi\tau\,\mathrm{Tr}a^2 + i\sum_{A=3}^{N}\pi^{A/2}\tau^A\mathrm{Tr}a^A\right]\,. \tag{3.38}$$

[19] $Z[S^4](\tau^i,\bar\tau^{\bar i},\tau^A,\bar\tau^{\bar A})$ should be thought of as a generating functional of correlators of chiral primary operators. We do not need to worry about its convergence properties at finite τ^A.

[20] It is trivial to extend this to any simple Lie group G. Then A takes values in the set of orders of the higher Casimirs of G. The formula easily extends when G is product of simple gauge group factors, each giving rise to an exactly marginal deformation and a set of higher Casimir couplings.

The one-loop determinant contribution is the same as in [140], as it arises from the $\mathcal{Q}$-exact deformation term

$$|Z_{\Omega,\mathrm{loop}}(a)|^2 = \frac{\prod_{\alpha>0} H^2(i\alpha\cdot a)}{\prod_{w\in\mathbf{r}} H(iw\cdot a)}\,, \tag{3.39}$$

where $H(x) = G(1+x)G(1-x)$ and $G(x)$ is the Barnes double-gamma-function, which obeys $G(1+x) = \Gamma(x)G(x)$, with $\Gamma(z)$ being Euler's gamma-function. The numerator is the contribution of the vectormultiplet, governed by a product over the positive roots of the Lie algebra of G.[21] The denominator is the hypermultiplet contribution. The product is over the weights of the representation $\mathbf{r}$ of $G\times G_F$, where G_F is the flavor symmetry acting on the hypermultiplet.[22]

$Z_{\Omega,\mathrm{inst}}(a,\tau^i,\tau^A)$ captures the contribution of point-like instantons to the path integral [134]. The fact that it depends on τ^A means that one cannot just evaluate the operator insertions on the saddle points of the undeformed SCFT. This is because the operators are inserted precisely where point-like instantons and anti-instantons are localized, thus changing the saddle points themselves. The instanton partition function is generally given by a series expansion over the instanton charge. Roughly speaking, the contribution at a given instanton charge is obtained by integrating a certain equivariant characteristic class of a vector bundle over the corresponding moduli space of instantons. Important subtleties arise because the moduli space of instantons has singularities, and the integrals must be properly defined. There is a canonical way of defining the integrals over instanton moduli space when the gauge group is $\mathrm{U}(N)$. In this case, singularities in the moduli space are resolved by turning on noncommutativity (see e.g. [134]). In general, it is an open problem to compute $Z_{\Omega,\mathrm{inst}}(a,\tau^i,\tau^A)$ for $\mathrm{SU}(N)$ with $N>2$. Solving this problem will have some applications for our study of extremal correlators, but one can make some significant mileage even before this problem is solved. In section 3 we study examples in which $Z_{\Omega,\mathrm{inst}}(a,\tau^i,\tau^A)$ is known as well as some examples where it is not known, but one can still study the perturbative series.

3.2.4 The Relation Between Correlators in $\mathbb{R}^4$ and S^4

As we have explained above, using the deformed partition function on S^4 (3.36) and the Ward identity (3.33), we can calculate, in particular, the two-point functions of arbitrary chiral primary operators on S^4

$$\langle \mathcal{O}_I(N)\overline{\mathcal{O}}_{\bar{J}}(S)\rangle_{S^4}\,. \tag{3.40}$$

[21] The Vandermonde determinant in terms of the roots is $\Delta(a) = \prod_{\alpha>0}(\alpha\cdot a)^2$.

[22] We set the equivariant parameters for G_F, i.e. the mass parameters, to zero.

In this section we explain how to obtain the two-point functions of chiral primary operators in flat space (3.19) from the explicit results of the correlation functions on S^4 .

As explained in subsection 1.3, in the dictionary between CFT sphere correlation functions and flat space correlation functions one expects operator mixing (3.26), induced by the background fields. In fact, in $\mathcal{N} = 2$ SCFTs we already know that such mixing must take place from the formula (3.4). This formula shows that the one-point function $\langle \mathcal{O}_i(N) \rangle_{S^4} = \frac{1}{Z[S^4]} \frac{\partial}{\partial \tau^i} Z[S^4]$, is non-vanishing. This is a special case of (3.26) since it can be interpreted as mixing of $\mathcal{O}_i$ with the identity operator $\mathbb{1}$. This mixing with the identity operator can be interpreted, in turn, as a conformal anomaly according to [82].

In complete generality, we should allow a chiral primary operator $\mathcal{O}_\Delta$ of dimension Δ to mix with lower dimensional chiral operators

$$\mathcal{O}_\Delta \longrightarrow \mathcal{O}_\Delta + \alpha_1(\tau^i, \bar{\tau}^{\bar{i}}) R\, \mathcal{O}_{\Delta-2} + \alpha_2(\tau^i, \bar{\tau}^{\bar{i}}) R^2\, \mathcal{O}_{\Delta-4} + \cdots \ , \tag{3.41}$$

and similarly for the anti-chiral operators. In (3.41) R^k stands schematically for some contraction of k Riemann tensors evaluated on the sphere. Note that the chiral operator $\mathcal{O}_\Delta$ can only mix with other chiral operators, and not anti-chiral or the Higgs branch operators $\mathcal{H}_I$ discussed in section 3.1.2. Indeed, while chiral operators are supersymmetric at the North pole of S^4, neither anti-chiral operators nor $\mathcal{H}_I$ are supersymmetric there. Anti-chiral operators are supersymmetric, instead, at the South pole of S^4, while the Higgs branch operators cannot be inserted anywhere on S^4 while preserving supersymmetry (just as operators in a long representation of the superconformal algebra). Since operator mixing is compatible with supersymmetry on S^4, chiral primary operators can only mix among themselves, and analogously for anti-chiral operators.

It is natural to conjecture that the mixing coefficient functions $\alpha_k(\tau^i, \bar{\tau}^{\bar{i}})$ are captured by some anomalies, in parallel with the origin of the mixing of $\mathcal{O}_i$ with the identity operator. Operator mixing of the type in (3.41) can only occur when the theory has operators with integer-spaced dimensions. We can then expect that the there would be various type-B "resonance" anomalies. See for example [10, 24, 55]. These anomalies generalize the Zamolodchikov anomaly studied in [82], which is responsible for the mixing of $\mathcal{O}_i$ with the identity operator. It would be very nice to understand this structure better.

Since the mixing functions $\alpha_k(\tau^i, \bar{\tau}^{\bar{i}})$ are expected to arise due to anomalies, they are expected to be universal. There is, however, a holomorphic ambiguity, which acts by

$$\alpha_k(\tau^i, \bar{\tau}^{\bar{i}}) \to \alpha_k(\tau^i, \bar{\tau}^{\bar{i}}) + \mathcal{F}_k(\tau^i) + \overline{\mathcal{F}}_k(\bar{\tau}^{\bar{i}})\,. \tag{3.42}$$

Of course, the holomorphic ambiguity is fixed when the renormalization scheme is fixed. These holomorphic ambiguities in operator mixing are due to $\mathcal{N} = 2$ supersymmetric counterterms. A special case of this holomorphic counterterm is responsible for the ambiguous mixing of $\mathcal{O}_i$ with the unit operator, which was already constructed in [81,82]. This counterterm is responsible for the Kähler ambiguity of the partition function of the SCFT on S^4 (3.4).

When mapping the S^4 correlation functions to the correlation functions on $\mathbb{R}^4$ we must deal with the operator mixing in (3.41). Let us first review how this is accomplished for the special case of chiral primaries of dimension 2, $\mathcal{O}_i$. We recall that their descendants are the exactly marginal deformations that generate the conformal manifold of the SCFT. On S^4, there is mixing of $\mathcal{O}_i$ with the unit operator, as follows from (3.4). In this special case, it is easy to disentangle the operator mixing: we simply subtract disconnected pieces in $\langle \mathcal{O}_i(N)\overline{\mathcal{O}}_{\bar{j}}(S)\rangle_{S^4}$ from the right hand side of (3.33). It is well known that this can be achieved by taking the logarithm of the sphere partition function (which indeed removes all the disconnected diagrams). After we have removed this mixing, we can straightforwardly relate the $\langle \mathcal{O}_i(N)\overline{\mathcal{O}}_{\bar{j}}(S)\rangle_{S^4}$ two-point functions with their flat space counterparts $\langle \mathcal{O}_i(0)\overline{\mathcal{O}}_{\bar{j}}(\infty)\rangle_{\mathbb{R}^4}$, from which the metric is extracted. Therefore, the mixed second derivatives of the $\ln Z[S^4]$ with respect to the moduli $\tau^i, \bar{\tau}^{\bar{i}}$ compute the Zamolodchikov metric on the conformal manifold. This is precisely the statement captured by (3.4).

In more generality, for higher-dimensional chiral primaries, there can be nontrivial mixing with all the chiral primary operators of lower dimension, and taking the logarithm of the sphere partition would not suffice to remove operator mixing. In this case, diagonalization of $\langle \mathcal{O}_I(N)\overline{\mathcal{O}}_{\bar{J}}(S)\rangle_{S^4}$ must be carried out, which can be implemented by a *Gram-Schmidt* procedure. This prescription is the appropriate generalization of the ideas leading to (3.4). As we will see, this approach to computing flat space correlation function successfully reproduces many perturbative results while providing many new results, and it satisfies nontrivial all-orders consistency checks. We now summarize the explicit algorithm to determine the chiral ring data of an $\mathcal{N}=2$ SCFT.

3.2.5 Summary of the Algorithm

We consider an $\mathcal{N}=2$ SCFT with exactly marginal couplings $\tau^i, \bar{\tau}^{\bar{i}}$. The chiral ring is finitely generated and we take the generators to be ϕ_α, $\alpha = 1, ..., \mathfrak{N}$, with $\mathfrak{N}$ the number of generators. $\mathfrak{N}$ is also the dimension of the Coulomb branch of the SCFT. We denote their dimensions by $\Delta(\phi_\alpha) = \Delta_\alpha$. Every element in the chiral ring can be uniquely represented as a linear combination of

$$\mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}} = \phi_1^{n_1}\phi_2^{n_2}\ldots\phi_{\mathfrak{N}}^{n_{\mathfrak{N}}} \ . \tag{3.43}$$

The Lagrangian of the SCFT is constructed from the ring generators with $\Delta = 2$. We now deform the SCFT using the chiral ring generators of $\Delta > 2$, which we denote by ϕ_A

$$S_{\text{SCFT}} \to S_{\text{SCFT}} - \frac{1}{32\pi^2} \int d^4x\, d^4\theta\, \mathcal{E} \sum_A \tau^A \phi_A + \text{c.c.} \tag{3.44}$$

This is appropriately supersymmetrized on S^4, as explained in subsection 2.1. The associated partition function (3.36) is denoted by

$$Z[S^4](\tau^i, \bar{\tau}^{\bar{i}}; \tau^A, \bar{\tau}^{\bar{A}}) \ . \tag{3.45}$$

Our goal is to compute the two-point functions in flat space

$$\langle \mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}}(0)\overline{\mathcal{O}}_{n'_1,\ldots,n'_{\mathfrak{N}}}(\infty)\rangle_{\mathbb{R}^4} \ .$$

These are possibly nonzero only if $\Delta \equiv \sum_{\alpha=1}^{\mathfrak{N}} n_\alpha \Delta_\alpha = \sum_{\alpha=1}^{\mathfrak{N}} n'_\alpha \Delta_\alpha$. Given $Z[S^4](\tau^i, \bar{\tau}^{\bar{i}}; \tau^A, \bar{\tau}^{\bar{A}})$, we must first disentangle the operator mixing of $\mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}}$ and $\mathcal{O}_{n'_1,\ldots,n'_{\mathfrak{N}}}$ on S^4 with the lower-dimensional chiral operators, as described in (3.41). In order to do this, we implement the following procedure:

1. List all chiral operators $\mathcal{O}_{m_1,\ldots,m_{\mathfrak{N}}}$ of dimension $\sum_{\alpha=1}^{\mathfrak{N}} n_\alpha \Delta_\alpha - 2$, $\sum_{\alpha=1}^{\mathfrak{N}} n_\alpha \Delta_\alpha - 4$ etc. We denote the number of operators up to dimension $\Delta - 2$ by $N_{\Delta-2}$.

2. Compute the $N_{\Delta-2} + 1$ dimensional matrix of two-point functions on the sphere

$$\langle \mathcal{O}_{m_1,\ldots,m_{\mathfrak{N}}}(N)\overline{\mathcal{O}}_{m'_1,\ldots,m'_{\mathfrak{N}}}(S)\rangle_{S^4} \equiv M_{m_1,\ldots,m_{\mathfrak{N}}|m'_1,\ldots,m'_{\mathfrak{N}}}$$

   for all the operators listed in the previous step and for the operator $\mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}}$ in question. This Hermitian matrix is generally nonzero in all its entries. Do the same for the operator $\overline{\mathcal{O}}_{n'_1,\ldots,n'_{\mathfrak{N}}}$.

3. From (3.45) we can extract the matrix $M_{m_1,\ldots,m_{\mathfrak{N}}|m'_1,\ldots,m'_{\mathfrak{N}}}$ by

$$M_{m_1,\ldots,m_{\mathfrak{N}}|m'_1,\ldots,m'_{\mathfrak{N}}} = \frac{1}{Z[S^4](\tau^i,\bar{\tau}^{\bar{i}})} \frac{\partial^{m_1}}{(\partial\tau^1)^{m_1}} \cdots \frac{\partial^{m_{\mathfrak{N}}}}{(\partial\tau^{\mathfrak{N}})^{m_{\mathfrak{N}}}} \frac{\partial^{m'_1}}{(\partial\bar{\tau}^1)^{m'_1}} \cdots \frac{\partial^{m'_{\mathfrak{N}}}}{(\partial\bar{\tau}^{\mathfrak{N}})^{m'_{\mathfrak{N}}}} Z[S^4]\Big|_{\tau^A=\bar{\tau}^{\bar{A}}=0} \tag{3.46}$$

4. The mixing of the operator $\mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}}$ on S^4 with lower-dimensional operators (3.41) is encoded in $N_{\Delta-2}$ coefficients $\alpha_k(\tau^i, \bar{\tau}^{\bar{i}})$. These can be determined uniquely by demanding that the two-point function of $\mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}}$ with each one of the $N_{\Delta-2}$ lower dimension operators vanishes. Do likewise for the operator $\overline{\mathcal{O}}_{n'_1,\ldots,n'_{\mathfrak{N}}}$.

5. This algorithm is equivalent to performing a Gram-Schmidt diagonalization procedure of the matrix $M_{m_1,\ldots,m_{\mathfrak{N}}|m'_1,\ldots,m'_{\mathfrak{N}}}$. After completing this procedure for $\mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}}$ and $\mathcal{O}_{n'_1,\ldots,n'_{\mathfrak{N}}}$, the two-point function of orthogonalized operators on S^4 are directly related to $\langle \mathcal{O}_{n_1,\ldots,n_{\mathfrak{N}}}(0)\overline{\mathcal{O}}_{n'_1,\ldots,n'_{\mathfrak{N}}}(\infty)\rangle_{\mathbb{R}^4}$.

Let us show that the formula (3.5) is a special case of the procedure outlined above. We are interested in the two-point functions of $\Delta = 2$ chiral operators in $\mathbb{R}^4$. Let us assume for notational simplicity that there is only one such $\Delta = 2$ operator. The matrix of two-point functions on the sphere is therefore a 2×2 matrix:

$$\frac{1}{Z[S^4]}\begin{pmatrix} Z[S^4] & \partial_\tau Z[S^4] \\ \partial_{\bar{\tau}} Z[S^4] & \partial_\tau \partial_{\bar{\tau}} Z[S^4] \end{pmatrix} . \tag{3.47}$$

We perform the Gram-Schmidt procedure and find the norm of the corresponding non-trivial orthogonal vector. This is given by the determinant of (3.47), namely,

$$\langle \mathcal{O}(0)\overline{\mathcal{O}}(\infty)\rangle_{\mathbb{R}^4} \sim \frac{1}{(Z[S^4])^2}\left(Z[S^4]\partial_\tau\partial_{\bar{\tau}} Z[S^4] - \partial_\tau Z[S^4]\partial_{\bar{\tau}} Z[S^4]\right) . \tag{3.48}$$

This combination coincides with $\partial_\tau\partial_{\bar{\tau}} \ln Z[S^4]$.

We now discuss several examples to further demonstrate the procedure and its various applications and consequences.

3.3 Examples

3.3.1 SU(2) Gauge Group

The first example we consider is $\mathcal{N} = 2$ SCFTs with gauge group SU(2). The discussion in this subsection applies both to superconformal SU(2) SQCD with four fundamental hypermultiplets and to $\mathcal{N} = 4$ SU(2) super-Yang-Mills.

The chiral ring in this case has one generator, $\phi_2 = -4\pi i \mathrm{Tr}\, \varphi^2$, where φ is the complex scalar in the vectormultiplet. Thus, the chiral ring operators are given by

$$\mathcal{O}_n = (\phi_2)^n \, , \quad n \in \mathbb{N} \, , \tag{3.49}$$

with $\mathcal{O}_0 \equiv \mathbb{1}$. The chiral ring OPE is

$$\mathcal{O}_n(x)\mathcal{O}_m(0) = \mathcal{O}_{n+m}(0) + \ldots \, . \tag{3.50}$$

Since in this case there is a single chiral primary with $\Delta = 2$, the conformal manifold is one-complex-dimensional. In gauge theory terms, the complex coordinate in the conformal manifold is given by the complexified gauge coupling $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$, where g is the Yang-Mills coupling and θ is the theta angle.

We now study the problem of computing all the flat space two-point functions (3.19)

$$G_{2n}(\tau, \bar{\tau}) = \left\langle \mathcal{O}_n(0)\overline{\mathcal{O}}_n(\infty)\right\rangle_{\mathbb{R}^4} . \tag{3.51}$$

This determines the chiral ring data and the extremal correlators (3.1). Obviously $G_0 = 1$, and it follows from (3.5) that

$$G_2 = 16\, \partial_\tau \partial_{\bar{\tau}} \ln Z[S^4] . \tag{3.52}$$

Alternatively, this formula can be derived from our Gram-Schmidt procedure as in (3.48).

We now follow the algorithm described in the previous section, and begin by studying the two-point functions of the operators (3.49) on S^4, $\left\langle \mathcal{O}_n(N)\overline{\mathcal{O}}_m(S)\right\rangle_{S^4}$. These two-point functions define an inner product on the chiral ring. As in (3.46), we express these two-point functions as derivatives of the sphere partition function[23]

$$\langle \mathcal{O}_n(N)\overline{\mathcal{O}}_m(S)\rangle_{S^4} = \frac{1}{Z[S^4]}\partial_\tau^n \partial_{\bar{\tau}}^m Z[S^4] . \tag{3.55}$$

The basis of operators $\{\mathcal{O}_n\}$ is not orthogonal with respect to this inner product. We diagonalize the mixing by carrying out the Gram-Schmidt construction in order to find a basis $\{\mathcal{O}_n\} \to \{\mathcal{O}'_n\}$, such that the new operators, given by

$$\mathcal{O}'_n = \mathcal{O}_n - \sum_{m=0}^{n-1} \frac{\langle \mathcal{O}_n(N)\overline{\mathcal{O}}'_m(S)\rangle}{\langle \mathcal{O}'_m(N)\overline{\mathcal{O}}'_m(S)\rangle}\mathcal{O}'_m \tag{3.56}$$

[23] For SU(2) SQCD with 4 hypermultiplets in the fundamental representation one finds (see subsection 3.2.3)

$$Z[S^4](\tau, \bar{\tau}) = \int_{-\infty}^{\infty} da\, e^{-4\pi\, \mathrm{Im}\tau\, a^2}(2a)^2 \frac{H(2ia)H(-2ia)}{[H(ia)H(-ia)]^4}|Z_{\Omega,\mathrm{inst}}(ia, \tau)|^2 , \tag{3.53}$$

and $Z_{\Omega,\mathrm{inst}}$ is Nekrasov's instanton partition function on the Ω-background [134]. By expanding the integrand in powers of g^2 we can compute $Z[S^4]$ to any order in perturbation theory, and we can also include instantons. In SU(2) gauge theory with an adjoint hypermultiplet, i.e. $\mathcal{N} = 4$ SU(2) super-Yang-Mills, the result is much simpler

$$Z[S^4](\tau, \bar{\tau}) = \int_{-\infty}^{\infty} da\, e^{-4\pi\, \mathrm{Im}\tau\, a^2}(2a)^2 . \tag{3.54}$$

are mutually orthogonal. The two-point functions on S^4 in this new basis can now be identified with the two-point functions in flat space $\big\langle \mathcal{O}_n(0)\overline{\mathcal{O}}_n(\infty)\big\rangle_{\mathbb{R}^4}$. Therefore,

$$\langle \mathcal{O}'_n(N)\overline{\mathcal{O}}'_m(S)\rangle_{S^4} \equiv \frac{1}{16^n} G_{2n}(\tau,\bar{\tau})\delta_{nm} \, . \tag{3.57}$$

The Gram-Schmidt diagonalization procedure (3.56) is recursive, and can be solved explicitly for arbitrary n. By virtue of (3.55), the orthogonal vectors can be expressed in terms of derivatives of $Z[S^4]$. Therefore, we can express the chiral ring data $G_{2n}(\tau,\bar{\tau})$ in terms of various derivatives of the S^4 partition function $Z[S^4]$. This suggests, in turn, that the various metrics $G_{2n}(\tau,\bar{\tau})$ can be related by differential equations. We will now prove this.

For the purpose of exhibiting the system of differential equations acting on the chiral ring data it is useful to organize the two-point functions on S^4 in (3.55) in an infinite dimensional matrix

$$M_{m,n} = \langle \mathcal{O}_m(N)\overline{\mathcal{O}}_n(S)\rangle_{S^4} \, , \qquad m,n = 0,1,\cdots \, . \tag{3.58}$$

Let us denote by $M_{(n)}$ the upper-left $(n+1)\times(n+1)$ submatrix of M, and

$$D_n \equiv \det M_{(n)} \, . \tag{3.59}$$

This submatrix captures the mixing of the operator $\mathcal{O}_n$ with all operators of smaller dimension, i.e. $\Delta < 2n$. Because the matrices that appear in the Gram-Schmidt procedure are triangular (operators can only mix with lower-dimensional operators), one can obtain $G_{2n}(\tau,\bar{\tau})$ in (3.57) as a ratio of determinants

$$G_{2n}(\tau,\bar{\tau}) = 16^n \frac{D_n}{D_{n-1}} \, . \tag{3.60}$$

In addition, we can prove that the determinant D_n satisfies the differential equation[24]

$$\partial_\tau \partial_{\bar{\tau}} \ln D_n = \frac{D_{n+1}D_{n-1}}{D_n^2} - (n+1)D_1 \, . \tag{3.65}$$

[24]To prove this, we first write the derivative of $\ln D_n$ in terms of derivatives of $M_{(n)}$ as follows:

$$\partial_\tau \partial_{\bar{\tau}} \ln D_n = \mathrm{Tr}\left(M_{(n)}^{-1}\partial_\tau\partial_{\bar{\tau}} M_{(n)} - M_{(n)}^{-1}\partial_\tau M_{(n)} M_{(n)}^{-1}\partial_{\bar{\tau}} M_{(n)}\right) . \tag{3.61}$$

Using (3.58) and (3.55), the derivatives of the components of M can be written as:

$$\partial_\tau M_{i,j} = M_{i+1,j} - M_{1,0}M_{i,j}, \tag{3.62a}$$
$$\partial_{\bar{\tau}} M_{i,j} = M_{i,j+1} - M_{0,1}M_{i,j}, \tag{3.62b}$$
$$\partial_\tau\partial_{\bar{\tau}} M_{i,j} = M_{i+1,j+1} - M_{1,0}M_{i,j+1} - M_{0,1}M_{i+1,j} + (2M_{1,0}M_{0,1} - M_{1,1})M_{i,j}. \tag{3.62c}$$

Combining (3.60) and (3.65) we find an equation directly for the two-point functions $G_{2n}(\tau,\bar{\tau})$

$$16\,\partial_\tau\partial_{\bar{\tau}}\ln G_{2n} = \frac{G_{2n+2}}{G_{2n}} - \frac{G_{2n}}{G_{2n-2}} - G_2\,, \qquad n = 1, 2\ldots\,. \tag{3.66}$$

Recall that $\{G_{2n}(\tau,\bar{\tau})\}$ obey the following boundary conditions: $G_0 = 1$ and $G_2 = 16\,\partial_\tau\partial_{\bar{\tau}}\ln Z[S^4]$. By defining $G_{2n} \equiv 16^n\, e^{q_n - \ln Z[S^4]}$, the differential equation (3.66) can be cast into the form of the semi-infinite Toda chain equation

$$\begin{aligned} &\partial_\tau\partial_{\bar{\tau}} q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}\,, \qquad n = 1, 2, \cdots \\ &\partial_\tau\partial_{\bar{\tau}} q_0 = e^{q_1 - q_0}\,. \end{aligned} \tag{3.67}$$

Therefore, the chiral ring data is governed by a system of coupled oscillators with a prescribed dependence on $\tau, \bar{\tau}$ for the leftmost oscillator, that is $q_0 = \ln Z[S^4]$. In this particle picture, we can think of $\mathrm{Im}\,\tau$ as physical time. Since $\mathrm{Re}\,\tau$ is compact, we can Fourier decompose in it and imagine that the lattice has two spatial dimensions.

We see that the Toda chain (3.67) arises essentially from the Gram-Schmidt procedure on S^4, with the ratio of some determinants (3.58)–(3.60) playing a central role. This is in fact reminiscent of the way solutions to the semi-infinite Toda system are actually constructed in the integrability literature [99].

In [7], the tt^* equations of four-dimensional $\mathcal{N} = 2$ SCFTs in the holomorphic gauge were exploited to arrive at the same equations (3.66) (the tt^* equations do not provide the boundary condition (3.52)). This agreement with the tt^* equations is therefore a nontrivial consistency check of our procedure.

In Appendix D.1 we show that the tt^* equations of an arbitrary four-dimensional $\mathcal{N} = 2$ SCFT are integrable. They can be written as the flatness condition of a one-parameter family of connections like the tt^* equations of a two-dimensional $(2,2)$ QFTs [28]. The tt^* equations are governed by a Hitchin integrable system.

Using these relations and noting that $D_1 = M_{1,1} - M_{1,0}M_{0,1}$, we arrive at

$$\partial_\tau\partial_{\bar{\tau}}\ln D_n = \left(M_{(n)}\right)^{-1}_{n,n}\left(M_{n+1,n+1} - \sum_{i,j=0}^{n} M_{n+1,i}\left(M_{(n)}\right)^{-1}_{i,j} M_{j,n+1}\right) - (n+1)D_1. \tag{3.63}$$

Using Schur's complement lemma

$$M_{n+1,n+1} - \sum_{i,j=0}^{n} M_{n+1,i}\left(M_{(n)}\right)^{-1}_{i,j} M_{j,n+1} = \frac{D_{n+1}}{D_n}, \quad \text{and} \quad \left(M_{(n)}\right)^{-1}_{n,n} = \frac{D_{n-1}}{D_n}\,, \tag{3.64}$$

we obtain (3.65).

SU(2) with an Adjoint Hypermultiplet

It is important to note that due to the simple form of the S^4 partition function given in (3.54) for $\mathcal{N} = 2$ SU(2) with an adjoint hypermultiplet, that is $\mathcal{N} = 4$ SU(2) super-Yang-Mills, the partition function evaluates to

$$Z_{S^4}[\tau, \bar{\tau}] = \frac{1}{4\pi(\mathrm{Im}\tau)^{3/2}} \,. \tag{3.68}$$

All the $G_{2n}(\tau, \bar{\tau})$ coincide with their tree-level expressions

$$G_{2n}(\tau, \bar{\tau}) = G_{2n}^{\mathrm{tree}}(\tau, \bar{\tau}) = \frac{(2n+1)!}{(\mathrm{Im}\tau)^{2n}} = (2n+1)! \left(\frac{g^2}{4\pi}\right)^{2n} \,. \tag{3.69}$$

One can easily verify that indeed these expressions obey the Toda equations (3.66).

SU(2) SQCD with Four Fundamental Hypermultiplets

In the case of SU(2) SQCD with four fundamental hypermultiplets, the S^4 partition function given in (3.53) has quite a non-trivial dependence on $\mathrm{Im}\tau = 4\pi/g^2$, and the $G_{2n}(\tau, \bar{\tau})$ receive both perturbative and non-perturbative corrections. To reproduce this expansion, one can start with (3.53) and expand the instanton partition function

$$Z_{\Omega,\mathrm{inst}}(ia, \tau) = 1 + \frac{1}{2} e^{2\pi i \tau}(a^2 - 3) + \cdots \,, \tag{3.70}$$

where the first term corresponds to the zero-instanton sector, the second term to the 1-instanton sector, etc., as well as expand the functions H in (3.53) at small a. Order by order in these expansions, the integrals in a are elementary. The first few terms are

$$\begin{aligned} Z_{S^4}[\tau, \bar{\tau}] = {} & \frac{1}{4\pi(\mathrm{Im}\tau)^{3/2}} \left[1 - \frac{45\zeta(3)}{16\pi^2(\mathrm{Im}\tau)^2} + \frac{525\zeta(5)}{64\pi^3(\mathrm{Im}\tau)^3} + \cdots\right] \\ & + \frac{e^{2\pi i\tau} + e^{-2\pi i\bar{\tau}}}{8\pi(\mathrm{Im}\tau)^{3/2}} \left[-3 + \frac{3}{8\pi\mathrm{Im}\tau} + \frac{135\zeta(3)}{16\pi^2(\mathrm{Im}\tau)^2} + \cdots\right] + \cdots \,, \end{aligned} \tag{3.71}$$

where the first line contains the perturbative contributions and the second line contains the non-perturbative ones starting with the 1-instanton result. As we have explained, this expression can be used to compute all the G_{2n} in SU(2) SQCD.

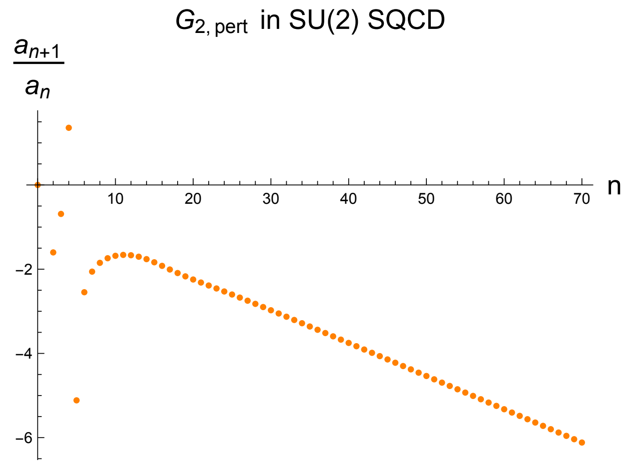


Figure 3.1: The ratio of consecutive coefficients appearing in the perturbative expansion (3.73) of G_2 in SU(2) SQCD plotted in terms of the loop order n.

For example, in a perturbative expansion around weak coupling, G_2 is

$$G_2(\tau,\bar{\tau})_{\text{pert}} = \frac{6}{(\text{Im}\tau)^2} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(\text{Im}\tau)^4} + \frac{1575\zeta(5)}{4\pi^3}\frac{1}{(\text{Im}\tau)^5} + \mathcal{O}\left(\frac{1}{(\text{Im}\tau)^6}\right) . \tag{3.72}$$

The first two terms in this result were checked against an explicit, two-loop computation in [9]. If we denote

$$G_2(\tau,\bar{\tau})_{\text{pert}} = \frac{6}{(\text{Im}\tau)^2}\sum_{n=0}^{\infty}\frac{a_n}{(\text{Im}\tau)^n} , \tag{3.73}$$

it is possible to calculate the coefficients a_n up to fairly high order—see Figure 3.1. From this figure it is clear that the ratio a_{n+1}/a_n asymptotically grows linearly with n with a negative coefficient. In [2,144] such behavior was established for the expansion coefficients of the S^4 partition function $Z_{S^4}[\tau,\bar{\tau}]$. Moreover, it was shown that the perturbative contribution to $Z_{S^4}[\tau,\bar{\tau}]$ is Borel summable. Since G_2 can be obtained by taking two derivatives of $\ln Z_{S^4}[\tau,\bar{\tau}]$, it follows that $G_{2,\text{pert}}$ is also Borel summable. The one-instanton correction to the perturbative result is non-trivial; it is given by

$$G_2(\tau,\bar{\tau})_{\text{1-inst}} = \cos\theta\, e^{-\frac{8\pi^2}{g^2}}\left(\frac{6}{(\text{Im}\tau)^2} + \frac{3}{\pi}\frac{1}{(\text{Im}\tau)^3} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(\text{Im}\tau)^4} + \mathcal{O}\left(\frac{1}{(\text{Im}\tau)^5}\right)\right) . \tag{3.74}$$

The perturbative expression (3.73) can be used to check the conjecture of [111], originally formulated for the case of QCD. The conjecture is that a Padé approximation of

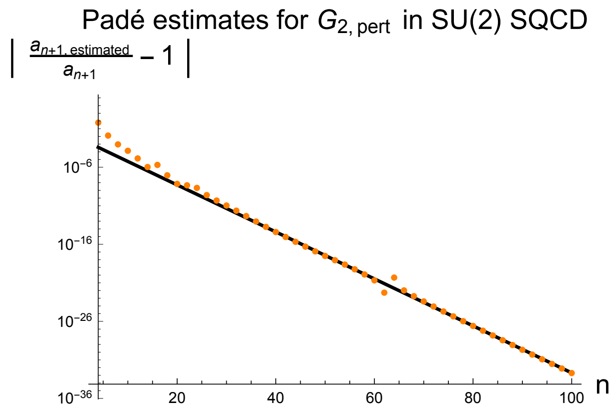


Figure 3.2: The relative difference between the Padé estimate of the coefficient a_{n+1} and its actual value in the case of G_2 in SU(2) SQCD. The black line is a linear fit for $n \geq 40$.

order $(n/2, n/2)$ obtained from the n-loop result (with n even) can be used to estimate the value of a_{n+1} with exponentially small error. If we denote the estimate of a_{n+1} using the symmetric Padé by $a_{n+1,\mathrm{estimated}}$, the conjecture is that

$$\left|\frac{a_{n+1,\mathrm{estimated}}}{a_{n+1}} - 1\right| < Ce^{-\sigma n} \tag{3.75}$$

for some $\sigma > 0$ and $C > 0$. As we show in Figure 3.2, the relation (3.75) is indeed true, with an exponent $\sigma \approx 0.7$ that can be determined from the slope of the logarithmic plot.[25]

3.3.2 SU(N) Gauge Group

SCFTs based on a single SU(N) gauge group have one exactly marginal coupling constant, τ, as in the SU(2) case. The chiral ring is generated by the $N-1$ operators $\phi_k = i^{k+1}(4\pi)^{k/2}\mathrm{Tr}\,(\varphi^k)$, $k = 2, \ldots, N$. The dimension-two operator ϕ_2, as usual, corresponds to the exactly marginal deformation. We can use the following basis in the space of chiral operators

$$\mathcal{O}_{\{n_i\}} = \prod_{k=2}^{N} (\phi_k)^{n_k} \ . \tag{3.76}$$

[25]If the perturbative series was simply $a_n = (-1)^n n!$, then (3.75) would have been satisfied with $\sigma = \ln(2)$. Even though the situation here is more complicated and there are in fact infinitely many poles on the negative axis of the Borel plane [2, 144], we still seem to find $\sigma \sim \ln(2)$. It would be interesting to understand this better.

In order to implement the algorithm of subsection 2.4. in these theories, we first deform the SCFT action on S^4 by

$$S_{\mathrm{SCFT}} \to S_{\mathrm{SCFT}} - \frac{1}{32\pi^2}\int d^4x\, d^4\theta\, \mathcal{E} \sum_{\mathfrak{a}=3}^{N} \tau^A \phi_A \ ,$$

and compute the S^4 partition function $Z[S^4](\tau,\bar\tau;\tau^A,\bar\tau^A)$ of the deformed SCFT.

We are interested in the two-point functions in flat space $\langle \mathcal{O}_{\{n_i\}}(0)\overline{\mathcal{O}}_{\{n_i'\}}(\infty)\rangle_{\mathbb{R}^4}$. These are potentially non-vanishing if $\sum_{k=2}^N kn_k = \sum_{k=2}^N kn_k'$. Note that unlike in the case of SU(2), for higher rank gauge group, there can be more than one operator of a given dimension and hence mixing already on $\mathbb{R}^4$, for example between $(\mathrm{Tr}\,(\varphi^3))^2$ and $(\mathrm{Tr}\,(\varphi^2))^3$.

As before, we begin by studying the matrix of two-point functions on S^4 $\mathcal{M}_{\{n_i\},\{n_i'\}} = \langle \mathcal{O}_{\{n_i\}}(N)\overline{\mathcal{O}}_{\{n_i'\}}(S)\rangle_{S^4}$. On S^4, this matrix is in general nonzero for all $\{n_i\},\{n_i'\}$. We could compute the correlators from the S^4 partition function $Z[S^4](\tau,\bar\tau;\tau^A,\bar\tau^A)$ by taking derivatives

$$\mathcal{M}_{\{n_i\},\{n_i'\}} = \frac{1}{Z[S^4]}\frac{\partial^{n_2}}{(\partial\tau)^{n_2}}\frac{\partial^{n_3}}{(\partial\tau^3)^{n_3}}\cdots\frac{\partial^{n_N}}{(\partial\tau^N)^{n_N}}\frac{\partial^{n_2'}}{(\partial\bar\tau)^{n_2'}}\frac{\partial^{n_3'}}{(\partial\bar\tau^3)^{n_3'}}\cdots\frac{\partial^{n_N'}}{(\partial\bar\tau^N)^{n_N'}}Z[S^4](\tau,\bar\tau;\tau^A,\bar\tau^A)\bigg|_{\tau^A=\bar\tau^A=0} . \tag{3.77}$$

Then, we perform the Gram-Schmidt procedure to extract the two-point functions in flat space $\langle \mathcal{O}_{\{n_i\}}(0)\overline{\mathcal{O}}_{\{n_i'\}}(\infty)\rangle_{\mathbb{R}^4}$.

We can explicitly determine the chiral ring data to all orders in perturbation theory. Unfortunately, the expression for $Z_{\Omega,\mathrm{inst}}(a,\tau^i,\tau^A)$ that appears in (3.36) and (3.37) in the localization computation of $Z[S^4](\tau,\bar\tau;\tau^A,\bar\tau^A)$ in SU(N) SQCD with $2N$ fundamental hypermultiplets is not yet available in the literature. The dependence of the instanton partition function on the higher Casimir couplings, τ^A ($A=3,..,N$) is unknown. (While it is available for U(N) theories [70, 126, 132, 133] it is an open problem to compute them for SU(N).) Ignoring the instantons, one can nevertheless use (3.77) to derive many interesting results independent of the specific expression for $Z[S^4](\tau,\bar\tau;\tau^A,\bar\tau^A)$. One application is the derivation of the coupled tt^* equations which are obtained from (3.77) in Appendix D.4. The general tt^* equations were first derived in [139]. In addition, we can say quite a bit about the structure of the solution to the tt^* equations in the case of SQCD.

SU(N) with an Adjoint Hypermultiplet

This corresponds to the maximally supersymmetric $\mathcal{N}=4$ super-Yang-Mills theory. In this theory, the four-sphere partition receives no instanton corrections [136]. The deformed partition function is given by a quadratic matrix model deformed by the higher Casimirs evaluated on the localization locus (our discussion here can be generalized to any gauge group):

$$Z[S^4](\tau,\bar{\tau};\tau^A,\bar{\tau}^A) = \int d^{N-1}a\, \Delta(a) |e^{i\pi\tau\, \mathrm{Tr}\,(a^2) + i\sum_{A=3}^{N} \pi^{A/2}\tau^A \mathrm{Tr}\,(a^A)}|^2 \ . \tag{3.78}$$

We show that the relatively simple form (3.78) leads to two main consequences which we now derive

- The flat-space two-point functions $\langle \mathcal{O}_{\{n_i\}}(0)\overline{\mathcal{O}}_{\{n'_i\}}(\infty)\rangle_{\mathbb{R}^4}$ are saturated by tree diagrams. This is a trivial consequence of the form of (3.78). This property of $\mathcal{N}=4$ is further discussed in [6, 56, 119].
- The chiral ring data can be organized in terms of infinitely many decoupled Toda chains.

Both of these conclusions are special to $\mathcal{N}=4$ super-Yang-Mills. As we will see below, the second conclusion is actually also valid in other theories up to two loops but not to higher orders.

In order to establish the second point we need to make some simple observations. The first observation is that multiplying two orthogonal operators that do not explicitly depend on τ by powers of ϕ_2 does not change the fact that they are orthogonal:

$$\left\langle \mathcal{O}_I(N)\overline{\mathcal{O}}_J(S)\right\rangle_{S^4} = 0 \quad \Rightarrow \quad \left\langle \phi_2^n\mathcal{O}_I(N)\overline{\phi_2^m\mathcal{O}_J}(S)\right\rangle_{S^4} = 0 \ . \tag{3.79}$$

This follows from (3.78). Indeed, if two operators are orthogonal and if they are independent of τ, then by taking derivatives with respect to $\tau, \bar{\tau}$ one finds (3.79).

Thus, if we choose a basis of the form

$$\mathcal{O}_n^{(m)} = \phi_2^n \mathcal{O}_0^{(m)} \ , \tag{3.80}$$

with the operators $\mathcal{O}_0^{(m)}$ constructed such that they are orthogonal to each other

$$\left\langle \mathcal{O}_0^{(m)}(N)\overline{\mathcal{O}_0^{(m')}}(S)\right\rangle_{S^4} = 0 \ , \quad \text{for } m \neq m' \ , \tag{3.81}$$

and such that $\mathcal{O}_0^{(m)}$ do not explicitly depend on τ, then in the basis (3.80) our system splits into orthogonal sectors:

$$\left\langle \mathcal{O}_n^{(m)}(N)\overline{\mathcal{O}_k^{(m')}}(S)\right\rangle_{S^4} = 0 \ , \quad \text{for } m \neq m' \ . \tag{3.82}$$

In [8], the operators $\mathcal{O}_0^{(m)}$ are called C_2 primaries because they have, in a sense, the minimal possible number of ϕ_2 factors.

It is easy to construct the basis (3.80) explicitly and verify that the operators in it are independent of τ. This is done as follows. We consider the set of operators of the form $\prod_{k=3}^{N}(\phi_k)^{n_k}$ (i.e. operators from the basis (3.80) with $n_2 = 0$), and choose an ordering on this set such that the operators are labeled as B_m, with $\Delta_m \leq \Delta_{m+1}$ (thus, $B_0 = \mathbb{1}$, $B_1 = \phi_3$,...). We can now define $\mathcal{O}_0^{(m)}$ by an inductive process. For $m = 0$, we choose $\mathcal{O}_0^{(0)} = B_0 = \mathbb{1}$. Assuming that we have defined $\mathcal{O}_0^{(m')}$ with m' ranging from 0 up to $m-1$, we define $\mathcal{O}_0^{(m)}$ to be a linear combination of B_m and operators of the form $\mathcal{O}_{n_{m'}}^{(m')} = \phi_2^{n_{m'}} O_0^{(m')}$, where $m' < m$ and $n_{m'} = \frac{\Delta_m - \Delta'_m}{2}$ is an integer. Note that B_m and $\mathcal{O}_{n_{m'}}^{(m')}$ have the same dimension Δ_m. This fact will be important to us soon. The coefficients in this linear combination are chosen such that $\left\langle \mathcal{O}_0^{(m)}(N)\overline{\mathcal{O}_0^{(m')}}(S)\right\rangle_{S^4} = 0$ will be obeyed for all $m' < m$, that is,

$$\mathcal{O}_0^{(m)} = B_m - \sum_{m'} \frac{\left\langle B_m(N)\overline{\mathcal{O}}_0^{(m')}(S)\right\rangle_{S^4}}{\left\langle \mathcal{O}_{n_{m'}}^{(m')}(N)\overline{\mathcal{O}}_0^{(m')}(S)\right\rangle_{S^4}} \ \mathcal{O}_{n_{m'}}^{(m')} \ , \tag{3.83}$$

where the sum above is only on m' such that $n_{m'} = \frac{\Delta_m - \Delta'_m}{2} \in \mathbb{N}$. This construction makes it obvious that the $\mathcal{O}_0^{(m)}$ are τ-independent, as required. Indeed, since the coefficients in (3.83) have the same dimension in the numerator and the denominator, and since these correlators in $\mathcal{N} = 4$ super-Yang-Mills are tree-level exact, the factors of τ cancel. In summary, we have constructed a basis of operators in the chiral ring that decouple into mutually orthogonal semi-infinite towers whose bottom operators are explicitly τ-independent.

For example, the first towers in SU(N) for $N \geq 4$ are

$$O_n^{(0)} = \phi_2^n \ , \quad O_n^{(1)} = \phi_2^n \phi_3 \ , \quad O_n^{(2)} = \phi_2^n \left(\phi_4 - \frac{\langle \phi_4(N) \rangle_{S^4}}{\langle \phi_2^2(N) \rangle_{S^4}} \phi_2^2 \right) \ \ldots . \tag{3.84}$$

By construction, this new basis satisfies (3.82) and as a result one can perform the Gram-Schmidt procedure of subsection 3.2.5 in each tower separately. This leads to a tremendous simplification. If we denote the matrix elements in this basis by $M_{i,j}^{(m)} = \left\langle O_i^{(m)}(N)\overline{O_j^{(m)}}(S)\right\rangle_{S^4}$, exactly the same derivation as the one presented in the case of SCFTs based on SU(2) proves that the chiral data, encoded in $G_{2n}^{(m)} \equiv \left\langle O_n^{(m)}(0)\overline{O_n^{(m)}}(\infty)\right\rangle_{\mathbb{R}^4}$, satisfies

$$
\begin{aligned}
16\,\partial_\tau\partial_{\bar\tau}\ln G_{2n}^{(m)} &= \frac{G_{2n+2}^{(m)}}{G_{2n}^{(m)}} - \frac{G_{2n}^{(m)}}{G_{2n-2}^{(m)}} - G_2\ , \\
16\,\partial_\tau\partial_{\bar\tau}\ln G_{0}^{(m)} &= \frac{G_{2}^{(m)}}{G_{0}^{(m)}} - G_2\ ,
\end{aligned}
\tag{3.85}
$$

and $G_2 = 16\,\partial_\tau\partial_{\bar\tau}\ln Z[S^4](\tau,\bar\tau;0,0)$.

Equation (3.85) describes decoupled semi-infinite Toda chains, in agreement with [8]. One can explicitly solve for the $G_{2n}^{(m)}$ using the fact that

$$
G_2 = \frac{2(N^2-1)}{(\mathrm{Im}\tau)^2}\ . \tag{3.86}
$$

One finds

$$
G_{2n}^{(m)}(\tau,\bar\tau) = 4^n \frac{n!\,\tilde G_0^{(m)}}{(\mathrm{Im}\tau)^{\Delta_m+2n}}\left(\frac{N^2-1}{2}+\Delta_m\right)_n\ , \tag{3.87}
$$

where $(x)_n$ is the Pochhammer symbol

$$
(x)_n = x(x+1)...(x+n-1) \tag{3.88}
$$

and $\tilde G_0^{(m)}$ encodes the normalization of the operator at the bottom of the m-th tower.

As we have already emphasized, this structure of decoupled Toda chains obviously exists at tree-level in $\mathcal{N}=2$ SU(N) SQCD as well (actually, in any SCFT at tree level). As we will show in the next subsection, it persists up to two-loops in SU(N) SQCD.

Decoupled Toda Chains at Two-Loops in SQCD

We now show that the decoupled Toda chain structure (3.85) remains in SU(N) SQCD up to two-loops. That is, the chiral ring data can be organized in terms of decoupled semi-infinite Toda chains up to that order in perturbation theory.

The operators $\mathcal{O}_0^{(m)}$ constructed in (3.83) are orthogonal at tree-level, but they are not guaranteed to stay orthogonal when higher-order corrections are included. If the operators were to stay orthogonal for all values of the coupling constant, then equation (3.85) would hold to all orders. Let us explain why the first non-trivial two-loop correction actually does not ruin the orthogonality that was achieved at tree-level.

The first non-trivial perturbative correction can be obtained by expanding the matrix integral representation (see section (3.2.3))[26] of the deformed SCFT partition function on S^4

$$\begin{aligned}&\frac{1}{Z[S^4]}\int d^{N-1}a\,\Delta(a)F(a)e^{-2\pi\,\mathrm{Im}\tau\,\mathrm{Tr}\,a^2}\left(1-3\zeta(3)(\mathrm{Tr}\,a^2)^2\right)=\\&\frac{1}{Z[S^4]}\left(1-3\zeta(3)\frac{\partial^2}{\partial(2\pi\,\mathrm{Im}\tau)^2}\right)\int d^{N-1}a\,\Delta(a)F(a)e^{-2\pi\,\mathrm{Im}\tau\,\mathrm{Tr}\,a^2}\ ,\end{aligned}\tag{3.90}$$

where $F(a)$ denotes some insertion in the localization formula.

The fact that the first non-trivial correction is obtained from the tree-level result by differentiating with respect to the coupling constant of the theory implies that the towers constructed to be orthogonal at tree-level (3.82) will remain orthogonal also up to two-loops in perturbation theory.

There is no reason to expect that this property will be true also for the next orders, and indeed the results that we present next contradict the decoupling conjecture already at the next order in perturbation theory.

3.3.3 SU(3) and SU(4) SQCD

SU(3) SQCD

We consider SU(3) SQCD with 6 fundamental hypermultiplets to three-loops. We show that at this order in perturbation theory the bottom operators of the towers we constructed before become explicitly τ-dependent. This indicates that there is no reason to expect decoupled Toda chains as in (3.85) anymore.

Let us consider the first few low-lying chiral operators in SU(3) SQCD: $\{\phi_2\}$, $\{\phi_3\}$, $\{{\phi_2}^2\}$, $\{\phi_3\phi_2\}$, $\{{\phi_2}^3,{\phi_3}^2\}$, $\{\phi_3{\phi_2}^2\}$ and $\{{\phi_2}^4,{\phi_3}^2\phi_2\}$. We are interested in their two-point functions in flat space $(G_\Delta)_{IJ}=\left\langle O_{\Delta I}(0)\overline{O_{\Delta J}}(\infty)\right\rangle_{\mathbb{R}^4}$. G_6 and G_8 are therefore 2×2 matrices while the rest are just functions of the gauge coupling g in perturbation theory.

[26]The perturbative matrix integral of SU(N) SQCD is

$$Z[S^4](\tau,\bar\tau,...)=\int d^{N-1}a\,e^{-2\pi\,\mathrm{Im}\tau\,\mathrm{Tr}\,a^2+\dots}\prod_{i\neq j}(w_{ij}\cdot a)\prod_{i\neq j}H(iw_{ij}\cdot a)\prod_{i}H(iw_i\cdot a)^{-2N}\ ,\tag{3.89}$$

where w_i, $i=1,..,N$, are the weights in the fundamental representation, and $w_{ij}=w_i-w_j$.

Following our Gram-Schmidt procedure we can compute these up to three-loops

$$G_2 = \left(\frac{g^2}{4\pi}\right)^2 \left(16 - \frac{45\,\zeta(3)}{2\pi^4}g^4 + \frac{425\,\zeta(5)}{8\pi^6}g^6 + O(g^8)\right) \,, \tag{3.91}$$

$$G_3 = \left(\frac{g^2}{4\pi}\right)^3 \left(40 - \frac{135\,\zeta(3)}{2\pi^4}g^4 + \frac{6275\,\zeta(5)}{48\pi^6}g^6 + O(g^8)\right) \,, \tag{3.92}$$

$$G_4 = \left(\frac{g^2}{4\pi}\right)^4 \left(640 - \frac{2160\,\zeta(3)}{\pi^4}g^4 + \frac{6375\,\zeta(5)}{\pi^6}g^6 + O(g^8)\right) \,, \tag{3.93}$$

$$G_5 = \left(\frac{g^2}{4\pi}\right)^5 \left(1120 - \frac{4410\,\zeta(3)}{\pi^4}g^4 + \frac{144725\,\zeta(5)}{12\pi^6}g^6 + O(g^8)\right) \,, \tag{3.94}$$

$$\begin{aligned} G_6 = \left(\frac{g^2}{4\pi}\right)^6 &\begin{pmatrix} 46080 - \frac{272160\,\zeta(3)}{\pi^4}g^4 + \frac{969000\,\zeta(5)}{\pi^6}g^6 & i\left(1920 - \frac{11340\,\zeta(3)}{\pi^4}g^4 + \frac{29875\,\zeta(5)}{\pi^6}g^6\right) \\ -i\left(1920 - \frac{11340\,\zeta(3)}{\pi^4}g^4 + \frac{29875\,\zeta(5)}{\pi^6}g^6\right) & 6800 - \frac{57645\,\zeta(3)}{2\pi^4}g^4 + \frac{1688875\,\zeta(5)}{24\pi^6}g^6 \end{pmatrix} \\ &+ O(g^{20}) \,, \end{aligned} \tag{3.95}$$

$$G_7 = \left(\frac{g^2}{4\pi}\right)^7 \left(71680 - \frac{483840\,\zeta(3)}{\pi^4}g^4 + \frac{4936400\,\zeta(5)}{3\pi^6}g^6 + O(g^8)\right) \,, \tag{3.96}$$

$$\begin{aligned} G_8 = \left(\frac{g^2}{4\pi}\right)^8 &\begin{pmatrix} 5160960 - \frac{46448640\,\zeta(3)}{\pi^4}g^4 + \frac{194208000\,\zeta(5)}{\pi^6}g^6 & i\left(215040 - \frac{1935360\,\zeta(3)}{\pi^4}g^4 + \frac{6412000\,\zeta(5)}{\pi^6}g^6\right) \\ -i\left(215040 - \frac{1935360\,\zeta(3)}{\pi^4}g^4 + \frac{6412000\,\zeta(5)}{\pi^6}g^6\right) & 277760 - \frac{2046240\,\zeta(3)}{\pi^4}g^4 + \frac{20027000\,\zeta(5)}{3\pi^6}g^6 \end{pmatrix} \\ &+ O(g^{24}) \,. \end{aligned} \tag{3.97}$$

This is in agreement with [8], where the same correlators were computed up to two-loops using standard perturbation theory. It would be interesting to verify our three-loop results by direct perturbative computations.

We now note that in performing the Gram-Schmidt procedure on the dimension 6 and 8 operators, we encounter the following ratios

$$\frac{G_6(2,1)}{G_6(1,1)} = -i\left(\frac{1}{24} - \frac{175\,\zeta(5)}{768\pi^6}\,g^6 + \mathcal{O}(g^8)\right) \,, \tag{3.98}$$

$$\frac{G_8(2,1)}{G_8(1,1)} = -i\left(\frac{1}{24} - \frac{125\,\zeta(5)}{384\pi^6}\,g^6 + \mathcal{O}(g^8)\right) \,. \tag{3.99}$$

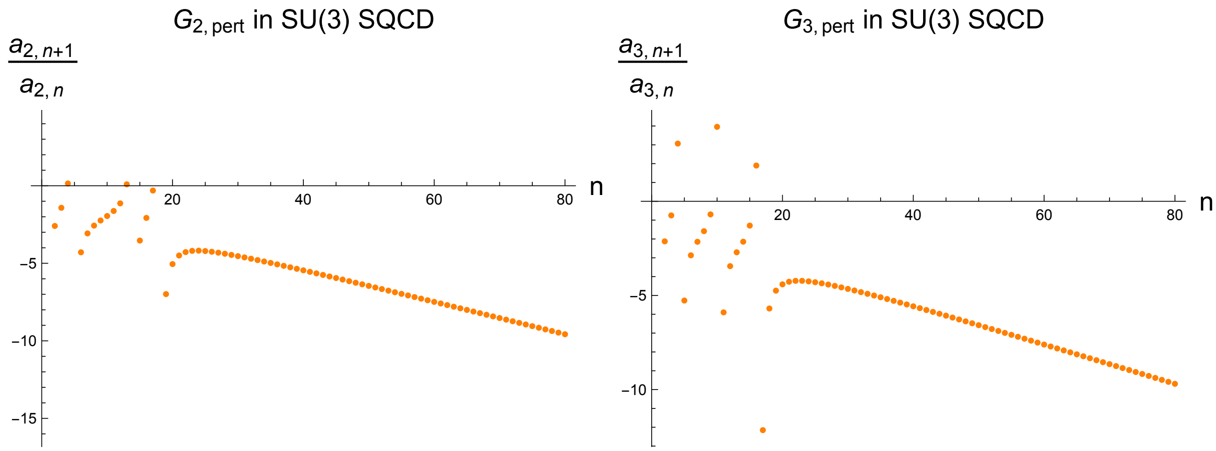


Figure 3.3: Ratios of consecutive coefficients in the series expansions (3.100) in the case of SU(3) SQCD.

