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Asymptotic analysis of the clamped-pinned elastica

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Asymptotic solutions for the clamped-pinned elastica when the displacement of the pinned end is small (immediately after buckling) and when it approaches its limiting displacement (when the force in the rod tends to infinity) are presented. Simple leading order relationships describing the force as a function of the pinned end's displacement are derived. Those approximate the force-displacement behaviour of the clamped-pinned elastica for small and limiting end displacements. All our results are valid for the clamped-pinned elastica in any mode.

Key words: elastica, clamped-pinned, asymptotic, force-displacement, post-buckling.

1. Introduction

The clamped-pinned elastica, whereby one end of a rod is displaced in a straight line towards the other end can be solved in a closed-form using elliptic integrals. Mikata [1] presents such a solution for the first mode of buckling that is valid for values of end displacement up to the point where the two ends meet and form a closed loop. Singh and Goss [2] present solutions for any mode in a closed-form for the full domain of end displacement, including the case where the ends have crossed through each other. For a more general review of elastica see Bigoni *et al.* [3].

Unfortunately, the closed-form solutions for the clamped-pinned elastica are rather unwieldy and do not provide simple expressions for either the configuration or for the end force as a function of end displacement, which may be useful in applications. The two extremities of end displacement are of particular interest; very small immediately after buckling and very large where the rod almost straightens out. Those may correspond respectively to failure of rod-like structures that are designed to elastically withstand small values of end displacement, and to rods and cables that are designed to withstand high tensile axial forces, but which may fail due to the large bending moments that arise. In such situations application of asymptotic analysis provides solutions which are easier to work with and understand.

Wang [4] has examined those two extremities for the first mode only using asymptotic analysis. In this paper, we extend the analysis to all modes for the clamped-pinned elastica. We additionally present approximate solutions for the configuration and leading order expressions for the end force as a function of the end displacement.

2. The clamped-pinned elastica model

We consider an elastic rod parameterised by its non-dimensional (normalised) arc length, $0 \leq s \leq 1$. The end points of the rod are fixed along the x-axis by a clamp at $s = 0$ and a pinned joint at $s = 1$ which can be displaced along the x-axis by amount d, where $-2 \leq d \leq 0$. The rod is inextensible and unshearable but can deform in the xy-plane, under rigid loading through the parameter d. The forces acting along the x and y axes are t and r, respectively. The tangent at any point $(x(s), y(s))$ on the deformed rod is given by $\psi(s)$, see Fig. 1.

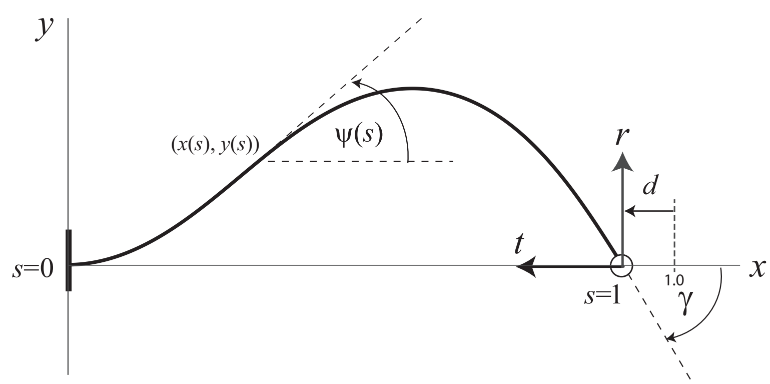

Fig. 1. General configuration of the rod for displacement of the pinned end along the x-axis by amount $d = x(1) - 1$.

The (non-dimensional) ordinary differential equations (odes) describing the clamped-pinned rod, which may be derived from force and moment balance arguments applied to an infinitesimal element of the deformed rod, are

$$\frac{d^2\psi}{ds^2} = t\sin(\psi) - r\cos(\psi), \tag{2.1}$$

$$\frac{dr}{ds} = 0, \tag{2.2}$$

$$\frac{dt}{ds} = 0, \tag{2.3}$$

$$\frac{dx}{ds} = \cos(\psi), \tag{2.4}$$

$$\frac{dy}{ds} = \sin(\psi). \tag{2.5}$$

The boundary conditions are

$$\psi(0) = 0, \tag{2.6}$$

$$\frac{d\psi(1)}{ds} = 0, \tag{2.7}$$

$$x(0) = 0, \tag{2.8}$$

$$x(1) = d + 1, \tag{2.9}$$

$$y(0) = 0, \tag{2.10}$$

$$y(1) = 0. \tag{2.11}$$

For convenience, we introduce $\psi(1) = \gamma$. Furthermore, from Eqs. (2.4), (2.8) and (2.9) we note that

$$x(1) = \int_0^1 ds \cos(\psi) = d + 1, \tag{2.12}$$

and from Eqs. (2.5), (2.10) and (2.11) we find that

$$\int_0^1 ds \sin(\psi) = 0. \tag{2.13}$$

3. Asymptotic analysis

In the asymptotic analysis of the clamped-pinned elastica we look for asymptotic solutions as $d \to 0^-$ and $d \to -2^+$. The former case deals with immediate post-buckling of the rod where the displacement d is small. In the later case, d approaches its maximum displacement and $t \to -\infty$.

3.1. Immediate post-buckling

First, consider immediate post-buckling, i.e. $d \to 0^-$, where we expect ψ and r to be small and t to be fairly close to the critical load for buckling. We use the following asymptotic expansions for ψ, t and r

$$\psi = \psi_0\varepsilon + \psi_1\varepsilon^3 + \mathrm{O}(\varepsilon^5), \tag{3.1}$$

$$t = t_0 + t_1\varepsilon^2 + \mathrm{O}(\varepsilon^4), \tag{3.2}$$

$$r = r_0\varepsilon + r_1\varepsilon^3 + \mathrm{O}(\varepsilon^5), \tag{3.3}$$

where $0 < \varepsilon \ll 1$ [4]. Substituting Eqs. (3.1)–(3.3) into Eq. (2.1) and equating coefficients of powers of ε, we obtain the following odes for ψ_0 and ψ_1

$$\frac{d^2\psi_0}{ds^2} - t_0\psi_0 = -r_0, \tag{3.4}$$

$$\frac{d^2\psi_1}{ds^2} - t_0\psi_1 = -\frac{1}{6}\psi_0^3 + t_1\psi_0 + \frac{1}{2}r_0\psi_0^2 - r_1. \tag{3.5}$$

Substituting Eq. (3.1) into Eqs. (2.6), (2.7) and (2.13) and equating coefficients of powers of ε, yields the boundary conditions that ψ_0 and ψ_1 must satisfy

$$\psi_0(0) = 0, \qquad \frac{d\psi_0(1)}{ds} = 0, \qquad \int_0^1 ds\psi_0 = 0, \tag{3.6}$$

$$\psi_1(0) = 0, \qquad \frac{d\psi_1(1)}{ds} = 0, \qquad \int_0^1 ds\left(\psi_1 - \frac{1}{6}\psi_0^3\right) = 0. \tag{3.7}$$

The general solution of Eq. (3.4) is

$$\psi_0 = a_0\cos((-t_0)^{\frac{1}{2}}s) + b_0\sin((-t_0)^{\frac{1}{2}}s) + \frac{r_0}{t_0}, \tag{3.8}$$

where a_0 and b_0 are constants of integration. Using the boundary conditions, given by Eq. (3.6), we find from Eq. (3.8) that

$$a_0 = -\frac{r_0}{t_0}, \qquad b_0 = -\frac{r_0}{t_0}(-t_0)^{\frac{1}{2}}, \qquad \tan((-t_0)^{\frac{1}{2}}) = (-t_0)^{\frac{1}{2}}. \tag{3.9}$$

The nature of the approximations makes it impossible to determine r_0 since we have only 3 boundary conditions, given by Eq. (3.6). However, since the amplitude of ψ must be small we may absorb $-\frac{r_0}{t_0}$ into ε, i.e. we scale ψ_0 given by Eq. (3.8) by $a_0 = -\frac{r_0}{t_0}$. This amounts to setting

$$r_0 = -t_0. \tag{3.10}$$

Hence, we can take

$$\psi_0 = \cos((-t_0)^{\frac{1}{2}}s) + (-t_0)^{\frac{1}{2}}\sin((-t_0)^{\frac{1}{2}}s) - 1, \tag{3.11}$$

where t_0 is determined by solving the last equation in the set given by Eq. (3.9). Those solutions give the critical loads for buckling. The first 3 solutions are

$$t_0 = -20.19072856, \qquad -59.67951594, \qquad -118.89986916 \tag{3.12}$$

and represent the critical loads for the first 3 buckling modes.

In order to simplify the calculations we have to perform to determine ψ_1, we define the parameter u by

$$u = (-t_0)^{\frac{1}{2}} s + u_0, \tag{3.13}$$

where u_0 is given by

$$\tan(u_0) = \frac{1}{(-t_0)^{\frac{1}{2}}}. \tag{3.14}$$

Using Eqs. (3.13) and (3.14), Eq. (3.11) gives the following form for $\psi_0(u)$

$$\psi_0 = (1 - t_0)^{\frac{1}{2}} \sin(u) - 1. \tag{3.15}$$

With respect to the parameter u given by Eq. (3.13), the ode for ψ_1 given by Eq. (3.5), with the assistance of Eq. (3.15), becomes

$$\frac{d^2\psi_1}{du^2} + \psi_1 = k_1 \sin(3u) + k_2 \sin(u) + k_3, \tag{3.16}$$

where

$$\begin{aligned} k_1 &= -\frac{1}{24}(1 - t_0)^{\frac{3}{2}}, \qquad k_2 = -\left(\frac{1}{8}t_0 + \frac{3}{8} + \frac{t_1}{t_0}\right)(1 - t_0)^{\frac{1}{2}}, \\ k_3 &= \frac{1}{3} + \frac{t_1}{t_0} + \frac{r_1}{t_0}. \end{aligned} \tag{3.17}$$

The general solution of Eq. (3.16) is

$$\psi_1 = a_1 \sin(3u) + b_1 u \cos(u) + c_1 + k_4 \cos(u) + k_5 \sin(u) \tag{3.18}$$

where a_1, b_1, c_1, k_4 and k_5 are constants of integration. We see that there are 5 constants of integration to find and only only 4 conditions to determine them, i.e. Eqs. (3.7) and (3.16). This is a consequence of the approximations made, as pointed out by TIMOSHENKO and GERE [5]. To get around this, scale ψ_1 by k_5 and note that $\frac{\psi_1}{k_5}$ is still a solution of Eq. (3.16). Effectively, we are absorbing k_5 into ε^3 and redefining $\frac{a_1}{k_5}$, $\frac{b_1}{k_5}$, $\frac{c_1}{k_5}$ and $\frac{k_4}{k_5}$ as a_1, b_1, c_1 and k_4. This is fine at this stage since ε is arbitrary at this stage and can be redefined. We now have

$$\psi_1 = a_1 \sin(3u) + b_1 u \cos(u) + c_1 + k_4 \cos(u) + \sin(u). \tag{3.19}$$

Substituting Eq. (3.19) into Eq. (3.16) gives

$$a_1 = -\frac{1}{8}k_1, \qquad b_1 = -\frac{1}{2}k_2, \qquad c_1 = k_3. \tag{3.20}$$

Applying the boundary conditions described by Eq. (3.7) to ψ_1 given by Eqs. (3.17), (3.19) and (3.20) gives the following final forms for all constants of integration

$$\begin{aligned} k_1 &= -\frac{1}{24}(1-t_0)^{\frac{3}{2}}, \qquad k_2 = \frac{1}{4}(1-t_0)^{\frac{1}{2}}, \\ k_3 &= \frac{1}{192}(27t_0+1) - \frac{1}{(1-t_0)^{\frac{1}{2}}}, \qquad k_4 = \frac{1}{8}((-t_0)^{\frac{1}{2}} + u_0)(1-t_0)^{\frac{1}{2}} \end{aligned} \tag{3.21}$$

and

$$t_1 = -\frac{1}{8}t_0(t_0+5), \qquad r_1 = \frac{1}{64}t_0(17t_0+19) - \frac{t_0}{(1-t_0)^{\frac{1}{2}}}, \tag{3.22}$$

where t_0 and r_0 are given by Eqs. (3.9) and (3.10), respectively. We present the values of all constants involved in the asymptotic expansions of ψ, t and r in Table 1 for the first 3 modes.

Table 1. Constants involved in the asymptotic expansions of ψ, t and r for the first 3 modes.

mode (n)	1	2	3	$\cdots$
t_0	−20.19072856	−59.67951594	−118.89986916	$\cdots$
r_0	20.19072856	59.67951594	118.89986916	$\cdots$
t_1	−38.33898461	−407.90588045	−1692.83494266	$\cdots$
r_1	106.67813633	936.00582799	3730.74829790	$\cdots$
k_1	−4.06450433	−19.69481564	−54.70371516	$\cdots$
k_2	1.15083471	1.94742644	2.73746997	$\cdots$
k_3	−3.05134650	−8.51559815	−16.80641097	$\cdots$
k_4	2.71159041	7.64752575	15.05002719	$\cdots$

Hence, our final form for ψ is

$$\begin{aligned} \psi &= (4k_2\sin(u) - 1)\varepsilon \\ &+ \left(-\frac{1}{8}k_1\sin(3u) - \frac{1}{2}k_2 u\cos(u) + k_3 + k_4\cos(u) + \sin(u)\right)\varepsilon^3 + \mathrm{O}(\varepsilon^5), \end{aligned} \tag{3.23}$$

where u is given by Eq. (3.13).