The presence of the g^6 term means that one cannot diagonalize G_4 and G_6 with a $\tau, \bar{\tau}$ independent basis. This contradicts the conjecture of [8]. Note that the absence of a term g^4 in (3.98),(3.99) is precisely as anticipated in 3.2.2. The decoupling of Toda chains therefore starts to fail at three-loop order in perturbation theory.

If we define

$$G_{m,\text{pert}}(g^2) = G_{m,\text{tree}} \sum_{n=0}^{\infty} a_{m,n} \left(\frac{g^2}{4\pi}\right)^n \tag{3.100}$$

where $G_{m,\text{tree}}$ is the tree-level contribution and so $a_0 = 1$, one can check that the ratio $a_{m,n+1}/a_{m,n}$ grows linearly at large n with a negative coefficient, just as was the case for SU(2) SQCD. See Figure 3.3 for plots of these ratios in the cases $m = 2, 3$. We expect that $G_{m,\text{pert}}$ is also Borel summable in this case, but we have not shown this conclusively.

As in the case of SU(2) SQCD, one can use the series expansions above to estimate whether the $(n/2, n/2)$ Padé, computed only from the first n terms, can be used to estimate the $(n+1)$th series coefficient with an exponentially small error. This is indeed the case, as can be seen from Figure 3.4 for $m = 2, 3$. Defining the exponents σ_m through

$$\left|\frac{a_{m,n+1,\text{estimated}}}{a_{m,n+1}} - 1\right| < C_m e^{-\sigma_m n} \tag{3.101}$$

linear fits of the log plots in Figure 3.4 give $\sigma_2 \approx 0.75$ and $\sigma_3 \approx 0.73$. These values are rather close to the corresponding values for SU(2) SQCD.

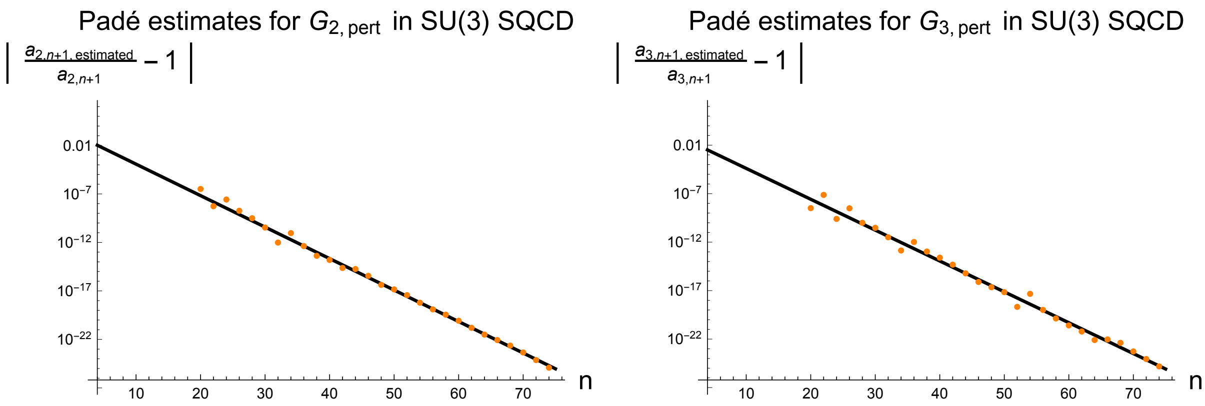


Figure 3.4: The relative difference between the Padé estimate of the coefficient $a_{m,n+1}$ and its actual value in the case of G_2 and G_3 in SU(3) SQCD. The black lines are linear fits for $n \geq 40$.

SU(4) SQCD

The conclusion from our study of SU(4) SQCD is the same as the conclusion from the study of SU(3) SQCD above. We present it just in order to demonstrate again the cancelation of the g^4 term and to provide additional data that can be compared with direct perturbative computations. We consider the operators $\{\phi_2\}$, $\{\phi_3\}$, $\{{\phi_2}^2, \phi_4\}$ and $\{\phi_2\phi_3\}$ and denote the corresponding two-point functions by G_2, G_3, G_4 and G_5, respectively. Using our Gram-Schmidt procedure we find

$$G_2 = \left(\frac{g^2}{4\pi}\right)^2 \left(30 - \frac{2295\,\zeta(3)}{32\pi^4} g^4 + \frac{118575\,\zeta(5)}{512\pi^6} g^6 + O(g^8)\right) \ , \tag{3.102}$$

$$G_3 = \left(\frac{g^2}{4\pi}\right)^3 \left(135 - \frac{23085\,\zeta(3)}{64\pi^4} g^4 + \frac{4100625\,\zeta(5)}{4096\pi^6} g^6 + O(g^8)\right) \ , \tag{3.103}$$

$$G_4 = \left(\frac{g^2}{4\pi}\right)^4 \begin{pmatrix} 2040 - \frac{43605\,\zeta(3)}{4\pi^4} g^4 + \frac{1304325\,\zeta(5)}{32\pi^6} g^6 & i\left(870 - \frac{74385\,\zeta(3)}{16\pi^4} g^4 + \frac{2351025\,\zeta(5)}{128\pi^6} g^6\right) \\ -i\left(870 - \frac{74385\,\zeta(3)}{16\pi^4} g^4 + \frac{2351025\,\zeta(5)}{128\pi^6} g^6\right) & \frac{1335}{2} - \frac{198045\,\zeta(3)}{64\pi^4} g^4 + \frac{5681925\,\zeta(5)}{512\pi^6} g^6 \end{pmatrix}$$
$$+ O(g^{16}) \tag{3.104}$$

$$G_5 = \left(\frac{g^2}{4\pi}\right)^5 \left(5670 - \frac{535815\,\zeta(3)}{16\pi^4} g^4 + \frac{248558625\,\zeta(5)}{2048\pi^6} g^6 + O(g^8)\right) \ . \tag{3.105}$$

Again, the two-loop results agree with those that were found by a direct Feynman diagrams computation in [8]. In performing the Gram-Schmidt procedure on the dimension 4 operators, we encounter the following ratios

$$\frac{G_4(2,1)}{G_4(1,1)} = -i\left(\frac{29}{68} + \frac{525\,\zeta(5)}{1088\pi^6} g^6 + \mathcal{O}(g^8)\right) \ . \tag{3.106}$$

As before, the g^4 piece cancels as anticipated but the g^6 piece contradicts the conjecture of [8]. Therefore, we do not expect decoupled semi-infinite Toda chains.

Chapter 4

An Application to Duality: Topological Holography

4.1 Introduction and Summary

Holography is a duality between two theories, referred to as a bulk theory and a boundary theory, in two different space-time dimensions that differ by one [95, 128, 155]. A familiar manifestation of the duality is an equality of the partition function of the two theories - the boundary partition function as a function of sources, and the bulk partition function as a function of boundary values of fields. This in turns implies that correlation functions of operators in the boundary theory can also be computed in the bulk theory by varying boundary values of its fields [95, 155]. This dictionary has been extended to include expectations values of non-local operators as well [84, 127, 142, 158]. This is a strong-weak duality, relating a strongly coupled boundary theory to a weakly coupled bulk theory. As is usual in strong-weak dualities, exact computations on both sides of the duality are hard. Topological theories have provided interesting examples of holographic dualities where exact computations are possible [34, 85, 86, 91, 137, 138].

Recently, Costello has shown that some instances of holography can be described as an algebraic relation, known as Koszul duality, between the operator algebras of the two dual theories [35]. It was previously known that the algebra of operators restricted to a line in the holomorphic twist of 4d $\mathcal{N} = 1$ gauge theory with the gauge group GL_K is the Koszul dual of the Yangian of $\mathfrak{gl}_K$ [33]. In light of the connection between Koszul duality and holography, this result suggests that if there is a theory whose local operator algebra is the Yangian of $\mathfrak{gl}_K$ then that theory could be a holographic dual to the twisted 4d theory.

Since the inception of holography, brane constructions played a crucial role in finding dual theories. It turns out that the particular twisted 4d theory is the world-volume theory of K D4-branes[1] embedded in a particular 6d topological string theory [36]. Since the operators whose algebra is the Koszul dual of the Yangian lives on a line, it is a reasonable guess that we need to include some other branes that intersect this stack of D4-branes along a line. Beginning from such motivations we eventually find (and demonstrate in this chapter) that the correct choice is to embed a stack of N D2-branes in the 6d topological string theory so that they intersect the D4-branes along a line. The world-volume theory of the D2-branes is 2d BF theory with GL_N gauge group coupled to a fermionic quantum mechanics along the D2-D4 intersection. The algebra of gauge invariant local operators along this D2-D4 intersection is precisely the Yangian of $\mathfrak{gl}_K$.

This connected the D2 world-volume theory and the D4 world-volume theory via holography in the sense of Koszul duality. The connection between these two theories via holography in the sense of [95, 155] was still unclear. In this chapter we begin to establish this connection. We take the D2-brane world-volume theory to be our boundary theory. This implies that the closed string theory in some background, including the D4-brane theory should give us the dual bulk theory. In the boundary theory, we consider the OPE (operator product expansion) algebra of gauge invariant local operators, we argue that this algebra can be computed in the bulk theory by computing a certain algebra of scatterings from the asymptotic boundary in the limit $N \to \infty$. Our computation of the boundary local operator algebra using the bulk theory follows closely the computation of boundary correlation functions using Witten diagrams [155].

The Feynman diagrams and Witten diagrams we compute in this chapter have at most two loops, however, we would like to emphasize that the identification we make between the operator algebras and the Yangian is true at *all* loop orders. In the boundary theory (D2-brane theory) this will follow from the simple fact that, for the operator product that we shall compute, there will be no non-vanishing diagrams beyond two loops. In the bulk theory this follows from a certain classification of anomalies in the D4-brane theory [43] and independently from the very rigid nature of the deformation theory of the Yangian. We explain some of these mathematical aspects underlying our results in appendix E.3.

A particular motivation for studying these topological/holomorphic theories and their duality is that these theories can be constructed from certain brane setup in string theory. We can identify these theories as certain supersymmetric subsectors of some theories on D-branes in type IIB string theory by applying supersymmetric twists and Ω-deformation.

[1] We are following the convention of [3], according to which, by a topological Dp-brane we mean a brane with a p-dimensional world-volume.

The organization of the chapter is as follows. In §4.2 we describe, in general terms, how holographic duality in the sense of [95, 155] leads to the construction of two isomorphic algebras from the two dual theories. In §4.3 we start from a brane setup involving N D2-branes and K D4-branes in a 6d topological string theory and describe the two theories that we claim to be holographic dual to each other. In §4.4 we compute the local operator algebra in the D2-brane theory, this algebra will be the Yangian $Y(\mathfrak{gl}_K)$ in the limit $N \to \infty$. In §4.5 we show that the same algebra can be computed using Witten diagrams in the D4-brane theory. In the last section, §4.6, we propose a string theory realization of the duality.

4.2 Isomorphic algebras from holography

In [95, 155], two theories, $\mathcal{T}_{\text{bd}}$ and $\mathcal{T}_{\text{bk}}$ were considered on two manifolds M_1 and M_2 respectively, with the property that M_1 was conformally equivalent to the boundary of M_2. The theory $\mathcal{T}_{\text{bd}}$ was considered with background sources, schematically represented by ϕ. The theory $\mathcal{T}_{\text{bk}}$ was such that the values of its fields at the boundary ∂M_2 can be coupled to the fields of $\mathcal{T}_{\text{bd}}$, then $\mathcal{T}_{\text{bk}}$ was quantized with the fields ϕ as the fixed profile of its fields at the boundary ∂M_2. These two theories were considered to be holographic dual when their partition functions were equal:

$$Z_{\text{bd}}(\phi) = Z_{\text{bk}}(\phi)\,. \tag{4.1}$$

This equality leads to an isomorphism of two algebras constructed from the two theories, as follows. Consider local operators O_i in $\mathcal{T}_{\text{bd}}$ with corresponding sources ϕ^i. The partition function $Z_{\text{bd}}(\phi)$ with these sources has the form:

$$Z_{\text{bd}}(\phi) = \int \mathcal{D}X \exp\left(-\frac{1}{\hbar}S_{\text{bd}} + \sum_i O_i\phi^i\right)\,, \tag{4.2}$$

where X schematically represents all the dynamical fields in $\mathcal{T}_{\text{bd}}$. Correlation functions of the operators O_i can be computed from the partition function by taking derivatives with respect to the sources:

$$\langle O_1(p_1)\cdots O_n(p_n)\rangle = \frac{1}{Z_{\text{bd}}(\phi)}\frac{\delta}{\delta\phi^1(p_1)}\cdots\frac{\delta}{\delta\phi^n(p_n)}Z_{\text{bd}}(\phi)\bigg|_{\phi=\phi_0}\,. \tag{4.3}$$

We can consider the algebra generated by the operators O_i using operator product expansion (OPE). However, this algebra is generally of singular nature, due to its dependence on the location of the operators and the possibility of bringing two operators too close to each other. In specific cases, often involving supersymmetry, we can consider sub-sectors of the operator spectrum that can generate algebras free from such contact singularity, so that a position independent algebra can be defined.[2] Suppose the set $\{O_i\}$ represents such a restricted set with an algebra:

$$O_i O_j = C_{ij}^k O_k \,. \tag{4.4}$$

Let us call this algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$. In terms of the partition function and the sources the relation (4.4) becomes:

$$\frac{\delta}{\delta\phi^i}\frac{\delta}{\delta\phi^j} Z_{\mathrm{bd}}(\phi)\bigg|_{\phi=0} = C_{ij}^k \, \frac{\delta}{\delta\phi^k} Z_{\mathrm{bd}}(\phi)\bigg|_{\phi=\phi_0} \,. \tag{4.5}$$

The statement of duality (4.1) then tells us that the above equation must hold if we replace Z_{bd} by Z_{bk}:

$$\frac{\delta}{\delta\phi^i}\frac{\delta}{\delta\phi^j} Z_{\mathrm{bk}}(\phi)\bigg|_{\phi=0} = C_{ij}^k \, \frac{\delta}{\delta\phi^k} Z_{\mathrm{bk}}(\phi)\bigg|_{\phi=\phi_0} \,. \tag{4.6}$$

This gives us a realization of the operator algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ in the dual theory $\mathcal{T}_{\mathrm{bk}}$.

This suggests a check for holographic duality. The input must be two theories, say $\mathcal{T}_{\mathrm{bd}}$ and $\mathcal{T}_{\mathrm{bk}}$, with some compatibility:

- $\mathcal{T}_{\mathrm{bd}}$ can be put on a manifold M_1 and $\mathcal{T}_{\mathrm{bk}}$ can be put on a manifold M_2 such that $\partial M_2 \cong M_1$, where equivalence between ∂M_2 and M_1 must be equivalence of whatever geometric/topological structure is required to define $\mathcal{T}_{\mathrm{bd}}$.[3]
- Quantum numbers of fields of the two theories are such that the boundary values of the fields in $\mathcal{T}_{\mathrm{bk}}$ can be coupled to the fields in $\mathcal{T}_{\mathrm{bd}}$.[4]

Suppose $\mathcal{T}_{\mathrm{bd}}$ has a sub-sector of its operator spectrum that generates a suitable algebra[5] $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$. We denote the operators in this algebra by $\{O_i\}$ with corresponding sources ϕ^i. According to the first compatibility condition these sources can be thought of as

[2]Various *chiral rings*, for example.

[3]In case of AdS/CFT, it is conformal equivalence, perfect for defining the CFT. In this chapter we shall only be concerned with topology.

[4]To clarify, this is merely a compatibility condition for the duality, the two dual theories are not supposed to be coupled, they are supposed to be alternative descriptions of the same dynamics.

[5]Ideally we should consider the OPE algebra of all the operators, but if that is too hard, we can restrict to smaller sub-sectors which may still provide a non-trivial check.

boundary values for the fields in $\mathcal{T}_{\mathrm{bk}}$, so that we can quantize $\mathcal{T}_{\mathrm{bk}}$ by fixing the values of the fields at the boundary to be ϕ. Then, we can define another algebra by taking functional derivatives of the partition function of $\mathcal{T}_{\mathrm{bk}}$ with respect to ϕ, as in (4.6). Let's call this algebra the *scattering algebra*, $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$. Now a check of holographic duality is the following isomorphism:

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) \cong \mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})\,. \tag{4.7}$$

This is the general idea that we employ in this chapter to check holographic duality. The operator algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ can be computed in perturbation theory using Feynman diagrams and we can use Witten diagrams, introduced in [155], to compute the scattering algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$. We will do this concretely in the rest of this chapter.

4.3 The dual theories

4.3.1 Brane construction

The quickest way to introduce the theories we claim to be holographic dual to each other is to use branes to construct them. Our starting point is a 6d topological string theory, in particular, the product of the A-twisted string theory on $\mathbb{R}^4$ and the B-twisted string theory on $\mathbb{C}$ [36]. The brane setup is the following:

	$\mathbb{R}_v$	$\mathbb{R}_w$	$\mathbb{R}_x$	$\mathbb{R}_y$	$\mathbb{C}_z$	No. of branes
D2	0	$\times$	$\times$	0	0	N
D4	0	0	$\times$	$\times$	$\times$	K

(4.8)

The subscripts denote the coordinates we use to parametrize the corresponding directions, and it is implied that the complex direction is parametrized by the complex variable z, along with its conjugate variable $\overline{z}$.

Our first theory, denoted by $\mathcal{T}_{\mathrm{bd}}$, is the theory of open strings on the stack of D2-branes. This is a 2d topological gauge theory with the complexified gauge group GL_N [36]. The intersection of the D2-branes with the D4-branes introduces a line operator in this theory. We describe this theory in §4.3.3.

Next, we consider the product of two theories, open string theory on the stack of D4-branes, and closed string theory on the 6d background sourced by the stack of D2-branes. The theory on the stack of D4-branes is a 4d analogue of Chern-Simons (CS) gauge theory with the complexified gauge group GL_K [36]. As it does in the theory on the D2-branes, the intersection between the D2 and the D4-branes introduces a line operator in this theory as well. This line sources a flux supported on the 3-sphere linking the line. Our bulk theory is the Kaluza-Klein compactification of the total 6d theory[6] on the 3-sphere. We describe the 4d CS theory in §4.3.4. Let us describe the closed sting theory in the next section.

4.3.2 The closed string theory

The closed string theory, denoted by $\mathcal{T}_{\mathrm{cl}}$, is a product of Kodira-Spencer (also known as BCOV) theory [17,41] on $\mathbb{C}$ and Kähler gravity [18] on $\mathbb{R}^4$, along with a 3-form flux sourced by the stack of D2-branes.[7] Fields[8] in this theory are given by:

$$\text{Set of fields,} \quad \mathcal{F} := \Omega^\bullet(\mathbb{R}^4) \otimes \Omega^{\bullet,\bullet}(\mathbb{C})\,, \tag{4.9}$$

i.e., the fields are differential forms on $\mathbb{R}^4$ and (p,q)-forms on $\mathbb{C}$.[9] The linearized BRST differential acting on these fields is a sum of the de Rham differential on $\mathbb{R}^4$ and the Dolbeault differential on $\mathbb{C}$, leading to the following equation of motion:

$$\left(\mathrm{d}_{\mathbb{R}^4} + \overline{\partial}_{\mathbb{C}}\right) \alpha = 0\,, \qquad \alpha \in \mathcal{F}\,. \tag{4.10}$$

The background field sourced by the D2-branes, let it be denoted by $F_3 \in \mathcal{F}$, measures the flux through a topological S^3 surrounding the D2-branes, it can be normalized as:

$$\int_{S^3} F_3 = N\,. \tag{4.11}$$

Note that the S^3 is only topological, i.e., continuous deformation of the S^3 should not affect the above equation. This is equivalent to saying that, the 3-form must be closed on the complement of the support of the D2-branes:

$$\mathrm{d}_{\mathbb{R}^4\times\mathbb{C}} F_3(p) = 0\,, \quad p \notin \mathrm{D2}\,. \tag{4.12}$$

Here the differential is the de Rham differential for the entire space, i.e., $\mathrm{d}_{\mathbb{R}^4\times\mathbb{C}} = \mathrm{d}_{\mathbb{R}^4} + \overline{\partial}_{\mathbb{C}} + \partial_{\mathbb{C}}$. Moreover, as a dynamically determined background it is also constrained by the equation of motion (4.10). In addition to satisfying these equations, F_3 must also

[6] 6d closed string theory coupled to 4d CS theory.

[7] This flux is analogous to the 5-form flux sourced by the stack of D4-branes in Maldacena's setup of AdS/CFT duality between $\mathcal{N}=4$ super Yang-Mills and supergravity on $\mathrm{AdS}_5 \times S^5$ [128].

[8] In the BV formalism, including ghosts and anti-fields.

[9] We are not being careful about the degree (ghost number) of the fields since this will not be used in this chapter.

be translation invariant corresponding to the directions parallel to the D2-branes. The solution is:

$$F_3 = \frac{iN}{2\pi(v^2+y^2+z\overline{z})^2}(v\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\overline{z} - y\,\mathrm{d}v \wedge \mathrm{d}z \wedge \mathrm{d}\overline{z} - 2\overline{z}\,\mathrm{d}v \wedge \mathrm{d}y \wedge \mathrm{d}z)\,. \tag{4.13}$$

In general, a closed string background like this might deform the theory on a brane, however, the pullback of the form (4.13) to the D4-branes vanishes:

$$\iota^* F_3 = 0\,, \tag{4.14}$$

where $\iota : \mathbb{R}^2_{x,y} \times \mathbb{C}_z \hookrightarrow \mathbb{R}^4_{v,w,x,y} \times \mathbb{C}_z$ is the embedding of the D4-branes into the entire space. So the closed string background leaves the D4-brane world-volume theory unaffected.[10]

The flux (4.13) signals a change in the topology of the closed string background:

$$\mathbb{R}^4_{v,w,x,y} \times \mathbb{C}_z \to \mathbb{R}^2_{w,x} \times \mathbb{R}_+ \times S^3\,, \tag{4.15}$$

where the $\mathbb{R}_+$ is parametrized by $r := \sqrt{v^2+y^2+z\overline{z}}$. This change follows from requiring translation symmetry in the directions parallel to the D2-branes and the existence of an S^3 supporting the flux F_3. This S^3 is analogous to the S^5 in the D4-brane geometry supporting the 5-form flux sourced by the said D4-branes in Maldacena's AdS/CFT [128]. The coordinate r measures distance[11] from the location of the D2-branes. The $r \to 0$ region would be analogous to Maldacena's near horizon geometry. In our topological setting there is no distinction between near and distant, and we treat the entire $\mathbb{R}^2_{w,x} \times \mathbb{R}_+ \times S^3$ as analogous to Maldacena's near horizon geometry. This makes $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$ analogous to the AdS geometry. We recall that, in the AdS/CFT correspondence the location of the black branes and the boundary of AdS correspond to two opposite limits of the non-compact coordinate transverse to the branes. In our case $r = 0$ corresponds to the location of the D2-branes, and we treat the plane at $r = \infty$, namely:

$$\mathbb{R}^2_{w,x} \times \{\infty\}\,, \tag{4.16}$$

as analogous to the asymptotic boundary of AdS.

[10] The flux (4.13) is the only background turned on in the closed string theory. This can be argued as follows: The D2-branes introduce a 4-form source (the Poincaré dual to the support of the branes) in the closed string theory. This form can appear on the right hand side of the equation of motion (4.10) only for a 3-form field α, which can then have a non-trivial solution, as in (4.13). Furthermore, since the equation of motion (4.10) is free, the non-trivial solution for the 3-form field does not affect any other field.

[11] In the absence of a metric "distance" should be taken lightly. We really only distinguish between the two extreme cases, $r = 0$ and $r = \infty$.

The D4-branes in (4.8) appear as a defect in the closed string theory, they are analogous to the D5-branes that were considered in [89] or the D3-branes considered in [89, 90], in Maldacena's setup of AdS/CFT, where they were presented as holographic duals of Wilson loops in 4d $\mathcal{N} = 4$ super Yang-Mills. For the world-volume of these branes, the transformation (4.15) corresponds to:

$$\mathbb{R}^2_{x,y} \times \mathbb{C}_z \to \mathbb{R}_x \times \mathbb{R}_+ \times S^2 \,, \tag{4.17}$$

where the $\mathbb{R}_+$ direction is parametrized by $r' := \sqrt{y^2 + z\bar{z}}$. The intersection of the boundary plane (4.16) and this world-volume is then the line:

$$\mathbb{R}_x \times \{\infty\}, \tag{4.18}$$

at infinity of r'. We draw a cartoon representing some aspects of the brane setup in figure 4.1.

We can now talk about two theories:

1. The 2d world-volume theory of the D2-branes. This is our analogue of the CFT (with a line operator) in AdS/CFT.
2. The effective[12] 3D theory on world-volume $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$ with a defect supported on $\mathbb{R}_x \times \mathbb{R}_+$. This is our analogue of the gravitational theory in AdS background (with defect) in AdS/CFT.

To draw parallels once more with the traditional dictionary of AdS/CFT [95, 128, 155], we should establish a duality between the operators in the D2-brane world-volume theory and variations of boundary values of fields in the "gravitational" theory on $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$ (the boundary is $\mathbb{R}_{w,x} \times \{\infty\}$). Both of these surfaces have a line operator/defect and this leads to two types of operators, ones that are restricted to the line, and others that can be placed anywhere. Local operators in a 2d surface are commuting, unless they are restricted to a line. Therefore, in both of our theories, we have non-commutative associative algebras whose centers consist of operators that can be placed anywhere in the 2d surface. For this chapter we are mostly concerned with the non-commuting operators:

1. Operators in the world-volume theory of the D2-branes that are restricted to the D2-D4 intersection.

[12]Effective, in the sense that this is the Kaluza-Klein reduction of a 6d theory with three compact directions, though we don't want to loose any dynamics, i.e., we don't throw away massive modes.

2. Variations of boundary values of fields in the effective theory along the intersection (4.18) of the boundary $\mathbb{R}^2_{w,x} \times \{\infty\}$ and the defect on $\mathbb{R}_x \times \mathbb{R}_+$.

In physical string theory, the analogue of the D4-branes would be coupled to the closed string modes. In an appropriate low energy limit such gravitational couplings can be ignored, leading to the notion of *rigid holography* [1]. Since we are working with topological theory, we are assuming such a decoupling.

The computations in the "gravitational" side will be governed by the effective dynamics on the defect on $\mathbb{R}_x \times \mathbb{R}_+$. This is the Kaluza-Klein compactification of the world-volume theory of the D4-branes (with a line operator due to D2-D4 intersection). This 4d theory (which we describe in §4.3.4) is familiar from previous works such as [43]. Therefore we use the 4d dynamics, instead of the effective 2d one for our computations. In terms of Witten diagrams (which we compute in §4.5) this means that while we have a 1D boundary, the propagators are from the 4d theory and the bulk points are integrated over the 4d world-volume $\mathbb{R}^2 \times \mathbb{C}$. We take the boundary line to be at $y = \infty$ with some fixed coordinate z in the complex direction. In future we shall refer to this line as $\ell_\infty(z)$:

$$\ell_\infty(z) := \mathbb{R}_x \times \{y = \infty\} \times \{z\}\,. \tag{4.19}$$

A cartoon of our setup

Let us make a diagrammatic summary of our brane setup in Fig 4.1. In the figure we draw

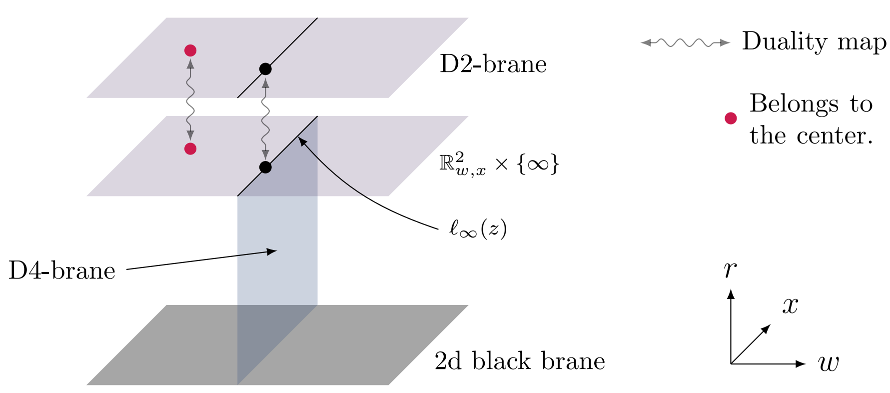


Figure 4.1: D2-brane, and the non-compact part of the backreacted bulk.

the non-compact part, namely $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$, of the closed string background (the right hand side of (4.15)). We identify the location of the 2d black brane and the defect D4-branes, the asymptotic boundary $\mathbb{R}^2_{w,x} \times \{\infty\}$, and the intersection between the boundary and the defect. At the top of the picture, parallel to the asymptotic boundary, we also draw the D2-branes. We draw the D2-branes independently of the rest of the diagram because the D2-branes do not exist in the backreacted bulk, they become the black brane. However, traditionally, parallels are drawn between the asymptotic boundary and the brane sourcing the bulk (the D2-brane in this case). The dots on the asymptotic boundary represent local variations of boundary values of fields in the bulk theory $\mathcal{T}_{\text{bk}}$. The corresponding dots on the D2-brane represent the local operators in the boundary theory $\mathcal{T}_{\text{bd}}$ that are dual to the aforementioned variations. By the duality map in the figure we schematically represent boundary excitations in the bulk theory corresponding to some local operators in the dual description of the same dynamics in terms of the boundary theory.

4.3.3 BF: The theory on D2-branes

This is a 2d topological gauge theory on the stack of N D2-branes (see (4.8)), supported on $\mathbb{R}^2_{w,x}$, with complexified gauge group GL_N. The field content of this theory is:

$$\begin{array}{cc} \text{Field} & \text{Valued in} \\ \hline \mathsf{B} & \Omega^0(\mathbb{R}^2) \times \mathfrak{gl}_N \\ \mathsf{A} & \Omega^1(\mathbb{R}^2) \times \mathfrak{gl}_N \end{array} . \tag{4.20}$$

A is a Lie algebra valued connection and B is a Lie algebra valued scalar, both complex. The curvature of the connection is denoted as $\mathsf{F} = \mathrm{d}\mathsf{A} + \mathsf{A} \wedge \mathsf{A}$. The action is given by:

$$S_{\text{BF}} := \int_{\mathbb{R}^2_{w,x}} \operatorname{tr}_{\mathbf{N}}(\mathsf{BF}) \,, \tag{4.21}$$

where the trace is taken in the fundamental representation of $\mathfrak{gl}_N$.

We consider this theory in the presence of a line operator supported on $\mathbb{R}_x \times \{0\}$, caused by the intersection of the D2 and D4-branes. The line operator is defined by a fermionic quantum mechanical system living on it.[13] The fields in the quantum mechanics (QM) are

[13]This closely resembles the D3-D5 system in type IIB string theory considered in [89], there too a fermionic quantum mechanics lived on the intersection, giving rise to Wilson lines upon integrating out the fermions. Note that we could have considered bosons, instead of fermions, living on the line, without any significant change to our following computations. This would be similar to the D3-D3 system considered in [89, 90].

K fundamental (of $\mathfrak{gl}_N$) fermions and their complex conjugates:

$$\begin{array}{cc} \text{Field} & \text{Valued in} \\ \hline \psi^i & \Omega^0(\mathbb{R}_x) \times \mathbf{N} \\ \overline{\psi}_i & \Omega^0(\mathbb{R}_x) \times \overline{\mathbf{N}} \end{array}, \qquad i \in \{1, \cdots, K\}, \tag{4.22}$$

where $\mathbf{N}$ refers to the fundamental representation of $\mathfrak{gl}_N$ and $\overline{\mathbf{N}}$ to the anti-fundamental. The fermionic system has a global symmetry $\mathrm{GL}_N \times \mathrm{GL}_K$. These fermions couple naturally to the $\mathfrak{gl}_N$ connection A of the BF theory. The action for the QM is given by:

$$S_{\mathrm{QM}} := \int_{\mathbb{R}_x} \left(\overline{\psi}_i \mathrm{d}\psi^i + \overline{\psi}_i \mathsf{A} \psi^i + \overline{\psi}_j A_i^j \psi^i \right) , \tag{4.23}$$

where we have introduced a *background* $\mathfrak{gl}_K$-valued gauge field $A \in \Omega^1(\mathbb{R}_x) \times \mathfrak{gl}_K$. Note that the terms in the above action are made $\mathfrak{gl}_N$ invariant by pairing up elements of $\mathbf{N}$ with elements of the dual space $\overline{\mathbf{N}}$.

Our first theory is this BF theory with the line operator, schematically:

$$\mathcal{T}_{\mathrm{bd}} := \mathrm{BF}_N \otimes_N \mathrm{QM}_{N \times K}, \tag{4.24}$$

where the subscripts on BF and QM refer to the symmetries (GL_N and $\mathrm{GL}_N \times \mathrm{GL}_K$ respectively) of the respective theories and the subscript on $\otimes$ implies that the GL_N is gauged. There are two types of gauge ($\mathfrak{gl}_N$) invariant operators in the theory:[14]

$$\text{for } n \in \mathbb{N}_{\geq 0}, \qquad \begin{array}{r} \text{operators restricted to } \mathbb{R}_x\text{:} \quad O_j^i[n] := \frac{1}{\hbar} \overline{\psi}_j \mathsf{B}^n \psi^i, \\ \text{operators } \textit{not} \text{ restricted to } \mathbb{R}_x\text{:} \quad O[n] := \frac{1}{\hbar} \mathrm{tr}_{\mathbf{N}} \mathsf{B}^n . \end{array} \tag{4.25}$$

Unrestricted local operators in two topological dimensions can be moved around freely, implying that for any $n \geq 0$, the operator $O[n]$ commutes with all of the operators defined above.[15] The operator algebra of the 2d BF theory consists of all theses operators but for this chapter we focus on the non-commuting ones, in other words we, focus on the quotient of the full operator algebra of the boundary theory by its center.[16] We shall compute their Lie bracket in §4.4, which will establish an isomorphism with the Yangian. Had we included the commuting operators as well we would have found a *central extension* of the Yangian. In sum, the operator algebra we construct from the theory $\mathcal{T}_{\mathrm{bd}}$ is:

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) := \left(O_j^i[n], O[n] \right) / (O[n]) . \tag{4.26}$$

[14] The $\hbar^{-1}$ appears in these definitions because the action (4.23) will appear in path integrals as $\exp\left(-\hbar^{-1} S_{\mathrm{QM}}\right)$, which means functional derivatives with respect to A_j^i inserts operators that carry $\hbar^{-1}$.

[15] These operators are represented by the red dot on the D2-brane in figure 4.1.

[16] We shall similarly quotient out the center in the bulk theory as well.

By the notation $(x, y, \cdots)$ we mean the algebra generated by the set of operators $\{x, y, \cdots\}$ over $\mathbb{C}$.

Remark 4.3.1 (A speculative link). Note that it is possible to lift our D2 and D4 branes to type IIB string theory while maintaining a one dimensional intersection. This results in a D3-D5 setup (studied in particular in [89]) where on the D3 brane we find the $\mathcal{N} = 4$ Yang-Mills theory with a Wilson line.[17] In [63, 77, 79], the authors considered local operators in the $\mathcal{N} = 4$ Yang-Mills that are restricted to certain Wilson lines. With the proper choice of Wilson lines, Localization reduces this setup to 2d Yang-Mills theory with Wilson lines – local operator insertions along the Wilson lines in 4d reduce to local operator insertions along the Wilson lines in 2d [78]. 2d BF theory is the zero coupling limit of 2d Yang-Mills theory. We therefore expect the algebra constructed in this section to be related to the algebra constructed in the aforementioned references, at least in some limit.[18] The algebra in [77] would correspond to the $K = 1$ instance of our algebra, it may be an interesting check to compute the analogue of the algebra in [77] for higher K. △

4.3.4 4d Chern-Simons: The theory on D4-branes

This is a 4d gauge theory on the stack of K D4-branes, supported on $\mathbb{R}^2_{x,y} \times \mathbb{C}_z$ with the line $L := \mathbb{R}_x \times (0, 0, 0)$ removed and with the (complexified) gauge group GL_K. The notation of distinguishing directions by $\mathbb{R}$ and $\mathbb{C}$ is meant to highlight the fact that observables in this theory depend only on the topology of the real directions and depend holomorphically on the complex directions.[19] Due to the removed line, we can represent the topology of the support of this theory as (c.f. (4.17)):

$$M := \mathbb{R} \times \mathbb{R}_+ \times S^2 \,. \tag{4.27}$$

The field of this theory is just a connection:

$$\begin{array}{cc} \text{Field} & \text{Valued in} \\ \hline A & \frac{\Omega^1(\mathbb{R}^2 \times \mathbb{C} \setminus L)}{(\mathrm{d}z)} \otimes \mathfrak{gl}_K \end{array} \,. \tag{4.28}$$

The above notation simply means that A is a $\mathfrak{gl}_K$-valued 1-form without a $\mathrm{d}z$ component. The theory is defined by the action:

$$S_{\mathrm{CS}} := \frac{i}{2\pi} \int_M \mathrm{d}z \wedge \mathrm{CS}(A) \,, \tag{4.29}$$

[17] It is also interesting to note that the D5 brane in an Omega background reproduces the 4d CS theory [44].

[18] We thank Shota Komatsu for pointing out this interesting possibility.

[19] In particular, they are independent of the coordinates x and y that parametrize the $\mathbb{R}^2$, and depend holomorphically on z which parametrizes the $\mathbb{C}$.

where $\mathrm{CS}(A)$ refers to the standard Chern-Simons Lagrangian:

$$\mathrm{CS}(A) = \mathrm{tr}_{\mathbf{K}} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) , \tag{4.30}$$

where the trace is taken over the fundamental representation of $\mathfrak{gl}_K$. This theory is a 4d analogue of the, perhaps more familiar, 3D Chern-Simons theory. We shall therefore refer to it as the 4d Chern-Simons theory and sometimes denote it by CS^4_K or just CS.

The removal of the line L from $\mathbb{R}^2 \times \mathbb{C}$ is caused by the D2-D4 brane intersection. Note that from the perspective of the CS theory, the D2-D4 intersection looks like a Wilson line. This means that we should be quantizing the CS theory on M with a background electric flux supported on the S^2 inside M. Alternatively, we can quantize the CS theory on $\mathbb{R}^4 \times \mathbb{C}$ with a Wilson line inserted along L.[20] The choice of representation for the Wilson line is determined by the number, N, of D2-branes, let us denote this representation as $\varrho : \mathfrak{gl}_K \to V$. With this choice, the Wilson line is defined as the following operator:

$$W_\varrho(L) := P \exp \left(\int_L \varrho(A) \right) , \tag{4.31}$$

where $P\exp$ implies path ordered exponentiation, made necessary by the fact that the exponent is matrix valued. The above operator is valued in $\mathrm{End}(V)$. This in general means that the following expectation value:

$$\langle W_\varrho(L) \rangle = \frac{\int \mathcal{D}A\, W_\varrho(L) \exp\left(-\frac{1}{\hbar} S_{\mathrm{CS}}\right)}{\int \mathcal{D}A\, \exp\left(-\frac{1}{\hbar} S_{\mathrm{CS}}\right)} , \tag{4.32}$$

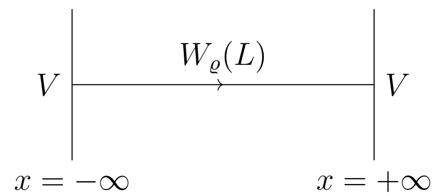
is valued in $\mathrm{Hom}(\mathcal{H}_{-\infty} \otimes V, \mathcal{H}_{+\infty} \otimes V)$, where $\mathcal{H}_{\pm\infty}$ are the Hilbert spaces of the CS^4_K theory on the Cauchy surfaces perpendicular to L at $x = \pm\infty$, in the absence of the Wilson line. However, for the particular CS theory, these Hilbert spaces are trivial and we end up with a map that transports vectors in V from $x = -\infty$ to $x = +\infty$:

$$\langle W_\varrho(L) \rangle : V_{-\infty} \to V_{+\infty} . \tag{4.33}$$

[20] Recall that in case of the BF theory the line operator at the D2-D4 intersection was described by a fermionic QM. We could do the same in this case. However, in this case it proves more convenient to integrate out the fermion, leaving a Wilson line in its place. The mechanism is the same that appeared for intersection of D3 and D5-branes in physical string theory [89].

In picture this operator may be represented as:

$$\langle W_\varrho(L)\rangle \ :$$



$$. \tag{4.34}$$

The CS theory is quantized with some fixed boundary profile of the connection along the boundary $\mathbb{R}_x \times \{\infty\} \times S^2$.[21] To express the dependence of expectations values on this boundary value we put a subscript, such as $\langle W_\varrho(L)\rangle_A$. Since we are essentially interested in the Kaluza-Klein reduced theory on $\mathbb{R}_x \times \mathbb{R}_+$ we mostly care about the value of the connection along the boundary line (defined in (4.19)) $\ell_\infty(z) \subset \mathbb{R}_x \times \{\infty\} \times S^2$.

To define our second theory, we start with the product of the closed string theory and the CS theory, $\mathcal{T}_{\rm cl} \otimes \mathrm{CS}^4_K$, supported on $\mathbb{R}^2_{w,x} \times \mathbb{R}_+ \times S^3$ and compactify on S^3, our notation for this theory is the following:

$$\mathcal{T}_{\rm bk} := \pi_*^{S^3}\left(\mathcal{T}_{\rm cl} \otimes \mathrm{CS}^4_K\right) . \tag{4.35}$$

We can put the theory $\mathcal{T}_{\rm bd}$ (4.24) on the plane $\mathbb{R}_{w,x}$ "at infinity" of $\mathbb{R}^2_{w,x} \times \mathbb{R}_+$. This plane has a distinguished line $\mathbb{R}_x \times \{\infty\}$ (4.18) where the D4-brane world volume intersects.[22] Along this line we have the $\mathfrak{gl}_K$ gauge field which couples to the fermions of the QM in $\mathcal{T}_{\rm bd}$ (this coupling corresponds to the last term in (4.23)). Boundary excitations from arbitrary points on $\mathbb{R}_{w,x} \times \{\infty\}$ will correspond to operators in the BF theory that are commuting, since these local excitations on a plane are not ordered. The non-commutative algebra we are interested in in the BF theory is the algebra of operators restricted to a particular line. Similarly, in the "gravitational" side of the setup, we are interested in boundary excitations restricted to the line $\ell_\infty(z)$. Let us look a bit more closely at the coupling between the connection A and the fermions:

$$I_z := \frac{1}{\hbar}\int_{\ell_\infty(z)} \overline{\psi}^i A_i^j \psi_j \, , \qquad \ell_\infty(z) = \mathbb{R}_x \times \{y=\infty\} \times \{z\} \, . \tag{4.36}$$

A small variation of z leads to coupling between the fermions and z-derivatives of the connection:

$$I_{z+\delta z} = \sum_{n=0}^{\infty} \frac{1}{\hbar}\int_{\ell_\infty(z)} \frac{(\delta z)^n}{n!} \overline{\psi}^i \partial_z^n A_i^j \psi_j \, . \tag{4.37}$$

[21] The boundary was chosen to respect the symmetry of the Wilson line along L.
[22] After aligning the v-coordinates of the plane and the D4-branes.

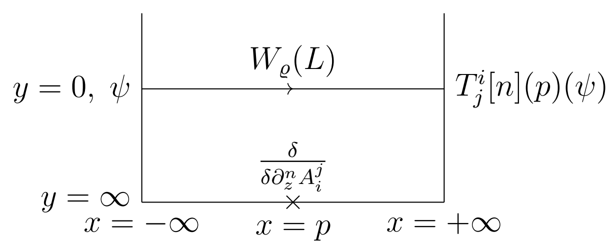
In the BF theory, the field B corresponds to the fluctuation of the D2-branes in the transverse $\mathbb{C}$ direction [36]. Therefore, we can interpret the above varied coupling term as saying that the operator in the boundary theory $\mathcal{T}_{\text{bd}}$ that couples to the derivative $\partial_z^n A_i^j$ is precisely the operator $O_j^i[n] = \hbar^{-1}\overline{\psi}^i \mathsf{B}^n \psi_j$ (c.f. (4.25), (4.26)). This motivates us to look at functional derivatives of $\langle W_\varrho(L)\rangle_A$ with respect to $\partial_z^n A_i^j$ at fixed points along $\ell_\infty(z)$, such as:

$$\frac{\delta}{\delta \partial_z^{n_1} A_{i_1}^{j_1}(p_1)} \cdots \frac{\delta}{\delta \partial_z^{n_m} A_{i_m}^{j_m}(p_m)} \left\langle W_\varrho(L)\right\rangle_A \,, \qquad p_1, \cdots, p_m \in \ell_\infty(z)\,. \tag{4.38}$$

Just as the expectation value $\langle W_\varrho(L)\rangle_A$ is $\mathrm{End}(V)$-valued, these functional derivatives are $\mathrm{End}(V)$-valued as well.[23] The action is given by applying the functional derivative on $\langle W_\varrho(L)\rangle_A(\psi)$ for any $\psi \in V$. Let us denote this operator as

$$\begin{gathered} T_j^i[n] : \ell_\infty(z) \times V \to V\,, \\ p \in \ell_\infty(z)\,, \qquad T_j^i[n](p) : \psi \mapsto \frac{\delta}{\delta \partial_z^n A_i^j(p)} \left\langle W_\varrho(L)\right\rangle_A(\psi)\,. \end{gathered} \tag{4.39}$$

which can be pictorially represented by slight modifications of (4.34):



(4.40)

Composition of these operators, such as $T_{j_1}^{i_1}(p_1) \cdots T_{j_m}^{i_m}(p_m)$, is defined by the expression (4.38). A more precise and computable characterization of these operators and their composition in terms of Witten diagrams [155] will be given in §4.5 (see (4.126)). Due to topological invariance along the x-direction, the operator $T_j^i[n](p)$ must be independent of the position p. However, since these operators are positioned along a line, their product should be expected to depend on the ordering, leading to a non-commutative associative algebra. We can now define the second algebra to appear in our example of holography:

$$\mathcal{A}^{\text{Sc}}(\mathcal{T}_{\text{bk}}) := \left(T_j^i[n]\right)\,, \tag{4.41}$$

i.e., the complex algebra generated by the set $\{T_j^i[n]\}$.

[23] After choosing a point along $\ell_\infty(z)$.

Remark 4.3.2 (Center of the algebra). In the BF theory we mentioned gauge invariant operators that belong to the center of the algebra. Clearly, the holographic dual of those operators do not come from the CS theory, rather they come from the closed string theory. A 2-form field $\phi = \phi_{wx} \mathrm{d}w \wedge \mathrm{d}x + \cdots$ from the closed string theory deforms the BF theory as:

$$S_{\mathrm{BF}} \to S_{\mathrm{BF}} + \int_{\mathbb{R}^2_{w,x}} \mathrm{d}w \wedge \mathrm{d}x \, (\partial_z^n \phi_{wx}) \, \mathrm{tr}_{\mathbf{N}} \, (\mathsf{B}^n) \; . \tag{4.42}$$

Functional derivatives with respect to the fields $\partial_z^n \phi_{w,x}$ placed at arbitrary locations on the asymptotic boundary $\mathbb{R}^2_{w,x} \times \{\infty\}$ correspond to inserting the operators $\mathrm{tr}_{\mathbf{N}} \mathsf{B}^n$ in the BF theory.[24] As we did in the BF theory, we are going to ignore these operators now as well. $\triangle$

After all this setup, we can present the main result of this chapter:

Theorem 4.3.3. *In the limit $N \to \infty$, both the algebra of local operators (4.26) along the line operator in the theory $\mathcal{T}_{\mathrm{bd}} = \mathrm{BF}_N \otimes_N \mathrm{QM}_{N\times K}$, and the algebra of scatterings from a line in the boundary (4.41) of the theory $\mathcal{T}_{\mathrm{bk}} = \pi_*^{S^3} \left(\mathcal{T}_{cl} \otimes \mathrm{CS}^4_K \right)$ are isomorphic to the Yangian of $\mathfrak{gl}_K$, i.e.:*

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K) \overset{N\to\infty}{\cong} \mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}}) \, . \tag{4.43}$$

The rest of the chapter is devoted to the explicit computations of these algebras.

4.4 $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ from $\mathrm{BF} \otimes \mathrm{QM}$ theory

In this section we prove the first half of our main result (Theorem 4.3.3):

Proposition 4.4.1. *The algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$, defined in the context of 2d BF theory with the gauge group GL_N coupled to a 1D fermionic quantum mechanics with global symmetry $\mathrm{GL}_N \times \mathrm{GL}_K$, is isomorphic to the Yangian of $\mathfrak{gl}_K$ in the limit $N \to \infty$:*

$$\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}}) \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K) \, . \tag{4.44}$$

The BF theory coupled to a fermionic quantum mechanics was defined in §4.3.3, let us repeat the actions here:

$$S_{\mathcal{T}_{\mathrm{bd}}} = S_{\mathrm{BF}} + S_{\mathrm{QM}} \, , \tag{4.45}$$

[24] These functional derivatives are represented by the red dot on the asymptotic boundary in figure 4.1.

where:

$$S_{\mathrm{BF}} = \int_{\mathbb{R}^2_{w,x}} \mathrm{tr}_{\mathbf{N}}(\mathsf{B}\mathrm{d}\mathsf{A} + \mathsf{B}[\mathsf{A},\mathsf{A}]) \tag{4.46}$$

$$\text{and} \quad S_{\mathrm{QM}} = \int_{\mathbb{R}_x} \left(\overline{\psi}_i \mathrm{d}\psi^i + \overline{\psi}_i \mathsf{A}\psi^i\right) . \tag{4.47}$$

We no longer need the source term, i.e., the coupling to the background $\mathfrak{gl}_K$ connection (c.f. (4.23)). Let us determine the propagators now.

The BF propagator is defined as the 2-point correlation function:

$$\mathsf{P}^{\alpha\beta}(p,q) := \left\langle \mathsf{B}^{\alpha}(p)\mathsf{A}^{\beta}(q)\right\rangle . \tag{4.48}$$

We choose a basis $\{\tau_\alpha\}$ of $\mathfrak{gl}_N$ which is orthonormal with respect to the trace $\mathrm{tr}_{\mathbf{N}}$:

$$\mathrm{tr}_{\mathbf{N}}(\tau_\alpha \tau_\beta) = \delta_{\alpha\beta} . \tag{4.49}$$

Then the two point correlation function becomes diagonal in the color indices:

$$\mathsf{P}^{\alpha\beta}(p,q) \equiv \delta^{\alpha\beta}\mathsf{P}(p,q) . \tag{4.50}$$

We shall often refer to just P as the propagator, it is determined by the following equation:[25]

$$\frac{1}{\hbar}\mathrm{d}\mathsf{P}(0,p) = \delta^2(p)\mathrm{d}w \wedge \mathrm{d}x . \tag{4.51}$$

Once we impose the following gauge fixing condition:[26]

$$\mathrm{d} \star \mathsf{P}(0,p) = 0 , \tag{4.52}$$

the solution is (using translation invariance to replace the 0 with an arbitrary point):

$$\mathsf{P}(p,q) = \frac{\hbar}{2\pi}\,\mathrm{d}\phi(p,q) , \tag{4.53}$$

where $\phi(p,q)$ is the angle (measured counter-clockwise) between the line joining p-q and any other reference line passing through p. In Feynman diagrams we shall represent this propagator as:

$$\mathsf{P}(p,q) = \ p \longrightarrow q \ . \tag{4.54}$$

[25] A minor technicality: $\mathsf{P}(p,q)$ is a 1-form on $\mathbb{R}^2_p \times \mathbb{R}^2_q$ and in (4.51), by $\mathsf{P}(0,p)$ we mean the pull-back of $\mathsf{P} \in \Omega^2(\mathbb{R}^4)$ by the diagonal embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^2$.

[26] This is the analogue of the Lorentz gauge.

Similarly, the propagator in the QM is defined by:

$$\frac{1}{\hbar}\partial_{x_2}\left\langle\overline{\psi}_i^a(x_1)\psi_b^j(x_2)\right\rangle = \delta_b^a\delta_i^j\delta^1(x_1-x_2)\,, \tag{4.55}$$

with the solution:

$$\left\langle\overline{\psi}_i^a(x_1)\psi_b^j(x_2)\right\rangle = \delta_b^a\delta_i^j\hbar\,\vartheta(x_2-x_1)\,, \tag{4.56}$$

where $\vartheta(x_2-x_1)$ is a unit step function. Anti-symmetry of the fermion fields dictates:

$$\left\langle\psi_b^j(x_1)\overline{\psi}_i^a(x_2)\right\rangle = -\left\langle\overline{\psi}_i^a(x_2)\psi_b^j(x_1)\right\rangle = -\delta_b^a\delta_i^j\hbar\,\vartheta(x_1-x_2)\,. \tag{4.57}$$

We take the step function to be:

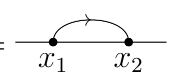
$$\vartheta(x) = \frac{1}{2}\mathrm{sgn}(x) = \begin{cases} 1/2 & \text{for } x>0 \\ 0 & \text{for } x=0 \\ -1/2 & \text{for } x<0 \end{cases}\,. \tag{4.58}$$

Then we can write:

$$\left\langle\overline{\psi}_i^a(x_1)\psi_b^j(x_2)\right\rangle = \left\langle\psi_b^j(x_1)\overline{\psi}_i^a(x_2)\right\rangle = \delta_b^a\delta_i^j\frac{\hbar}{2}\,\mathrm{sgn}(x_2-x_1)\,. \tag{4.59}$$

This propagator does not distinguish between ψ and $\overline{\psi}$ and it depends only on the order of the fields, not their specific positions. In Feynman diagrams we shall represent this propagator as:

$$\frac{\hbar}{2}\mathrm{sgn}(x_2-x_1) = \tag{4.60}$$



where the curved line refers to the propagator itself and the horizontal line refers to the support of the QM, i.e., the line $w=0$. We now move on to computing operator products that will give us the algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$.