It is often useful to see the shape of the rod immediately after buckling. We therefore substitute Eq. (3.23) into Eqs. (2.4) and (2.5) and perform some very tedious integrations to obtain

$$\begin{aligned}(3.24)\qquad x = s + \Bigg(&\frac{2k_2^2}{(-t_0)^{\frac{1}{2}}}(\sin(2u) - \sin(2u_0)) \\ &- \frac{4k_2}{(-t_0)^{\frac{1}{2}}}(\cos(u) - \cos(u_0)) - (4k_2^2 + \frac{1}{2})s\Bigg)\varepsilon^2 + \mathrm{O}(\varepsilon^4)\end{aligned}$$

and

$$\begin{aligned}(3.25)\qquad y = &\left(\frac{4k_2}{(-t_0)^{\frac{1}{2}}}(\cos(u_0) - \cos(u)) - s\right)\varepsilon \\ &+ \Bigg(\frac{3k_1}{8(-t_0)^{\frac{1}{2}}}(\cos(3u) - \cos(3u_0)) - \frac{k_2}{2(-t_0)^{\frac{1}{2}}}(u\sin(u) - u_0\sin(u_0)) \\ &+ \frac{k_4}{(-t_0)^{\frac{1}{2}}}(\sin(u) - \sin(u_0)) + \frac{1}{(-t_0)^{\frac{1}{2}}}(\frac{1}{2}k_2(4 - t_0) - 1)(\cos(u) - \cos(u_0)) \\ &- \frac{2k_2^2}{(-t_0)^{\frac{1}{2}}}(\sin(2u) - \sin(2u_0)) + (k_3 + \frac{5}{12} - \frac{1}{4}t_0)s\Bigg)\varepsilon^3 + \mathrm{O}(\varepsilon^5),\end{aligned}$$

where u is given by Eq. (3.13).

The approach above to asymptotic analysis immediately after buckling is due to KOITER [6] and [7], and similar to that of WANG [4]. We differ from Wang in our calculation of ψ_1 by preferring to strictly observe linear independence of terms of different powers of ε. Therefore we do not dispose of the $k_5\sin(u)$ term in ψ_1 given by Eq. (3.18), in order to avoid calculating k_5, by absorbing it into $\sin(u)$ term in ψ_0 given by Eqs. (3.11) or (3.15). Instead, we avoid calculating k_5 by scaling ψ_1 by k_5 and using ψ_1 given by Eq. (3.19). Apart from this minor difference, our work is in agreement with Wang's for the first mode ($n = 1$).

One may ask, what is ε? We know it is an arbitrary small parameter. However, it can't be plucked out of a magician's top hat like a rabbit. It could be related to some small physical quantity in the clamped-pinned elastica problem such as: the maximum deflection of the rod, α (see Eq. (3.26)) or γ. DYM [8] suggests the maximum deflection of the rod as a choice for ε. We found α works very well. In the plots shown in Fig. 2 we have used α obtained from the numerical solution of the odes for the clamped-pinned elastica for ε in the asymptotic solution. Those plots compare numerical solutions (solid lines) and asymptotic solutions as $d \to 0^-$ (dashed lines) for the first 3 modes ($n = 1, 2, 3$) and $d = -0.01$ and $d = -0.1$. We observe that the asymptotic solution becomes more accurate with respect to the numerical solution as $|d|$ becomes smaller. This is as expected

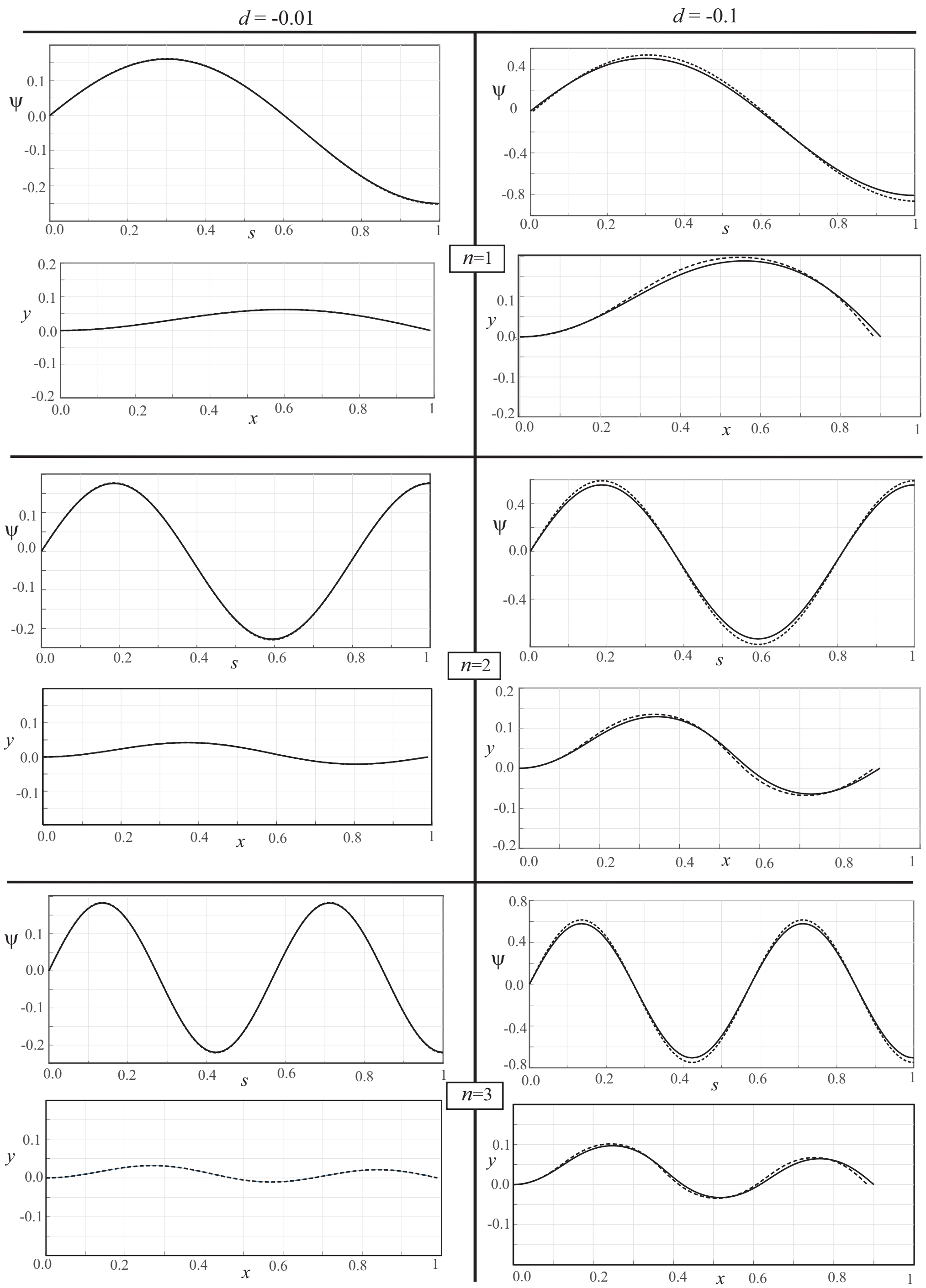

Fig. 2. Numerical solutions (solid lines) and asymptotic solutions as $d \to 0^-$ (dashed lines) for the first 3 modes ($n = 1, 2, 3$) and $d = -0.01$ and $d = -0.1$.

since the asymptotic solution is valid as $d \to 0^-$. The accuracy of the asymptotic solution depends very much on $|d|$, but seems to be independent of the mode n.