Remark 4.4.1 (Fermion vs. Boson - Propagator). We might as well have considered a bosonic QM instead of a fermionic QM. At present, this is an arbitrary choice, however, if one starts from some brane setup in physical string theory and reduce it to the topological setup we are considering by twists and Ω-deformations,[27] then depending on the starting setup one might end up with either statistics. Let us make a few comments about the bosonic case. In the first order formulation of bosonic QM the action looks exactly as in the fermionic action 4.47 except the fields would be commuting – let us denote the bosonic

[27]We describe one such specific procedure in §4.6.

counterpart of $\overline{\psi}$ and ψ by $\overline{\phi}$ and ϕ respectively. Then, instead of the propagator (4.59), we would have the following propagator:[28]

$$-\left\langle \overline{\phi}_i^a(x_1)\phi_b^j(x_2)\right\rangle = \left\langle \phi_b^j(x_1)\overline{\phi}_i^a(x_2)\right\rangle = \delta_b^a\delta_i^j\frac{\hbar}{2}\text{sgn}(x_2-x_1)\,. \tag{4.61}$$

Note that the extra sign in the first term (compared to (4.59)) is consistent with the commutativity of the bosonic fields:

$$\left\langle \overline{\phi}_i^a(x_1)\phi_b^j(x_2)\right\rangle = \left\langle \phi_b^j(x_2)\overline{\phi}_i^a(x_1)\right\rangle \,. \tag{4.62}$$

The bosonic propagator (4.61) distinguishes between ϕ and $\overline{\phi}$, in that, the propagator is positive if $\phi(x_1)$ is placed before $\overline{\phi}(x_2)$, i.e., $x_1 < x_2$, and negative otherwise. $\triangle$

4.4.1 Free theory limit, $\mathcal{O}(\hbar^0)$

Interaction in the quantum mechanics is generated via coupling to the $\mathfrak{gl}_N$ gague field (see (4.47)). Without this coupling, the quantum mechanics is free. In this section we compute the operator product between $O_j^i[m]$ and $O_l^k[n]$ in this free theory, which will give us the classical algebra.

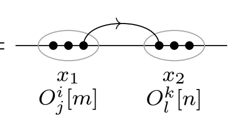
Let us denote the operator product by $\star$, as in:

$$O_j^i[m]\star O_l^k[n]\,. \tag{4.63}$$

The classical limit of this product has an expansion in Feynman diagrams where we ignore all diagrams with BF propagators. Before evaluating this product let us illustrate the computations of the relevant diagrams by computing one exemplary diagram in detail.

Consider the following diagram:[29]

$$G_{jl}^{ik}[\triangle\cdot\blacktriangle](x_1,x_2) := \tag{4.64}$$



[28]We have chosen the overall sign of the propagator to make comparision between Feynman diagrams involving bosonic operators and fermionic operators as simple as possible. However, the overall sign is not important for the determination of the algebra. The parameter $\hbar$ enters the algebra as the formal variable deforming the universal enveloping algebra $U(\mathfrak{gl}_K[z])$ to its Yangian, and the sign of $\hbar$ is irrelevant for this purpose.

[29]The reader can ignore the elaborate symbols (triangles and as such) that we use to refer to a diagram. They are meant to systematically identify a diagram, but for practical purposes the entire expression can be thought of as an unfortunately long unique symbol assigned to a diagram, just to refer to it later on.

We are representing the operator $O^i_j[m] = \frac{1}{\hbar}\overline{\psi}^a_j(\mathsf{B}^m)^b_a\psi^i_b$ by the symbol  where the three dots represent the three fields $\overline{\psi}^a_j$, $(\mathsf{B}^m)^b_a$, and ψ^i_b respectively. The coordinate below an operator in (4.64) represents the position of that operator and the lines connecting different dots are propagators. Depending on which dots are being connected a propagator is either the BF propagator (4.53) or the QM propagator (4.59). The value of the diagram is then given by:

$$\begin{aligned} G^{ik}_{jl}[\triangle\cdot\blacktriangle](x_1,x_2) &= \frac{1}{\hbar}\overline{\psi}^a_j(x_1)(\mathsf{B}(x_1)^m)^b_a\frac{1}{2}\hbar\delta^c_b\delta^i_l\frac{1}{\hbar}(\mathsf{B}(x_2)^n)^d_c\psi^k_d(x_2)\,, \\ &= \frac{1}{2\hbar}\delta^i_l\overline{\psi}_j(x_1)\mathsf{B}(x_1)^m\mathsf{B}(x_2)^n\psi^k(x_2)\,. \end{aligned} \tag{4.65}$$

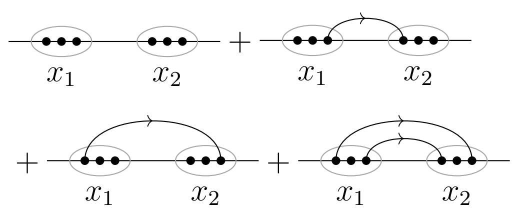
In the second line we have hidden away the contracted $\mathfrak{gl}_N$ indices. In computing the operator product (4.63) only the following limit of the diagram is relevant:

$$\lim_{x_2\to x_1} G^{ik}_{jl}[\triangle\cdot\blacktriangle](x_1,x_2) = \frac{1}{2\hbar}\delta^i_l\overline{\psi}_j\mathsf{B}^{m+n}\psi^k = \frac{1}{2}\delta^i_l O^k_j[m+n]\,. \tag{4.66}$$

We have ignored the positions of the operators, because the algebra we are computing must be translation invariant. Reference to position only matters when we have different operators located at different positions.

We can now give a diagramatic expansion of the operator product (4.63) in the free theory:

$$O^i_j[m]\star O^k_l[n] \overset{x_2\to x1}{=} \tag{4.67}$$



We have omitted the labels for the operators in the diagrams. It is understood that the first operator is $O^i_j[m]$ and the second one is $O^k_l[n]$. Summing these four diagrams we find:

$$O^i_j[m]\star O^k_l[n] = O^i_j[m]O^k_l[n] + \frac{1}{2}\delta^i_l O^k_j[m+n] - \frac{1}{2}\delta^k_j O^i_l[m+n] + \frac{1}{4}\delta^i_l\delta^k_j \mathrm{tr}_{\mathbf{N}}\mathsf{B}^{m+n}\,. \tag{4.68}$$

The product in the first term on the right hand side of the above equation is a c-number product, hence commuting. The sign of the third term comes from the first diagram in the second line in (4.67). In short, this comes about by commuting two fermions, as follows:

$$\lim_{x_2\to x_1} G^{ik}_{jl}[\blacktriangle\cdot\triangle](x_1,x_2) = \frac{1}{2\hbar}\delta^k_j\psi^i\mathsf{B}^{m+n}\overline{\psi}_l = -\frac{1}{2\hbar}\delta^k_j\overline{\psi}_l\mathsf{B}^{m+n}\psi^i = -\frac{1}{2}\delta^k_j O^i_l[m+n]\,. \tag{4.69}$$

Using (4.68) we can compute the Lie bracket of the algebra $\mathcal{A}^{\mathrm{Op}}(\mathcal{T}_{\mathrm{bd}})$ in the classical limit:

$$\left[O^i_j[m], O^k_l[n]\right]_\star = \delta^i_l O^k_j[m+n] - \delta^k_j O^i_l[m+n]\,. \tag{4.70}$$

This is the Lie bracket in the loop algebra $\mathfrak{gl}_K[z]$.[30]

Remark 4.4.2 (Fermion vs. Boson - Classical Algebra). How would the bracket (4.70) be affected if we had a bosonic QM? It would not. The first and the fourth diagrams from (4.67) would still cancel with their counterparts when we take the commutator. The value of the second diagram, (4.66), remains unchanged. In computing the value of the third diagram (see (4.69)) we get an extra sign compared to the fermionic case because we don't pick up any sign by commuting bosonos, however, we pick up yet another sign from the propagator relative to the fermionic propagator (see Remark 4.4.1 – compare the bosonic (4.61) and fermionic (4.59) propagators).

4.4.2 Loop corrections from BF theory

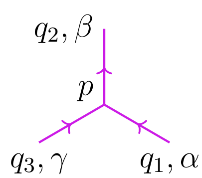
Interaction in the BF theory comes from the following term in the BF action (4.46):

$$f_{\alpha\beta\gamma}\int_{\mathbb{R}^2} \mathsf{B}^\alpha \mathsf{A}^\beta \wedge \mathsf{A}^\gamma\,, \tag{4.72}$$

where the structure constant $f_{\alpha\beta\gamma}$ comes from the trace in our orthonormal basis (4.49):

$$f_{\alpha\beta\gamma} = \mathrm{tr}_{\mathbf{N}}(\tau_\alpha[\tau_\beta, \tau_\gamma])\,. \tag{4.73}$$

In Feynman diagrams this interaction will be represented by a trivalent vertex with exactly 1 outgoing and 2 incoming edges. Including the propagators for the edges, such a vertex will look like:



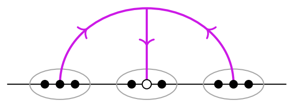
$$\begin{aligned} &= \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma}\int_{p\in\mathbb{R}^2} \mathrm{d}_{q_1}\phi(p,q_1)\wedge \mathrm{d}_{q_2}\phi(p,q_2)\wedge \mathrm{d}_{q_3}\phi(p,q_3)\,, \\ &=: V^{\alpha\beta\gamma}(q_1,q_2,q_3)\,. \end{aligned} \tag{4.74}$$

[30]The isomorphism is given by: $O^i_j[m] \mapsto z^m \mathrm{e}^j_i$, where e^j_i are the elementary matrices of dimension $K\times K$ satisfying the relation:

$$[\mathrm{e}^j_i, \mathrm{e}^l_k] = \delta^l_i \mathrm{e}^j_k - \delta^j_k \mathrm{e}^l_i\,. \tag{4.71}$$

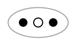
We have given the name $V^{\alpha\beta\gamma}$ to this vertex function.

Possibilities of Feynman diagrams are rather limited in the BF theory. In particular, there are no cycles.[31] This means that there is only one possible BF diagram that will appear in our computations, which is the following:



$$. \tag{4.75}$$

The middle operator looks slightly different because this operator involves the connection A and an integration, as opposed to just the B field, to be specific,



$$= \frac{1}{\hbar}\int_{\mathbb{R}} \overline{\psi}_i \mathsf{A} \psi^i \,. \tag{4.76}$$

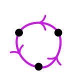
This term is the result of the insertion of the term coupling the fermions to the $\mathfrak{gl}_N$ connection in the QM action (4.47). In doing the above integrationover $\mathbb{R}$ we shall take $\overline{\psi}$ and ψ to be constant. In other words, we are taking derivatives of the fermions to be zero. The reason is that, the equations of motion for the fermions (derived from the action (4.47)), namely $\mathrm{d}\psi^i = -\mathsf{A}\psi^i$ and $\mathrm{d}\overline{\psi}_i = \mathsf{A}\overline{\psi}_i$, tell us that derivatives of the fermions are not gauge-invariant quantities – and we want to expand the operator product of gauge invariant operators in terms of other gauge invariant operators only.[32]

In the following we shall consider the diagram (4.75) with all possible fermionic propagators added to it.

0 fermionic propagators

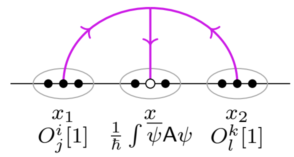
We are mostly going to compute products of level 1 operators, i.e., $O^i_j[1]$, this is because together with the level 0 operators, they generate the entire algebra. Without any fermionic

[31] By cycle we mean loop in the sense of graph theory. In this chapter when we write loop without any explanation, we mean the exponent of $\hbar$, as is customary in physics. This exponent is related but not always equal to the number of loops (graph theory). Therefore, we reserve the word loop for the exponent of $\hbar$, and the word cycle for what would be loop in graph theory.

Let us illustrate why there are no cycles in BF Feynman diagrams. Consider the cycle  . The three propagators in the cycle contribute the 3-form $\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_3$ to a diagram containing the cycle, where the ϕ's are the angles between two successive vertices. However, due to the constraint $\phi_1 + \phi_2 + \phi_3 = 2\pi$, only two out of the three propagators are linearly independent. Therefore, their product vanishes.

[32] An alternative, and perhaps more streamlined, way to say this would be to formulate all the theories in the BV/BRST formalism, where operators are defined, a priori, to be in the cohomology of the BRST operator, which would exclude derivatives of the fermions to begin with.

propagators, we just have the diagram (4.75):

$$G^{ik}_{jl}[\cdot\cdot](x_1, x_2) := $$ 

. (4.77)

In future, we shall omit the labels below the operators to reduce clutter. In terms of the BF vertex function (4.74), the above diagram can be expressed as:

$$G^{ik}_{jl}[\cdot\cdot](x_1, x_2) = \frac{1}{\hbar^3}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}\tau_\beta\psi\overline{\psi}_l\tau_\gamma\psi^k\int_{\mathbb{R}_x} V^{\alpha\beta\gamma}(x_1, x, x_2)\,. \tag{4.78}$$

We have used the expansions of $\mathsf{B} = \mathsf{B}^\alpha\tau_\alpha$ and $\mathsf{A} = \mathsf{A}^\beta\tau_\beta$ in the orthonormal $\mathfrak{gl}_N$ basis $\{\tau_\alpha\}$. As defined in (4.74), the vertex function $V^{\alpha\beta\gamma}$ is a 2d integral of a 3-form, therefore, the integration of the vertex function on a line gives us a number. It will be convenient to divide up the integral of the vertex function into three integrals depending on the location of the point x relative to x_1 and x_2:

$$\int_{\mathbb{R}_x} V^{\alpha\beta\gamma}(x_1, x, x_2) = \mathcal{V}^{\alpha\beta\gamma}_{\cdot||}(x_1, x_2) + \mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}(x_1, x_2) + \mathcal{V}^{\alpha\beta\gamma}_{||\cdot}(x_1, x_2)\,, \tag{4.79}$$

where,

$$\mathcal{V}^{\alpha\beta\gamma}_{\cdot||}(x_1, x_2) := \int_{x<x_1} V^{\alpha\beta\gamma}(x_1, x, x_2) = \frac{\hbar^2}{24}f^{\alpha\beta\gamma}\,, \tag{4.80a}$$

$$\mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}(x_1, x_2) := \int_{x_1<x<x_2} V^{\alpha\beta\gamma}(x_1, x, x_2) = \frac{\hbar^2}{24}f^{\alpha\beta\gamma}\,, \tag{4.80b}$$

$$\mathcal{V}^{\alpha\beta\gamma}_{||\cdot}(x_1, x_2) := \int_{x_2<x} V^{\alpha\beta\gamma}(x_1, x, x_2) = \frac{\hbar^2}{24}f^{\alpha\beta\gamma}\,. \tag{4.80c}$$

We evaluate these integrals in Appendix §E.1. Adding them up and substituting in (4.78) we get from the diagram (4.77):

$$G^{ik}_{jl}[\cdot\cdot](x_1, x_2) \overset{x_1\rightarrow x_2}{=} \frac{1}{8\hbar}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}\tau_\beta\psi\overline{\psi}_l\tau_\gamma\psi^k f^{\alpha\beta\gamma}\,. \tag{4.81}$$

Since the $\mathfrak{gl}_N$ indices are all contracted, we can choose a particular basis to get an expression independent of any reference to $\mathfrak{gl}_N$. Choosing the elementary matrices as the basis we get the following expression:

$$G^{ik}_{jl}[\cdot\cdot] = \frac{\pi^2}{2\hbar}\overline{\psi}_j\mathrm{e}^a_b\psi^i\overline{\psi}\mathrm{e}^c_d\psi\overline{\psi}_l\mathrm{e}^e_f\psi^k f^{bdf}_{ace}\,. \tag{4.82}$$

Using the definition of the elementary matrices $(\mathrm{e}^a_b)^c_d = \delta^a_d \delta^c_b$ we get $\overline{\psi}_j \mathrm{e}^a_b \psi^i = \overline{\psi}^d_j \, (\mathrm{e}^a_b)^c_d \, \psi^i_c = \overline{\psi}^a_j \psi^i_b$ and in this basis the structure constant is:

$$f^{bdf}_{ace} = \delta^d_a \delta^f_c \delta^b_e - \delta^b_c \delta^d_e \delta^f_a \,. \tag{4.83}$$

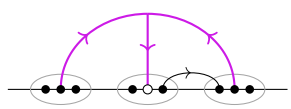
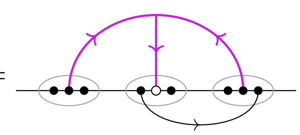
Using these expressions in (4.82) we get:

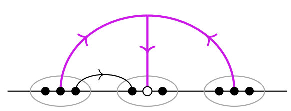
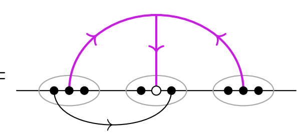
$$\begin{aligned} G^{ik}_{jl}[\cdot\cdot] &= \frac{1}{8\hbar} \left(\overline{\psi}_j \psi^m \, \overline{\psi}_m \psi^k \, \overline{\psi}_l \psi^i - \overline{\psi}_l \psi^m \, \overline{\psi}_m \psi^i \, \overline{\psi}_j \psi^k \right) \,, \\ &= \frac{1}{8} \hbar^2 \left(O^m_j[0] O^k_m[0] O^i_l[0] - O^m_l[0] O^i_m[0] O^k_j[0] \right) \,. \end{aligned} \tag{4.84}$$

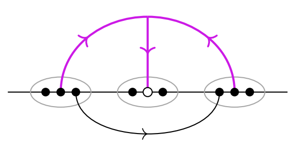
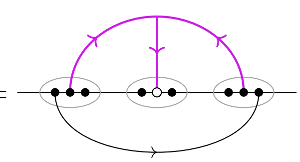
The above expression is anti-symmetric under the exchange $(i, j) \leftrightarrow (k, l)$, therefore, the contribution of this diagram to the Lie bracket (4.70) is twice the value of the diagram.

1 **fermionic propagator**

We have the following six diagrams:

$G^{ik}_{jl}[\cdot \vartriangle \cdot \blacktriangle] =$  , $G^{ik}_{jl}[\cdot \blacktriangle \cdot \vartriangle] =$  ,

$G^{ik}_{jl}[\vartriangle \cdot \blacktriangle \cdot] =$  , $G^{ik}_{jl}[\blacktriangle \cdot \vartriangle \cdot] =$  , (4.85)

$G^{ik}_{jl}[\vartriangle \cdot \cdot \blacktriangle] =$  , $G^{ik}_{jl}[\blacktriangle \cdot \cdot \vartriangle] =$  .

In all the above diagrams, the left and the right most operators are $O^i_j[1]$ and $O^k_l[1]$ respectively, and all the graphs are functions of x_1 and x_2, where these two operators are located. Let us explain the evaluation of the top left diagram in detail. Written explicitly, this diagram is:

$$\begin{aligned} G^{ik}_{jl}[\cdot \vartriangle \cdot \blacktriangle](x_1, x_2) = \frac{1}{\hbar^3} \int_{\mathbb{R}_x} & \overline{\psi}_j(x_1) \tau_\alpha \psi^i(x_1) \overline{\psi}^a_m(x) \, (\tau_\beta)^b_a \left\langle \psi^m_b(x) \overline{\psi}^c_l(x_2) \right\rangle \\ & \times (\tau_\gamma)^d_c \, \psi^k_d(x_2) \, V^{\alpha\beta\gamma}(x_1, x, x_2) \,, \end{aligned} \tag{4.86}$$

where the two point correlation function is the QM propagator (4.59). The integrand above depends on the position only to the extend that they depend on the ordering of the positions, since we are only quantizing the constant modes of the fermions.[33] The propagator between the two fermions gives a propagator which depends on the sign of $x_2 - x$ (see (4.59), (4.60)), since we are integrating over x, this propagator will change sign depending on whether x is to the left or to the right of x_2.[34] Therefore, we can write this graph as:

$$
\begin{aligned}
G^{ik}_{jl}[\cdot\,\triangle\cdot\blacktriangle] &= \frac{1}{\hbar^2}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}_l\tau_\beta\tau_\gamma\psi^k\left(\mathcal{V}^{\alpha\beta\gamma}_{\cdot||}+\mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}-\mathcal{V}^{\alpha\beta\gamma}_{||\cdot}\right)\,,\\
&= \frac{1}{24}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}_l\tau_\beta\tau_\gamma\psi^k\, f^{\alpha\beta\gamma} = \frac{1}{24}\overline{\psi}_j\tau_\alpha\psi^i\overline{\psi}_l\tau_\delta\psi^k\, f_{\beta\gamma}{}^{\delta} f^{\alpha\beta\gamma}\,.
\end{aligned} \tag{4.87}
$$

Due to the symmetry $f_{\beta\gamma}{}^{\delta}f^{\alpha\beta\gamma} = f_{\beta\gamma}{}^{\alpha}f^{\delta\beta\gamma}$, the above expression is symmetric under the exchange $(i,j)\leftrightarrow(k,l)$, therefore this diagram does not contribute to the Lie bracket (4.70). The diagrams $G^{ik}_{jl}[\cdot\blacktriangle\cdot\triangle]$, $G^{ik}_{jl}[\triangle\cdot\blacktriangle\cdot]$, and $G^{ik}_{jl}[\blacktriangle\cdot\triangle\cdot]$ do not contribute to the Lie bracket for exactly the same reason. The remaining two diagrams evaluate to the following expressions:

$$
G^{ik}_{jl}[\triangle\cdot\cdot\blacktriangle] = \frac{1}{8\hbar}f^{\alpha\beta\gamma}\delta^i_l\overline{\psi}_j\tau_\alpha\tau_\gamma\psi^k\overline{\psi}\tau_\beta\psi\,, \tag{4.88a}
$$

$$
G^{ik}_{jl}[\blacktriangle\cdot\cdot\triangle] = -\frac{1}{8\hbar}f^{\alpha\beta\gamma}\delta^k_j\overline{\psi}_l\tau_\gamma\tau_\alpha\psi^i\overline{\psi}\tau_\beta\psi\,. \tag{4.88b}
$$

Their sum is symmetric under the exchange $(i,j)\leftrightarrow(k,l)$,[35] and therefore these diagrams do not contribute to the Lie bracket either.

None of the diagrams with one fermionic propagator contributes to the Lie bracket.

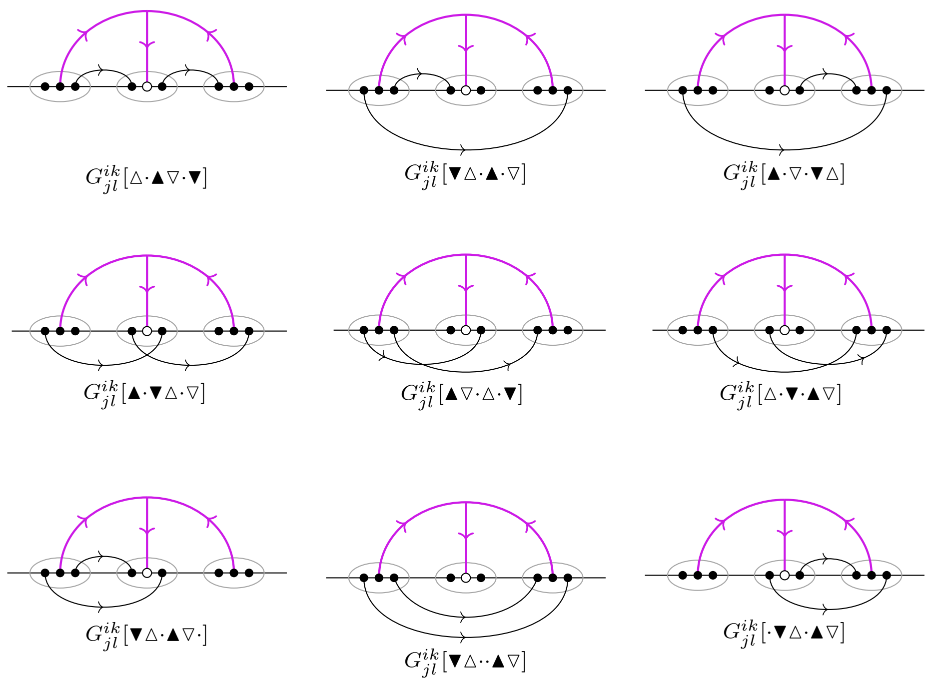
[33] Derivatives of the fermions are not gauge invariant.

[34] This is the reason why we computed the integrals (4.80) separately depending on the position of x.

[35] The opposite ordering of τ_α and τ_γ cancels the sign, using the anti-symmetry of the indices on the structure constant.

2 fermionic propagators

There are nine ways to join two pairs of fermions with propagators:



(4.89)

The left and the right most operators in all of the above diagrams are $O^i_j[1]$ and $O^k_l[1]$ respectively.

All three of the diagrams in the bottom line vanish. This is because joining all the fermions in two operators with propagators introduces a trace $\mathrm{tr}_{\mathbf{N}}(\tau_\alpha\tau_\beta)$ of $\mathfrak{gl}_N$ generators when the same color indices, α and β in this case, are contracted with the structure constant coming form the BF interaction vertex, as in $\mathrm{tr}_{\mathbf{N}}(\tau_\alpha\tau_\beta)f^{\alpha\beta\gamma}$. Since the trace is symmetric and the structure constant is anti-symmetric, these three diagrams vanish.

Computation also reveals the following relations:[36]

$$G^{ik}_{jl}[\blacktriangledown\,\triangle\cdot\blacktriangle\triangledown] = G^{ik}_{jl}[\blacktriangle\cdot\triangledown\cdot\blacktriangledown\,\triangle]\,,\quad G^{ik}_{jl}[\blacktriangle\triangledown\cdot\triangle\cdot\blacktriangledown] = G^{ik}_{jl}[\triangle\cdot\blacktriangledown\cdot\blacktriangle\triangledown]\,, \tag{4.90}$$

together with the fact that $G^{ik}_{jl}[\blacktriangledown\,\triangle\cdot\blacktriangle\cdot\triangledown]+G^{ik}_{jl}[\blacktriangle\triangledown\cdot\triangle\cdot\blacktriangledown]$ is symmetric under the exchange $(i,j)\leftrightarrow(k,l)$. The above relations and symmetry implies that when anti-symmetrized with respect to $(i,j)\leftrightarrow(k,l)$, the sum of the four diagrams appearing in the above relations

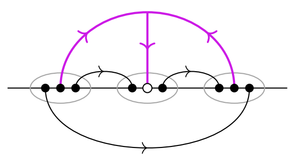
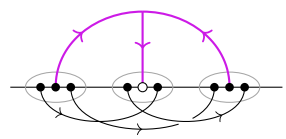
[36]Among the four diagrams at the top right 2×2 corner of (4.89).

vanish. In a similar vein, the sum $G^{ik}_{jl}[\triangle \cdot \blacktriangle\triangledown \cdot \blacktriangledown] + G^{ik}_{jl}[\blacktriangle \cdot \blacktriangledown\ \triangle \cdot \triangledown]$ also turns out to be symmetric under $(i,j) \leftrightarrow (k,l)$ and therefore these two diagrams do not contribute to the Lie bracket either.

None of the diagrams with two fermionic propagators contributes to the Lie bracket.

3 fermionic propagators

There are two ways to join all the fermions with propagators:

 ,  (4.91)

As before, the left and the right most operators are $O^i_j[1]$ and $O^k_l[1]$ respectively. Both of these diagrams are proportional to $\delta^i_l\delta^k_j$, in particular, they are symmetric under the exchange $(i,j) \leftrightarrow (k,l)$, and therefore do not contribute to the Lie bracket.

Lie bracket

Since only the diagram with zero fermionic propagator (4.84) survives the anti-symmetrization, the Lie bracket (4.70) up to $\mathcal{O}(\hbar^2)$ corrections becomes:

$$\begin{aligned}\left[O^i_j[1], O^k_l[1]\right]_\star &= \delta^i_l O^k_j[2] - \delta^k_j O^i_l[2] + G^{ik}_{jl}[\cdot\cdot] - G^{ki}_{lj}[\cdot\cdot]\,, \\ &= \delta^i_l O^k_j[2] - \delta^k_j O^i_l[2] + \frac{\hbar^2}{4}\left(O^m_j[0]O^k_m[0]O^i_l[0] - O^m_l[0]O^i_m[0]O^k_j[0]\right)\,. \end{aligned} \tag{4.92}$$

Though we have only computed up to 2-loops diagrams, this result is exact, because there are no more non-vanishing Feynman diagrams that can be drawn.

Since (4.92) is not among the standard relations of the Yangian that are readily available in the literature, we shall now make a change of basis to get to a standard relation. First note that, the product of operators in the right hand side of the above equation is not the operator product, this product is commutative (anti-commutative for fermions) and therefore we can write it in an explicitly symmetric form, such as:

$$O^m_j[0]O^k_m[0]O^i_l[0] = \left\{O^m_j[0], O^k_m[0], O^i_l[0]\right\}\,, \tag{4.93}$$

where the bracket means complete symmetriazation, i.e., for any three symbols O_1, O_2 and O_3 with a product we have:

$$\{O_1, O_2, O_3\} = \frac{1}{3!}\sum_{s\in S_3} O_{s(1)}O_{s(2)}O_{s(3)}\,, \tag{4.94}$$

where S_3 is the symmetric group of order 3!. With this symmetric bracket, let us now define:

$$Q^{ik}_{jl} := f^{iun}_{jvm} f^{vpq}_{uor} f^{rtk}_{qsl} \left\{O^m_n[0], O^o_p[0], O^s_t[0]\right\}\,, \tag{4.95}$$

where f^{ijk}_{lmn} are the $\mathfrak{gl}_K$ structure constants in the basis of elementary matrices. Using the form of the $\mathfrak{gl}$ structure constant in the basis of elementary matrices (c.f. (4.83)) we can write:

$$Q^{ik}_{jl} = 3\left\{O^i_l, O^m_j, O^k_m\right\} - 3\left\{O^k_j, O^m_l, O^i_m\right\} + \delta^k_j\left\{O^m_l, O^n_m, O^i_n\right\} - \delta^i_l\left\{O^m_j, O^n_m, O^k_n\right\}\,. \tag{4.96}$$

We have ignored to write the [0] for each of the operators. Using the above expression we can re-write (4.92) as:

$$\left[O^i_j[1], O^k_l[1]\right]_\star = \delta^i_l \widetilde{O}^k_j[2] - \delta^k_j \widetilde{O}^i_l[2] + \frac{\hbar^2}{12}Q^{ik}_{jl}\,, \tag{4.97}$$

with the redefinition:

$$\widetilde{O}^k_j[2] := O^k_j[2] - \frac{\hbar^2}{12}\left\{O^m_j, O^n_m, O^k_n\right\}\,. \tag{4.98}$$

Note that, $\left\{O^m_j, O^n_m, O^k_n\right\}$ does indeed transform as an element of $\mathfrak{gl}_K$, since it only has a pair of fundamental-anti-fundamental $\mathfrak{gl}_K$ indices free. This makes the redefinition of $O^k_j[2]$ possible. The Lie bracket (4.97) is how the Yangian was presented in [43].

Remark 4.4.3 (Fermion vs. Boson - Quantum Algebra)*.* In Remark 4.4.2 we pointed out that the classical part of the algebra (4.97) remains unchanged if we replace the fermionic QM on the defect with a bosonic QM. This remains true at the quantum level – though a bit tedious, it can be readily verified by using the bosonic propagator (4.61) and keeping track of signs through the computations of this section without any other modifications. △

4.4.3 Large N limit: The Yangian

For finite N, there are some extra relations among the operators $O^i_j[n]$ that are not part of the Yangian algebra. These relations are simply a result of having finite dimensional matrices. We start by noting that the operators $O^i_j[m]$ act on the Hilbert space $\mathcal{H}^{\text{fer}}_{\text{QM}}$ of the quantum mechanics. This is a finite dimensional Hilbert space constructed by acting with the fermionic zero modes on the vacuum of the theory:

$$\mathcal{H}^{\text{fer}}_{\text{QM}} = \mathbb{C}|\Omega\rangle \oplus \bigoplus_{i,a} \mathbb{C}\psi^i_a|\Omega\rangle \oplus \bigoplus_{i,j,a,b} \mathbb{C}\psi^i_a\psi^j_b|\Omega\rangle + \cdots . \tag{4.99}$$

Considering the GL_N and GL_K indices on the fermions this Hilbert space can be decomposed into tensor products of representations of GL_K and GL_N as follows (see (E.9)):

$$\mathcal{H}^{\text{fer}}_{\text{QM}} = \bigoplus_{Y} \mathcal{H}^N_{Y^T} \otimes \overline{\mathcal{H}^K_Y} , \tag{4.100}$$

where Y is a Young tableaux, Y^T is the transpose of Y, $\mathcal{H}^N_{Y^T}$ (resp. $\mathcal{H}^K_Y$) is the GL_N (resp. GL_K) representation associated to the tableaux Y^T (resp. Y), and a bar over a representation denotes its dual. Any $d \times d$ matrix X satisfies a degree d polynomial equation:[37]

$$X^d = \sum_{i=0}^{d-1} c_i X^i . \tag{4.101}$$

Therefore, all the operators $O^i_j[m]$ satisfy some polynomial equation of degree $\dim \mathcal{H}^{\text{fer}}_{\text{QM}}$. Since the matrix B is an $N \times N$ matrix there are relations among its different powers, which can lead to relations among operators of the QM as well. In the limit $N \to \infty$ we do not need to worry about such truncations of the Yangian and we have the full Yangian. This positively concludes the first half of our main result (Theorem 4.3.3).

Remark 4.4.4 (Fermion vs. Boson – Hilbert Space). The Hilbert space as a representation of $\mathrm{GL}_N \times \mathrm{GL}_K$ differs between the fermionic description of the defect QM and the bosonic description. The fermionic Hilbert space (4.100) is finite dimensional because of the anti-symmetry of the fermionic generators. There is no such exclusion principle for the bosons and the bosonic Hilbert space is infinite dimensional. The bosonic Hilbert space is (see (E.13)):

$$\mathcal{H}^{\text{bos}}_{\text{QM}} = \bigoplus_{Y} \mathcal{H}^N_{Y} \otimes \overline{\mathcal{H}^K_Y} , \tag{4.102}$$

where $\mathcal{H}^N_Y$ and $\mathcal{H}^K_Y$ are representations of GL_N and GL_K denoted by the same tableaux Y. △

[37] The relation between the coefficients appearing in (4.101) and X is the following [148]: if X has the characteristic polynomial $\sum_{i=0}^{d} a_i x^{d-i}$ with $a_0 = 1$ and u_i satisfy the recurrence relation $\sum_{i=0}^{d} a_i u_{d-i} = 0$, then $c_i = \sum_{j=i}^{d-1} a_{j-i} u_{d-j}$.

4.5 $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ from 4d Chern-Simons Theory

In this section we prove the second half of our main result (Theorem 4.3.3):

Proposition 4.5.1. *The algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, defined in (4.41) in the context of 4d Chern-Simons theory, is isomorphic to the Yangian $Y_\hbar(\mathfrak{gl}_K)$:*

$$\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}}) \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K)\,. \tag{4.103}$$

The 4d Chern-Simons theory with gauge group GL_K, also denoted by CS^4_K, is defined by the action (4.29), which we repeat here for convenience:

$$S_{\mathrm{CS}} := \frac{i}{2\pi}\int_{\mathbb{R}^2_{x,y}\times\mathbb{C}_z} \mathrm{d}z\wedge \mathrm{tr}_{\mathbf{K}}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right)\,. \tag{4.104}$$

The trace in the fundamental representation defines a positive-definite metric on $\mathfrak{gl}_K$, moreover, we choose a basis of $\mathfrak{gl}_K$, denoted by $\{t_\mu\}$, in which the metric becomes diagonal:

$$\mathrm{tr}_{\mathbf{K}}(t_\mu t_\nu) \propto \delta_{\mu\nu}\,. \tag{4.105}$$

We consider this theory in the presence of a Wilson line in some representation $\varrho : \mathfrak{gl}_K \to \mathrm{End}(V)$, supported along the line L defined by $y = z = 0$:

$$W_\varrho(L) = P\exp\left(\int_L \varrho(A)\right)\,. \tag{4.106}$$

Consideration of fusion of Wilson lines to give rise to Wilson lines in tensor product representation shows that it is not only the connection A that couples to a Wilson line but also its derivatives $\partial_z^n A$ [43]. Furthermore, gauge invariance at the classical level requires that $\partial_z^n A$ couples to the Wilson line via a representation of the loop algebra $\mathfrak{gl}_K[z]$. So the line operator that we consider is the following:

$$P\exp\left(\sum_{n\geq 0}\varrho_{\mu,n}\int_L \partial_z^n A^\mu\right)\,, \tag{4.107}$$

where the matrices $\varrho_{\mu,n}\in\mathrm{End}(V)$ satisfy:

$$[\varrho_{\mu,m},\varrho_{\nu,n}] = f_{\mu\nu}{}^{\xi}\varrho_{\xi,m+n}\,. \tag{4.108}$$

The structure constant ${f_{\mu\nu}}^{\xi}$ is that of $\mathfrak{gl}_K$. In particular, we have $\varrho_{\mu,0} = \varrho(t_\mu)$.

In (4.28), A was defined to not have a $\mathrm{d}z$ component. The reason is that, due to the appearance of $\mathrm{d}z$ in the above action (4.104), the $\mathrm{d}z$ component of the connection A never appears in the action anyway.[38]

Though the theory is topological, in order to do concrete computations, such as imposing gauge fixing conditions, computing propagator, and evaluating Witten diagrams etc. we need to make a choice of metric on $\mathbb{R}^2_{x,y} \times \mathbb{C}_z$, we choose:[39]

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z\mathrm{d}\overline{z}\,. \tag{4.111}$$

For the GL_K gauge symmetry we use the following gauge fixing condition:

$$\partial_x A_x + \partial_y A_y + 4\partial_z A_{\overline{z}} = 0\,. \tag{4.112}$$

The propagator is defined as the two-point correlation function:

$$P^{\mu\nu}(v_1, v_2) := \langle A^\mu(v_1) A^\nu(v_2)\rangle\ . \tag{4.113}$$

Since in the basis of our choice the Lie algebra metric is diagonal (4.105), this propagator is proportional to a Kronecker delta in the Lie algebra indices:

$$P^{\mu\nu}(v_1, v_2) = \delta^{\mu\nu} P(v_1, v_2)\,, \tag{4.114}$$

where P is a 2-form on $\mathbb{R}^4_{v_1} \times \mathbb{R}^4_{v_2}$. We can fix one of the coordinates to be the origin, this amounts to taking the projection:

$$\varpi : \mathbb{R}^4_{v_1} \times \mathbb{R}^4_{v_1} \to \mathbb{R}^4_v\,, \qquad \varpi : (v_1, v_2) \mapsto v_1 - v_2 =: v\,. \tag{4.115}$$

Due to translation invariance, P can be written as a pullback of some 2-form on $\mathbb{R}^4$ by ϖ, i.e., $P = \varpi^* \overline{P}$ for some $\overline{P} \in \Omega^2(\mathbb{R}^4)$. The propagator P can be characterized as the

[38] Had we defined the space of connections to be $\Omega^1(\mathbb{R}^2_{x,y} \times \mathbb{C}_z) \otimes \mathfrak{gl}_K$, then, in addition to the usual GL_K gauge symmetry, we would have to consider the following additional gauge transformation:

$$A \to A + f\mathrm{d}z\,, \tag{4.109}$$

for arbitrary function $f \in \Omega^0(\mathbb{R}^2 \times \mathbb{C})$. We could fix this gauge by imposing:

$$A_z = 0\,. \tag{4.110}$$

This would get us back to the space $\big(\Omega^1(\mathbb{R}^2_{x,y} \times \mathbb{C}_z)/(\mathrm{d}z)\big) \otimes \mathfrak{gl}_K$.

[39] For this theory we follow the choices of [43] whenever possible.

Green's function for the differential operator $\frac{i}{2\pi\hbar}\mathrm{d}z \wedge \mathrm{d}$ that appears in the kinetic term of the action S_{CS}. For $\overline{P}$ this results in the following equation:

$$\frac{i}{2\pi\hbar}\mathrm{d}z \wedge \mathrm{d}\overline{P}(v) = \delta^4(v)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}\overline{z}\,, \tag{4.116}$$

The propagator P, and in turns $\overline{P}$, must also satisfy the gauge fixing condition (4.112):

$$\partial_x \overline{P}_x + \partial_y \overline{P}_y + 4\partial_z \overline{P}_{\overline{z}} = 0\,. \tag{4.117}$$

The solution to (4.116) and (4.117) is given by:

$$\overline{P}(x,y,z,\overline{z}) = \frac{\hbar}{2\pi}\frac{x\,\mathrm{d}y \wedge \mathrm{d}\overline{z} + y\,\mathrm{d}\overline{z} \wedge \mathrm{d}x + 2\overline{z}\,\mathrm{d}x \wedge \mathrm{d}y}{(x^2+y^2+z\overline{z})^2}\,. \tag{4.118}$$

The propagator $P(v_1, v_2)$ will be referred to as the *bulk-to-bulk* propagator, since the points v_1 and v_2 can be anywhere in the world-volume $\mathbb{R}^2_{x,y} \times \mathbb{C}_z$ of CS theory. To compute Witten diagrams we also need a *boundary-to-bulk* propagator. We will denote it as $\mathsf{K}_\mu(v,x) \equiv \mathsf{K}(v,x)t_\mu$, where $v \in \mathbb{R}^2_{x,y} \times \mathbb{C}_z$ and $x \in \ell_\infty(z)$ is restricted to the boundary line. The boundary-to-bulk propagator is a 1-form defined as a solution to the classical equation of motion:

$$\mathrm{d}z_v \wedge \mathrm{d}_v \mathsf{K}(v,x) = 0\,, \tag{4.119}$$

and by the condition that when pulled back to the boundary, in this case $\ell_\infty(z)$, it must become a delta function supported at x:

$$\varepsilon^* \mathsf{K}(x', x) = \delta^1(x'-x)\mathrm{d}x'\,, \qquad x' \in \ell_\infty(z) \tag{4.120}$$

where $\varepsilon : \ell_\infty(z) \hookrightarrow \mathbb{R}^2 \times \mathbb{C}$ is the embedding of the line in the larger 4d world-volume. As our boundary-to-bulk propagator we choose the following:

$$\mathsf{K}(v,x) = \mathrm{d}_v \theta(x_v - x) = \delta^1(x_v - x)\mathrm{d}x_v\,, \tag{4.121}$$

where x_v refers to the x-coordinate of the bulk point v. The function θ is the following step function:

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases}\,. \tag{4.122}$$

Note that we have functional derivatives with respect to $\partial_z^n A$ for $n \in \mathbb{N}_{\geq 0}$. The propagator (4.121) corresponds to the functional derivative with $n = 0$. Let us denote the propagator corresponding to $\frac{\delta}{\delta \partial_z^n A}$, more generally, as K_n, and for $n \geq 0$, we modify the condition (4.120) by imposing:

$$\lim_{v \to x'} \varepsilon^* \partial_z^n \mathsf{K}(v,x) = \delta^1(x'-x)\mathrm{d}x' \,, \qquad x' \in \ell_\infty(z) \,. \tag{4.123}$$

This leads us to the following generalization of (4.121):

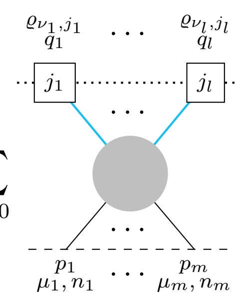
$$\mathsf{K}_n(v,x) = z_v^n \delta^1(x_v - x)\mathrm{d}x_v \,. \tag{4.124}$$

Apart from the two propagators, we shall need the coupling constant of the theory to compute Witten diagrams. The coupling constant of this theory can be read off from the interaction term in the action S_{CS}, it is:

$$\frac{i}{2\pi\hbar} f_{\mu\nu}{}^{\xi} \mathrm{d}z \,. \tag{4.125}$$

Now we can give a diagrammatic definition of the operators in the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, namely the ones defined in (4.39), and their products:

$$T_{\mu_1}[n_1](p_1) \cdots T_{\mu_m}[n_m](p_m) = \sum_{l=1}^{\infty} \sum_{j_i \geq 0}$$



$. \qquad (4.126)$

Let us clarify some points about the picture. We have replaced the pair of fundamental-anti-fundamental indices on T with a single adjoint index. The bottom horizontal line represents the boundary line $\ell_\infty(z)$, and the top horizontal line represents the Wilson line in representation $\varrho : \mathfrak{gl}_K \to V$ at $y = 0$. The sum is over the number of propagators attached to the Wilson line and all possible derivative couplings. The orders of the derivatives are mentioned in the boxes. The points $q_1 \leq \cdots \leq q_l$ on the Wilson line are all integrated along the line without changing their order. The gray blob represents a sum over all possible graphs consistent with the external lines. We use different types of lines to represent different entities:

$$\begin{aligned}
\text{Bulk-to-bulk propagator, } P(v_1, v_2) &= \; v_1 \text{ ——— } v_2 \;, \\
\text{Boundary-to-bulk propagator, } \mathsf{K}(v,x) &= \; v \text{ ——— } x \;, \\
\text{The boundary line } \ell_\infty(z) &\; : \; \text{- - - - -} \,, \\
\text{Wilson line} &\; : \; \cdots\cdots\cdots \,.
\end{aligned} \tag{4.127}$$

The labels μ_i, n_i below the points along the boundary line implies that the corresponding boundary-to-bulk propagator is $\mathsf{K}_{n_i} = z^{n_i}\mathsf{K}$ and that it carries a $\mathfrak{gl}_K$-index μ_i. Finally, the jth derivative of A^ν couples to the Wilson line via the matrix $\varrho_{\nu,j}$. Such a diagram with m boundary-to-bulk propagators and l bulk-to-bulk propagators attached to the Wilson lines will be evaluated to an element of $\mathrm{End}(V)$ which will schematically look like:

$$(\Gamma_{m\to l})^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_m}\,\varrho_{\mu_1,j_1}\cdots\varrho_{\mu_l,j_l}\,, \tag{4.128}$$

where $(\Gamma_{m\to l})^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_l}$ is a number that will be found by evaluating the Witten diagram. Since the bulk-to-bulk propagator (4.118) is proportional to $\hbar$ and the interaction vertex (4.125) is proportional to $\hbar^{-1}$, each diagram will come with a factor of $\hbar$ that will be related to the Euler character of the graph.[40] In the following we start computing diagrams starting from $\mathcal{O}(\hbar^0)$ and up to $\mathcal{O}(\hbar^2)$, by the end of which we shall have proven the main result (Proposition 4.5.1) of this section.

Remark 4.5.1 (Diagrams as $m \to l$ maps, and deformation). Each $m \to l$ Witten diagram that appears in sums such as (4.126) can be interpreted as a map whose image is the value of the diagram:

$$\begin{aligned}\Gamma_{m\to l} &: \bigotimes_{i=1}^{m} z^{n_i}\mathfrak{gl}_K \to \bigotimes_{i=1}^{l} z^{j_i}\mathfrak{gl}_K \to \mathrm{End}(V)\,,\\ \Gamma_{m\to l} &: \bigotimes_{i=1}^{m} z^{n_i}t_{\mu_i} \mapsto (\Gamma_{m\to l})^{\mu_1\cdots\mu_l}_{\nu_1\cdots\nu_m}\,\varrho_{\mu_1,j_1}\cdots\varrho_{\mu_l,j_l}\,.\end{aligned} \tag{4.129}$$

As we shall see explicitly in our computations, diagrams in (4.126) without loops (diagrams of $\mathcal{O}(\hbar^0)$) define an associative product that leads to classical algebras such as $\mathrm{U}(\mathfrak{gl}_K[z])$. However, there are generally more diagrams in (4.126) involving loops (diagrams of $\mathcal{O}(\hbar)$ and higher order) that change the classical product to something else. Since loops in Witten or Feynman diagrams are the essence of the quantum interactions, classical algebras deformed by such loop diagrams are aptly called quantum groups (of course, why they are called groups is a different story entirely [30].) $\triangle$

[40] In a Feynman diagram all propagators are proportional to $\hbar$ and the power of $\hbar$ of a diagram relates simply to the number of faces of the diagram, which is why $\hbar$ is called the loop counting parameter. In a Witten diagram the boundary-to-bulk propagators do not carry any $\hbar$ and therefore the power of $\hbar$ depends also on the number of boundary-to-bulk propagators. However, we are going to ignore this point and simply refer to the power of $\hbar$ in a diagram as the loop order of that diagram.

4.5.1 Relation to anomaly of Wilson line

As we shall compute relevant Witten diagrams of the 4d Chern-Simons theory in detail in later sections, we shall find that the computations are essentially similar to the computations of gauge anomaly of the Wilson line [43] in this theory. This of course is not a coincidence. To see this, let us consider the variation of the expectation value of the Wilson line, $\langle W_\varrho(L)\rangle_A$, as we vary the connection A along the boundary line $\ell_\infty(z)$:

$$\delta \langle W_\varrho(L)\rangle_A = \sum_{n=0}^{\infty} \int_{p\in\ell_\infty(z)} \frac{\delta}{\delta\partial_z^n A^\mu(p)} \langle W_\varrho(L)\rangle_A \,\delta\partial_z^n A^\mu(p)\,. \tag{4.130}$$

Let us make the following variation:

$$\delta\partial_z A^\mu(x) = \delta^1(x-p)\eta^\mu = \mathrm{d}_x\theta(x-p)\eta^\mu\,, \tag{4.131}$$

for some fixed Lie algebra element $\eta^\mu t_\mu \in \mathfrak{gl}_K$. Then we find:

$$\delta \langle W_\rho(L)\rangle_A = \frac{\delta}{\delta\partial_z A^\mu(p)} \langle W_\varrho(L)\rangle_A \,\eta^\mu\,. \tag{4.132}$$

An exact variation of the boundary value of the connection is like a gauge transformation that does not vanish at the boundary. In [43] it was proved that such a variation of the connection leads to a variation of the Wilson line which is a local functional supported on the line:

$$\delta \langle W_\varrho(L)\rangle_A = \left([\varrho_{\mu,1},\varrho_{\nu,1}] + \Theta_{\mu,1,\nu,1}\right) \int_L \partial_z A^\mu \partial_z \mathsf{c}^\nu\,, \tag{4.133}$$

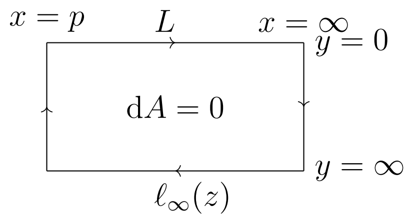
where c was the generator of the gauge transformation:

$$\partial_z \mathrm{d}\mathsf{c}^\mu = \delta\partial_z A^\mu\,, \tag{4.134}$$

$\rho_{\mu,1} \in \mathrm{End}(V)$ is part of the representation of $\mathfrak{gl}_K[z]$ that couples $\partial_z A^\mu$ to the Wilson line (see (4.107)), and $\Theta_{\mu,1,\nu,1}$, which is anti-symmetric in μ and ν, is a matrix that acts on V. Variations such as the above measure gauge anomaly associated to the line, though in our case it is not an anomaly since we are varying the connection at the boundary, and such "large gauge" transformations are not actually part of the gauge symmetry of the theory. The matrix $\Theta_{\mu,1,\nu,1}$ which signals the presence of anomaly is not an arbitrary matrix and in [43], all constraints on this matrix were worked out, we shall not need them at the moment. Comparing with (4.131) we see that for us $\partial_z c^\mu(x) = \theta(x-p)\eta^\mu$, which leads to:

$$\delta \langle W_\varrho(L)\rangle_A = \left(f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\right) \int_{x>p} \partial_z A^\mu \eta^\nu\,, \tag{4.135}$$

where we have used the fact that the matrices $\varrho_{\mu,1}$ satisfy the loop algebra (4.108). The integral above is along L. The connection A above is a background connection satisfying the equation of motion, i.e., it is flat. Since the D4 world-volume, even in the presence of a Wilson line, has no non-contractible loop, all flat connections are exact. Symmetry of world-volume dictates in particular that the connection must also be translation invariant along the direction of the Wilson line L. By considering the integral of A along the following rectangle:



(4.136)

and using translation invariance in the x-direction along with Stoke's theorem, we can change the support of the integral in (4.135) from L to $\ell_\infty(z)$, to get:

$$\delta \left\langle W_\varrho(L)\right\rangle_A = \left(f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\right)\int_{\ell_\infty(z)\ni x>p} \partial_z A^\mu \eta^\nu \,. \tag{4.137}$$

Comparing with (4.132) we find:

$$\frac{\delta}{\delta\partial_z A^\nu(p)} \left\langle W_\varrho(L)\right\rangle_A = \left(f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\right)\int_{x>p} \partial_z A^\mu \,, \tag{4.138}$$

where the integral is now along the boundary line $\ell_\infty(z)$. This leads to the following relation between our algebra and anomaly:

$$\begin{aligned}&[T_\mu[1], T_\nu[1]]\\ &= \lim_{\iota\to 0}\left[\frac{\delta}{\delta\partial_z A^\mu(p+\iota)}\frac{\delta}{\delta\partial_z A^\nu(p)} - \frac{\delta}{\delta\partial_z A^\nu(p)}\frac{\delta}{\delta\partial_z A^\mu(p+\iota)}\right]\left\langle W_\varrho(L)\right\rangle_A\\ &= f_{\mu\nu}{}^{\xi}\varrho_{\xi,2} + \Theta_{\mu,1,\nu,1}\,.\end{aligned} \tag{4.139}$$

The first term with the structure constant gives us the loop algebra $\mathfrak{gl}_K[z]$, which is the classical result. The anomaly term is the result of 2-loop dynamics [43], i.e., it is proportional to $\hbar^2$. This term gives the quantum deformation of the classical loop algebra. This also explains why our two loop computation of the algebra is similar to the two loop computation of anomaly from [43].

At this point, we note that we can actually just use the result of [43] to find out what $\Theta_{\mu,1,\nu,1}$ is and we would find that the deformed algebra of the operators $T^\mu[n]$ is indeed the Yangian $Y_\hbar(\mathfrak{gl}_K)$. However, we think it is illustrative to derive this result from a direct computation of Witten diagrams.

4.5.2 Classical algebra, $\mathcal{O}(\hbar^0)$

Lie bracket

We denote a diagram by $\Gamma^d_{n\to m}$ when there are n boundary-to-bulk propagators, m propagators attached to the Wilson line, and the diagram is of order $\hbar^d$. If there are more than one such diagrams we denote them as $\Gamma^d_{n\to m,i}$ with $i = 1, \cdots$.

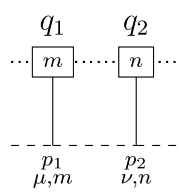
Our aim in this section is to compute the product $T_\mu[m](p_1)T_\nu[n](p_2)$ and eventually the commutator

$$[T_\mu[m], T_\nu[n]] := \lim_{p_2\to p_1} \left(T_\mu[m](p_1)T_\nu[n](p_2) - T_\nu[n](p_1)T_\mu[m](p_2)\right) , \tag{4.140}$$

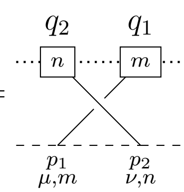
at 0-loop.[41]

We have the following two $2 \to 2$ diagrams:

$$\Gamma^0_{2\to2,1}\left(\begin{smallmatrix}p_1\\ \mu,m\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,n\end{smallmatrix}\right) =$$



$$, \qquad \Gamma^0_{2\to2,2}\left(\begin{smallmatrix}p_1\\ \mu,m\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,n\end{smallmatrix}\right) =$$



$$, \tag{4.141}$$

where a label m in a box on the Wilson line refers to the coupling between the Wilson line and the mth derivative of the connection. The first diagram evaluates to (note that $p_1 < p_2$ and $q_1 < q_2$):

$$\begin{aligned}\Gamma^0_{2\to2,1}\left(\begin{smallmatrix}p_1\\ \mu,m\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,n\end{smallmatrix}\right) &= \int_{q_1<q_2} \mathrm{d}q_1\mathrm{d}q_2\, \delta^1(q_1-p_1)\delta^1(q_2-p_2)\varrho^\mu_m\varrho^\nu_n\,,\\ &= \varrho_{\mu,m}\varrho_{\nu,n}\,,\end{aligned} \tag{4.142}$$

and the second one (with $p_1 < p_2$ and $q_1 > q_2$):

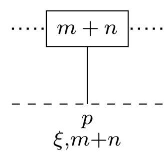
$$\begin{aligned}\Gamma^0_{2\to2,2}\left(\begin{smallmatrix}p_1\\ \mu,m\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,n\end{smallmatrix}\right) &= \int_{q_1>q_2} \mathrm{d}q_2\mathrm{d}q_1\, \delta^1(q_1-p_1)\delta^1(q_2-p_2)\varrho_{\nu,n}\varrho_{\mu,m}\,,\\ &= 0\,.\end{aligned} \tag{4.143}$$

[41] $[T_\mu[m](p_1), T_\nu[n](p_1)]$ may be a more accurate notation but this algebra must be position invariant and therefore we shall ignore the position. Reference to the position only matters when different operators are positioned at different locations.