3.2. Large t

We now look for asymptotic solutions for $t \to -\infty$, i.e. $d \to -2^+$. In this configuration the pinned end is pulled well beyond the clamped end. As we will see, these solutions are characterised by sections which are very nearly straight or exhibit significant concentrations of curvature.

For convenience, we introduce f, α and θ

$$f = (t^2 + r^2)^{\frac{1}{2}}, \qquad \alpha = \arctan\left(\frac{r}{-t}\right), \qquad \theta = \psi + \alpha. \tag{3.26}$$

where f is the (non-dimensional) resultant force. Consequently, $t \to -\infty$ is equivalent to $f \to \infty$ and the clamped-pinned rod described by Eqs. (2.1)–(2.5) is now described by the following odes

$$\frac{d^2\theta}{ds^2} = -f\sin(\theta), \tag{3.27}$$

$$\frac{df}{ds} = 0, \tag{3.28}$$

$$\frac{d\alpha}{ds} = 0, \tag{3.29}$$

$$\frac{dx}{ds} = \cos(\alpha)\cos(\theta) + \sin(\alpha)\sin(\theta), \tag{3.30}$$

$$\frac{dy}{ds} = \cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta), \tag{3.31}$$

and the boundary conditions given by Eqs. (2.6) and (2.7) become

$$\theta(0) = \alpha, \tag{3.32}$$

$$\frac{d\theta(1)}{ds} = 0. \tag{3.33}$$

The boundary conditions given by Eqs. (2.8)–(2.11) are unchanged by our decision to use f, α and θ instead of t, r and ψ, and still hold. In terms of θ, Eqs. (2.12) and (2.13) become

$$x(1) = \cos(\alpha)\int_0^1 ds\cos(\theta(s)) + \sin(\alpha)\int_0^1 ds\sin(\theta(s)) = d + 1, \tag{3.34}$$

and

$$\cos(\alpha)\int_0^1 ds\sin(\theta(s)) - \sin(\alpha)\int_0^1 ds\cos(\theta(s)) = 0, \tag{3.35}$$

respectively.

The first integral of Eq. (3.27) subject to the boundary condition given by Eq. (3.33) is

$$\begin{aligned} \frac{d\theta}{ds} &= \pm 2 f^{\frac{1}{2}} \left(\sin^2\left(\frac{\gamma+\alpha}{2} \right) - \sin^2\left(\frac{\theta}{2} \right) \right)^{\frac{1}{2}} \\ &= \pm 2 f^{\frac{1}{2}} \left(\sin^2\left(\frac{(-1)^n(\gamma+\alpha)}{2} \right) - \sin^2\left(\frac{\theta}{2} \right) \right)^{\frac{1}{2}} \\ &= \pm 2 f^{\frac{1}{2}} \left(\sin^2\left(\frac{\theta_\pi}{2} \right) - \sin^2\left(\frac{\theta}{2} \right) \right)^{\frac{1}{2}}. \end{aligned} \tag{3.36}$$

We have introduced the mode number n, which takes positive integer values $1, 2, 3, \ldots$, to deal with inflection points in higher modes. The cumbersome looking term $(-1)^n(\gamma+\alpha)$ is defined in more compact notation as

$$\theta_\pi = (-1)^n(\gamma+\alpha) \tag{3.37}$$

and for later use define the angle ϕ_θ by

$$\sin\left(\frac{\theta_\pi}{2} \right) \sin(\phi_\theta) = \sin\left(\frac{\theta}{2} \right). \tag{3.38}$$

In the nth mode, for $d < -1$, which includes the case $d \to -2^+$, there are n inflection points at s_i where $\theta_i = \theta(s_i) = (-1)^{n+i}(\gamma+\alpha)$ for $i = 1, \ldots, n$. For consistency with the boundary condition given by Eq. (3.33), we have $s_n = 1$ where $\theta_n = \theta(s_n) = \theta(1) = \gamma+\alpha$. Since the sign of $d\theta/ds$ given by Eq. (3.36) alternates between consecutive pairs of inflection points, care must be taken to ensure that we choose the correct sign when performing our calculations. With this in mind, we choose a configuration of the clamped-pinned rod in which $d\theta/ds < 0$ for $0 \le s < s_1$, $d\theta/ds > 0$ for $s_1 \le s < s_2$, $d\theta/ds < 0$ for $s_2 \le s < s_3, \ldots$, and $d\theta/ds < 0$ for n odd and $d\theta/ds > 0$ for n even for $s_{n-1} \le s \le s_n$.

We integrate Eq. (3.36) and obtain

$$\begin{aligned} f^{\frac{1}{2}} &= \int_0^{\phi_\alpha} d\phi \frac{1}{(1 - \sin^2(\frac{\theta_\pi}{2}) \sin^2(\phi))^{\frac{1}{2}}} \\ &\quad + (2n-1) \int_0^{\frac{\pi}{2}} d\phi \frac{1}{(1 - \sin^2(\frac{\theta_\pi}{2}) \sin^2(\phi))^{\frac{1}{2}}}. \end{aligned} \tag{3.39}$$

Substituting Eq. (3.35) into Eq. (3.34) and integrating gives

$$(d+1) f^{\frac{1}{2}} = -\frac{2}{\sin(\alpha)} \left(\sin^2\left(\frac{\theta_\pi}{2} \right) - \sin^2\left(\frac{\alpha}{2} \right) \right)^{\frac{1}{2}}. \tag{3.40}$$

Finally, integrating Eq. (3.35) yields

$$
\begin{aligned}
(3.41)\qquad & 2\cos(\alpha)\left(\sin^2\left(\frac{\theta_\pi}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\right)^{\frac{1}{2}} \\
&+\sin(\alpha)\left(2\int_0^{\phi_\alpha} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}}-\int_0^{\phi_\alpha} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) \\
&+(2n-1)\sin(\alpha)\left(2\int_0^{\frac{\pi}{2}} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}}\right. \\
&\qquad\qquad\left.-\int_0^{\frac{\pi}{2}} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right)=0.
\end{aligned}
$$

It is plain that all the integrals in Eqs. (3.39)–(3.41) are either complete or incomplete elliptic integrals of the first and second kind.

As $d \to -2^+$, i.e. $f \to \infty$,

$$
(3.42)\qquad \theta_\pi = (-1)^n(\gamma+\alpha) \to \pi^-,
$$

i.e. the rod straightens out. Using Eq. (3.42), Eq. (3.38) to leading order becomes

$$
(3.43)\qquad \phi_\theta = \frac{\theta}{2} + \mathrm{O}\left(\cos^2\left(\frac{\theta_\pi}{2}\right)\right).
$$

The following formulae (see Gradshteyn and Ryzhik [9]) will prove useful in subsequent calculations

$$
(3.44)\qquad \int_0^{\frac{\pi}{2}} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}} = \log_e\left(\frac{4}{\cos(\frac{\theta_\pi}{2})}\right)+\mathrm{O}\left(\cos^2\left(\frac{\theta_\pi}{2}\right)\right),
$$

$$
(3.45)\qquad \int_0^{\frac{\pi}{2}} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} = 1+\mathrm{O}\left(\cos^2\left(\frac{\theta_\pi}{2}\right)\right),
$$

$$
\begin{aligned}
(3.46)\qquad & \int_0^{\phi_\theta} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}} \\
&\qquad = \frac{1}{2}\log_e\left(\frac{1+\sin(\phi_\theta)}{1-\sin(\phi_\theta)}\right)+\mathrm{O}\left(\cos^2\left(\frac{\theta_\pi}{2}\right)\right),
\end{aligned}
$$

$$
(3.47)\qquad \int_0^{\phi_\theta} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} = \sin(\phi_\theta)+\mathrm{O}\left(\cos^2\left(\frac{\theta_\pi}{2}\right)\right).
$$

Note that k and k' in [9] are equivalent to $\sin(\frac{\theta_\pi}{2})$ and $\cos(\frac{\theta_\pi}{2})$, respectively. Using Eqs. (3.43)–(3.47), Eqs. (3.39)–(3.41) simplify greatly to give the following to leading order as $d \to -2^+$

$$\begin{aligned}
(3.48)\quad (2n-1)\log_e\left(\frac{4}{\cos(\frac{\theta_\pi}{2})}\right) &= 2(2n-1) + \frac{1}{\sin(\frac{\alpha}{2})} - \frac{1}{2}\log_e\left(\frac{1+\sin(\frac{\alpha}{2})}{1-\sin(\frac{\alpha}{2})}\right) \\
&= \frac{2(2n-1)}{(d+2)} - \frac{1}{2}\log_e\left(\frac{(4n-3)d+4(n-1)}{(4n-1)d+4n}\right),
\end{aligned}$$

$$(3.49)\qquad \sin\left(\frac{\alpha}{2}\right) = -\frac{(d+2)}{2(2n-1)(d+1)},$$

$$(3.50)\qquad f^{\frac{1}{2}} = \frac{2(2n-1)}{(d+2)}.$$

Equations (3.49) and (3.50) for the first mode ($n = 1$) are in complete agreement with Wang's results [4].

We now derive solutions for the nth mode, θ, x and y. Those are presented piece-wise between consecutive pairs of inflection points s_{i-1} and s_i for $i = 1, \ldots, n$. We denote $s_0 = 0$ and point out that it is not an inflection point, merely a convenient choice for our notation. At the ith inflection point s_i, let $\theta_i = \theta(s_i) = (-1)^{n+i}(\gamma + \alpha)$. We also denote the values of x and y at the ith inflection point s_i by $x_i = x(s_i)$ and $y_i = y(s_i)$, respectively. Integrating Eq. (3.36) for $s_0 \le s \le s_i$ we find

$$(3.51)\qquad s_i = \frac{1}{f^{\frac{1}{2}}}\Bigg(\int_0^{\phi_\alpha} d\phi \frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}} + (2i-1)\int_0^{\frac{\pi}{2}} d\phi \frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\Bigg).$$

As $d \to -2^+$, using Eqs. (3.42), (3.43), (3.44), (3.46) and (3.48), Eq. (3.51) becomes

$$(3.52)\qquad s_i = \frac{1}{f^{\frac{1}{2}}}\left(\frac{(n-i)}{(2n-1)}\log_e\left(\frac{1+\sin(\frac{\alpha}{2})}{1-\sin(\frac{\alpha}{2})}\right) + \frac{(2i-1)}{(2n-1)\sin(\frac{\alpha}{2})} + 2(2i-1)\right).$$

Setting $i = n$ in Eq. (3.52) gives $s_n = 1$, as it should do.

Integrating Eq. (3.30) for $s_0 \le s \le s_i$ gives

$$x_i = \frac{\cos(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\phi_\alpha} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\phi_\alpha} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) + \frac{(2i-1)\cos(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\frac{\pi}{2}} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\frac{\pi}{2}} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) - \frac{2\sin(\alpha)}{f^{\frac{1}{2}}}\left(\sin^2\left(\frac{\theta_\pi}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\right)^{\frac{1}{2}}. \tag{3.53}$$

As $d \to -2^+$, using Eqs. (3.42)–(3.48), Eq. (3.53) becomes

$$x_i = -\frac{\cos(\alpha)}{f^{\frac{1}{2}}}\left(\frac{(n-i)}{(2n-1)}\log_e\left(\frac{1+\sin(\frac{\alpha}{2})}{1-\sin(\frac{\alpha}{2})}\right)+\frac{(2i-1)}{(2n-1)\sin(\frac{\alpha}{2})}-2\sin\left(\frac{\alpha}{2}\right)\right) - \frac{2\sin(\alpha)}{f^{\frac{1}{2}}}\cos\left(\frac{\alpha}{2}\right). \tag{3.54}$$

Setting $i=n$ in Eq. (3.54) gives $x_n = d+1$, as it should do since $x_n = x(s_n) = x(1) = d+1$.