Therefore their contribution to the commutator is:

$$[T_\mu[m], T_\nu[n]] = \lim_{p_2 \to p_1} \left(\Gamma^0_{2\to 2,1} \left({}^{p_1}_{\mu,m} ; {}^{p_2}_{\nu,n} \right) - \Gamma^0_{2\to 2,1} \left({}^{p_1}_{\nu,n} ; {}^{p_2}_{\mu,m} \right) \right) ,$$
$$= [\varrho_{\mu,m}, \varrho_{\nu,n}] = f_{\mu\nu}{}^{\xi} \varrho_{\xi,m+n} = f_{\mu\nu}{}^{\xi} T_\xi[m+n] , \qquad (4.144)$$

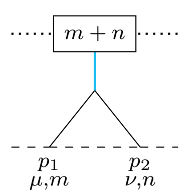
where the last equality is established by evaluating the diagram:



. (4.145)

The bracket (4.144) is precisely the Lie bracket in the loop algebra $\mathfrak{gl}_K[z]$. Note in passing that had we considered the same diagrams as the ones in (4.141) except with different derivative couplings at the Wilson line then the diagrams would have vanished, either because there would be more z-derivatives than z, or there would be less, in which case there would be z's floating around which vanish along the Wilson line located at $y = z = 0$.

There is one $2 \to 1$ diagram as well:



, (4.146)

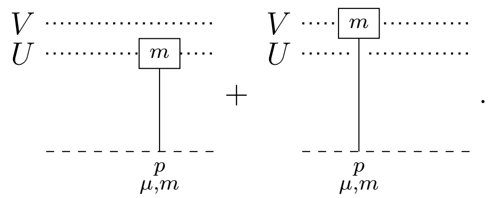
however, since the two boundary-to-bulk propagators are two parallel delta functions,[42] they never meet in the bulk and therefore the diagram vanishes. There are no more classical diagrams, so the Lie bracket in the classical algebra is just the bracket in (4.144).

Coproduct

Apart from the Lie algebra structure, the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ also has a coproduct structure. This can be seen by considering the Wilson line in a tensor product representation, say $U \otimes V$. Such a Wilson line can be produced by considering two Wilson lines in representations U and V respectively and bringing them together, and asking how $T_\mu[n]$ acts on $U \otimes V$.

[42] i.e., their support are restricted to $x = p_1$ and $x = p_2$ respectively with $p_1 \neq p_2$, so they never intersect.

Since there are going to be multiple vector spaces in this section, let us distinguish the actions of $T_\mu[n]$ on them by a superscript, such as, $T^U_\mu[n]$, $T^V_\mu[n]$, etc. At the classical level the answer to the question we are asking is simply given by computing the following diagrams:



(4.147)

Evaluation of these diagrams is very similar to that of the diagrams in (4.141) and the result is:

$$T^{U\otimes V}_\mu[m] = T^U_\mu[m] \otimes \mathrm{id}_V + \mathrm{id}_U \otimes T^V_\mu[m]\,. \tag{4.148}$$

This is the same coproduct structure as that of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_K[z])$.

Combining the results of this section and the previous one we find that, at the classical level we have an associative algebra with generators $T_\mu[n]$ with a Lie bracket and coproduct given by the Lie bracket of the loop algebra $\mathfrak{gl}_K[z]$ and the coproduct of its universla enveloping algebra. This identifies $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, clasically, as the universal enveloping algebra itself:

Lemma 4.5.2. *The large N limit of the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ at the classical level is the universal enveloping algebra $U(\mathfrak{gl}_K[z])$:*

$$\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})/\hbar \overset{N\to\infty}{\cong} U(\mathfrak{gl}_K[z]) \cong Y_\hbar(\mathfrak{gl}_K)/\hbar\,. \tag{4.149}$$

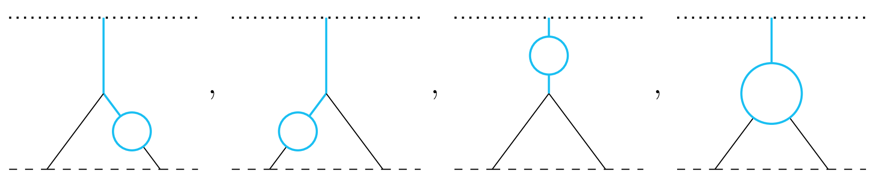
The reason why we need to take the large N limit is that, the operators $T_\mu[m]$ acts on a vector space which is finite dimensional for finite N. This leads to some extra relations in the algebra, which we can get rid of in the large N limit. A similar argument was presented for the operator algebra coming from the BF theory in §4.4.3 and the argument in the context of the CS theory will be explained in more detail in §4.5.4.

4.5.3 Loop corrections

1**-loop,** $\mathcal{O}(\hbar)$

Now we want to compute 1-loop deformation to both the Lie algebra structure and the coproduct structure of $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$.

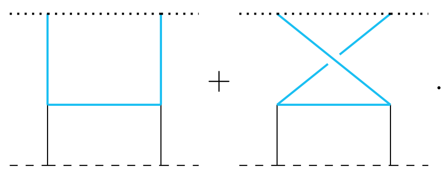
Lie bracket. The $2 \to 1$ diagrams at this loop order are:[43]



(4.150)

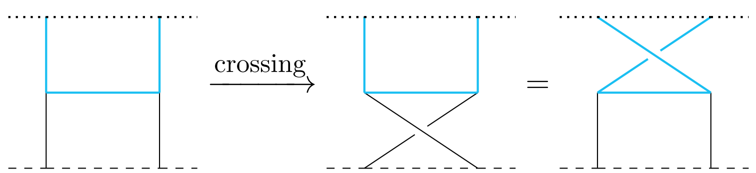
All of these vanish due to Lemma E.4.1 of §E.4.1.

The $2 \to 2$ diagrams at this loop order are:



(4.151)

Note that, since the bulk points are being integrated over, crossing the boundary-to-bulk propagators does not produce any new diagram, it just exchanges the two diagrams that we have drawn:



(4.152)

For this reason, in future we shall only draw diagrams up to crossing of the boundary-to-bulk propagators that are connected to bulk interaction vertices.

Now let us comment on the evaluation of the diagrams in (4.151). We start by doing integration by parts with respect the differential corresponding to either one of the two boundary-to-bulk propagators. As mentioned in §E.4.2, this gives two kinds of contributions, one coming from collapsing a bulk-to-bulk propagator, the other coming from boundary terms. Collapsing any of the bulk-to-bulk propagators leads to a configuration which will vanish due to Lemma E.4.2 (§E.4.1). Therefore, doing integration by parts will only result in a boundary term. Recall from the general discussion in §E.4.2 that only the boundary component of the integrals along the Wilson line can possibly contribute. Since there are two points on the Wilson line, let us call them p_1 and p_2, the domain of integration is:

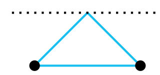
$$\Delta_2 = \{(p_1, p_2) \in \mathbb{R}^2 \,|\, p_1 < p_2\} . \tag{4.153}$$

[43]Sometimes we ignore to specify the derivative couplings at the Wilson line, when the diagrams we draw are vanishing regardless.

The boundary of this domain is:

$$\partial\Delta_2 = \{(p_1, p_2) \in \mathbb{R}^2 \,|\, p_1 = p_2\}\,. \tag{4.154}$$

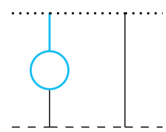
Once restricted to this boundary, both of the diagrams in (4.151) will involve a configuration such as the following:



, (4.155)

which vanishes due to Lemma E.4.1.[44] The diagrams (4.151) thus vanish.

There are four other $2 \to 2$ diagrams at 1-loop, they can be generated by starting with:



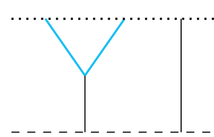
, (4.156)

and then

1. Permuting the two points on the Wilson line.
2. Permuting the two points on the boundary.
3. Simultaneously permuting the two points on the Wilson line and the two points on the boundary.

All of these diagrams vanish due to Lemma E.4.1.

There are also six $2 \to 3$ diagrams. All of these can be generated from the following:



, (4.157)

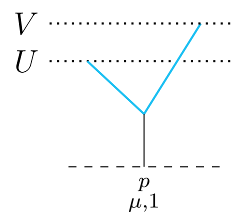
by permuting the points along the Wilson line and the boundary. However, due to Lemma E.4.2, all of these diagrams vanish.

[44]These diagrams actually require a UV regularization due to logarithmic divergence coming from the two points on the Wilson line being coincident. To regularize, the domain of integration needs to be restricted from Δ_2 to $\widetilde{\Delta}_2 := \{(p_1, p_2) \in \mathbb{R}^2 \,|\, p_1 \leq p_2 - \epsilon\}$ for some small positive number ϵ, which leads to the modified boundary equation $p_1 = p_2 - \epsilon$, however, this does not affect the arguments presented in the proof of Lemma E.4.1 (essentially because ϵ is a constant and $\mathrm{d}\epsilon = 0$, resulting in no new forms other than the ones considered in the proof), and therefore we are not going to describe the regularization of these diagrams in detail.

There are no more $2 \to m$ diagrams at 1-loop. Thus, we conclude that there is no 1-loop contribution to the Lie bracket in $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$.

Coproduct. We use the same superscript notation we used in §4.5.2 to distinguish between the actions of $T_\mu[m]$ on different vector spaces. The 1-loop diagram that deforms the classical coproduct is the following:

$$\Gamma^1_{1\to 2}\left(\begin{smallmatrix} p \\ \mu,1 \end{smallmatrix}\right) = \tag{4.158}$$



Happily for us, precisely this diagram was computed in eq. 5.6 of [43] to answer the question "how does a background connection couple to the product Wilson line?". The result of that paper involved an arbitrary background connection where we have our boundary-to-bulk propagator, so we just need to replace that with $\mathsf{K}_1(v,p) = z_v \delta^1(x_v - p)$ and we find:

$$\Gamma^1_{1\to 2}\left(\begin{smallmatrix} p \\ \mu,1 \end{smallmatrix}\right) = -\frac{\hbar}{2} f_\mu{}^{\nu\xi} T^U_\nu[0] \otimes T^V_\xi[0]\,. \tag{4.159}$$

This deforms the classical coproduct (4.148) as follows:

$$T^{U\otimes V}_\mu[1] = T^U_\mu[1] \otimes \mathrm{id}_V + \mathrm{id}_U \otimes T^V_\mu[1] - \frac{\hbar}{2} f_\mu{}^{\nu\xi} T^U_\nu[0] \otimes T^V_\xi[0]\,. \tag{4.160}$$

The exact same computation with K_0 instead of K_1 shows that $\Gamma^1_{1\to 2}\left(\begin{smallmatrix} p \\ \mu,0 \end{smallmatrix}\right) = 0$, i.e., the classical algebra of the 0th level operators remain entirely undeformed at this loop order.[45]

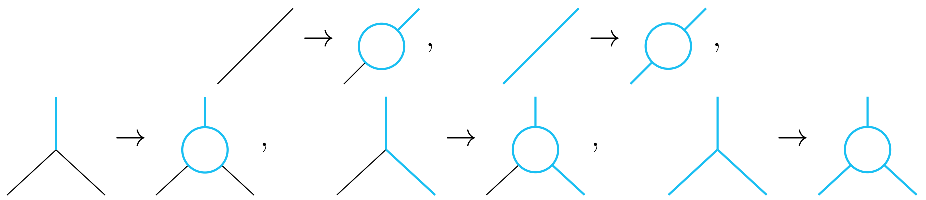
Thus we see that at 1-loop, the Lie algebra structure in $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ remains undeformed, but there is a non-trivial deformation of the coalgebra structure. At this point, there is a mathematical shortcut to proving that the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, including all loop corrections, is the Yangian. The proof relies on a uniqueness theorem (Theorem 12.1.1 of [30]) concerning the deformation of $\mathrm{U}(\mathfrak{gl}_K[z])$. Being able to use the theorem requires satisfying some technical conditions, we discuss this proof in Appendix E.3. This proof is independent of the rest of the chapter, where we compute two loop corrections to the commutator (4.144) which will directly show that the algebra is the Yangian.

[45] Note that the 0th level operators form a closed algebra which is nothing but the Lie algebra $\mathfrak{gl}_K$. Reductive Lie algebras belong to discrete isomorphism classes and therefore they are robust against continuous deformations. So the algebra of $T_\mu[0]$ will in fact remain undeformed at all loop orders. We will not make more than a few remarks about them in the future.

2-loops, $\mathcal{O}(\hbar^2)$

The number of 2-loop diagrams is too large to list them all, and most of them are zero. Instead of drawing all these diagrams let us mention how we can quickly identify a large portion of the diagrams that end up being zero.

Consider the following transformations that can be performed on a propagator or a vertex in any diagram:

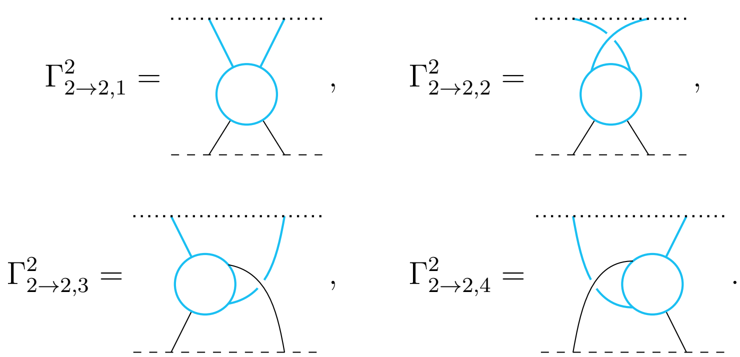


(4.161)

All these transformations increase the order of $\hbar$ by one, however, all the diagrams constructed using these modifications are zero due to Lemma E.4.1. We will therefore ignore such diagrams. Let us now identify potentially non-zero $2 \to m$ diagrams at 2-loops.

All 2-loop $2 \to 1$ diagrams are created from lower order diagrams using modifications such as (4.161). All of them vanish.

For $2 \to 2$ diagrams, ignoring those that are results of modifications such as (4.161) or that are product of lower order vanishing diagrams, we are left with the sum of the following diagrams:

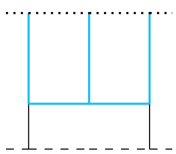
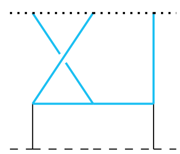
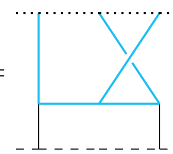


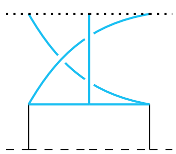
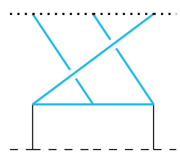
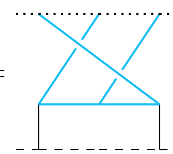
(4.162)

Let us first consider the first two diagrams $\Gamma^2_{2\to2,1}$ and $\Gamma^2_{2\to2,2}$. Collapsing any of the bulk-to-bulk propagators will result in a configuration where either Lemma E.4.1 or E.4.2 is applicable. Therefore, when we do integration by parts with respect to the differential in one of the two boundary-to-bulk propagators we only get a boundary term. The boundary corresponds to the boundary of Δ_2 (defined in (4.153)), and when restricted to this boundary, the integrand vanishes due to Lemma E.4.2, in the same way as for the diagrams in (4.151).[46]

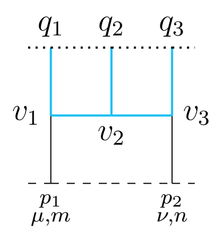
The diagrams $\Gamma^2_{2\to 2,3}$ and $\Gamma^2_{2\to 2,4}$ are symmetric under the exchange of the color labels associated to the boundary-to-bulk propagators, for a proof see the discussion following (E.40). So these diagrams don't contribute to the anti-symmetric commutator we are computing.

Now we come to the most involved part of our computations, $2 \to 3$ diagrams at 2-loops. We have the sum of the following diagrams:

$\Gamma^2_{2\to 3,1} =$  , $\Gamma^2_{2\to 3,2} =$  , $\Gamma^2_{2\to 3,3} =$  ,

$\Gamma^2_{2\to 3,4} =$  , $\Gamma^2_{2\to 3,5} =$  , $\Gamma^2_{2\to 3,6} =$  . (4.163)

All of these diagrams are in fact non-zero. We proceed with the evaluation of the diagram $\Gamma^2_{2\to 3,1}$:

$\Gamma^2_{2\to 3,1}\left({p_1 \atop \mu,1};{p_2 \atop \nu,1}\right) =$ 

(4.164)

The $\mathfrak{gl}_K$ factor of the diagram is easily evaluated to be:

$$f_\mu{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu\pi}{}^{\sigma} \varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)\,. \tag{4.165}$$

The numerical factor takes a bit more care. Each of the bulk points $v_i = (x_i, y_i, z_i, \overline{z}_i)$ is integrated over $M = \mathbb{R}^2 \times \mathbb{C}$ and the points q_i on the Wilson line take value in the

[46]These diagrams are linearly divergent when the two points on the Wilson line are coincident and they require similar UV regularization as their 1-loop counterparts.

simplex $\Delta_3 = \{(q_1, q_2, q_3) \in \mathbb{R}^3 \,|\, q_1 < q_2 < q_3\}$. For the sake of integration we can partially compactify the bulk to $M = \mathbb{R} \times S^3$. So the domain of integration for this diagram is:

$$M^3 \times \Delta_3\,. \tag{4.166}$$

However, this domain needs regularization due to UV divergences coming from the points q_i all coming together. As in [43], we use a point splitting regulator, by restricting integration to the domain:

$$\widetilde{\Delta}_3 := \{(q_1, q_2, q_3) \in \Delta_3 \,|\, q_1 < p_3 - \epsilon\}\,, \tag{4.167}$$

for some small positive number ϵ. We are not going to discuss the regulator here, as it would be identical to the discussion in [43]. We shall now do integration by parts with respect to the differential in the propagator connecting p_1 and v_1. Note that collapsing any of the bulk-to-bulk propagators leads to a configuration where the vanishing Lemma E.4.2 applies. Therefore, contribution to the integral only comes from the boundary $M^3 \times \partial\widetilde{\Delta}_3$. The boundary of the simplex has three components, respectively defined by the constraints $q_1 = q_2$, $q_2 = q_3$, and $q_1 = q_3 - \epsilon$. However, when $q_1 = q_2$ or $q_2 = q_3$, we can use the vanishing Lemma E.4.1 and the integral vanishes. Therefore the contribution to the diagram comes only from integration over:

$$M^3 \times \{(q_1, q_2, q_3) \in \widetilde{\Delta}_3 \,|\, q_1 = q_3 - \epsilon\}\,. \tag{4.168}$$

Further simplification can be made using the fact that the propagator connecting p_2 and v_3 is $z\delta^1(x_3 - p_2)$. This restricts the integration over v_3 to $\{p_2\} \times S^3$. However, using translation symmetry in the x-direction we can fix the position of q_1 at $(0, 0, 0, 0)$ and allow the integration of v_3 over all of M. However, due to the presence of the delta function $\delta^1(x_3 - p_2)$ in the boundary-to-bulk propagator, x_3 and $p_1 = p_2 - \delta$ are rigidly tied to each other. This way, we end up with the following integration for the numerical factor:[47]

$$\begin{aligned}\frac{1}{2}\left(\frac{i}{2\pi\hbar}\right)^3 \int_{\substack{0<q_2<\epsilon \\ v_1, v_2, v_3}} & \mathrm{d}q_2 \mathrm{d}^4v_1 \mathrm{d}^4v_2 \mathrm{d}^4v_3 \theta(x_1 - x_3^-) z_1 z_3 P(v_2, v_1) \\ & \times P(v_3, v_2) P(q_1, v_1) P(q_2, v_2) P(q_3, v_3)\,,\end{aligned} \tag{4.169}$$

where $q_1 = (0, 0, 0, 0)$, $q_2 = (p_2, 0, 0, 0)$, $q_3 = (\epsilon, 0, 0, 0)$, and $x_3^- := x_3 - \delta$, and since all the forms that appear are even we have ignored the wedge product symbols to be economic.

Before evaluating the above integral, we note that the diagram $\Gamma^2_{2\to 3,4}$ evaluates to the

[47] The factor of 1/2 comes from diagram automorphisms.

same color factor and almost same numerical factor, except for a different step function:

$$\begin{aligned}\frac{1}{2}\left(\frac{i}{2\pi\hbar}\right)^3 \int_{\substack{0<q_2<\epsilon\\ v_1,v_2,v_3}} & \mathrm{d}q_2 \mathrm{d}^4v_1 \mathrm{d}^4v_2 \mathrm{d}^4v_3 \theta(x_3 - x_1^-) z_1 z_3 P(v_2, v_1) \\ & \times P(v_3, v_2) P(q_1, v_1) P(q_2, v_2) P(q_3, v_3)\,,\end{aligned} \tag{4.170}$$

Since we have to sum over all the diagrams, we use the fact that:

$$\lim_{\delta\to 0}\big(\theta(x_1 - x_3^-) + \theta(x_3 - x_1^-)\big) = 1\,, \tag{4.171}$$

to write:

$$\begin{aligned}&\lim_{p_2\to p_1}\left(\Gamma^2_{2\to 3,1}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right) + \Gamma^2_{2\to 3,4}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)\right) \\ &= f_\mu{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu\pi}{}^\sigma \varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)\left(\frac{i}{2\pi\hbar}\right)^3 \frac{1}{2}\int_{\substack{0<q_2<\epsilon\\ v_1,v_2,v_3}} \mathrm{d}q_2 \mathrm{d}^4v_1 \mathrm{d}^4v_2 \mathrm{d}^4v_3 \\ &\quad \times z_1 z_3 P(v_2,v_1) P(v_3,v_2) P(q_1,v_1) P(q_2,v_2) P(q_3,v_3)\,,\end{aligned} \tag{4.172}$$

Let us refer to the above integral by $\hbar^2 I_1$, so that we can write the right hand side of the above equation as:

$$\hbar^2 f_\mu{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu\pi}{}^\sigma \varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)\, I_1\,. \tag{4.173}$$

Similar considerations for the rest of the diagrams in (4.163) lead to similar expressions:

$$\lim_{p_2\to p_1}\left(\Gamma^2_{2\to 3,2}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right) + \Gamma^2_{2\to 3,5}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)\right) = \hbar^2 f_\mu{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu\pi}{}^\sigma \varrho(t_\rho)\varrho(t_o)\varrho(t_\sigma)\, I_2\,, \tag{4.174a}$$

$$\lim_{p_2\to p_1}\left(\Gamma^2_{2\to 3,2}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right) + \Gamma^2_{2\to 3,5}\left(\begin{smallmatrix}p_1\\ \mu,1\end{smallmatrix};\begin{smallmatrix}p_2\\ \nu,1\end{smallmatrix}\right)\right) = \hbar^2 f_\mu{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu\pi}{}^\sigma \varrho(t_o)\varrho(t_\sigma)\varrho(t_\rho)\, I_3\,, \tag{4.174b}$$

for two integrals I_2 and I_3 that are only slightly different from I_1.[48] To get the 2-loop contributions to the commutator $[T_\mu[1], T_\nu, [1]]$ we need only to anti-symmetrize the expressions (4.173), (4.174). Putting them together with the classical result (4.144) we get the Lie bracket up to 2-loops:

$$\begin{aligned}[T_\mu[1], T_\nu, [1]] = f_{\mu\nu}{}^\xi T_\xi[2] + 2\hbar^2 f_{[\mu}{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu]\pi}{}^\sigma \big(& T_o[0]T_\rho[0]T_\sigma[0]\, I_1 \\ & + T_\rho[0]T_o[0]T_\sigma[0]\, I_2 + T_o[0]T_\sigma[0]T_\rho[0]\, I_3\big)\,,\end{aligned} \tag{4.175}$$

where we have replaced matrix products such as $\varrho(t_\rho)\varrho(t_o)\varrho(t_\sigma)$ with $T_\rho[0]T_o[0]T_\sigma[0]$ which is accurate up to the loop order shown. Thus we see that quantum corrections deform the classical Lie algebra of $\mathfrak{gl}_K[z]$.

[48]These integrals can be performed and their values are $I_2 = I_3 = \frac{1}{72}\left(2 - \frac{3}{\pi^2}\right)$, $I_1 = \frac{1}{36}\left(1 + \frac{3}{\pi^2}\right)$ though we postpone computing them until we no longer need to compute them.

4.5.4 Large N limit: The Yangian

The deformed Lie bracket (4.175) may not look quite like the standard relations of the Yangian found in the literature, but we can choose a different basis to get to the standard relations, which we shall do momentarily.[49] However, for finite N, our algebra has more relations. Recall that the generators $T_\mu[1]$ act on the space V where classically V is a representation space, $\varrho : \mathfrak{gl}_K[z] \to \mathrm{End}(V)$, of the loop algebra $\mathfrak{gl}_K[z]$ and the representation ρ was determined by the number N. The representation ϱ depends on N because ρ is the representation that couples the $\mathfrak{gl}_K$ connection A to the Wilson line generated by integrating out $N \times K$ fermions. The representation is essentially the Hilbert space (4.100) of the fermionic QM that lives on the line. The important point for us is that, for finite N, ϱ is finite dimensional. This implies that, as we discussed at the end of §(4.4.2), the generators $T_\mu[1]$ satisfy degree d polynomial equations where $d = \dim(V)$. In the limit $N \to \infty$ these relations disappear and we have our isomorphism with the Yangian $Y(\mathfrak{gl}_K)$.

The Yangian in a more standard basis

To get to a standard defining bracket for the Yangian, we change basis as follows. There is an ambiguity in $T_\xi[2]$. In (4.144) it was equal to $\varrho_{\xi,2}$ at the classical level, but it can be shifted at 2-loops (i.e., by a term proportional to $\hbar^2$) by the image $\vartheta(t_\xi)$ for an arbitrary $\mathfrak{gl}_K$-equivariant map $\vartheta : \mathfrak{gl}_K \to \mathrm{End}(V)$. This shift simply corresponds to a different choice for the counterterm that couples $\partial_z^2 A^\xi$ to the Wilson line. Using this freedom we want to replace products such as $\varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)$ with the totally symmetric product $\{\varrho(t_o), \varrho(t_\rho), \varrho(t_\sigma)\}$ (defined in (4.94)). To this end, Consider the difference:

$$\Delta_{\mu\nu} := 2\hbar^2 f_{[\mu}{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu]\pi}{}^{\sigma} \left(\varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma) - \{\varrho(t_o), \varrho(t_\rho), \varrho(t_\sigma)\}\right) . \tag{4.176}$$

The square brackets around μ and ν in the above equation implies anti-symmetrization with respect to μ and ν. The difference $\Delta_{\mu\nu}$ can be viewed as the image of the following $\mathfrak{gl}_K$-equivariant map:

$$\Delta : \wedge^2 \mathfrak{gl}_K \to \mathrm{End}(V) , \qquad \Delta : t_\mu \wedge t_\nu \mapsto \Delta_{\mu\nu} . \tag{4.177}$$

We now propose the following lemma:

Lemma 4.5.3. *The map Δ factors through $\mathfrak{gl}_K$, i.e., $\Delta : \wedge^2\mathfrak{gl}_K \to \mathfrak{gl}_K \to \mathrm{End}(V)$.*

[49] We can also appeal to the uniqueness theorem 12.1.1 of [30], in conjunction with the result of Appendix E.3, to conclude that the deformed algebra must be the Yangian $Y_\hbar(\mathfrak{gl}_K)$.

The proof of this lemma involves some algebraic technicalities which we relegate to the Appendix §E.5. The utility of this lemma is that, it establishes the difference (4.176) as the image of an element of $\mathfrak{gl}_K$ which, according to our previous argument, can be absorbed into a redefinition of $\varrho_{\xi,2}$ (equivalently $T_\xi[2]$). Therefore, with a new $T_\xi^{\text{new}}[2]$ we can rewrite (4.175) as:

$$[T_\mu[1], T_\nu, [1]] = f_{\mu\nu}{}^\xi T_\xi^{\text{new}}[2] + \hbar^2(I_1 + I_2 + I_3)Q_{\mu\nu}\,, \tag{4.178}$$

where we have also defined:

$$Q_{\mu\nu} := 2f_{[\mu}{}^{\xi o} f_\xi{}^{\pi\rho} f_{\nu]\pi}{}^\sigma \left\{T_o[0], T_\rho[0], T_\sigma[0]\right\}\,. \tag{4.179}$$

The reason why we have postponed presenting the evaluations of the individual integrals I_1, I_2, and I_3 is that we don't need their individual values, only the sum, and precisely this sum was evaluated in eq. (E.23) of [43] with the result:

$$I_1 + I_2 + I_3 = \frac{1}{12}\,. \tag{4.180}$$

We can therefore write (ignoring the "new" label on $T_\xi[2]$):

$$[T_\mu[1], T_\nu[1]] = f_{\mu\nu}{}^\xi T_\xi[2] + \frac{\hbar^2}{12} Q_{\mu\nu}\,. \tag{4.181}$$

This is the relation for the Yangian that was presented in §8.6 of [43] and how to relate this to other standard relations of the Yangian was also discussed there. This is also the exact relation we found in the boundary theory (c.f. (4.97)). Note furthermore that, if we had used the relation between our algebra and anomaly (4.139) to derive the algebra Lie bracket, we would have arrived at precisely the same conclusion, as the second term in (4.181) is indeed the anomaly of a Wilson line (c.f. eq. 8.35 of [41]).

Thus we see that the algebra $\mathcal{A}^{\text{Sc}}(\mathcal{T}_{\text{bk}})$, defined in (4.41), at 2-loops, is the Yangian $Y_\hbar(\mathfrak{gl}_K)$:

$$\mathcal{A}^{\text{Sc}}(\mathcal{T}_{\text{bk}})/\hbar^3 \overset{N\to\infty}{\cong} Y_\hbar(\mathfrak{gl}_K)/\hbar^3\,. \tag{4.182}$$

The two loop result in the BF theory was exact. The above two loop result is exact as well. Though we do not prove this by computing Witten diagrams, we can argue using the form of the algebra in terms of anomaly (4.139). In [43] it was shown that there are no anomalies beyond two loops. This concludes our second proof of Proposition 4.5.1.[50]

[50]The first one, which is significantly more abstract, being in Appendix E.3.

4.6 String Theory Construction of The Duality

The topological theories we have considered so far can be constructed from a certain brane setup in type IIB string theory and then applying a twist and an omega deformation. This brane construction will show that the algebras we have constructed are infact certain supersymmetric subsectors of the well studied $\mathcal{N} = 4$ SYM theory with defect and its holographic dual. We dscribe this construction below.

4.6.1 Brane Configuration

Our starting brane configuration involves a stack of N D3 branes and K D5 branes in type IIB string theory on a 10d target space of the form $\mathbb{R}^8 \times C$ where C is a complex curve which we take to be just the complex plane $\mathbb{C}$. The D5 branes wrap $\mathbb{R}^4 \times C$ and the D3 branes wrap an $\mathbb{R}^4$ which has a 3d intersection with the D5 branes. Let us express the brane configuraiton by the following table:

$$
\begin{array}{c|cccc|cc|cccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
 & & \mathbb{R}^4 & & & & C & & \mathbb{R}^4 & & \\
\hline
D5 & \times & \times & \times & \times & \times & \times & & & & \\
D3 & \times & & \times & \times & & & & \times & &
\end{array}
\tag{4.183}
$$

The world-volume theory on the D5 branes is the 6d $\mathcal{N} = (1, 1)$ SYM theory coupled to a 3d defect preserving half of the supersymmetry. Similarly, the world-volume theory on the D3 branes is the 4d SYM theory coupled to a 3d defect preserving half of the supersymmetry. To this setup we apply a particular twist, i.e., we choose a nilpotent supercharge and consider its cohomology.

4.6.2 Twisting Supercharge

From the 6d Perspective

We use Γ_i with $i \in \{0, \cdots, 9\}$ for 10d Euclidean gamma matrices. We also use the notation:

$$\Gamma_{i_1 \cdots i_n} := \Gamma_{i_1} \cdots \Gamma_{i_n} \, . \tag{4.184}$$

Type IIB has 32 supercharges, arranged into two Weyl spinors of the same 10 dimensional chirality – let us denote them as Q_l and Q_r. A general linear combination of them is written as $\epsilon_L Q_l + \epsilon_R Q_r$ where ϵ_L and ϵ_R are chiral spinors parametrizing the supercharge. The chirality constraints on them are:

$$i\Gamma_{0\cdots 9}\epsilon_L = \epsilon_L\,, \qquad i\Gamma_{0\cdots 9}\epsilon_R = \epsilon_R\,. \tag{4.185}$$

We shall discuss constraints on the supercharge by describing them as constraints on the parametrizing spinors.

The supercharges preserved by the D5 branes are constrained by:

$$\epsilon_R = i\Gamma_{012345}\epsilon_L\,. \tag{4.186}$$

This reduces the number of supercharges to 16. The D3 branes imposes the further constraint:

$$\epsilon_R = i\Gamma_{0237}\epsilon_L\,. \tag{4.187}$$

This reduces the number of supercharges by half once more. Therefore the defect preserves just 8 supercharges. Since ϵ_R is completely determined given ϵ_L, in what follows we refer to our choice of supercharge simply by referring to ϵ_L.

We want to perform a twist that makes the D5 world-volume theory topological along $\mathbb{R}^4$ and holomorphic along C. This twist was described in [45]. Let us give names to the two factors of $\mathbb{R}^4$ in the 10d space-time:

$$M := \mathbb{R}^4_{0123}\,, \qquad M' := \mathbb{R}^4_{6789}\,. \tag{4.188}$$

The spinors in the 6d theory transform as representations of Spin(6) under space-time rotations. $\mathcal{N} = (1,1)$ algebra has two left handed spinors and two right handed spinors transforming as $\mathbf{4}_l$ and $\mathbf{4}_r$ respectively.[51] The subgroup of Spin(6) preserving the product structure $\mathbb{R}^4 \times C$ is $\mathrm{Spin}(4)_M \times \mathrm{U}(1)$. Under this subgroup $\mathbf{4}_l$ and $\mathbf{4}_r$ transform as $(\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_{+1}$ and $(\mathbf{2},\mathbf{1})_{+1} \oplus (\mathbf{1},\mathbf{2})_{-1}$ respectively, where the subscripts denote the U(1) charges. Rotations along M' act as R-symmetry on the spinors – the spinors transform as representations of $\mathrm{Spin}(4)_{M'}$ such that $\mathbf{4}_+$ transforms as $(\mathbf{2},\mathbf{1})$ and $\mathbf{4}_-$ transforms as $(\mathbf{1},\mathbf{2})$. In total, under the symmetry group $\mathrm{Spin}(4)_M \times \mathrm{U}(1) \times \mathrm{Spin}(4)_{M'}$) the 16 supercharges of the 6d theory transform as:

$$((\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_{+1}) \otimes (\mathbf{2},\mathbf{1}) \oplus ((\mathbf{2},\mathbf{1})_{+1} \oplus (\mathbf{1},\mathbf{2})_{-1}) \otimes (\mathbf{1},\mathbf{2})\,. \tag{4.189}$$

[51] There are two of each chirality because the R-symmetry is $\mathrm{Sp}(1) \times \mathrm{Sp}(1) = \mathrm{Spin}(4)_{M'}$ such that the two left handed spinors transform as a doublet of one Sp(1) and the two right handed spinors transform as a doublet of the other Sp(1).

The twist we seek is performed by redefining the the space-time isometry:

$$\mathrm{Spin}(4)_M \rightsquigarrow \mathrm{Spin}(4)_M^{\mathrm{new}} \subseteq \mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_{M'}\,, \tag{4.190}$$

where the subgroup $\mathrm{Spin}(4)_M^{\mathrm{new}}$ of $\mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_{M'}$ consists of elements $(x, \theta(x))$ which is defined by the isomorphism $\theta : \mathrm{Spin}(4)_M \xrightarrow{\sim} \mathrm{Spin}(4)_{M'}$. More, explicitly, the isomorphism acts as:

$$\theta(\Gamma_{\mu\nu}) = \Gamma_{\mu+6,\nu+6}\,, \qquad \mu,\nu \in \{0,1,2,3\}\,. \tag{4.191}$$

The generators of the new $\mathrm{Spin}(4)_M^{\mathrm{new}}$ are then:

$$\Gamma_{\mu\nu} + \Gamma_{\mu+6,\nu+6}\,. \tag{4.192}$$

After this redefinition, the symmetry $\mathrm{Spin}(4)_M \times \mathrm{U}(1) \times \mathrm{Spin}(4)_{M'}$ of the 6d theory reduces to $\mathrm{Spin}(4)_M^{\mathrm{new}} \times \mathrm{U}(1)$ and under this group the representation (4.189) of the supercharges becomes:

$$2(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1} \oplus 2(\mathbf{2},\mathbf{2})_{+1}\,. \tag{4.193}$$

We thus have two supercharges that are scalars along M, both of them have charge -1 under the U(1) rotation along C. We take the generator of this rotation to be $-i\Gamma_{45}$, then if ϵ is one of the scalar (on M) supercharges that means:

$$i\Gamma_{45}\epsilon = \epsilon\,. \tag{4.194}$$

We identify the supercharge ϵ by imposing invariance under the new rotation generators on M, namely (4.192):

$$(\Gamma_{\mu\nu} + \Gamma_{\mu+6,\nu+6})\epsilon = 0\,. \tag{4.195}$$

The constraints (4.186) and (4.187) put by the D-branes and the U(1)-charge on C (4.194) together are equivalent to the following four independent constraints:

$$i\Gamma_{\mu,\mu+6}\epsilon = \epsilon\,, \qquad \mu\{0,1,2,3\}\,. \tag{4.196}$$

Together with the chirality constraint (4.185) in 10d we therefore have 5 equations, each reducing the number degrees of freedom by half. Since a Dirac spinor in 10d has 32 degrees of freedom, we are left with $32 \times 2^{-5} = 1$ degree of freedom, i.e., we have a unique supercharge,[52] which we call Q. It was shown in [45] that the supercharge Q is nilpotent:

$$Q^2 = 0\,, \tag{4.197}$$

[52]Note that without using the constraint put by the D3 branes we would get *two* supercharges that are scalar on M, i.e., there are two superhcarges in the 6d theory (by itself) that are scalar on M.

and the 6d theory twisted by this Q is topological along M – which is simply a consequence of (4.195) – and it is holomorphic along C. The latter claim follows from the fact that there is another supercharge in the 2d space of scalar (on M) supercharges in the 6d theory, let's call it Q', which has the following commutator with Q:

$$\{Q, Q'\} = \partial_{\overline{z}}\,, \tag{4.198}$$

where $z = \frac{1}{2}(x^4 - ix^5)$ is the holomorphic coordinate on C. This shows that $\overline{z}$-dependence is trivial (Q-exact) in the Q-cohomology.

From the 4d Perspective

What is new in our setup compared to the setup considered in [45] is the stack of D3 branes. We can figure out what happens to the world-volume theory of the D3 branes – we get the *Kapustin-Witten (KW)* twist [110], as we now show. The equations (4.196) can be used to to get the following six (three of which are independent) equations:

$$\begin{aligned} &(\Gamma_{02} + \Gamma_{68})\epsilon = 0\,, \qquad &(\Gamma_{03} + \Gamma_{69})\epsilon = 0\,, \qquad &(\Gamma_{23} + \Gamma_{89})\epsilon = 0\,, \\ &(\Gamma_{07} + \Gamma_{16})\epsilon = 0\,, \qquad &(\Gamma_{27} + \Gamma_{18})\epsilon = 0\,, \qquad &(\Gamma_{37} + \Gamma_{19})\epsilon = 0\,. \end{aligned} \tag{4.199}$$

These are in fact the equations that defines a scalar supercharge in the KW twist of $\mathcal{N} = 4$ theory on $\mathbb{R}^4_{0237}$ for a particular homomorphism from space-time ismoetry to the R-symmetry.[53] Space-time isometry of the theory on $\mathbb{R}^4_{0237}$ acts on the spinors as $\mathrm{Spin}(4)_{\mathrm{iso}}$, generated by the six generators:

$$\Gamma_{\mu\nu}\,, \qquad \mu, \nu \in \{0, 2, 3, 7\} \text{ and } \mu \neq \nu\,. \tag{4.200}$$

Rotations along the transverse directions act as R-symmetry, which is Spin(6), though the subgroup of the R-symmetry preserving the product structure $C \times \mathbb{R}^4_{1689}$ is $\mathrm{U}(1) \times \mathrm{Spin}(4)_{\mathrm{R}}$. The KW twist is constructed by redefining space-time isometry to be a Spin(4) subgroup of $\mathrm{Spin}(4)_{\mathrm{iso}} \times \mathrm{Spin}(4)_{\mathrm{R}}$ consisting of elements $(x, \vartheta(x))$ where $\vartheta : \mathrm{Spin}(4)_{\mathrm{iso}} \xrightarrow{\sim} \mathrm{Spin}(4)_{\mathrm{R}}$ is an isomorphism. The particular isomorphism that leads to the equations (4.199) is:

$$\begin{aligned} &\Gamma_{02} \mapsto \Gamma_{68}\,, \qquad &\Gamma_{03} \mapsto \Gamma_{69}\,, \qquad &\Gamma_{23} \mapsto \Gamma_{89}\,, \\ &\Gamma_{07} \mapsto \Gamma_{16}\,, \qquad &\Gamma_{27} \mapsto \Gamma_{18}\,, \qquad &\Gamma_{37} \mapsto \Gamma_{19}\,. \end{aligned} \tag{4.201}$$

[53] Note that we ar using subscripts simply to refer to particular directions.

Remark 4.6.1 (A member of a $\mathbb{CP}^1$ family of twists). In [110] it was shown that there is a family of KW twists parametrized by $\mathbb{CP}^1$. The unique twist (by the supercharge Q) we have found is a specific member of this family. Let us identify which member that is.

The $\mathbb{CP}^1$ family comes from the fact that there is a 2d space of scalar (on M) supercharges (in (4.193)) in the twisted theory.[54] Also note from the original representation of the spinors (4.189) that the two scalar supercharges come from spinors transforming as $(\mathbf{1}, 2)$ and $(\mathbf{2}, 1)$ under the original isometry $\mathrm{Spin}(4)^{\mathrm{old}}$.[55] Let us choose two $\mathrm{Spin}(4)^{\mathrm{new}}$ scalar spinors with opposite $\mathrm{Spin}(4)^{\mathrm{old}}$ chiralities and call them ϵ_l and ϵ_r. The $\mathrm{Spin}(4)^{\mathrm{old}}$ chirality operator is $\Gamma^{\mathrm{old}} := \Gamma_{0237}$. Let us choose ϵ_l and ϵ_r in such a way that they are related by the following equation:

$$\epsilon_r = N\epsilon_l \quad \text{where} \quad N = \frac{1}{4}(\Gamma_{06} + \Gamma_{28} + \Gamma_{39} + \Gamma_{17})\,. \tag{4.202}$$

This relation is consistent with the spinors being $\mathrm{Spin}(4)^{\mathrm{new}}$ invariant because N anti-commutes with $\mathrm{Spin}(4)^{\mathrm{new}}$ (thus invariant spinors are still invariant after being operated with N), as well as with Γ^{old} (chaning $\mathrm{Spin}(4)^{\mathrm{old}}$ chirality). An arbitrary scalar supercharge in the twisted theory is a complex linear combination of ϵ_l and ϵ_r, such as $\alpha\epsilon_l + \beta\epsilon_r$, however, since the overall normalization of the spinor is irrelevant, the true parameter identifying a spinor is the ratio $t := \beta/\alpha \in \mathbb{CP}^1$. Furthermore, due to the equations (4.199), N^2 acts as -1 on any $\mathrm{Spin}(4)^{\mathrm{new}}$ scalar, leading to:

$$\epsilon_l = -N\epsilon_r\,. \tag{4.203}$$

To see the value of the twisting parameter t for the supercharge identified by the equations (4.196) (in addition to the 10d chirality (4.185)), we first pick a linear combination $\epsilon := \epsilon_l + t\epsilon_r$ with $t \in \mathbb{CP}^1$. Then using (4.203) and (4.196) we get:

$$-i\epsilon = N\epsilon = \epsilon_r - t\epsilon_l\,, \tag{4.204}$$

where the first equality follows from (4.196) and the second from (4.203). Equating the two sides we find the twisting parameter:

$$t = i\,. \tag{4.205}$$

$\triangle$

[54] Though we began the discussion with a view to identifying topological-holomorphic twist of 6d $\mathcal{N} = (1,1)$ theory, what we found in the process in particular are supercharges that are scalar on M. If we forget that we had a 6d theory on $M \times C$ and just consider a theory on M with rotations on C being part of the R-symmetry then, first of all, we find a $\mathcal{N} = 4$ SYM theory on M and the twist we described is precisely the KW twist.

[55] We are writing $\mathrm{Spin}(4)^{\mathrm{old}}$ instead of $\mathrm{Spin}(4)_M$ since the support of the 4d theory is not $M \equiv \mathbb{R}^4_{0123}$ but $\mathbb{R}^4_{0237}$.

From the 3d Perspective

Finally, at the 3 dimensional D3-D5 intersection lives a 3d $\mathcal{N} = 4$ theory consisting of bifundamental hypermultiplets coupled to background gauge fields which are restrictions of the gauge fields from the D3 and the D5 branes [80]. Considering Q-cohomology for the 3d theory reduces it to a topological theory as well. To identify the topological 3d theory we note that for the twisting parameter $t = i$, the 4d theory is an analogue of a 2d B-model[56] [110] and this can be coupled to a 3d analogue of the 2d B-model[57] – a B-type topological twist of 3d $\mathcal{N} = 4$ is called a *Rozansky-Witten (RW)* twist [143]. The flavor symmetry of the theory is $\mathrm{U}(N) \times \mathrm{U}(K)$ which acts on the hypers and is gauged by the background connections.

We can reach the same conclusion by analyzing the constraints on the twisting supercharge viewed from the 3d point of view. The bosonic symmetry of the 3d theory includes $\mathrm{SU}(2)_{\mathrm{iso}} \times \mathrm{SU}(2)_H \times \mathrm{SU}(2)_C$ where $\mathrm{SU}(2)_{\mathrm{iso}}$ is the isometry of the space-time $\mathbb{R}^3_{023}$, $\mathrm{SU}(2)_C$ are rotations in $\mathbb{R}^3_{689}$, and $\mathrm{SU}(2)_H$ are rotations in $\mathbb{R}^3_{145}$. The hypers in the 3d theory come from strings with one end attached to the D5 branes and another end attached to the D3 branes. Rotations in $\mathbb{R}^3_{145}$ – the R-symmetry $\mathrm{SU}(2)_H$ – therefore act on the hypers. This means that $\mathrm{SU}(2)_H$ acts on the Higgs branch of the 3d theory. This leaves the other R-symmetry group $\mathrm{SU}(2)_C$ which would act on the coulomb branch of the theory if the theory had some dynamical 3d vector multiplets. We now note that the topological twist, from the 3d perspective, involves twisting the isometry $\mathrm{SU}(2)_{\mathrm{iso}}$ with the R-symmetry group $\mathrm{SU}(2)_C$, as evidenced explicitly by the three equations in the first line of (4.199). This particular topological twist (as opposed to the topological twist using the other R-symmetry $\mathrm{SU}(2)_H$) of 3d $\mathcal{N} = 4$ is indeed the RW twist [38].

To summerize, taking cohomology with respect to the supercharge Q leaves us with the KW twist (twisting parameter $t = i$) of $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^4$ with gauge group $\mathrm{U}(N)$ and a topological-holomorphic twist of $\mathcal{N} = (1, 1)$ theory on $\mathbb{R}^4 \times C$ with gauge group $\mathrm{U}(K)$, and these two theories are coupled via a 3d RW theory of bifundamental hypers with flavor symmetry $\mathrm{U}(N) \times \mathrm{U}(K)$ gauged by background connections.[58]

---

[56] In particular, the 4d Theory on $\mathbb{R}^2 \times T^2$ can be compactified on the two-torus T^2 to get a B-model on $\mathbb{R}^2$.

[57] We want to be able to take the 3d theory on $\mathbb{R}^2 \times S^1$ and compactify it on S^1 to get a B-model on $\mathbb{R}^2$. If we have a 4d theory on $\mathbb{R}^2 \times T^2$ coupled to a 3d theory on $\mathbb{R}^2 \times S^1$, compactifying on T^2 should not make the two systems incompatible.

[58] Though it is customary to decouple the central $\mathrm{U}(1)$ subgroup from the gauge groups as it doesn't interact with the non-abelian part, our computations look somewhat simpler if we keep the $\mathrm{U}(1)$.

4.6.3 Omega Deformation

We start by noting that the dimensional reduction of the topological-holomorphic 6d theory from $\mathbb{R}^4 \times C$ to $\mathbb{R}^4$ reduces it to the KW twist of $\mathcal{N} = 4$ SYM on the $\mathbb{R}^4$.[59] This observation allows us to readily tailor the results obtained in [45] about omega deformation of the 6d theory to the case of omega deformation of 4d KW theory.

The fundamental bosonic field in the 10d $\mathcal{N} = 1$ SYM theory is the connection A_I where $I \in \{0, \cdots, 9\}$. When dimensionally reduced to 6d, this becomes a 6d connection A_M with $M \in \{0, \cdots, 5\}$ and four scalar fields ϕ_0, ϕ_1, ϕ_2, and ϕ_3 which are just the remaining four components of the 10d connection. The $\mathrm{Spin}(4)_M$ space-time isometry acts on the first four components of the connection, namely A_0, A_1, A_2, and A_3 via the vector representation. The four scalars – ϕ_0, ϕ_1, ϕ_2, and ϕ_3 – transform under the vector representation of the R-symmetry $\mathrm{Spin}(4)_{M'}$. Once twisted according to (4.190), only the diagonal subgroup $\mathrm{Spin}(4)_M^{\mathrm{new}}$ of $\mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_{M'}$ acts on the fields, under which the first four components of the connection and the four scalars transform in the same way[60] and therefore we can package them together into one complex valued gauge field:

$$\mathcal{A}_\mu := A_\mu + i\phi_\mu\,, \qquad \mu \in \{0, 1, 2, 3\}\,. \tag{4.206}$$

We also write the remaining components of the connection in complex coordinates on C:

$$A_z := A_4 + iA_5 \quad \text{and} \quad A_{\bar{z}} := A_4 - iA_5\,. \tag{4.207}$$

It was shown in [45] that this topological-holomorphic 6d theory can be viewed as a 2d gauged B-model on $\mathbb{R}^2_{23}$ where the fields are valued in maps $\mathrm{Map}(\mathbb{R}^2_{01} \times \mathbb{C}, \mathfrak{gl}_K)$. This is a vector space which plays the role of the Lie algebra of the 2d gauge theory. From the 2d point of view $\mathcal{A}_2$ and $\mathcal{A}_3$ are part of a connection on $\mathbb{R}^2_{23}$ and there are four chiral multiplets with the bottom components $\mathcal{A}_0, \mathcal{A}_1, A_z$, and $A_{\bar{z}}$. The 2d theory consists of a superpotential which is a holomorphic function of these chiral multiplets – the superpotential can be written conveniently in terms of a one form $\widetilde{\mathcal{A}} := \mathcal{A}_0 \mathrm{d}x^0 + \mathcal{A}_1 \mathrm{d}x^1 + A_z \mathrm{d}z + A_{\bar{z}} \mathrm{d}\bar{z}$ on $\mathbb{R}^2_{01} \times C$ consisting of these chiral fields:[61]

$$W(\mathcal{A}_0, \mathcal{A}_1, A_z, A_{\bar{z}}) = \int_{\mathbb{R}^2_{01} \times C} \mathrm{d}z \wedge \mathrm{tr}\left(\widetilde{\mathcal{A}} \wedge \mathrm{d}\widetilde{\mathcal{A}} + \frac{2}{3}\widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}} \wedge \widetilde{\mathcal{A}}\right)\,. \tag{4.208}$$

[59] Both the 6d $\mathcal{N} = (1, 1)$ SYM and the 4d $\mathcal{N} = 4$ SYM are dimensional reductions of the 10d $\mathcal{N} = 1$ SYM and dimensional reduction commutes with the twisting procedure.

[60] Apart from the inhomogeneous transformation of the connection.

[61] Up to some overall numerical factors.

The superpotential is the action functional of a 4d CS theory on $\mathbb{R}^2_{01} \times C$ for the connection $\widetilde{\mathcal{A}}$.

One of the results of [45] is the following: Ω-deformation applied to this topological-holomorphic 6d theory with respect to rotation on $\mathbb{R}^2_{23}$ reduces the the theory to a 4d CS theory on $\mathbb{R}^2_{01} \times C$ with *complexified* gauge group GL_K .

The twisted 4d theory (the D3 world-volume theory) wraps the plane $\mathbb{R}^2_{23}$ as well and therefore is affected by the Ω-deformation. By noting that the 4d theory is a dimensional redcution of the 6d theory from $\mathbb{R}^4 \times C$ to $\mathbb{R}^4$ and assuming that Ω-deformation commutes with dimensional reduction,[62] we can deduce what the Ω-deformed version of the twisted 4d theory is. This will be a 2d gauge theory with complexified gauge group GL_N and the action will be the dimensional reduction of the 4d CS action (4.208) from $\mathbb{R}^2 \times C$ to $\mathbb{R}^2$ – this is the 2d BF theory where $A_{\overline{z}}$ plays the role of the B field:

$$\begin{aligned}\int_{\mathbb{R}^2\times C} \mathrm{d}z \wedge \mathrm{CS}(A_{\mathbb{R}^2\times C}) &\xrightarrow{\text{Reduce on } C} \int_{\mathbb{R}^2} \mathrm{tr}\, A_{\overline{z}} \left(\mathrm{d}A_{\mathbb{R}^2} + \frac{1}{2} A_{\mathbb{R}^2} \wedge A_{\mathbb{R}^2} \right) \\ &= \int_{\mathbb{R}^2} \mathrm{tr}\, A_{\overline{z}} F(A_{\mathbb{R}^2})\,,\end{aligned} \tag{4.209}$$

where, as before, $\overline{z}$ is the anti-holomorphic coordinate on C.

Finally, it was shown in [157] that the RW twist of a 3d $\mathcal{N} = 4$ theory on $\mathbb{R}^2_\Omega \times \mathbb{R}$ with only hypers reduces, upon Ω-deformation with respect to rotation in the plane $\mathbb{R}^2_\Omega$, to a free quantum mechanics on $\mathbb{R}$. A slight modification of this result, involving background connections gauging the flavor symmetry of the hypers leads to the result that the omega deformed theory is a gauged quantum mechanics, the kind of theory we have considered on the defect in the 2d BF theory.[63]

4.6.4 Takeaway from the Brane Construction

Via supersymmetric twists and Ω-deformation, we have made contact with precisely the setup we have considered in this chapter. We have a 4d CS theory with gauge group GL_K and a 2d BF theory with gauge group GL_N and they intersect along a topological line supporting a gauged quantum mechanics with $\mathrm{GL}_K \times \mathrm{GL}_N$ symmetry. We thus claim that

[62] Alternatively, one can redo the localization computations of [45] for the 4d case, confirming that Ω-deformation does indeed commute with dimensional reduction.

[63] The bosonic version, which leads to the same Yangian with minor modifications to the computations as remarked in 4.4.1, 4.4.2, and 4.4.3.

the topological holographic duality that we have established in this chapter is indeed a topological subsector of the standard holographic duality involving defect $\mathcal{N} = 4$ SYM. As mentioned in Remark 1.4.4, it would be nice to have an equivariant BV formulation to formally describe this topological holography as a certain cohomology of the duality of $\mathcal{N} = 4$.

Chapter 5

Conclusion

In the introduction (§1) we tried to motivate, in general terms, that cohomological algebras can potentially be interesting objects to study in supersymmetric QFTs and that they can play significant roles in establishing otherwise complicated dualities. Throughout this thesis, working on exemplary theories, we have concretely demonstrated how such algebras can be computed and how they can arise as supersymmetric subsectors of much more complicated theories such as $\mathcal{N} = 4$ SYM and its holgraphic dual. We think that this shows evidence for two things:

1. Cohomological algebras are particularly amenable to concrete computations.
2. They can be used effectively to probe dualities that are hard to establish in full generality, such as holography.

We would also like to emphasize the exact nature of our computations, in the sense that our computations were not limited in loop order. Though we have focused our attention to limited sets of observables compared to the full QFTs, we think the concrete results that we have been able to find in these limited contexts is motivating. There are two *broad* categories of investigations that we plan to pursue from here:

1. Identify various cohomological subsectors of other supersymmetric theories of interest, possiblity in the context of dualities.
2. Study deformations of QFTs preserving some given cohomology, this will give us an estimate of the extent to which a duality established at the level of cohomology can be expected to hold when lifted to the full QFT.

We note that the first line of investigation we mentioned above has been popular and successful in the ethos of physics for decades – cohomological algebras (precisely the kind we studied in chapter 2) in 2d $\mathcal{N} = (2, 2)$ theories were used to study mirror symmetry [25, 115], and certain supersymmetric boundary conditions and line operators in $\mathcal{N} = 4$ SYM theory played a significant role in studying the geometric Langlands correspondence [110]. However, application of cohomological algebras to study holography is a relatively new concept, suggested originally in [42] and later studied in [35, 37, 105]. We think that this is a particularly interesting and potentially fruitful topic and we intend to pursue this research further.

Bibliography

[1] Ofer Aharony, Micha Berkooz, and Soo-Jong Rey. Rigid holography and six-dimensional $\mathcal{N} = (2,0)$ theories on $\mathrm{AdS}_5 \times \mathbb{S}^1$. *JHEP*, 03:121, 2015.

[2] Inês Aniceto, Jorge G. Russo, and Ricardo Schiappa. Resurgent Analysis of Localizable Observables in Supersymmetric Gauge Theories. *JHEP*, 03:172, 2015.

[3] Paul S. Aspinwall. D-branes on Calabi-Yau manifolds. In *Progress in string theory. Proceedings, Summer School, TASI 2003, Boulder, USA, June 2-27, 2003*, pages 1–152, 2004.

[4] Benjamin Assel, Davide Cassani, and Dario Martelli. Supersymmetric counterterms from new minimal supergravity. *JHEP*, 11:135, 2014.

[5] Jinbeom Bae, Camillo Imbimbo, Soo-Jong Rey, and Dario Rosa. New Supersymmetric Localizations from Topological Gravity. *JHEP*, 03:169, 2016.

[6] Marco Baggio, Jan de Boer, and Kyriakos Papadodimas. A non-renormalization theorem for chiral primary 3-point functions. *JHEP*, 07:137, 2012.

[7] Marco Baggio, Vasilis Niarchos, and Kyriakos Papadodimas. Exact correlation functions in $SU(2)\mathcal{N} = 2$ superconformal QCD. *Phys. Rev. Lett.*, 113(25):251601, 2014.

[8] Marco Baggio, Vasilis Niarchos, and Kyriakos Papadodimas. On exact correlation functions in SU(N) $\mathcal{N} = 2$ superconformal QCD. *JHEP*, 11:198, 2015.

[9] Marco Baggio, Vasilis Niarchos, and Kyriakos Papadodimas. tt* equations, localization and exact chiral rings in 4d $\mathcal{N}$ =2 SCFTs. *JHEP*, 02:122, 2015.

[10] M. Banados, A. Schwimmer, and S. Theisen. Remarks on Resonant Scalars in the AdS/CFT Correspondence. *JHEP*, 09:058, 2006.

[11] I. A. Batalin and G. A. Vilkovisky. Gauge Algebra and Quantization. *Phys. Lett.*, 102B:27–31, 1981. [,463(1981)].

[12] I. A. Batalin and G. A. Vilkovisky. Quantization of Gauge Theories with Linearly Dependent Generators. *Phys. Rev.*, D28:2567–2582, 1983. [Erratum: Phys. Rev.D30,508(1984)].

[13] Christopher Beem, Madalena Lemos, Pedro Liendo, Wolfger Peelaers, Leonardo Rastelli, and Balt C. van Rees. Infinite Chiral Symmetry in Four Dimensions. *Commun. Math. Phys.*, 336(3):1359–1433, 2015.

[14] Christopher Beem, Madalena Lemos, Pedro Liendo, Leonardo Rastelli, and Balt C. van Rees. The $\mathcal{N} = 2$ superconformal bootstrap. 2014.

[15] Francesco Benini and Stefano Cremonesi. Partition Functions of $\mathcal{N} = (2, 2)$ Gauge Theories on S^2 and Vortices. *Commun. Math. Phys.*, 334(3):1483–1527, 2015.

[16] Francesco Benini, Daniel S. Park, and Peng Zhao. Cluster Algebras from Dualities of 2d $\mathcal{N}$ = (2, 2) Quiver Gauge Theories. *Commun. Math. Phys.*, 340:47–104, 2015.

[17] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. *Commun. Math. Phys.*, 165:311–428, 1994.

[18] M. Bershadsky and V. Sadov. Theory of Kahler gravity. *Int. J. Mod. Phys.*, A11:4689–4730, 1996.

[19] Peter Breitenlohner and Martin F. Sohnius. An Almost Simple Off-shell Version of SU(2) Poincare Supergravity. *Nucl.Phys.*, B178:151, 1981.

[20] Matthew Buican, Takahiro Nishinaka, and Constantinos Papageorgakis. Constraints on chiral operators in $\mathcal{N} = 2$ SCFTs. *JHEP*, 12:095, 2014.

[21] Daniel Butter, Bernard de Wit, Sergei M. Kuzenko, and Ivano Lodato. New higher-derivative invariants in N=2 supergravity and the Gauss-Bonnet term. *JHEP*, 12:062, 2013.

[22] Daniel Butter, Bernard de Wit, and Ivano Lodato. Non-renormalization theorems and N=2 supersymmetric backgrounds. *JHEP*, 03:131, 2014.

[23] Daniel Butter and Sergei M. Kuzenko. New higher-derivative couplings in 4D N = 2 supergravity. *JHEP*, 1103:047, 2011.

[24] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies. 2015.

[25] Philip Candelas, Xenia C. De La Ossa, Paul S. Green, and Linda Parkes. A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nucl. Phys.*, B359:21–74, 1991. [AMS/IP Stud. Adv. Math.9,31(1998)].

[26] John L. Cardy. Continuously Varying Exponents and the Value of the Central Charge. *J. Phys.*, A20:L891–L896, 1987.

[27] S. Cecotti, L. Girardello, and A. Pasquinucci. Nonperturbative Aspects and Exact Results for the $N = 2$ Landau-ginzburg Models. *Nucl. Phys.*, B328:701–722, 1989.

[28] Sergio Cecotti and Cumrun Vafa. Topological antitopological fusion. *Nucl. Phys.*, B367:359–461, 1991.

[29] Sergio Cecotti and Cumrun Vafa. On classification of N=2 supersymmetric theories. *Commun. Math. Phys.*, 158:569–644, 1993.

[30] V. Chari and A. Pressley. *A guide to quantum groups*. 1994.

[31] Cyril Closset and Stefano Cremonesi. Comments on $\mathcal{N} = (2, 2)$ supersymmetry on two-manifolds. *JHEP*, 07:075, 2014.

[32] Cyril Closset, Stefano Cremonesi, and Daniel S. Park. The equivariant A-twist and gauged linear sigma models on the two-sphere. *JHEP*, 06:076, 2015.

[33] Kevin Costello. Supersymmetric gauge theory and the Yangian. 2013.

[34] Kevin Costello. M-theory in the Omega-background and 5-dimensional non-commutative gauge theory. 2016.

[35] Kevin Costello. Holography and Koszul duality: the example of the $M2$ brane. 2017.

[36] Kevin Costello. String theory for mathematicians. http://pirsa.org/C17014, 2017. Accessed Aug 31, 2018.

[37] Kevin Costello and Davide Gaiotto. Twisted Holography. 2018.

[38] Kevin Costello and Davide Gaiotto. Vertex Operator Algebras and 3d $\mathcal{N} = 4$ gauge theories. *JHEP*, 05:018, 2019.

[39] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory*, volume 2. Cambridge University Press, 20**. To be published.

[40] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory*, volume 1 of *New Mathematical Monographs*. Cambridge University Press, 2016.

[41] Kevin Costello and Si Li. Quantization of open-closed BCOV theory, I. 2015.

[42] Kevin Costello and Si Li. Twisted supergravity and its quantization. 2016.

[43] Kevin Costello, Edward Witten, and Masahito Yamazaki. Gauge Theory and Integrability, I. 2017.

[44] Kevin Costello and Junya Yagi. In preparation.

[45] Kevin Costello and Junya Yagi. Unification of integrability in supersymmetric gauge theories. 2018.

[46] Jan de Boer, Jan Manschot, Kyriakos Papadodimas, and Erik Verlinde. The Chiral ring of AdS(3)/CFT(2) and the attractor mechanism. *JHEP*, 03:030, 2009.

[47] M. de Roo, J.W. van Holten, B. de Wit, and Antoine Van Proeyen. Chiral Superfields in $N = 2$ Supergravity. *Nucl.Phys.*, B173:175, 1980.

[48] B. de Wit, R. Philippe, and Antoine Van Proeyen. The Improved Tensor Multiplet in $N = 2$ Supergravity. *Nucl.Phys.*, B219:143, 1983.

[49] B. de Wit, J. W. van Holten, and Antoine Van Proeyen. Structure of N=2 Supergravity. *Nucl. Phys.*, B184:77, 1981. [Erratum: Nucl. Phys.B222,516(1983)].

[50] B. de Wit and J.W. van Holten. Multiplets of Linearized SO(2) Supergravity. *Nucl.Phys.*, B155:530, 1979.

[51] B. de Wit, J.W. van Holten, and Antoine Van Proeyen. Transformation Rules of N=2 Supergravity Multiplets. *Nucl.Phys.*, B167:186, 1980.

[52] B. de Wit, J.W. van Holten, and Antoine Van Proeyen. Structure of N=2 Supergravity. *Nucl.Phys.*, B184:77, 1981.

[53] Bernard de Wit, Stefanos Katmadas, and Maaike van Zalk. New supersymmetric higher-derivative couplings: Full N=2 superspace does not count! *JHEP*, 1101:007, 2011.

[54] Pierre Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift*, pages 111–195. Springer, 2007.

[55] Stanley Deser, M. J. Duff, and C. J. Isham. Nonlocal Conformal Anomalies. *Nucl. Phys.*, B111:45, 1976.

[56] Eric D'Hoker, Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Extremal correlators in the AdS / CFT correspondence. 1999.

[57] Robbert Dijkgraaf, Herman L. Verlinde, and Erik P. Verlinde. Notes on topological string theory and 2-D quantum gravity. In *Cargese Study Institute: Random Surfaces, Quantum Gravity and Strings Cargese, France, May 27-June 2, 1990*, 1990.

[58] Robbert Dijkgraaf, Herman L. Verlinde, and Erik P. Verlinde. Topological strings in d ¡ 1. *Nucl. Phys.*, B352:59–86, 1991.

[59] F. A. Dolan and H. Osborn. On short and semi-short representations for four-dimensional superconformal symmetry. *Annals Phys.*, 307:41–89, 2003.

[60] Nima Doroud and Jaume Gomis. Gauge theory dynamics and Kahler potential for Calabi-Yau complex moduli. *JHEP*, 12:99, 2013.

[61] Nima Doroud, Jaume Gomis, Bruno Le Floch, and Sungjay Lee. Exact Results in D=2 Supersymmetric Gauge Theories. *JHEP*, 05:093, 2013.

[62] Nadav Drukker, Davide Gaiotto, and Jaume Gomis. The Virtue of Defects in 4D Gauge Theories and 2D CFTs. *JHEP*, 06:025, 2011.

[63] Nadav Drukker, Simone Giombi, Riccardo Ricci, and Diego Trancanelli. Wilson loops: From four-dimensional SYM to two-dimensional YM. *Phys. Rev.*, D77:047901, 2008.

[64] S. Ferrara and P. van Nieuwenhuizen. Consistent Supergravity with Complex Spin 3/2 Gauge Fields. *Phys.Rev.Lett.*, 37:1669, 1976.

[65] Guido Festuccia and Nathan Seiberg. Rigid Supersymmetric Theories in Curved Superspace. *JHEP*, 06:114, 2011.

[66] Bartomeu Fiol, Efrat Gerchkovitz, and Zohar Komargodski. The Exact Bremsstrahlung Function in N=2 Superconformal Field Theories. 2015.

[67] Stefan Forste and Daniel Roggenkamp. Current current deformations of conformal field theories, and WZW models. *JHEP*, 05:071, 2003.

[68] Daniel Z. Freedman and Antoine Van Proeyen. *Supergravity*. Cambridge University Press, New York, USA, 2012.

[69] Daniel Z. Freedman and Antoine Van Proeyen. *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 2012.

[70] F. Fucito, J. F. Morales, and R. Poghossian. Wilson loops and chiral correlators on squashed spheres. *JHEP*, 11:064, 2015.

[71] W. Fulton, W.F.J. Harris, and J. Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics. Springer New York, 1991.

[72] Davide Gaiotto. N=2 dualities. *JHEP*, 08:034, 2012.

[73] Davide Gaiotto and Joel Lamy-Poirier. Irregular Singularities in the H_3^+ WZW Model. 2013.

[74] S. J. Gates, Jr., Marcus T. Grisaru, and M. E. Wehlau. A Study of general 2-D, N=2 matter coupled to supergravity in superspace. *Nucl. Phys.*, B460:579–614, 1996.

[75] Efrat Gerchkovitz, Jaume Gomis, Nafiz Ishtiaque, Avner Karasik, Zohar Komargodski, and Silviu S. Pufu. Correlation Functions of Coulomb Branch Operators. *JHEP*, 01:103, 2017.

[76] Efrat Gerchkovitz, Jaume Gomis, and Zohar Komargodski. Sphere Partition Functions and the Zamolodchikov Metric. *JHEP*, 11:001, 2014.

[77] Simone Giombi and Shota Komatsu. Exact Correlators on the Wilson Loop in $\mathcal{N} = 4$ SYM: Localization, Defect CFT, and Integrability. *JHEP*, 05:109, 2018.

[78] Simone Giombi and Vasily Pestun. Correlators of local operators and 1/8 BPS Wilson loops on S^2 from 2d YM and matrix models. *JHEP*, 10:033, 2010.

[79] Simone Giombi, Radu Roiban, and Arkady A. Tseytlin. Half-BPS Wilson loop and $\mathrm{AdS}_2/\mathrm{CFT}_1$. *Nucl. Phys.*, B922:499–527, 2017.

[80] Amit Giveon and David Kutasov. Brane dynamics and gauge theory. *Rev. Mod. Phys.*, 71:983–1084, 1999.

[81] Jaume Gomis and Nafiz Ishtiaque. Kahler potential and ambiguities in 4d $\mathcal{N} = 2$ SCFTs. *JHEP*, 04:169, 2015.

[82] Jaume Gomis, Zohar Komargodski, Po-Shen Hsin, Adam Schwimmer, Nathan Seiberg, and Stefan Theisen. Anomalies, Conformal Manifolds, and Spheres. 2015.

[83] Jaume Gomis and Sungjay Lee. Exact Kahler Potential from Gauge Theory and Mirror Symmetry. *JHEP*, 04:019, 2013.

[84] Jaume Gomis, Shunji Matsuura, Takuya Okuda, and Diego Trancanelli. Wilson loop correlators at strong coupling: From matrices to bubbling geometries. *JHEP*, 08:068, 2008.

[85] Jaume Gomis and Takuya Okuda. D-branes as a Bubbling Calabi-Yau. *JHEP*, 07:005, 2007.

[86] Jaume Gomis and Takuya Okuda. Wilson loops, geometric transitions and bubbling Calabi-Yau's. *JHEP*, 02:083, 2007.

[87] Jaume Gomis and Takuya Okuda. S-duality, 't Hooft operators and the operator product expansion. *JHEP*, 09:072, 2009.

[88] Jaume Gomis, Takuya Okuda, and Vasily Pestun. Exact Results for 't Hooft Loops in Gauge Theories on S4. *JHEP*, 05:141, 2012.

[89] Jaume Gomis and Filippo Passerini. Holographic Wilson Loops. *JHEP*, 08:074, 2006.

[90] Jaume Gomis and Filippo Passerini. Wilson Loops as D3-Branes. *JHEP*, 01:097, 2007.

[91] Rajesh Gopakumar and Cumrun Vafa. On the gauge theory / geometry correspondence. *Adv. Theor. Math. Phys.*, 3:1415–1443, 1999. [AMS/IP Stud. Adv. Math.23,45(2001)].

[92] Daniel Green, Zohar Komargodski, Nathan Seiberg, Yuji Tachikawa, and Brian Wecht. Exactly Marginal Deformations and Global Symmetries. *JHEP*, 06:106, 2010.

[93] Marcus T. Grisaru and M. E. Wehlau. Superspace measures, invariant actions, and component projection formulae for (2,2) supergravity. *Nucl. Phys.*, B457:219–239, 1995.

[94] Marcus T. Grisaru and Marcia E. Wehlau. Prepotentials for (2,2) supergravity. *Int. J. Mod. Phys.*, A10:753–766, 1995.

[95] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett.*, B428:105–114, 1998.

[96] Sergei Gukov and Edward Witten. Gauge Theory, Ramification, And The Geometric Langlands Program. 2006.

[97] Naofumi Hama and Kazuo Hosomichi. Seiberg-Witten Theories on Ellipsoids. *JHEP*, 1209:033, 2012.

[98] Naofumi Hama, Kazuo Hosomichi, and Sungjay Lee. Notes on SUSY Gauge Theories on Three-Sphere. *JHEP*, 03:127, 2011.

[99] Ryogo Hirota, Yasuhiro Ohta, and Junkichi Satsuma. Solutios of the kadomtsev-petviashvili equation and the two-dimensional toda equations. *Journal of the Physical Society of Japan*, 57(6):1901–1904, 1988.

[100] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow. *Mirror symmetry*, volume 1 of *Clay mathematics monographs*. AMS, Providence, RI, 2003. With a preface by Vafa.

[101] Kazuo Hosomichi, Sungjay Lee, and Jaemo Park. AGT on the S-duality Wall. *JHEP*, 12:079, 2010.

[102] P. S. Howe and G. Papadopoulos. N=2, d=2 supergeometry. *Classical and Quantum Gravity*, 4:11–21, January 1987.

[103] Kenneth A. Intriligator. Bonus symmetries of N=4 superYang-Mills correlation functions via AdS duality. *Nucl. Phys.*, B551:575–600, 1999.

[104] Nafiz Ishtiaque. 2D BPS Rings from Sphere Partition Functions. *JHEP*, 04:124, 2018.

[105] Nafiz Ishtiaque, Seyed Faroogh Moosavian, and Yehao Zhou. Topological Holography: The Example of The D2-D4 Brane System. 2018.

[106] Daniel L. Jafferis. The Exact Superconformal R-Symmetry Extremizes Z. *JHEP*, 05:159, 2012.

[107] Hans Jockers, Vijay Kumar, Joshua M. Lapan, David R. Morrison, and Mauricio Romo. Two-Sphere Partition Functions and Gromov-Witten Invariants. *Commun. Math. Phys.*, 325:1139–1170, 2014.

[108] André Joyal and Ross Street. An introduction to tannaka duality and quantum groups. In *Category theory*, pages 413–492. Springer, 1991.

[109] Anton Kapustin, Brian Willett, and Itamar Yaakov. Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter. *JHEP*, 03:089, 2010.

[110] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. 2006.

[111] Marek Karliner. Precise estimates of high orders in QCD. *Acta Phys. Polon.*, B29:1505–1520, 1998.

[112] Sergei V. Ketov. 2-d, N=2 and N=4 supergravity and the Liouville theory in superspace. *Phys. Lett.*, B377:48–54, 1996.

[113] Justin Kinney, Juan Martin Maldacena, Shiraz Minwalla, and Suvrat Raju. An Index for 4 dimensional super conformal theories. *Commun. Math. Phys.*, 275:209–254, 2007.

[114] Claudius Klare and Alberto Zaffaroni. Extended Supersymmetry on Curved Spaces. *JHEP*, 1310:218, 2013.

[115] Maxim Kontsevich. Homological Algebra of Mirror Symmetry. 1994.

[116] David Kutasov. Geometry on the Space of Conformal Field Theories and Contact Terms. *Phys. Lett.*, B220:153, 1989.

[117] Sergei M. Kuzenko. Super-Weyl anomalies in N=2 supergravity and (non)local effective actions. *JHEP*, 1310:151, 2013.

[118] Sergei M. Kuzenko and Gabriele Tartaglino-Mazzucchelli. Different representations for the action principle in 4D N = 2 supergravity. *JHEP*, 04:007, 2009.

[119] Sangmin Lee, Shiraz Minwalla, Mukund Rangamani, and Nathan Seiberg. Three point functions of chiral operators in D = 4, N=4 SYM at large N. *Adv. Theor. Math. Phys.*, 2:697–718, 1998.

[120] Robert G. Leigh and Matthew J. Strassler. Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory. *Nucl. Phys.*, B447:95–136, 1995.

[121] Madalena Lemos and Pedro Liendo. $\mathcal{N} = 2$ central charge bounds from $2d$ chiral algebras. 2015.

[122] Madalena Lemos and Pedro Liendo. Bootstrapping $\mathcal{N} = 2$ chiral correlators. *JHEP*, 01:025, 2016.

[123] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner. Chiral Rings in N=2 Superconformal Theories. *Nucl. Phys.*, B324:427–474, 1989.

[124] Pedro Liendo, Israel Ramirez, and Jihye Seo. Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories. *JHEP*, 02:019, 2016.

[125] Gabriel Lopes Cardoso, Bernard de Wit, and Thomas Mohaupt. Corrections to macroscopic supersymmetric black hole entropy. *Phys.Lett.*, B451:309–316, 1999.

[126] Andrei S. Losev, Andrei Marshakov, and Nikita A. Nekrasov. Small instantons, little strings and free fermions. 2003.

[127] Juan Martin Maldacena. Wilson loops in large N field theories. *Phys. Rev. Lett.*, 80:4859–4862, 1998.

[128] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)].

[129] W.G. McKay, J. Patera, and D.W. Rand. *Tables of Representations of Simple Lie Algebras: Exceptional simple lie algebras*. Tables of Representations of Simple Lie Algebras. Centre de Recherches Mathématiques, Université de Montréal, 1990.

[130] David R. Morrison and M. Ronen Plesser. Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties. *Nucl. Phys.*, B440:279–354, 1995.

[131] G. Mussardo, G. Sotkov, and M. Stanishkov. N=2 SUPERCONFORMAL MINIMAL MODELS. *Int. J. Mod. Phys.*, A4:1135, 1989.

[132] Hiraku Nakajima and Kota Yoshioka. Lectures on instanton counting. In *CRM Workshop on Algebraic Structures and Moduli Spaces Montreal, Canada, July 14-20, 2003*, 2003.

[133] Nikita Nekrasov and Andrei Okounkov. Seiberg-Witten theory and random partitions. *Prog. Math.*, 244:525–596, 2006.

[134] Nikita A. Nekrasov. Seiberg-Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.*, 7(5):831–864, 2003.

[135] Jihwan Oh and Junya Yagi. Chiral algebras from Ω-deformation. 2019.

[136] Takuya Okuda and Vasily Pestun. On the instantons and the hypermultiplet mass of N=2* super Yang-Mills on S^4. *JHEP*, 03:017, 2012.

[137] H. Ooguri and C. Vafa. Knot invariants and topological strings. *Nucl. Phys. B*, 577:419, 2000.

[138] Hirosi Ooguri and Cumrun Vafa. World sheet derivation of a large N duality. *Nucl. Phys.*, B641:3–34, 2002.

[139] Kyriakos Papadodimas. Topological Anti-Topological Fusion in Four-Dimensional Superconformal Field Theories. *JHEP*, 08:118, 2010.

[140] Vasily Pestun. Localization of gauge theory on a four-sphere and supersymmetric Wilson loops. *Commun. Math. Phys.*, 313:71–129, 2012.

[141] Antoine Van Proeyen. N = 2 supergravity in d = 4, 5, 6 and its matter couplings. http://itf.fys.kuleuven.be/~toine/LectParis.pdf.

[142] Soo-Jong Rey and Jung-Tay Yee. Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity. *Eur. Phys. J.*, C22:379–394, 2001.

[143] L. Rozansky and Edward Witten. HyperKahler geometry and invariants of three manifolds. *Selecta Math.*, 3:401–458, 1997.

[144] Jorge G. Russo. A Note on perturbation series in supersymmetric gauge theories. *JHEP*, 06:038, 2012.

[145] Nathan Seiberg. Observations on the Moduli Space of Superconformal Field Theories. *Nucl. Phys.*, B303:286, 1988.

[146] Nathan Seiberg. Naturalness versus supersymmetric nonrenormalization theorems. *Phys.Lett.*, B318:469–475, 1993.

[147] S. J. Summers. Yet More Ado About Nothing: The Remarkable Relativistic Vacuum State. *ArXiv e-prints*, February 2008.

[148] Zhihong Sun. Linear recursive sequences and powers of matrices. *The Fibonacci Quarterly*, 39, 08 2001.

[149] Yuji Tachikawa. A pseudo-mathematical pseudo-review on 4d n=2 supersymmetric quantum field theories. https://member.ipmu.jp/yuji.tachikawa/tmp/review-rebooted7.pdf. Accessed: 2019-05-21.

[150] Eric W. Weisstein. Exotic sphere. MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/ExoticSphere.html. Accessed: 2019-06-12.

[151] Edward Witten. Topological Sigma Models. *Commun. Math. Phys.*, 118:411, 1988.

[152] Edward Witten. On the Structure of the Topological Phase of Two-dimensional Gravity. *Nucl. Phys.*, B340:281–332, 1990.

[153] Edward Witten. Mirror manifolds and topological field theory. pages 121–160, 1991. [AMS/IP Stud. Adv. Math.9,121(1998)].

[154] Edward Witten. Phases of N=2 theories in two-dimensions. *Nucl. Phys.*, B403:159–222, 1993. [AMS/IP Stud. Adv. Math.1,143(1996)].

[155] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.

[156] Edward Witten. Analytic Continuation Of Chern-Simons Theory. *AMS/IP Stud. Adv. Math.*, 50:347–446, 2011.

[157] Junya Yagi. Ω-deformation and quantization. *JHEP*, 08:112, 2014.

[158] Satoshi Yamaguchi. Bubbling geometries for half BPS Wilson lines. *Int. J. Mod. Phys.*, A22:1353–1374, 2007.

[159] A. B. Zamolodchikov. Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory. *JETP Lett.*, 43:730–732, 1986. [Pisma Zh. Eksp. Teor. Fiz.43,565(1986)].

Appendix A

BV in Finite Dimensions

The Batalin-Vilkovisky (BV) formalism imposes relations among the observables of a field theory coming from equations of motion. In mathematical terms, it constructs the derived critical locus of the action on the field space. It is significantly easier to describe the BV formalism for functions on a finite dimensional manifold, which corresponds to operators of a 0-dimensional QFT. For this reason, putting aside the issue that the structure of a factorization algebra is lost on observables of a 0-dimensional field theory, we introduce the BV method of constructing the critical locus of the action for such a theory.

Consider a 0-dimensional QFT of maps from a point to a finite dimensional target space X, which we take to be $\mathbb{R}^n$. The field space is the space of such maps, which is X itself:

$$\text{Space of fields, } \mathcal{E} = X = \mathbb{R}^n \,. \tag{A.1}$$

The action of the theory is a polynomial functional $S : X \to \mathbb{R}$ which is bounded from bellow and grows at least quadratically at infinity. Space of observables is the space of functions on the field space which does not grow too fast at infinity,[1] for convenience we consider polynomial functions:

$$\text{Space of observables, Obs} = \mathbb{R}[x^1, \cdots, x^n] =: \mathrm{Pol}(X)\,. \tag{A.2}$$

Expectation value of an observable $O \in \mathrm{Obs}$ is given by the integral:

$$\langle O \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n x \, e^{-S(x)/\hbar} O(x)\,. \tag{A.3}$$

[1]We want functions f such that $e^{-S/\hbar} f$ decays exponentially at infinity.

For an integration of the form $\int \mathrm{d}^n x\, O$, Stokes' theorem tells us that if O is the divergence of a vector field then the integral vanishes. In a completely analogous way, for the modified integration measure $\mathrm{d}^n x\, e^{-S/\hbar}$ one can define a modified divergence map whose images vanish under integration. For any real function f, let us define the integration measure:

$$\omega_f := \mathrm{d}^n x\, e^{-f}\,. \tag{A.4}$$

Then we can define a modified divergence map $\mathrm{Div}_{\omega_f} : \mathrm{Vec}(X) \to \mathrm{Pol}(X)$ from polynomial vector fields to polynomial functions whose image vanishes under integration. For a polynomial vector field $V = V^i \partial_{x^i}$ we define:

$$\mathrm{Div}_{S/\hbar}(V) = \partial_{x^i} V^i - \frac{1}{\hbar} V^i \frac{\partial S}{\partial x^i}\,. \tag{A.5}$$

The first term is the usual divergence of the measure $\mathrm{d}^n x$ and the second term is the modification due to the modified emasure $\mathrm{d}^n x\, e^{-S/\hbar}$. Stoke's theorem for this modified measure tells us that we have an exact sequence

$$0 \to \mathrm{Vec}(X) \xrightarrow{\mathrm{Div}_{S/\hbar}} \mathrm{Pol}(X) \xrightarrow{\int_{\mathbb{R}^n} \mathrm{d}^n x\, e^{-S/\hbar}} \mathbb{R} \to 0\,. \tag{A.6}$$

This is a motivation to construct the cokernel of $\mathrm{Div}_{S/\hbar}$:

$$\operatorname{coker} \mathrm{Div}_{S/\hbar} = \frac{\mathrm{Pol}(X)}{\operatorname{im} \mathrm{Div}_{S/\hbar}}\,. \tag{A.7}$$

BV is a homological method of implementing this quotient.

Remark A.0.1 (A peculiarity in 0-dimension). Naively, we would have liked to define the above quotient of $\mathrm{Pol}(X)$, instead of $\mathrm{Pol}(X)$, as the space of observables, since such observables are guaranteed to have zero expectation value. However, this does not make sense in 0-dimension, because the quotient is not closed under multiplication, i.e., for $O_1, O_2 \in \operatorname{im} \mathrm{Div}_{S/\hbar}$, their product $O_1 O_2$ is not necessarily a divergence. This non-closure disappears in higher dimension if we consider the factorization algebraic structure[2] on the space of observables [40] and we would indeed define the higher dimensional analogue of the above quotient as the physical space of observables. △

[2] Which we can not do in 0-dimension, indeed, all operators in a 0-dimensional QFT must be placed at the same 0-dimensional point.

A.0.1 Divergence Complex

The divergence map $\mathrm{Vec}(\mathbb{R}^n) \xrightarrow{\mathrm{Div}_{S/\hbar}} \mathrm{Pol}(X)$ can be thought of as the tail end of a cochain complex.

Let us denote by $\mathrm{Pol}(X, \wedge^k TX)$ the space of polyvector fields on X of degree k with polynomial coefficients. In particular, $\mathrm{Pol}(X, \wedge^1 TX)$ is the space of polynomial vector fields and $\mathrm{Pol}(X, \wedge^0 TX)$ is the space of polynomial functions. There is an isomorphism between k-polyvector fields $\mathrm{Pol}(X, \wedge^n TX)$ and $(n-k)$-forms $\Omega^{n-k}(X)$:

$$\begin{gathered} \mathrm{Pol}(X, \wedge^k TX) \xrightarrow{\sim} \Omega^k \,, \\ \partial_{x^{i_1}} \wedge \cdots \wedge \partial_{x^{i_k}} \mapsto \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^{i_1}} \wedge \cdots \wedge \widehat{\mathrm{d}x^{i_k}} \wedge \cdots \wedge \mathrm{d}x^n \,. \end{gathered} \tag{A.8}$$

The hat over a form means that the form is absent. This isomorphism can be used to identify the action of the divergence map on the vector fields with the action of a "twisted" de Rham differential

$$\mathrm{d}_{S/\hbar} := \mathrm{d} - \frac{1}{\hbar}\mathrm{d}S\wedge$$

on $(n-1)$-forms, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathrm{Pol}(X, TX) \arrow[r, "\mathrm{Div}_{S/\hbar}"] \arrow[d, "\wr"'] & \mathrm{Pol}(X) \arrow[r, "\mathrm{Div}_{S/\hbar}"] \arrow[d, "\wr"'] & 0 \\ \Omega^{n-1}(X) \arrow[r, "\mathrm{d}_{S/\hbar}"'] & \Omega^n(X) \arrow[r, "\mathrm{d}_{S/\hbar}"'] & 0 \end{tikzcd} \tag{A.9}$$

In fact, the isomorphism (A.8) can be used to define the action of the divergence map on polyvector fields of arbitrary degree by demanding commutativity of the following diagram:

$$\begin{tikzcd} \cdots \arrow[r, "\mathrm{Div}_{S/\hbar}"] & \mathrm{Pol}(X, \wedge^2 TX) \arrow[r, "\mathrm{Div}_{S/\hbar}"] \arrow[d, "\wr"'] & \mathrm{Pol}(X, TX) \arrow[r, "\mathrm{Div}_{S/\hbar}"] \arrow[d, "\wr"'] & \mathrm{Pol}(X) \arrow[r, "\mathrm{Div}_{S/\hbar}"] \arrow[d, "\wr"'] & 0 \\ \cdots \arrow[r, "\mathrm{d}_{S/\hbar}"'] & \Omega^{n-2}(X) \arrow[r, "\mathrm{d}_{S/\hbar}"'] & \Omega^{n-1}(X) \arrow[r, "\mathrm{d}_{S/\hbar}"'] & \Omega^n(X) \arrow[r, "\mathrm{d}_{S/\hbar}"'] & 0 \end{tikzcd} \tag{A.10}$$

By construction, the divergence map is nilpotent:

$$\mathrm{Div}_{S/\hbar} \circ \mathrm{Div}_{S/\hbar} = 0 \,. \tag{A.11}$$

The top row of the above diagram is called *the divergence complex*. The cokernel from (A.7) appears as its cohomology at degree zero:

$$\frac{\mathrm{Pol}(X)}{\mathrm{im}\,\mathrm{Div}^1_{S/\hbar}} = H^0_{\mathrm{Div}_{S/\hbar}}(\mathrm{Pol}(X, \wedge^\bullet TX))\,, \tag{A.12}$$

where the superscript on $\mathrm{Div}_{S/\hbar}$ labels the degree of the polyvector fields on which the divergence map acts. The above formula already gives a homological description of the critical locus of the action. The only remaining question from the physical perspective is how to interpret a general polyvector on X in the field theory language – the way we say that functions on X are observables in the field theory. We can in fact interpret the polyvectors as observalbes as well if we extend the space of fields from appropriately – by introducing anti-fields.

A.0.2 Anti-field Formalism

We choose local coordinates x^i on X, these x^i will appear as fields of the theory. For each field x^i we introduce an *anti-field* θ_i which we take to be fermionic.[3] These anti-fields are simply a way of writing polyvector fields as ordinary fields as follows:

$$\begin{array}{cc} \text{Polyvector} & \text{Operator with fields and anti-fields} \\ f^{i_1\cdots i_k}(x)\partial_{x^{i_1}} \wedge \cdots \wedge \partial_{x^{i_k}} & f^{i_1\cdots i_k}(x)\theta_{x^{i_1}} \cdots \theta_{x^{i_k}} \end{array} \tag{A.13}$$

In order to be able to establish a cochain complex, we assign cohomological degree -1 to the anti-fields and 0 to the fields as usual. Now note that, just as operators involving only x^i are polynomial functions on X, we can interpret operators involving x^i and θ_i as polynomial functions on $T^*[-1]X$. The notation means that we consider the cotangential directions to have cohomological degree 1, coordinate functions along the cotangential directions are *dual* to the cotangent vectors and therefore they have cohomological degree -1, these are precisely the θ_i's. The space of fields in the BV formulation is this extend space including the anti-fields:

$$\text{Space of fields, } \mathcal{E} := T^*[-1]X\,, \tag{A.14}$$

and, the observables are the polynomial functions on this space:

$$\text{Space of observables, Obs} = \mathrm{Pol}^\bullet(T^*[-1]X)\,. \tag{A.15}$$

[3]This is assuming bosonic statistics for the fields x^i. If x^i has fermionic statistics then θ_i will be bosonic.

Operators in $\mathrm{Pol}^i(T^*[-1]X)$ contain i anti-fields. The divergence map from the divergence complex in (A.10) acting on these observables looks like:

$$\Delta^{\mathrm{BV}}_{S/\hbar} = \Delta^{\mathrm{BV}} - \frac{1}{\hbar}\frac{\partial S}{\partial x^i}\frac{\partial}{\partial \theta_i}\,, \tag{A.16}$$

where Δ^{BV} is the following differential operator:

$$\Delta^{\mathrm{BV}} = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \theta_i}\,. \tag{A.17}$$

Δ^{BV} is called the *BV Laplacian*. The divergence complex, written in terms of the anti-fields and the BV differential, is called the BV complex:

$$\cdots \xrightarrow{\Delta^{\mathrm{BV}}_{S/\hbar}} \mathrm{Pol}^1(T^*[-1]X) \xrightarrow{\Delta^{\mathrm{BV}}_{S/\hbar}} \mathrm{Pol}^0(T^*[-1]X) \xrightarrow{\Delta^{\mathrm{BV}}_{S/\hbar}} 0\,. \tag{A.18}$$

This is the desired homological model for the critical locus of the action:

$$\text{Critical locus of the action: } H^0_{\Delta^{\mathrm{BV}}_{S/\hbar}}\left(\mathrm{Pol}^\bullet(T^*[-1]X)\right)\,. \tag{A.19}$$

The final ingredient we are going to introduce from the BV formalism is the *shifted symplectic structure* on the field space $T^*[-1]X$, defined by a degree 1 Poisson bracket – for two *homogeneous* functions $F, G \in \mathrm{Pol}^\bullet(T^*[-1]X)$ their bracket is defined using the BV Laplacian:[4]

$$\{F, G\} = \Delta^{\mathrm{BV}}(FG) - \Delta^{\mathrm{BV}}(F)G - (-1)^{|F|}F\Delta^{\mathrm{BV}}(G)\,. \tag{A.20}$$

where $|F|$ and $|G|$ denote the degree (number of θ's) of the respective functions. The above bracket is graded anti-symmetric, satisfies a graded Jacobi identity, and imposes the following commutation relation on the coordinate functions:

$$\{x^i, \theta_j\} = \delta^i_j\,. \tag{A.21}$$

In terms of this bracket the differential operator $\Delta^{\mathrm{BV}}_{S/\hbar}$ can be written as:

$$\Delta^{\mathrm{BV}}_{S/\hbar} = \Delta^{\mathrm{BV}} - \frac{1}{\hbar}\{S, -\}\,. \tag{A.22}$$

[4]This is simply the *Schouten-Nijenhuis (SN) bracket* of polyvector fields. The relation between the SN bracket and the divergence map (the BV Laplacian in terms of the anti-fields) follows form the isomorphism between the divergence complex of polyvector fields and the de Rham complex using the fact that the de Rham differential is a derivation for the wedge product.

Finally, for us to be able to use the complex (A.18) as a homological model for the critical locus of the action[5] it must be a complex to begin with, i.e., the differential must be nilpotent:

$$\Delta^{\mathrm{BV}}_{S/\hbar} \circ \Delta^{\mathrm{BV}}_{S/\hbar} = 0 \,. \tag{A.23}$$

This is automatic, by construction, in this finite dimensional case.

The key points to remember from the finite dimensional case is the following: A field space is a shifted symplectic manifold $\mathcal{E}$ with a degree 1 differential Δ^{BV}, which defines a degree one Poisson bracket $\{-,-\}$. The theory is defined by an action functional S. The degree 1 differential $\Delta^{\mathrm{BV}}_{S/\hbar} := \Delta^{\mathrm{BV}} - \frac{1}{\hbar}\{S,-\}$ which acts on a function space $\mathscr{O}(\mathcal{E})$ is nilpotent. To this data we assign a cochain complex $(\mathscr{O}(\mathcal{E}), \Delta^{\mathrm{BV}}_{S/\hbar})$ and the observables of the theory are elements of its 0th cohomology, namely $H^0_{\Delta^{\mathrm{BV}}_{S/\hbar}}(\mathscr{O}(\mathcal{E}))$.

Remark A.0.2 (Path integrals in the BV formulation)*.* It may seem odd that we have extended the field space by brute force and it is then a legitimate question as to how path integrals in the BV formulation relates to path integrals in the original formulation. The answer comes from the following observation[6] – The original space of fields X is a particular Lagrangian subspace – namely the zero section – of the symplectic manifold $T^*[-1]X$. One can extend the action functional $S : X \to \mathbb{R}$ to this entire symplectic space $S \rightsquigarrow S' : T^*[-1]X \to \mathbb{R}$ in such a way that the restriction of S' to the zero section X is the original action S and furthermore, provided that the new action S' satisfies certain consistency conditions, integrals of the form $\int_{\mathcal{L}} e^{-S'/\hbar}\mathcal{O}$ over Lagrangian subspaces $\mathcal{L} \subseteq T^*[-1]X$ are invariant under smooth deformations of the Lagrangian. Path integrals in the BV formalism are therefore performed over suitably chosen Lagrangians which are smooth deformations of the original field space and all correlation functions remain invariant. $\triangle$

If we consider a QFT on the space-time M of dimension $d > 0$, then the space of fields is going to be some infinite dimensional space, such as the space of functions on M in case of a scalar field theory (1.7). Divergences arise when we try to perform integrals on the field space and we are forced to talk about effective field theories where interactions are allowed only above some certain length scale. The structure we found in the previous section picks up this scale dependence and because we are moving from a 0d space-time to a higher dimensional space-time, the whole structure generalizes as a (co)sheaf over the space-time.

[5] In other words, we want to be able to use the generalized Stoke's theorem.

[6] Which we do not prove here but refer to [40].

Appendix B

Backgraound Materials on 2d BPS Rings

B.1 The $\mathcal{N} = (2,2)$ superconformal Algebra

We use complex coordinates on $\mathbb{R}^2$:

$$z = x + iy\,, \quad \bar{z} = x - iy\,. \tag{B.1}$$

The $\mathcal{N} = (2,2)$ superconformal algebra[1] $\mathfrak{su}(2|2)$ contains eight supercharges:

$$Q_+\,, Q_-\,, \overline{Q}_+\,, \overline{Q}_-\,, \quad \text{and,} \quad S_+\,, S_-\,, \overline{S}_+\,, \overline{S}_-\,, \tag{B.2}$$

and the following bosonic generators:

$$\begin{aligned} \text{Rotation in } \mathbb{R}^2,\ \mathfrak{u}(1)_L :&\quad 2(\overline{L}_0 - L_0) =: 2J_L \\ \text{Dilatation :}&\quad L_0 + \overline{L}_0 =: \Delta \\ \text{Translations :}&\quad L_{-1}, \overline{L}_{-1} \\ \text{Special conformal transformations :}&\quad L_1, \overline{L}_1 \\ \text{Vector R-symmetry, } \mathfrak{u}(1)_V :&\quad J_V \\ \text{Axial R-symmetry, } \mathfrak{u}(1)_A :&\quad J_A \end{aligned} \tag{B.3}$$

[1] We are only concerned with algebra that is globally defined on $\mathbb{R}^2$, i.e., the globally defined subalgebra of the super Virasoro algebra.

where $L_s := -z^{s+1}\partial_z$ and $\overline{L}_s := -\overline{z}^{s+1}\partial_{\overline{z}}$ gnerate the conformal algebra $\mathrm{conf}(\mathbb{R}^2) = \mathfrak{so}(3,1)$:

$$[L_r, L_s] = (r-s)L_{r+s}\,, \quad [\overline{L}_r, \overline{L}_s] = (r-s)\overline{L}_{r+s}\,, \qquad r,s \in \{-1,0,1\} \tag{B.4}$$

The nonzero anti-commutation relations of the supercharges (B.2) are:

$$\{Q_+, \overline{Q}_+\} = 2L_{-1}\,, \qquad \{Q_-, \overline{Q}_-\} = 2\overline{L}_{-1}\,, \tag{B.5a}$$

$$\{S_+, \overline{S}_+\} = 2L_1\,, \qquad \{S_-, \overline{S}_-\} = 2\overline{L}_1\,, \tag{B.5b}$$

$$\{Q_+, \overline{S}_+\} = 2L_0 + \frac{1}{2}(J_V + J_A)\,, \qquad \{Q_-, \overline{S}_-\} = 2\overline{L}_0 + \frac{1}{2}(J_V - J_A)\,, \tag{B.5c}$$

$$\{\overline{Q}_+, S_+\} = 2L_0 - \frac{1}{2}(J_V + J_A)\,, \qquad \{\overline{Q}_-, S_-\} = 2\overline{L}_0 - \frac{1}{2}(J_V - J_A)\,. \tag{B.5d}$$

The commutators of the supercharges with the $\mathfrak{u}(1)$'s and the dilatation are conveniently expressed by specifying the charges of the supercharges under the respective generators:

	$Q_\pm$	$\overline{Q}_\pm$	$S_\pm$	$\overline{S}_\pm$
$2J_L$	$\mp$	$\mp$	$\pm$	$\pm$
J_V	$-$	$+$	$-$	$+$
J_A	$\mp$	$\pm$	$\mp$	$\pm$
2Δ	$+$	$+$	$-$	$-$

(B.6)

The rest of the nonzero commutators of $\mathfrak{su}(2|2)$ are:

$$\begin{array}{llll} [L_1, Q_+] = S_+\,, & [L_1, \overline{Q}_+] = \overline{S}_+\,, & [L_{-1}, S_+] = -Q_+\,, & [L_{-1}, \overline{S}_+] = -\overline{Q}_+\,, \\ [\overline{L}_1, Q_-] = S_-\,, & [\overline{L}_1, \overline{Q}_-] = \overline{S}_-\,, & [\overline{L}_{-1}, S_-] = -Q_-\,, & [\overline{L}_{-1}, \overline{S}_-] = -\overline{Q}_-\,. \end{array} \tag{B.7}$$

B.2 Supersymmetry on the sphere

A theory with $\mathcal{N} = (2,2)$ superconformal symmetry, namely the symmetry algebra $\mathfrak{su}(2|2)$, can be put on the two-sphere by a Weyl transformation, classically preserving the full superconformal symmetry. Though UV regularization will break the $\mathfrak{su}(2|2)$ symmetry to an $\mathfrak{su}(2|1)$ subalgebra. On the other hand, a nonconformal theory, such as a guage theory, can preserve even classically only an $\mathfrak{su}(2|1)$ subalgebra of the full $\mathfrak{su}(2|2)$ superconformal algebra.

$\mathfrak{su}(2|1)$ subalgebras of $\mathfrak{su}(2|2)$ consist of the isometries of the two-sphere, supercharges that generate these isometries and a $\mathfrak{u}(1)$ subalgebra of the $\mathfrak{u}(1)_V \times \mathfrak{u}(1)_A$ R-symmetry algebra of $\mathfrak{su}(2|2)$.[2] There are two non-equivalent[3] $\mathfrak{su}(2|1)$ subalgebras of $\mathfrak{su}(2|2)$, one

[2]If the theory is to flow to a nontrivial CFT in the IR the theory must preserve both of the U(1) R-symmetries.

[3]Not related by any inner automorphism of $\mathfrak{su}(2|2)$.

of them contains the vector R-symmetry $\mathfrak{u}(1)_V$ and the other one contains the axial R-symmetry $\mathfrak{u}(1)_A$. They are referred to as $\mathfrak{su}(2|1)_A$ and $\mathfrak{su}(2|1)_B$ respectively.[4]

The two supersymmetric sphere backgrounds can be derived as two different supergravity backgrounds preserving four supercharges. We will refer to the $\mathfrak{su}(2|1)_A$ and $\mathfrak{su}(2|1)_B$ preserving sphere backgrounds as "background-A" and "background-B" respectively. The conformal Killing spinor equations are:

$$\nabla_m \epsilon(x) = \eta(x)\,, \qquad \nabla_m \widetilde{\epsilon}(x) = \widetilde{\eta}(x)\,, \tag{B.8}$$

where ϵ, $\widetilde{\epsilon}$, η and $\widetilde{\eta}$, parametrize the $\overline{Q}$, Q, $\overline{S}$, and S transformations. The two sphere backgrounds are defined by imposing constraints on the S-supersymmetris as we discuss in the following.

Background-A

The Killing spinor equation of the supersymmetric S^2 background preserving the vector R-symmetry is found by imposing on η and $\widetilde{\eta}$ the follwoing constraints [5, 15, 31, 76]:

$$\eta = \frac{i}{2r}\epsilon\,, \qquad \widetilde{\eta} = \frac{i}{2r}\widetilde{\epsilon}\,. \tag{B.9}$$

So that the Killing spinor equations end up being:

$$\nabla_m \epsilon(x) = \frac{i}{2r}\gamma_m \epsilon(x) \qquad \nabla_m \widetilde{\epsilon}(x) = \frac{i}{2r}\gamma_m \widetilde{\epsilon}(x)\,, \tag{B.10}$$

where r is the radius of the sphere, and the covariant derivative ∇_m does not contain any background field other than the spin connection. The Killing spinor equations (B.10) have a (complex) four dimensional space of solutions that can be written as:[5]

$$\epsilon^A_{\chi_0,\widetilde{\chi}_0}(x) = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\left(\mathbb{1} + \frac{i}{2r}x^m\Gamma_m\right)\chi_0\,, \tag{B.11a}$$

$$\widetilde{\epsilon}^A_{\chi_0,\widetilde{\chi}_0}(x) = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\left(\mathbb{1} + \frac{i}{2r}x^m\Gamma_m\right)\widetilde{\chi}_0\,. \tag{B.11b}$$

Here χ_0 and $\widetilde{\chi}_0$ are two constant Dirac spinors parametrizing the space of solutions.

[4] Note the slightly unfortunate notation that $\mathfrak{u}(1)_A$ is contained in $\mathfrak{su}(2|1)_B$ and not in $\mathfrak{su}(2|1)_A$.

[5] We are using γ and Γ to refer to the curved space and flat space gamma matrices respectively. In stereographic coordinate the metric on the sphere of radius r is $\left(1+\frac{x^2}{4r^2}\right)^{-2}\mathrm{diag}(1,1)$.

Background-B

Analogously, the axial R-symmetry preserving background is defined by imposing in the conformal Killing spinor equation (B.8) [76]:

$$\eta = \frac{i}{2r}\widetilde{\epsilon}, \qquad \widetilde{\eta} = \frac{i}{2r}\epsilon\,, \tag{B.12}$$

so that we have the following Killing spinor equations:

$$\nabla_m \epsilon(x) = \frac{i}{2r}\gamma_m \widetilde{\epsilon}(x) \qquad \nabla_m \widetilde{\epsilon}(x) = \frac{i}{2r}\gamma_m \epsilon(x)\,. \tag{B.13}$$

These can be solved by defining:

$$\varepsilon := \epsilon_+ + \widetilde{\epsilon}_-\,, \qquad \widetilde{\varepsilon} := \epsilon_- + \widetilde{\epsilon}_+ \tag{B.14}$$

which satisfy the already solved equations (B.10):

$$\nabla_m \varepsilon = \frac{i}{2r}\gamma_m \varepsilon \qquad \nabla_m \widetilde{\varepsilon} = \frac{i}{2r}\gamma_m \widetilde{\varepsilon}\,. \tag{B.15}$$

Thus we find that the solutions to (B.13) are given by:

$$\epsilon^B_{\chi_0,\widetilde{\chi}_0}(x) = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\left(\chi_{0+} + \widetilde{\chi}_{0-} + \frac{i}{2r}x^m\Gamma_m(\chi_{0-} + \widetilde{\chi}_{0+})\right)\,, \tag{B.16a}$$

$$\widetilde{\epsilon}^B_{\chi_0,\widetilde{\chi}_0}(x) = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\left(\chi_{0-} + \widetilde{\chi}_{0+} + \frac{i}{2r}x^m\Gamma_m(\chi_{0+} + \widetilde{\chi}_{0-})\right)\,, \tag{B.16b}$$

parametrized by two constant Dirac spinors χ_0 and $\widetilde{\chi}_0$.

B.3 Ward identity

Let us present the proof of the Ward identity from §2.3.1 for twisted chiral multiplets in background-A.

Supersymmetric deformations of the action

For a twisted chiral primary Y of Weyl weight w, the $\mathfrak{su}(2|1)_A$ variations of the twisted chiral multiplet $\Psi = (Y, \zeta, G)$, generated by the Killing spinors ϵ and $\widetilde{\epsilon}$, are [76]:

$$\delta Y = \widetilde{\epsilon}_+ \zeta_- - \epsilon_- \zeta_+ \,, \tag{B.17a}$$

$$\delta \zeta_+ = -i \not{\partial} Y \widetilde{\epsilon}_- + \left(G + \frac{w}{r} Y \right) \widetilde{\epsilon}_+ \,, \tag{B.17b}$$

$$\delta \zeta_- = i \not{\partial} Y \epsilon_+ - \left(G + \frac{w}{r} Y \right) \epsilon_- \,, \tag{B.17c}$$

$$\delta G = -i \widetilde{\epsilon}_- \not{\nabla} \zeta_- + i \epsilon_+ \not{\nabla} \zeta_+ + \frac{w}{r} \left(\zeta_+ \epsilon_- - \zeta_- \widetilde{\epsilon}_+ \right) . \tag{B.17d}$$

The twisted F-term action for Ψ on the sphere invariant under these variations is given by [31, 83]:

$$I_w^{\text{tc}-\text{F}}(\Psi) := \int_{S^2} \mathrm{d}^2 x \sqrt{g(x)}\, \mathcal{G}(\Psi) \,, \qquad \mathcal{G}(\Psi) = G + \frac{w-1}{r} Y. \tag{B.18}$$

The subscript on I refers to the Weyl weight of Y, the Weyl weight of the integral $I_w^{\text{tc}-\text{F}}$ is $w-1$, therefore a deformation of an action S by this term can be introduced by simply introducing a coupling τ of Weyl weight $(1-w)$:

$$S \to S - \frac{i\tau}{4\pi} I_w^{\text{tc}-\text{F}}(\Psi) \,. \tag{B.19}$$

We want to show that the integrated operator $I_w^{\text{tc}-\text{F}}(\Psi)$ localizes to a point inside an extremal correlator. To proceed, let us pick a particular supercharge $Q_A \in \mathfrak{su}(2|1)_A$ by restricting $\widetilde{\chi}_0$ in (B.11) to be chiral:

$$\widetilde{\chi}_{0+} = 0 \,. \tag{B.20}$$

Then the Killing spinor (B.11b), which we write simply as $\widetilde{\epsilon}$, becomes:

$$\widetilde{\epsilon}_+ = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} \frac{i}{2r} x^m \Gamma_m \widetilde{\chi}_{0-} \,, \qquad \widetilde{\epsilon}_- = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} \widetilde{\chi}_{0-} \,. \tag{B.21}$$

Solving (B.17b) for G and substituting it in the expression for $\mathcal{G}$ in (B.18), we find:

$$\mathcal{G}(\Psi) = \delta \left(\frac{\widetilde{\epsilon}_+^\dagger \zeta_+}{\|\widetilde{\epsilon}_+\|^2} \right) + \frac{i}{\|\widetilde{\epsilon}_+\|^2} \widetilde{\epsilon}_+^\dagger \not{\nabla} (Y \widetilde{\epsilon}_-) \,. \tag{B.22}$$

were $\|\widetilde{\epsilon}_+\|^2 := \widetilde{\epsilon}_+^\dagger \widetilde{\epsilon}_+$. Using complex coordinates $z = x^1 + ix^2$ and $\overline{z} = x^1 - ix^2$ (so that $\widetilde{\epsilon}_+$ satisfies the simple equations $\nabla_z \widetilde{\epsilon}_+ = 0$ and $\nabla_z \frac{\widetilde{\epsilon}_+^\dagger}{\|\widetilde{\epsilon}_+\|^2} = 0$) we can turn the second term into a total derivative:

$$\mathcal{G}(\Psi) = \delta \left(\frac{\widetilde{\epsilon}_+^\dagger \zeta_+}{\|\widetilde{\epsilon}_+\|^2} \right) + 2i \nabla_z \left(\frac{1 + \frac{z\overline{z}}{4r^2}}{\|\widetilde{\epsilon}_+\|^2} \, \widetilde{\epsilon}_+^\dagger \Gamma_1 \widetilde{\epsilon}_- Y \right) . \tag{B.23}$$

The key point here is that the norm of $\widetilde{\epsilon}_+$ vanishes at $z = \overline{z} = 0$ (see (B.21)) and therefore the space-time integral of the derivative localizes at the origin, which we call the North pole N. When inserted in a correlator with Q_A-closed operators, such as an extremal correlator, the Q_A-exact term in (B.23) can be ignored. In [76] the integral of the derivative was computed to be $-4\pi r Y(N)$. Therefore we reach the conclusion that inside an extremal correlator:

$$\left\langle \int_{S^2} \mathrm{d}^2 x \sqrt{g(x)} \, \mathcal{G}(\Psi) \cdots \right\rangle_{S^2} = -4\pi r \, \langle Y(N) \cdots \rangle_{S^2} \, . \tag{B.24}$$

This is the equation (2.34). The equation (2.35) can be proven by starting from the $\mathfrak{su}(2|1)_A$-variation of a twisted anti-chiral multiplet. The analogous equations for the chiral and the anti-chiral multiplets in background-B are proven similarly.