Integrating Eq. (3.31) for $s_0 \le s \le s_i$ gives

$$y_i = -\frac{\sin(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\phi_\alpha} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\phi_\alpha} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) - \frac{(2i-1)\sin(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\frac{\pi}{2}} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\frac{\pi}{2}} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) - \frac{2\cos(\alpha)}{f^{\frac{1}{2}}}\left(\sin^2\left(\frac{\theta_\pi}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\right)^{\frac{1}{2}}. \tag{3.55}$$

As $d \to -2^+$, using Eqs. (3.42)–3.48), Eq. (3.55) becomes

$$y_i = \frac{\sin(\alpha)}{f^{\frac{1}{2}}}\left(\frac{(n-i)}{(2n-1)}\log_e\left(\frac{1+\sin(\frac{\alpha}{2})}{1-\sin(\frac{\alpha}{2})}\right)+\frac{(2i-1)}{(2n-1)\sin(\frac{\alpha}{2})}-2\sin\left(\frac{\alpha}{2}\right)\right) - \frac{2\cos(\alpha)}{f^{\frac{1}{2}}}\cos\left(\frac{\alpha}{2}\right). \tag{3.56}$$

Setting $i = n$ in Eq. (3.56) gives $y_n = 0$, as it should do since $y_n = y(s_n) = y(1) = 0$ at the pinned end.

Integrating Eq. (3.36) from $s = 0$ to s where $s_{i-1} < s < s_i$ we find

$$s = 2(s_{i-1} - s_{i-2} + s_{i-3} - \cdots + (-1)^i s_1) + \frac{(-1)^i}{f^{\frac{1}{2}}}\left(\int_0^{\phi_\theta} d\phi \frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}} - \int_0^{\phi_\alpha} d\phi \frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right). \tag{3.57}$$

As $d \to -2^+$, using Eqs. (3.42), (3.43) and (3.46), Eq. (3.57) becomes

$$s = \delta s_{i-1} + \frac{(-1)^i}{2f^{\frac{1}{2}}}\left(\log_e\left(\frac{1+\sin(\frac{\theta}{2})}{1-\sin(\frac{\theta}{2})}\right) - \log_e\left(\frac{1+\sin(\frac{\alpha}{2})}{1-\sin(\frac{\alpha}{2})}\right)\right), \tag{3.58}$$

where we define

$$\delta s_{i-1} = 2(s_{i-1} - s_{i-2} + s_{i-3} - \cdots + (-1)^i s_1) \tag{3.59}$$

with $\delta s_0 = 0$. We can rearrange Eq. (3.58) to obtain θ as a function of s for $s_{i-1} < s < s_i$

$$\theta = 2\arcsin\left(\frac{\sin(\frac{\alpha}{2}) + (-1)^i\tanh(f^{\frac{1}{2}}(s-\delta s_{i-1}))}{1+(-1)^i\sin(\frac{\alpha}{2})\tanh(f^{\frac{1}{2}}(s-\delta s_{i-1}))}\right). \tag{3.60}$$

Again, for the first mode ($n = 1$), there is complete agreement between Eq. (3.60) and WANG [4].

Integrating Eq. (3.30) from $s = 0$ to s where $s_{i-1} < s < s_i$ we find

$$\begin{aligned} x = {} & 2(x_{i-1}-x_{i-2}+x_{i-3}-\cdots+(-1)^i x_1) \\ & + \frac{(-1)^i\cos(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\phi_\theta} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\phi_\theta} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) \\ & - \frac{(-1)^i\cos(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\phi_\alpha} d\phi\left(1-\sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\phi_\alpha} d\phi\frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) \\ & + \frac{(-1)^i 2\sin(\alpha)}{f^{\frac{1}{2}}}\left(\left(\sin^2\left(\frac{\theta_\pi}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\right)^{\frac{1}{2}} - \left(\sin^2\left(\frac{\theta_\pi}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)\right)^{\frac{1}{2}}\right). \end{aligned} \tag{3.61}$$

As $d \to -2^+$, using Eqs. (3.42), (3.43), (3.46) and (3.47), Eq. (3.61) gives x as a function of s for $s_{i-1} < s < s_i$

$$x = -\frac{(-1)^i 2\cos(\alpha)}{f^{\frac{1}{2}}}\left(\sin\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right) + \frac{(-1)^i 2\sin(\alpha)}{f^{\frac{1}{2}}}\left(\cos\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\theta}{2}\right)\right) - \cos(\alpha)(s - \delta s_{i-1}) + \delta x_{i-1}, \tag{3.62}$$

where we define

$$\delta x_{i-1} = 2(x_{i-1} - x_{i-2} + x_{i-3} - \cdots + (-1)^i x_1) \tag{3.63}$$

with $\delta x_0 = 0$.

Integrating Eq. (3.31) from $s = 0$ to s where $s_{i-1} < s < s_i$ we find

$$\begin{aligned} y = {} & 2(y_{i-1} - y_{i-2} + y_{i-3} - \cdots + (-1)^i y_1) \\ & - \frac{(-1)^i \sin(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\phi_\theta} d\phi \left(1 - \sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\phi_\theta} d\phi \frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) \\ & + \frac{(-1)^i \sin(\alpha)}{f^{\frac{1}{2}}}\left(2\int_0^{\phi_\alpha} d\phi \left(1 - \sin^2\left(\frac{\theta_\pi}{2}\right)\sin^2(\phi)\right)^{\frac{1}{2}} - \int_0^{\phi_\alpha} d\phi \frac{1}{(1-\sin^2(\frac{\theta_\pi}{2})\sin^2(\phi))^{\frac{1}{2}}}\right) \\ & + \frac{(-1)^i 2\cos(\alpha)}{f^{\frac{1}{2}}}\left(\left(\sin^2\left(\frac{\theta_\pi}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right)^{\frac{1}{2}} - \left(\sin^2\left(\frac{\theta_\pi}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)^{\frac{1}{2}}\right). \end{aligned} \tag{3.64}$$

As $d \to -2^+$, using Eqs. (3.42), (3.43), (3.46) and (3.47), Eq. (3.64) gives y as a function of s for $s_{i-1} < s < s_i$

$$y = \frac{(-1)^i 2\cos(\alpha)}{f^{\frac{1}{2}}}\left(\cos\left(\frac{\alpha}{2}\right) - \cos\left(\frac{\theta}{2}\right)\right) + \frac{(-1)^i 2\sin(\alpha)}{f^{\frac{1}{2}}}\left(\sin\left(\frac{\alpha}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right) + \sin(\alpha)(s - \delta s_{i-1}) + \delta y_{i-1}, \tag{3.65}$$

where we define

$$\delta y_{i-1} = 2(y_{i-1} - y_{i-2} + y_{i-3} - \cdots + (-1)^i y_1) \tag{3.66}$$

with $\delta y_0 = 0$.