B.4 Contour integrals

On the two-sphere, the extremal correlation functions of chiral operators in a LG model of type A_{k+1} involve the following integrals (c.f. §2.4.2):

$$\int_{\mathbb{C}} \mathrm{d}X \mathrm{d}\overline{X} \, X^m \overline{X}^n e^{-4\pi i X^{k+2} - 4\pi i \overline{X}^{k+2}} = (4\pi)^{-\frac{2(m+1)}{k+2} - q} (-i)^q \int_{\mathbb{C}} \mathrm{d}X \mathrm{d}\overline{X} \, X^m \overline{X}^n e^{X^{k+2} - \overline{X}^{k+2}} \, , \tag{B.25}$$

where $q = \frac{n-m}{k+2}$. After defining:

$$\omega_m := X^m e^{X^{k+2}} \, \mathrm{d}X \, , \qquad \widetilde{\omega}_m := \overline{X}^m e^{-\overline{X}^{k+2}} \, \mathrm{d}\overline{X} \, , \tag{B.26}$$

we can write:[6]

$$\int_{\mathbb{C}} \mathrm{d}X \mathrm{d}\overline{X} \, X^m \overline{X}^n e^{X^{k+2} - \overline{X}^{k+2}} = \frac{i}{2} \int_{\mathbb{C}} \omega_m \wedge \widetilde{\omega}_n \, . \tag{B.27}$$

[6] The factor of $i/2$ is there because by $\mathrm{d}X\mathrm{d}\overline{X}$ we mean $\mathrm{d}x\mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y$ where $X = x + iy$ and $\overline{X} = x - iy$, whereas $\mathrm{d}X \wedge \mathrm{d}\overline{X} = (\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) = -2i \mathrm{d}x \wedge \mathrm{d}y$.

We will evaluate this integral by writing it as a sum of integrals of ω_m and $\widetilde{\omega}_n$ over one-cycles that we will define momentarily. This procedure will use a generalization of the Riemann bilinear identity as explained in Appendix C of [73].[7]

We denote by $\theta_\uparrow$ a ray originating from the origin of the complex plane and going off to infinity at an angle θ with the x-axis. And by $\mathcal{C}_{\theta_1,\theta_2}$ we will refer to a curve that originates at infinity, comes near the origin and then goes off to infinity again in a way such that it is wedged between the rays $\theta_{1\uparrow}$ and $\theta_{2\uparrow}$ and approaches these two rays asymptotically. Such curves will be thought of as noncompact cycles in the complex plane. Our convention is such that $\theta_{2\uparrow}$ follows $\theta_{1\uparrow}$ in the anticlockwise direction in $\mathcal{C}_{\theta_1,\theta_2}$. To denote the same contour with opposite orientation we will use the superscript " $-$ ", e.g., $\mathcal{C}^-_{\theta_1,\theta_2}$.

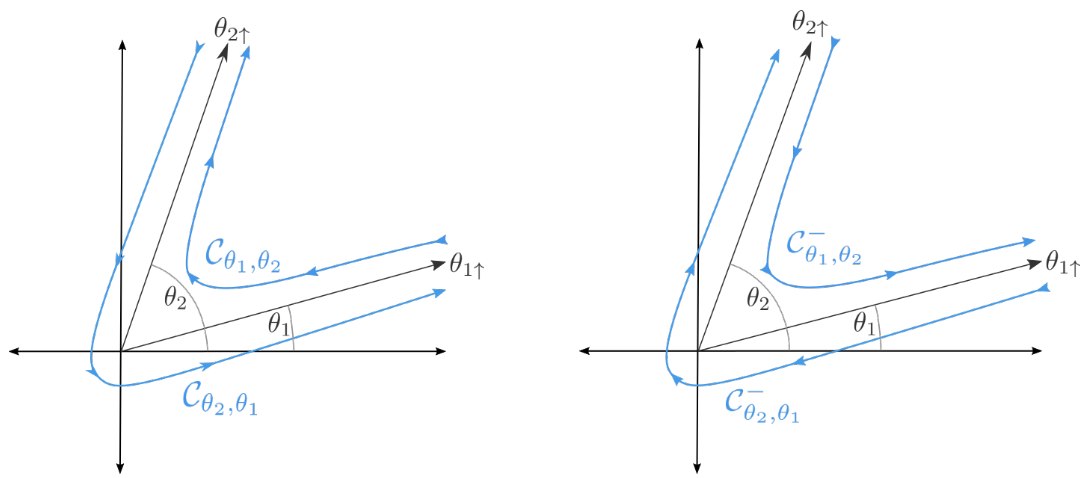


Figure: Some exemplary contours.

We define the following angles and cycles:

$$\vartheta_a := \frac{\pi(2a-1)}{k+2}\,, \qquad \varphi_a := \frac{2\pi a}{k+2}$$
$$C_a := \mathcal{C}_{\vartheta_a,\vartheta_{a+1}}\,, \qquad \widetilde{C}_a := \mathcal{C}_{\varphi_a,\varphi_{a+1}}\,, \qquad \text{for} \quad a \in \mathbb{Z}\,, \tag{B.28}$$

In the figure bellow we draw a couple of these cycles and point out their intersection numbers. If two cycles C_a and $\widetilde{C}_b$ intersect at a point then we denote the intersection

[7] The general idea behind evaluating certain integrals by decomposing them over some cycles comes from Picard–Lefschetz theory, which has been used in the past to compute integrals similar to ours [29,100,156]. Also note that the integration (B.25) seems to be computable using integral identity involving Bessel function of the first kind, but we were unable to confirm that all convergence conditions relevant for the integral identity are satisfied in the present case. Still, we note that a straightforward application of the Bessel function identity produces exactly the same result as the one given by the Riemann bilinear identity.

number of that point simply by $C \circ \widetilde{C}$ when the point being referred to is understood.

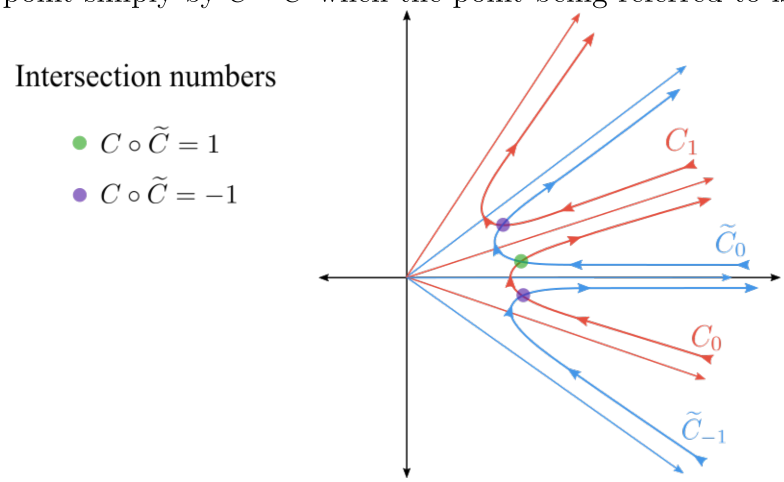


Figure: The cycles C and $\widetilde{C}$.

Now we have:

$$\oint_{C_a} \omega_m = e^{\frac{2\pi i a(m+1)}{k+2}} \oint_{C_0} \omega_m \,, \qquad \oint_{\widetilde{C}^-_{-a-1}} \widetilde{\omega}_n = e^{-\frac{2\pi i a(n+1)}{k+2}} \oint_{\widetilde{C}^-_{-1}} \widetilde{\omega}_n \,, \tag{B.29}$$

which follows after redefining integration variables as $X \to X e^{\frac{2\pi i a}{k+2}}$ and $\overline{X} \to \overline{X} e^{-\frac{2\pi i a}{k+2}}$ respectively. Note that if we set $X = re^{i\vartheta_a}$ and $\overline{X} = re^{i\varphi_a}$ with $r > 0$ then we get the following asymptotic behaviours for the one forms:

$$r \to \infty : \quad \omega_m \sim r^m e^{i(m+1)\vartheta_a} e^{-r^{k+2}} \, \mathrm{d}r \,, \qquad \widetilde{\omega}_n \sim r^n e^{i(n+1)\varphi_a} e^{-r^{k+2}} \, \mathrm{d}r \,, \tag{B.30}$$

making the integrals well defined. We now proceed to evaluate $\oint_{C_0} \omega_m$. We define a new variable:

$$Y := X^{k+2} \tag{B.31}$$

in terms of which we can write:

$$X = Y^{\frac{1}{k+2}} \,, \qquad \mathrm{d}X = \frac{1}{k+2} Y^{\frac{-k-1}{k+2}} \mathrm{d}Y \,. \tag{B.32}$$

and the angles defining the contour C_0 change as:

$$\vartheta_0 = -\frac{\pi}{k+2} \to -\pi \,, \qquad \vartheta_1 = \frac{\pi}{k+2} \to \pi \,. \tag{B.33}$$

Since they represent the same direction in the complex plane we will write $\pi^{\pm} := \pi \pm \epsilon$ for the angles, where $\epsilon > 0$ is infinitesimal, to keep track of the orientation of the resulting contour. Therefore,

$$\oint_{C_0} \omega_m = \frac{1}{k+2} \oint_{\mathcal{C}_{\pi^+,\pi^-}} Y^{-\frac{k-m+1}{k+2}} e^Y \, \mathrm{d}Y = \frac{1}{k+2} \frac{2\pi i}{\Gamma\left(\frac{k-m+1}{k+2}\right)} \,. \tag{B.34}$$

To get the last equality we used the integral form of the reciprocal Gamma function:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \oint_{\mathcal{C}_{\pi^+,\pi^-}} \mathrm{d}t \, t^{-z} e^t \,. \tag{B.35}$$

Similarly by defining $\overline{Y} = -\overline{X}^{k+2}$ we find:

$$\oint_{\widetilde{C}^-_{-1}} \widetilde{\omega}_n = -\frac{e^{\frac{\pi i}{k+2}(n+1)}}{k+2} \oint_{\mathcal{C}_{\pi^+,\pi^-}} \overline{Y}^{-\frac{k-n+1}{k+2}} e^{\overline{Y}} \mathrm{d}\overline{Y} = -2i \frac{e^{\frac{\pi i}{k+2}(n+1)}}{k+2} \sin\left(\frac{\pi(n+1)}{k+2}\right) \Gamma\left(\frac{n+1}{k+2}\right) \,. \tag{B.36}$$

The last equality is a combination of (B.35) and Euler's reflection formula:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \,. \tag{B.37}$$

The cycles defined in (B.28) are distinct for $a = 0, \cdots, k+1$ and they satisfy:

$$\sum_{a=0}^{k+1} C_a = \sum_{a=0}^{k+1} \widetilde{C}_a = 0 \,. \tag{B.38}$$

The cycles $\widetilde{C}_a$ are dual to the cycles C_a with the intersection form:

$$I_{ab} := C_a \circ C_b = \delta_{a,b} - \delta_{a,b+1} \,, \tag{B.39}$$

with inverse (restricting to $a, b = 1, \cdots, k+1$ for independence):

$$I^{-1}_{ab} = \left\{ \begin{array}{ll} 1 & \text{when } a \geq b \\ 0 & \text{otherwise} \end{array} \right. \,. \tag{B.40}$$

Complex conjugation acts on the contours as follows:

$$\widetilde{C}^*_a = \mathcal{C}^*_{\varphi_a,\varphi_{a+1}} = \mathcal{C}^-_{-\varphi_{a+1},-\varphi_a} = \mathcal{C}^-_{\varphi_{-a-1},\varphi_{-a}} = \widetilde{C}^-_{-a-1} \tag{B.41}$$

Now, the generalization of Riemann bilinear identity [73] gives us:

$$\begin{aligned}\int_{\mathbb{C}} \omega_m \wedge \widetilde{\omega}_n &= -\sum_{a,b=1}^{k+1} I^{-1}_{ab} \oint_{C_a} \omega_m \oint_{\widetilde{C}^*_b} \widetilde{\omega}_n = -\sum_{a=1}^{k+1}\sum_{b=1}^{a} \oint_{C_a} \omega_m \oint_{\widetilde{C}^-_{-b-1}} \widetilde{\omega}_n \\ &= -\sum_{a=1}^{k+1}\sum_{b=1}^{a} e^{\frac{2\pi i}{k+2}[a(m+1)-b(n+1)]} \oint_{C_0} \omega_m \oint_{\widetilde{C}^-_{-1}} \widetilde{\omega}_n \,, \qquad \text{using (B.29)}\,. \end{aligned} \tag{B.42}$$

The integrals are independent of a and b, so we can evaluate the sum separately:

$$\sum_{a=1}^{k+1}\sum_{b=1}^{a} e^{\frac{2\pi i}{k+2}[a(m+1)-b(n+1)]} = \frac{e^{-\frac{2\pi i}{k+2}(n+1)}}{1-e^{-\frac{2\pi i}{k+2}(n+1)}}\left[\sum_{a=1}^{k+1} e^{\frac{2\pi i a}{k+2}(m+1)} - \sum_{a=1}^{k+1} e^{\frac{2\pi i a}{k+2}(m-n)}\right] \tag{B.43}$$

Note that for $m+1 \equiv 0\,(\mathrm{mod}(k+2))$ (B.34) vanishes since the Gamma function in the denominator acquires a pole, and therefore the expression (B.42) will vanish as well. Assuming $m+1 \not\equiv 0\,(\mathrm{mod}(k+2))$ we see that the first sum inside the parentheses in (B.43) is a sum over roots of unity excluding 1, therefore the first sum is -1. The second sum is also a sum over roots of unity excluding 1 unless $(m-n) \equiv 0\,(\mathrm{mod}(k+2))$. Therefore, whenever $(m-n) \not\equiv 0\,(\mathrm{mod}(k+2))$ the second sum is -1 and the expression (B.43), and consequently (B.42), vanish. From now on we assume that there exists a $q \in \mathbb{Z}$ such that $m-n+q(k+2)=0$. Then (B.43) reduces to:

$$\sum_{a=1}^{k+1}\sum_{b=1}^{a} e^{\frac{2\pi i}{k+2}[a(m+1)-b(n+1)]} = \frac{-e^{-\frac{\pi i}{k+2}(n+1)}}{2i\sin\left(\frac{\pi(n+1)}{k+2}\right)}(k+2) \tag{B.44}$$

Substituting (B.44), (B.34) and (B.36) in (B.42) we find:

$$\frac{i}{2}\int_{\mathbb{C}} \omega_m \wedge \widetilde{\omega}_n = \frac{\pi}{k+2}\frac{\Gamma\left(\frac{m+1}{k+2}+q\right)}{\Gamma\left(\frac{k-m+1}{k+2}\right)} \tag{B.45}$$

Substituting this into (B.27) and using (B.25) we get the extremal correlators on S^2 (2.93b).

Appendix C

Kähler Ambiguities in 4d $\mathcal{N}=2$ SCFTs

C.1 Introduction

Recent years have witnessed remarkable progress in obtaining the exact partition function of supersymmetric field theories in various background geometries. When the geometry is $S^1 \times \mathcal{M}_{d-1}$ the partition function admits a standard Hilbert space interpretation as a supertrace over the states of the theory on $\mathcal{M}_{d-1}$. In other geometries, such as on a sphere S^d, the physical interpretation of the partition function must be sought.

In [76] it has been shown that the partition function of 4d $\mathcal{N}=2$ superconformal field theories (SCFTs) on S^4 computes the exact Kähler potential K on the space of exactly marginal couplings, also referred to as the conformal manifold. This result was proven both by using supersymmetric localization [140] and by conformal dimension regularization on S^4, and extends the proof in [83] that the S^2 partition function of 2d $\mathcal{N}=(2,2)$ SCFTs computes the exact Kähler potential on the conformal manifold, as conjectured by [107] based on the exact formulae in [15, 61] (see also [60, 83]). In detail, [76] demonstrated that

$$Z_{S^4} = e^{K/12}\,. \tag{C.1}$$

These identifications provide a physical and geometrical interpretation of the sphere partition function of 4d $\mathcal{N}=2$ and 2d $\mathcal{N}=(2,2)$ SCFTs. These results also provide a computational pathway for obtaining the exact metric in the conformal manifold, which are interesting new observables in these theories, acted on by dualities (see e.g. recent work [9] [7]).

Here we present an elementary proof of the formula (C.1) using supersymmetry Ward identities. This new proof does not require localization nor that the 4d $\mathcal{N} = 2$ SCFT admits a Lagrangian description. By virtue of the relation (C.1) identifying the S^4 partition function with the Kähler potential K on the conformal manifold, it follows that the partition function is subject to the Kähler ambiguity transformations

$$K(\tau, \bar{\tau}) \to K(\tau, \bar{\tau}) + \mathcal{F}(\tau) + \bar{\mathcal{F}}(\bar{\tau}) \,, \tag{C.2}$$

where $\mathcal{F}$ is an arbitrary holomorphic function and τ are holomorphic coordinates on the conformal manifold. This ambiguity implies that the partition function is a section over the space of exactly marginal couplings.

We also give the microscopic realization of the Kähler ambiguity (C.2) by constructing the local supergravity counterterm in 4d $\mathcal{N} = 2$ off-shell supergravity that when evaluated on the supersymmetric S^4 background yields (C.2). This is the 4d counterpart of the Kähler ambiguity counterterm for 2d $\mathcal{N} = (2,2)$ SCFTs constructed in [76].

The plan is as follows. In section C.2 we use supersymmetry Ward identities to show that the S^4 partition function of 4d $\mathcal{N} = 2$ SCFTs computes the Kähler potential in the conformal manifold. In section C.3 we identify the off-shell 4d $\mathcal{N} = 2$ Poincaré supergravity theory in which the S^4 is a supersymmetric background. In section C.4 we construct the supergravity invariant in the relevant Poincaré supergravity theory that once evaluated on S^4 provides a first principles realization of the Kähler transformation (C.2).

C.2 Kähler Potential from S^4 Partition Function

An exactly marginal operator in a four dimensional $\mathcal{N} = 2$ SCFT is a scalar operator of dimension four which is a superconformal descendant of a scalar chiral primary operator of $\mathrm{U}(1)_R$ charge $w = 2$. An $\mathcal{N} = 2$ SCFT can be deformed while preserving superconformal invariance by[1]

$$\frac{1}{\pi^2} \int d^4x \sum_I \left(\tau_I O_I + \bar{\tau}_{\bar{I}} \bar{O}_{\bar{I}} \right) \,. \tag{C.3}$$

The exactly marginal couplings τ_I are holomorphic coordinates in the space of exactly marginal deformations, known as the conformal manifold. The canonical metric in the conformal manifold $g_{I\bar{J}}$ is the Zamolodchikov metric

$$\langle O_I(x) \bar{O}_{\bar{J}}(0) \rangle = \frac{g_{I\bar{J}}}{x^8} \,, \tag{C.4}$$

[1] We use the same conventions as in [76].

which in four dimensional $\mathcal{N}=2$ SCFTs is Kähler, that is

$$g_{I\bar{J}} = \frac{\partial}{\partial \tau_I}\frac{\partial}{\partial \bar{\tau}_{\bar{J}}} K(\tau,\bar{\tau}) \equiv \partial_I \partial_{\bar{J}} K(\tau,\bar{\tau})\,. \tag{C.5}$$

An $\mathcal{N}=2$ SCFT can be canonically placed on S^4 by the stereographic projection. The $\mathcal{N}=2$ superconformal transformations on S^4 are parametrized by chiral conformal Killing spinors ϵ^i and ϵ_i of opposite chirality transforming as doublets of the $\mathrm{SU}(2)_R$ R-symmetry, which obey[2]

$$\nabla_m \epsilon^i = \gamma_m \eta^i \qquad \nabla_m \epsilon_i = \gamma_m \eta_i\,, \tag{C.6}$$

so that $\eta^i = \frac{1}{4}\not{\nabla}\epsilon^i$ and $\eta_i = \frac{1}{4}\not{\nabla}\epsilon_i$.

An exactly marginal operator in an $\mathcal{N}=2$ SCFT can be represented as the top component of a four dimensional $\mathcal{N}=2$ *chiral multiplet* of R-charge $w=2$, whose bottom component realizes the parent chiral primary operator. The holomorphic coordinates on the conformal manifold can be promoted to supersymmetric background chiral superfields with vanishing R-charge $w=0$. The $\mathcal{N}=2$ superconformal transformations of a chiral multiplet with R-charge w on S^4 are given by [19] (we use [141]):[3]

$$\begin{aligned}
\delta A &= \frac{1}{2}\bar{\epsilon}^i \Psi_i \\
\delta \Psi_i &= \not{\nabla}(A\epsilon_i) + \frac{1}{2}B_{ij}\epsilon^j + \frac{1}{4}\Gamma^{ab}F^-_{ab}\varepsilon_{ij}\epsilon^j + (2w-4)A\eta_i \\
\delta B_{ij} &= \bar{\epsilon}_{(i}\not{\nabla}\Psi_{j)} - \bar{\epsilon}^k \Lambda_{(i}\varepsilon_{j)k} + 2(1-w)\bar{\eta}_{(i}\Psi_{j)} \\
\delta F^-_{ab} &= \frac{1}{4}\varepsilon^{ij}\bar{\epsilon}_i \not{\nabla}\Gamma_{ab}\Psi_j + \frac{1}{4}\bar{\epsilon}^i\Gamma_{ab}\Lambda_i - \frac{1}{2}(1+w)\varepsilon^{ij}\bar{\eta}_i\Gamma_{ab}\Psi_j \\
\delta \Lambda_i &= -\frac{1}{4}\Gamma^{ab}\not{\nabla}(F^-_{ab}\epsilon_i) - \frac{1}{2}\not{\nabla}B_{ij}\varepsilon^{jk}\epsilon_k + \frac{1}{2}C\varepsilon_{ij}\epsilon^j - (1+w)B_{ij}\varepsilon^{jk}\eta_k + \frac{1}{2}(3-w)\Gamma^{ab}F^-_{ab}\eta_i \\
\delta C &= -\nabla_m(\varepsilon^{ij}\bar{\epsilon}_i\gamma^m\Lambda_j) + (2w-4)\varepsilon^{ij}\bar{\eta}_i\Lambda_j\,,
\end{aligned} \tag{C.7}$$

where in Euclidean signature F^-_{ab} is a self-dual rank-two tensor. Indeed, for $w=2$, the integrated top component is superconformal invariant and we have the identification

$$C_I = O_I \qquad \text{for } w=2\,. \tag{C.8}$$

[2] Γ^a denotes tangent space gamma matrices while $\gamma^m = e^m_a\Gamma^a$ denotes curved space ones.
[3] Throughout a barred spinor is $\bar{\lambda} = \lambda^T\mathcal{C}$, where $\mathcal{C}$ is the charge conjugation matrix.

For $w = 0$, an arbitrary covariantly constant background value for the bottom component of the chiral multiplet[4] is superconformal invariant, and serves as the spurion field for the holomorphic coordinates on the conformal manifold

$$A_I = \tau_I \qquad \text{for } w = 0\,. \tag{C.9}$$

We denote by $\mathcal{A}_I$ the chiral multiplets to which the coordinates in the conformal manifold have been promoted.

Consider now the SCFT partition function on S^4 as a function of the exactly marginal couplings $Z_{S^4}(\tau, \bar\tau)$. The second derivative

$$\partial_I \partial_{\bar J} \log Z_{S^4} = \frac{1}{\pi^4} \left\langle \int_{S^4} d^4x \sqrt{g}\, C_I(x) \ \int_{S^4} d^4y \sqrt{g}\, \bar C_{\bar J}(y) \right\rangle \tag{C.10}$$

is the *integrated* connected two-point function of exactly marginal operators. This correlator is ultraviolet divergent, divergences arising when the operators collide. These ultraviolet divergences can be regularized by introducing a massive deformation. Regulating divergences in a supersymmetric manner leads us to consider the OSp(2|4) massive subalgebra of the $\mathcal{N} = 2$ superconformal algebra on S^4, which is the supersymmetry algebra of an arbitrary massive four dimensional $\mathcal{N} = 2$ theory on S^4.

The OSp(2|4) massive subalgebra on S^4 is generated by supercharges that anticommute to the SO(5) isometries of S^4 and an $\mathrm{SO}(2)_R \subset \mathrm{SU}(2)_R$ R-symmetry. Conformal generators and $\mathrm{U}(1)_R$ are projected out. The OSp(2|4) transformations are generated by Killing spinors which obey

$$\nabla_m \chi^j = \frac{i}{2r} \gamma_m \chi^j\,, \tag{C.11}$$

where

$$\chi^j = \epsilon^j + \tau_1^{jk} \epsilon_k \tag{C.12}$$

so that[5]

$$\epsilon^i = \chi^i_L \qquad \epsilon_i = \tau_{1ij} \chi^j_R \tag{C.13}$$

and $\tau_p^{jk} = (i\sigma_3, -1, -i\sigma_1) = (\tau_{p_{jk}})^*$, where σ_p are the Pauli matrices. In stereographic coordinates, where $ds^2 = \frac{1}{\left(1+\frac{x^2}{4r^2}\right)^2} dx_m dx^m$, we have

$$\chi^j = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} \left(1 + \frac{i}{2r} x_m \Gamma^m\right) \chi^j_0\,. \tag{C.14}$$

[4] All other components in multiplet must vanish.

[5] P_L and P_R are the spinor chirality projectors: $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L + P_R = 1$. The Killing spinors obey $P_L \epsilon^i = \epsilon^i$ and $P_R \epsilon_i = \epsilon_i$.

The constant spinors χ_0^j parametrize the transformations of the eight supercharges in OSp(2|4). If these parameters are chiral

$$P_L\chi_0^j = 0\,, \tag{C.15}$$

the corresponding spinors generate an OSp(2|2) subalgebra OSp(2|4). The chiral components of these spinors χ_L^j and χ_R^j

$$\chi_L^j = P_L\chi^j = \frac{i/2r}{\sqrt{1+\frac{x^2}{4r^2}}}x_m\Gamma^m\chi_{0R}^j \qquad \chi_R^j = P_R\chi^j = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\chi_{0R}^j\,, \tag{C.16}$$

vanish at the North and the South poles of the sphere respectively. If the parameters are further constrained by

$$\chi_0^i = \tau_1^{ij}\varepsilon_{jk}\Gamma_1\Gamma_2\chi_0^k\,, \tag{C.17}$$

the corresponding spinors generate a further SU(1|1) subalgebra

$$Q^2 = J + R \tag{C.18}$$

of $\mathrm{OSp}(2|2) \subset \mathrm{OSp}(2|4)$, where $J = J_{12} + J_{34}$ is a self-dual rotation on S^4 and R is the $\mathrm{SO}(2)_R \subset \mathrm{SU}(2)_R$ R-symmetry.

Our strategy is to first prove that the *integrated* top component of the chiral multiplet in (C.10) can be written as an $\mathrm{SU}(1|1) \subset \mathrm{OSp}(2|4)$ supersymmetry transformation δ everywhere except at the North pole of S^4, where the corresponding Killing spinor vanishes. The proof is completed by showing that the correlator of the integrated top component C with an arbitrary operator $\mathcal{O}$ invariant under the SU(1|1) supersymmetry transformation δ reduces to the correlator of the bottom component A at the North pole with $\mathcal{O}$. In detail

$$\left\langle \int_{S^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle = 32\pi^2r^2\left\langle A(N)\,\mathcal{O}\right\rangle\,. \tag{C.19}$$

The supersymmetry transformation of the fermions in a chiral multiplet with R-charge $w = 2$ can be written as (C.7)[6]

$$\delta\Psi_i = \tau_{1ij}\nabla\!\!\!/(A\chi_R^j) + \frac{1}{2}\vec{B}\cdot\vec{\tau}_{ij}\chi_L^j + \frac{1}{4}\Gamma^{ab}F_{ab}^-\varepsilon_{ij}\chi_L^j \tag{C.20a}$$

$$\begin{aligned}\delta\Lambda_i = &-\frac{1}{4}\Gamma^{ab}\nabla\!\!\!/F_{ab}^-\tau_{1ij}\chi_R^j - \frac{i}{4r}\Gamma^{ab}F_{ab}^-\tau_{1ij}\chi_L^j + \frac{1}{2}C\varepsilon_{ij}\chi_L^j\\ &-\frac{1}{2}\nabla\!\!\!/\vec{B}\cdot\vec{\tau}_{ij}\tau_1^{jk}\varepsilon_{kl}\chi_R^l - \frac{3i}{2r}\vec{B}\cdot\vec{\tau}_{ij}\tau_1^{jk}\varepsilon_{kl}\chi_L^l\end{aligned} \tag{C.20b}$$

[6] $\vec{B} = (B_1, B_2, B_3)$ such that $B_{ij} = \vec{B}\cdot\vec{\tau}_{ij} = \sum_p B_p\tau_{pij}$.

Using the SU(1|1) supersymmetry transformation δ obtained by imposing the constraints (C.15) and (C.17) on the Killing spinors, we get after multiplying (C.20a) by $\tau_{2ij}\tau_1^{jk}{\chi_L^i}^\dagger$ and (C.20b) by $\tau_{2ij}\varepsilon^{jk}{\chi_L^i}^\dagger$ that

$$\begin{aligned} B_1 =& -\delta\left(\frac{{\chi_L^i}^\dagger\Psi_k}{\|\chi_L\|^2}\right)\tau_{2ij}\tau_1^{jk} + \frac{{\chi_L^i}^\dagger}{\|\chi_L\|^2}\not{\nabla}(A\chi_R^j)\tau_{2ij} - \frac{1}{4}\frac{{\chi_L^i}^\dagger\Gamma^{ab}\chi_L^j}{\|\chi_L\|^2}\tau_{3ij}F^-_{ab} \qquad \text{(C.21a)}\\ C =& -\delta\left(\frac{{\chi_L^i}^\dagger\Lambda_k}{\|\chi_L\|^2}\right)\tau_{2ij}\varepsilon^{jk} - \frac{1}{2}\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2}\tau_{2ij}\nabla_m B_1 + \frac{3i}{r}B_1\\ &+\frac{1}{4}\frac{{\chi_L^i}^\dagger\Gamma^{ab}\gamma^m\chi_R^j}{\|\chi_L\|^2}\tau_{3ij}\nabla_m F^-_{ab} + \frac{i}{4r}\frac{{\chi_L^i}^\dagger\Gamma^{ab}\chi_L^j}{\|\chi_L\|^2}\tau_{3ij}F^-_{ab} \qquad \text{(C.21b)} \end{aligned}$$

where we have used that for the SU(1|1) Killing spinors $\|\chi_L\|^2 := \|\chi_L^1\|^2 = \|\chi_L^2\|^2$, where $\|\lambda\|^2 = \lambda^\dagger\lambda$. The terms proportional to F^-_{ab} and $\nabla_\mu F^-_{ab}$ in (C.20a) (C.20b) also vanish. Their coefficients are anti-self-dual in the tangent space indices since

$${\chi_L^i}^\dagger\Gamma^{ab}\gamma^{(r)}\chi_{L/R}^j = {\chi_L^i}^\dagger\Gamma_*\Gamma^{ab}\gamma^{(r)}\chi_{L/R}^j = -\frac{1}{2}\varepsilon^{ab}{}_{cd}{\chi_L^i}^\dagger\Gamma^{cd}\gamma^{(r)}\chi_{L/R}^j\,, \qquad \text{(C.22)}$$

where $\gamma^{(r)}$ is the product of r distinct gamma matrices. Since F^-_{ab} is self-dual in Euclidean signature, all the terms involving F^-_{ab} vanish. We can eliminate B_1 from (C.21b) by using (C.21a), which yields

$$\begin{aligned} C =& -\frac{1}{2}\frac{{\chi_L^i}^\dagger}{\|\chi_L\|^2}\not{\nabla}\left(\left[\frac{{\chi_L^k}^\dagger}{\|\chi_L\|^2}\not{\nabla}(A\chi_R^l)\right]\chi_R^j\right)\tau_{2ij}\tau_{2kl} + \frac{i}{r}\frac{{\chi_L^i}^\dagger}{\|\chi_L\|^2}\not{\nabla}(A\chi_R^j)\tau_{2ij}\\ &+\delta\left(\Xi(\Lambda_i,\Psi_i,\chi^i)\right)\,, \qquad \text{(C.23)} \end{aligned}$$

where, for brevity, we have defined

$$\Xi(\Lambda_i,\Psi_i,\chi^i) := -\frac{{\chi_L^i}^\dagger\Lambda_k}{\|\chi_L\|^2}\tau_{2ij}\varepsilon^{jk} + \frac{1}{2}\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2}\nabla_m\left(\frac{{\chi_L^k}^\dagger\Psi_t}{\|\chi_L\|^2}\right)\tau_{2ij}\tau_{2kl}\tau_1^{lt} - \frac{3i}{r}\frac{{\chi_L^i}^\dagger\Psi_k}{\|\chi_L\|^2}\tau_{2ij}\tau_1^{jk}\,. \qquad \text{(C.24)}$$

We now show that the sum of the terms in (C.23) involving A are a total derivative.

For any OSp(2|2) supersymmetry parameter χ^j and any scalar quantity X we have that[7]

$$\frac{{\chi_L^j}^\dagger}{\|\chi_L^j\|^2}\not{\nabla}(X\chi_R^j) = \nabla_m\left(\frac{{\chi_L^j}^\dagger\gamma^m\chi_R^j}{\|\chi_L^j\|^2}X\right) + \frac{4irX}{x^2}\,. \qquad \text{(C.25)}$$

[7] By using that $\nabla_m\chi_L^\dagger = -\frac{i}{2r}\chi_R^\dagger\gamma_m$.

Using this, the top component C of a chiral multiplet with $w = 2$ can be written locally as the sum of an SU(1|1) supersymmetry transformation δ and total derivatives

$$C = \delta\left(\Xi(\Lambda_i, \Psi_i, \chi^i)\right) - \frac{1}{2}\nabla_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2}\nabla_n\left[\frac{\chi_L^{k\,\dagger}\gamma^n\chi_R^l}{\|\chi_L\|^2}A\right]\right)\tau_{2ij}\tau_{2kl}$$
$$+ 8ir\nabla_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j A}{\|\chi_L\|^2 x^2}\right)\tau_{2ij} + \frac{i}{r}\nabla_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2}A\right)\tau_{2ij}\,. \tag{C.26}$$

This formula fails at the North pole, where $\|\chi_L^1\| = \|\chi_L^2\| = 0$ and Ξ diverges. Therefore the integrated top component is non-trivial in correlation functions, as it is not supersymmetry-exact globally, but the entire contribution localizes to the North pole, just as in the analysis of 2d $\mathcal{N} = (2,2)$ SCFTs in [76].[8]

Let us consider the integrated correlator with an operator $\mathcal{O}$ obeying $\delta\mathcal{O} = 0$

$$\left\langle\int_{S^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle = \lim_{R\to 0}\left[\left\langle\int_{S^4\backslash B_R^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle + \left\langle\int_{B_R^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle\right]. \tag{C.27}$$

We have divided S^4 into two-regions: a four-dimensional ball B_R^4 of radius R around the North pole and its complement $S^4\backslash B_R^4$. In the $R \to 0$ limit the ball contribution vanishes[9] and we are left with

$$\lim_{R\to 0}\left\langle\int_{S^4\backslash B_R^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle. \tag{C.28}$$

Using (C.26), which is valid in $S^4\backslash B_R^4$, and $\delta\Phi = 0$, we can replace C by the last three terms in (C.26), which inside (C.28) can be written as an integral over the three-sphere S_R^3 of radius R at the boundary of $S^4\backslash B_R^4$. For any OSp(2|2) Killing spinor χ^j (C.16), we have that in the $R \to 0$ limit

$$\chi_L^i(R) \sim O(R),\ \chi_R^i(R) \sim O(1) \Rightarrow \frac{\chi_L^{i\,\dagger}\gamma^\mu\chi_R^i}{\|\chi_L\|^2} \sim O\left(\frac{1}{R}\right). \tag{C.29}$$

Therefore, a simple scaling argument shows that the last term in (C.26) cannot compensate for the R^3 measure factor coming from S_R^3 and gives a vanishing contribution in the $R \to 0$

[8] We note that had we assumed that the partition function can be regulated while preserving full $\mathcal{N} = 2$ superconformal invariance, we would have concluded that the partition function is independent of the moduli, as the top component C is globally superconformal-exact.

[9] The R^4 measure factor suppresses the ball contribution in the $R \to 0$ limit.

limit. Therefore, we have shown that in the presence of δ-closed operators

$$\begin{aligned}\int_{S^4} d^4x\sqrt{g}\,C(x) &= \lim_{R\to 0}\int_{S^4\setminus B_R^4} d^4x\sqrt{g}\,C(x)\\ &= -\frac{1}{2}\lim_{R\to 0}\int_{S^4\setminus B_R^4}\mathrm{d}^4x\,\partial_m\left(\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2}\partial_n\left[\frac{{\chi_L^k}^\dagger\gamma^n\chi_R^l}{\|\chi_L\|^2}A(x)\sqrt{g}\right]\right)\tau_{2ij}\tau_{2kl}\\ &\quad + 8ir\lim_{R\to 0}\int_{S^4\setminus B_R^4}\mathrm{d}^4x\,\partial_m\left(\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2x^2}A(x)\sqrt{g}\right)\tau_{2ij}\,.\end{aligned} \tag{C.30}$$

In the limit $R\to 0$ we can replace the bottom component $A(x)$ by its value at the North pole $A(N)$, as higher order terms in the expansion in R vanish in the limit, and using Stoke's theorem

$$\int_{S^4} d^4x\sqrt{g}\,C(x) = \lim_{R\to 0}\int_{S_R^3} V\cdot\hat{\eta}\,, \tag{C.31}$$

where

$$V^m := -\frac{1}{2}\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2}\partial_n\left(\frac{{\chi_L^k}^\dagger\gamma^n\chi_R^l}{\|\chi_L\|^2}\sqrt{g}\right)A(N)\tau_{2ij}\tau_{2kl} + 8ir\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2x^2}A(N)\sqrt{g}\tau_{2ij}\,, \tag{C.32}$$

and $\hat{\eta}$ is the unit vector towards the North pole of S^4 along the radial direction.[10] Going to spherical coordinates, where R is the radial coordinate, we find that

$$V\cdot\hat{\eta} = \frac{512A(N)r^6(R^2-2r^2)}{R^3(R^2+4r^2)^3} + \frac{2048A(N)r^8}{R^3(R^2+4r^2)^3} = \frac{512A(N)r^6(R^2+2r^2)}{R^3(R^2+4r^2)^3}\,. \tag{C.33}$$

The integration in (C.31) is over S_R^3, therefore

$$\int_{S_R^3} V\cdot\hat{\eta} = \frac{512A(N)r^6(R^2+2r^2)}{R^3(R^2+4r^2)^3}2\pi^2R^3\,, \tag{C.34}$$

and

$$\lim_{R\to 0}\frac{512A(N)r^6(R^2+2r^2)}{R^3(R^2+4r^2)^3}2\pi^2R^3 = 32A(N)\pi^2r^2\,. \tag{C.35}$$

[10]The unit radial vector in cartesian coordinates is given by $\hat{\eta}^a = -\frac{x^a}{\sqrt{x^2}}$.

This yields the desired formula

$$\left\langle \int_{S^4} d^4x\sqrt{g}\, C(x)\, \mathcal{O} \right\rangle = 32\pi^2 r^2 \left\langle A(N)\, \mathcal{O} \right\rangle \, . \tag{C.36}$$

The integrated top component C of a chiral multiplet is equivalent to inserting the bottom component A at the North pole. A very similar analysis yields

$$\left\langle \int_{S^4} d^4x\sqrt{g}\, \bar{C}(x)\, \mathcal{O} \right\rangle = 32\pi^2 r^2 \left\langle \bar{A}(S)\, \mathcal{O} \right\rangle \, . \tag{C.37}$$

The integrated top component $\bar{C}$ of an anti-chiral multiplet is equivalent to inserting the bottom component $\bar{A}$ at the South pole.

We can now use (C.36) and (C.37) to express the derivative of the partition function in (C.10) as an *unintegrated* two-point function

$$\partial_I \partial_{\bar{J}} \log Z_{S^4} = \frac{1}{\pi^4} \left\langle \int_{S^4} d^4x\sqrt{g}\, C_I(x) \ \int_{S^4} d^4y\sqrt{g}\, \bar{C}_{\bar{J}}(y) \right\rangle = \left(32r^2\right)^2 \left\langle A_I(N)\bar{A}_{\bar{J}}(S) \right\rangle \, . \tag{C.38}$$

It follows from the first equation in (C.7) that the correlator $\left\langle A_I(N)\bar{A}_{\bar{J}}(S) \right\rangle$ is δ invariant, since SU(1|1) supersymmetry parameters ϵ^{j} and ϵ_{j} vanish at the North pole and South pole respectively, and therefore $\delta A_I(N) = \delta \bar{A}_{\bar{J}}(S) = 0$.

Using the supersymmetry Ward identity $\left\langle A_I(N)\bar{A}_{\bar{J}}(S) \right\rangle = \frac{r^4}{48} \left\langle C_I(N)\bar{C}_{\bar{J}}(S) \right\rangle$ [76], that $\left\langle C_I(N)\bar{C}_{\bar{J}}(S) \right\rangle = \frac{1}{(2r)^8} g_{I\bar{J}}$ defines the Zamolodchikov metric $g_{I\bar{J}}$ and that the metric is Kähler (C.5) we arrive at

$$\partial_I \partial_{\bar{J}} \log Z_{S^4} = \frac{1}{12} g_{I\bar{J}} = \frac{1}{12} \partial_I \partial_{\bar{J}} K \, . \tag{C.39}$$

Therefore, the four sphere partition function of a four dimensional $\mathcal{N}=2$ SCFT computes the Kähler potential in the conformal manifold (C.1), and is subject to Kähler transformation ambiguities (C.2), which do not affect the Zamolodchikov metric.

C.3 Off-shell $\mathcal{N}=2$ Poincaré Supergravity for S^4

The partition function of a field theory in a curved geometry can be ambiguous. These ambiguities are encoded in finite counterterms for the background fields that capture the background geometry and the parameters of the theory. When the partition function of a supersymmetric theory can be regulated in a diffeomorphism invariant and supersymmetric manner, the counterterms are *supergravity* invariants constructed out of the supergravity multiplet encoding the background geometry and the supersymmetry multiplets to which the other parameters of the theory can be promoted, since all parameters in a supersymmetric field theory can be promoted to background supermultiplets [146].

Constructing these supergravity invariants requires identifying first the supergravity theory in which the curved geometry over which the partition function is computed is a supersymmetric background. This can be analyzed in the framework of off-shell supergravity [65]. In this section we identify the four dimensional $\mathcal{N}=2$ off-shell Poincaré supergravity theory and the background fields in that supergravity multiplet that give rise to the OSp(2|4)-invariant four-sphere background geometry.

A conceptual way of constructing off-shell Poincaré supergravity theories is to start with off-shell conformal supergravity and partially gauge fix the conformal symmetries down to Poincaré by adding compensating supermultiplets. Different choices of compensating multiplets give rise to different off-shell Poincaré supergravity theories, with different sets of auxiliary fields.[11] The Poincaré supersymmetry transformations of the gauge fixed theory are constructed by combining the Poincaré supersymmetry transformations in conformal supergravity with field dependent superconformal transformations that are needed to preserve the gauge choice.[12]

Our starting point is four dimensional $\mathcal{N}=2$ conformal supergravity [52] (we refer to [68] for more details). Off-shell $\mathcal{N}=2$ superconformal transformations are realized on the Weyl multiplet, whose independent fields are

$$\begin{aligned} \text{bosonic:}\, & e_m^a, b_m, V_{m\,i}^{\;\;j}, A_m^R, T_{ab}^-, D \\ \text{fermionic:}\, & \psi_m^i, \chi^i\,. \end{aligned} \tag{C.40}$$

The fields $e_m^a, b_m, V_{m\,i}^{\;\;j}, A_m^R, \psi_m^i$ are the gauge fields for translations, dilatations, $\mathrm{SU}(2)_R$, $\mathrm{U}(1)_R$ and Poincaré supersymmetry generators in the $\mathcal{N}=2$ superconformal algebra. The Weyl multiplet is completed by the bosonic auxiliary fields T_{ab}^- and D, and the fermionic auxiliary field χ^i. In Euclidean signature T_{ab}^- is a self-dual rank-two tensor. The embedding of the OSp(2|4)-invariant S^4 in conformal supergravity appeared in [97] [114].

[11] For instance, old and new minimal four dimensional $\mathcal{N}=1$ Poincaré supergravity arises from $\mathcal{N}=1$ conformal supergravity by using a compensating chiral and tensor multiplet respectively.

[12] We refer to the [68] for more background material and references, in particular for 4d $\mathcal{N}=2$ supergravity.

Four dimensional $\mathcal{N} = 2$ Poincaré supergravity [64] contains a graviphoton gauge field A_m. This field is furnished in the conformal approach by coupling an abelian vector multiplet to the Weyl multiplet [51] [50]. An $\mathcal{N} = 2$ vector multiplet, also known as a restricted chiral multiplet, is an $\mathcal{N} = 2$ chiral multiplet (C.7) with $w = 1$ subject to constraints, and consists of

$$\begin{aligned} \text{bosonic:}\, & X, A_m, Y_{ij} \\ \text{fermionic:}\, & \Omega_i \end{aligned} \tag{C.41}$$

a complex scalar X, a gauge field A_m, a triplet of real auxiliary fields $Y_{ij} = Y_{ji}$ and gauginos Ω_i. The vielbein e_m^a and gravitino ψ_m^i of the Weyl multiplet and the gauge field A_m in the vector multiplet complete the on-shell content of four dimensional $\mathcal{N} = 2$ Poincaré supergravity multiplet.

The first step in constructing a Poincaré supergravity theory is to gauge fix special conformal transformations. This can be accomplished by setting

$$b_m = 0\,. \tag{C.42}$$

In order to preserve this gauge, supersymmetry transformations must be accompanied by a compensating special conformal transformation, which acts nontrivially on b_m. Fortunately, all elementary fields in conformal supergravity and all fields in $\mathcal{N} = 2$ matter multiplets transform trivially under special conformal transformations, and therefore the supersymmetry transformations of these fields are not modified by the gauge choice (C.42).

Dilatations and $\mathrm{U}(1)_R$ are gauge fixed by setting [52]

$$X = \mu\,, \tag{C.43}$$

where μ is an arbitrary mass scale, while [52]

$$\Omega_i = 0 \tag{C.44}$$

fixes the special conformal supersymmetry transformations. Under supersymmetry [68]

$$\delta X = \frac{1}{2}\bar{\epsilon}^i \Omega_i \tag{C.45}$$

$$\delta\Omega_i = \not{D} X \epsilon_i + \frac{1}{4}\Gamma^{ab}\mathcal{F}_{ab}\varepsilon_{ij}\epsilon^j + \frac{Y_{ij}}{2}\epsilon^j + 2X\eta_i\,, \tag{C.46}$$

where $\delta \equiv \delta_\epsilon + \delta_\eta$, and (ϵ^i, ϵ_i) and (η^i, η_i) parametrize the Poincaré and conformal supersymmetry transformations. $\mathcal{F}_{ab}$ is the superconformal covariant field strength (see equation (20.77) in [68]) and

$$\mathcal{D}_\mu X = (\partial_\mu - b_\mu - iA_\mu^R)X - \frac{1}{2}\overline{\psi}^i_\mu \Omega_i \tag{C.47}$$

is the superconformal covariant derivative acting on the scalar field X. In order to preserve the gauge choice (C.43)(C.44), we must accompany the Poincaré supersymmetry transformations δ_ϵ with a field dependent compensating conformal supersymmetry transformation δ_η with parameter[13]

$$\eta_i = \frac{i}{2}\not{A}^R \epsilon_i - \frac{1}{2\mu}\left(\frac{1}{4}\Gamma^{ab}\mathcal{F}_{ab}\varepsilon_{ij} + \frac{Y_{ij}}{2}\right)\epsilon^j \,. \tag{C.48}$$

Different Poincaré supergravity theories depend on the choice of a second multiplet which gauge fixes the remaining $\mathrm{SU}(2)_R$ symmetry. Three choices for this compensating multiplet have been considered in the literature (see [48]): a non-linear multiplet, a hypermultiplet and a tensor multiplet. We now demonstrate that the OSp(2|4)-invariant S^4 is a supersymmetric background of the $\mathcal{N}=2$ Poincaré supergravity theory constructed with a tensor multiplet (and not with the non-linear or hypermultiplet).

Consider the off-shell $\mathcal{N}=2$ Poincaré supergravity multiplet constructed by coupling a vector multiplet and a tensor multiplet to the Weyl multiplet. An $\mathcal{N}=2$ tensor multiplet [48]

$$\begin{aligned} \text{bosonic} &: L_{ij}, G, E_{mn} \\ \text{fermionic} &: \phi^i \end{aligned} \tag{C.49}$$

consists of a triplet of real scalars $L_{ij} = L_{ji}$, a tensor gauge field E_{mn}, a complex scalar G and a doublet of spinors ϕ^i. The $\mathrm{SU}(2)_R$ symmetry can be gauge fixed by setting

$$L_{ij} = \tau_{1ij}\varphi \,, \tag{C.50}$$

which breaks $\mathrm{SU}(2)_R$ down to $\mathrm{SO}(2)_R$. The supersymmetry transformation [141]

$$\delta L_{ij} = \overline{\epsilon}_{(i}\phi_{j)} + \varepsilon_{ik}\varepsilon_{jl}\overline{\epsilon}^{(k}\phi^{l)} \tag{C.51}$$

implies that to preserve (C.50), we must accompany the Poincaré supersymmetry transformation δ_ϵ with a compensating $\mathrm{SU}(2)_R$ transformation $\delta_{\mathrm{SU}(2)_R}(\Lambda^k{}_j)$ with parameter[14]

$$\Lambda^k{}_j = -\tau_1^{km}\frac{(\overline{\epsilon}_m\phi_j - \varepsilon_{im}\varepsilon_{jl}\overline{\epsilon}^i\phi^l)}{\varphi} \,. \tag{C.52}$$

[13]Since (C.44) preserves δX, no other compensating transformation in required.

[14]The parameter is determined only up to an $\mathrm{SO}(2)_R$ transformation.

In summary, this off-shell Poincaré supergravity multiplet constructed by gauge fixing a Weyl, vector and tensor multiplet completes the on-shell multiplet e_m^a, ψ_m^i, A_m with bosonic auxiliary fields and fermionic auxiliary fields χ^i, ϕ^i. The Poincaré supersymmetry transformations in this $\mathcal{N}=2$ Poincaré supergravity theory are given by the following combination of superconformal transformations

$$\delta_\epsilon + \delta_\eta + \delta_{\mathrm{SU}(2)_R}(\Lambda^k{}_j) \tag{C.53}$$

with η in (C.48) and $\Lambda^k{}_j$ in (C.52).

In this $\mathcal{N}=2$ Poincaré supergravity theory the supersymmetric backgrounds where the background values of all fermions vanish are solutions to the following equations

$$(\delta_\epsilon + \delta_\eta)\,\psi_m^i = 0 \qquad (\delta_\epsilon + \delta_\eta)\,\chi^i = 0 \qquad (\delta_\epsilon + \delta_\eta)\,\phi^i = 0 \tag{C.54}$$

with η in (C.48), since $\Lambda^k{}_j = 0$ vanish on bosonic backgrounds. The explicit form of these transformations are [51] [50] [48] (we use [141])

$$\begin{aligned}
\delta\psi_m^i &= \left(\partial_m + \frac{1}{2}b_m + \frac{1}{4}\Gamma^{ab}\omega_{mab} - \frac{1}{2}iA_m^R\right)\epsilon^i + V_m{}^i{}_j\epsilon^j - \frac{1}{16}\Gamma^{ab}T_{ab}^-\varepsilon^{ij}\gamma_m\epsilon_j - \gamma_m\eta^i \\
\delta\chi^i &= \frac{1}{2}D\epsilon^i + \frac{1}{6}\Gamma^{ab}\left[-\frac{1}{4}\not{\mathcal{D}}T_{ab}^-\varepsilon^{ij}\epsilon_j - \widehat{R}_{ab}(U_j{}^i)\epsilon^j + i\widehat{R}_{ab}(T)\epsilon^i + \frac{1}{2}T_{ab}^-\varepsilon^{ij}\eta_j\right] \\
\delta\phi^i &= \frac{1}{2}\not{D}L^{ij}\epsilon_j + \frac{1}{2}\varepsilon^{ij}\not{E}\epsilon_j - \frac{1}{2}G\epsilon^i + 2L^{ij}\eta_j\,,
\end{aligned} \tag{C.55}$$

with η in (C.48). $\mathcal{D}$ is superconformal covariant derivative and $\widehat{R}_{ab}(T)$ and $\widehat{R}_{ab}(U_j{}^i)$ are covariant curvatures for $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$.

The OSp(2|4)- supersymmetric S^4 background is described by the following Killing spinor equations (C.11)

$$\nabla_m\epsilon^i = \frac{i}{2r}\gamma_m\tau_1^{ij}\epsilon_j \qquad \nabla_m\epsilon_i = \frac{i}{2r}\gamma_m\tau_{1ij}\epsilon^j\,. \tag{C.56}$$

From (C.55) we find that S^4 is a supersymmetric background of this supergravity theory with the following non-vanishing background fields turned on

$$e_m^a = e_m^a|_{S^4} \qquad Y^{ij} = -\frac{2i\mu}{r}\tau_1^{ij} \qquad Y_{ij} = -\frac{2i\mu}{r}\tau_{1ij} \qquad \text{other} = 0\,. \tag{C.57}$$

With these background fields turned on $\delta\psi_m^i$ realizes the S^4 Killing spinor equations (C.56), while $\delta\xi^i$ and $\delta\phi^i$ vanish identically.[15] The algebra of supergravity transformations when evaluated on the background (C.57) realizes the OSp(2|4) symmetry of S^4.