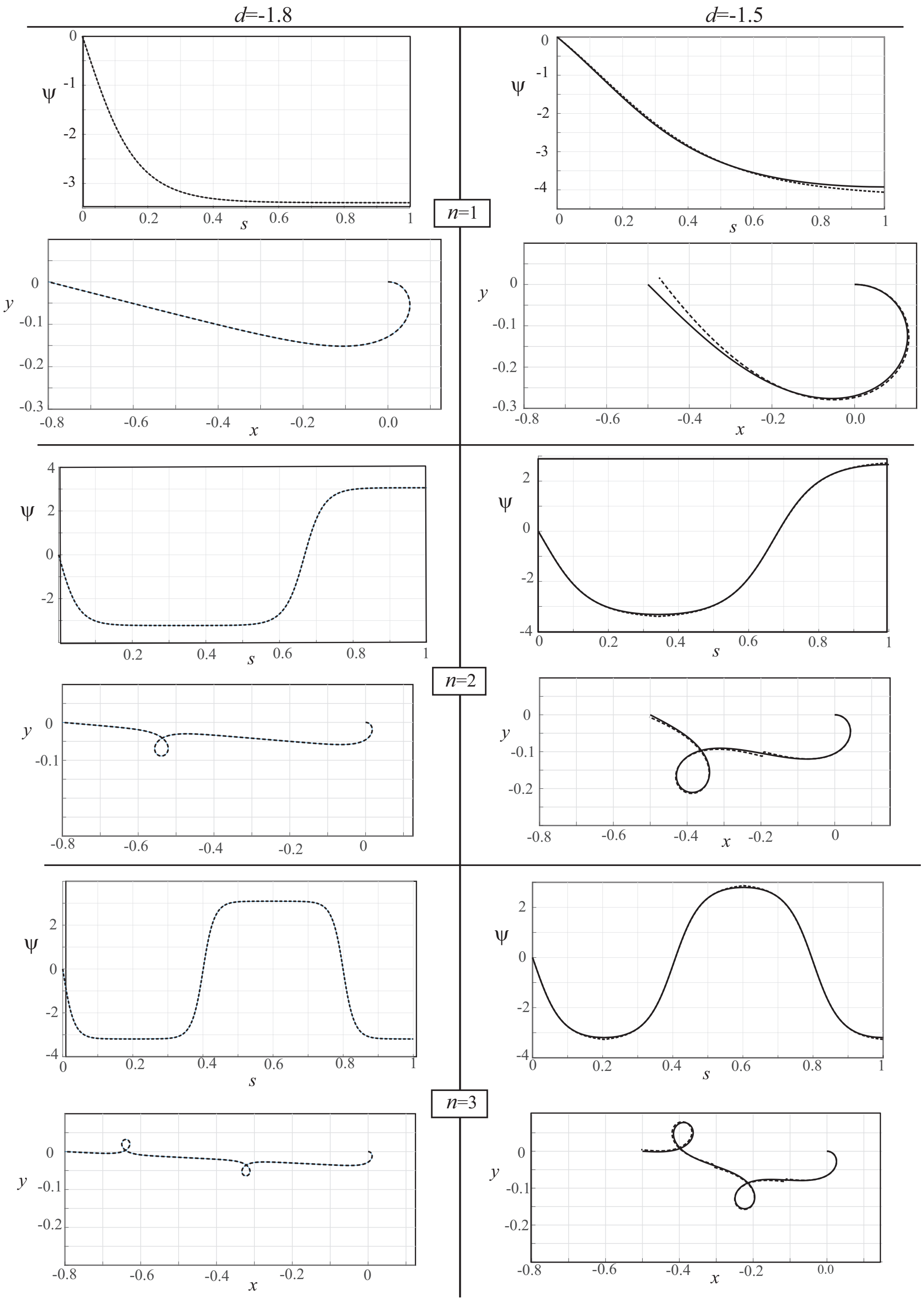

FIG. 3. Numerical solutions (solid lines) and asymptotic solutions as $d \to -2^{+}$ (dashed lines) for the first 3 modes ($n = 1, 2, 3$) and $d = -1.8$ and $d = -1.5$.

Equations (3.60), (3.62) and (3.65) are the asymptotic solution of the clamped-pinned elastica for the nth mode as $d \to -2^+$. In Fig. 3 we have plotted the numerical solutions (solid lines) and the asymptotic solutions (dashed lines) for the first 3 modes ($n = 1, 2, 3$) and $d = -1.8$ and $d = -1.5$. Using the numerical solution as the benchmark, it may be seen that the asymptotic solution is more accurate for $d = -1.8$ than it is for $d = -1.5$, particularly near the inflection points. Again, this is to be expected since the asymptotic solution is valid as $d \to -2^+$. As for the case $d \to 0^-$, we observe that the accuracy of the asymptotic solution depends very much on $|d|$, but seems to be independent of the mode n.

4. Horizontal force-displacement plot

Now let us consider the horizontal force-displacement plot, i.e. td plot. We are particularly interested in the td plot as $d \to 0^-$ and $d \to -2^+$.

As $d \to 0^-$, Eq. (3.24) at $s = 1$ gives

(4.1) $$x(1) = 1 + \frac{1}{4}t_0\varepsilon^2 + \mathrm{O}(\varepsilon^4).$$

Then by ignoring all terms of $\mathrm{O}(\varepsilon^4)$ we have

(4.2) $$d = x(1) - 1 = \frac{1}{4}t_0\varepsilon^2.$$

Using Eqs. (3.2) and (3.22) and ignoring all terms of $\mathrm{O}(\varepsilon^4)$ we find

(4.3) $$t = t_0 - \frac{1}{8}t_0(t_0 + 5)\varepsilon^2.$$

Combining these Eqs. (4.2) and (4.3) gives a nice simple linear relationship between t and d as follows

(4.4) $$t = t_0 - \frac{1}{2}(t_0 + 5)d.$$

This is the asymptotic form of t as a function of d as $d \to 0^-$, i.e. immediately after buckling.

Now consider $d \to -2^+$. We know from Eq. (3.26) that

(4.5) $$t = -f\cos(\alpha) = -f\left(1 - 2\sin^2\left(\frac{\alpha}{2}\right)\right).$$

Using Eqs. (3.49) and (3.50), Eq. (4.5) becomes

(4.6) $$t = -\frac{4(2n-1)^2}{(d+2)^2}\left(1 - \frac{(d+2)^2}{2(2n-1)^2(d+1)^2}\right).$$

This is the asymptotic form of t as a function of d as $d \to -2^+$, i.e. as $t \to -\infty$.