C.4 The Kähler ambiguity Supergravity Counterterm

In this section we construct the $\mathcal{N} = 2$ Poincaré supergravity invariant constructed out of the supergravity multiplet and the $w = 0$ chiral multiplets $\mathcal{A}_I$ (see below (C.9)) which when evaluated on the OSp(2|4)-supersymmetric background (C.63) realizes the Kähler ambiguity (C.2).

Our approach is to construct a superconformal invariant constructed out of the Weyl multiplet, the compensating vector multiplet Φ, the compensating tensor multiplet and the chiral multiplets $\mathcal{A}_I$, the supermultiplets to which the coordinates in the conformal manifold τ_I have been promoted. This invariant, when evaluated on the Poincaré gauge fixing choice described in the previous section yields an invariant in the associated $\mathcal{N} = 2$ Poincaré supergravity theory. We first recall some facts about the construction of superconformal invariants.

Consider an abstract chiral multiplet (C.7) with $w = 2$, which we denote by $\hat{\mathcal{A}}$, coupled to the Weyl multiplet (C.40). The following superconformal invariant can be constructed from such a chiral multiplet [52]

$$I[\hat{\mathcal{A}}] = \int d^4x\sqrt{g}\left[\hat{C}(x) - \frac{1}{4}\hat{A}\left(T_{ab}^{+}\right)^2 + \text{fermions}\right], \tag{C.58}$$

where $\hat{C}$ and $\hat{A}$ denote the top and bottom components of the multiplet $\hat{\mathcal{A}}$. The coupling of the chiral multiplet to the Weyl multiplet is responsible for the appearance of the terms after $\hat{C}$ in (C.58). The product of two chiral multiplets with R-charge w_1 and w_2 yields another chiral multiplet of R-charge $w_1 + w_2$. Therefore, superconformal invariants can be constructed from products of chiral multiplets with total R-charge $w = 2$.

Consider now the compensating vector multiplet that appears in the construction of $\mathcal{N} = 2$ Poincaré supergravity, which we denote by Φ. It is important to note that an $\mathcal{N} = 2$ vector multiplet is a chiral multiplet with $w = 1$ subject to reducibility constraints [47],

[15] A similar analysis for the Poincaré supergravity theories constructed with a compensating non-linear multiplet and hypermultiplet demonstrates that the background fields that yield the S^4 Killing spinor equations are incompatible with the vanishing of the supersymmetry variations of the fermions in these multiplets. Therefore, S^4 is not a supersymmetric background of these supergravity theories.

which express the last two components of the chiral multiplet in terms of the previous ones. It is also known as a restricted chiral multiplet. The components of a chiral multiplet (C.7) are given in terms of the fields in the abelian vector multiplet (C.41) by

$$\begin{aligned} A|_\Phi &= X \\ \Psi_i|_\Phi &= \Omega_i \\ B_{ij}|_\Phi &= Y_{ij} \\ F^-_{ab}|_\Phi &= \mathcal{F}^-_{ab} \\ \Lambda_i|_\Phi &= -\varepsilon_{ij} \not{D}\Omega^j \\ C|_\Phi &= -2D_a D^a \bar{X} - \frac{1}{2}\mathcal{F}^+_{ab}T^{ab+} - 3\bar{\chi}_i\Omega^i \end{aligned} \tag{C.59}$$

where $\mathcal{F}_{ab}$ is the superconformal covariant field strength. Expressing a vector multiplet as a $w=1$ chiral multiplet provides a way of constructing a superconformal invariant out of Φ using (C.58).

We will now construct the supergravity counterterm that realizes the Kähler ambiguity by writing down a supergravity invariant constructed out of a composite chiral multiplet with $w=2$. From the compensating vector multiplet Φ, which has $w=1$, it is possible to construct two chiral multiplets with $w=2$

$$\mathbb{T}(\log\bar{\Phi}) \quad \text{and} \quad \Phi^2\,, \tag{C.60}$$

where the first is the so called non-linear kinetic multiplet [49] [21].[16] Given a chiral multiplet $\mathcal{A}$ with bottom component A and R-charge w, the multiplet $\log\mathcal{A}$ is a chiral multiplet whose bottom component, namely $\log A$, transforms inhomogeneously under dilatations but its higher components in the multiplet (in particular the top component) transform as if they belonged to a chiral multiplet with $w=0$ [21]. The usefulness of this multiplet comes from the fact that the top component of a chiral multiplet with $w=0$ is the bottom component of an anti-chiral multiplet with $w=2$, and in particular a superconformal primary (i.e. invariant under S-supersymmetry). Taking the CPT conjugate we find that the top component of $\log\bar{\Phi}$ is a chiral primary with $w=2$. Therefore we can build a chiral multiplet with $w=2$ by applying the Q-supersymmetry generators on the top component of $\log\bar{\Phi}$ and this multiplet is precisely $\mathbb{T}(\log\bar{\Phi})$ [21].

The supergravity counterterm responsible for the Kähler ambiguity is the superconformal invariant (C.58) constructed from the $w=2$ composite chiral multiplet

$$\mathcal{F}(\mathcal{A}_I)\mathbb{T}(\log\bar{\Phi})\,, \tag{C.61}$$

[16] Given an anti-chiral multiplet $\bar{\mathcal{A}}$ with $w=0$, the corresponding kinetic multiplet $\mathbb{T}(\bar{\mathcal{A}})$, which has $w=2$, is defined as: $\mathbb{T}(\bar{\mathcal{A}}) \propto \bar{D}^4\bar{\mathcal{A}}$, where $\bar{D}^4$ involves all four anti-chiral covariant superspace derivatives.

where $\mathcal{F}$ is an arbitrary holomorphic function of the $w=0$ chiral multiplets $\mathcal{A}_I$ describing the coordinates in the conformal manifold.[17] The associated $\mathcal{N}=2$ Poincaré supergravity invariant is

$$I\left[\mathcal{F}(\mathcal{A}_I)\mathbb{T}(\log\bar{\Phi})\right]\,. \tag{C.62}$$

We will now evaluate this invariant in the OSp(2|4) invariant background field configuration:

$$\begin{aligned}
&\text{Weyl: } e_m^a = e_m^a|_{S^4}\\
&\text{vector: } A|_\Phi = X = \mu\,, \qquad B_{ij}|_\Phi = -\frac{2i\mu}{r}\tau_{1ij}\,, \qquad C|_\Phi = \frac{4\mu}{r^2}\\
&\text{chiral: } A|_{\mathcal{F}(\mathcal{A}_I)} = \mathcal{F}(\tau_I)\,,
\end{aligned} \tag{C.63}$$

Since the only field with nonzero expectation value in the Weyl multiplet is the vierbein, the only term from (C.58) that survives in this background is the top component of the product chiral multiplet $\mathcal{F}(\mathcal{A}_I)\mathbb{T}(\log\bar{\Phi})$. The product of two chiral multiplets with bosonic components (A, B_{ij}, F_{ab}^-, C) and (a, b_{ij}, f_{ab}^-, c) yields a new chiral multiplet with bosonic components (setting all fermions to zero, as they vanish in the OSp(2|4)-invariant background (C.63))

$$\left(Aa, Ab_{ij}+aB_{ij}, Af_{ab}^- + aF_{ab}^-, Ac + aC - \frac{1}{2}\varepsilon^{ik}\varepsilon^{jl}B_{ij}b_{kl} + F_{ab}^- f^{-ab}\right)\,. \tag{C.64}$$

We need the bosonic components of the chiral multiples $\mathcal{F}(\mathcal{A}_I)$ and $\mathbb{T}(\log\bar{\Phi})$ to compute the top component of their product. Since only the bottom component of $\mathcal{F}(\mathcal{A}_I)$ is nonzero in the background (C.63), to compute the top component of the product $\mathcal{F}(\mathcal{A}_I)\mathbb{T}(\log\bar{\Phi})$ according to (C.64) we only need to know the top component of $\mathbb{T}(\log\bar{\Phi})$. The components of $\mathbb{T}(\log\bar{\Phi})$ were computed in terms of the components of $\bar{\Phi}$ in [21]. Using their expressions, in the background (C.63), the top component becomes:

$$C|_{\mathbb{T}(\log\bar{\Phi})} = 4D^2D^2\log\bar{\mu} - 8\mathcal{R}^{ab}D_aD_b\log\bar{\mu} + \frac{8}{3}\mathcal{R}D^2\log\bar{\mu} + \frac{2}{3}D^2\mathcal{R} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{2}{3}\mathcal{R}^2\,, \tag{C.65}$$

[17] An analogous $w=2$ chiral multiplet constructed out of Φ^2, namely $\mathcal{F}(\mathcal{A}_I)\Phi^2$ can be used to construct another counterterm but when evaluated on the OSp(2|4) invariant background (C.63) the invariant becomes: $I[\mathcal{F}(\mathcal{A}_I)\Phi^2] = 32\pi^2\mu^2r^2$. Yet another natural guess for the Kähler counterterm can be constructed from the $w=2$ chiral multiplet $W^{ab}W_{ab}\mathcal{F}(\mathcal{A}_I)$, where W_{ab} is a chiral multiplet that encodes the covariant Weyl multiplet (C.40). However, upon evaluating these terms on the background (C.63) they all vanish, as these terms involve the Weyl tensor, which vanishes on S^4. Supergravity couplings involving W^2 have been considered in the literature [125]. For other higher derivative invariants in $\mathcal{N}=2$ supergravity see e.g. [23] [53] [21] [117] [22].

where D_a is space-time covariant derivative, $D^2 \equiv D_a D^a$, $\mathcal{R}_{ab}$ is the Ricci curvature on the sphere, and $\mathcal{R}$ is the scalar curvature. Since $\log \bar{\mu}$ and $\mathcal{R}$ are covariantly constant scalars all the terms in (C.65) with derivatives vanish and after substituting the values for $\mathcal{R}_{ab}$ and $\mathcal{R}$, (C.65) becomes:

$$C|_{\mathbb{T}(\log \bar{\Phi})} = \frac{24}{r^4}\,, \tag{C.66}$$

and finally

$$C|_{\mathcal{F}(\mathcal{A}_I)\mathbb{T}(\log \bar{\Phi})} = \mathcal{F}(\tau_I)\frac{24}{r^4}\,. \tag{C.67}$$

Thus, we find that the invariant (C.62) is:

$$I\left[\mathcal{F}(\mathcal{A}_I)\mathbb{T}(\log \bar{\Phi})\right] = \int_{S^4} \sqrt{g}\,\mathcal{F}(\tau_I)\frac{24}{r^4} = 64\pi^2\mathcal{F}(\tau_I)\,. \tag{C.68}$$

Therefore, the marginal supergravity counterterm

$$\frac{1}{768\pi^2}\left(I[\mathcal{F}(\mathcal{A}_I)\mathbb{T}(\log \bar{\Phi})] + I[\bar{\mathcal{F}}(\bar{\mathcal{A}}_I)\mathbb{T}(\log \Phi)]\right) \tag{C.69}$$

is responsible for the Kähler ambiguity (C.2) in the four sphere partition function of four dimensional $\mathcal{N} = 2$ SCFTs

$$Z_{S^4} \simeq Z_{S^4} e^{\frac{1}{12}\left(\mathcal{F}(\tau)+\bar{\mathcal{F}}(\bar{\tau})\right)}\,. \tag{C.70}$$

This provides a microscopic realization of Kähler ambiguities in these SCFTs.

Appendix D

Background Materials on 4d $\mathcal{N} = 2$ SCFTs

D.1 Integrability of tt^* Equations

In this appendix we will show that the tt^* equations together with the WDVV equations for any 4d $\mathcal{N} = 2$ SCFT are integrable, in the sense that these equations can be written as the flatness condition of a one parameter family of connections on a certain vector bundle over the conformal manifold $\mathcal{M}$, which is equivalent to the Lax representation (with spectral parameter) of a classically integrable system. Recall that the WDVV equations [57,58,152] and the tt^* equation are [139]

$$\nabla_i C^L_{jK} = \nabla_j C^L_{iK}, \qquad \overline{\nabla}_{\bar{i}} \overline{C}^{\bar{L}}_{\bar{j}\bar{K}} = \overline{\nabla}_{\bar{j}} \overline{C}^{\bar{L}}_{\bar{i}\bar{K}}, \tag{D.1a}$$

$$[\nabla_i, \nabla_j]^L_K = [\overline{\nabla}_{\bar{i}}, \overline{\nabla}_{\bar{j}}]^{\bar{L}}_{\bar{K}} = 0, \tag{D.1b}$$

$$[\nabla_i, \overline{\nabla}_{\bar{j}}]^L_K = -[C_i, \overline{C}_{\bar{j}}]^L_K + g_{i\bar{j}} \delta^L_K \left(1 + \frac{R}{4c}\right), \tag{D.1c}$$

where i, j run over chiral primaries of $\Delta = 2$, K, L run over all chiral primaries, and C^K_{IJ} are OPE coefficients defined as:

$$\mathcal{O}_I(x)\mathcal{O}_J(0) = C^K_{IJ}\mathcal{O}_K(0) + \cdots . \tag{D.2}$$

C_I can be thought of as an operator acting on chiral primaries whose matrix components are C^K_{IJ}. In (D.1c), R is the R-charge of the chiral primaries that comprise the fibre of the bundle $\mathcal{V}_R \to \mathcal{M}$ on which the covariant derivatives and the C_i's act, c is the central charge of the SCFT, and $g_{i\bar{j}}$ is the Zamolodchikov metric on $\mathcal{M}$. We also note the fact that C_I ($\overline{C}_{\bar{I}}$) is covariantly holomorphic (antiholomorphic) [139]

$$\overline{\nabla}_{\bar{i}} C_I = \nabla_i \overline{C}_{\bar{I}} = 0\,. \tag{D.3}$$

Now consider the holomorphic vector bundle $\mathcal{V}_R \otimes \mathcal{L}^{\otimes n}$ with $n = -(4c+R)$, where $\mathcal{L}$ is the supercharge bundle.[1] $\mathcal{L}$ is a holomorphic line bundle over $\mathcal{M}$ whose curvature is given by [139]

$$F_{ij} = F_{\bar{i}\bar{j}} = 0, \qquad F_{i\bar{j}} = \frac{1}{4c} g_{i\bar{j}}. \tag{D.4}$$

This nontrivial curvature encodes the ambiguity in defining the phase of the supercharges, i.e., the following automorphism of the $\mathcal{N}=2$ superconformal algebra:

$$Q^i_\alpha \to e^{i\theta} Q^i_\alpha, \quad \overline{Q}^{\bar{i}}_{\dot{\alpha}} \to e^{-i\theta} \overline{Q}^{\bar{i}}_{\dot{\alpha}}, \quad S^i_\alpha \to e^{-i\theta} S^i_\alpha, \quad \overline{S}^{\bar{i}}_{\dot{\alpha}} \to e^{i\theta} \overline{S}^{\bar{i}}_{\dot{\alpha}}. \tag{D.5}$$

(D.4) implies that the curvature of $\mathcal{L}^{\otimes n}$, let it be denoted by F^n, is given by:

$$F^n_{ij} = F^n_{\bar{i}\bar{j}} = 0, \qquad F^n_{i\bar{j}} = \frac{n}{4c} g_{i\bar{j}} = -\left(1 + \frac{R}{4c}\right) g_{i\bar{j}}. \tag{D.6}$$

Let the covariant derivative on $\mathcal{L}^{\otimes n}$ be denoted by $\nabla^{\mathcal{L}}_i$ and define the following one parameter family of connections on $\mathcal{V}_R \otimes \mathcal{L}^{\otimes n}$:

$$\nabla^\xi_i \equiv \nabla_i + \xi C_i + \nabla^{\mathcal{L}}_i, \qquad \overline{\nabla}^\xi_{\bar{i}} \equiv \overline{\nabla}_{\bar{i}} + \xi^{-1} \overline{C}_{\bar{i}} + \overline{\nabla}^{\mathcal{L}}_{\bar{i}}, \tag{D.7}$$

where ∇_i and C_i are the same operators that appear in (D.1). The flatness condition of this connection for any value of the parameter ξ is:

$$[\nabla^\xi_i, \nabla^\xi_j] = [\overline{\nabla}^\xi_{\bar{i}}, \overline{\nabla}^\xi_{\bar{j}}] = [\nabla^\xi_i, \overline{\nabla}^\xi_{\bar{j}}] = 0, \qquad \forall\, \xi \in \mathbb{C}. \tag{D.8}$$

Using (D.7), (D.6) and noting that operators on $\mathcal{V}_R$ commute with operators on $\mathcal{L}^{\otimes n}$ we get:

$$[\nabla^\xi_i, \nabla^\xi_j] = [\nabla_i, \nabla_j] + \xi(\nabla_i C_j - \nabla_j C_i) + \xi^2 [C_i, C_j] \tag{D.9a}$$

$$[\overline{\nabla}^\xi_{\bar{i}}, \overline{\nabla}^\xi_{\bar{j}}] = [\overline{\nabla}_{\bar{i}}, \overline{\nabla}_{\bar{j}}] + \xi^{-1}(\overline{\nabla}_{\bar{i}} \overline{C}_{\bar{j}} - \overline{\nabla}_{\bar{j}} \overline{C}_{\bar{i}}) + \xi^{-2} [\overline{C}_{\bar{i}}, \overline{C}_{\bar{j}}] \tag{D.9b}$$

$$[\nabla^\xi_i, \overline{\nabla}^\xi_{\bar{j}}] = [\nabla_i, \overline{\nabla}_{\bar{j}}] + [C_i, \overline{C}_{\bar{j}}] - \left(1 + \frac{R}{4c}\right) g_{i\bar{j}} + \xi^{-1} [\nabla_i, \overline{C}_{\bar{j}}] - \xi [\overline{\nabla}_{\bar{j}}, C_i]\,. \tag{D.9c}$$

[1] Negative power of a line bundle is defined as the positive power of the dual bundle, i.e., if $E \to M$ is a line bundle then for some negative real number $m < 0$ we have $E^{\otimes m} \equiv (E^*)^{\otimes |m|}$.

Equation (D.8) must be satisfied at each order in ξ. The C_i's commute among themselves and so do the $\overline{C}_{\bar{i}}$'s [139], so the $\mathcal{O}(\xi^2)$ and $\mathcal{O}(\xi^{-2})$ terms vanish. The $\mathcal{O}(\xi)$ and $\mathcal{O}(\xi^{-1})$ terms of (D.9c) vanish due to (D.3). By imposing (D.8) order by order on the rest of terms in (D.9) we recover precisely (D.1), thus proving integrability of WDVV and tt^* equations of four-dimensional $\mathcal{N}=2$ SCFTs. They are governed by a Hitchin integrable system.

D.2 Deforming $\mathcal{N}=2$ SCFT on S^4 by Chiral Operators

When we place an $\mathcal{N}=2$ SCFT on S^4 via the stereographic map, then the Lagrangian preserves the full superconformal symmetry. However, the partition function and various other observables need to be regulated in the ultraviolet. The maximal subalgebra that can be preserved by the regulator is $\mathfrak{osp}(2|4)$. This is because this subgroup does not include conformal transformations but only isometries of the sphere. In this appendix we discuss F-term deformations of the action that preserve $\mathfrak{osp}(2|4)$, i.e.

$$S \to S - \tau_{\mathcal{U}} \int_{S^4} \mathrm{d}^4x\sqrt{g}\,\mathcal{U}(x)\,, \tag{D.10}$$

such that the deformation term is $\mathfrak{osp}(2|4)$ invariant, i.e.

$$\delta\left(\tau_{\mathcal{U}} \int_{S^4} \mathrm{d}^4x\sqrt{g}\,\mathcal{U}(x)\right) = 0\,, \tag{D.11}$$

where δ represents an $\mathfrak{osp}(2|4)$ transformation. Here $\tau_{\mathcal{U}}$ is the coupling constant corresponding to the operator $\mathcal{U}$. If $\tau_{\mathcal{U}}$ has Weyl weight 0 then such a deformation is marginal but here we are interested in more general deformations where $\tau_{\mathcal{U}}$ can have arbitrary Weyl weight. The systematic way to find such deformations is to start with an $\mathcal{N}=2$ superfield and integrate it over the chiral superspace ($\int \mathrm{d}^4\theta$) with the appropriate measure and evaluate the resulting term on the S^4 background.

In order to achieve this, we begin by promoting the coupling constant $\tau_{\mathcal{U}}$ to an $\mathcal{N}=2$ chiral multiplet with Weyl weight $(2-w)$ whose bottom component is $\tau_{\mathcal{U}}$. We also consider another chiral multiplet of Weyl weight w whose bottom component will be called $\mathcal{U}$. We will denote these two multiplets as $\underline{\tau_{\mathcal{U}}}$ and $\underline{\mathcal{U}}$ and denote the component fields of these two multiplets as $(\tau_{\mathcal{U}}, \psi_i, b_{ij}, f^-_{ab}, \lambda_i, c)$ and $(\mathcal{U}, \Psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C)$, respectively.[2] b_{ij}, B_{ij} are

[2] i, j are $\mathfrak{su}(2)_R$ indices and a, b are local frame indices on S^4.

symmetric and in Euclidean signature f^-_{ab}, F^-_{ab} are selfdual tensors. Now the following term is manifestly $\mathfrak{osp}(2|4)$ invariant:

$$\int_{S^4} \mathrm{d}^4x \int \mathrm{d}^4\theta\, \mathcal{E}\, \underline{\tau_{\mathcal{U}}}(x,\theta)\underline{\mathcal{U}}(x,\theta)\,, \tag{D.12}$$

where $\mathcal{E}$ is the chiral density. Since we want $\underline{\tau_{\mathcal{U}}}$ to be a background multiplet, we need to restrict its components in such a way that the required supersymmetry algebra is unbroken. First, in order to preserve rotational invariance on S^4 we can give the spacetime scalars $\tau_{\mathcal{U}}$, b_{ij} and c constant expectation values and let all the other fields in $\underline{\tau_{\mathcal{U}}}$ vanish. Supersymmetry is preserved if the supersymmetry variations of all the background fields vanish.

The Supersymmetry (SUSY) variations of a chiral multiplet of Weyl weight w, with component fields written as $(A, \Psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C)$, under an $\mathcal{N}=2$ superconformal transformation are given by (see e.g. [76]):

$$\delta A = \frac{1}{2}\bar{\epsilon}^i\Psi_i \tag{D.13a}$$

$$\delta\Psi_i = \nabla\!\!\!/(A\epsilon_i) + \frac{1}{2}B_{ij}\epsilon^j + \frac{1}{4}\gamma^{ab}F^-_{ab}\varepsilon_{ij}\epsilon^j + (2w-4)A\eta_i \tag{D.13b}$$

$$\delta B_{ij} = \bar{\epsilon}_{(i}\nabla\!\!\!/\Psi_{j)} - \bar{\epsilon}^k\Lambda_{(i}\varepsilon_{j)k} + 2(1-w)\bar{\eta}_{(i}\Psi_{j)} \tag{D.13c}$$

$$\delta F^-_{ab} = \frac{1}{4}\varepsilon^{ij}\bar{\epsilon}_i\nabla\!\!\!/\gamma_{ab}\Psi_j + \frac{1}{4}\bar{\epsilon}^i\gamma_{ab}\Lambda_i - \frac{1}{2}(1+w)\varepsilon^{ij}\bar{\eta}_i\gamma_{ab}\Psi_j \tag{D.13d}$$

$$\delta\Lambda_i = -\frac{1}{4}\gamma^{ab}\nabla\!\!\!/(F^-_{ab}\epsilon_i) - \frac{1}{2}\nabla\!\!\!/B_{ij}\varepsilon^{jk}\epsilon_k + \frac{1}{2}C\varepsilon_{ij}\epsilon^j - (1+w)B_{ij}\varepsilon^{jk}\eta_k + \frac{1}{2}(3-w)\gamma^{ab}F^-_{ab}\eta_i \tag{D.13e}$$

$$\delta C = -\nabla_m(\varepsilon^{ij}\bar{\epsilon}_i\gamma^m\Lambda_j) + (2w-4)\varepsilon^{ij}\bar{\eta}_i\Lambda_j\,. \tag{D.13f}$$

where δ is a generic $\mathcal{N}=2$ superconformal transformation being generated by the chiral conformal Killing spinors ϵ^i, η_i and ϵ_i, η^i, and we use the matrices $\tau_p^{ij} \equiv \{i\sigma_3, -\mathbb{1}_{2\times 2}, -i\sigma_1\} =: \tau^*_{pij}$. γ_m are curved space gamma matrices defined in terms of the vierbein e^a_m and the flat space gamma matrices Γ_a as $\gamma_m(x) \equiv e^a_m(x)\Gamma_a$. The conformal Killing spinors satisfy the equations:

$$\nabla_m\epsilon^i = \gamma_m\eta^i\,, \qquad \nabla_m\epsilon_i = \gamma_m\eta_i\,. \tag{D.14}$$

The $\mathfrak{osp}(2|4)$ transformations can be generated by imposing the following constraints on the conformal Killing spinors:

$$\eta^j = \frac{i}{2r}\tau_1^{jk}\epsilon_k\,, \qquad \eta_j = \frac{i}{2r}\tau_{1jk}\epsilon^k\,. \tag{D.15}$$

In the background where the fermions are all vanishing the variations of the bosonic fields automatically vanish. So for the background fields in $\underline{\tau_{\mathcal{U}}}$ we demand that the fermionic variations in (D.13) vanish:

$$\delta\psi_i = \frac{1}{2}b_{ij}\epsilon^j + \frac{i}{r}(2-w)\tau_{\mathcal{O}}\tau_{1ij}\epsilon^j = 0 \tag{D.16}$$

$$\delta\lambda_i = \frac{1}{2}c\epsilon^j\varepsilon_{ij} - \frac{i}{2r}(3-w)b_{ij}\varepsilon^{jk}\tau_{1kl}\epsilon^l = 0\,. \tag{D.17}$$

These equations are satisfied when:

$$b_{jk} = \frac{2i}{r}(w-2)\tau_{1jk}\tau_{\mathcal{U}}\ , \tag{D.18a}$$

$$c = \frac{2}{r^2}(w-2)(w-3)\tau_{\mathcal{U}}\ . \tag{D.18b}$$

Now, the product of two chiral multiplets is another chiral multiplet whose bottom component is the product of the bottom components of the individual chiral multiplets and this multiplication is defined in a way such that the integration over the chiral superspace in (D.12) will simply pick out the top component of the product chiral multiplet $\underline{\tau_{\mathcal{U}}}\,\underline{\mathcal{U}}$. The general expression for the top component of $\underline{\tau_{\mathcal{U}}}\,\underline{\mathcal{U}}$ is given by:

$$\tau_{\mathcal{O}}C + \mathcal{O}c - \frac{1}{2}\varepsilon^{ik}\varepsilon^{jl}B_{ij}b_{kl} + F^-_{ab}f^-_{ab} + \varepsilon^{ij}\left(\overline{\Psi}_i\lambda_j + \overline{\psi}_i\Lambda_j\right). \tag{D.19}$$

When we use the background values where $\tau_{\mathcal{U}}$ is a constant, f_{ab} and all the fermions in $\underline{\tau_{\mathcal{U}}}$ vanish and the rest of the fields satisfy (D.18), this becomes:

$$\tau_{\mathcal{U}}\,\mathcal{C}(x) \equiv \tau_{\mathcal{U}}\left[C(x) + \frac{2}{r^2}(w-2)(w-3)A(x) - \frac{i}{r}(w-2)\tau_1^{ij}B_{ij}(x)\right], \tag{D.20}$$

and (D.12) reduces to:

$$\tau_{\mathcal{U}}\int_{S^4}\mathrm{d}^4x\sqrt{g}\,\mathcal{C}(x). \tag{D.21}$$

D.3 Ward Identity

For a chiral multiplet $(A, \Psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C)$ of weight w recall the combination (D.20):

$$\mathcal{C}(x) \equiv C(x) + \frac{2}{r^2}(w-2)(w-3)A(x) - \frac{i}{r}(w-2)\tau_1^{ij}B_{ij}(x)\,. \tag{D.22}$$

In this appendix we prove the following identity: If $\mathcal{U}$ is some $\mathfrak{osp}(2|4)$ supersymmetric operator, i.e. $\delta_{\text{SUSY}}\mathcal{U} = 0$, then

$$\left\langle \left(\int_{S^4} \mathrm{d}^4x \sqrt{g}\, \mathcal{C}(x) \right) \mathcal{U} \right\rangle = 32\pi^2 r^2 \langle A(N)\, \mathcal{U} \rangle \,, \tag{D.23}$$

where N is the North Pole of the sphere. Similarly, for an anti-chiral multiplet we have:

$$\left\langle \mathcal{U} \left(\int_{S^4} \mathrm{d}^4x \sqrt{g}\, \overline{\mathcal{C}}(x) \right) \right\rangle = 32\pi^2 r^2 \langle \mathcal{U}\, \overline{A}(S) \rangle \,, \tag{D.24}$$

where S is the South Pole.

From (D.14) and (D.15) we see that the nonchiral Killing spinors generating the $\mathfrak{osp}(2|4)$ algebra preserved on S^4 satisfy the equation:

$$\nabla_m \chi^j = \frac{i}{2r} \gamma_m \chi^j \,, \tag{D.25}$$

where, $\chi^j \equiv \epsilon^j + \tau_1^{jk} \epsilon_k$. In steregraphic coordinates the solutions to (D.25) are given by:

$$\chi^j = \frac{1}{\sqrt{1 + \frac{x^2}{4r^2}}} \left(\mathbb{1} + \frac{i}{2r} x_m \Gamma^m \right) \chi_0^j \,. \tag{D.26}$$

The constant spinors χ_0^j parametrize the eight supercharges of $\mathfrak{osp}(2|4)$. We choose an $\mathfrak{su}(1|1) \subset \mathfrak{osp}(2|4)$ by imposing the following constraints:

$$P_L \chi_0^i = 0, \qquad \chi_0^i = \tau_1^{ij} \varepsilon_{jk} \Gamma_1 \Gamma_2 \chi_0^k \,. \tag{D.27}$$

The chosen Killing spinors and the supersymmetry transformation they generate will henceforth be denoted by χ^i and δ respectively. χ^i satisfy the following equations:

$$\frac{\chi_L^{i\,\dagger}}{\|\chi_L\|^2} \not{\nabla} (A \chi_R^j) \tau_{2ij} = \nabla_m (U^m A) - \frac{8ir}{x^2} A \,, \qquad \nabla_m U^m = \frac{8ir}{x^2} - \frac{4i}{r} \,, \tag{D.28}$$

where we have defined:

$$\|\chi_L\|^2 \equiv \|\chi_L^1\|^2 = \|\chi_L^2\|^2 \quad \text{and,} \quad U^m \equiv \frac{\chi_L^{i\,\dagger} \gamma^m \chi_R^j}{\|\chi_L\|^2} \tau_{2ij} \,. \tag{D.29}$$

Now, using the supersymmetry transformation of a chiral multiplet $(A, \Psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C)$ of weight w (D.13, D.15) we can write[3]

$$\frac{1}{2}\tau_1^{ij}B_{ij} \overset{\text{mod}\,\delta}{=} \nabla_m(U^m A) - \frac{8ir}{x^2}A - \frac{2i}{r}(w-2)A \tag{D.30a}$$

$$C \overset{\text{mod}\,\delta}{=} -\frac{1}{4}U^m\nabla_m B_{ij}\tau_1^{ij} + \frac{3i}{2r}\tau_1^{ij}B_{ij} + \frac{i}{2r}(w-2)\tau_1^{ij}B_{ij}\,. \tag{D.30b}$$

For a chiral multiplet with $w = 2$ this calculation was done in more detail in [81] and it was shown that we have the following schematic form:

$$C_{w=2}(x) \overset{\text{mod}\,\delta}{=} f(A_{w=2}(x)) \tag{D.31}$$

where f is a function that satisfies:

$$\int_{S^4} \mathrm{d}^4x\sqrt{g}\, f(A(x)) = 32\pi^2 r^2 A(N)\,. \tag{D.32}$$

We want to repeat this computation now for arbitrary w. We define:

$$\Delta C(x) \equiv C(x) - f(A(x))\,. \tag{D.33}$$

To compute ΔC we can use (D.30a) in (D.30b) to write C entirely in terms of A and then if we consider the expression for C as a polynomial in $(w-2)$ then ΔC is given by the terms that depend on $(w-2)$. After some simplifications using (D.28) we find:

$$\Delta C \overset{\text{mod}\,\delta}{=} \frac{2i}{r}(w-2)\nabla_m(U^m A) + \frac{16}{x^2}(w-2)A + \frac{2}{r^2}(w-2)(w-1)A\,. \tag{D.34}$$

Multiplying (D.30a) by $-\frac{2i}{r}(w-2)$ and adding it to the above equation we find the desired result

$$\Delta C + \frac{2}{r^2}(w-2)(w-3)A - \frac{i}{r}(w-2)\tau_1^{ij}B_{ij} \overset{\text{mod}\,\delta}{=} 0\,, \tag{D.35}$$

or equivalently:

$$C(x) + \frac{2}{r^2}(w-2)(w-3)A(x) - \frac{i}{r}(w-2)\tau_1^{ij}B_{ij}(x) = \mathcal{C}(x) \overset{\text{mod}\,\delta}{=} f(A(x))\,. \tag{D.36}$$

Integrating the two sides of $\overset{\text{mod}\,\delta}{=}$ on S^4 and putting them inside a correlator with $\mathcal{U}$ gives us the desired identity (D.23). The proof of (D.24) follows similarly.

[3] Using (D.13) will also result in some terms proportional to F^-_{ab} and $\nabla_m F^-_{ab}$ in (D.30), but these terms are vanishing, because while F^-_{ab} is selfdual in Euclidean signature, their coefficients will be proportional to ${\chi_L^i}^\dagger\Gamma^{ab}\gamma^{(r)}\chi^j_{L/R}\tau_{3ij}$, where $\gamma^{(r)}$ is a product of r distinct gamma matrices, and these terms are anti-selfdual as they satisfy: ${\chi_L^i}^\dagger\Gamma^{ab}\gamma^{(r)}\chi^j_{L/R} = {\chi_L^i}^\dagger\Gamma_*\Gamma^{ab}\gamma^{(r)}\chi^j_{L/R} = -\frac{1}{2}\varepsilon^{ab}{}_{cd}{\chi_L^i}^\dagger\Gamma^{ab}\gamma^{(r)}\chi^j_{L/R}$, where Γ_* is the chirality matrix.

D.4 tt^* Equations from Sphere Partition Function

In this appendix we prove that the two-point functions in SU(N) $\mathcal{N}=2$ SQCD (with $2N$ fundamental hypermultiplets) satisfy the coupled tt^* equation. We denote by $\tau, \bar{\tau}$ the marginal coupling which parametrizes the conformal manifold. The chiral ring is generated by the $N-1$ generators

$$\phi_k \propto \mathrm{Tr}\left(\varphi^k\right) \ , \ k = 2, ..., N \tag{D.37}$$

and a convenient basis for the chiral primaries is

$$\mathcal{O}_i \equiv \mathcal{O}_{i_2,i_3,\ldots,i_N} = \prod_{k=2}^{N} \phi_k^{i_k} \ . \tag{D.38}$$

We will define the matrix of two-point functions on the sphere (dropping the S^4 subscript)

$$M_{ab} = \left\langle \mathcal{O}_a(N)\overline{\mathcal{O}}_b(S)\right\rangle \ . \tag{D.39}$$

As a consequence of the mixing explained in section 3.3, M_{ab} is in general not zero even when $\mathcal{O}_a$ and $\mathcal{O}_b$ are not of the same dimension. The physical operators $\{\mathcal{O}'_a\}$ can be obtained by doing a Gram-Schmidt procedure with respect to all the lower-dimensional CPOs (chiral primary operators):

$$\mathcal{O}'_a = \mathcal{O}_a - \sum_{\Delta_i<\Delta_a} \frac{\left\langle \mathcal{O}_a(N)\overline{\mathcal{O}'}_i(S)\right\rangle}{\left\langle \mathcal{O}'_i(N)\overline{\mathcal{O}'}_i(S)\right\rangle}\mathcal{O}'_i \ . \tag{D.40}$$

The physical two-point functions which correspond to the flat space two-point functions are obtained from

$$G_{ab} = \left\langle \mathcal{O}'_a(N)\overline{\mathcal{O}'}_b(S)\right\rangle , \tag{D.41}$$

which is non zero only if $\Delta_a = \Delta_b$.

We will define the matrix $M^{ij}_{\Delta'}$ to be the inverse of the submatrix of M_{ij} that includes all the operators up to dimension Δ'. Another useful notation is to denote operators of the form $\phi_2\mathcal{O}_a$ by $\mathcal{O}_{\partial a}$, and the corresponding matrix elements are

$$M_{\partial i,j} = \left\langle \phi_2\mathcal{O}_i(N)\bar{\mathcal{O}}_j(S)\right\rangle \ , \ M_{i,\partial j} = \left\langle \mathcal{O}_i(N)\overline{\phi_2\mathcal{O}}_j(S)\right\rangle \ . \tag{D.42}$$

Derivatives with respect to $\tau, \bar{\tau}$ bring down insertions of $\phi_2, \bar{\phi}_2$ such that the following relations between the matrix elements hold

$$\begin{aligned}\partial_\tau M_{IJ} &= M_{\partial I,J} - M_{10}M_{IJ}\\ \partial_{\bar{\tau}} M_{IJ} &= M_{I,\partial J} - M_{01}M_{IJ}\end{aligned} \tag{D.43}$$

where

$$M_{10} = \langle \phi_2(N) \rangle \ , \ M_{01} = \left\langle \overline{\phi_2}(S) \right\rangle \ . \tag{D.44}$$

In the proceeding of this section, we will use the indices a, b, c to denote operators of dimension Δ, indices i, j, k, l to denote operators of dimension smaller than Δ and I, J, K to denote operators up to dimension Δ. Contracted indices are summed over all their possible values unless specified differently. Due to the Gram-Schmidt procedure, we can write G_{ab} in the following way ($\Delta_a = \Delta_b = \Delta$)

$$G_{ab} = M_{ab} - M_{ai} M^{ij}_{\Delta-2} M_{jb} \ . \tag{D.45}$$

It will be useful to show that the inverse of G_{ab} denoted by G^{bc} is equal to $G^{bc} = M^{bc}_{\Delta}$. Proof:

$$\begin{aligned} G_{ab}G^{bc} &= (M_{ab} - M_{ai}M^{ij}_{\Delta-2}M_{jb})M^{bc}_{\Delta} = M_{ab}M^{bc}_{\Delta} - M_{ai}M^{ij}_{\Delta-2}M_{jb}M^{bc}_{\Delta} \\ &= M_{ab}M^{bc}_{\Delta} + M_{ai}M^{ij}_{\Delta-2}M_{jk}M^{kc}_{\Delta} = M_{ab}M^{bc}_{\Delta} + M_{ai}M^{ic}_{\Delta} = \delta^c_a \ . \end{aligned} \tag{D.46}$$

The tt^* equations (D.1c) in the holomorphic gauge and in these notations take the form

$$\partial_{\bar\tau}(\partial_\tau G_{ab}G^{bc}) = G_{\partial a,\partial b}G^{bc} - G_2\delta^c_a - G_{a\partial i}G^{ij}\delta^c_{\partial j} \ . \tag{D.47}$$

In order to prove (D.47), we need to compute $\partial_{\bar\tau}(\partial_\tau G_{ab}G^{bc})$. Do it in steps:

$$\begin{aligned} &\partial_\tau G_{ab} = \partial_\tau(M_{ab} - M_{ai}M^{ij}_{\Delta-2}M_{jb}) \\ &= M_{\partial a,b} - M_{10}M_{ab} - M_{\partial a,i}M^{ij}_{\Delta-2}M_{jb} - M_{ai}M^{ij}_{\Delta-2}M_{\partial j,b} + \\ &+ M_{ai}M^{ik}_{\Delta-2}M_{\partial k,l}M^{lj}_{\Delta-2}M_{jb} + M_{10}M_{ai}M^{ij}_{\Delta-2}M_{jb}M_{ai}M^{ij}_{\Delta-2}M_{jb} \\ &= M_{\partial a,b} - M_{10}G_{ab} - M_{\partial a,i}M^{ij}_{\Delta-2}M_{jb} - M_{ai}M^{ij}_{\Delta-2}M_{\partial j,b} + M_{ai}M^{ik}_{\Delta-2}M_{\partial k,l}M^{lj}_{\Delta-2}M_{jb} \\ &= M_{\partial a,b} - M_{10}G_{ab} - M_{\partial a,i}M^{ij}_{\Delta-2}M_{jb} - \sum_{\partial k\in\Delta} M_{ai}M^{ik}_{\Delta-2}M_{\partial k,b} + \sum_{\partial k\in\Delta} M_{ai}M^{ik}_{\Delta-2}M_{\partial k,l}M^{lj}_{\Delta-2}M_{jb} \\ &= M_{\partial a,b} - M_{10}G_{ab} - M_{\partial a,i}M^{ij}_{\Delta-2}M_{jb} - \sum_{\partial k\in\Delta} M_{ai}M^{ik}_{\Delta-2}G_{\partial k,b} \end{aligned} \tag{D.48}$$

and

$$\begin{aligned} \partial_\tau G_{ab}G^{bc} &= \left(M_{\partial a,b} - M_{10}G_{ab} - M_{\partial a,i}M^{ij}_{\Delta-2}M_{jb} - \sum_{\partial j\in\Delta} M_{ai}M^{ij}_{\Delta-2}G_{\partial j,b} \right) M^{bc}_{\Delta} \\ &= M_{\partial a,b}M^{bc}_{\Delta} - M_{10}\delta^c_a + M_{\partial a,i}M^{ij}_{\Delta-2}M_{jk}M^{kc}_{\Delta} - \sum_{\partial j\in\Delta} M_{ai}M^{ij}_{\Delta-2}\delta^c_{\partial j} \\ &= M_{\partial a,b}M^{bc}_{\Delta} - M_{10}\delta^c_a + M_{\partial a,i}M^{ic}_{\Delta} - \sum_{\partial j\in\Delta} M_{ai}M^{ij}_{\Delta-2}\delta^c_{\partial j} \ . \end{aligned} \tag{D.49}$$

and finally

$$\partial_{\bar{\tau}}(\partial_\tau G_{ab} G^{bc}) = \partial_{\bar{\tau}} \left(M_{\partial a,b} M_\Delta^{bc} - M_{10}\delta_a^c + M_{\partial a,i} M_\Delta^{ic} - \sum_{\partial j \in \Delta} M_{ai} M_{\Delta-2}^{ij} \delta_{\partial j}^c \right) \ . \tag{D.50}$$

Compute the different terms:

$$\begin{aligned}
&\partial_{\bar{\tau}} M_{\partial a,I} M_\Delta^{Ic} = M_{\partial a,\partial I} M_\Delta^{Ic} - M_{01} M_{\partial a,I} M_\Delta^{Ic} - M_{\partial a,I} M_\Delta^{IJ} (M_{J,\partial K} - M_{01} M_{JK}) M^{Kc} \\
&= M_{\partial a,\partial I} M_\Delta^{Ic} - M_{01} M_{\partial a,I} M_\Delta^{Ic} - M_{\partial a,I} M_\Delta^{IJ} M_{J,\partial K} M^{Kc} + M_{\partial a,I} M_\Delta^{IJ} M_{01} M_{JK} M^{Kc} \\
&= M_{\partial a,\partial b} M_\Delta^{bc} + M_{\partial a,\partial i} M_\Delta^{ic} - M_{\partial a,\partial i} M_\Delta^{ic} - M_{\partial a,I} M_\Delta^{IJ} M_{J\partial b} M_\Delta^{bc} = G_{\partial a,\partial b} G^{bc} \ .
\end{aligned} \tag{D.51}$$

Second term:

$$\partial_{\bar{\tau}}(-M_{10}\delta_a^c) = -(M_{11} - M_{10}M_{01})\delta_a^c = -G_2 \delta_a^c \ . \tag{D.52}$$

Last term:

$$\begin{aligned}
&- \sum_{\partial j \in \Delta} \delta_{\partial j}^c \partial_{\bar{\tau}} (M_{ai} M_{\Delta-2}^{ij}) = - \sum_{\partial j \in \Delta} \delta_{\partial j}^c (M_{a\partial i} M_{\Delta-2}^{ij} - M_{al} M_{\Delta-2}^{lk} M_{k,\partial i} M_{\Delta-2}^{ij}) \\
&= - \sum_{\partial i, \partial j \in \Delta} \delta_{\partial j}^c (M_{a\partial i} M_{\Delta-2}^{ij} - M_{al} M_{\Delta-2}^{lk} M_{k,\partial i} M_{\Delta-2}^{ij}) = - \sum_{\partial j \in \Delta} \delta_{\partial j}^c G_{a\partial i} G^{ij} \ .
\end{aligned} \tag{D.53}$$

Putting everything together we get exactly (D.47).

D.5 Scheme Independence of the Results

The sphere partition function is subject to Kähler ambiguity transformations

$$\ln Z[S^4] \to \ln Z[S^4] + f(\tau^i) + \bar{f}(\bar{\tau}^{\bar{i}}) \ . \tag{D.54}$$

That is, sphere partition functions that were computed in different regularization schemes may differ by holomorphic functions in the exactly marginal couplings [81]. More generally, the deformed partition function $Z[S^4](\tau^i, \bar{\tau}^{\bar{i}}, \tau^A, \bar{\tau}^{\bar{A}})$ is subject to holomorphic ambiguities, as discussed in section 3.2.4.

The expressions obtained for the extremal correlators in our prescription are, by construction, unambiguous. The effect of the holomorphic ambiguities on sphere correllators is in holomorphic contributions to the mixing of chiral primaries with lower dimensional chiral primaries (see equations (3.41-3.42)), and the Gram-Schmidt procedure subtracts these holomorphic contributions. The algorithm described in section 3.2.5 is therefore guaranteed to yield results that are scheme independent. Here we would like to demonstrate how this works.

Let us start with the example of gauge group SU(2). Using the recursive formula (3.66), the invariance of the extremal two-point functions follows from the invariance of the boundary condition $G_2 = 16\,\partial_\tau \partial_{\bar{\tau}} \ln Z[S^4]$ under Kähler transformations. Alternatively, consider the formula (3.3) and note that

$$\partial_\tau^l \partial_{\bar{\tau}}^j \left(e^{f(\tau)} Z[S^4] \right) = e^{f(\tau)} \partial_\tau^l \partial_{\bar{\tau}}^j Z[S^4] + \sum_{k=0}^{l-1} \binom{l}{k} \left(\partial_\tau^{l-k} e^{f(\tau)} \right) \partial_\tau^k \partial_{\bar{\tau}}^j Z[S^4] \,. \tag{D.55}$$

The second term in the right hand side of the equation above is a linear combination of the first l columns of the matrix defined by the first term, and therefore does not affect the determinant,

$$\det \left(\partial_\tau^l \partial_{\bar{\tau}}^j \left(e^{f(\tau)} Z[S^4] \right) \right) = \det \left(e^{f(\tau)} \partial_\tau^l \partial_{\bar{\tau}}^j Z[S^4] \right) . \tag{D.56}$$

It follows that equation (3.3) is invariant under holomorphic transformations (and similarly under antiholomorphic transformations.)

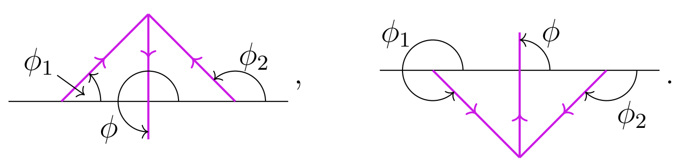
More generally, every extremal two-point function that we would like to compute is given in our prescription in terms of determinants of the Gram-Schmidt matrix of two-point functions on the sphere. The holomorphic mixing can always be canceled by subtracting from columns linear combinations of the previous columns, and therefore the holomorphic ambiguities do not affect the (appropriately normalized) determinants. Importantly, non-holomorphic contributions to $\ln Z[S^4]$, such as the one due to the anomaly discussed in [82], do not simply mix columns and rows with the previous ones, and they do affect the result of the Gram-Schmidt procedure.

Appendix E

Accompanying Computations for Topological Holography

E.1 Integrating the BF interaction vertex

In this appendix we evaluate the integrals in (4.80).



(E.1)

We split up each integral into two, based on whether the bulk point is above or below the line operator. We use angular coordinates defined as in the above diagrams. One subtlety is that, from the definition of the propagators in the Cartesian coordinate we can see that the integrand[1] is even under reflection with respect to the line. So, we just have to make sure that when we divide up the integral in the aforementioned way, even when written in angular coordinates, the integrand does not change sign under reflection. With this in

[1]including the measure

mind, the integrals we have to evaluate are:

$$
\begin{aligned}
\mathcal{V}^{\alpha\beta\gamma}_{\cdot||}(x_1,x_2) &= \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma} \int_0^{2\pi} \mathrm{d}\phi_1 \int_{\phi_1}^{\pi} \mathrm{d}\phi_2 \left(\int_{\pi}^{\phi_1+\pi} \mathrm{d}\phi + \int_{\pi}^{\phi_1-\pi} \mathrm{d}\phi \right) , \\
\mathcal{V}^{\alpha\beta\gamma}_{|\cdot|}(x_1,x_2) &= \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma} \int_0^{2\pi} \mathrm{d}\phi_1 \int_{\phi_1}^{\pi} \mathrm{d}\phi_2 \left(\int_{\phi_1+\pi}^{\phi_2+\pi} \mathrm{d}\phi + \int_{\phi_1-\pi}^{\phi_2-\pi} \mathrm{d}\phi \right) , \\
\mathcal{V}^{\alpha\beta\gamma}_{||\cdot}(x_1,x_2) &= \frac{\hbar^2}{(2\pi)^3} f^{\alpha\beta\gamma} \int_0^{2\pi} \mathrm{d}\phi_1 \int_{\phi_1}^{\pi} \mathrm{d}\phi_2 \left(\int_{\phi_2+\pi}^{2\pi} \mathrm{d}\phi + \int_{\phi_2-\pi}^{0} \mathrm{d}\phi \right) .
\end{aligned}
$$

All three terms are equal to $\frac{\hbar^2}{24} f^{\alpha\beta\gamma}$.

E.2 Quantum Mechanical Hilbert Spaces

E.2.1 Fermionic

The quantum mechanical action (4.23) is written in terms of fermions $\overline{\psi}$ and ψ that transform under $\mathrm{GL}_N \times \mathrm{GL}_K$ according to the representations $V := \overline{\mathbf{N}} \otimes \mathbf{K}$ and $\overline{V} := \mathbf{N} \otimes \overline{\mathbf{K}}$ respectively. The kinetic term in the action is first order in derivative, which establishes $\overline{\psi}$ and ψ as canonically conjugate variables, in other words, the phase space of the QM is:

$$
V \oplus \overline{V} = T^* V \,. \tag{E.2}
$$

The Hilbert space of this theory can now be written as the space of functions on V – since V is a *fermionic* vector space, functions on this space can be written as *anti-symmetric* polynomials in the dual vectors:

$$
\mathcal{H}^{\text{fer}} = \mathscr{O}(V) = \wedge^{\bullet}(\overline{V}) \,. \tag{E.3}
$$

Let us look at the anti-symmetric polynomials of degree n, which can be defined as the subspace of $\overline{V}^{\otimes n}$ where S_n acts by sign – S_n being the permutation group of n objects:

$$
\wedge^n \left(\overline{V}^{\otimes n} \right) = \mathrm{Hom}_{S_n} \left(\varepsilon, \overline{V}^{\otimes n} \right) \cong \mathrm{Hom}_{S_n} \left(\varepsilon, \mathbf{N}^{\otimes n} \otimes \overline{\mathbf{K}}^{\otimes n} \right) . \tag{E.4}
$$

Here ε is the one dimensional sign representation of the symmetric group S_n. Using *Schur-Weyl duality* we can decompose spaces such as $\mathbf{N}^{\otimes n}$ into irreducible representations of $S_n \times \mathrm{GL}_N$:

$$
\wedge^n \left(\overline{V}^{\otimes n} \right) = \bigoplus_{|Y|=|Y'|=n} \mathrm{Hom}_{S_n} \left(\varepsilon, \pi_Y \otimes \mathcal{H}^N_Y \otimes \pi_{Y'} \otimes \overline{\mathcal{H}^K_{Y'}} \right) , \tag{E.5}
$$

where Y and Y' are Young tableau, the sum is over tableau containing n boxes, π_Y is the irreducible representation of S_n parametrized by the tableaux Y, $\mathcal{H}^K_{Y'}$ is the irreducible representation of GL_K parametrized by the tableaux Y' and $\overline{\mathcal{H}^K_{Y'}}$ is its dual. Since we are computing S_n equivariant Hom, we can focus on the S_n representations:

$$\mathrm{Hom}_{S_n}(\varepsilon, \pi_Y \otimes \pi_{Y'}) \cong \mathrm{Hom}_{S_n}(\varepsilon \otimes \overline{\pi_Y}, \pi_{Y'}) = \mathrm{Hom}_{S_n}(\varepsilon \otimes \pi_Y, \pi_{Y'}) \,, \tag{E.6}$$

where we have used the fact that representations of S_n are self-dual. Now, tensoring with the sign representation exchanges the role of rows and columns in a Yaoung tableau parametrizing a representation of S_n [71], and by *Schur's lemma*, there is exactly one (up to scalar multiples) map of representations between two irreducible representations if they are isomorphic and no such map if they are not. These two facts tell us that:

$$\mathrm{Hom}_{S_n}(\varepsilon \otimes \pi_Y, \pi_{Y'}) = \delta_{Y^T,Y'}\mathbb{C} \,, \tag{E.7}$$

where Y^T denotes the transpose of the tableaux Y. This leaves just one sum in (E.5):

$$\wedge^n \left(\overline{V}^{\otimes n}\right) = \bigoplus_{|Y|=n} \mathcal{H}^N_{Y^T} \otimes \overline{\mathcal{H}^K_Y} \,. \tag{E.8}$$

The full fermionic Hilbert space (E.3) is then the following sum:

$$\mathcal{H}^{\mathrm{fer}} = \bigoplus_{Y} \mathcal{H}^N_{Y^T} \otimes \overline{\mathcal{H}^K_Y} \,. \tag{E.9}$$

Note that, this is a finite sum, since the tableau Y can have at most K rows and at most N columns – this is of course a consequence of exclusion principle for fermions.

E.2.2 Bosonic

Let us replace the fermions in the action (4.23) with bosons and change nothing else. Representations of the bosons are the same as their fermionic counterpart and therefore we still have the phase space $T^*(V)$ where $V = \overline{\mathbf{N}} \oplus \mathbf{K}$. The difference, compared to the fermionic case, is that the Hilbert space now consists of *symmetric* polynomials in $\overline{V}$ (c.f. (E.3)):

$$\mathcal{H}^{\mathrm{bos}} = \mathrm{Sym}^{\bullet}(\overline{V}) \,. \tag{E.10}$$

Then, instead of (E.4) we have:

$$\mathrm{Sym}^n \left(\overline{V}^{\otimes n}\right) = \mathrm{Hom}_{S_n}\left(\mathbb{C}, \mathbf{N}^{\otimes n} \otimes \overline{\mathbf{K}}^{\otimes n}\right) \,, \tag{E.11}$$

where $\mathbb{C}$ is the trivial representation of S_n. Following a similar computation as we did for the fermionic case we now end up with the following Hom between representations of S_n (c.f. (E.7)):

$$\mathrm{Hom}_{S_n}(\pi_Y, \pi_{Y'}) = \delta_{Y,Y'}\mathbb{C}\,, \tag{E.12}$$

which leads to the following description of the bosonic Hilbert space:

$$\mathcal{H}^{\mathrm{bos}} = \bigoplus_Y \mathcal{H}_Y^N \otimes \overline{\mathcal{H}_Y^K}\,. \tag{E.13}$$

Note that, as opposed to the fermionic case, we now have no restriction on the number of columns of Y (number of rows is restricted to be at most $\min(N, K)$) and therefore the Hilbert space is infinite dimensional, as expected given the lack of any exclusion principle for bosons.