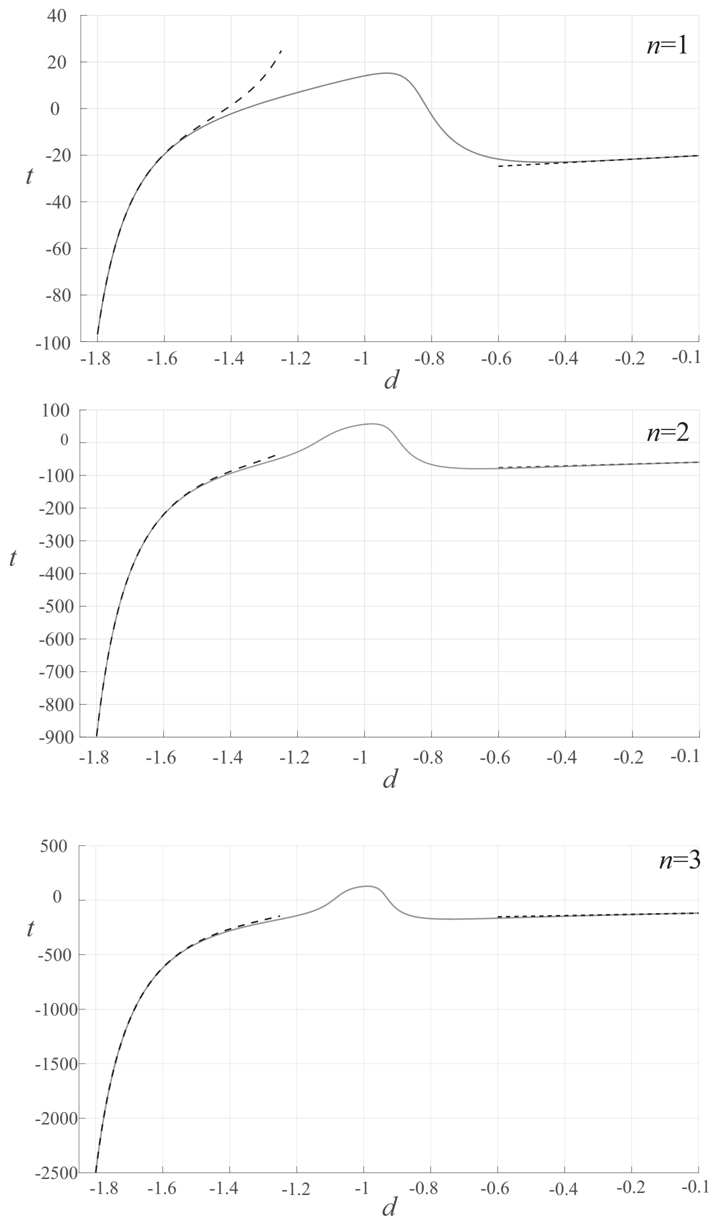

FIG. 4. td plots for the first 3 modes ($n = 1, 2, 3$). The td plots obtained numerically are shown as solid lines and the td plots obtained from asymptotic solutions as $d \to 0^-$ and $d \to -2^+$ are shown as dashed lines.

In Fig. 4 we have shown td plots for the first 3 modes ($n = 1, 2, 3$). The td plots obtained by numerically solving the odes for the clamped-pinned elastica are shown as solid lines and the td plots obtained from asymptotic solutions as

$d \to 0^-$ and $d \to -2^+$ given by Eqs. (4.4) and (4.6), respectively, are shown as dashed lines. For each of the modes shown, we observe that the asymptotic forms given by Eqs. (4.4) and (4.6) are good approximations to the td plot as $d \to 0^-$ and $d \to -2^+$, respectively.

5. Conclusions

There are few problems in the elastic rod theory which have closed-form solutions for post-buckling. Therefore, it is useful and important to develop asymptotic techniques for problems with known closed-form solutions in order to verify these asymptotic techniques by comparing known closed-form solutions directly with asymptotic solutions. Also, approximate asymptotic solutions are often simpler in form than complete closed-form solutions and therefore much easier to understand and plot, making the former very useful in their own right. Finally, for the clamped-pinned elastica, the singularity at $d = -2$ makes the numerical solution of the odes difficult without resorting to asymptotic methods. It is therefore very useful to have asymptotic solutions for the clamped-pinned elastica as $d \to -2^+$.

We have presented a complete asymptotic analysis of the clamped-pinned elastica for any mode $n = 1, 2, 3, \ldots$. For the first mode ($n = 1$), our results agree with those of Wang [4]. The higher modes ($n = 2, 3, \ldots$) are unstable, but may be realised in the presence of suitable constraints applied at certain inflection points that suppress lower modes. In Bigoni *et al.* [3] those constraints are referred to as "clamped-guided". A motive for studying higher modes is that it may explain phenomena like snake locomotion and be useful in designing snake-like robots, see [3] and references cited therein. Figures 2, 3 and 4 show good agreement between numerical solutions and asymptotic solutions near the bifurcation point at $d = 0$ and singularity at $d = -2$. Furthermore, we have also derived simple leading order relationships between t and d near the bifurcation point at $d = 0$ and singularity at $d = -2$ which could be useful for discussing td plots.

It would be interesting to apply the asymptotic analysis techniques investigated in our work to boundary value problems involving elastic rods where closed-form solutions are not generally available, e.g. rods with non-uniform taper and/or intrinsic curvature. Asymptotic solutions near the bifurcation point could be useful in understanding the immediate post-buckling behaviour of tapered columns in structural engineering [5] and asymptotic solutions for large forces could describe loop formation in cables [10]. In robotics rod-like structures are used as sensors to navigate environments [11]. Those rod-like sensors could have taper and/or intrinsic curvature. Asymptotic solutions for small deflections, i.e. immediately after buckling, could be very useful for modelling the robot's sensors.

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References

1. Y. Mikata, *Complete solution of elastica for a clamped-hinged beam, and its applications to a carbon nanotube*, Acta Mechanica, **190**, 1, 133–150, 2007.
2. P. Singh, V.G.A. Goss, *Critical Points of the Clamped-Pinned Elastica*, submitted, 2017.
3. D. Bigoni, F. Bosi, D. Misseroni, F. Dal Corso, G. Noselli, *New phenomena in nonlinear elastic structures: from tensile buckling to configurational forces*, [in:] D. Bigoni [ed.], *Extremely Deformable Structures*, Springer, Vienna, pp. 55–135, 2015.
4. C.Y. Wang, *Post-buckling of a clamped-simply supported elastica*, International Journal of Non-Linear Mechanics, **32**, 6, 1115–1122, 1997.
5. S.P. Timoshenko, J.M. Gere, *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.
6. W.T. Koiter, *On The Stability of Elastic Equilibrium*, (a translation of a doctoral dissertation, Techische Hooge School, Delft, 1945), Technical Report AFFDL-TR-70-25, Wright-Patterson Air Force Base, Ohio, 1970.
7. A.M.A. van der Heijden, *W.T. Koiter's Elastic Stability of Solids and Structures*, Cambridge University Press, Cambridge, 2012.
8. C.L. Dym, *Stability Theory and Its Application to Structural Mechanics*, Noordhoff International Publishing, The Netherlands, 1974.
9. I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Elsevier Academic Press, San Diego, 2007.
10. J. Coyne, *Analysis of the formation and elimination of loops in twisted cable*, IEEE Journal of Oceanic Engineering, **15**, 72–83, 1990.
11. M. Kaneko, N. Kanayama, T. Tsuji, *Active antenna for contact sensing*, IEEE Transactions on Robotics and Automation, **14**, 2, 278–291, 1998.

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