E.3 Yangian from 1-loop Computations

At the end of §4.5.3, by computing 1-loop diagrams, we concluded that quantum corrections deform the coalgebra structure of the classical Hopf algebra $U(\mathfrak{gl}_K[z])$. Since $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is an algebra to begin with, we conclude that at one loop, we have a deformation of the classical algebra as a Hopf algebra. We are using the term "deformation" (alternatively, "quantization") in the sense of Definition 6.1.1 of [30], which essentially means that:

- $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bd}})$ becomes the classical algebra $U(\mathfrak{gl}_K[z])$ in the classical limit $\hbar \to 0$.
- $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is isomorphic to $U(\mathfrak{gl}_K[z])[\![\hbar]\!]$ as a $\mathbb{C}[\![\hbar]\!]$-module.
- $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is a *topological* Hopf algebra (with respect to $\hbar$-adic topology).

The reason that we adhere to these conditions is that, there is a well known uniqueness theorem (Theorem 12.1.1 of [30]) which says that the Yangian is the unique deformation of $U(\mathfrak{gl}_K[z])$ in the above sense. Therefore, if we can show that our algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ satisfies all these conditions and it is a nontrivial deformation of $U(\mathfrak{gl}_K)$ then we can conclude that it is the Yangian. From 1-loop computations we already know that it is a non-trivial deformation. That the first condition in the list above is satisfied is the content of Lemma 4.5.2. The second condition is satisfied because $\hbar$ acts on the generators of our algebra by simply multiplying the external propagators by $\hbar$ in the relevant Witten diagrams, this action does not distinguish between classical diagrams and higher loop diagrams. Satisfying the last condition is less trivial. While it seems known to people working in the field, we were unable to find a reference to cite, therefore, for the sake of completion, we provide a proof in this appendix, that the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ is indeed an ($\hbar$-adic)*topological* Hopf algebra.

We shall prove this by reconstructing the algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$ from its representations. As mentioned in §4.3.4, representations of this algebra are carried by Wilson lines, which form an abelian monoidal category. A morphism between two representations V and U in this category is constructed by computing the expectation value of two Wilson lines in representations U and $V^\vee$ and providing a state at one end of each of the lines. For example, if ϱ and ϱ' are two homomorphisms from $\mathfrak{gl}_K$ to U and $V^\vee$ respectively, then for two lines L and L' in the topological plane of the CS theory and any $\psi \otimes \chi^\vee \in U \otimes V^\vee$, the expectation value $\langle W_\varrho(L) W_{\varrho'}(L') \rangle (\psi \otimes \chi^\vee)$ is a morphism $V \to U$.

Classically, these same Wilson lines carry representations of the classical algebra $U(\mathfrak{gl}_K[z])$. When viewed as representations of the deformed (alternatively, quantized) algebra $\mathcal{A}^{\mathrm{Sc}}(\mathcal{T}_{\mathrm{bk}})$, we shall call the category of Wilson lines as the *quantized category* and viewed as representations of $U(\mathfrak{gl}_K[z])$ we shall refer to the category as the *classical category*. Given any two Wilson lines U and V, any non-trivial morphism between them in the quantized category is a quantization of a non-trivial morphism in the classical category.[2] In fact, there is a one-to-one correspondence between morphisms between two lines in the classical category and the morphisms between the same lines in the quantized category.

For the sake of proof, let us abstract the information we have. We start with a $\mathbb{C}$-linear rigid abelian monoidal category $\mathcal{C} = \mathrm{Rep}_{\mathbb{C}}(H)$ which is the representation category of a Hopf algebra H. We then find a $\mathbb{C}[\![\hbar]\!]$-linear abelian monoidal category $\mathcal{C}_\hbar$, whose objects are representations of some, yet unknown, Hopf algebra $H_\hbar$, with the following properties:

- $\mathrm{ob}(\mathcal{C}_\hbar) = \mathrm{ob}(\mathcal{C})\,,$
- $\mathrm{Hom}_{\mathcal{C}_\hbar}(U, V) \cong \mathrm{Hom}_{\mathcal{C}}(U, V)[\![\hbar]\!]$ as $\mathbb{C}[\![\hbar]\!]$-modules.

Given this information we shall now prove that $H_\hbar$ is unique and that it is topological with respect to $\hbar$-adic topology.

[2] Recall that a morphism between two Wilson lines is the expectation value of the lines when provided with a state at one end. A classical morphism is computed with classical diagrams and its quantization amounts to adding loop diagrams. A zero morphism is constructed by providing zero states, this is independent of quantization, i.e., a quantized morphism is zero, if the provided states are zero, but then so is the original classical morphism.

E.3.1 Tannaka formalism

The aim of this formalism is to realize certain abelian rigid monoidal categories as the representation (or corepresentation) categories of Hopf algebras (possibly with extra structures). To avoid running into some subtlety in the beginning (we shall explain the subtlety later in this section), we first consider the reconstruction from the category of corepresentations.

Reconstruction from corepresentation. Let k be a field, $\mathcal{C}$ an abelian (resp. abelian monoidal and $\mathrm{End}(1) = k$) category such that morphisms are k-bilinear, and let R be a commutative algebra over k – if there is an exact faithful (resp. monoidal) functor ω from $\mathcal{C}$ to $\mathrm{Mod}_f(R)$[3] such that the image of ω is inside the full subcategory $\mathrm{Proj}_f(R)$[4], then we shall say that $\mathcal{C}$ has a *fiber functor* ω to $\mathrm{Mod}_f(R)$.

Theorem E.3.1 (Tannaka Reconstruction for Coalgebra and Bialgebra)**.** *With the notation above, if moreover R is a local ring or a PID*[5]*, then there exists a unique flat R-coalgebra (resp. R-bialgebra) A, up to unique isomorphism, such that A represents the endomorphism of ω in the sense that $\forall M \in \mathit{IndProj}_f(R)$* [6]

$$\mathit{Hom}_R(A, M) \cong \mathit{Nat}(\omega, \omega \otimes M)\,.$$

Moreover, there is a functor $\phi : \mathcal{C} \to \mathit{Corep}_R(A)$ which makes the following diagram commutative

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[dr, "\omega"'] & \mathit{Corep}_R(A) \arrow[d, "forget"] \\ & \mathit{Mod}_f(R) \end{tikzcd}$$

and ϕ is an equivalence if $R = k$.

Our strategy in proving this theorem basically follows [54]. First of all, we need the following

Lemma E.3.2. *$\mathcal{C}$ is both Noetherian and Artinian.*

[3]finitely generated modules of R

[4]finitely generated projective modules of R

[5]PID=Principal Ideal Domain

[6]$\mathrm{IndProj}_f(R)$ means category of inductive limit of finite projective R-modules, which is equivalent to category of flat R-modules.

Proof. Take $X \in \mathrm{ob}(\mathcal{C})$, and an ascending chain X_i of subobjects of X, apply the functor ω to this chain, so that $\omega(X_i)$ is an ascending chain of finitely generated projective submodules of finitely generated projective module $\omega(X)$, thus there is an index j such that $\mathrm{rank}(\omega(X_j)) = \mathrm{rank}(\omega(X))$. Now the quotient of $\omega(X)$ by $\omega(X_j)$ is $\omega(X/X_j)$, which is again finitely generated projective, so it has zero rank, hence trivial. Faithfulness of ω implies that X/X_j is zero, i.e. $X = X_j$, so $\mathcal{C}$ is Noetherian. It follows similarly that $\mathcal{C}$ is Artinian as well. □

Next, we define a functor

$$\otimes : \mathrm{Proj}_f(R) \times \mathcal{C} \to \mathcal{C}$$

by sending (R^n, X) to X^n, recall that every finitely generated projective module over a local ring or a PID is free, thus isomorphic to R^n for some n. Define $\underline{\mathrm{Hom}}(M, X)$ to be $M^\vee \otimes X$. For $V \subset M$ and $Y \subset X$, we define the transporter of V to Y to be

$$(Y : V) := \mathrm{Ker}(\underline{\mathrm{Hom}}(M, X) \to \underline{\mathrm{Hom}}(V, X/Y))$$

We now have the following:

Lemma E.3.3. *Take the full abelian subcategory $\mathcal{C}_X$ of $\mathcal{C}$ generated by subquotients of X^n, consider the largest subobject P_X of $\underline{Hom}(\omega(X), X)$ whose image in $\underline{Hom}(\omega(X)^n, X^n)$ under diagonal embedding is contained in $(Y : \omega(Y))$ for all subobjects Y of X^n and all n. Then the Theorem (E.3.1) is true for $\mathcal{C}_X$ with coalgebra defined by $A_X := \omega(P_X)^\vee$.*

Proof. P_X exists because $\mathcal{C}$ is Artinian. Notice that ω takes $\underline{\mathrm{Hom}}(M, X)$ to $\mathrm{Hom}_R(M, X)$ and $(Y : V)$ to $(\omega(Y) : V)$, so it takes P_X, which is defined by

$$\bigcap (\underline{\mathrm{Hom}}(\omega(X), X) \cap (Y : \omega(Y)))$$

to

$$\bigcap (\mathrm{End}_R(\omega(X)) \cap (\omega(Y) : \omega(Y))) \ .$$

Hence $\omega(P_X)$ is the largest subring of $\mathrm{End}_R(\omega(X))$ stabilizing $\omega(Y)$ for all $Y \subset X^n$ and all n. It's a finitely generated projective R module by construction, and so is A_X. Note that only finitely many intersection occurs because $\underline{\mathrm{Hom}}(\omega(X), X)$ is Artinian.

Next, take any flat R module M,[7] since $\mathcal{C}_X$ is generated by subquotients of X, an element $\lambda \in \mathrm{Nat}(\omega, \omega \otimes M)$ is completely determined by it is value on X, so $\lambda \in \mathrm{End}_R(\omega(X)) \otimes M$. Since $- \otimes_R M$ is an exact functor, we have:

$$\begin{aligned}
&\bigcap \left(\mathrm{Hom}_R(\omega(X), \omega(X) \otimes_R M) \cap (\omega(Y) \otimes_R M : \omega(Y))\right) \\
&= \left(\bigcap \left(\mathrm{End}_R(\omega(X)) \cap (\omega(Y) : \omega(Y))\right)\right) \otimes_R M\,.
\end{aligned}$$

This follows because there are only finitely many intersections and finite limit commutes with tensoring with flat module. Therefore,

$$\lambda \in \omega(P_X) \underset{R}{\otimes} M\,.$$

Conversely, every element in $\omega(P_X) \otimes_R M$ gives rise to a natural transform in the way described above. Hence we establish the isomorphism

$$\mathrm{Nat}(\omega, \omega \otimes M) \cong \omega(P_X) \otimes_R M \cong \mathrm{Hom}_R(A_X, M)\,.$$

A_X is unique up to unique isomorphism (as a flat R module) because it represents the functor $M \mapsto \mathrm{Nat}(\omega, \omega \otimes M)$.

Next, we shall define a co-action of A_X on ω, a counit and a coproduct on A_X which makes A_X an R-coalgebra and ω a corepresentation:

$$\rho \in \mathrm{Nat}(\omega, \omega \otimes A_X) \cong \mathrm{End}_R(A_X)$$

corresponds to the identity map of A_X, and

$$\epsilon \in \mathrm{Hom}_R(A_X, R) \cong \mathrm{Nat}(\omega, \omega)$$

corresponds to Id_ω. The co-action ρ tensored with Id_{A_X} gives a natural transform between $\omega \otimes A_X$ and $\omega \otimes A_X \otimes A_X$, whose composition with ρ gives the following commutative diagram:

$$\begin{tikzcd}
\omega \arrow[r, "\rho"] \arrow[dr, "\psi"'] & \omega \otimes A_X \arrow[d, "\rho \otimes \mathrm{Id}_{A_X}"] \\
& \omega \otimes A_X \otimes A_X
\end{tikzcd}\,.$$

[7]Recall that a R module is flat if and only if it is a filtered colimit of finitely generated projective modules.

Take Δ to be the image of ψ in $\mathrm{Hom}_R(A_X, A_X \otimes_R A_X)$. It follows from definition that A_X is counital and $\rho : \omega \to \omega \otimes A_X$ is a corepresentation. It remains to check that Δ is coassociative.

Observe that the essential image of $\omega \otimes A_X$ is a subcategory of the essential image of ω, hence every functor that shows up here can be restricted to $\omega \otimes A_X$, in particular, ρ, whose restriction to $\omega \otimes A_X$ is obviously $\rho \otimes \mathrm{Id}_{A_X}$. It follows from the definition that

$$(\rho \otimes \mathrm{Id}_{A_X}) \circ \rho = (\mathrm{Id}_\omega \otimes \Delta) \circ \rho \in \mathrm{Nat}(\omega, \omega \otimes A_X \otimes A_X)\,.$$

Restrict this equation to $\omega \otimes A_X$ and we get

$$(\rho \otimes \mathrm{Id}_{A_X} \otimes \mathrm{Id}_{A_X}) \circ (\rho \otimes \mathrm{Id}_{A_X}) = (\mathrm{Id}_\omega \otimes \mathrm{Id}_{A_X} \otimes \Delta) \circ (\rho \otimes \mathrm{Id}_{A_X})\,.$$

Composing with ρ, the LHS corresponds to $(\Delta \otimes \mathrm{Id}_{A_X}) \circ \Delta$ and the RHS corresponds to $(\mathrm{Id}_{A_X} \otimes \Delta) \circ \Delta$ whose equality is exactly the coassociativity of A_X.

It follows that $\forall Z \in \mathcal{C}_X$,

$$\rho(Z) : \omega(Z) \to \omega(Z) \otimes_R A_X$$

gives $\omega(Z)$ a A_X corepresentation structure and this is functorial in Z, thus ω factors through a $\phi : \mathcal{C}_X \to \mathrm{Corep}_R(A_X)$.

Back to the uniqueness of A_X. It has been shown that it is unique up to unique isomorphism as a flat R module. Additionally, if $\phi : A_X \to A'_X$ is an isomorphism such that it induces identity transformation on the functor $M \mapsto \mathrm{Nat}(\omega, \omega \otimes M)$ then, ϕ automatically maps the triple (Δ, ϵ, ρ) to $(\Delta', \epsilon', \rho')$, so ϕ is a coalgebra isomorphism.

Finally, it remains to show that when $R = k$, ϕ is essentially surjective[8] and full:

- Essentially Surjective: If $M \in \mathrm{Corep}_k(A_X)$, then define

$$\widetilde{M} := \mathrm{Coker}(M \otimes \omega(P_X) \otimes P_X \rightrightarrows M \otimes P_X)\,,$$

  where two arrows are $\omega(P_X)$ representation structure of M and P_X respectively, then

$$\omega(\widetilde{M}) = M \underset{\omega(P_X)}{\otimes} \omega(P_X) = M\,.$$

[8] In fact, ϕ is essentially surjective even without the assumption that $R = k$.

- Full: If $f : M \to N$ is a A_X-corepresentation morphism, then by the k-linearlity of $\mathcal{C}_X$, f lifts to morphisms

$$f \otimes \mathrm{Id}_{P_X} : M \otimes P_X \to N \otimes P_X\,,$$

and

$$f \otimes \mathrm{Id}_{\omega(P_X)} \otimes \mathrm{Id}_{P_X} : M \otimes \omega(P_X) \otimes P_X \to N \otimes \omega(P_X) \otimes P_X\,.$$

Thus, passing to cokernel gives rise to $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ which is mapped to f by ω.

□

Next we move on to recover the category $\mathcal{C}$ by its subcategories $\mathcal{C}_X$. Define an index category I such that its objects are isomorphism classes of objects in $\mathcal{C}$, denoted by X_i for each index i, and a unique arrow from i to j if X_i is a subobject of X_j. I is directed because for any two objects Z and W, they are subobjects of $Z \oplus W$. Observe that if X is a subobject of Y, then $\mathcal{C}_X$ is a full subcategory of $\mathcal{C}_Y$, so a functorial restriction

$$\mathrm{Hom}_R(A_Y, M) \cong \mathrm{Nat}(\omega_Y, \omega_Y \otimes M) \to \mathrm{Nat}(\omega_X, \omega_X \otimes M) \cong \mathrm{Hom}_R(A_Y, M)\,,$$

gives rise to a coalgebra homomorphism $A_X \to A_Y$. Futhermore, this homomorphism is injective because $\omega(P_Y) \to \omega(P_X)$ is surjective, otherwise $\mathrm{Coker}(\omega(P_Y) \to \omega(P_X))$ will be mapped to the zero object in $\mathrm{Corep}_R(A_Y)$, which contradicts with ω being faithful.

Lemma E.3.4. *Define the coalgbra*

$$A := \varinjlim_{i \in I} A_{X_i}\,,$$

then it is the desired coalgebra in Theorem E.3.1.

Proof. A is flat because it is an inductive limit of flat R modules. Moreover

$$\mathrm{Hom}_R(A, M) = \varprojlim_{i \in I} \mathrm{Hom}_R(A_{X_i}, M) \cong \varprojlim_{i \in I} \mathrm{Nat}(\omega_{X_i}, \omega_{X_i} \otimes M) = \mathrm{Nat}(\omega, \omega \otimes M)\,,$$

which gives the desired functorial property and this implies that A is unique up to unique isomorphism. Finally, when $R = k$, the functor ϕ is defined and it is fully faithful because it is fully faithful on each subcategory $\mathcal{C}_{X_i}$. It's also essentially surjective because every corepresentation V of A comes from a corepresentation of a finite dimensional sub-coalgebra of A,[9] and A is a filtered union of sub-coalgebras A_{X_i}, so V comes from a corepresentation of some A_{X_i}. □

[9]Take a basis $\{e_i\}$ for V, the co-action ρ takes e_i to $\sum_j e_j \otimes a_{ji}$, then it is easy to see that $\mathrm{span}\{a_{ji}\}$ is a finite dimensional sub-coalgebra of A.

Proof of Theorem E.3.1. It remains to prove the theorem when $\mathcal{C}$ is monoidal. This amounts to including $m : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ and $e : \mathbb{1} \to \mathcal{C}$ with associativity and unitarity constrains, where $\mathbb{1}$ is the trivial tensor category with objects $\{0, 1\}$ and only nontrivial morphisms are $\mathrm{End}(1) = k$. Using the isomorphism:

$$\mathrm{Hom}_R(A \otimes_R A, A \otimes_R A) \cong \mathrm{Nat}(\omega \boxtimes \omega, \omega \boxtimes \omega \otimes A \otimes_R A)\,,$$

we get a homomorphism

$$\tau : \mathrm{Hom}_R(A \otimes_R A, M) \to \mathrm{Nat}(\omega \boxtimes \omega, \omega \boxtimes \omega \otimes M)\,.$$

It is an isomorphism because for each pair of subcategories $(\mathcal{C}_X, \mathcal{C}_Y)$

$$\begin{aligned}\mathrm{Hom}_R(A_X \otimes_R A_Y, M) &\cong \mathrm{Hom}_R(A_X, R) \otimes_R \mathrm{Hom}_R(A_Y, M)\\ &\cong \mathrm{Nat}(\omega_X, \omega_X) \otimes_R \mathrm{Nat}(\omega_Y, \omega_Y \otimes M)\\ &\cong \mathrm{Nat}(\omega_X \boxtimes \omega_Y, \omega_X \boxtimes \omega_Y \otimes M)\end{aligned}$$

and it is compatible with the homomorphism given above, so after taking limit, τ is an isomorphism. We also have a homomorphism:

$$\mathrm{Nat}(\omega, \omega \otimes M) \to \mathrm{Nat}(\omega \boxtimes \omega, \omega \boxtimes \omega \otimes M)\,,$$

by taking any $\alpha \in \mathrm{Nat}(\omega, \omega \otimes M)$, and composing with the isomorphism $\omega \boxtimes \omega(X \boxtimes Y) \cong \omega(X \otimes Y)$. This homomorphism in turn becomes a homomorphism

$$\mu : A \otimes_R A \to A\,.$$

And the obvious isomorphism

$$\mathrm{Hom}_R(R, M) = M \to \mathrm{Nat}(\omega_{\mathbb{1}}, \omega_{\mathbb{1}} \otimes M)\,,$$

together with the unit functor $e : \mathbb{1} \to \mathcal{C}$ give a homomorphism

$$\iota : R \to A\,.$$

All of the homomorphisms are functorial with respect to M so μ and ι are homomorphisms between coalgebras. Now the associativity and unitarity of monoidal category $\mathcal{C}$ translates into associativity and unitarity of μ and ι, which are exactly conditions for A to be a bialgebra. This concludes the proof of Theorem E.3.1. $\square$

Remark E.3.5. In the statement of Theorem E.3.1, it is assumed that R is a local ring or a PID, for the following technical reason: we want to introduce the functor

$$\otimes : \mathrm{Proj}_f(R) \times \mathcal{C} \to \mathcal{C}$$

which is defined by sending (R^n, X) to X^n. This is feasible only if every finite projective module is free, which is not always true for an arbitary ring. Nevertheless, this is true when R is local or a PID. It is tempting to eliminate this assumption when $\mathcal{C}$ is rigid, since we only use the $\underline{\mathrm{Hom}}(\omega(X), X)$ to define the crucial object P_X, and there is no need to define a $\underline{\mathrm{Hom}}$ when the category is rigid. In fact, there is no loss of information if we define P_X by

$$\bigcap \left(\underline{\mathrm{Hom}}(X, X) \cap (Y : Y)\right) ,$$

then the fiber functor ω takes P_X to

$$\bigcap \left(\mathrm{End}_R(\omega(X)) \cap (\omega(Y) : \omega(Y))\right) ,$$

since ω is monoidal by definition and a monoidal functor between rigid monoidal categories preserves duality and thus preserves inner Hom. $\triangle$

Following the above remark, we drop the assumption on ring R and state the following version of Tannaka reconstruction for Hopf algebras:

Theorem E.3.6 (Tannaka Reconstruction for Hopf Algebra)**.** *Let R be a commutative k-algebra, $\mathcal{C}$ a k-linear abelian rigid monoidal category (resp. abelian rigid braided monoidal) with a fiber functor ω to $Mod_f(R)$, then there exists a unique flat R-Hopf algebra A (resp. R-coquasitriangular Hopf algebra), up to unique isomorphism, such that A represents the endomorphism of ω in the sense that $\forall M \in IndProj_f(R)$*

$$Hom_R(A, M) \cong Nat(\omega, \omega \otimes M) .$$

Moreover, there is a functor $\phi : \mathcal{C} \to Corep_R(A)$ which makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[dr, "\omega"'] & Corep_R(A) \arrow[d, "forget"] \\ & Mod_f(R) \end{tikzcd}$$

and ϕ is an equivalence if $R = k$.

Sketch of proof. The idea of proof basically follows [108]. Accoring to Remark E.3.5 and Theorem E.3.1, there exists a bialgebra A which satisfies all conditions in the theorem, so it remains to prove that there are compatible structures on A when $\mathcal{C}$ has extra structures.

(a) **$\mathcal{C}$ is rigid.** This means that there is an equivalence between k-linear abelian monoidal categories

$$\sigma : \mathcal{C} \to \mathcal{C}^{op} ,$$

by taking the right dual of each object, so it turns into an isomophism between R modules

$$\sigma : \mathrm{Nat}(\omega, \omega \otimes M) \to \mathrm{Nat}(\omega^{op}, \omega^{op} \otimes M) .$$

According to the functoriality of the construction of the bialgebra A, there is a bialgebra isomorphism:

$$\mathcal{S} : A \to A^{op} ,$$

put it in another way, a bialgebra anti-automorphism of A. To prove that it satisfies the required compatibility:

$$\mu \circ (\mathcal{S} \otimes \mathrm{Id}) \circ \Delta = \iota \circ \epsilon = \mu \circ (\mathrm{Id} \otimes \mathcal{S}) \circ \Delta ,$$

we observe that $\iota \circ \epsilon$ gives the natural transformation

$$\mathrm{Id} \otimes \rho_{\omega(1)} : \omega(X) = \omega(X) \otimes \omega(1) \mapsto \omega(X) \otimes \rho(\omega(1)) ,$$

but 1 is the trivial corepresentation of A, so $\rho(\omega(1))$ is canonically identified with $\omega(1)$, so $\iota \circ \epsilon$ is just the identity morphism on $\omega(X)$. On the other hand, $\mu \circ (\mathcal{S} \otimes \mathrm{Id}) \circ \Delta$ corresponds to the homomorphism

$$\omega(X) \to \omega(X) \otimes \omega(X)^\vee \otimes \omega(X) \to \omega(X) \otimes \omega(X^\vee \otimes X) \to \omega(X) \otimes \omega(1) = \omega(X)$$

which is identity by the rigidity of $\mathcal{C}$, hence $\mu \circ (\mathcal{S} \otimes \mathrm{Id}) \circ \Delta = \iota \circ \epsilon$. The other equation is similiar.

(b) $\mathcal{C}$ **is rigid braided.** This means that there is a natural transformation:

$$r : \omega \boxtimes \omega \to \omega \boxtimes \omega \, ,$$

which gives the braiding. This corresponds to a homomorphism of R-modules

$$\mathcal{R} : A \otimes A \to R \, ,$$

let's define it to be the universal R-matrix. The fact that r is a natural transformation is equivalent to the diagram below being commutative

$$\begin{tikzcd}
\omega(U) \otimes \omega(V) \arrow[r, "\rho\otimes\rho"] \arrow[d, "r"] & \omega(U) \otimes \omega(V) \otimes A \otimes A \arrow[r, "\mathrm{Id}\otimes\mathrm{Id}\otimes\mu"] & \omega(U) \otimes \omega(V) \otimes A \arrow[d, "r\otimes\mathrm{Id}"] \\
\omega(V) \otimes \omega(U) \arrow[r, "\rho\otimes\rho"] & \omega(V) \otimes \omega(U) \otimes A \otimes A \arrow[r, "\mathrm{Id}\otimes\mathrm{Id}\otimes\mu"] & \omega(V) \otimes \omega(U) \otimes A
\end{tikzcd}$$

which in turn translates to the following equation of $\mathcal{R}$:

$$\mathcal{R}_{12} \circ \mu_{24} \circ (\Delta \otimes \Delta) = \mathcal{R}_{23} \circ \mu_{13} \circ \tau_{13} \circ (\Delta \otimes \Delta) \, ,$$

where $\tau : A \otimes A \to A \otimes A$ sends $x \otimes y$ to $y \otimes x$. The compactibility of r with the identity

$$\begin{tikzcd}
\omega(X) \arrow[r] \arrow[d, "\mathrm{Id}"] & \omega(X) \otimes \omega(1) \arrow[d, "r"] \\
\omega(X) & \omega(1) \otimes \omega(X) \arrow[l]
\end{tikzcd} \, ,$$

translates to $\mathcal{R} \circ (\mathrm{Id}_A \otimes 1) = \epsilon$. And symmetrically $\mathcal{R} \circ (1 \otimes \mathrm{Id}_A) = \epsilon$.

Finally, the hexagon axiom of braiding:

$$\begin{tikzcd}
 & (\omega(X) \otimes \omega(Y)) \otimes \omega(Z) \arrow[dl, "r\otimes 1"] \arrow[dr] & \\
(\omega(Y) \otimes \omega(X)) \otimes \omega(Z) \arrow[d] & & \omega(X) \otimes (\omega(Y) \otimes \omega(Z)) \arrow[d, "r"] \\
\omega(Y) \otimes (\omega(X) \otimes \omega(Z)) \arrow[dr, "1\otimes r"] & & (\omega(Y) \otimes \omega(Z)) \otimes \omega(X) \arrow[dl] \\
 & \omega(Y) \otimes (\omega(Z) \otimes \omega(X)) &
\end{tikzcd} \, ,$$

translates to the commutativity of the diagram

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{Id}\otimes\mathrm{Id}\otimes\Delta"] \arrow[d, "\mu\otimes\mathrm{Id}"] & A \otimes A \otimes A \otimes A \arrow[d, "\mathcal{R}_{13}\cdot\mathcal{R}_{24}"] \\
A \otimes A \arrow[r, "\mathcal{R}"] & R
\end{tikzcd},$$

and the same hexagon but with r^{-1} instead of r gives another one:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\Delta\otimes\mathrm{Id}\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\mu"] & A \otimes A \otimes A \otimes A \arrow[d, "\mathcal{R}_{14}\cdot\mathcal{R}_{23}"] \\
A \otimes A \arrow[r, "\mathcal{R}"] & R
\end{tikzcd}.$$

So we end up confirming all the properties that universal R-matrix should satisfy, and we conclude that A is indeed a coquasitriangular Hopf algebra.

$\square$

Reconstruction from representation It is tempting to dualize everything above to formalize the Tannaka reconstruction for the category of representations. In other words, we can take the dual of A instead of A itself, and a corepresentation becomes the representaion, and when the category has extra structures, those structures will be dualized, for example, when $\mathcal{C}$ is a k-linear abelian rigid braided monoidal category, it should come from the representation category of a flat R-quasitriangular Hopf algebra, since the dual of those diagrams involved in the proof of Theorem E.3.6 are exactly properties of universal R-matrix of a quasitriangular Hopf algebra.

This is naive because the statement:

$$\mathrm{Hom}_R(U, V \otimes A) \cong \mathrm{Hom}_R(U \otimes A^*, V)\,,$$

is not true in general, since A can be infinite dimensional, thus the naive dualizing procedure is not feasible. To resolve this subtlety, we observe that A is constructed from a filtered colimit of finite projective R-modules, each is an R-coalgebra, and any finitely generated corepresentation of A comes from a corepresentation of a finite coalgebra, so it is natural to define the action of A^* on those modules by factoring through some finite quotient A_X^*

for some $X \in \mathrm{ob}(\mathcal{C})$. Similiarly, the multiplication structure on A^* can be defined by first projecting down to some finite quotient and taking multiplication

$$A^* \otimes A^* = \varprojlim_{i \in I} A_{X_i} \otimes \varprojlim_{i \in I} A_{X_i} \to A_{X_i} \otimes A_{X_i} \to A_{X_i}$$

which is compatible with transition map $A_{X_j} \to A_{X_i}$ then taking the inverse limit gives the multiplication of A^*. For antipode $\mathcal{S}$, its dual is a map $A^* \to A^*$.

On the other hand, the comultiplication on A^*, is still subtle. If we dualize the multiplication of A, cut-off at some finite submodule

$$A_{X_i} \otimes A_{X_j} \to A\,,$$

we only get an inverse system of morphisms from A^* to $A^*_{X_i} \otimes A^*_{X_j}$ and the latter's inverse limit is $A^* \widehat{\otimes} A^*$, instead of $A^* \otimes A^*$. So we actually get a *topological Hopf algebra* with topological basis

$$N_i := \ker(A^* \to A^*_{X_i})\,,$$

so that the comultiplication is continuous. Similiarly the counit, multiplication, and anipode are continuous as well. Finally when $\mathcal{C}$ is braided, there exists an invertible element $\mathcal{R} \in A^* \widehat{\otimes} A^*$, and the dual of the structure homomorphism in A is exactly the condition that $\mathcal{R}$ is the universal R-matrix of a topological quasitriangular Hopf algebra.

So we can restate Theorem E.3.6 in terms of representations of topological Hopf algebras:

Theorem E.3.7. *Let R be a commutative k-algebra, $\mathcal{C}$ a k-linear abelian rigid monoidal category (resp. abelian rigid braided monoidal) with a fiber functor ω to $Mod_f(R)$, then there exists a unique topological R-Hopf algebra H (resp. R-quasitriangular Hopf algebra) which is an inverse limit of finite projective R-modules endowed with discrete topology, up to unique isomorphism, such that H represents the endomorphism of ω in the sense that*

$$H \cong Nat(\omega, \omega)\,.$$

Moreover, there is a functor $\phi : \mathcal{C} \to Rep_R(H)$ which sends an object in $\mathcal{C}$ to a continuous representation of H and makes the following diagram commutative:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\phi"] \arrow[dr, "\omega"'] & Rep_R(H) \arrow[d, "forget"] \\
& Mod_f(R)
\end{tikzcd},$$

and ϕ is an equivalence if $R = k$.

Application to Quantization We now consider the case that we have a category $\mathcal{C}_\hbar$, which is a *quantization* of the category of representations of some Hopf algebra H over $\mathbb{C}$. The quantization, namely $\mathcal{C}_\hbar$, of $\mathrm{Rep}_\mathbb{C}(H)$ is a $\mathbb{C}$-linear abelian monoidal category which has the same set of generators as $\mathrm{Rep}_\mathbb{C}(H)$, together with a fiber functor $\omega_\hbar : \mathcal{C}_\hbar \to \mathrm{Mod}_f(\mathbb{C}[\![\hbar]\!])$ which acts on generators of $\mathrm{Rep}_\mathbb{C}(H)$ by tensoring with $\mathbb{C}[\![\hbar]\!]$, and

$$\mathrm{Hom}_{\mathcal{C}_\hbar}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}_\hbar}(X, Y)/\hbar = \mathrm{Hom}_{\mathrm{Rep}_\mathbb{C}(H)}(X, Y)$$

for any pair of generators X and Y. For example, the classical algebra of local observables in 4D Chern-Simons theory is $U(g[z])$, the universal enveloping algebra of Lie algebra $g[z]$, which has the category of representations generated by classical Wilson lines. Quantized Wilson lines naturally generated a $\mathbb{C}$-linear abelian monoidal category.

Applying Theorem E.3.7, $(\mathcal{C}_\hbar, \omega_\hbar)$ gives us a (topological) $\mathbb{C}[\![\hbar]\!]$-Hopf algebra $H_\hbar$. Since $\mathcal{C}_\hbar$ and $\mathcal{C}$ shares the same set of generators, and the construction of those Hopf algebras as $\mathbb{C}[\![\hbar]\!]$-modules only involves generators of corresponding categories, so $H_\hbar$ is isomorphic to the completion of $H \otimes \mathbb{C}[\![\hbar]\!]$ in the $\hbar$-adic topology:

$$\begin{aligned}
H_\hbar := \varprojlim_{i \in I} H_{X_i} \otimes \mathbb{C}[\![\hbar]\!] &\cong \varprojlim_{i \in I} \varprojlim_{n} H_{X_i} \otimes \mathbb{C}[\hbar]/(\hbar^n) \\
&\cong \varprojlim_{n} \varprojlim_{i \in I} H_{X_i} \otimes \mathbb{C}[\hbar]/(\hbar^n) \\
&\cong \varprojlim_{n} H \otimes \mathbb{C}[\hbar]/(\hbar^n)\,.
\end{aligned}$$

For the same reason, tensor product of two copies of $H_\hbar$ and completed in the inverse limit topology is isomorphic to the completion of $H_\hbar \otimes_{\mathbb{C}[\![\hbar]\!]} H_\hbar$ in the $\hbar$-adic topology:

$$H_\hbar \widehat{\otimes} H_\hbar \cong \varprojlim_{n} H_\hbar \otimes_{\mathbb{C}[\![\hbar]\!]} H_\hbar/(\hbar^n)$$

From the construction of those Hopf algebras and the condition that a morphism in $\mathcal{C}_\hbar$ modulo $\hbar$ is a morphism in $\mathrm{Rep}_\mathbb{C}(H)$, it is easy to see that modulo $\hbar$ respects all structure homomorphisms, thus $H_\hbar$ modulo $\hbar$ and H are isomorphic as Hopf algebras. Finally, structure homomorphisms of $H_\hbar$ are continuous in the $\hbar$-adic topology because they are $\hbar$-linear. Thus we conlude that:

Theorem E.3.8. *$H_\hbar$ is a quantization of H in the sense of Definition 6.1.1 of [30], i.e. it is a topological Hopf algebra over $\mathbb{C}[\![\hbar]\!]$ with $\hbar$-adic topology, such that*

(i) $H_\hbar$ is isomorphic to $H[\![\hbar]\!]$ as a $\mathbb{C}[\![\hbar]\!]$-module;

(ii) $H_\hbar$ modulo $\hbar$ is isomorphic to H as Hopf algebras.

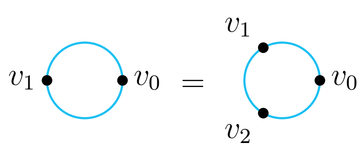
In our case, $H = U(g[z])$ for $g = \mathfrak{gl}_K[z]$, so $H_\hbar$ is a quantization of $U(\mathfrak{gl}_K[z])$, and according to Theorem 12.1.1 of [30], this is unique up to isomorphisms. This proves Proposition (4.5.1).

E.4 Technicalities of Witten Diagrams

E.4.1 Vanishing lemmas

We introduce some lemmas to allow us to readily declare several Witten diagrams in the 4D Chern-Simons theory to be zero.

Lemma E.4.1. *The product of two or three bulk-to-bulk propagators vanish when attached cyclically, diagrammatically this means:*



$$= 0\,. \qquad \text{(E.14)}$$

Proof. Two propagators: We can choose one of the two bulk points, say v_0, to be at the origin and denote v_1 simply as v. This amounts to taking the projection (4.115), namely: $\mathbb{R}^4_{v_0} \times \mathbb{R}^4_{v_1} \ni (v_0, v_1) \mapsto v_1 - v_0 =: v \in \mathbb{R}^4$. Then the product of the two propagators become:

$$P(v_0, v_1) \wedge P(v_1, v_0) \mapsto \overline{P}(v) \wedge \overline{P}(-v) = -\overline{P}(v) \wedge \overline{P}(v)\,. \qquad \text{(E.15)}$$

This is a four form at v, however, P does not have any $\mathrm{d}z$ component, therefore the four form $P(v) \wedge P(v)$ necessarily contains repetition of a one form and thus vanishes.

Three propagators: By choosing v_0 to be the origin of our coordinate system we can turn the product to the following:

$$\overline{P}(v_1) \wedge \overline{P}(v_2) \wedge P(v_1, v_2)\,. \tag{E.16}$$

We now need to look closely at the propagators (see (4.115) and (4.118)):

$$\overline{P}(v_i) = \frac{\hbar}{2\pi} \frac{x_i\, \mathrm{d}y_i \wedge \mathrm{d}\overline{z}_i + y_i\, \mathrm{d}\overline{z}_i \wedge \mathrm{d}x_i + 2\overline{z}_i\, \mathrm{d}x_i \wedge \mathrm{d}y_i}{d(v_i, 0)^4}\,, \tag{E.17a}$$

$$P(v_1, v_2) = \frac{\hbar}{2\pi} \frac{x_{12}\, \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} + y_{12}\, \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} + 2\overline{z}_{12}\, \mathrm{d}x_{12} \wedge \mathrm{d}y_{12}}{d(v_1, v_2)^4}\,, \tag{E.17b}$$

where $v_i := (x_i, y_i, z_i, \overline{z}_i)$, $x_{ij} := x_i - x_j$, $y_{ij} := y_i - y_j, \cdots$, and $d(v_i, v_j)^2 := (x_{ij}^2 + y_{ij}^2 + z_{ij}\overline{z}_{ij})$. Since the propagators don't have any $\mathrm{d}z$ component the product (E.16) must be proportional to $\omega := \bigwedge_{i\in\{1,2\}} \mathrm{d}x_i \wedge \mathrm{d}y_i \wedge \mathrm{d}\overline{z}_i$. In the product there are six terms that are proportional to ω. For example, we can pick $\mathrm{d}x_1 \wedge \mathrm{d}y_1$ from $\overline{P}(v_1)$, $\mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2$ from $\overline{P}(v_2)$ and $\mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12}$ from $P(v_1, v_2)$, this term is proportional to:

$$\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} = -\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_1 = +\omega\,. \tag{E.18}$$

The other five such terms are:

$$\begin{aligned}
\mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_{12} \wedge \mathrm{d}y_{12} &= -\,\omega\,,\\
\mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} &= +\,\omega\,,\\
\mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_{12} \wedge \mathrm{d}y_{12} &= +\,\omega\,,\\
\mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} &= -\,\omega\,,\\
\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_2 \wedge \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} &= -\,\omega\,.
\end{aligned} \tag{E.19}$$

These signs can be determined from a determinant, stated differently, we have the following equation:

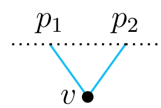
$$\det \begin{pmatrix} \mathrm{d}y_1 \wedge \mathrm{d}\overline{z}_1 & \mathrm{d}\overline{z}_1 \wedge \mathrm{d}x_1 & \mathrm{d}x_1 \wedge \mathrm{d}y_1 \\ \mathrm{d}y_2 \wedge \mathrm{d}\overline{z}_2 & \mathrm{d}\overline{z}_2 \wedge \mathrm{d}x_2 & \mathrm{d}x_2 \wedge \mathrm{d}y_2 \\ \mathrm{d}y_{12} \wedge \mathrm{d}\overline{z}_{12} & \mathrm{d}\overline{z}_{12} \wedge \mathrm{d}x_{12} & \mathrm{d}x_{12} \wedge \mathrm{d}y_{12} \end{pmatrix} = -6\omega\,, \tag{E.20}$$

where the product used in taking determinant is the wedge product. The above equation implies that in the product (E.16) the coefficient of $-\omega$ is given by the same determinant if we replace the two forms with their respective coefficients as they appear in (E.17). Therefore, the coefficient is:

$$\frac{1}{8\pi^3 d(v_1,0)^4 d(v_2,0)^4 d(v_1,v_2)^4} \det \begin{pmatrix} x_1 & y_1 & \overline{z}_1 \\ x_2 & y_2 & \overline{z}_2 \\ x_{12} & y_{12} & \overline{z}_{12} \end{pmatrix} = 0\,. \tag{E.21}$$

The determinant vanishes because the three rows of the matrix are linearly dependent. Thus we conclude that the product (E.16) vanishes. ☐

Lemma E.4.2. *The product of two bulk-to-bulk propagators joined at a bulk vertex where the other two endpoints are restricted to the Wilson line, vanishes, i.e., in any Witten diagram:*



$$= 0\,. \tag{E.22}$$

Proof. This simply follows from the explicit form of the bulk-to-bulk propagator. Computation verifies that:

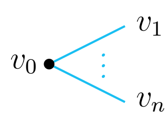
$$\iota_{\partial_{x_1}\wedge\partial_{x_2}}\left(P(v,p_1)\wedge P(v,p_2)\right) = 0\,, \tag{E.23}$$

where x_1 and x_2 are the x-coordinates of the points p_1 and p_2 respectively. ☐

The world-volume on which the CS theory is defined is $\mathbb{R}^2_{x,y}\times\mathbb{C}_z$, which in the presence of the Wilson line at $y = z = 0$ we view as $\mathbb{R}_x\times\mathbb{R}_+\times S^2$. When performing integration over this space we approximate the non-compact direction by a finite interval and then taking the length of the interval to infinity. In doing so we introduce boundaries of the world-volume, namely the two components $B_{\pm D} := \{\pm D\}\times\mathbb{R}_+\times S^2$ at the two ends of the interval $[-D, D]$. Our next lemma concerns some integrals over these boundaries.

Lemma E.4.3. *The integral over a bulk point vanishes when restricted to the spheres at infinity, in diagram:*

$$\lim_{D\to\infty}\int_{v_0\in B_{\pm D}}$$



$$= 0\,. \tag{E.24}$$

Proof. Symbolically, the integration can be written as:

$$\lim_{D\to\infty}\int_{B_{\pm D}} \mathrm{dvol}_{B_{\pm D}}\,\iota_{\partial_y\wedge\partial_{\overline{z}}}\left(P(v_0,v_1)\wedge\cdots\wedge P(v_0,v_n)\right)\,, \tag{E.25}$$

where y and $\overline{z}$ are coordinates of v_0. Note that the $\mathrm{d}z$ required for the volume form on $B_{\pm D}$ comes from the structure constant at the interaction vertex, not from the propagators. In the above integration the x-component of v_0 is fixed at $\pm D$, which introduces D dependence in the integrand. The bulk-to-bulk propagator has the following asymptotic scaling behavior:[10]

$$P((D,y,z,\overline{z}),v_j) \overset{D\to\infty}{\sim} D^{-2} + \mathcal{O}(D^{-3})\,. \tag{E.26}$$

[10] Keep in mind that $\hbar$ has a (length) scaling dimension 1.

The integration measure on $B_{\pm D}$ is independent of D, therefore the integral behaves as D^{-2n} for large D, and consequently vanishes in the limit $D \to \infty$. □

E.4.2 Comments on integration by parts

Finally, let us make a few general remarks about the integrals involved in computing Witten diagrams. Since the boundary-to-bulk propagators are exact and the bulk-to-bulk propagators behave nicely when acted upon by differential (see (4.116)), we want to use Stoke's theorem to simplify any given Witten diagram. Suppose we have a Witten diagram with m propagators connected to the boundary, n propagators connected to the Wilson line, and l bulk points. Let us denote the bulk points by v_i for $i = 1, \cdots, l$, the points on the Wilson line by p_j for $j = 1, \cdots, n$, and the points on the boundary as x_k for $k = 1, \cdots, m$. The domain of integration for the diagram is then $M^l \times \Delta_n$, where $M = \mathbb{R} \times \mathbb{R}_+ \times S^2$ and Δ_n is an n-simplex defined as:

$$\Delta_n := \{(p_1, \cdots, p_n) \in \mathbb{R}^n \,|\, p_1 \leq p_2 \leq \cdots \leq p_n\}\,. \tag{E.27}$$

This domain may need to be modified in some Witten diagrams due to the integral over this domain having UV divergences. UV divergences can occur when some points along the Wilson line collide with each other. To avoid such divergences we shall use a *point splitting* regulator, i.e., we shall cut some corners from the simplex Δ_n. Let us denote the regularized simplex as $\widetilde{\Delta}_n$. The exact description of $\widetilde{\Delta}_n$ will vary from diagram to diagram, and we shall describe them as we encounter them.

When we do integration by parts with respect to the differential in a boundary-to-bulk propagator, we get the following three types of terms:

1. A boundary term. Boundaries of our integration domain comes from boundaries of M and $\widetilde{\Delta}_n$. For M we get:
$$\partial M = B_{+\infty} \sqcup B_{-\infty}\,. \tag{E.28}$$
Due to Lemma E.4.3, integrations over ∂M will vanish. Therefore, nonzero contribution to the boundary integration, when we do integration by parts, will only come from the boundary of the regularized simplex, namely $\partial\widetilde{\Delta}_n$. Schematically, the appearance of such a boundary integral will look like:
$$\int_{M^l \times \widetilde{\Delta}_n} \mathrm{d}\theta \wedge (\cdots) = \int_{M^l \times \partial\widetilde{\Delta}_n} \theta \wedge (\cdots) + \cdots\,. \tag{E.29}$$

2. The differential acts on a bulk-to-bulk propagator. Due to (4.116), this identifies the two end points of the propagator, schematically:

$$\mathsf{b} \in \{0,1\}\,, \qquad \int_{M^l \times \partial^{\mathsf{b}} \widetilde{\Delta}_n} \mathrm{d}\theta \wedge P \wedge (\cdots) = \int_{M^{l-1} \times \partial^{\mathsf{b}} \widetilde{\Delta}_n} \theta \wedge (\cdots) + \cdots\,. \tag{E.30}$$

3. The differential acts on a step function left by a previous integration by parts. This does not change the domain of integration.

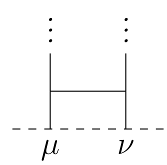
The third option does not to lead a simplification of the domain of integration. Therefore, at the present abstract level, our strategy to simplify an integration is: first go to the boundary of the simplex, and then keep collapsing bulk-to-bulk propagators until we have no more differential left or when no more bulk-to-bulk propagator can be collapsed without the diagram vanishing due the vanishing lemmas from §E.4.1.

E.5 Proof of Lemma 4.5.3

All the diagrams that we draw in this section only exist to represent color factors, their numerical values are irrelevant. Which is why we also ignore the color coding we used in the diagrams in chapter 4.

We start with yet another lemma:

Lemma E.5.1. *The color factor of any Witten diagram with two boundary-to-bulk propagators connected by a single bulk-to-bulk propagator, that is any Witten diagrams with the following configuration:*



(E.31)

upon anti-symmetrizing the color labels of the boundary-to-bulk propagators, involves the following factor:

$$f_{\mu\nu}{}^{\xi} X_{\xi}\,, \tag{E.32}$$

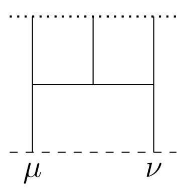
for some matrix X_ξ that transforms under the adjoint representation of $\mathfrak{gl}_K$. In particular, this color factor is the image in $\mathrm{End}(V)$ of some element of $\mathfrak{gl}_K$ where V is the representation of some distant Wilson line.

Proof. The two bulk vertices in the diagram results in the following product of structure constants: ${f_{\mu o}}^{\pi}{f_{\nu\rho}}^{o}$ where the indices π and ρ are contracted with the rest of the diagram. Anti-symmetrizing the indices μ and ν we get ${f_{\mu o}}^{\pi}{f_{\nu\rho}}^{o} - {f_{\nu o}}^{\pi}{f_{\mu\rho}}^{o}$, which using the Jacobi identity becomes $-{f_{\mu\nu}}^{o}{f_{\rho o}}^{\pi}$. Once π and ρ are contracted with the rest of the diagram we get an expression of the general form (E.32). Furthermore, any expression of the form (E.32) is an image in $\mathrm{End}(V)$ of some element in $\mathfrak{gl}_K$, since the structure constant ${f_{\mu\nu}}^{\xi}$ can be viewed as a map:

$$f : \wedge^2\mathfrak{gl}_K \to \mathfrak{gl}_K\,, \qquad f : t_\mu \wedge t_\nu \mapsto {f_{\mu\nu}}^{\xi} t_\xi\,. \tag{E.33}$$

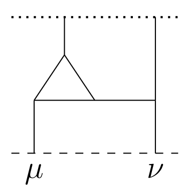
Now composing the above map with a representation of $\mathfrak{gl}_K$ on V gives the aforementioned image. □

Let us now look at the color factor (4.165) of the diagram (4.164), both of which we repeat here:



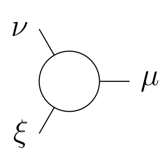
$$, \qquad {f_\mu}^{\xi o}{f_\xi}^{\pi\rho}{f_{\nu\pi}}^{\sigma}\varrho(t_o)\varrho(t_\rho)\varrho(t_\sigma)\,. \tag{E.34}$$

By commuting $\varrho(t_o)$ and $\varrho(t_\rho)$ in the color factor we create a difference which is the color factor of the following diagram:



$$. \tag{E.35}$$

The key feature of the above diagram is the loop with three propagators attached to it. Such a loop produces a color factor which is a $\mathfrak{gl}_K$-invariant inside $(\mathfrak{gl}_K)^{\otimes 3}$, explicitly we can write a loop and its associated color factor respectively as:



$$\text{and} \quad {f_{\mu o}}^{\pi}{f_{\nu\rho}}^{o}{f_{\xi\pi}}^{\rho}\,. \tag{E.36}$$

The color factor is $\mathfrak{gl}_K$-invariant since the structure constant itself is such an invariant. To find the invariants in $(\mathfrak{gl}_K)^{\otimes 3}$ we start by writing $\mathfrak{gl}_K$ as:

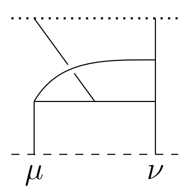
$$\mathfrak{gl}_K = \mathfrak{sl}_K \oplus \mathbb{C}\,, \tag{E.37}$$

where by $\mathfrak{sl}_K$ we mean the complexified algebra $\mathfrak{sl}(K,\mathbb{C})$. This gives us the decomposition

$$(\mathfrak{gl}_K)^{\otimes 3} = (\mathfrak{sl}_K)^{\otimes 3} \oplus \cdots \,, \tag{E.38}$$

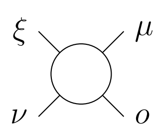
where the "$\cdots$" contains summands that necessarily include at leas one factor of the center $\mathbb{C}$. However, none of the three indices that appear in the diagram in (E.36) can correspond to the center, because each of these indices belong to an instance of the structure constant, which vanishes whenever one of its indices correspond to the center.[11] This means that the $\mathfrak{gl}_K$ invariant we are looking for must lie in $(\mathfrak{sl}_K)^{\otimes 3}$. For $K > 2$, there are exactly two such invariants [129], one of them is the structure constant itself, which is totally anti-symmetric. The other invariant is totally symmetric. However the structure constant is even (invariant) under the $\mathbb{Z}_2$ outer automorphism of $\mathfrak{sl}_K$ whereas the symmetric invariant is odd. Since our theory has this $\mathbb{Z}_2$ as a symmetry, only the structure constant can appear as the invariant in a diagram.[12] This means, as far as the color factor is concerned, we can collapse a loop such as the one in (E.36) to an interaction vertex. As soon as we do this operation to the diagram (E.35), Lemma E.5.1 tells us that the color factor of the diagram is an image in $\mathrm{End}(V)$ of an element in $\mathfrak{gl}_K$. This shows that we can swap the positions of any of the two pairs of the adjacent matrices in the color factor in (E.34) and the difference we shall create is an image of a map $\mathfrak{gl}_K \to \mathrm{End}(V)$. To achieve all permutations of the three matrices wee need to be able to keep swaping positions, let us therefore keep looking forward.

Suppose we commute $\varrho(t_o)$ and $\varrho(t_\rho)$ in (E.34), then we end up with the color factor of the diagram (4.163). Now if we commute $\varrho(t_o)$ and $\varrho(t_\sigma)$, we create a difference that corresponds the color factor of the following diagram:



. (E.39)

The key feature of this diagram is a loop with four propagators attached to it. The loop and its associated color factor can be written as:



$$, \qquad f_{\mu\pi}{}^{\tau} f_{o\tau}{}^{\sigma} f_{\nu\sigma}{}^{\rho} f_{\xi\rho}{}^{\pi} \,. \tag{E.40}$$

[11] In other words, the central abelian photon in $\mathfrak{gl}_K$ interacts with neither itself nor the non-abelian gluons and therefore can not contribute to the diagrams we are considering.

[12] This is also apparent from the way this invariant is written in (E.36), since the structure constant is invariant under this $\mathbb{Z}_2$, certainly a product of them is invariant as well.

As before, the color factor is a $\mathfrak{gl}_K$-invariant in $(\mathfrak{gl}_K)^{\otimes 4}$. This time, it will be more convenient to write the color factor as a trace. Noting that the structure constants are the adjoint representations of the generators of the algebra we can write the above color factor as:

$$\mathrm{tr}_{\,\mathrm{ad}}(t_\mu t_o t_\nu t_\xi)\,. \tag{E.41}$$

The adjoint representation of $\mathfrak{gl}_K$ factors through $\mathfrak{sl}_K$, and the adjoint representation of $\mathfrak{sl}_K$ has a non-degenerate metric with which we can raise and lower adjoint indices. Suitably changing positions of some of the indices in the color factor we can conclude:

$$\mathrm{tr}_{\,\mathrm{ad}}(t_\mu t_o t_\nu t_\xi) = \mathrm{tr}_{\,\mathrm{ad}}(t_\mu t_\xi t_\nu t_o)\,. \tag{E.42}$$

Using the cyclic symmetry of the trace we then find that the color factor is symmetric under the exchange of μ and ν, therefore when we anti-symmetrize the diagram with respect to μ and ν it vanishes.

In summary, starting from the color factor in (E.34), we can keep swapping any two adjacent matrices and the difference can always be written as an image of some map $\mathfrak{gl}_K \to \mathrm{End}(V)$. The same argument applies to the color factors of all the diagrams in (4.163). This proves the lemma.